Link Invariants and Combinatorial Quantization of Hamiltonian Chern Simons Theory

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Abstract: We define and study the properties of observables associated to any link in $\Sigma \times \mathbf{R}$ (where Σ is a compact surface) using the combinatorial quantization of hamiltonian Chern–Simons theory. These observables are traces of holonomies in a non-commutative Yang–Mills theory where the gauge symmetry is ensured by a quantum group. We show that these observables are link invariants taking values in a non-commutative algebra, the so-called Moduli Algebra. When $\Sigma = S^2$ these link invariants are pure numbers and are equal to Reshetikhin–Turaev link invariants.

1. Introduction

Since the fundamental discovery by V. Jones in 1984 of a new link invariant, there has been a tremendous interest and activity in low dimension topology using field theory techniques. The original definition of the Jones Polynomial was purely combinatorial and a geometrical understanding of it was finally given by E. Witten in 1989 [22]. He showed that the Jones polynomial could be interpreted as the correlation function of Wilson loops (i.e. traces of holonomies) in Chern–Simons theory. His work opened a new area of research in what is now called three dimensional topological field theory. Although this theory is purely topological (i.e. in a hamiltonian picture the hamiltonian is zero) and therefore contains no dynamics, the quantization of this theory is not at all a trivial task, mainly because there is no direct procedure to quantify this theory. The original method of E. Witten is a brilliant use of path integrals, heuristic regularization (by a framing) of Wilson loops and relations with conformal field theory. Although very appealing and having far reaching consequences, his formalism is not at all mathematically well defined and this is one of the reasons why many researchers in this field have used other approaches. These methods can be roughly divided in two classes: perturbative and non-perturbative methods. On the one hand perturbative methods

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have continuously attracted interest [15] and have provided many interesting recent results· generalization of Gauss invariants, connections with Vassiliev invariants [6] etc. .. On the other hand, the geometrical quantization program and combinatorial methods are the main approaches to quantize non-perturbatively Chern–Simons theory. Combinatorial methods, introduced by Reshetikhin–Turaev [17] and Turaev–Viro [20] give explicit representations of abstract amplitudes satisfying algebraic relations of a topological field theory. The essential ingredient in their approach is the representation theory of modular Hopf algebras which provides family of numbers satisfying Yang Baxter equation, 6-j identities, etc   These combinatorial methods, although completely well defined, are losing completely the relationship with Chern–Simons theory

The phase space of Hamiltonian Chern–Simons with gauge group G on $\Sigma \times \mathbf{R}$ is the space of flat connections $\mathscr{A}$ on Σ moded out by the action of the gauge group $\mathscr{G}$ $\mathscr{A}/\mathscr{G}$ is homeomorphic to $\mathrm{Hom}(\pi_1(\Sigma), G)^G$ as a manifold. Hence $\mathrm{Hom}(\pi_1(\Sigma), G)^G$ is endowed with a natural structure of Poisson manifold. Therefore quantization of Hamiltonian Chern–Simons theory can be tackled by a direct quantization of the Poisson manifold $\mathrm{Hom}(\pi_1(\Sigma), G)^G$ Attempts in this direction, have been first pursued by V. Turaev [19 and 11] and more recently further investigated in [1, 2].

In our work we will continue the study of a different type of quantization, named combinatorial quantization of Hamiltonian Chern Simons theory, which has been introduced by V V Fock and A.A Rosly [14] and further developed by A.Y. Alekseev et al in [3–5] and ourself [10] This quantization can be thought of as being a lattice regularization of Chern Simons theory in the spirit of Wilson After quantization, gauge invariance is replaced by gauge invariance under a quantum group, and the lattice variables, group elements before quantization, are replaced by elements of a non-commutative algebra. The final and central object of our study is a two dimensional non-commutative Yang Mills theory. Elements of this program was already described in the abelian case in [13].

In Sect 2 of this work we give a summary of works on non-commutative two dimensional Yang Mills theory We associate to each compact triangulated surface Σ a lattice gauge theory which is covariant under a quantum gauge group. The algebra Λ of gauge fields is non-commutative because matrix elements of gauge fields associated to arbitrary edges are non-commuting. Locality is however preserved in the sense that matrix elements of gauge fields associated to edges having no boundary points are commuting elements. Wilson loops associated to non-self-intersecting loops on the surface are defined and it is shown that these Wilson loops are gauge invariant elements A non-commutative analogue of the Yang Mills action is built following the lines of A A. Migdal. In the weak coupling regime, the exponential of this action is an analogue of the Dirac delta function which selects gauge fields with zero curvature. This theory is therefore a topological field theory and the algebra of observables (the Moduli algebra) Λ_{CS} of this theory is expected to be a new description of the algebra of observables of Hamiltonian Chern Simons theory, i.e. when the three manifold is equal to $\Sigma \times \mathbf{R}$.

In Sect 3 we generalize this construction to the case where the loop is an arbitrary framed link L in $\Sigma \times [0, 1]$. We obtain observables associated to these framed links which behave as desired they are gauge invariant and are invariant under ambient isotopy As a result we obtain a new type of ribbon invariants which are not pure numbers but take their values in the algebra Λ_{CS}. This algebra is

non-commutative except in the case $\Sigma = S^2$, where it is one dimensional. In that case the ribbon invariants take their value in the field **C**.

The last part of our work gives the proof that these invariants in the case where $\Sigma = S^2$ are the Reshetikhin–Turaev invariants.

2. Summary of Works on Noncommutative Two Dimensional Yang Mills Theory

In this section we will make constant use of results obtained in [3, 10, 4]. Let Σ be a compact connected oriented surface and let $\mathcal{T}$ be a triangulation of Σ. Let us denote by $\mathcal{F}$ the faces of $\mathcal{T}$, by $\mathcal{L}$ the set of oriented edges and $\mathcal{V}$ the set of points (vertices) of this triangulation. If l is an edge, $-l$ will denote the opposite edge and we have $\{l, -l\} \subset \mathcal{L}$.

If l is an oriented edge it will be convenient to write $l = xy$, where y is the departure point of l and x the end point of l. We will write $y = d(l)$ and $x = e(l)$.

Let A be a quasitriangular Hopf algebra such that each finite dimensional A-module is semisimple. Let $\mathrm{Irr}(A)$ be the set of all equivalency classes of finite dimensional irreducible representations, in each of these classes $\dot{\alpha}$ we will pick out a particular representative α. Let us denote by $\overset{\alpha}{V}$ the vector space on which the representation α acts.

We will denote by $\bar{\alpha}$ (resp. $\tilde{\alpha}$) the right (resp. left) contragredient representation associated to α acting on $\overset{\alpha}{V}{}^{\star}$ and defined by: $\bar{\alpha} = {}^{t}\alpha \circ S$ (resp. $\tilde{\alpha} = {}^{t}\alpha \circ S^{-1}$). We will also denote by 0 the representation of dimension 1 related to the counit ε.

As usual, let $R = \sum_i a_i \otimes b_i$ denote the universal R matrix of A and let us define the invertible element u of A by $u = \sum_i S(b_i)a_i$ (properties of u can be found in [12].) Two important elements of A are the ribbon central element v defined by $v^2 = uS(u)$ and the element $\mu = uv^{-1}$. It will be convenient to define the endomorphism $\overset{\alpha}{\mu} = \alpha(\mu)$ and the complex number v_α by $\alpha(v) = v_\alpha 1$. If $(\beta) \in \mathrm{Irr}(A)^{\times n}$ we will use the notation $\overset{(\beta)}{V}$ to denote the space $\overset{(\beta)}{V} = \bigotimes_{i=1}^{n} \overset{\beta_i}{V}$ and $\overset{(\beta)}{\mu} = \bigotimes_{i=1}^{n} \overset{\beta_i}{\mu}$.

