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A Variational Problem for a System of Magnetic Monopoles Joined by Abrikosov Vortices

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Abstract: An action functional, related to the Higgs model to field theory, depending on a complex scalar field and a $U(1)$ connection is defined. The complex scalar field is a section of a line bundle associated to a principal $U(1)$-bundle with base space $\mathbb{R}^3\backslash\{x_1,\ldots,x_n\}$. The points $x_1,\ldots,x_n$ are the positions of n magnetic monopoles of magnetic charges $m_1,\ldots,m_n$, with $\sum_{i=1}^n m_i = 0$. The existence of minimizers of the action functional is proven using direct methods of the calculus of variation. Regularity and decay properties of the minimizers are obtained. By constructing explicit comparison field configurations, we establish accurate upper and lower bounds for the action of the minimizers in a variety of special situations, e.g. $n = 2$ and $m_1 = -m_2$.

Contents

1. Introduction

The variational problem studied in this paper arises in the description of the quantum counterparts of classical vortex configurations in the $U(1)$-Higgs model in $2+1$ space-time dimensions. Using Euclidean functional integral methods to construct the Green functions of the $U(1)$-Higgs model one is led to study the classical variational problem described in the abstract: In attempting to calculate these Green functions within a semi-classical approximation one finds that the leading terms can be expressed in terms of the solutions to that variational problem [5]

In the following we first review some results about the $U(1)$-Higgs model and its classical vortex configurations Then we introduce the notions required to state the variational problem in a mathematically precise manner.

The Lagrange density of the $U(1)$-Higgs model in $2+1$ space-time dimensions is given by

$$\mathscr{L}(\Phi, A) = \frac{1}{2e^2} dA \wedge \star dA + \frac{1}{2} D_A \Phi \wedge \star \overline{D_A \Phi} - \star V(|\Phi|) \,,$$

where $A = (A_\mu dx^\mu)$ is a $U(1)$-connection (the gauge field) on a complex line bundle over three-dimensional Minkowski space, and Φ (the Higgs field) denotes a section of this bundle. The symbol $\star$ denotes the Hodge star operation on forms, d is the exterior derivative, and $D_A = d - iA$ denotes the covariant derivative. Finally, $V(|\Phi|)$ (the Higgs potential) is a polynomial in $|\Phi|$ bounded from below. It is given, for example, by

$$V(|\Phi|) = \frac{\lambda}{8}(|\Phi|^2 - \rho^2)^2 \,,$$

where λ (the coupling parameter) is a positive constant. Since we are using units in which the velocity of light and Planck's constant are unity, we are left with only one basic unit, that of length. The action $\int \mathscr{L}(\Phi, A)$, is dimensionless. Thus ρ^2, e^2 and λ have dimension $(\text{length})^{-1}$. Passing to dimensionless variables,

$$\frac{1}{\rho}\Phi \to \Phi, \qquad \frac{1}{\rho e} A_\mu \to A_\mu, \qquad (\rho e) x^\mu \to x^\mu, \; \frac{1}{e^2}\lambda \to \lambda \,,$$

and choosing suitable units, we end up with

$$\mathscr{L}(\Phi, A) = \frac{1}{2} dA \wedge \star dA + \frac{1}{2} D_A \Phi \wedge \star \overline{D_A \Phi} - \star V(|\Phi|) \tag{1 1}$$

and

$$V(|\Phi|) = \frac{\lambda}{8}(|\Phi|^2 - 1)^2 \,. \tag{1.2}$$

Time-independent configurations (Φ, A) with the property that the time-component of A vanishes are called *static* The energy of a static configuration is given by

$$E(\phi, a) = \int_{\mathbb{R}^2} \left[\frac{1}{2}|da|^2 + \frac{1}{2}|D_a \phi|^2 + V(|\phi|)\right] dx \,, \tag{1.3}$$

where $a_i(x) = A_i(t, x)$ $(i = 1, 2)$, $\phi(x) = \Phi(t, x)$ and $|D_a\phi|^2 := \star(D_a \phi \wedge \star \overline{D_a \phi})$. This energy functional has been studied in the mathematical literature, see e g. [7, 1] and references therein We summarize some key results.

Let a be a continuous connection and ϕ a C^1-section Assume that

$$\lim_{r \to \infty} \sup_{|x| = r} |1 - |\phi|| = 0 \,,$$

$$|x|^{1+\delta} |D_a \phi| \leqq \text{const} \,, \text{ for some } \delta > 0 \tag{1.4}$$

Then the configuration (ϕ, a) defines a homotopy class given by the winding number of the map

$$\left.\frac{\phi(x)}{|\phi(x)|}\right|_{|x|=r} : S^1 \to S^1 ,$$

provided r is large enough. This winding number, m, coincides with the vorticity of the gauge field a which is defined by

$$m = \lim_{r\to\infty} \frac{1}{2\pi} \int_{|x|\leqq r} da \in \mathbb{Z} . \tag{1.5}$$

It is expected (conjecture by Schoen and Yau'79) that the homotopy class of (ϕ, a) is well defined under the only assumption of *finite energy*. Thus the space of classical field configurations (ϕ, a) of finite energy very likely decomposes into infinitely many, topologically distinct classes labeled by their vorticity. We call a finite-energy configuration with properties (1.4) and (1.5) a *m-vortex configuration.*

Further, existence of finite-energy solutions to the variational equations derived from (1.3), i.e., static solutions to the classical Euler–Lagrange equations derived from (1.1), have been established in [7] and [1]. More precisely, let $m \in \mathbb{Z}$ and $\lambda > 0$. Then there exists a smooth, finite-energy critical point (ϕ, a) of the energy functional $E(\phi, a)$ defined in (1.3), with $\phi(0) = 0$. (ϕ, a) is rotationally symmetric in the sense that

$$\begin{aligned} a &= m\alpha(r) d\theta , \\ \phi &= \varphi(r) e^{im\theta} , \end{aligned} \tag{1.6}$$

where $\alpha(r)$, $\varphi(r) \in C^\infty(0,\infty)$, and (r,θ) are polar coordinates in $\mathbb{R}^2$. Moreover, φ and α are strictly increasing from 0 to 1 on $(0,\infty)$, and we have the following decay properties for $r \geqq 0$:

(D1) $$1 - |\phi|^2 \leqq Me^{-\mu r} ,$$

(D2) $$|da| \leqq Me^{-\mu r} ,$$

(D3) $$|D_a\phi| \leqq Me^{-\mu r} ,$$

where μ and M denote positive constants depending only on λ and m. For this reason we can think of these solutions as describing "extended classical" objects (*vortices*). In the Bogomol'nyi limit, $\lambda = 1$, the solutions satisfy first order, "self-dual" equations, and one has a rather detailed picture of all finite-energy solutions. For $\lambda \neq 1$, however, only the existence of rotationally symmetric solutions has been established. One has the heuristic picture that vortices (of vorticity $|m| = 1$) attract or repel one another, for $\lambda < 1$ or $\lambda > 1$, respectively.

An attempt to understand the quantum counterparts of these classical solutions (or, more generally, of the different homotopy classes of vortex configurations) within a functional integral formalism leads to the variational problem which is the subject of this paper. This is described in [5]. In order to state the problem in a mathematically precise manner, we require some definitions:

We choose a set of n distinct points in $\mathbb{R}^3$, $\underline{x} := \{x_1, \ldots, x_n\}$, and define

$$M_{\underline{x}} := \mathbb{R}^3 \setminus \{x_1, \ldots, x_n\}$$

equipped with the Euclidean metric. $U(1)$-bundles over $M_{\underline{x}}$ are classified by the second cohomology group $H^2(M_{\underline{x}}, \mathbb{Z})$,

$$H^2(M_{\underline{x}}, \mathbb{Z}) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (n \text{ summands}) . \tag{1.7}$$

Let $\underline{m} := \{m_1, \ldots, m_n\}$ be a set of n non-vanishing integers We then denote by $P_{\underline{x},\underline{m}}$ the $U(1)$-bundle over $M_{\underline{x}}$ specified by $\underline{m}$ according to (1.7). The n integers $m_1, \ldots, m_n$ can be interpreted as magnetic charges of n magnetic (Dirac) monopoles located at the points $x_1, \ldots, x_n$ of $\mathbb{R}^3$.

Let A_0 be a connection on $P_{\underline{x},\underline{m}}$ and $F_0 := dA_0$ its corresponding curvature or field strength Then every other connection $\tilde{A}$ on $P_{\underline{x},\underline{m}}$ is of the form

$$\tilde{A} = A_0 + A , \tag{1.8}$$

where A is a globally defined 1-form on $M_{\underline{x}}$. We choose an explicit reference connection A_0: On a ball containing the punctures $x_1, \ldots, x_n$ we choose A_0 to be given by A_0^h, such that its curvature, F_0^h, is harmonic on $M_{\underline{x}}$, i e., given by

$$F_0^h(x) = 2\pi \sum_{j=1}^{n} m_j \bigstar dE(x - x_j) , \tag{1.9}$$

where $E(x) = \frac{-1}{4\pi}|x|^{-1}$ is the fundamental solution of the Laplacian in three-dimensional Euclidean space Let $\{\mathcal{O}^{(j)}\}_{j=1}^{n+1}$ be the open cover of $M_{\underline{x}}$, as indicated in Fig 1 below Then A_0^h is locally given as a family of 1-forms $\{A_0^{h(j)}(x) : j = 1, \ldots, n+1, \text{ and } \operatorname{supp} A_0^{h(j)} \subset \mathcal{O}^{(j)}\}$, where

$$A_0^{h(j)}(x) := \sum_{i=1}^{n} \frac{m_i}{2|x - x_i|} \, \frac{(x^1 - x_i^1)dx^2 - (x^2 - x_i^2)dx^1}{(x^3 - x_i^3) + \eta_i^{(j)}|x - x_i|} ,$$

with

$$\eta_i^{(j)} := \begin{cases} -1 & \text{for } 1 \leqq i < j \leqq n+1 \\ 1 & \text{for } 1 \leqq j \leqq i \leqq n \end{cases} \tag{1.10}$$

Furthermore, on the intersections $\mathcal{O}^{(j)} \cap \mathcal{O}^{(j+1)}, j = 1, \ldots, n$, $A_0^{h(j)}$ and $A_0^{h(j+1)}$ are related by the transition conditions

$$A_0^{h(j+1)}(x) = A_0^{h(j)}(x) - d\psi^{(j)}(x), \quad \text{where } \psi^{(j)}(x) = m_j \arctan\left(\frac{x^2 - x_j^2}{x^1 - x_j^1}\right) .$$

This corresponds to transition functions $g_{j,j+1} : \mathcal{O}^{(j)} \cap \mathcal{O}^{(j+1)} \to U(1)$ given by

$$g_{j,j+1} := \exp(i\psi^{(j)}(x)), \quad \text{for } j = 1, \ldots, n .$$

Similarly, we have transition functions $g_{1,k} : \mathcal{O}^{(1)} \cap \mathcal{O}^{(k)} \to U(1)$, for $k = 3, \ldots, n+1$, given by

$$g_{1,k} := \exp\left(i \sum_{j=1}^{k-1} \psi^{(j)}(x)\right) .$$

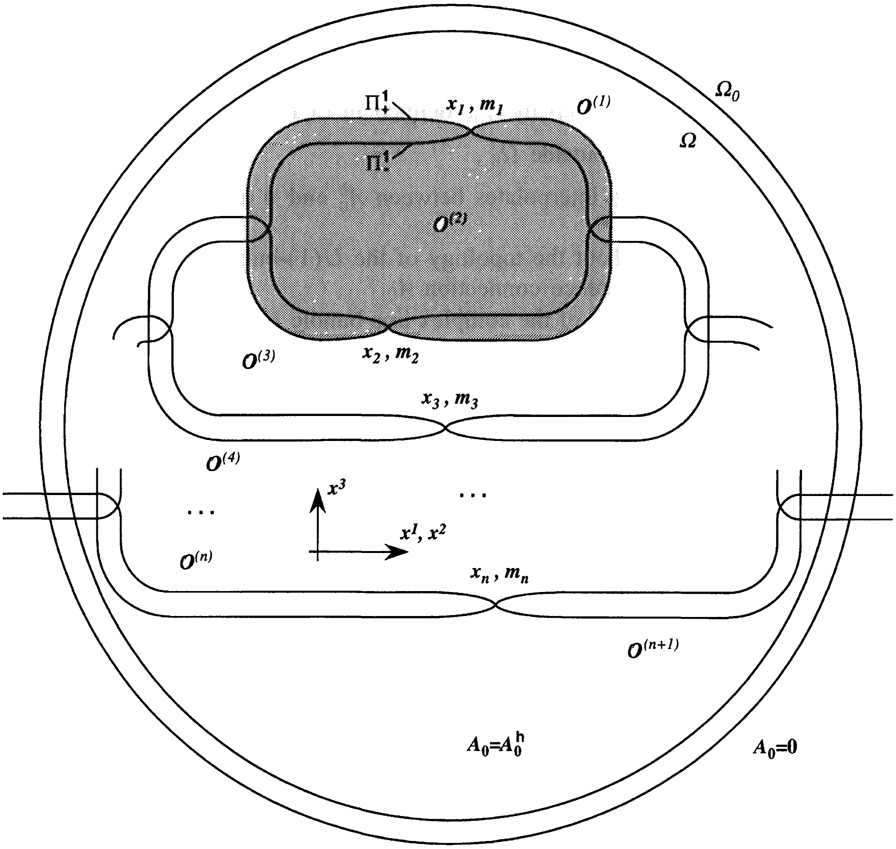

Fig. 1. (Choice of the open cover $\{\mathcal{O}^{(j)}\}_{j=1}^{n+1}$ of $M_{\underline{x}}$). Let $\varepsilon > 0$ be small and denote by (r_i, Θ_i, z_i) cylindrical coordinates centered at the point $x_i = (x_i^1, x_i^2, x_i^3)$, $i = 1, \ldots, n$, with z_i-axis parallel to the x^3-axis. In the neighbourhood of each puncture x_i of $M_{\underline{x}} = \mathbb{R}^3 \backslash \{x_1, \quad , x_n\}$ we take two (smooth) surfaces $\Pi_\pm^i$ such that, for $r_i < \varepsilon$, $\Pi_\pm^i = \{r_i = \pm z_i\}$, and, for $r_i > 3\varepsilon$, $\Pi_\pm^i = \{x^3 = x_i^3 \pm 2\varepsilon\}$. Thus we obtain a pair of surfaces meeting each other only in the point x_i. In the exterior of a sphere containing x_1 and x_2, but not x_i, $i \geqq 3$, we deform the pairs of surfaces associated with x_1 and x_2 in an axially symmetric way, as indicated in the figure Thus we obtain a closed surface and a new pair of surfaces, which in turn is combined in a similar manner with the pair of surfaces associated with x_3, and so on $\mathcal{O}^{(2)}$, $\quad ., \mathcal{O}^{(n)}$ denote the domains bounded by the closed surfaces constructed in the process above $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(n+1)}$ denote the remaining unbounded domains, which overlap each other outside some sphere containing all punctures $x_1, \quad , x_n$

One easily checks that on the intersections $\mathcal{O}^{(1)} \cap \mathcal{O}^{(j)} \cap \mathcal{O}^{(j+1)}, j = 2, \ldots, n$, the co-cycle conditions $g_{1,j}(x) g_{j,j+1}(x) = g_{1,j+1}(x)$ are satisfied. Henceforth we require *neutrality* in the sense that

$$\sum_{i=1}^{n} m_i = 0 \,. \tag{1.11}$$

Let $\Omega \subset \mathbb{R}^3$ be a closed ball containing the punctures $x_1, \ldots, x_n$ in its interior, and let $\Omega_0 := \{x \in \mathbb{R}^3 \quad \mathrm{dist}(x, \Omega) \leqq 1\}$ Then we can choose the reference connection such that

$$\begin{cases} \text{on } \Omega,\, A_0 \text{ is given by } A_0^h, \text{ defined in (1.10)}, \\ A_0 \text{ vanishes outside } \Omega_0\,, \\ A_0 \text{ smoothly interpolates between } A_0^h \text{ and } 0 \text{ on } \Omega_0 \backslash \Omega\,. \end{cases} \tag{1 12}$$

Note that all information about the topology of the $U(1)$-bundle $P_{\underline{x},\underline{m}}$ is encoded in the curvature F_0 of the reference connection A_0.

Next, we consider sections of the complex line bundle $E_{\underline{x},\underline{m}}$, the bundle associated to $P_{\underline{x},\underline{m}}$ With respect to the open cover $\{\mathcal{O}^{(j)}\}_{j=1}^{n+1}$, a section Φ of $E_{\underline{x},\underline{m}}$ is given by a family of complex-valued functions $\{\Phi^{(j)} . \mathcal{O}^{(j)} \to \mathbb{C} | j = 1, \ldots, n+1\}$. On all non-empty intersections $\mathcal{O}^{(j)} \cap \mathcal{O}^{(k)}$, $1 \leqq j,\ k \leqq n+1$, the transition conditions

$$\Phi^{(j)}(x) = g_{j,k}(x)\Phi^{(k)}(x) \tag{1.13}$$

are satisfied. Finally, for a fixed connection $\tilde{A}$ on $P_{\underline{x},\underline{m}}$, the covariant derivative on $E_{\underline{x},\underline{m}}$ restricted to $\mathcal{O}^{(j)}$ reads

$$\begin{aligned} D_{\tilde{A}}\Phi|_{\mathcal{O}^{(j)}}(x) &= \sum_{i=1}^{3} (\nabla_{\tilde{A}}\Phi|_{\mathcal{O}^{(j)}})_i(x)dx^i \\ &= d\Phi^{(j)}(x) - i\tilde{A}^{(j)}(x)\Phi^{(j)}(x) \end{aligned}$$

On all non-empty intersections transition conditions analogous to (1 13) hold. As a consequence, $|\Phi|$ and $|D_{\tilde{A}}\Phi|$ are globally defined, non-negative functions on $M_{\underline{x}}$

In the following we identify forms and vectors by the canonical isomorphism provided by the Euclidean metric, i e., if $\alpha(x) = \sum_{i=1}^3 \alpha_i(x)dx^i$ is a one-form and $\beta = \frac{1}{2}\sum_{i,j=1}^3 \beta_{ij}(x)dx^i \wedge dx^j$ is a two-form we identify $\alpha(x)$ with the vector $(\alpha_1(x), \alpha_2(x), \alpha_3(x))$ and $\beta(x)$ with the axial vector $(\beta_{23}(x), \beta_{31}(x), \beta_{12}(x))$. Furthermore, dx stands for the Lebesgue volume element on $\mathbb{R}^3$ or $\mathbb{R}^2$, respectively.