Let $R' = \sigma(R)$, where σ is the permutation operator acting on $A \otimes A$. We will use the standard notation:

$$R^{(+)} = R, \qquad R^{(-)} = R'^{-1}. \tag{1}$$

The q-dimension of α is defined by $[d_\alpha] = \mathrm{tr}(\alpha(\mu))$.

Let $(\overset{\alpha}{e}_i | i = 1 \cdots \dim \overset{\alpha}{V})$ be a particular basis of $\overset{\alpha}{V}$, and $(\overset{\alpha}{e}{}^i | i = 1 \cdots \dim \overset{\alpha}{V})$ its dual basis. We will define the linear forms $\overset{\alpha}{g}{}^j_i = \langle \overset{\alpha}{e}{}^j | \alpha(\,\cdot\,) | \overset{\alpha}{e}_i \rangle$.

The existence of R implies that they satisfy the exchange relations:

$$\overset{\alpha\beta}{R}_{12} \, \overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \overset{\beta}{g}_2 \overset{\alpha}{g}_1 \, \overset{\alpha\beta}{R}_{12}, \tag{2}$$

also equivalent to:

$$\overset{\alpha\beta}{R}{}^{(-)} \, \overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \overset{\beta}{g}_2 \overset{\alpha}{g}_1 \, \overset{\alpha\beta}{R}{}^{(-)}, \tag{3}$$

where $\overset{\alpha\beta}{R} = (\alpha \otimes \beta)(R)$ and $\overset{\alpha\beta}{R}{}^{(-)} = (\alpha \otimes \beta)(R^{(-)})$.

Let Γ be the restricted dual of A: it is by definition the Hopf algebra generated as a vector space by the elements $\overset{\alpha}{g}{}^i_j$.

The action of the coproduct on these elements is:

$$\Delta(\overset{\alpha}{g}{}^i_j) = \sum_k \overset{\alpha}{g}{}^i_k \otimes \overset{\alpha}{g}{}^k_j .$$ (4)

$\overset{\alpha}{V}$ can be endowed with a structure of right comodule over Γ:

$$\Delta_\alpha(\overset{\alpha}{e}_i) = \sum_J \overset{\alpha}{e}_j \otimes \overset{\alpha}{g}{}^j_i .$$ (5)

Let α, β be two fixed elements of $\mathrm{Irr}(A)$. By assumption finite dimensional representations are completely reducible, therefore we can write:

$$\alpha \otimes \beta = \bigoplus_{\gamma \in \mathrm{Irr}(A)} N^\gamma_{\alpha\beta} \gamma .$$ (6)

Let us define, for each γ, $(\psi^{\alpha,\beta}_{\gamma,m})_{m=1, ,N^\gamma_{\alpha\beta}}$ a basis of $\mathrm{Hom}_A(\overset{\alpha}{V} \otimes \overset{\beta}{V}, \overset{\gamma}{V})$ and $(\phi^{\gamma,m}_{\alpha,\beta})_{m=1, ,N^\gamma_{\alpha\beta}}$ a basis of $\mathrm{Hom}_A(\overset{\gamma}{V}, \overset{\alpha}{V} \otimes \overset{\beta}{V})$:

$$\overset{\alpha}{V} \otimes \overset{\beta}{V} \xrightarrow{\psi^{\alpha,\beta}_{\gamma,m}} \overset{\gamma}{V} \xrightarrow{\phi^{\gamma,m'}_{\alpha,\beta}} \overset{\alpha}{V} \otimes \overset{\beta}{V} .$$ (7)

We have the relation:

$$\overset{\alpha}{g}_1 \overset{\beta}{g}_2 = \sum_{\gamma,m} \phi^{\gamma,m}_{\alpha,\beta} \overset{\gamma}{g} \psi^{\alpha,\beta}_{\gamma,m} .$$ (8)

We can always assume that these interwiners satisfy the following relations·

$$\sum_{m,\gamma} \phi^{\gamma,m}_{\alpha,\beta} \psi^{\alpha,\beta}_{\gamma,m} = \mathrm{id}_{\overset{\alpha}{V} \otimes \overset{\beta}{V}} ,$$ (9)

$$\psi^{\alpha,\beta}_{\gamma',m'} \phi^{\gamma,m}_{\alpha,\beta} = \mathrm{id}_{\overset{\gamma}{V}} \delta^m_{m'} \delta^\gamma_{\gamma'} ,$$ (10)

$$\phi^{\gamma,m}_{\beta,\alpha} = \lambda_{\alpha\beta\gamma} P_{12} \overset{\alpha\beta}{R}{}^{-1}_{21} \phi^{\gamma,m}_{\alpha,\beta} ,$$ (11)

$$\psi^{\beta,\alpha}_{\gamma,m} = \lambda^{-1}_{\alpha\beta\gamma} \psi^{\alpha,\beta}_{\gamma,m} \overset{\alpha\beta}{R}_{21} P_{12} ,$$ (12)

$$\psi^{\beta,\alpha}_{\gamma,m} = \sum_{m'} (M^{-1})^{m'}_m \frac{[d_\gamma]^{\frac{1}{2}}[d_\beta]^{\frac{1}{2}}}{[d_\alpha]^{\frac{1}{2}}} (\psi^{\beta,\bar\beta}_0 \otimes \mathrm{id}_{\overset{\gamma}{V}})(\mathrm{id}_{\overset{\beta}{V}} \otimes \phi^{\alpha,m'}_{\bar\beta,\gamma}) ,$$ (13)

$$\phi^{\gamma,m}_{\beta,\alpha} = \sum_{m'} M^m_{m'} \frac{[d_\gamma]^{\frac{1}{2}}[d_\beta]^{\frac{1}{2}}}{[d_\alpha]^{\frac{1}{2}}} (\mathrm{id}_{\overset{\beta}{V}} \otimes \psi^{\bar\beta,\gamma}_{\alpha,m'})(\phi^0_{\beta,\bar\beta} \otimes \mathrm{id}_{\overset{\gamma}{V}}) ,$$ (14)

where $\lambda_{\alpha\beta\gamma} = (v_\alpha v_\beta v^{-1}_\gamma)^{1/2}$ and $M \in GL(N^\gamma_{\alpha\beta})$.

Definition 1 (Gauge symmetry algebra). *Let us define for* $z \in \mathscr{V}$, *the Hopf algebra* $\Gamma_z = \Gamma \times \{z\}$ *and* $\hat\Gamma = \bigotimes_{z \in \mathscr{V}} \Gamma_z$. *This Hopf algebra is "the gauge symmetry algebra."*

If z is a vertex we shall write $\overset{\alpha}{g}_z$ to denote the embedding of the element $\overset{\alpha}{g}$ in Γ_z.

In order to define the non-commutative analogue of algebra of gauge fields we have to endow the triangulation with an additional structure [14], an order between edges incident to each vertex, the *cilium order*.

Definition 2 (Ciliation). *A ciliation of the triangulation is an assignment of a cilium c_z to each vertex z which consists in a non-zero tangent vector at z. The orientation of the surface defines a canonical cyclic order of the links admitting z as departure or end point. Let l_1, l_2 be links incident to a common vertex z, the strict partial cilium order $<_c$ is defined by:*
$l_1 <_c l_2$ if $l_1 \neq l_2, -l_2$ and the unoriented edges c_z, l_1, l_2 appear in the cyclic order defined by the orientation.