In the following we consider the renormalized action functional

$$\begin{aligned} \tilde{S}(\Phi, A) &= \pi \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|} + \frac{1}{2} \int_{\mathbb{R}^3 \backslash \Omega} [|F_0|^2(x) - |F_0^h|^2(x)]\, dx \\ &\quad + \int_{M_{\underline{x}}} \left[\frac{1}{2}|\mathrm{curl}\, A|^2(x) + (\mathrm{curl}\, A \cdot F_0)(x) \right. \\ &\quad \left. + \frac{1}{2}|\nabla_{A_0 + A}\Phi|^2(x) + V(|\Phi|)(x)\right] dx\,, \end{aligned} \tag{1 14}$$

which arises in the description of the quantum counterparts of classical vortex configurations This action functional is well defined (Lemma 1 1 (ii)) on a space,

$\tilde{\mathscr{F}}$, of pairs, (Φ, A), where Φ is a Sobolev section of $E_{\underline{x},\underline{m}}$ and $\tilde{A} = A_0 + A$ are the components of a connection of $P_{\underline{x},\underline{m}}$. The space $\tilde{\mathscr{F}}$ is defined as follows:

$$\begin{aligned} \tilde{\mathscr{F}} := \{(\Phi, A),\ & \Phi \text{ a section of } E_{\underline{x},\underline{m}} \text{ with the properties (1.13)},\\ & A_0 + A \text{ a connection on } P_{\underline{x},\underline{m}} \text{ with } A_0 \text{ defined in (1.12)}:\\ & (|\Phi|, A) \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^+ \times \mathbb{R}^3), \operatorname{curl} A \in L^2(\mathbb{R}^3; \mathbb{R}^3),\\ & |\nabla_{A_0+A}\Phi| \in L^2(\mathbb{R}^3; \mathbb{R}^+), V(|\Phi|) \in L^1(\mathbb{R}^3; \mathbb{R}^+)\}. \end{aligned} \tag{1.15}$$

We remark that the action functional (1.14) defined on the space $\tilde{\mathscr{F}}$ is invariant under gauge transformations

$$A \to A + \nabla\psi,\ \Phi \to \Phi e^{i\psi},$$

with $\psi \in H^{2,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R})$.

The subject of this paper is to minimize the action functional $\tilde{S}(\Phi, A)$ on $\tilde{\mathscr{F}}$ and prove regularity and other properties of the minimizers. Unfortunately, this variational problem is not well posed, since the term proportional to $|\operatorname{curl} A|^2$ in $\tilde{S}(\Phi, A)$ is not coercive. This difficulty can be avoided by choosing a fixed gauge.

Lemma 1.1. (i) *Assume that* $A \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ *and* $\operatorname{curl} A \in L^2(\mathbb{R}^3; \mathbb{R}^3)$. *Then there exists a gauge transformation* $A \to \hat{A} := A + \nabla\psi$ *with* $\psi \in H^{2,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R})$, *such that* $\nabla \cdot \hat{A} = 0$ *a.e., and the following identity holds*:

$$\int_{\mathbb{R}^3} |\operatorname{curl} \hat{A}|^2\, dx = \int_{\mathbb{R}^3} |\nabla\hat{A}|^2\, dx\,, \tag{1.16}$$

where $|\nabla\hat{A}|^2 = \sum_{i,j=1}^3 |\partial_i \hat{A}_j|^2$.

(ii) *Let* A_0 *be the reference connection defined in* (1.12). *Assume that* $\hat{A} \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$. *Then the following identity holds*:

$$\int_{M_{\underline{x}}} \operatorname{curl} \hat{A} \cdot F_0 dx = \int_{\Omega_0 \setminus \Omega} \hat{A} \cdot \operatorname{curl} F_0 dx\,. \tag{1.17}$$

The proof of Lemma 1.1 will be given in the Appendix.

Now, for any $(\Phi, A) \in \tilde{\mathscr{F}}$, we may assume that A satisfies the Coulomb gauge condition $\nabla \cdot A = 0$ and (1.16); as guaranteed by Lemma 1.1, (i). Then the variational problem above reduces to a variational problem with the following coercive action functional

$$\tilde{S}(\Phi, A) = \pi \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|} + \frac{1}{2} \int_{\mathbb{R}^3 \setminus \Omega} [|F_0|^2(x) - |F_0^h|^2(x)]\, dx + S(\Phi, A)\,,$$

where

$$S(\Phi, A) := \int_{M_{\underline{x}}} \left[\frac{1}{2}|\nabla A|^2(x) + A \cdot \operatorname{curl} F_0(x) + \frac{1}{2}|\nabla_{A_0+A}\Phi|^2(x) + V(|\Phi|)(x)\right] dx\,. \tag{1.18}$$

The functional $S(\Phi,A)$ is invariant under gauge transformations

$$A \to A + \nabla\psi, \qquad \Phi \to \Phi e^{i\psi},$$

where $\nabla\psi$ is constant For the second term on the r.h s. in (1.18) this follows by Lemma 1.1, (ii) Thus we may impose the condition $\int_K A dx = 0$, where $K \subset \Omega_0$ is a compact set with Lebesgue measure $|K| > 0$. This additional gauge condition is important in our analysis as it permits us to apply the Poincaré inequality. Then we enlarge the space of admissible sections and 1-forms by setting

$$\mathscr{F} := \Big\{(\Phi,A),\ \Phi \text{ a section of } E_{\underline{x},\underline{m}} \text{ with the properties (1 13)},$$
$$A_0 + A \text{ a connection on } P_{\underline{x},\underline{m}} \text{ with } A_0 \text{ defined in (1.12)}.$$
$$(|\Phi|,A) \in H^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^+ \times \mathbb{R}^3), |\nabla_{A_0+A}\Phi| \in L^2_{\text{loc}}(\mathbb{R}^3),$$
$$\int_K A dx = 0 \text{ for the compact set } K \subset \Omega_0, \text{ with } |K| > 0\Big\}. \tag{1 19}$$

A minimizer for the functional $S(\Phi,A)$ on the enlarged space $\mathscr{F}$ turns out to be a minimizer for $\tilde{S}(\Phi,A)$ on $\tilde{\mathscr{F}}$

We briefly summarize our main results· In Sect 2 we prove the existence of minimizers, $(\underline{\Phi},\underline{A})$, for $S(\Phi,A)$ on $\mathscr{F}$ under very general hypotheses concerning the potential V (Theorem 2 1) In Sect 3, we study regularity and decay properties of the minimizers: The section $\underline{\Phi}(x)$ and the form $\underline{A}(x)$ are smooth on $\mathbb{R}^3\setminus\{x_1,\dots,x_n\}$ (Theorem 3 1) The function $|\underline{\Phi}|$ is bounded above by 1, and, in the neighbourhood of any puncture x_i, $\underline{A}$ is Hölder continuous (Lemma 3 1), and $\underline{\Phi}$ possesses a Hölder continuous extension to x_i (Theorem 3.2) with a zero at x_i The functions $1 - |\underline{\Phi}|^2$, $|\text{curl}\,(A_0 + \underline{A})|$ and $|\bar{\underline{\Phi}}\nabla_{A_0+\underline{A}}\underline{\Phi}|$ decay to zero exponentially fast, as $|x|$ tends to infinity (Theorem 3 3). In the last section, we derive accurate upper (Theorems 4.1, 4 2) and lower bounds (Theorem 4 3) on the action $\tilde{S}$ of the minimizers in the special situation $M_{\underline{x}} = \mathbb{R}^3\setminus\{x_1,x_2\}$, i.e., for two magnetic monopoles of opposite magnetic charges The action essentially grows linearly with the distance $|x_1 - x_2|$ and in the monopole charge.

Independently, T. Rivière [10] worked on the same minimization problem He gives a direct and short proof of the existence of minimizers and then focusses on an asymptotic analysis of the minimizers when the coupling parameter λ tends to infinity.

2. Existence Results

Our main result in this section is the existence of minimizers for S on $\mathscr{F}$ (where S and $\mathscr{F}$ are defined in (1.18), (1.19)), under very general hypotheses concerning the potential V

Theorem 2.1. *Suppose $V : \mathbb{R} \to \mathbb{R}$ is continuous, non-negative and coercive in the sense that*

$$V(x) \geqq C^{-1}|x|^2 - C, \quad \textit{for some constant } C > 0 \textit{ and all } x \in \mathbb{R} \tag{2.1}$$

Also suppose that there exists an element $(\Phi',A') \in \mathscr{F}$ such that $s' := S(\Phi',A') < \infty$ Then there exists an element $(\underline{\Phi},\underline{A}) \in \mathscr{F}$ which minimizes S on $\mathscr{F}$

Remark. For a choice of $M_{\underline{x}}$, with $\underline{x} = \{x_1, x_2\}$ and $\underline{m} = \{-m, m\}$, the existence of $(\Phi', A') \in \mathscr{F}$ follows from Theorems 4.1, 4.2 and Lemma 1.1. The proofs can easily be generalized to the general situation.

The proof of Theorem 2.1 is based on the techniques presented in [5]. We remark that, if the bundles $P_{\underline{x},\underline{m}}$ and $E_{\underline{x},\underline{m}}$ are trivial (i.e. $A_0 \equiv 0$) and if the term $\int_{M_{\underline{x}}} A \cdot \operatorname{curl} F_0 dx$ in (1.18) is replaced by $-\int_{\mathbb{R}^3} \operatorname{curl} A \cdot H_{\text{ext}} dx$, where $H_{\text{ext}} \in L^2 \cap H^{1,2}_{\text{loc}}$, this minimization problem has already been solved [14]. In our proof we fill in the details how to handle the difficulties arising from the fact that $A_0^{(j)}(x) = O(|x - x_i|^{-1})$, for $x \to x_i$, and that Φ is only a local section.

Proof. Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be a sequence of compact, smooth domains (balls, for instance) exhausting $\mathbb{R}^3$, with $K \subset \Omega_0 \subset \Omega_k$. The proof comprises three steps: (i) First we investigate the coercivity properties of S restricted to the compact sets Ω_k. (ii) Then, we study the convergence behaviour of a minimizing sequence in $\mathscr{F}$ and extract a subsequence, which converges, in a sufficiently strong sense, to an element $(\underline{\Phi}, \underline{A}) \in \mathscr{F}$. (iii) By weak lower semi-continuity of S and a monotone convergence argument, we show that $(\underline{\Phi}, \underline{A})$ minimizes S on $\mathscr{F}$.

Step (i). Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be given, and denote by $S(\Phi, A; \Omega_k)$ the functional in (1.18), but with integration over Ω_k instead of $M_{\underline{x}}$, then

$$S(\Phi, A; \Omega_k) \geqq c\left[\|\nabla A\|^2_{L^2(\Omega_k)} - \left|\int_{\Omega_0 \backslash \Omega} A \cdot \operatorname{curl} F_0\, dx\right| + \|\nabla_{A_0 + A} \Phi\|^2_{L^2(\Omega_k)} + \int_{\Omega_k} V(|\Phi|)\, dx\right].$$

For the second term we have used that, due to our choice of A_0, $\operatorname{supp} \operatorname{curl} F_0 \subset \Omega_0 \backslash \Omega$, and we can further estimate it by

$$\begin{aligned} \left|\int_{\Omega_0 \backslash \Omega} A \cdot \operatorname{curl} F_0\, dx\right| &\leqq c' \|A\|_{L^2(\Omega_0 \backslash \Omega)} \leqq C(K; \Omega_0) \|\nabla A\|_{L^2(\Omega_0)} \\ &\leqq C(K; \Omega_0) \|\nabla A\|_{L^2(\Omega_k)}, \end{aligned} \tag{2.2}$$

where we have used Poincaré's inequality and the hypothesis that $\int_K A dx = 0$ in $\mathscr{F}$. With the coercivity hypothesis (2.1) for the potential V we can estimate the last term and get

$$\begin{aligned} S(\Phi, A; \Omega_k) &\geqq c[\|\nabla A\|^2_{L^2} - C(K; \Omega_0) \|\nabla A\|_{L^2} + \|\nabla_{A_0 + A} \Phi\|^2_{L^2} + C^{-1} \|\Phi\|^2_{L^2} - C|\Omega_k|] \\ &\geqq c\left[\frac{1}{2} \|\nabla A\|^2_{L^2(\Omega_k)} - \frac{1}{2} C(K; \Omega_0)^2 + \|\nabla |\Phi|\|^2_{L^2(\Omega_k)} \right. \\ &\quad \left. + C^{-1} \|\Phi\|^2_{L^2(\Omega_k)} - C|\Omega_k|\right], \end{aligned}$$

where we have used Kato's inequality for the third term. Splitting the first term and applying Poincaré's inequality again, we obtain:

$$S(\Phi, A; \Omega_k) \geqq c_1(K; \Omega_k) \|A\|^2_{H^{1,2}(\Omega_k)} + c_2 \|\, |\Phi|\, \|^2_{H^{1,2}(\Omega_k)} - c_3(K; \Omega_k), \tag{2.3}$$

where c_i, $i = 1, 2, 3$, are positive constants.

Step (ii) We have that $\inf_{(\Phi,A)\in\mathscr{F}} S(\Phi,A) > -\infty$, due to inequality (2.2) and the non-negativity of the potential V. Let $((\Phi_m,A_m))$ be a minimizing sequence for S in $\mathscr{F}$, i e, $S(\Phi_m,A_m) \to \inf_{(\Phi,A)\in\mathscr{F}} S(\Phi,A)$, as $m\to\infty$. We may assume that $S(\Phi_m,A_m) \leqq s' < \infty$, uniformly in $m\in\mathbb{N}$. By (2.3) and since $\operatorname{supp}\operatorname{curl}F_0 \subset \Omega_0\backslash\Omega \subset \Omega_k, \forall k\in\mathbb{N}$, we have that

$$\begin{aligned} c_1(K,\Omega_k)\|A_m\|^2_{H^{1,2}(\Omega_k)} + c_2\|\,|\Phi_m|\,\|^2_{H^{1,2}(\Omega_k)} &\leqq S(\Phi_m,A_m) + c_3(K,\Omega_k) \\ &\leqq s' + c_3(K;\Omega_k) < \infty\,, \end{aligned}$$

uniformly in $m\in\mathbb{N}$. Hence (A_m) is bounded in $H^{1,2}(\Omega_k,\mathbb{R}^3)$, and $(|\Phi_m|)$ in $H^{1,2}(\Omega_k;\mathbb{R}^+)$, for any k. Since $H^{1,2}$-spaces are reflexive, we may assume – if necessary extracting a diagonal sequence – that for any k

$$A_m \rightharpoonup \underline{A} \text{ weakly in } H^{1,2}(\Omega_k;\mathbb{R}^3) \quad\text{and}\quad |\Phi_m| \rightharpoonup \underline{\phi} \text{ weakly in } H^{1,2}(\Omega_k;\mathbb{R}^+)\,, \tag{2.4}$$

hence $(\underline{\phi},\underline{A}) \in H^{1,2}_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^+\times\mathbb{R}^3)$. Using Rellich's theorem, we conclude that $A_m \to \underline{A}$ in $L^2(\Omega_k;\mathbb{R}^3)$ and $|\Phi_m| \to \underline{\phi}$ in $L^2(\Omega_k,\mathbb{R}^+)$, for all $k\in\mathbb{N}$. Furthermore, we may assume – if necessary again by extracting a diagonal sequence – that

$$|\Phi_m| \to \underline{\phi} \text{ pointwise, a.e. on } \Omega_k, \text{ for all } k\,. \tag{2 5}$$

Next, we show that we may assume the sequence of sections (Φ_m) to be such that $\Phi_m^{(j)} \to \underline{\Phi}^{(j)}$ in $L^2(\mathscr{O}_k^{(j)},\mathbb{C})$, for all $j=1,\dots,n+1$ and $k\in\mathbb{N}$, where $\mathscr{O}_k^{(j)} := \mathscr{O}^{(j)}\cap\Omega_k$, and that $\underline{\Phi}$ is also a section of the complex line bundle $E_{\underline{x},\underline{m}}$

Let j,k be an arbitrary but fixed pair. (We note, that our construction of the cover $\{\mathscr{O}^{(j)}\}$ in Fig. 1, is such that the bounded set $\mathscr{O}_k^{(j)}$ has Lipschitz boundary, for every j and k) Then we have that

(a) $(\Phi_m^{(j)})$ is bounded in $L^p(\mathscr{O}_k^{(j)};\mathbb{C}), 1\leqq p\leqq 6$, because of the boundedness of $(|\Phi_m|)$ in $H^{1,2}(\Omega_k;\mathbb{R}^+)$ and the Sobolev imbedding on $\mathscr{O}_k^{(j)}$. By extracting a subsequence, if necessary, we may assume that

$$\Phi_m^{(j)} \rightharpoonup \underline{\Phi}^{(j)} \text{ weakly in } L^p(\mathscr{O}_k^{(j)};\mathbb{C}), \quad 1\leqq p\leqq 6\,. \tag{2 6}$$

(b) Since $A_0^{(j)} \in L^2(\mathscr{O}_k^{(j)})$, Hölder's inequality and (2.6) yield that

$$\|A_0^{(j)}\Phi_m^{(j)}\|_{L^q(\mathscr{O}_k^{(j)},\mathbb{C}^3)} \text{ is uniformly bounded in } m\in\mathbb{N}, \text{ for } 1\leqq q\leqq\frac{3}{2}\,.$$

(c) We claim that $A_m\Phi_m^{(j)} \rightharpoonup \underline{A}\,\underline{\Phi}^{(j)}$ weakly in $L^q(\mathscr{O}_k^{(j)};\mathbb{C}^3)$, for $1\leqq q\leqq\frac{3}{2}$ This implies that

$$\|A_m\Phi_m^{(j)}\|_{L^q(\mathscr{O}_k^{(j)};\mathbb{C}^3)} \text{ is uniformly bounded in } m\in\mathbb{N}, \text{ for } 1\leqq q\leqq\frac{3}{2}\,.$$

Indeed, for $\tau \in L^{q'}(\mathcal{O}_k^{(j)}; \mathbb{C}^3), q' := \frac{q}{q-1} \geqq 3$, we derive the bound

$$\begin{aligned}
&\left| \int_{\mathcal{O}_k^{(j)}} (A_m \Phi_m^{(j)} - \underline{A}\, \underline{\Phi}^{(j)}) \cdot \tau \, dx \right| \\
&\leqq \left| \int_{\mathcal{O}_k^{(j)}} (A_m - \underline{A}) \Phi_m^{(j)} \cdot \tau \, dx \right| + \left| \int_{\mathcal{O}_k^{(j)}} \underline{A} (\Phi_m^{(j)} - \underline{\Phi}^{(j)}) \cdot \tau \, dx \right| \\
&\leqq \|A_m - \underline{A}\|_{L^r(\mathcal{O}_k^{(j)}; \mathbb{R}^3)} \|\Phi_m^{(j)}\|_{L^p(\mathcal{O}_k^{(j)}; \mathbb{C})} \|\tau\|_{L^{q'}(\mathcal{O}_k^{(j)}; \mathbb{C}^3)} + \left| \int_{\mathcal{O}_k^{(j)}} (\underline{A} \cdot \tau)(\Phi_m^{(j)} - \underline{\Phi}^{(j)}) \, dx \right| ,
\end{aligned}$$

for $r \in [1,6)$, $p \in [1,6]$ and $\underline{A} \cdot \tau \in L^2(\mathcal{O}_k^{(j)}; \mathbb{C})$. By the compact imbedding $H^{1,2}(\mathcal{O}_k^{(j)}) \hookrightarrow L^r(\mathcal{O}_k^{(j)})$ of Rellich–Kondrachov, we conclude that $\|A_m - \underline{A}\|_{L^r(\mathcal{O}_k^{(j)}; \mathbb{R}^3)} \to 0$. Using (2.6) the claim follows.

(d) Due to (2.2) and the non-negativity of the potential V, we conclude that the sequence $(\nabla_{A_0^{(j)} + A_m} \Phi_m^{(j)})$ is bounded in $L^2(\mathcal{O}_k^{(j)}; \mathbb{C}^3)$, and again – if necessary passing to a subsequence we have that

$$\chi_m^{(j)} := \nabla_{A_0^{(j)} + A_m} \Phi_m^{(j)} \rightharpoonup \underline{\chi}^{(j)} \quad \text{weakly in } L^2(\mathcal{O}_k^{(j)}; \mathbb{C}^3) . \tag{2.7}$$

Next, using (2.6) and $A_m \to \underline{A}$ in $L^2(\Omega_k; \mathbb{R}^3)$ we find that, for any $\tau \in C_0^\infty(\mathcal{O}_k^{(j)}; \mathbb{C}^3)$,

$$\begin{aligned}
\int_{\mathcal{O}_k^{(j)}} \chi_m^{(j)} \cdot \tau dx &= \int_{\mathcal{O}_k^{(j)}} [(-\nabla \cdot \tau) \Phi_m^{(j)} - i A_0^{(j)} \cdot \tau \Phi_m^{(j)} - i A_m \cdot \tau \Phi_m^{(j)}] \, dx \\
&\underset{(m \to \infty)}{\longrightarrow} \int_{\mathcal{O}_k^{(j)}} \nabla_{A_0^{(j)} + \underline{A}} \underline{\Phi}^{(j)} \cdot \tau \, dx ,
\end{aligned}$$

where we have used that $(-\nabla \cdot \tau)$ and $A_0^{(j)} \cdot \tau$ are in $L^2(\mathcal{O}_k^{(j)}; \mathbb{C})$. Since $C_0^\infty(\mathcal{O}_k^{(j)}; \mathbb{C}^3)$ is dense in $L^2(\mathcal{O}_k^{(j)}; \mathbb{C}^3)$, we conclude, by the uniqueness of weak limits, that

$$\underline{\chi}^{(j)} = \nabla_{A_0^{(j)} + \underline{A}} \underline{\Phi}^{(j)}, \quad \text{a.e. on } \mathcal{O}_k^{(j)} .$$

Note that, because of the imbedding $L^2(\mathcal{O}_k^{(j)}) \hookrightarrow L^p(\mathcal{O}_k^{(j)})$, for $1 \leqq p \leqq 2$, the sequence $(\chi_m^{(j)})$ is bounded in $L^p(\mathcal{O}_k^{(j)}; \mathbb{C}^3)$. Statements (b), (c) and (d) together imply that, for $1 \leqq q \leqq \frac{3}{2}$,

$$\|\nabla \Phi_m^{(j)}\|_{L^q(\mathcal{O}_k^{(j)}; \mathbb{C}^3)} \leqq \|\chi_m^{(j)}\|_{L^q(\mathcal{O}_k^{(j)}; \mathbb{C}^3)} + \|A_0^{(j)} \Phi_m^{(j)}\|_{L^q(\mathcal{O}_k^{(j)}; \mathbb{C}^3)} + \|A_m \Phi_m^{(j)}\|_{L^q(\mathcal{O}_k^{(j)}; \mathbb{C}^3)}$$

is uniformly bounded in $m \in \mathbb{N}$. Together with (2.4), this shows that the sequence $(\Phi_m^{(j)})$ is bounded in $H^{1,q}(\mathcal{O}_k^{(j)}; \mathbb{C})$, for $1 \leqq q \leqq \frac{3}{2}$, and, by appealing to Rellich–Kondrachov imbedding, we get that $(\Phi_m^{(j)})$ is relatively compact in $L^r(\mathcal{O}_k^{(j)}; \mathbb{C})$, for $1 \leqq r < 3$. In particular, $(\Phi_m^{(j)})$ is relatively compact in $L^2(\mathcal{O}_k^{(j)}; \mathbb{C})$, and we conclude (passing to a subsequence and comparing with (2.6)) that

$$\Phi_m^{(j)} \to \underline{\Phi}^{(j)} \in L^2(\mathcal{O}_k^{(j)}; \mathbb{C}) . \tag{2.8}$$

The chain of arguments from (a) to (d) is valid for any $\mathscr{C}_k^{(j)}, j = 1, \ldots, n+1$, and $k \in \mathbb{N}$ Thus, by applying a double-diagonal sequence process in $j = 1, \ldots, n+1$ and $k = 1, 2, \ldots$, we may assume that (Φ_m) actually possesses properties (2.5), (2.7) and (2 8). The section property (1.13) of $\underline{\Phi}$ follows from (2 8) and the section properties of the Φ_m's

We conclude this step by showing that $(\underline{\Phi}, \underline{A}) \in \mathscr{F}$, see (1.19). The regularity properties of $(\underline{\Phi}, \underline{A})$ are already established. That the gauge condition $\int_k \underline{A} dx = 0$ is satisfied follows from the $L^2(\Omega_k, \mathbb{R}^3)$-convergence of $A_m \to \underline{A}$ and the fact that $K \subset \Omega_k$, for any k.

Step (iii) By weak lower semi-continuity of the L^2-norm and (2.4), we have that

$$\int_{\Omega_k} |\nabla \underline{A}|^2 dx \leqq \liminf_{m \to \infty} \int_{\Omega_k} |\nabla A_m|^2 dx .$$

Since $\operatorname{curl} F_0 \in C_0^\infty(\Omega_0 \backslash \Omega; \mathbb{R}^3)$, we conclude from the L^2-convergence of $A_m \to \underline{A}$ that

$$\int_{\Omega_k} \underline{A} \cdot \operatorname{curl} F_0 \, dx = \lim_{m \to \infty} \int_{\Omega_k} A_m \cdot \operatorname{curl} F_0 \, dx ,$$

for all Ω_k We denote by $\{h^{(i)}\}$ a locally finite partition of unity subordinate to the open cover $\{\mathscr{C}^{(j)}\}_{j=1}^{n+1}$, i.e., $h^{(i)} \in C_0^\infty(\mathscr{C}^{(j)})$, for some $j = j(i)$, $0 \leqq h^{(i)} \leqq 1$ and $\sum_{i \geqq 1} h^{(i)} = 1$. Again using the weak lower semi-continuity of the L^2-norm, we get that

$$\begin{aligned} \int_{\Omega_k} |\nabla_{A_0 + \underline{A}} \underline{\Phi}|^2 \, dx &= \sum_{i \geqq 1} \int_{\mathscr{C}_k^{(j)}} |\nabla_{A_0^{(j)} + \underline{A}} \underline{\Phi}^{(j)}|^2 \, h^{(i)} \, dx \\ &\leqq \sum_{i \geqq 1} \liminf_{m \to \infty} \int_{\mathscr{C}_k^{(j)}} |\nabla_{A_0^{(j)} + A_m} \Phi_m^{(j)}|^2 h^{(i)} dx \\ &= \liminf_{m \to \infty} \int_{\Omega_k} |\nabla_{A_0 + A_m} \Phi_m|^2 dx , \end{aligned}$$

for all Ω_k Finally, since V is continuous and non-negative, we obtain, by using (2 5) and Fatou's lemma, that

$$\int_{\Omega_k} V(|\underline{\Phi}|) \, dx \leqq \liminf_{m \to \infty} \int_{\Omega_k} V(|\Phi_m|) \, dx ,$$

for all Ω_k. These facts imply that, for any k,

$$S(\underline{\Phi}, \underline{A}, \Omega_k) \leqq \liminf_{m \to \infty} S(\Phi_m, A_m, \Omega_k) \leqq \liminf_{m \to \infty} S(\Phi_m, A_m) = \inf_{(\Phi, A) \in \mathscr{F}} S(\Phi, A)$$

By the Monotone Convergence theorem, and letting $k \to \infty$, we see that $(\underline{\Phi}, \underline{A})$ minimizes S on $\mathscr{F}$. □

Corollary 2.2. *Let $(\underline{\Phi}, \underline{A})$ be a minimizer of S on $\mathscr{F}$ Then $(\underline{\Phi}, \underline{A})$ minimizes $\tilde{S}$ on $\tilde{\mathscr{F}}$ and $\underline{A}$ satisfies the Coulomb gauge condition $\nabla \cdot \underline{A} = 0$, a e on $\mathbb{R}^3$*

Remark. For special configurations of the punctures $x_1, \ldots, x_n$, we expect the minimizers to be unique (up to gauge tranformations) In general, however, the minimizers will *not* be unique. This is described at the end of Sect 4

Proof. Since $S(\underline{\Phi},\underline{A}) < \infty$, $\operatorname{supp}\operatorname{curl} F_0 \subset \Omega_0 \backslash \Omega$ and V is non-negative, we conclude that $|\nabla A| \in L^2(\mathbb{R}^3)$. Then one easily derives that

$$\|\operatorname{curl}\underline{A}\|^2_{L^2(\mathbb{R}^3)} + \|\nabla\cdot\underline{A}\|^2_{L^2(\mathbb{R}^3)} = \|\nabla\underline{A}\|^2_{L^2(\mathbb{R}^3)}\,. \tag{2.9}$$

On the one hand, $S(\underline{\Phi},\underline{A}) < \infty$, and (2.9) imply that $(\underline{\Phi},\underline{A}) \in \tilde{\mathscr{F}}$, and, moreover,

$$\tilde{S}(\underline{\Phi},\underline{A}) \leqq S_0 + S(\underline{\Phi},\underline{A})\,,$$

where $S_0 := \pi \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|} + \frac{1}{2}\int_{\mathbb{R}^3\backslash\Omega}[|F_0|^2(x) - |F_0^h|^2(x)]dx$, by Lemma 1.1, (ii).

On the other hand, we infer by construction of S (see Introduction) that

$$\inf_{(\Phi,A)\in\tilde{\mathscr{F}}} \tilde{S}(\Phi,A) \geqq S_0 + \inf_{(\Phi,A)\in\mathscr{F}} S(\Phi,A) = S_0 + S(\underline{\Phi},\underline{A})\,.$$

Thus, inserting $(\underline{\Phi},\underline{A})$, we conclude that $(\underline{\Phi},\underline{A})$ minimizes $\tilde{S}$ on $\tilde{\mathscr{F}}$, and by Lemma 1.1, (ii), that

$$\|\operatorname{curl}\underline{A}^2\|_{L^2(\mathbb{R}^3)} = \|\nabla\underline{A}\|^2_{L^2(\mathbb{R}^3)}\,. \tag{2.10}$$

Finally, (2.9) and (2.10) show that $\nabla\cdot\underline{A} = 0$, a.e. on $\mathbb{R}^3$. □

3. Regularity Results and Exponential Decay

In this section we study the regularity of the minimizers $(\underline{\Phi},\underline{A})$. This is done in two steps: First, the regularity is discussed in a domain excluding the singularities $x_1,\dots,x_n$ of A_0, i.e., on $\Omega_R^c = \mathbb{R}^3\backslash\Omega_R$, where

$$\Omega_R := \overline{\bigcup_{i=1}^{n} B_R(x_i)}, \quad \text{for some arbitrarily small } R > 0\,.$$

Second, regularity properties in the neighbourhoods of these singularities, i.e. on Ω_R, are established. We recall that the integers m_i, for $i = 1,\dots,n$, are non-zero.

3.1. Regularity away from the singularities.

Theorem 3.1. *For a potential $V : \mathbb{R} \to \mathbb{R}^+$, given by*

$$V(x) := \frac{\lambda}{8}(x^2 - 1)^2\,, \tag{3.1}$$

a minimizer $(\underline{\Phi},\underline{A})$ of S on $\mathscr{F}$ has the regularity properties:

$\underline{A} \in C^\infty(\Omega_R^c;\mathbb{R}^3)$, and $\underline{\Phi}$ restricted to Ω_R^c is a C^∞-section, i.e., on any chart $\mathcal{O}^{(j)}$, $j = 1,\dots,n+1$, $\underline{\Phi}^{(j)} \in C^\infty(\mathcal{O}^{(j)} \cap \Omega_R^c;\mathbb{C})$.

Remark. For definiteness we have chosen the potential $V(x)$ as in (3.1). But the following proof can easily be *generalized* (with the help of Lemma 3.1) to other potentials sharing the qualitative properties of the potential in (3.1). These properties are: V is non-negative, $V(x) = 0$ if and only if $|x| = 1$ and $V(x) = \tilde{V}(x^2)$ for some smooth $\tilde{V}$.

Proof. Set $\mathcal{C} := \mathcal{C}^{(j)}$, and $\Phi := \underline{\Phi}^{(j)}$, $A_0 := A_0^{(j)}$, $A := \underline{A}$ and $\tilde{A} := A_0 + A$ on $\mathcal{C}$. From Corollary 2.2 we recall that $\nabla \cdot A = 0$. Thus, minimality of (Φ, A) implies that (Φ, A) is a weak solution of the variational equations

$$0 = -\Delta\Phi + 2i\tilde{A} \cdot \nabla\Phi + |\tilde{A}|^2\Phi + i(\nabla \cdot A_0)\Phi + \frac{\lambda}{2}(|\Phi|^2 - 1)\Phi\,, \tag{V1}$$

$$0 = -\Delta A + |\Phi|^2 A + \operatorname{curl} F_0 - \operatorname{Im}[(\nabla_{A_0}\Phi)\bar{\Phi}] \tag{V2}$$

on $\mathcal{C}$. In (V1) and (V2) the explicit potential (3 1) has been inserted

By standard regularity theory of elliptic equations (see for instance [6]) and by using the iterative bootstrap argument the regularity results stated in the theorem are established. □

3 2 Regularity in the neighbourhood of the singularities In this subsection we discuss the regularity of the minimizers $(\underline{\Phi}, \underline{A})$ in neighbourhoods $B_R(x_i) \subset \Omega$ of the singularities x_i of A_0 For definiteness we again consider the potential

$$V(x) = \frac{\lambda}{8}(x^2 - 1)^2\,.$$

Lemma 3.1. *Let $(\underline{\Phi}, \underline{A})$ be a minimizer of S on $\mathcal{F}$ Then, for any $i = 1, \ldots, n$ and $R > 0$ with $B_R(x_i) \subset \Omega$, we have that*

(i) $|\underline{\Phi}| \leqq 1$, *a e on* $\mathbb{R}^3$,
(ii) $\underline{A} \in C^{0,\alpha}(\overline{B_R(x_i)}; \mathbb{R}^3)$, *with* $\alpha \leqq \frac{1}{2}$

Proof Let $\Phi^{(j)} = \underline{\Phi}^{(j)}$ and $A := \underline{A}$.
(i) We define a comparison section $\tilde{\Phi}$ by

$$\tilde{\Phi}^{(j)}(x) = \begin{cases} \Phi^{(j)}(x), & \text{if } |\Phi|(x) < 1 \\ \frac{\Phi^{(j)}}{|\Phi|}(x), & \text{if } |\Phi|(x) \geqq 1\,. \end{cases}$$

Then, $(\tilde{\Phi}, A) \in \mathcal{F}$, see (1.19). By minimality of (Φ, A) we infer that

$$\begin{aligned} 0 &\geqq S(\Phi, A) - S(\tilde{\Phi}, A) \\ &= \int_{\{x:|\Phi|(x)>1\}} \left[\frac{1}{2}|\nabla_{A_0+A}\Phi|^2(x) - \frac{1}{2}|\nabla_{A_0+A}\tilde{\Phi}|^2(x) + V(|\Phi|)(x)\right] dx \\ &\geqq \int_{\{x:|\Phi|(x)>1\}} V(|\Phi|)(x)dx\,, \end{aligned}$$

where we have used that on $\{x \; |\Phi|(x) > 1\} \cap \mathcal{C}^{(j)}, j = 1, \ldots, n+1$,

$$\begin{aligned} |\nabla_{A_0+A}\Phi|^2(x) - |\nabla_{A_0+A}\tilde{\Phi}|^2(x) &= |\nabla(\tilde{\Phi}^{(j)}|\Phi|) - i(A_0^{(j)} + A)\tilde{\Phi}^{(j)}|\Phi||^2 - |\nabla_{A_0+A}\tilde{\Phi}|^2 \\ &= |(\nabla_{A_0^{(j)}+A}\tilde{\Phi}^{(j)})|\Phi| + \tilde{\Phi}^{(j)}\nabla|\Phi||^2 - |\nabla_{A_0+A}\tilde{\Phi}|^2 \\ &= |\nabla_{A_0+A}\tilde{\Phi}|^2(|\Phi|^2 - 1) + |\nabla|\Phi||^2 \\ &\geqq 0 \end{aligned}$$