If l_1, l_2 are incident to a same vertex z we define:

$$\varepsilon(l_1, l_2) = \begin{cases} +1 & \text{if } l_1 <_c l_2 \\ -1 & \text{if } l_2 <_c l_1 \end{cases} .$$

Definition 3 (Gauge fields algebra). *The algebra of gauge fields [3] Λ is the algebra generated by the elements $\overset{\alpha}{u}(l)^i_j$ with $l \in \mathscr{L}, \alpha \in \mathrm{Irr}(A), i,j = 1 \cdots \dim \overset{\alpha}{V}$ and satisfying the following determining relations:*

Commutation rules.

$$\overset{\alpha}{u}(xy)_1 \overset{\beta}{u}(zy)_2 \overset{\alpha\beta}{R}_{12} = \overset{\beta}{u}(zy)_2 \overset{\alpha}{u}(xy)_1 , \tag{15}$$

$$\overset{\alpha}{u}(xy)_1 \overset{\alpha\beta}{R}_{12}^{-1} \overset{\beta}{u}(yz)_2 = \overset{\beta}{u}(yz)_2 \overset{\alpha}{u}(xy)_1 , \tag{16}$$

$$\overset{\alpha\beta}{R}_{12} \overset{\alpha}{u}(yx)_1 \overset{\beta}{u}(yz)_2 = \overset{\beta}{u}(yz)_2 \overset{\alpha}{u}(yx)_1 , \tag{17}$$

$$\forall (yx), (yz) \in \mathscr{L} x \neq z \quad \text{and} \quad xy <_c yz ,$$

$$\overset{\alpha}{u}(l) \overset{\alpha}{u}(-l) = 1 , \tag{18}$$

$$\forall \, l \in \mathscr{L},$$

$$\overset{\alpha}{u}(xy)_1 \overset{\beta}{u}(zt)_2 = \overset{\beta}{u}(zt)_2 \overset{\alpha}{u}(xy)_1 , \tag{19}$$

$$\forall \, x, y, z, t \text{ pairwise distinct in } \mathscr{V} .$$

Decomposition rule.

$$\overset{\alpha}{u}(l)_1 \overset{\beta}{u}(l)_2 = \sum_{\gamma,m} \phi^{\gamma,m}_{\alpha,\beta} \overset{\gamma}{u}(l) \psi^{\beta,\alpha}_{\gamma,m} P_{12} , \tag{20}$$

$$\overset{0}{u}(l) = 1, \quad \forall l \in \mathscr{L} . \tag{21}$$

Gauge covariance of gauge fields comes from the property that Λ is a right $\hat{\Gamma}$ algebra comodule defined by the morphism of algebra $\Omega : \Lambda \to \Lambda \otimes \hat{\Gamma}$:

$$\Omega(\overset{\alpha}{u}(xy)) = \overset{\alpha}{g}_x \overset{\alpha}{u}(xy) S(\overset{\alpha}{g}_y) . \tag{22}$$

The subalgebra of gauge coinvariant elements of Λ is denoted Λ^{inv}. Moreover it can be shown that $\overset{\alpha}{u}(-l) = \overset{\bar{\alpha}}{u}^{-1t}\overset{\bar{\alpha}}{u}(l)$. If z is a vertex we will define $\Omega_z : \Lambda \to \Lambda \otimes \Gamma_z$ to be equal to $\Omega_z = (\mathrm{id} \otimes p_z)\Omega$, where $p_z : \Gamma \to \Gamma_z$ is the morphism of the algebra defined by $p_z = \bigotimes_{x \in \mathscr{V}, x \neq z} \varepsilon_x$.

It was shown (provided some assumption on the existence of a basis of Λ of a special type) that there exists a unique non-zero linear form $h \in \Lambda^{\star}$ satisfying:

1 (invariance) $(h \otimes \mathrm{id})\Omega(a) = h(a) \otimes 1 \ \forall a \in \Lambda$,
2. (factorisation) $h(ab) = h(a)h(b)$
   $\forall a \in \Lambda_X, \ \forall b \in \Lambda_Y, \ \forall X, \ Y \subset L, \ (X \cup -X) \cap (Y \cup -Y) = \emptyset$

(we have used the notation Λ_X for $X \subset \mathscr{L}$ to denote the subalgebra of Λ generated as an algebra by $\overset{\alpha}{u}(l)$ with $l \in X$ and $\alpha \in \mathrm{Irr}(A)$).

It can be evaluated on any element using the formula·

$$h(\overset{\alpha}{u}(l)^{i}_{j}) = \delta_{\alpha,0} \, , \tag{23}$$

where 0 denotes the trivial representation of dimension 1, corresponding to the counit.

It is convenient to use the notation $\int dh$ instead of h. The following formula is quite important:

$$h(\overset{\alpha}{u}(l)_1 \ \overset{\alpha}{\mu}_2\overset{\alpha}{u}(-l)_2) = \frac{1}{[d_\alpha]}P_{12} \, . \tag{24}$$

We will use this linear form h in Sect. 4 to compute link invariants.

A path P (resp. a loop P) is a connected path (resp. a loop) in the graph attached to the triangulation of Σ, it will also denote equivalently the continuous curve (resp. loop) in Σ defined by the links of P. Following the definition for links, the departure point of P is denoted $d(P)$ and its endpoint $e(P)$. A colored path is a couple (P,α), where P is a path and α is an element of $\mathrm{Irr}(A)$. In the rest of this work, we will use as a shortcut the word path instead of colored path This should cause no confusion

Properties of paths and loops such as self intersections, transverse intersections will always be understood as properties satisfied by the corresponding curves on Σ

Let $x_0, \ldots, x_n$ be points of $\mathscr{V}$ such that $x_{i+1}x_i$ is an edge of the triangulation. This collection of points defines a path $P = [x_n, \ldots, x_0]$, with departure point x_0 and end point x_n. In [10] we defined the sign $\varepsilon(x_i, P) = \varepsilon((x_{i+1}x_i),(x_ix_{i-1}))$.

If P is a simple path $P = [x_n, \quad, x_0]$ with $x_0 \neq x_n$, we can define the holonomy along P by

$$\overset{\alpha}{u}_P = v_\alpha^{\frac{1}{2}\sum_{i=1}^{n-1}\varepsilon(x_i,P)} \prod_{p=n}^{1} \overset{\alpha}{u}(x_px_{p-1}) \tag{25}$$

When C is a simple loop $C = [x_{n+1} = x_0, x_n, \ . \ , x_0]$, we will define the holonomy along C by

$$\overset{\alpha}{u}_C = v_\alpha^{\frac{1}{2}(\sum_{i=1}^{n}\varepsilon(x_i,C)-\varepsilon(x_0,C))} \prod_{p=n}^{0} \overset{\alpha}{u}(x_{p+1}x_p) \, . \tag{26}$$

In [10] we defined an element of Λ, which we called the *Wilson loop* attached to C.

$$\overset{\alpha}{W}_C = \mathrm{tr}(\overset{\alpha}{\mu}\overset{\alpha}{u}_C) \tag{27}$$

This element is gauge invariant and moreover it does not depend on the departure point of the loop C This last property can be easily shown using another equivalent

expression of $\overset{\alpha}{W}_C$:

$$\overset{\alpha}{W}_C = \omega_\alpha(C) \mathrm{tr}_{\underset{V}{\alpha \otimes n+1}} \left(\overset{\alpha}{\mu}^{\otimes n+1} \prod_{i=n}^{1} P_{ii-1} \left(\prod_{j=n}^{1} \overset{\alpha}{u}(x_{j+1}x_j)_j A_j \right) \overset{\alpha}{u}(x_1 x_0)_0 \right), \qquad (28)$$

where A_j is the matrix $R_{jj-1}^{(\varepsilon(x_j,C))-1}$ and $\omega_\alpha(C) = v_\alpha^{-\frac{1}{2}(\sum_{x \in C} \varepsilon(x,C))}$.