Since $V(x)$ is positive for $|\Phi| > 1$, it follows that $\{x : |\Phi|(x) > 1\}$ is of measure zero.
(ii) Recall that $\operatorname{supp}\operatorname{curl} F_0 \subset \Omega_0 \backslash \Omega$ due to our choice of A_0, see (1.12). Thus, similarly as in (V2), we find that A weakly solves the variational equation

$$(\Delta - |\Phi|^2)A = -\operatorname{Im}[(\nabla_{A_0}\Phi)\bar{\Phi}]$$

on $B_R(x_i)$. Note that $|\Phi|(x)$ and $(\nabla_{A_0}\Phi)\bar{\Phi}(x)$ are well-defined, a.e. on $B_R(x_i)$, due to (1.13). Using the regularity result in (i) standard elliptic regularity theory (see for instance [6]) yields that $A \in H^{2,2}(B_R(x_i); \mathbb{R}^3) \hookrightarrow C^{0,\alpha}(\overline{B_R(x_i)}; \mathbb{R}^3)$, for $0 < \alpha \leqq \frac{1}{2}$.□

The next step is to improve the regularity properties of the section $\underline{\Phi}$ in the balls $B_R(x_i)$. This can be accomplished by studying the variational equation (V1), i.e.,

$$-\Delta_{A_0^{(j)}+\underline{A}}\underline{\Phi}^{(j)} + \frac{\lambda}{2}(|\underline{\Phi}|^2 - 1)\underline{\Phi}^{(j)} = 0\,, \tag{3.2}$$

where $\Delta_{A_0^{(j)}+\underline{A}} = \nabla^2_{A_0^{(j)}+\underline{A}}$. Equation (3.2) raises hopes that one can develop a "covariant" L^p-theory. However, this is a rather delicate business, since x_i is a boundary point of $\mathcal{O}^{(j)}(j = i, i+1)$ and $A_0^{(j)}(x) = O(|x - x_i|^{-1})$, for $x \to x_i$. The approach we present here is based on an expansion in *monopole harmonics.*

Let x_i be the origin of our coordinate system and let $\mathcal{O}^a := B_R(x_i) \cap \mathcal{O}^{(i)}, \mathcal{O}^b := B_R(x_i) \cap \mathcal{O}^{(i+1)}$ and $\Phi^{a,b} := \underline{\Phi}^{(i),(i+1)}$. Extracting the part of Eq. (1.10) singular in x_i we rewrite the connection in the form

$$A_0^{(i),(i+1)} + \underline{A} =: A + a_0^{a,b}\,, \tag{3.3}$$

where $a_0^{a,b} := q\frac{x^1dx^2 - x^2dx^1}{|x|(x^3 \pm |x|)}$, with $q := \frac{m_i}{2} \in \frac{1}{2}\mathbb{Z}\backslash\{0\}$, and A comprises the terms of $A_0^{(i),(i+1)} + \underline{A}$ regular on $B_R(x_i)$. The form A is Hölder continuous, due to Lemma 3.1, and Corollary 2.2 implies that $\nabla \cdot A = 0$, a.e. on $B_R(x_i)$. Hence, Eq. (3.2) reads

$$-\Delta_{a_0^{a,b}}\Phi^{a,b} = -2iA \cdot \nabla_{a_0^{a,b}}\Phi^{a,b} - |A|^2\Phi^{a,b} - \frac{\lambda}{2}(|\Phi|^2 - 1)\Phi^{a,b} =: H_0^{a,b}\,.$$

Clearly we have that $|H_0| \in L^2(B_R(x_i))$, as a consequence of Lemma 3.1 and the fact that $|\nabla_{a_0}\Phi| \in L^2(B_R(x_i))$. For technical reasons we introduce a cut-off function, $\chi \in C_0^\infty(B_R(x_i)), 0 \leqq \chi(x) \leqq 1$, and $\chi(x) = 1$, for $|x| \leqq \frac{3}{4}R$. Hence $u^{a,b} := \chi\Phi^{a,b}$ weakly solves the equation

$$-\Delta_{a_0^{a,b}}u^{a,b} = \chi H_0^{a,b} - (\Delta\chi)\Phi^{a,b} - 2\nabla\chi \cdot \nabla_{a_0^{a,b}}\Phi^{a,b} =: H^{a,b}\,, \tag{3.4}$$

on $\mathcal{O}^{a,b}$, where $|u| \in H_0^{1,2}(B_R(x_i)) \cap L^\infty$ and $|H| \in L^2(B_R(x_i))$.

We introduce spherical coordinates (r, θ, φ), with $(dr, d\theta, r\sin\theta d\varphi)$ the corresponding orthonormal frame of 1-forms. In these coordinates we have that

$$a_0^{a,b} = \frac{q}{r\sin\theta}(\pm 1 - \cos\theta)r\sin\theta d\varphi$$

and

$$-\Delta_{a_0^{a,b}} = -\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{1}{r^2 \sin\theta}\partial_\theta \sin\theta \partial_\theta - \frac{1}{r^2 \sin^2\theta}\partial_\varphi^2$$
$$+2iq\frac{\pm 1 - \cos\theta}{r^2 \sin^2\theta}\partial_\varphi + q^2 \frac{(\pm 1 - \cos\theta)^2}{r^2 \sin^2\theta} . \tag{3.5}$$

Next, we recall a result of [13] (Wu and Yang): Let $q \in \frac{1}{2}\mathbb{Z}\backslash\{0\}$. The monopole harmonics $Y_{qlm}^{a,b}$,

$$\{Y_{qlm}^{a,b}(\theta,\varphi) := \Theta_{qlm}(\theta)e^{i(m\pm q)\varphi} : l = |q|, |q|+1, \dots \text{ and } m = -l, -l+1, \dots, l\} ,$$

are real analytic sections of the complex line bundle (restricted to S^2) around a monopole of charge $2q$. They form a complete and orthonormal set with respect to the scalar product

$$\langle Y_{qlm}, Y_{ql'm'}\rangle_{S^2} := \int_0^\pi \int_0^{2\pi} (\overline{Y}_{qlm} Y_{ql'm'})(\theta,\varphi) d\varphi \sin\theta d\theta = \delta_{ll'}\delta_{mm'} . \tag{3.6}$$

The functions Θ_{qlm} satisfy the ordinary differential equations

$$\left[-\frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{1}{\sin^2\theta}(m + q\cos\theta)^2\right]\Theta_{qlm} = [l(l+1) - q^2]\Theta_{qlm} . \tag{3.7}$$

We remark that the scalar product in (3.6) is well-defined, since

$$(\overline{Y}_{qlm}^a Y_{ql'm'}^a)(\theta,\varphi) = (\overline{Y}_{qlm}^b Y_{ql'm'}^b)(\theta,\varphi) ,$$

on the intersection $\mathcal{O}^a \cap \mathcal{O}^b$.

We expand $u^{a,b}$ and $H^{a,b}$ in monopole harmonics:

$$u^{a,b}(r,\theta,\varphi) = \sum_{l=|q|}^{\infty} \sum_{m=-l}^{l} u_{qlm}(r) Y_{qlm}^{a,b}(\theta,\varphi) \tag{3.8}$$

and

$$H^{a,b}(r,\theta,\varphi) = \sum_{l=|q|}^{\infty} \sum_{m=-l}^{l} h_{qlm}(r) Y_{qlm}^{a,b}(\theta,\varphi) , \tag{3.9}$$

where $u_{qlm}(r) := \langle Y_{qlm}, u(r, \cdot, \cdot)\rangle_{S^2}$ and $h_{qlm}(r) := \langle Y_{qlm}, H(r, \cdot, \cdot)\rangle_{S^2}$ are complex-valued functions. Both sums converge in the norm $\| \ \|_{S^2}$, induced by the scalar product in (3.6), and Parseval's identity yields that

$$\sum_{l=|q|}^{\infty} \sum_{m=-l}^{l} |u_{qlm}(r)|^2 = \|u\|_{S^2}^2(r), \qquad \sum_{l=|q|}^{\infty} \sum_{m=-l}^{l} |h_{qlm}(r)|^2 = \|H\|_{S^2}^2(r) , \tag{3.10}$$

for a.e. $r \in [0,R]$. Moreover, we conclude that the functions $u_{qlm}(r)$ and $h_{qlm}(r)$ are in $L^2([0,R], r^2 dr)$, since

$$\int_0^R |h_{qlm}(r)|^2 r^2 dr \leqq \int_0^R \|H\|_{S^2}^2(r) r^2 dr = \|H\|_{L^2(B_R(x_i))}^2 .$$

Let $f \in C_0^\infty([0,R])$. Using (3.5), (3.7) and $\nabla \cdot a_0^{a,b} = 0$, we obtain that

$$-\Delta_{a_0^{a,b}}(f Y_{qlm}^{a,b}) = (L_{ql} f(r)) Y_{qlm}^{a,b}, \tag{3.11}$$

where $L_{ql} := -\frac{1}{r^2}\partial_r r^2 \partial_r + \frac{1}{r^2}[l(l+1) - q^2]$. Thus, testing (3.4) with $f(r)\overline{Y_{qlm}^{a,b}}(\theta,\varphi)$ on $B_R(x_i)$, we conclude, using (3.8)–(3.11), that u_{qlm} (or more precisely, the real- and imaginary parts of u_{qlm}) weakly (in $L^2([0,R], r^2 dr)$) solves the following (singular) Sturm–Liouville problem

$$L_{ql} u_{qlm}(r) = h_{qlm}(r),$$
$$u_{qlm}(R) = 0.$$

In order to exploit this fact, we consider

$$\tilde{u}_{qlm}(r) := u_{qlm}(r) r \quad \text{and} \quad \tilde{h}_{qlm}(r) := h_{qlm}(r) r. \tag{3.12}$$

The function $\tilde{u}_{qlm}$ is a weak solution of the (singular) Sturm–Liouville problem

$$\begin{aligned} \tilde{L}_{ql} \tilde{u}_{qlm}(r) &= \tilde{h}_{qlm}(r), \\ \tilde{u}_{qlm}(R) &= 0 \end{aligned} \tag{3.13}$$

in $L^2([0,R], dr)$ with $\tilde{L}_{ql} := -\partial_r^2 + \frac{1}{r^2}[l(l+1) - q^2]$. The homogeneous (real) differential equation $\tilde{L}_{ql} v(r) = 0$ is an Euler equation and has the two solutions

$$v_{ql}^{(1)}(r) := r^{\alpha_{ql}}, \quad v_{ql}^{(2)}(r) := r^{1-\alpha_{ql}}, \tag{3.14}$$

where $\alpha_{ql} := \frac{1}{2} + \sqrt{(l+\frac{1}{2})^2 - q^2}$. Since $\alpha_{\pm\frac{1}{2}\frac{1}{2}} = \frac{1+\sqrt{3}}{2}$ and $\alpha_{ql} > \frac{3}{2}$, for all $l \geqq |q| > \frac{1}{2}$, it follows that $v_{\pm\frac{1}{2}\frac{1}{2}}^{(1),(2)}, v_{ql}^{(1)} \in L^2([0,R], dr)$, but $v_{ql}^{(2)} \notin L^2([0,R], dr)$, for all $l \geqq |q| > \frac{1}{2}$. By general Sturm–Liouville theory [3, 11], $\tilde{L}_{\pm\frac{1}{2}\frac{1}{2}}$ is said to be of the limit-circle type, $\tilde{L}_{ql}$ ($l \geqq |q| > \frac{1}{2}$) of the limit-point type at the singular endpoint $r = 0$. In the limit-circle case, one has to impose a boundary condition at $r = 0$ to make (3.13) well-defined. This is achieved by Lemma 3.1, which, by (3.10), rules out that $\tilde{u}_{\pm\frac{1}{2}\frac{1}{2}m}(r)$ diverges, as $r \to 0$. Thus, for a $\mathbb{C}$-valued function $\tilde{h} \in L^2([0,R], dr)$ we define the Green's operator G_{ql} by

$$G_{ql}\tilde{h}(r) := \int_0^R G_{ql}(r,\rho)\tilde{h}(\rho)d\rho, \tag{3.15}$$

with the kernel

$$G_{ql}(r,\rho) := \begin{cases} (2\alpha_{ql} - 1)^{-1}\rho^{1-\alpha_{ql}} r^{\alpha_{ql}}, & \text{if } 0 < r \leqq \rho \leqq R \\ (2\alpha_{ql} - 1)^{-1}\rho^{\alpha_{ql}} r^{1-\alpha_{ql}}, & \text{if } 0 < \rho \leqq r \leqq R. \end{cases}$$

It is easy to show that

$$|G_{ql}\tilde{h}(r)| \leqq \begin{cases} C(R)\|\tilde{h}\|_{L^2([0,R],dr)} \cdot r^{\frac{1+\sqrt{3}}{2}}, & \text{if } \alpha_{ql} = \alpha_{\pm\frac{1}{2}\frac{1}{2}} \\ 4(2\alpha_{ql} - 3)^{-\frac{3}{2}}\|\tilde{h}\|_{L^2([0,R],dr)} \cdot r^{\frac{3}{2}}, & \text{if } \alpha_{ql} > \frac{3}{2}, \end{cases} \tag{3.16}$$

and, that $G_{ql}\tilde{h} \in C^0([0,R];\mathbb{C})$. Hence G_{ql} $L^2([0,R],dr) \to L^2([0,R],dr) \cap C^0$ is a bounded, self-adjoint operator. The solution of problem (3.13) is continuous and given by

$$\tilde{u}_{qlm}(r) = G_{ql}\tilde{h}_{qlm}(r) + R^{-\alpha_{ql}} G_{ql}\tilde{h}_{qlm}(R) r^{\alpha_{ql}} \tag{3 17}$$

We prove regularity properties for $u^{a,b} = \chi\Phi^{a,b}$ in two steps:
(i) By the smooth gauge transformation

$$\underline{A} \to \underline{A} + \nabla\psi, \ \underline{\Phi} \to \underline{\Phi} e^{i\psi}, \tag{3.18}$$

with $\psi(x) := A(0) \cdot x$, we may impose (besides the Coulomb gauge) the additional gauge condition $A(0) = 0$ Since A is Hölder continuous with Hölder exponent $\frac{1}{2}$, we conclude from Lemma 3 1 and from the fact that $|\nabla_{a_0}\Phi| \in L^2(B_R)$ that

$$|x|^{-\frac{1}{2}} H^{a,b}(x) \in L^2(B_R(x_i)). \tag{3.19}$$

Thus, Eqs (3 9), (3 10) and (3.12) imply that $r^{-\frac{1}{2}}\tilde{h}_{qlm}(r) \in L^2([0,R],dr)$ Exploiting this fact one can improve inequality (3.16), i e.,

$$|G_{ql}\tilde{h}_{qlm}(r)| \leqq \begin{cases} C(q,R)\|r^{-\frac{1}{2}}\tilde{h}_{qlm}\|_{L_2([0,R],dr)} \cdot r^{\alpha_{ql}}, & \text{if } \alpha_{ql} < 2 \\ C(q,R,\varepsilon)\|r^{-\frac{1}{2}}\tilde{h}_{qlm}\|_{L^2([0,R],dr)} \cdot r^{2-\varepsilon} & \text{if } \alpha_{ql} = 2 \\ 4(2\alpha_{ql}-4)^{-\frac{3}{2}}\|r^{-\frac{1}{2}}\tilde{h}_{qlm}\|_{L^2([0,R],dr)} \cdot r^2, & \text{if } \alpha_{ql} > 2, \end{cases} \tag{3 20}$$

for $0 \leqq r \leqq R$, where ε is some arbitrarily small positive constant
(ii) We prove that the sum in (3 8), i e.,

$$\chi\Phi^{a,b}(r,\theta,\varphi) = u^{a,b}(r,\theta,\varphi) = \sum_{l=|q|}^{\infty} \sum_{m=-l}^{l} u_{qlm}(r) Y^{a,b}_{qlm}(\theta,\varphi),$$

converges uniformly (on natural domains specified below) Inequality (3.20) is useful in reaching this goal. In addition, suitable uniform estimates on the monopole harmonics are required, which we derive in the Appendix; see Theorem A 1.

Let $I^a := \{0 \leqq \theta \leqq \frac{\pi}{2} + \varepsilon, \varphi \in S^1\}$ and $I^b := \{\frac{\pi}{2} - \varepsilon \leqq \theta \leqq \pi, \varphi \in S^1\}$, such that $(0,R] \times I^{a,b} \subset \mathcal{O}^{a,b}$, and let $0 \leqq \delta < \delta_0 := \delta_0(|q|)$. We define an auxiliary section

$$\phi^{a,b}_{ql}(r,\theta,\varphi) := \sum_{m=-l}^{l} \frac{\tilde{u}_{qlm}(r)}{r^{1+\delta}} Y^{a,b}_{qlm}(\theta,\varphi),$$

for $r \in (0,R]$ and $(\theta,\varphi) \in I^{a,b}$ From (3.17) and (3.20) we infer that if $|q| \leqq 2$ then $\phi^{a,b}_{ql}$ is continuous on $[0,R] \times I^{a,b}$ for $\delta < \delta_0 = \alpha_{q|q|} - 1$, and if $|q| > 2$ then $\phi^{a,b}_{ql}$ is continuous on $[0,R] \times I^{a,b}$ for $\delta < \delta_0 = 1$ Moreover, since $\alpha_{ql} > 2$, for $l \geqq |q| + 1$, one finds that

$$l^{\frac{3}{2}}|\tilde{u}_{qlm}(r)| \leqq C(\delta,q,R)\|r^{-\frac{1}{2}}\tilde{h}_{qlm}\|_{L^2([0,R],dr)} \cdot r^{1+\delta}, \tag{3 21}$$

for $0 \leqq r \leqq R$ From (3 21) and Theorem A 1 it follows that

$$\sup_{B_R(x_i)} |\phi_{ql}(r,\theta,\varphi)| \leqq C(\delta,q,R) l^{-\frac{3}{2}} \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} \left(\sum_{m=-l}^{l} \|r^{-\frac{1}{2}}\tilde{h}_{qlm}\|^2_{L^2([0,R],dr)}\right)^{\frac{1}{2}} =: M_{ql}$$

Since $\sum_{l=|q|}^{\infty}\sum_{m=-l}^{l}\|r^{-\frac{1}{2}}\tilde{h}_{qlm}\|^2_{L^2([0,R],dr)} = \||x|^{-\frac{1}{2}}H\|^2_{L^2(B_R(x_i))}$ (by (3.12), (3.10), (3.19) and the Lebesgue Convergence theorem), we conclude that

$$\sum_{l=|q|}^{\infty} M_{ql} \leqq C\left(\sum_{l=|q|}^{\infty}\frac{1}{l^2}\right)^{\frac{1}{2}} \||x|^{-\frac{1}{2}}H\|_{L^2(B_R(x_i))} < \infty\,.$$

Thus, we have a majorizing series for $\phi_{ql}^{a,b}(r,\theta,\varphi)$, and, by the Weierstrass theorem, (3.12) and (3.8), it follows that

$$u^{a,b}(r,\theta,\varphi) = r^{\delta}\phi_{\delta}^{a,b}(r,\theta,\varphi)\,, \tag{3.22}$$

with $\phi_{\delta}^{a,b} := \sum_{l=|q|}^{\infty}\phi_{ql}^{a,b}$ a continuous section on $[0,R]\times I^{a,b}$.