The equivalence between relations (27) and (28) uses the simple identity:

$$\mathrm{tr}_1(\overset{\alpha}{\mu}_1 P_{12}(\alpha \otimes \alpha)(R^{(\varepsilon)-1})) = v_\alpha^\varepsilon \mathrm{id}_{\underset{V}{\alpha}}, \qquad (29)$$

where $\varepsilon = \pm 1$.

Remark 1. Compared to our first definition of $\omega_\alpha(C)$ in [10], we have used a different normalisation; they are related by a simple factor v_α. The normalisation of the Wilson loops is discussed in [10] in great details.

Remark 2. Expression of the type (28) is reminiscent of the formulas of [16] for the conserved charges in the context of quantum lax pairs.

It can be shown that $\overset{\alpha}{W}_C$ satisfies the following fusion relation:

$$\overset{\alpha}{W}_C \overset{\beta}{W}_C = \sum_{\gamma \in \mathrm{Irr}(A)} N_{\alpha\beta}^\gamma \overset{\gamma}{W}_C, \qquad (30)$$

where C is any simple loop.

It was also shown that the following commutation relations hold:

$$[\overset{\alpha}{W}_C, \overset{\beta}{W}_{C'}] = 0 \qquad (31)$$

when C, C' are simple loops without transverse intersections.

The geometrical content of this last result is very natural and explained in the sequel.

Remark. We can define the algebra Λ for any type of graph provided that the graph is endowed with a total order of the link incident to each vertex. In particular we can consider any triangulation of any manifold of dimension greater than three, and define a non-commutative lattice gauge theory associated to it. Unfortunately in that case we lose the property (31) which is of central importance to define a non-commutative Yang Mills action commuting with gauge invariant elements. This is the reason which prevents us to extend the present formalism to higher dimensions.

Although the structure of the algebra Λ depends on the ciliation, it has been shown in [3] that the algebra Λ^{inv} does not depend on it up to isomorphism. This is completely consistent with the approach of V.V. Fock and A.A. Rosly: in their work the graph needs to be endowed with a structure of ciliated fat graph in order to put on the space of graph connections $\mathcal{A}^l$ a structure of Poisson algebra compatible with the action of the gauge group G^l. However, as a Poisson algebra $\mathcal{A}^l/G^l$ is canonically isomorphic to the space $\mathcal{M}^G$ of flat connections modulo the gauge group, the Poisson structure of the latter being independent of any choice of r-matrix [14].

In [10] we introduced a Boltzmann weight attached to any simple loop C and defined by:

$$\delta_C = \sum_{\alpha \in \mathrm{Irr}(A)} [d_\alpha] \overset{\alpha}{W}_C \ . \tag{32}$$

It was shown that this element satisfies a delta function property:

$$\delta_C \overset{\alpha}{u}_C{}^i_j = \delta_{1j} \delta_C \ . \tag{33}$$

We were led to define an element a_{YM}, generalizing to our setting the exponential of the Yang Mills action in the topological limit and defined by:

$$a_{YM} = \prod_{f \in \mathscr{F}} \delta_{\partial f}$$

(note that from the relation (31) the elements of this product are pairwise commuting).

This element satisfies the equation:

$$a_{YM} \overset{\alpha}{u}_C = 1 a_{YM} \ , \tag{34}$$

for each homologically trivial simple loop on Σ. This element is the non-commutative analogue of the projector on the space of flat connections. The argument leading to commutation relation (31) can be generalized, and it was proved in [4] that $\delta_{\partial f}$ for $f \in \mathscr{F}$ is a central element of Λ^{inv}.

The algebra $\Lambda_{CS} = \Lambda^{\mathrm{inv}} a_{YM}$ was shown [4] to be independent, up to isomorphism, of the triangulation As a result it was advocated that Λ_{CS} is the algebra of observables of the Chern Simons theory on the manifold $\Sigma \times \mathbf{R}$ This is supported by the topological invariance of Λ_{CS} (i.e. this algebra depends only on the topological structure of the surface Σ and not on the triangulation) and the flatness of the connection

This geometrical representation of Λ_{CS} is particularly appealing. In particular the element $\overset{\alpha}{W}_C$ that we already built should be interpreted as being the observable associated to the Wilson loop of horizontal curves, i.e. loops in $\Sigma \times \{t\}$. The time t, which is the third coordinate in $\Sigma \times \mathbf{R}$, manifests itself in the algebraic point of view as an element used to order the observables if $C_1, \ldots, C_n$ are colored loops on Σ, the element $W_{C_1} \ldots W_{C_n}$ is the observable associated to the link $L = \bigcup_{i=1}^n \{(C_i, t_i)\}$, where $t_1 < \cdots < t_n$. Note that we are free to choose any time t_i provided that they respect the order $t_1 < \cdots < t_n$ This relative independence on the time variable is a simple consequence of the vanishing of the hamiltonian of Chern–Simons theory.

In particular, if C and C' are curves on Σ with no intersection points, the curves (C, t) and (C', t') never intersect. As a result we obtain that

$$\overset{\alpha}{W}_C \overset{\beta}{W}_{C'} = \overset{\beta}{W}_{C'} \overset{\alpha}{W}_C \ , \tag{35}$$

which is the result (31) (note that this last result was proved in the more general case where there are no transverse intersections).

Our aim is now to construct in the algebra Λ_{CS} the observables related to the Wilson loop associated to any link in $\Sigma \times [0, 1]$

Remarks 1. In order to simplify our work we have assumed that the surface has no punctures This situation can be handled using vertical lines as shown in [4].

2. The definition of the element δ_C assumes that the algebra A has a finite number of irreducible representations. Unfortunately we want to apply our formalism to the case where $A = \mathcal{U}_q(\mathcal{G})$. When q is generic we can however formally bypass this technical problem using the formal properties of δ_C such as (33). The only infinity which can possibly occur comes from the square of δ_C due to the relation:

$$\delta_C^2 = \left(\sum_{\alpha \in \mathrm{Irr}(A)} [d_\alpha]^2 \right) \delta_C . \tag{36}$$

The only sensible way to cure this problem seems to work with q being a root of unity and to truncate the spectrum either by quotienting by an appropriate ideal or by using the formalism of weak quasi Hopf algebra as shown in [3, 4].

3. Construction of Observables W_L Associated to Links L in $\Sigma \times [0, 1]$

3.1. Links and chord diagrams. A link in $\Sigma \times [0, 1]$ is an embedding of $(S^1)^{\cup p}$ into $\Sigma \times [0, 1]$. On the set of links we can define a composition law [19], denoted $*$ defined as follows: let $j_{[a,b]}$ be any increasing diffeomorphism from $[0, 1]$ to $[a, b]$, and let us denote $\bar{j}_{[a,b]} = \mathrm{id}_\Sigma \times j_{[a,b]}$. The composition $L * L'$ is defined by

$$L * L' = \bar{j}_{[0,\frac{1}{2}]}(L) \cup \bar{j}_{[\frac{1}{2},1]}(L') , \tag{37}$$

which is an element of

$$\Sigma \times \left[0, \frac{1}{2}\right] \cup_{\Sigma \times \{\frac{1}{2}\}} \Sigma \times \left[\frac{1}{2}, 1\right] = \Sigma \times [0, 1] .$$

When the links are considered up to an ambient isotopy this composition is associative and admits the empty link as a unit element. This composition law is commutative if and only if Σ is homeomorphic to the sphere.

In the sequel we will use as a shortcut the term link to denote a colored link in $\Sigma \times [0, 1]$ (a link in $\Sigma \times [0, 1]$ with an element of $\mathrm{Irr}(A)$ attached to each connected component of L) such that the projection of L on Σ is a union of loops on Σ in generic position (i.e. no more than double points and transverse intersections at these points). Let us denote by $(\overset{i}{L})_{i=1 \cdots p}$ the connected components of the link L, $\alpha_i \in \mathrm{Irr}(A)$ the color of this component, and by $\overset{i}{P}$ the colored loop obtained by projecting $\overset{i}{L}$ on Σ. It is very convenient to associate to the link L a colored chord diagram [21, 9, 1] which will encode intersections of the loops $\overset{i}{P}$. This chord diagram is constructed as follows: the projection of the link on Σ defines p colored loops on Σ with transverse intersections. This configuration of loops defines uniquely a colored chord diagram by the standard construction. Let us denote by $(\overset{i}{S})_{i=1 \cdots p}$ the coloured circles of the chord diagram. Each circle $\overset{i}{S}$ is oriented. We will denote by $(\overset{i}{y}_j)_{j=1 \cdots n_i}$ the intersection points of the circle $\overset{i}{S}$ with the chords. We will assume that they are labelled in such a way that $\overset{i}{y}_j$ appears before $\overset{i}{y}_{j+1}$ with respect to the cyclic order defined by the orientation of the circles. Let $Y = \bigcup_{i=1}^{p} \{\overset{i}{y}_j, j = 1 \cdots n_i\}$.

We define a relation $\sim$ on the set Y by:

$$y \sim y' \quad \text{if and only if } y \text{ and } y' \text{ are connected by a chord} . \qquad (38)$$

We will denote by φ the immersion of the chord diagram in Σ, in particular we have $\overset{i}{P} = \varphi(\overset{i}{S})$. Each intersection point of the projection of L on Σ has exactly two inverse images by φ in the chord diagram and these points are linked by a unique chord If p, q are two points of $\overset{i}{S}$ we will use the notation $[pq]$ to denote the oriented arc segment of $\overset{i}{S}$ having q as departure point and p as endpoint. In the rest of this work we will assume that $\varphi[\overset{i}{y}_{j+1}\overset{i}{y}_j]$ contains at least two edges, for all i, j. This allows us to find a point $\overset{i}{z}_j \in \overset{i}{S}$ such that $\overset{i}{z}_j \in] \overset{i}{y}_j \overset{i}{y}_{j-1}$ [and $\varphi(\overset{i}{z}_j)$ is a vertex of the triangulation. This is a purely technical assumption which could be easily removed. Each circle $\overset{i}{S}$ is the union of $2n_i$ oriented arc segments of type $[\overset{i}{y}_j \overset{i}{z}_j]$ and $[\overset{i}{z}_j \overset{i}{y}_{j-1}]$. Let us denote by $\mathscr{S}_i$ this family of segments, $\mathscr{S} = \bigcup_{i=1}^{p} \mathscr{S}_i$ and $Z = \bigcup_{i=1}^{p} \{ \overset{i}{z}_j, j = 1 \cdots n_i \}$

To each segment $s = [pq]$ of $\mathscr{S}_i$ we will as usual denote its end point $e(s) = p$, its departure point $d(s) = q$, and associate to s the vector spaces V_{q^-} and V_{p^+} such that $V_{q^-} = V_{p^+} = \overset{\alpha_i}{V}$.

If a is a point of $Y \cup Z$, we define $s(a^+)$ (resp $s(a^-)$) to be the unique element of $\mathscr{S}$ such that $e(s(a^+)) = a$ (resp $d(s(a^-)) = a$) We shall also use the arc segment $s(a) = s(a^+) \cup s(a^-)$.

Let $a \in Y \cup Z$, and $\xi \in \{+, -\}$, we define $l(a^\xi)$ to be the link of the path $\phi(s(a^\xi))$ incident to $\phi(a)$.

$\mathscr{S}$ being a finite set, we choose on it a total ordering and denote by $\lhd$ the strict ordering associated to it. This ordering allows us to define two vector spaces V_- and V_+ :

$$V_- = \bigotimes_{s \in \mathscr{S}} V_{d(s)^-} = \bigotimes_{x \in Y \cup Z} V_{x^-} \quad \text{and} \quad V_+ = \bigotimes_{s \in \mathscr{S}} V_{d(s)^+} = \bigotimes_{x \in Y \cup Z} V_{x^+} , \qquad (39)$$

where the order in the tensor product is taken with respect to $\lhd$.

Let $a, b \in Y \cup Z$ and $\xi, \eta \in \{+, -\}$, and assume that $\phi(a) = \phi(b)$. We will use as a shortcut the notation:

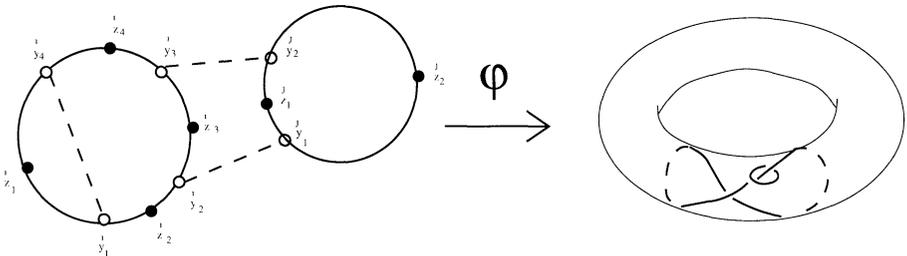$$\varepsilon(a^\xi b^\eta) = \varepsilon(l(a^\xi), l(b^\eta)) . \qquad (40)$$

Fig. 1.

We define the space $\Lambda_{\mathscr{S}}$ by:

$$\Lambda_{\mathscr{S}} = \Lambda \otimes \bigotimes_{s \in \mathscr{S}} \mathrm{Hom}(V_{d(s)^-}, V_{e(s)^+}),\qquad(41)$$

where the order in the tensor product is taken with respect to $\lhd$.

Let $a, b \in Y \cup Z$ and $\xi, \eta \in \{+, -\}$. We shall always use the notation $P_{a^\xi b^\eta}$ to denote the permutation operator exchanging the vector spaces V_{a^ξ} and V_{b^η} in a tensor product of vector spaces.

If s is an element of $\mathscr{S}_i$ we denote by j_s the canonical injection $j_s : \Lambda \otimes \mathrm{Hom}({}^{r}_{d(s)^-}, V_{e(s)^+}) \hookrightarrow \Lambda_{\mathscr{S}}$.

Let us define two types of holonomy along s:

- $u_s \in \Lambda \otimes \mathrm{Hom}(V_{d(s)^-}, V_{e(s)^+})$ is defined by $u_s = u_{\varphi(s)}$, (the right-hand side has already been defined so that there is no risk of confusion).
- $U_s \in \Lambda_{\mathscr{S}}$ is defined by $U_s = j_s(u_{\varphi(s)})$.

We have to introduce both of these holonomies because in the rest of this work some of the constructions are easier to formulate with U_s whereas some are easier to work with u_s. These two points of view were already appearing in our previous work. Indeed if C is a closed path then W_C which is defined by (27) can also be written as (28). The expression (27) only uses the holonomies $u(l)$ (playing the role of variables u_s) whereas expression (28) uses only the variables $u(l)_j$ (playing the role of variables U_s.) The definitions and notations we are explaining in this section have been designed to include automatically the auxiliary space previously labelled by a number. As a result the holonomy U_s is labelled by a segment s and the auxiliary space is also labelled by the same segment. This has the advantage to greatly simplify the notations. The price to pay is that we have to choose a total order on $\mathscr{S}$ (the order $\lhd$).