This regularity result also holds for the original section $\underline{\Phi}$ (before the smooth gauge transformation (3.18) is applied) in the neighbourhood of any singularity x_i. We summarize these results in the following theorem:

Theorem 3.2. *Let* $(\underline{\Phi},\underline{A})$ *be a minimizer of* S *on* $\mathscr{F}$ *with* V *as in Eq.* (3.1)*. Then, for any monopole located at* x_i*, with integer magnetic charge* $m_i =: 2q \neq 0$*, and* $R>0$*, with* $B_{2R}(x_i)\subset\Omega$*, the section* $\underline{\Phi}^{(i),(i+1)}$ *has a Hölder continuous extension to* x_i*, i.e.,*

$$\underline{\Phi}^{(i),(i+1)}(r,\theta,\varphi) = r^{\delta}\phi_{\delta}^{(i),(i+1)}(r,\theta,\varphi)\,,$$

where $\phi_{\delta}^{(i),(i+1)}$ *is a continuous section on* $[0,R]\times I^{a,b}$, $I^a := \{0 \leqq \theta \leqq \frac{\pi}{2}+\varepsilon, \varphi\in S^1\}$ *and* $I^b := \{\frac{\pi}{2}-\varepsilon \leqq \theta \leqq \pi, \varphi\in S^1\}$*. The Hölder exponent* δ *depends on* $|q|$*, i.e.* $\delta < \alpha_{q|q|}-1$*, if* $|q|\leqq 2$*, and* $\delta<1$*, if* $|q|>2$*.*

Remark. These regularity results are *not* optimal. However, they give good support to the conjecture that the statement above holds for $\delta < \alpha_{q|q|}-1$, for all q, where $\alpha_{q|q|}-1 = \sqrt{|q|+\frac{1}{4}}-\frac{1}{2}$.

3.3. Exponential Decay. In order to arrive at a better picture of the properties of minimizers, $(\underline{\Phi},\underline{A})$, we propose to study their decay properties. Let $\Omega_0^e := \overline{\Omega_0^c}$ denote the exterior of Ω_0. The neutrality condition in (1.11) and the choice of A_0 in (1.12), $A_0=0$ on Ω_0^e, imply that $\Phi := \underline{\Phi}$ and $A := \underline{A} = A_0 + \underline{A}$ are well-defined on Ω_0^e. From Theorem 3.1, (V1), (V2) and $\nabla\cdot A = 0$, by Corollary 2.2, we infer that (Φ,A) smoothly solves the variational equations

$$0 = -\Delta\Phi + 2iA\cdot\nabla\Phi + |A|^2\Phi + i(\nabla\cdot A)\Phi + \frac{\lambda}{2}(|\Phi|^2-1)\Phi\,, \tag{V1$''$}$$

$$0 = -\Delta A + \nabla(\nabla\cdot A) - \mathrm{Im}\,[(\nabla_A\Phi)\bar{\Phi}] \tag{V2$''$}$$

on Ω_0^e. Further, we recall from Lemma 3.1 that $|\Phi|\leqq 1$ on Ω_0^e. In the following theorem we state the resulting exponential decay for $1-|\Phi|^2$, $|\mathrm{curl}\,A|$ and $|\bar{\Phi}\nabla_A\Phi|$, whenever (Φ,A) is a smooth, finite-action solution to (V1$''$) and (V2$''$) with $|\Phi|\leqq 1$.

Theorem 3.3. *Assume that* $(\Phi, A) \in \tilde{\mathscr{F}}$ *is a smooth solution to* (V1″), (V2″) *on* Ω_0^e *Further assume that* $|\Phi| \leqq 1$ *on* Ω_0^e *Then either* $|\Phi| \equiv 1$ (*and* $\nabla_A \Phi \equiv 0$, $\operatorname{curl} A \equiv 0$), *or else* $|\Phi| < 1$ *on* Ω_0^c *For every* $\lambda > 0$, *given* $\varepsilon > 0$, *there exists* $M = M(\varepsilon, \lambda) < \infty$ *such that*

$$1 - |\Phi|^2, \ |\mathrm{Re}(\bar{\Phi} \nabla_A \Phi)| \leqq M e^{-(1-\varepsilon) m_L |x|} , \tag{3.23}$$

$$|\operatorname{curl} A|, \ |\mathrm{Im}\,(\bar{\Phi} \nabla_A \Phi)| \leqq M e^{-(1-\varepsilon)|x|} \tag{3.24}$$

on Ω_0^e, *where* $m_L := \min(\lambda^{\frac{1}{2}}, 2)$

The proof to establish exponential decay is based on the method presented in [7, Sects. III.7–III.9]. A detailed proof can be found in [8].

Assume that a (neutral) system of n magnetic monopoles, given by $(\underline{x}, \underline{m})$, can be decomposed into κ (neutral) subsystems, given by $(\underline{x}^1, \underline{m}^2), \ldots, (\underline{x}^\kappa, \underline{m}^\kappa)$. Let Ω^k denote a closed ball with center ω_k containing all punctures of the set $\underline{x}^k$. If $\Omega_0^k := \{x : \mathrm{dist}(x, \Omega^k) \leqq 1\}$, we further assume that the subsystems are separated by

$$\inf_{k \neq l} \mathrm{dist}(\Omega_0^k, \Omega_0^l) \gg 1 .$$

The theorem above implies that the action $\tilde{S}_{\underline{x},\underline{m}}$ *of a minimizer* $(\underline{\Phi}, \underline{A})$ *for the system* $(\underline{x}, \underline{m})$ *is bounded from above by the actions* $\tilde{S}_{\underline{x}^k,\underline{m}^k}$ *of minimizers* $(\underline{\Phi}^k, \underline{A}^k)$ *for the subsystems* $(\underline{x}^k, \underline{m}^k)$, *i e* ,

$$\tilde{S}_{\underline{x},\underline{m}}(\underline{\Phi}, \underline{A}) \leqq \sum_{k=1}^{\kappa} \tilde{S}_{\underline{x}^k,\underline{m}^k}(\underline{\Phi}^k, \underline{A}^k) + C d_{\underline{x}}^2 e^{-c d_{\underline{x}}} , \tag{3.25}$$

where $d_{\underline{x}} := \inf_{k \neq l} |\omega_k - \omega_l|$ *and* C, c *are some positive constants.*

Indeed, Theorem 3.3 implies that $1 - |\underline{\Phi}^k(x)|^2 \leqq M_k e^{-(1-\varepsilon) m_L |x - \omega_k|}$, on $\mathbb{R}^3 \backslash \Omega_0^k$. Thus, for R_k sufficiently large, we can choose a gauge such that $\underline{\Phi}^k(x) > 0$ on Ω_{R_k}, where $\Omega_{R_k} := \mathbb{R}^3 \backslash \{x \cdot |x - \omega_k| \leqq R_k\} \subset \mathbb{R}^3 \backslash \Omega_0^k$. In this gauge (3.24) implies exponential decay for $|\underline{A}^k|$. Hence, for $R > R_k$, we modify $(\underline{\Phi}^k, \underline{A}^k)$ on Ω_{R_k} by

$$1 - \Phi_R^k := (1 - \underline{\Phi}^k) \chi_R \quad \text{and} \quad A_R^k := \underline{A}^k \chi_R ,$$

where χ_R is a smooth cut-off function, with $\chi_R(x) \equiv 1$, if $|x - \omega_k| \leqq R$ and $\chi_R \equiv 0$, if $|x - \omega_k| \geqq R + 1$. This defines an admissible comparison configuration (Φ_R^k, A_R^k) "localized" in $B_{R+1}(\omega_k)$. By (3.23) and (3.24) it follows that

$$0 \leqq \tilde{S}_{\underline{x}^k,\underline{m}^k}(\Phi_R^k, A_R^k) - \tilde{S}_{\underline{x}^k,\underline{m}^k}(\underline{\Phi}^k, \underline{A}^k) \leqq C(M_k, \lambda) R^2 e^{-2(1-\varepsilon)\bar{m} R} ,$$

for $R > R_k$, where $\bar{m} = \min(\lambda^{\frac{1}{2}}, 1)$

Take $R = \frac{1}{2}(d_{\underline{x}} - 2) \geqq \max\{R_k \cdot 1 \leqq k \leqq \kappa\}$ If A_0^k denote the reference connections for the subsystems $(\underline{x}^k, \underline{m}^k)$, then $\hat{A}_0 := \sum_{k=1}^{\kappa} A_0^k$ is, in addition to A_0, a reference connection on the bundle $P_{\underline{x},\underline{m}}$ w.r.t. the system $(\underline{x}, \underline{m})$. Define

$$\Phi := \prod_{k=1}^{\kappa} \Phi_R^k \quad \text{and} \quad \hat{A} := \sum_{k=1}^{\kappa} A_R^k .$$

Then Φ is a section of $E_{\underline{x},\underline{m}}$ and $\hat{A}_0 + \hat{A}$ a connection on $P_{\underline{x},\underline{m}}$. Let $A := \hat{A}_0 + \hat{A} - A_0$. Since the configurations (Φ_R^k, A_R^k) are localized in $B_{R+1}(\omega_k)$ one easily derives that

$$\tilde{S}_{\underline{x},\underline{m}}(\Phi, A) = \sum_{k=1}^{K} \tilde{S}_{\underline{x}^k,\underline{m}^k}(\Phi_k^R, A_k^R)\,,$$

where $R = \frac{1}{2}(d_{\underline{x}} - 2)$, which implies (3.25).

Note that the accuracy of the upper bound in (3.25) depends essentially on the *choice* of the subsystems. A lower bound for $\tilde{S}_{\underline{x},\underline{m}}(\underline{\Phi}, \underline{A})$ of the type of (3.25) will *only* exist for an appropriate choice of the subsystems. This is discussed at the end of Sect. 4.

4. Bounds on the Action of the Minimizers

In this section we focus on the special situation $M_{\underline{x}}$, where

$$\underline{x} = \{x_1, x_2\} \quad \text{and} \quad \underline{m} = \{-m, m\}, \quad \text{for } m \text{ a positive integer}\,,$$

i.e. an anti-monopole-monopole pair located at positions x_1 and x_2, respectively. We establish accurate upper and lower bounds for the action $\tilde{S}$ of the minimizer $(\underline{\Phi}, \underline{A})$. As a consequence, the action essentially grows linearly with the distance $|x_1 - x_2|$ and the monopole charge m. Thus, in addition to the exponential decay, this confirms the heuristic picture, that the action is concentrated in m vortex tubes joining both monopoles. That means, in any plane orthogonal to the symmetry-axis, the minimizer describes a vortex configuration consisting of m vortices. Since vortices exhibit different types of behaviour for $\lambda < 1$ and $\lambda > 1$, respectively, the following arrangements of the vortex tubes will occur: For $\lambda < 1$, all vortex tubes are concentrated on the symmetry-axis, whereas, for $\lambda > 1$, they repel each other, forming a spindle. For $\lambda = 1$, there is no interaction between the vortex tubes. Thus they are concentrated on the symmetry-axis.

For $\lambda \leqq 1$ we have the following upper bound:

Theorem 4.1. *Let* $0 < \lambda \leqq 1$. *Consider two monopoles of integer magnetic charge* $-m$ *and* m *located at positions* x_1, x_2, *respectively, with* $|x_1 - x_2| =: l$. *Let* $(\underline{\Phi}, \underline{A})$ *denote the minimizer of the functional* $\tilde{S}(\Phi, A)$ *on* $\tilde{\mathscr{F}}$, *with* V *as in Eq.* (3.1). *Then, given* $l_0 > 0$, *there exists some constant* s_0 *such that* $\tilde{S}(\underline{\Phi}, \underline{A})$ *is bounded above by*

$$\tilde{S}(\underline{\Phi}, \underline{A}) \leqq s_0 + l e_{m,\lambda}, \quad \textit{for } l \geqq l_0\,, \tag{4.1}$$

where $e_{m,\lambda}$ *is the energy of a rotationally symmetric, critical point with vorticity* m *of the energy functional* E *defined in Eq.* (1.3).

For repelling vortex tubes we have a slightly weaker result.

Theorem 4.2. *Let* $\lambda > 1$. *Consider two monopoles of integer magnetic charge* $-m$ *and* m *located at positions* x_1, x_2, *respectively, with* $|x_1 - x_2| =: l$. *Let* $(\underline{\Phi}, \underline{A})$ *denote the minimizer of the functional* $\tilde{S}(\Phi, A)$ *on* $\tilde{\mathscr{F}}$, *with* V *as in Eq.* (3.1), *and let* δ *be an arbitrarily small positive constant. Then there exists a constant* s_0 *such that* $\tilde{S}(\underline{\Phi}, \underline{A})$ *is bounded above by*

$$\tilde{S}(\underline{\Phi}, \underline{A}) \leqq s_0 + O(l^{2\delta}) + m l e_\lambda, \quad \textit{as } l \to \infty\,. \tag{4.2}$$

where e_λ is the energy of a rotationally symmetric, critical point with vorticity 1 *of the energy functional E*

Proof (Theorem 4 1) Let us consider the situation in Fig. 2. Since any $(\Phi, A) \in \tilde{\mathscr{F}}$ yields an upper bound on $\tilde{S}(\underline{\Phi}, \underline{A})$ we construct some Φ and $A := A_0 + A$ according to the heuristic picture sketched at the beginning of this section. That is, on the slab V we choose Φ and $\tilde{A}$ such that $(\Phi, \tilde{A})$ is, in any $\{x^3\text{=const.}\}$-plane, a rotationally symmetric ciritical point centered on the x^3-axis. In the ball $B_R(x_i)$, we set $\Phi \equiv 0$ and $\tilde{A}$ to be approximately equal to the connection describing a magnetic monopole located in x_i with charge m_i. Finally, on the domains V_i, appropriate interpolations are constructed.

We now present the details of our construction.

(i) Given m and λ, there exists a smooth critical point, (ϕ, a), of E with $e_{m,\lambda} := E(\phi, a) < \infty$ (ϕ, a) is rotationally symmetric in the sense of (1 6). On the slab V we set

$$\begin{aligned} \Phi(x^1, x^2, x^3) &:= \phi(x^1, x^2)\,, \\ \tilde{A}(x^1, x^2, x^3) &:= a(x^1, x^2)\,, \end{aligned} \tag{4.3}$$

where we identify the 1-form a with the vector $a = (a_1, a_2, 0)$. Hence we obtain that $(\Phi, \tilde{A}) \in C^\infty(V, \mathbb{C} \times \mathbb{R}^3)$, and, furthermore that

$$\tilde{S}(\Phi, \tilde{A}; V) := \frac{1}{2} \int_V \left[|\tilde{F}|^2 + |\nabla_{\tilde{A}} \Phi|^2 + \frac{\lambda}{4}(|\Phi|^2 - 1)^2 \right] dx = (l - 4R) e_{m,\lambda}\,, \tag{4.4}$$

with $\tilde{F} := \operatorname{curl} \tilde{A}$.

(ii) In the neighbourhood of x_i, $i = 1, 2$, we introduce

$$A_i^{(j)}(x) := \frac{m_i}{2|x - x_i|} \frac{(x^1 - x_i^1)dx^2 - (x^2 - x_i^2)dx^1}{(x^3 - x_i^3) + \eta_i^{(j)}|x - x_i|}, \quad \text{with } F_i := \operatorname{curl} A_i\,, \tag{4.5}$$

defined on $\tilde{\mathscr{C}}^{(j)}$, and

$$W_i(x) := \frac{-m_i}{2|x - y_i|} \frac{(x^1 - y_i^1)dx^2 - (x^2 - y_i^2)dx^1}{(x^3 - y_i^3) - (-1)^i|x - y_i|}, \quad \text{with } G_i := \operatorname{curl} W_i\,, \tag{4.6}$$

where $\eta_i^{(j)}$ was defined in (1.10) and y_i is the mirror image of x_i, see Fig. 2

The connection associated with the mirror-monopole, W_i, plays a crucial role for the estimates in the domain V_i. On $B_R(x_i)\backslash\{x_i\}$, we set

$$\begin{aligned} \Phi^{(j)}(x) &\equiv 0\,, \\ \tilde{A}^{(j)}(x) &:= A_i^{(j)}(x) + W_i(x)\,, \end{aligned} \tag{4 7}$$