To formulate the exchange relations satisfied by the U_s we will introduce one more definition:

Definition 4. *Let a, b be two points of $Y \cup Z$ such that $a \in \overset{i}{S}$, $b \in \overset{j}{S}$ and $\phi(a) = \phi(b)$, let us define the endomorphism:*
$R^{(a^\xi b^\eta)} \in \mathrm{End}(V_{a^\xi} \otimes V_{b^\eta})$ *(resp.* $\mathrm{End}(V_{b^\eta} \otimes V_{a^\xi})$*) if $s(a^\xi) \lhd s(b^\eta)$ (resp. if $s(b^\eta) \lhd s(a^\xi)$) by:*

$$R^{(a^\xi b^\eta)} = \begin{cases} (\alpha_i \otimes \alpha_j)(R^{(\varepsilon(a^\xi b^\eta))}) & \text{if } s(a^\xi) \lhd s(b^\eta) \\ P_{a^\xi b^\eta}(\alpha_i \otimes \alpha_j)(R^{(\varepsilon(a^\xi b^\eta))})P_{a^\xi b^\eta} & \text{if } s(b^\eta) \lhd s(a^\xi) \end{cases}.$$

From the previous definitions, the endomorphism $R^{(a^\xi b^\eta)}$ acts on the space $V_{a^\xi} \otimes V_{b^\eta}$ or $V_{b^\eta} \otimes V_{a^\xi}$ depending of the order of $s(a^\xi)$ and $s(b^\xi)$ with respect to $\lhd$. It will be sometimes useful to use the notation $R^{(a^\xi b^\eta)}_{c^\rho d^\sigma}$ to denote the element R or R'^{-1} (depending on the order of $s(a^\xi)$ and $s(b^\eta)$ with respect to $<_c$) represented on the space $V_{c^\rho} \otimes V_{d^\sigma}$ or $V_{d^\sigma} \otimes V_{c^\rho}$ (depending on the order of $s(c^\rho)$ and $s(d^\sigma)$ with respect to $\lhd$. Note that the relation $R^{(a^\xi b^\eta)}_{c^\rho d^\sigma} R^{(b^\eta a^\xi)}_{d^\sigma c^\rho} = 1 \otimes 1$ always holds true.

Proposition 1 (General exchange relations). *The elements U_s, $s \in \mathscr{S}$ satisfy the following exchange relations in $\Lambda_{\mathscr{S}}$.*

$$R^{(v_1^+ v_2^+)} U_{[y_1 z_1]} U_{[y_2 z_2]} = U_{[y_2 z_2]} U_{[z_1 v_1]}, \quad \forall [y_1 z_1], [y_2 z_2] \in \mathscr{S}, y_1 \sim y_2 \,,$$

$$U_{[z_1 y_1]} R^{(y_1^- y_2^+)-1} U_{[y_2 z_2]} = U_{[y_2 z_2]} U_{[y_1 z_1]}, \quad \forall [y_1 z_1], [y_2 z_2] \in \mathscr{S}, y_1 \sim y_2 \,,$$

$$U_{[z_1 y_1]} U_{[z_2 y_2]} R^{(y_1^- v_2^-)} = U_{[z_2 y_2]} U_{[z_1 y_1]}, \quad \forall [z_1 y_1], [z_2 y_2] \in \mathscr{S}, y_1 \sim y_2 \,,$$

$$U_{s_1} R^{(d(s_1)^- e(s_2)^+)-1} U_{s_2} = U_{s_2} U_{s_1}, \quad \forall s_1, s_2 \in \mathscr{S}, \ e(s_2) = d(s_1), \varphi(d(s_2)) \neq \varphi(e(s_1)) \,,$$

$$U_{[y_1 z_1]} R^{(z_1^+ z_1^+)-1} U_{[z_1 y_2]} = U_{[z_1 y_2]} R^{(y_2^- y_1^+)-1} U_{[y_1 z_1]}, \quad \forall [y_1 z_1], [z_1 y_2] \in \mathscr{S}, y_1 \sim y_2 \,,$$

$$U_{s_1} U_{s_2} = U_{s_2} U_{s_1}, \quad \forall s_1, s_2 \in \mathscr{S}, \phi(s_1) \cap \phi(s_2) = \emptyset \,.$$

Proof. It follows straightforwardly from the definition of U_s and the exchange relations of the gauge fields $u(l)$. □

Up to now we have not used the information coming from the topology of the link, i.e over and under crossings of the projected paths $\overset{l}{P}$. This will be encoded in the following definition·

Definition 5. *Let $<$ be any fixed strict total order on Y, we define a family $\{\mathscr{R}^{(y)}\}_{y \in Y}$ of elements of $\bigotimes_{s \in \mathscr{S}} \mathrm{Hom}(V_{d(s)^-}, V_{e(s)^+})$ as follows: let $\{y, y'\}$ be any pair of points of Y such that $y \sim y'$ We can always assume (otherwise we just exchange y and y') that $y < y'$,*

$$\mathscr{R}^{(y)} = \begin{cases} R^{(y^- v^+)-1} & \text{if } \varphi(s(y)) \text{ is above } \varphi(s(y')) \\ R^{(y^+ v'^+)} R^{(y^- y^+)-1} R^{(y^- y'^+)-1} & \text{if } \varphi(s(y)) \text{ is under } \varphi(s(y')) \end{cases}, \quad (42)$$

$$\mathscr{R}^{(y')} = \begin{cases} R^{(v'^- y'^+)-1} & \text{if } \varphi(s(y)) \text{ is above } \varphi(s(y')) \\ R^{(y'^- v^+)-1} R^{(y^- y'^-)} R^{(v'^- v^+)}_{y'^- y^-} & \text{if } \varphi(s(y)) \text{ is under } \varphi(s(y')) \end{cases} \quad (43)$$

This definition defines completely the elements $\mathscr{R}^{(y)}$ for any $y \in Y$. Similarly if z is an element of Z we will define $\mathscr{R}^{(z)} = R^{(z^- z^+)-1}$.

The reader can legitimately find the definition of $\mathscr{R}^{(v)}$ obscure. This definition should be clear after reading the next lemmas.

At this point it is very important to understand that $\mathscr{R}^{(v)}$ depends on numerous orderings namely·

- the total order $<$ on Y
- the total order $\vartriangleleft$ on $\mathscr{S}$

- the partial order coming from the ciliation
- the under and overcrossings of the projection of the link.

We will sometimes write $\mathcal{R}^{(y)}(<)$ to make explicit the dependence in $<$.

Let us define the cyclic permutation operator:

$$\overset{i}{\sigma} = P_{\overset{i}{z_1} + \overset{i}{y_1} +} \prod_{j=2}^{n_i} P_{\overset{i}{z_1} + \overset{i}{z_j} +} P_{\overset{i}{z_1} + \overset{i}{y_j} +} \, .$$

We now have defined the framework necessary to associate to L an element of Λ denoted W_L which generalizes the construction of Wilson loops. We denote by $<_l$ the strict lexicographic order induced on Y by the enumeration of the connected components of L and a choice of departure point for each of these components, i.e. $\overset{i}{y}_p <_l \overset{j}{y}_q$ if and only if $i < j$ or $(i = j$ and $p > q$.)

Let us denote the holonomy along the circle $\overset{i}{S}$ by

$$\overset{i}{\mathcal{U}} = \omega(\overset{i}{S}) U_{[\overset{i}{z_1} \overset{i}{y}_{n_i}]} \mathcal{R}^{(\overset{i}{y}_{n_i})}(<_l) U_{[\overset{i}{y}_{n_i} \overset{i}{z}_{n_i}]} \mathcal{R}^{(\overset{i}{z}_{n_i})}(<_l) U_{[\overset{i}{z}_{n_i} \overset{i}{y}_{n_{i-1}}]} \cdots U_{[\overset{i}{y}_1 \overset{i}{z}_1]}, \qquad (44)$$

where $\omega(\overset{i}{S}) = v_{\alpha_i}^{-\frac{1}{2} \sum_{p=1}^{n_i} \varepsilon(\phi(\overset{i}{y}_p), \overset{i}{P}) + \varepsilon(\phi(\overset{i}{z}_p), \overset{i}{P})}$.

This holonomy generalizes the holonomy we associated in [10] to simple loops: the elements $\mathcal{R}^{(y)}$ contain all the information on the relative crossings of the projection of the link L.

Let y be a point of Y such that $y \in \overset{i}{S}$, let $[zy] = s(y^-)$ and $[yt] = s(y^+)$, we can define the holonomy

$$\mathcal{U}^{(y)} = v_{\alpha_i}^{-\frac{1}{4}(\varepsilon(\varphi(z), \overset{i}{P}) + 2\varepsilon(\varphi(y), \overset{i}{P}) + \varepsilon(\varphi(t), \overset{i}{P}))} U_{[zy]} \mathcal{R}^{(y)} U_{[yt]}, \qquad (45)$$

which is such that

$$\overset{i}{\mathcal{U}} = \prod_{m=n_i}^{2} (\mathcal{U}^{(\overset{i}{y}_m)} \mathcal{R}^{(\overset{i}{z}_m)}) \mathcal{U}^{(\overset{i}{y}_1)} \, . \qquad (46)$$

The following important lemmas describe the commutation relations of the elements $\mathcal{U}^{(y)}$ which explain the definition of $\mathcal{R}^{(y)}$.

Lemma 1. *Let y be a point of Y in $\overset{i}{S}$ and let us denote by y' the point of Y connected by a chord to y, and assume that $y < y'$. We have the following commutation relations:*

- *If $s(y) \cap s(y') = \emptyset$ and $\varphi(s(y))$ is above $\varphi(s(y'))$, then*

$$\mathcal{U}^{(y)} \mathcal{U}^{(y')} = \sum_r \beta_{y'+}^r R^{(y'^+ y^+)} \mathcal{U}^{(y')} R^{(y'^- y^+) - 1} \mathcal{U}^{(y)} R^{(y'^- y^-)} S^2(\alpha_{y-}^r), \qquad (47)$$

where $R^{(y^- y'^+)} = \sum_r \alpha_{y-}^r \otimes \beta_{y'+}^r$,

- if $s(y) \cap s(y') = \{z\}$, with $z = d(s(y)) = e(s(y'))$ and $\varphi(s(y))$ is above $\varphi(s(y'))$ then:

$$\mathcal{U}^{(y)} R^{(z^- z^+) - 1} \mathcal{U}^{(y')} = \sum_r \beta'_{y'+} R^{(y'^+ y^+)} \mathcal{U}^{(y')} R^{(y'^- y^+) - 1} \mathcal{U}^{(y)} R^{(y'^- y^-)} S^2(\alpha'_{y-}) \qquad (48)$$

Let a, b be two distinct points of Y we denote by $\tau_{a,b} <$ the total order on Y obtained from $<$ by permuting a and b.

The first relation can also be written·

- if y and y' are points of Y linked by a chord such that $s(y) \cap s(y') = \emptyset$ with $y < y'$ and $\varphi(s(y))$ above $\varphi(s(y'))$:

$$\mathcal{U}^{(y)}(<) \mathcal{U}^{(y')}(<) = \sum_r \beta'_{y'+} \mathcal{U}^{(y')}(\tau_{y,y'} <) \mathcal{U}^{(y)}(\tau_{y,y'} <) R^{(y^- y'^+) - 1}_{y^- y'^-} S^2(\alpha'_{y-}), \qquad (49)$$

where $R^{(y^- y'^+)} = \sum_l \alpha'_{y-} \otimes \beta'_{y'+}$.

- if y and y' are points of Y linked by a chord such that $s(y) \cap s(y') = \emptyset$ with $y < y'$ and $\varphi(s(y))$ under $\varphi(s(y'))$:

$$\mathcal{U}^{(y)}(<) \mathcal{U}^{(y')}(<) = \sum_l \beta^r_{y'+} \mathcal{U}^{(y')}(\tau_{y,y'} <) \mathcal{U}^{(y)}(\tau_{y,y'} <) S(\alpha'_{y'-}) R^{(y'^- y^+)}_{y'^- y^-}. \qquad (50)$$

(Similarly if $s(y) \cap s(y') = \{z\}$ with $z = d(s(y)) = e(s(y'))$ the last two equations hold true if one replaces the left-hand side by $\mathcal{U}^{(y)} R^{(z^- z^+) - 1} \mathcal{U}^{(y')}$).

Proof   This lemma is a direct consequence of the following computation:

$$U_{[zv]} R^{(y^- y^+) - 1} U_{[yt]} U_{[z'y']} R^{(y'^- y'^+) - 1} U_{[y't']} = U_{[yt]} U_{[zy]} U_{[y't']} U_{[z'y']}$$

$$= \sum_i U_{[yt]} U_{[zy]} \beta^i_{y'+} R^{(y^- y^+) - 1} S^2(\alpha^i_{y-}) U_{[y't']} U_{[z'v']}$$

$$= \sum_i \beta^i_{y'+} U_{[yt]} U_{[y't']} U_{[zy]} U_{[z'y']} S^2(\alpha^i_{y-})$$

$$= \sum_l \beta^i_{y'+} R^{(y'^+ y^+)} U_{[y't']} U_{[yt]} U_{[z'y']} U_{[zy]} R^{(y^- y'^-)} S^2(\alpha^i_{y-})$$

$$= \sum_i \beta^i_{y'+} R^{(y'^+ y^+)} U_{[y't']} U_{[z'y']} R^{(y'^- v^+) - 1} U_{[yt]} U_{[zy]} R^{(y^- y'^-)} S^2(\alpha^i_{y-})$$

$$= \sum_i \beta^i_{y'+} R^{(y'^+ v^+)} U_{[z'y']} R^{(y'^- y'^+) - 1} U_{[y't']} R^{(y'^- y^+) - 1} U_{[zy]} R^{(y^- y^+) - 1} U_{[yt]}$$

$$\times R^{(y'^- y^-)} S^2(\alpha^i_{y-}).$$

Equations (47, 48) are a straightforward consequence of the previous computations and the definition of $\mathcal{R}^{(y)}$.   □

Lemma 2. Let y_1, y_2 be two points of Y not connected by a chord and let y'_1, y'_2 be the points of Y such that $y_1 \sim y'_1$, $y_2 \sim y'_2$ If we assume $s(y_1) \cap s(y_2) = \emptyset$, we have the following relation:

$$\mathcal{U}^{(y_1)}(<) \mathcal{U}^{(y_2)}(<) = \mathcal{U}^{(y_2)}(<) \mathcal{U}^{(y_1)}(<). \qquad (51)$$

Moreover if we assume $y_1 < y_2$ and $\{y'_1, y'_2\} \cap \{y \in Y, \ y_1 < y < y_2\} = \emptyset$ the last relation can also be written:

$$\mathcal{U}^{(y_1)}(<)\mathcal{U}^{(y_2)}(<) = \mathcal{U}^{(y_2)}(\tau_{y_1, y_2} <)\mathcal{U}^{(y_1)}(\tau_{y_1, y_2} <). \tag{52}$$

Of course if $s(y_1) \cap s(y_2) = \{z\}$ with $z = d(s(y_1)) = e(s(y_2))$ the previous equations hold true if we replace the left-hand side by: $\mathcal{U}^{(y_1)}(<)R^{(z^- z^+) - 1}\mathcal{U}^{(y_2)}(<)$.