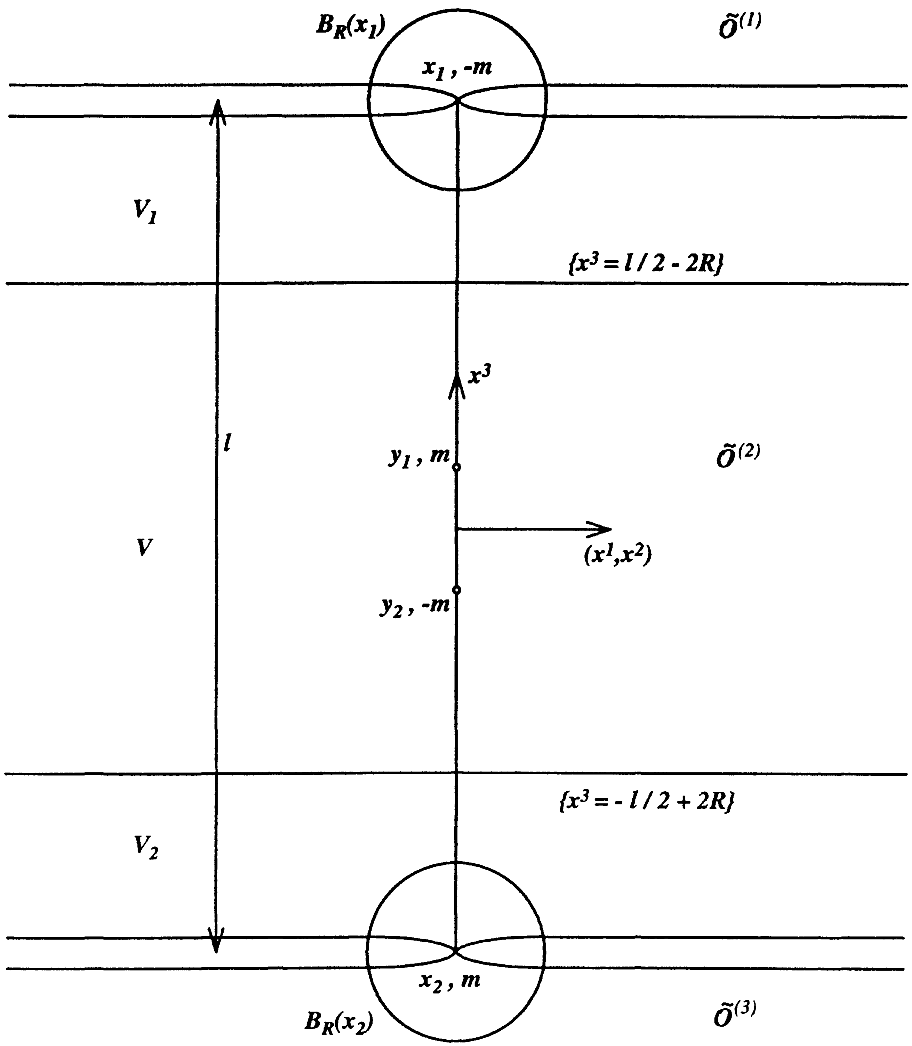

Fig. 2. (Configuration of the anti-monopole-monopole pair). Given $l \geqq l_0 > 0$, let $x_1 := (0, 0, \frac{l}{2})$, $m_1 := -m$ and $x_2 := (0, 0, -\frac{l}{2})$, $m_2 := m$, respectively. Denote by $\{\tilde{\mathcal{O}}^{(j)}\}_{j=1}^{3}$ the open cover, as indicated, and let $R > 0$ be such that $l_0 - 4R > 0$ Then V is the open slab bounded by the planes $\Pi_1 := \{x^3 = \frac{l}{2} - 2R\}$, and $\Pi_2 := \{x^3 = -\frac{l}{2} + 2R\}$, and, V_i is the closed domain bounded by the two-sphere $\partial B_R(x_i)$ and the plane Π_i. Finally, $y_1 := (0, 0, \frac{l}{2} - 4R)$ and $y_2 := (0, 0, -\frac{l}{2} + 4R)$ are mirror images of x_1 and x_2 at the planes Π_1, Π_2, respectively

for $j = i, i+1$. Hence $(\Phi^{(j)}, \tilde{A}^{(j)}) \in C^\infty(B_R(x_i) \cap \tilde{\mathcal{O}}^{(j)}; \mathbb{C} \times \mathbb{R}^3)$, and the transition conditions are satisfied. Recall that $F_i = 2\pi m_i \nabla E(x - x_i)$ and $G_i = -2\pi m_i \nabla E(x - y_i)$, see (1.9). Thus, using (4.7), integrating over $B_R(x_i) \backslash B_\varepsilon(x_i)$ and passing to the limit $\varepsilon \to 0$ yields that

$$\tilde{S}(\Phi,\tilde{A},B_R(x_i)) := \frac{1}{2}\int_{B_R(x_i)\setminus\{x_i\}} \left[(|\tilde{F}|^2 - |F_i|^2) + |\nabla_{\tilde{A}}\Phi|^2 + \frac{\lambda}{4}(|\Phi|^2-1)^2\right] dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^3\setminus B_R(x_i)} |F_i|^2 dx$$
$$= \frac{1}{2}\int_{B_R(x_i)} \left[|G_i|^2 + \frac{\lambda}{4}\right] dx - \frac{1}{2}\int_{\mathbb{R}^3\setminus B_R(x_i)} |F_i|^2 dx \leqq C_1 , \tag{4.8}$$

where $C_1 = C_1(m, \lambda, R)$ is a positive constant depending only on m, λ and R

(iii) In the following we construct an appropriate interpolation in V_1 Let x_1 be the origin of our coordinate system Denote by χ a smooth function in $C_0^\infty((-2R,2R),\mathbb{R}^+)$ with $\chi(0)=R, \chi(t)=\chi(-t)$ and $\chi(t) \geqq \sqrt{R^2-t^2}$, for $|t| \leqq R$ Extend the definition of χ by setting

$$\chi(t) := \begin{cases} \chi(t), & \text{if } t \leqq 2R \\ -t+3R, & \text{if } t \geqq 4R , \end{cases} \tag{4.9}$$

and by a smooth interpolation between 0 and $-R$ in $[2R,4R]$. Since the critical point (ϕ,a) from (i) satisfies (1 6) we define, for $j=1,2$,

$$\Phi^{(j)}(x) := \begin{cases} \varphi(r-\chi(x^3)), & \text{in } V_1 \cap \tilde{\mathcal{C}}^{(1)} \\ \varphi(r-\chi(x^3))e^{im\Theta}, & \text{in } V_1 \cap \tilde{\mathcal{C}}^{(2)} , \end{cases} \tag{4 10}$$

where (r,Θ,x^3) are cylindrical coordinates, and $\varphi(t)$ is required to vanish for $t<0$ Similarly one defines $\Phi^{(j)}$ on $V_2 \cap \tilde{\mathcal{C}}^{(j)}$, for $j=2,3$ Hence $\Phi^{(j)} \in C(V_i \cap \tilde{\mathcal{C}}^{(j)}, \mathbb{C})$; moreover, $\nabla\Phi^{(j)} \in L^2_{\text{loc}}(V_i \cap \tilde{\mathcal{C}}^{(j)}; \mathbb{C}^3)$, $|\Phi| \in H^{1,2}_{\text{loc}}(V_i, \mathbb{R}^+)$, and the transition conditions (1 13) are satisfied Using (4.10), (4.9) and decay property (D1) we obtain the bound

$$\frac{\lambda}{8}\int_{V_i} (|\Phi|^2-1)^2 dx \leqq C_2(m,\lambda,R) . \tag{4 11}$$

Next, we construct $\tilde{A}$ in V_1 Let

$$W(x) := \xi(x^3)[a(x^1,x^2) - A_1^{(2)}(x^1,x^2,-2R) - W_1(x^1,x^2,-2R)] , \tag{4.12}$$

with $G := \operatorname{curl} W$, and ξ is a function in $C^\infty(\mathbb{R};\mathbb{R}^+)$, with $\xi(t)=1$ if $t \leqq -2R$, and $\xi(t)=0$ if $t \geqq -R$ Then we define, for $j=1,2$,

$$\tilde{A}^{(j)}(x) := A_1^{(j)}(x) + W_1(x) + W(x), \text{ in } V_1 \cap \tilde{\mathcal{C}}^{(j)} , \tag{4 13}$$

and similarly in $V_2 \cap \tilde{\mathcal{C}}^{(j)}$, for $j=2,3$. Hence, $\tilde{A}^{(j)} \in C^\infty(V_i \cap \tilde{\mathcal{C}}^{(j)}, \mathbb{R}^3)$. We conclude this step by proving some important estimates By (4 13) we obtain that

$$\int_{V_1} |\tilde{F}|^2 dx \leqq 4\left(\int_{\mathbb{R}^3\setminus B_R(x_1)} |F_1|^2 dx + \int_{\mathbb{R}^3\setminus B_R(y_1)} |G_1|^2 dx + \int_{V_1} |G|^2 dx\right)$$
$$\leqq \tfrac{8\pi m^2}{R} + 4\int_{V_1} |G|^2 dx ,$$

where we have used the explicit expressions in (4.5) and (4.6). Inserting (4.12) in the last term on the r.h.s., we get

$$\begin{aligned}\int_{V_1} |G|^2 dx &\leqq 2\int_{-2R}^{-R} \xi^2 \left(\int_{x^3=-2R} [|f|^2 + |F_1 + G_1|^2] d^2x \right) dx^3 \\ &+ \int_{-2R}^{-R} \dot{\xi}^2 \left(\int_{x^3=-2R} |a - A_1^{(2)} - W_1|^2 d^2x \right) dx^3 \\ &\leqq c_1 e_{m,\lambda} + c_2 + c_3 \int_{x^3=-2R} |a - A_1^{(2)} - W_1|^2 d^2x \,,\end{aligned}$$

where we have abbreviated $f := \operatorname{curl} a$ and $\dot{\xi}$ denotes the derivative of ξ. Here and in the following c_i denotes a positive constant depending only on m, λ and R. We claim that

$$\int_{x^3=-2R} |a - A_1^{(2)} - W_1|^2 d^2x \leqq c_4 \,, \tag{4.14}$$

and hence the above estimates imply

$$\int_{V_1} |\tilde{F}|^2 dx \leqq C_3(m, \lambda, R) \,. \tag{4.15}$$

Proof of the claim. Expressing (4.5) and (4.6) in cylindrical coordinates yields

$$A_1^{(j)} + W_1 = \begin{cases} \frac{m}{2}(\cos\theta_1 - \cos\theta_2)\, d\Theta, & \text{if } j = 1 \\ m d\Theta + \frac{m}{2}(\cos\theta_1 - \cos\theta_2)\, d\Theta, & \text{if } j = 2 \,, \end{cases} \tag{4.16}$$

where θ_1 and θ_2 are given by $\cos\theta_1 := \frac{t}{\sqrt{r^2+t^2}}$ and $\cos\theta_2 := \frac{4R+t}{\sqrt{r^2+(t+4R)^2}}$, respectively, with $t = x^3 \geqq \frac{l}{2} - 2R$. Equation (1.6) and (D1) imply $\varphi(r) \geqq \frac{1}{2}$, for $r \geqq r_0$, where r_0 depends only on m and λ. Thus, on one hand we find

$$\int_{r \leqq r_0} |a - A_1^{(2)} - W_1|^2 d^2x \leqq c_5 \,,$$

since $a - A_1^{(2)} - W_1$ is smooth for $t = -2R$. On the other hand, using (1.6) and (4.16), with $t = -2R$, leads to

$$\begin{aligned}\int_{r > r_0} |a - A_1^{(2)} - W_1|^2 d^2x &= \int_{r > r_0} \left| \frac{m}{r}(\alpha - 1) + \frac{m}{2r} \frac{4R}{\sqrt{r^2 + 4R^2}} \right|^2 d^2x \\ &\leqq 8 \int_{r > r_0} |\nabla_a \phi|^2 d^2x + c_6 \leqq 16 e_{m,\lambda} + c_6 \,.\end{aligned}$$

This proves our claim. Next we show that

$$\frac{1}{2} \int_{V_1} |\nabla_{\tilde{A}} \Phi|^2 dx \leqq C_4(m, \lambda, R) \,. \tag{4.17}$$

Expressions (4.10) and (4.13) yield

$$\frac{1}{2}\int_{V_1} |\nabla_{\tilde{A}}\Phi|^2 dx \leqq \int_{V_1} (|\nabla_{A_1+W_1}\Phi|^2 + |W|^2) dx$$
$$\leqq \int_{|x^3|\leqq 2R} |\nabla_{A_1+W_1}\Phi|^2 dx + \int_{2R<x^3} |\nabla_{A_1+W_1}\Phi|^2 dx + c_1 \,, \quad (4.18)$$

where we have used (4 12) and (4.14). By (4.10) and (4 16), the first term on the r.h s. of (4 18) reads

$$2\pi \int_{-2R}^{2R} dt \int_{\chi}^{\infty} \left[(\varphi')^2(r-\chi)(1+\dot{\chi}^2) + \frac{m^2}{r^2}\varphi(r-\chi)^2 \sin^2\left(\frac{\theta_1+\theta_2}{2}\right)\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)\right] r dr$$
$$\leqq 2\pi \int_{-2R}^{2R} dt \int_{0}^{\infty} [|\nabla_a\phi|^2(1+\dot{\chi}^2)](r+\chi)dr + 2\pi \int_{-2R}^{2R} dt \int_{\chi}^{\infty} \frac{m^2}{r^2}\varphi(r-\chi)^2 \frac{4R^2}{r^2+4R^2} r dr \,,$$

where $\chi = \chi(t)$ and $\dot{\chi} = \dot{\chi}(t)$ as defined in (4.9). Due to (D3) and (4.9), the first term in the expression above is bounded by a constant depending only on m, λ and R. The same holds for the second term. Indeed, since $\lim_{r\to 0} \alpha(r) = 0$, there exists r_α (depending only on m and λ) such that

$$\frac{m}{r}\varphi(r) \leqq \begin{cases} 2|\nabla_a\phi|(x), & \text{if } r \leqq r_\alpha \\ \frac{m}{r}, & \text{if } r \geqq r_\alpha \,. \end{cases}$$

Thus we conclude that

$$\int_{|x^3|\leqq 2R} |\nabla_{A_1+W_1}\Phi|^2 dx \leqq c_2 \quad (4.19)$$

Similarly, the second term on the r.h.s. of (4.18) can be bounded by

$$\int_{2R<x^3} |\nabla_{A_1+W_1}\Phi|^2 dx = 2\pi \int_{2R}^{\infty} dt \int_{0}^{\infty} [(\varphi')^2(r-\chi)(1+\dot{\chi}^2)]\, r dr$$
$$+ \int_{2R<x^3} |(A_1^{(1)}+W_1)\Phi^{(1)}|^2 dx \leqq c_3 + \int_{2R<x^3} |A_1^{(1)}+W_1|^2 dx \,, \quad (4.20)$$

where we have used (D3) and (4.9). Finally, we claim that

$$\int_{2R<x^3} |A_1^{(1)}+W_1|^2 dx \leqq c_4 \,, \quad (4.21)$$

which, together with (4.18)–(4.20), yields the desired inequality (4 17)

Proof of the claim From (4 16) it follows that $|A_1^{(1)}+W_1|^2 \leqq \frac{m^2}{r^2}\sin^2(\frac{\theta_1-\theta_2}{2})$. In order to bound the r.h.s , we choose a positive constant r_0 and find that, for $r \geqq r_0$ and $t \geqq 2R$,

$$\frac{m^2}{r^2}\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \leqq \frac{m^2}{r^2}\,\frac{4R^2}{r^2+t^2} \,.$$

For $r \leqq r_0$ and $t \geqq 2R$, however, we find that

$$\frac{m^2}{r^2}\sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \leqq \frac{2m^2}{r^2}\left[\sin^2\left(\frac{\theta_1}{2}\right)+\sin^2\left(\frac{\theta_2}{2}\right)\right]$$
$$\leqq 2m^2\left[\frac{1}{r^2+t^2}+\frac{1}{r^2+(t+4R)^2}\right].$$

Using these bounds in the integration of $|A_1^{(1)}+W_1|^2$ over $\{x : 2R < x^3\}$ our claim follows. All estimates derived in V_1 can equally be derived in V_2. Thus from (4.11), (4.15) and (4.17) it follows that

$$\tilde{S}(\Phi,\tilde{A};V_i) := \frac{1}{2}\int_{V_i}\left[|\tilde{F}|^2+|\nabla_{\tilde{A}}\Phi|^2+\frac{\lambda}{4}(|\Phi|^2-1)^2\right]dx \leqq C_5(m,\lambda,R). \quad (4.22)$$

(iv) On the open cover $\{\tilde{\mathcal{O}}^{(j)}\}$, $\Phi^{(j)}$ and $\tilde{A}^{(j)}$ are locally given by the expressions in (4.3), (4.7), (4.10) and (4.13). Thus, Φ is a section of $E_{\underline{x},\underline{m}}$ and $\tilde{A}$ is a connection on $P_{\underline{x},\underline{m}}$. With respect to the open cover $\{\mathcal{O}^{(j)}\}$ in Fig. 1, let $A := \tilde{A}^{(j)} - A_0^{(j)}$, where $A_0^{(j)}$ is the reference connection defined in (1.12). Then

$$(|\Phi|,A) \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^+\times\mathbb{R}^3),\ |\nabla_{A_0+A}\Phi| \in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^+).$$

From (4.4), (4.8) and (4.22) it follows that

$$\frac{1}{2}\int_{M_{\underline{x}}}\left[|\mathrm{curl}A+F_0|^2-\sum_{i=1}^{2}|F_i|^2+|\nabla_{A_0+A}\Phi|^2+\frac{\lambda}{4}(|\Phi|^2-1)^2\right]dx \leqq s_0 + le_{m,\lambda}, \quad (4.23)$$

where s_0 is a positive constant depending only on m, λ and R. Since $A_0^{h(j)} = A_1^{(j)} + A_2^{(j)}$, by (1.10) and (4.5), we have that

$$|\mathrm{curl}A+F_0|^2-\sum_{i=1}^{2}|F_i|^2 = |\mathrm{curl}A|^2+2\mathrm{curl}A\cdot F_0+|F_0|^2-|F_0^h|^2+2F_1\cdot F_2. \quad (4.24)$$

Let $M_{\underline{x}}^{\delta,R} := B_R(0)\setminus\bigcup_{i=1}^{n}\overline{B_\delta(x_i)}$, for $\delta > 0$ small and $R > 0$ large. Integrating (4.24) over $M_{\underline{x}}^{\delta,R}$ and passing to the limits $\delta \to 0$ and $R \to \infty$ yields

$$\int_{M_{\underline{x}}}\left[|\mathrm{curl}A+F_0|^2-\sum_{i=1}^{2}|F_i|^2\right]dx = \int_{M_{\underline{x}}}[|\mathrm{curl}A|^2+2\mathrm{curl}A\cdot F_0]\,dx$$
$$+\int_{\mathbb{R}^3\setminus\Omega}[|F_0|^2-|F_0^h|^2]\,dx + 2\pi\frac{m_1m_2}{|x_1-x_2|}. \quad (4.25)$$

Equations (4.25) and (4.23) imply that $\tilde{S}(\Phi,A) \leqq s_0 + le_{m,\lambda}$. Finally, by Lemma 1.1, (ii), $\mathrm{curl}A\cdot F_0 \in L^1(M_{\underline{x}};\mathbb{R})$, and therefore $(\Phi,A)\in\tilde{\mathscr{F}}$, as desired. □

The proof of Theorem 4.2 is similar, but more delicate. We again construct some Φ and $\tilde{A} = A_0 + A$ according to the heuristic picture sketched at the beginning of this section. We need an appropriate multi-vortex configuration in the domain V for the repulsive case. The following tool proves to be useful.

Lemma 4.1. *Let $(\underline{\phi},\underline{a})$ be a smooth critical point of the energy functional E defined in* (1 3) *with $\underline{\phi}=\underline{\varphi}(r)e^{i\Theta}$, $\underline{a}=\underline{\alpha}(r)d\Theta$, and with energy $e_\lambda := E(\underline{\phi},\underline{a}) < \infty$ Let r_0 be such that $|\underline{\phi}|(x) \geqq \frac{1}{2}$, for $|x| \geqq r_0$, and let $L \geqq r_0+1$ Then there exist some constants M', μ and c and a smooth, rotationally symmetric* 1*-vortex configuration (ϕ,a) with the properties:*

(i) $\phi=\varphi(r)\exp(i\Theta)$, *with*

$$\varphi(r)=\begin{cases}\underline{\varphi}(r) & \text{if } r \leqq L-1\\ 1 & \text{if } r \geqq L\,,\end{cases}$$

such that $\underline{\varphi}(r) \leqq \varphi(r) \leqq 1$ and $|\varphi'(r)| \leqq M'e^{-\mu r}$, for all $r \geqq 0$

(ii) $a=\alpha(r)d\Theta$, *with*

$$\alpha(r)=\begin{cases}\underline{\alpha}(r) & \text{if } r \leqq L-1\\ 1 & \text{if } r \geqq L\,,\end{cases}$$

such that $|1-\alpha(r)| \leqq M're^{-\mu r}$ and $|\alpha'(r)| \leqq M're^{-\mu r}$, for all $r \geqq r_0$

(iii) $|E(\phi,a)-e_\lambda| \leqq cLe^{-2\mu L}$

Proof Let ξ be a function in $C^\infty(\mathbb{R};\mathbb{R}^+)$ with $\xi(t)=1$, if $t \leqq 0$ and $\xi(t)=0$, if $t \geqq 1$ Define $1-\varphi(r) := \xi(r-L+1)(1-\underline{\varphi}(r))$ and $1-\alpha(r) = \xi(r-L+1)(1-\underline{\alpha}(r))$ Then $\phi := \varphi(r)e^{i\Theta}$ and $a := \alpha(r)d\Theta$ defines a smooth 1-vortex configuration Using decay properties (D1)–(D3) the properties stated in the lemma are easily established. □

Proof (Theorem 4 2) Let $\delta > 0$ be arbitrarily small, and consider the situation in Fig 2. On the slab V we choose Φ and $\tilde{A}$ such that $(\Phi,\tilde{A})$ is an m-vortex configuration with its zeros located along the x^2-axis at distance $2(l^\delta+1)$, in every $\{x^3=\text{const}\}$-plane On the ball $B_R(x_i)$ we choose Φ and $\tilde{A}$ as in the previous proof, and on the domain V_i, we again construct appropriate interpolations.

(i) Given $\lambda > 1$, there exists a smooth critical point $(\underline{\phi},\underline{a})$ of E with vorticity 1 and $e_\lambda := E(\underline{\phi},\underline{a}) < \infty$. Let r_0 be as in Lemma 4 1 and assume that $l_0 \geqq r_0^{1/\delta}$ We denote by $z_1,\ldots,z_m$ m points on the x^2-axis at distance $2L$, where $L := l^\delta+1$ More precisely, $z_1,\ldots,z_m$ are given by

$$(0,-(m-1)2L),\quad \ldots,(0,(m-1)2L),\quad \text{if } m \text{ is odd and}$$

$$(0,-\left(m-\tfrac{3}{2}\right)2L),\quad \ldots,(0,\left(m-\tfrac{3}{2}\right)2L),\quad \text{if } m \text{ is even} \tag{4.26}$$

Let (ϕ,a) be the 1-vortex configuration of Lemma 4 1 We introduce

$$\tilde{\phi}(x) := \prod_{k=1}^{m}\phi_k(x) \quad \text{and} \quad \tilde{a}(x) := \sum_{k=1}^{m} a_k(x)\,, \tag{4 27}$$

where (ϕ_k,a_k) is given by $\phi_k(x) := \phi(x-z_k)$ and $a_k(x) := a(x-z_k)$. Then $(\tilde{\phi},\tilde{a})$ is a smooth m-vortex configuration with zeros $z_1,\ldots,z_m$ Moreover, by Lemma 4 1,

$$|E(\tilde{\phi},\tilde{a})-me_\lambda| = m|E(\phi,a)-e_\lambda| \leqq mcLe^{-2\mu L}\,. \tag{4 28}$$

Thus, on the slab V we set

$$\Phi(x^1,x^2,x^3) := \tilde{\phi}(x^1,x^2)\,,$$
$$\tilde{A}(x^1,x^2,x^3) := \tilde{a}(x^1,x^2)\,, \tag{4.29}$$

where we identify the 1-form $\tilde{a}$ with the vector $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, 0)$. Hence we obtain that $(\Phi,\tilde{A}) \in C^\infty(V; \mathbb{C} \times \mathbb{R}^3)$, and (4.28) yields, as in (4.4),

$$\tilde{S}(\Phi,\tilde{A}; V) \leqq C_1 + C_2 L e^{-2\mu L} + C_3 l L e^{-2\mu L} + l m e_\lambda\,. \tag{4.30}$$

Here and in the sequel, C_i or c_i denote positive constants depending only on m, λ and R.

(ii) On $B_R(x_i)\backslash\{x_i\}$, $i = 1,2$, we set, as in (4.7),

$$\Phi^{(j)}(x) :\equiv 0\,,$$
$$\tilde{A}^{(j)}(x) := A_i^{(j)}(x) + W_i(x)\,, \tag{4.31}$$

for $j = i,\ i+1$. This yields the estimate in (4.8), namely

$$\tilde{S}(\Phi,\tilde{A}; B_R(x_i)) \leqq C_4\,. \tag{4.32}$$

(iii) Let x_1 be the origin of our coordinate system. To get an appropriate interpolation between (4.29) and (4.31) in V_1, we have to tie together all "vortices" of $(\tilde{\phi},\tilde{a})$ on the x^3-axis. For this purpose we introduce smooth functions

$$\xi_k(t) := z_k^2 \xi(t), \quad \text{for } k = 1,\ldots,m\,, \tag{4.33}$$

where $z_k = (z_k^1, z_k^2)$ is given by (4.26) and ξ is a function in $C^\infty(\mathbb{R};\mathbb{R}^+)$ with $\xi(t) = 1$, if $t \leqq -2R$ and $\xi(t) = 0$, if $t \geqq -\frac{3R}{2}$. Further, on V_1, let

$$H_k^{(j)}(x) := \frac{1}{m}\begin{pmatrix} (A_1^{(j)} + W_1)_1(x^1, x^2 - \xi_k(x^3), x^3) \\ (A_1^{(j)} + W_1)_2(x^1, x^2 - \xi_k(x^3), x^3) \\ -\dot{\xi}_k(x^3)(A_1^{(j)} + W_1)_2(x^1, x^2 - \xi_k(x^3), x^3) \end{pmatrix}, \tag{4.34}$$

for $k = 1,\ldots,m$ and $j = 1,2$, and let

$$H(x) := \xi(x^3)\left[\tilde{a}(x^1,x^2) - \sum_{k=1}^m H_k^{(2)}(x^1,x^2,-2R)\right]. \tag{4.35}$$

Then we define, for $j = 1,2$,

$$\tilde{A}^{(j)}(x) := \sum_{k=1}^m H_k^{(j)}(x) + H(x) = \begin{cases} A_1^{(1)}(x) + W_1(x) & \text{in } V_1 \cap \tilde{\mathcal{O}}^{(1)} \\ \sum_{k=1}^m H_k^{(2)}(x) + H(x) & \text{in } V_1 \cap \tilde{\mathcal{O}}^{(2)}\,, \end{cases} \tag{4.36}$$

and similarly in $V_2 \cap \tilde{\mathcal{O}}^{(j)}$, for $j = 2,3$. Hence $\tilde{A}^{(j)} \in C^\infty(V_i \cap \tilde{\mathcal{O}}^{(j)}; \mathbb{R}^3)$. Let $V_R := \{x \in \mathbb{R}^3 : -2R \leqq x^3 \leqq -\frac{3R}{2}\}$ and let $\tilde{R} := \frac{3R}{2}$. From (4.36) we obtain that

$$\int_{V_1} |\tilde{F}|^2 dx \leqq \frac{4\pi m^2}{\tilde{R}} + c_1 \sum_{k=1}^m \int_{V_R} |\operatorname{curl} H_k|^2 dx + 2\int_{V_R} |\operatorname{curl} H|^2 dx\,, \tag{4.37}$$

where we have used the explicit expressions in (4.5) and (4.6). The second term on the r.h.s. in (4.37) can be estimated further by (4.34), (4.33) and (4.26). This leads to

$$\int_{V_R} |\operatorname{curl} H_k|^2 dx \leqq \int_{V_R} \frac{2}{m^2}(\dot{\xi}_k^2 + 1)|(F_1 + G_1)(x^1, x^2 - \xi_k, x^3)|^2 dx \leqq c_2 L^2\,,$$

where $\xi_k = \xi_k(x^3)$, with derivative $\dot{\xi}_k = \dot{\xi}_k(x^3)$. For the third term on the r.h.s. in (4.37) one finds, using (4 35), (4.34), (4 33) and (4.27), that

$$\begin{aligned}\int_{V_R} |\operatorname{curl} H|^2 dx &\leqq c_3 \sum_{k=1}^{m} \int_{V_R} \left(\left| f_k(x^1,x^2) - \frac{1}{m}(F_1 + G_1)(x^1, x^2 - z_k^2, -2R) \right|^2 \right. \\ &\quad \left. + c_4 \left| a_k(x^1,x^2) - \frac{1}{m}(A_1^{(2)} + W_1)(x^1, x^2 - z_k^2, -2R) \right|^2 \right) dx \\ &\leqq c_5 \left[\int_{x^3=-2R} \left| f - \frac{1}{m}(F_1 + G_1) \right|^2 d^2x \right. \\ &\quad \left. + \int_{x^3=-2R} \left| a - \frac{1}{m}(A_1^{(2)} + W_1) \right|^2 d^2x \right],\end{aligned}$$

where we identify the 1-forms a_k and a with vectors in $\mathbb{R}^3$ with $f_k := \operatorname{curl} a_k$ and $f := \operatorname{curl} a$ On one hand, using the explicit expressions in (4 5), (4.6) and applying Lemma 4 1, we find that

$$\int_{x^3=-2R} \left| f - \frac{1}{m}(F_1 + G_1) \right|^2 d^2x \leqq c_6(e_\lambda + cLe^{-2\mu L}) + c_7 .$$

On the other hand, one proves with the help of Lemma 4.1 that

$$\int_{x^3=-2R} \left| a - \frac{1}{m}(A_1^{(2)} + W_1) \right|^2 d^2x \leqq c_8 \tag{4.38}$$

The proof is analogous to the one of (4.14). Thus, the estimates from (4 37) to (4 38) imply

$$\int_{V_1} |\tilde{F}|^2 dx \leqq C_5 + C_6 L^2 + C_7 L e^{-2\mu L} . \tag{4.39}$$

In order to define Φ on V_1 we introduce smooth functions $\psi_k, k = 1, \ldots, m$,

$$\psi_k(x) := \phi(x^1, x^2 - \xi_k(x^3)) , \tag{4.40}$$

where ξ_k is given by (4.33) and ϕ by Lemma 4.1 in step (i). Furthermore, we reintroduce the smooth function χ defined in (4.9), replacing $2R$ by $\tilde{R} = \frac{3R}{2}$. We extend the definition of χ by setting

$$\chi(t) = \begin{cases} \chi(t), & \text{if } t \leqq \tilde{R} \\ -t + 2R, & \text{if } t \geqq 3R , \end{cases} \tag{4.41}$$

and by a smooth interpolation between 0 and $-R$ on $[\tilde{R}, 3R]$ Then, for $j = 1, 2$, we set

$$\Phi^{(j)}(x) = \begin{cases} \varphi(r - \chi(x^3))^m & \text{in } V_1 \cap \tilde{\mathscr{C}}^{(1)} \\ \varphi(r - \chi(x^3))^m e^{im\Theta} & \text{in } V_1 \cap \tilde{\mathscr{C}}^{(2)}, \text{ with } x^3 > -\tilde{R} \\ \prod_{k=1}^m \psi_k(x) & \text{in } V_1 \cap \tilde{\mathscr{C}}^{(2)}, \text{ with } x^3 \leqq -\tilde{R} , \end{cases} \tag{4.42}$$

where $\varphi(t)$ is required to vanish for $t < 0$. Similarly one defines $\Phi^{(j)}$ on $V_2 \cap \tilde{\mathcal{O}}^{(j)}$, for $j = 2, 3$. Hence $\Phi^{(j)} \in C(V_i \cap \tilde{\mathcal{O}}^{(j)}; \mathbb{C})$; moreover $\nabla\Phi^{(j)} \in L^2_{\text{loc}}(V_i \cap \tilde{\mathcal{O}}^{(j)}; \mathbb{C}^3)$, $|\Phi| \in H^{1,2}_{\text{loc}}(V_i; \mathbb{R}^+)$, and the transition conditions (1.13) are satisfied. We claim that

$$\frac{\lambda}{8} \int_{V_i} (|\Phi|^2 - 1)^2 dx \leqq C_8 \,. \tag{4.43}$$

Proof of the claim. Lemma 4.1 implies $1 - \varphi^{2m} \leqq m(1 - \underline{\varphi}^2)$, for all $r \geqq 0$, and therefore

$$\int_{V_1, x^3 > -\tilde{R}} (1 - \varphi^{2m}(r - \chi))^2 dx \leqq c_1 \,,$$

where we have inserted (4.41) and used (D1). Next, we choose $\tilde{r}_0 \geqslant r_0$ so large that, by Lemma 4.1 and (D1), $\varphi^2(r) \geqq \underline{\varphi}^2(r) \geqq 1 - Me^{-\mu r} > 0$, for $r \geqq \tilde{r}_0$. Let $r_k := \sqrt{(x^1)^2 + (x^2 - \xi_k(x^3))^2}$. With ψ_k defined in (4.40), we then obtain that

$$\left(1 - \prod_{k=1}^{m} |\psi_k(x)|^2\right)^2 \leqq \begin{cases} 1, & \text{if } r_k \leqq \tilde{r}_0, \text{ for some } k \in \{1, \ldots, m\} \\ c_2 \sum_{k=1}^{m} e^{-2\mu r_k}, & \text{otherwise}\,. \end{cases}$$

Using these bounds, it follows that

$$\int_{V_1, x^3 \leqq \tilde{R}} \left(\left| \prod_{k=1}^{m} \psi_k(x) \right|^2 - 1 \right)^2 dx \leqq c_3 \,.$$

This proves our claim. Next we show that

$$\frac{1}{2} \int_{V_1} |\nabla_{\tilde{A}} \Phi|^2 dx \leqq C_9 + C_{10} L^2 \,. \tag{4.44}$$

Recalling (4.36), the l.h.s. of (4.44) can be bounded from above by

$$\int_{V_R} \left| \nabla\Phi^{(2)} - i \sum_{k=1}^{m} H_k^{(2)} \Phi^{(2)} \right|^2 dx + \int_{V_1, |x^3| < \tilde{R}} |\nabla\Phi^{(j)} - i(A_1^{(j)} + W_1)\Phi^{(j)}|^2 dx$$
$$+ \int_{x^3 \geqq \tilde{R}} |\nabla\Phi^{(1)} - i(A_1^{(1)} + W_1)\Phi^{(1)}|^2 dx + \int_{V_1} |H|^2 dx =: I_d + I_c + I_b + I_a \,. \tag{4.45}$$

(a) By (4.35), (4.34), (4.33), (4.27) and (4.38) it follows that

$$I_a \leqq \int_{V_R} \left| \sum_{k=1}^{m} a_k(x^1, x^2) - \sum_{k=1}^{m} \frac{1}{m} (A_1^{(2)} + W_1)(x^1, x^2 - z_k^2, -2R) \right|^2 dx \leqq c_1 \,.$$

(b) From (4.42) and (4.41) it follows that

$$I_b \leqq 2\pi \int_{\tilde{R}}^{\infty} dx^3 \int_0^{\infty} [2m^2 (\varphi')^2 (r - \chi)] \, r dr + \int_{x^3 \geqq \tilde{R}} |A_1^{(1)} + W_1|^2 dx \leqq c_2 \,,$$

where we have used Lemma 4.1 and the same argumentation as for (4 21).
(c) Using (4.16), (4.42) and (4.41), it follows that

$$I_c \leqq c_3 \int_{-\tilde{R}}^{\tilde{R}} dt \int_{\chi}^{\infty} (\varphi')^2 (r-\chi)\, r dr + 2\pi \int_{-\tilde{R}}^{\tilde{R}} dt \int_{\chi}^{\infty} \frac{m^2}{r^2} \varphi^2 (r-\chi) \sin^2\left(\frac{\theta_1-\theta_2}{2}\right) r dr \leqq c_4\,,$$

where we have used Lemma 4.1 and the same argumentation as above (4 19).
(d) By (4 42), (4.40) and (4.34) it follows that

$$I_d \leqq c_5 \sum_{k=1}^{m} \int_{V_R} |\nabla \psi_k - i H_k^{(2)} \psi_k|^2 dx \leqq c_5 \sum_{k=1}^{m} \int_{-2R}^{-\tilde{R}} dx^3 \int_{\mathbb{R}^2} \left[\left| \nabla \phi(x^1,x^2) - \frac{i}{m}(A_1^{(2)} + W_1)\phi(x^1,x^2) \right|^2 (1+\xi_k^2(x^3)) \right] d^2x\,,$$

and, using (4 33), (4 26), (4 16) and Lemma 4.1, one shows, as above (4 19), that

$$\leqq (L^2+1) c_6 \int_{-2R}^{-\tilde{R}} dt \int_{0}^{\infty} \left[(\varphi')^2 + \frac{1}{r^2} \varphi^2 \sin^2\left(\frac{\theta_1-\theta_2}{2}\right) \right] r dr \leqq c_7 L^2\,.$$

Thus, combining (a)–(d) with (4 45), we obtain (4.44). Finally, combining (4 39), (4.43) and (4.44), one finds, as in (4.22), that

$$\tilde{S}(\Phi, \tilde{A}; V_i) \leqq C_{11} + C_{12} L^2 + C_{13} L e^{-2\mu L}\,. \tag{4.46}$$

(iv) On the open cover $\{\tilde{\mathcal{O}}^{(j)}\}$, $\Phi^{(j)}$ and $\tilde{A}^{(j)}$ are locally given by the expressions in (4.29), (4.31), (4 36) and (4 42) Following (iv) in the proof of Theorem 4.1, let $A := \tilde{A}^{(j)} - A_0^{(j)}$, then (4.30), (4.32), (4.46) imply that $\tilde{S}(\Phi, A) \leqq s_0 + O(l^{2\delta}) + m l e_\lambda$, for $l_0 \leqq l \to \infty$, where $L = l^\delta + 1$, and s_0 is a positive constant depending only on m, λ and R Hence $(\Phi, A) \in \tilde{\mathscr{F}}$, as desired. □

Theorem 4.3. *Let $\lambda > 0$ Consider two monopoles of integer magnetic charge $-m$ and m located at positions x_1, x_2, respectively, with $|x_1 - x_2| =: l$ Let $(\underline{\Phi}, \underline{A})$ denote the minimizer of the functional $\tilde{S}(\Phi, A)$ on $\tilde{\mathscr{F}}$, with V as in Eq* (3.1) *Then, given $l_0 > 0$, there exists some constant s_0' such that $\tilde{S}(\underline{\Phi}, \underline{A})$ is bounded below by*

$$\tilde{S}(\underline{\Phi}, \underline{A}) \geqq s_0' + l e'_{m,\lambda}, \quad \text{for } l \geqq l_0\,, \tag{4.47}$$

where $e'_{m,\lambda}$ is the infimum over all m-vortex configurations of the energy functional E defined in Eq (1 3)

Proof We consider the situation in Fig. 2. Then in any cross-section of V, $\underline{\Phi}$ and $\tilde{A} = A_0 + \underline{A}$ describe a m-vortex configuration The energy of this configuration is bounded from below by $e'_{m,\lambda}$.

More precisely, for t fixed, with $-\frac{l}{2}+2R \leqq t \leqq \frac{l}{2}-2R$, we define

$$\begin{aligned} \phi_t(x^1,x^2) &:= \underline{\Phi}(x^1,x^2,t) \\ a_t(x^1,x^2) &:= (\tilde{A}_1(x^1,x^2,t),\tilde{A}_2(x^1,x^2,t))\,. \end{aligned} \tag{4.48}$$

By Theorem 3.1, we have that $(\phi_t,a_t) \in C^\infty(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$. Theorem 3.3 implies (1.4) and ensures that (ϕ_t,a_t) is a vortex configuration in the homotopy class given by the vorticity of a_t. Since $\frac{1}{2\pi}\int_\Sigma \operatorname{curl} A_0 \cdot n d\sigma = -m$, where Σ is any hemisphere enclosing x_1, but not x_2, it follows from the exponential decay of $\operatorname{curl}\underline{A}$ (Theorem 3.3) that $\frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl} a_t dx = m$. Thus, (ϕ_t,a_t) is a m-vortex configuration. Moreover, we have that

$$E(\phi_t,a_t) \geqq \inf\{E(\phi,a) : (\phi,a) \text{ a } m\text{-vortex configuration}\} =: e'_{m,\lambda}\,. \tag{4.49}$$

Next, inserting (4.25) in (1.14), we obtain that

$$\begin{aligned} \tilde{S}(\underline{\Phi},\underline{A}) \geqq\ & \frac{1}{2}\int_V \left[|\operatorname{curl}\underline{A}+F_0|^2+|\nabla_{A_0+\underline{A}}\underline{\Phi}|^2+\frac{\lambda}{4}(|\underline{\Phi}|^2-1)^2\right]dx \\ & -\frac{1}{2}\sum_{i=1}^{2}\int_{\mathbb{R}^3\setminus B_R(x_i)} |F_{3-i}|^2 dx + \frac{1}{2}\sum_{i=1}^{2}\Bigg[\int_{B_R(x_i)\setminus\{x_i\}} [|\operatorname{curl}\underline{A}+F_0|^2-|F_i|^2]\,dx \\ & + \int_{V_i} [|\operatorname{curl}\underline{A}+F_0|^2]\,dx\Bigg] \\ \geqq\ & (l-4R)e'_{m,\lambda} - \frac{\pi m^2}{R} + \frac{1}{2}\sum_{i=1}^{2}\int_{B_R(x_i)\setminus\{x_i\}} [|\operatorname{curl}\underline{A}+F_0|^2-|F_i|^2]\,dx\,, \end{aligned} \tag{4.50}$$

where we have used (4.49) and (4.5). Since we may assume that $B_R(x_i) \subset \Omega$, (1.10) and (4.5) imply that $A_0^{(j)} = A_1^{(j)} + A_2^{(j)}$ in $B_R(x_i)\cap\tilde{\mathcal{O}}^{(j)}$, for $j=i,i+1$. Thus we find

$$\int_{B_R(x_i)\setminus\{x_i\}} [|\operatorname{curl}\underline{A}+F_0|^2-|F_i|^2]\,dx \geqq 2\int_{B_R(x_i)\setminus\{x_i\}} [\operatorname{curl}\underline{A}+F_{3-i}]\cdot F_i dx = 0\,,$$

where the last step follows, by an integration by parts, from $\operatorname{curl} F_i \equiv 0$ in $B_R(x_i)\setminus\{x_i\}$, and the fact that F_i is parallel to the normal on $\partial B_R(x_i)$ and $\partial B_\delta(x_i)$, for $\delta \to 0$, respectively. Inserting the above inequality into (4.50) completes the proof of the theorem. □

Remark. In the Bogomol'nyi limit $\lambda = 1$ the infimum $e'_{m,\lambda}$ *equals* $e_{m,\lambda} = m\pi$. Thus, to leading order in the distance l, the upper and lower bound on the action $\tilde{S}(\underline{\Phi},\underline{A})$ coincide. For $\lambda < 1$ or $\lambda > 1$, it has not been rigorously established, yet, that $e'_{m,\lambda} = e_{m,\lambda}$ or $e'_{m,\lambda} = me_\lambda$, respectively. See [7].

The results we have proven so far provide a fairly detailed picture of the properties of a minimizer. Let us consider the following two situations in Fig. 3 where $d \gg l \gg 1$ and (x_i,m_i) denotes a monopole of magnetic charge m_i located at the position x_i.

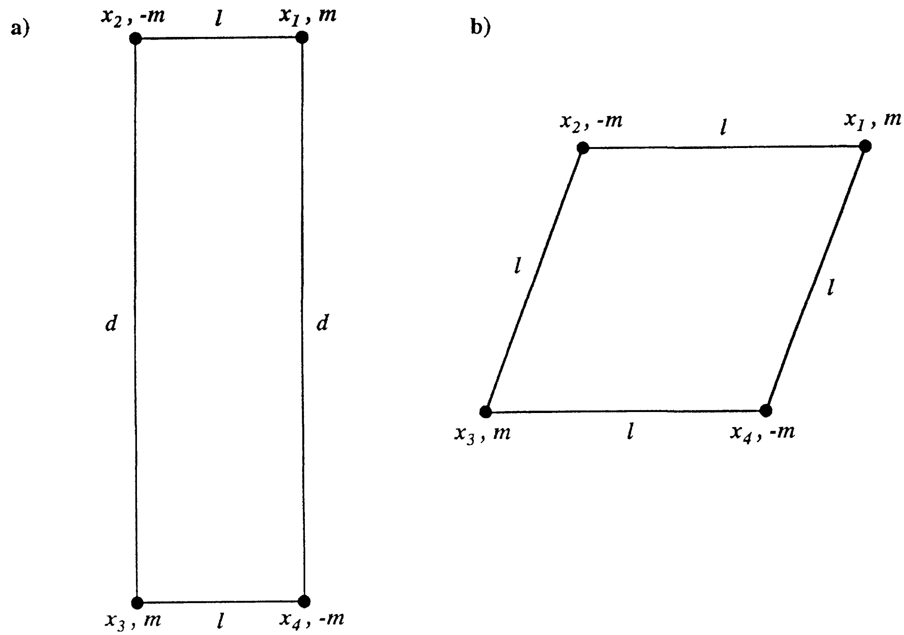

Fig. 3.

In situation a) (Fig. 3) the minimizer is expected to be *unique*. Its vortex tubes should join the anti-monopole-monopole pairs separated by the distance l and carry vorticity m. The reasoning is as follows A minimizer in situation a) describes vortices joining the four monopoles. There can be a cluster of vortices joining x_1 to x_2 with total vorticity given by $m - k$, $k = 0, 1, \ldots, m$, and a cluster of vortices joining x_1 to x_4 with total vorticity given by k. There must then be a cluster of vortices joining x_3 to x_2 of total vorticity k and one joining x_3 to x_4 of total vorticity $m - k$. For $d \gg l \gg 1$, the total action of the putative minimizer described here is expected to be given by

$$\tilde{S} = \tilde{S}^{(l)} + \tilde{S}^{(d)} + \text{const} \ ,$$

where $\tilde{S}^{(l)} \leqq \text{const}(m-k)l$, and $\tilde{S}^{(d)} \geqq \text{const}\, kd$ (see inequality (3 25) and Theorems 4.1 and 4.2) Thus if $d \gg l$, then the total action is minimized by setting $k = 0$. Since $d \gg 1$, the minimizer is then obtained by gluing together two minimizers for the subsystems $(\{x_1, x_2\}, \{m, -m\})$ and $(\{x_3, x_4\}, \{m, -m\})$, respectively.

In situation b) (Fig 3) and for m an odd integer, we expect that there are at least two distinct minimizers (of equal total action) corresponding to two distinct configurations of vortices joining the four monopoles, whereas if m is an even integer and equally oriented vortices joining one pair of monopoles repel each other the minimizer is expected to be *unique*.

Although we have not attempted to establish this picture rigorously, we expect that situation a) and the first part of situation b) could be proven

The arguments above can be extended to more general situations. *If all monopoles have magnetic charges* ± 1 *the problem of minimizing the action functional appears to reduce to a problem of connecting monopoles of opposite charges with a family of curves of minimal total length*

A. Appendix

We first prove Lemma 1.1 of the introduction. Then we derive uniform estimates on the monopole harmonics, required for the regularity results near the monopoles.

Proof of Lemma 1.1. (i) Let us write $F := \operatorname{curl} A$. Then $\operatorname{div} F = 0$ in $\mathscr{D}'$, the space of distributions.

According to [4, Theorem A.1], there exists a unique vector field $\hat{A}$ with $\hat{A} \in L^6(\mathbb{R}^3; \mathbb{R}^3)$, $\operatorname{curl}\hat{A} = F$, $\operatorname{div}\hat{A} = 0$ in $\mathscr{D}'$ and $\partial_j \hat{A}_i$ L^2-functions satisfying (1.16). We claim that $\hat{A} = A + \nabla\psi$, for an appropriate $\psi \in H^{2,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R})$.

Let $\alpha := \hat{A} - A$. Then $\alpha \in L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ and $\operatorname{curl}\alpha = 0$, a.e. on $\mathbb{R}^3$. Let $j_\varepsilon(x)$ be an approximate identity, i.e., let $j_\varepsilon(x) := \varepsilon^{-3} j(x/\varepsilon)$, where $j \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^+)$ and $\int j(x)dx = 1$. Let $\alpha_\varepsilon := j_\varepsilon * \alpha$, the convolution of j_ε with α. It is easy to see that $\alpha_\varepsilon \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$, $\operatorname{curl}\alpha_\varepsilon = 0$ and $\alpha_\varepsilon \to \alpha$ in $L^2(U)$, as $\varepsilon \to 0$, for any $U \Subset \mathbb{R}^3$. By $\{\Omega_k\}_{k \in \mathbb{N}}$ we denote a sequence of closed balls exhausting $\mathbb{R}^3$. Let K be a set with Lebesgue measure $|K| > 0$ and with $K \subset \Omega_k$, for all $k \in \mathbb{N}$. Since $\operatorname{curl}\alpha_\varepsilon = 0$, there exists $\psi_\varepsilon \in C^\infty(\mathbb{R}^3; \mathbb{R})$ with $\nabla\psi_\varepsilon = \alpha_\varepsilon$ and $\int_K \psi_\varepsilon dx = 0$, for any $\varepsilon > 0$. By Poincaré's inequality,

$$\|\psi_\varepsilon - \psi_{\varepsilon'}\|^2_{H^{1,2}(\Omega_k)} \leqq C\|\nabla\psi_\varepsilon - \nabla\psi_{\varepsilon'}\|^2_{L^2(\Omega_k)} = C\|\alpha_\varepsilon - \alpha_{\varepsilon'}\|^2_{L^2(\Omega_k)} \,.$$

Thus, $\psi_\varepsilon \to \psi$ in $H^{1,2}(\Omega_k)$, as $\varepsilon \to 0$, for any $k \in \mathbb{N}$. Hence, $\psi \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R})$ and $\nabla\psi = \alpha$, a.e. on $\mathbb{R}^3$. Since $A \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$, we conclude that $\alpha \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ and hence $\psi \in H^{2,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R})$ as claimed.
(ii) According to (1.12) and (1.9) we may decompose F_0 as follows:

$$F_0 = \sum_{i=1}^n F_{0i}, \text{ with } F_{0i} \in C_0^\infty(\Omega_0; \mathbb{R}^3), \qquad F_{0i}(x) = 2\pi m_i \nabla E(x - x_i) \text{ on } \Omega \,. \tag{A.1}$$

Let $\delta_0 := \frac{1}{2}\min\{|x_i - x_j| : 1 \leqq i < j \leqq n\}$ and $R_0 := \inf\{R : \Omega_0 \subset B_{R-\delta_0}(0)\}$. Then, for $\delta < \delta_0$ and $R > R_0$, let $M^{\delta,R}_{\underline{x}} := B_R(0) \setminus \bigcup_{i=1}^n \overline{B_\delta(x_i)}$, $B_i^{\delta,R} := B_R(0) \setminus \overline{B_\delta(x_i)}$ and $b_i^\delta := \bigcup_{j \neq i}^n B_\delta(x_j)$. Since $\hat{A} \in H^{1,2}_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ we infer that $\hat{A} \wedge F_{0i} \in H^{1,2}(B_i^{\delta,R})$ and that $\hat{A}$, with $\hat{F} := \operatorname{curl}\hat{A}$, satisfies the identity

$$\hat{F} \cdot F_{0i} = \hat{A} \cdot \operatorname{curl} F_{0i} + \operatorname{div}(\hat{A} \wedge F_{0i}) \,, \tag{A.2}$$

a.e. on $B_i^{\delta,R}$. Thus, using (A.1) and (A.2) we find that

$$\begin{aligned} \int_{M^{\delta,R}_{\underline{x}}} \hat{F} \cdot F_0 dx &= \sum_{i=1}^n \left(\int_{B_i^{\delta,R}} \hat{F} \cdot F_{0i} dx - \int_{b_i^\delta} \hat{F} \cdot F_{0i} dx \right) \\ &= \int_{\Omega_0 \setminus \Omega} \hat{A} \cdot \operatorname{curl} F_0 dx - \sum_{i=1}^n \left(\int_{\partial B_\delta(x_i)} (\hat{A} \wedge F_{0i}) \cdot n_i d\sigma + \int_{b_i^\delta} \hat{F} \cdot F_{0i} dx \right), \end{aligned} \tag{A.3}$$

where $n_i = \frac{x - x_i}{\delta}$ and $d\sigma$ is the scalar surface element on $\partial B_\delta(x_i)$. Note that, the boundary term in (A.3) is well-defined, due to the imbedding $H^{1,2}(U) \hookrightarrow L^2(\partial U)$

for any bounded domain U of class C^1, [12, Theorem A.8]. But it vanishes since $(\hat{A} \wedge F_{0i}) \cdot n_i \equiv 0$, on $\partial B_\delta(x_\iota)$. Thus, using $\hat{F}$ and $F_{0\iota} \in L^2(b_\iota^\delta)$, we infer from (A.3) that

$$\lim_{\delta, R \to 0, \infty} \int_{M_{\underline{x}}^{\delta, R}} \hat{F} \cdot F_0 dx = \int_{\Omega_0 \backslash \Omega} \hat{A} \cdot \operatorname{curl} F_0 dx . \qquad \square$$

Theorem A.1. *Given a monopole of integer charge* $2q$, $q \neq 0$, *with monopole harmonics* $Y_{qlm}^{a,b}(\theta, \varphi) := \Theta_{qlm}(\theta) e^{i(m \pm q)\varphi}$ ($l = |q|, |q| + 1, \ldots$ *and* $m = -l, -l + 1, \ldots, l$) *the following addition formula holds*:

$$\sum_{m=-l}^{l} |Y_{qlm}(\theta, \varphi)|^2 = \frac{2l + 1}{4\pi}, \quad \textit{for all } \theta \textit{ and } \varphi . \tag{A.4}$$

Proof. We first prove that the l.h.s. in (A.4) is constant in θ and φ.

For $l = 0, \frac{1}{2}, 1, \ldots$, let D_l denote the representation space of a representation of $SU(2)$ of spin l. (When l is an integer, the usual spherical harmonics $Y_{q=0lm}$ form an orthonormal basis of D_l.) Given a rotation of three dimensional Euclidean space mapping the unit vector with spherical coordinates (θ', φ') in the one with coordinates (θ, φ), let $\{t_{mm'}^{(l)}\}$ denote the matrix elements of a unitary matrix representing that rotation. For the monopole harmonics ($q \neq 0$) one then has the formula (see [13]).

$$Y_{qlm}^{a,b}(\theta, \varphi) \times \text{phase factor} = \sum_{m'=-l}^{l} Y_{qlm'}^{a,b}(\theta', \varphi') t_{m'm}^{(l)} .$$

Hence, for a (fixed) $(\theta', \varphi') \in \mathcal{C}^a$ and $(\theta, \varphi) \in \mathcal{C}^{a,b}, (\theta, \varphi) \neq (\pi, \cdot)$, we obtain that

$$\sum_{m=-l}^{l} |Y_{qlm}(\theta, \varphi)|^2 = \sum_{m', m''=-l}^{l} Y_{qlm'}^{a}(\theta', \varphi') \overline{Y_{qlm''}^{a}(\theta', \varphi')} \sum_{m=-l}^{l} t_{m'm}^{(l)} \overline{t_{m''m}^{(l)}}$$

$$= \sum_{m'=-l}^{l} |Y_{qlm'}^{a}(\theta', \varphi')|^2 .$$

Therefore the l h.s. in (A 4) is constant for all $(\theta, \varphi) \in \mathcal{C}^{a,b}$. Exploiting this fact, we conclude that

$$\sum_{m=-l}^{l} |Y_{qlm}(\theta, \varphi)|^2 = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \sum_{m=-l}^{l} |Y_{qlm}(\theta, \varphi)|^2 d\varphi \sin\theta d\theta = \frac{2l + 1}{4\pi} ,$$

where we have used that $\langle Y_{qlm}, Y_{ql'm'} \rangle_{S^2} = \delta_{ll'} \delta_{mm'}$, see (3.6). $\square$

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