Proof. The first part follows straightforwardly from the definition of $\mathscr{R}^{(y)}$ and the assumption $s(y_1) \cap s(y_2) = \emptyset$. The second part comes from the fact that the relative order of y_i and y'_i with respect to $<$ is the same as the one with respect to $\tau_{y_1, y_2} <$. $\square$

3.2. Definition and first properties of the observables W_L associated to a link $L \subset \Sigma \times [0, 1]$

Definition 6 (Generalized Wilson loops). *To each link in $\Sigma \times [0, 1]$ satisfying the assumptions of Sect. (3.1) we associate an element W_L of the algebra Λ by the following procedure: let us denote by $\mathscr{W}_L$ the element*

$$\mathscr{W}_L = \mu_{\mathscr{S}} \prod_{i=1}^{p} \overset{i}{\sigma} \prod_{i=1}^{p} \overset{i}{\mathcal{U}}, \tag{53}$$

where $\mu_{\mathscr{S}} = \bigotimes_{s \in \mathscr{S}} \mu_{e(s)^+}$.

From the definition of $\overset{i}{\sigma}$ it follows that: $\mathscr{W}_L \in \Lambda \otimes \mathrm{Hom}(V_-, V_+)$.

The Wilson loop associated to the link L is defined by $W_L = \mathrm{tr}_{V_-} \mathscr{W}_L$, where tr_{V_-} means the partial trace over the space V_- after the natural identification $V_+ = V_-$. It will be convenient to write

$$W_L = \mathrm{Tr}_q \left(\prod_{i=1}^{p} \overset{i}{\mathcal{U}} \right). \tag{54}$$

When L is a simple loop in $\Sigma \times \{t\}$, the element W_L we just defined is equal to the Wilson loop we already defined by Eq. (27).

This element satisfies very important properties which are contained in the following theorem:

Theorem 1. *Let L be a link in $\Sigma \times [0, 1]$, then W_L does not depend on the labelling of the components nor on the choice of departure points of the components. As a result W is a function on the space of links with values in Λ. Moreover this mapping takes its value in Λ^{inv}.*

If L and L' are two links, we have the morphism property:

$$W_{L \star L'} = W_L W_{L'}. \tag{55}$$

Proof. We will first show that W_L is invariant under relabelling of the connected components of the link. It suffices to show that W_L is invariant under the exchange

of $\overset{i}{S}$ and $\overset{i+1}{S}$ for all i. Let $j = i + 1$, we can write

$$W_L = \mathrm{Tr}_q(A\,\overset{i}{\mathscr{U}}\overset{i+1}{\mathscr{U}}\,B)$$

$$= \mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\quad \mathscr{U}^{(\overset{i}{y}_1)}\mathscr{U}^{(\overset{j}{y}_{n_j})}\mathscr{R}^{(\overset{j}{z}_{n_j})}\cdots \mathscr{U}^{(\overset{j}{y}_1)}B)\,.$$

Let $\overset{(ij)}{<_l}$ denote the lexicographic order associated to the labelling of the connected components after the exchange of $\overset{i}{S}$ and $\overset{i+1}{S}$, it is obtained from the first lexicographic order $<_l$ by exchanging the points $\overset{i}{y}_{n_i} <_l \quad \cdot <_l \overset{i}{y}_1$ with the points $\overset{j}{y}_{n_j} <_l \quad <_l \overset{j}{y}_1$ In order to prove invariance under the permutation of the connected components it suffices to show that:

$$\mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}(<_l)\mathscr{R}^{(\overset{i}{z}_{n_i})}\,\cdots\mathscr{U}^{(\overset{i}{y}_1)}(<_l)\mathscr{U}^{(\overset{j}{v}_{n_j})}(<_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\cdots\mathscr{U}^{(\overset{j}{y}_1)}(<_l)B)$$

$$= \mathrm{Tr}_q(A\mathscr{U}^{(\overset{j}{y}_{n_j})}(\overset{(ij)}{<_l})\mathscr{R}^{(\overset{j}{z}_{n_j})}\cdots\quad\mathscr{U}^{(\overset{j}{y}_1)}(\overset{(ij)}{<_l})\mathscr{U}^{(\overset{i}{y}_{n_i})}(\overset{(ij)}{<_l})\mathscr{R}^{(\overset{i}{z}_{n_i})}\cdot\quad\mathscr{U}^{(\overset{i}{y}_1)}(\overset{(ij)}{<_l})B)\,.$$

The proof goes as follows if $\overset{i}{y}_1$ is not connected to any $\overset{j}{y}_m$, then we deduce from the second lemma that·

$$\mathscr{U}^{(\overset{i}{y}_1)}(<_l)\mathscr{U}^{(\overset{j}{v}_{n_j})}(<_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\,\cdots\mathscr{U}^{(\overset{j}{y}_1)}(<_l)$$

$$= \mathscr{U}^{(\overset{j}{y}_{n_j})}(\tau_{\overset{i}{y}_1,\overset{j}{v}_{n_j}}<_l)\mathscr{U}^{(\overset{i}{y}_1)}(\tau_{\overset{i}{y}_1,\overset{j}{v}_{n_j}}<_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\cdot\quad\mathscr{U}^{(\overset{j}{y}_1)}(<_l)$$

$$= \mathscr{U}^{(\overset{j}{v}_{n_j})}(\tau_{\overset{i}{y}_1,\overset{j}{v}_{n_j}}<_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\mathscr{U}^{(\overset{i}{y}_1)}(\tau_{\overset{i}{y}_1,\overset{j}{y}_{n_j}}<_l)\cdot\mathscr{U}^{(\overset{j}{y}_1)}(\tau_{\overset{i}{y}_1,\overset{j}{y}_{n_j}}<_l)$$

$$= \mathscr{U}^{(\overset{j}{y}_{n_j})}\left(\prod_{p=1}^{n_j}\tau_{\overset{i}{y}_1,\overset{j}{y}_p}<_l\right)\mathscr{R}^{(\overset{j}{z}_{n_j})}\cdots\mathscr{U}^{(\overset{j}{y}_1)}\left(\prod_{p=1}^{n_j}\tau_{\overset{i}{y}_1,\overset{j}{y}_p}<_l\right)\mathscr{U}^{(\overset{i}{y}_1)}\left(\prod_{p=1}^{n_j}\tau_{\overset{i}{y}_1,\overset{j}{y}_p}<_l\right)\,.$$

Let τ_q denote the permutation $\tau_q = \prod_{p=1}^{n_j}\tau_{\overset{i}{y}_q,\overset{j}{y}_p}$ We have just proved that if $\overset{i}{y}_1$ is not connected to any $\overset{j}{y}_m$ we have:

$$\mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\quad\mathscr{U}^{(\overset{i}{y}_1)}\mathscr{U}^{(\overset{j}{y}_{n_j})}\mathscr{R}^{(\overset{j}{z}_{n_j})}\quad\mathscr{U}^{(\overset{j}{y}_1)}B)$$

$$= \mathrm{Tr}_q(A\mathscr{U}^{(\overset{j}{y}_{n_j})}(\tau_1<_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\cdot\quad\mathscr{U}^{(\overset{j}{y}_1)}(\tau_1<_l)\mathscr{U}^{(\overset{i}{y}_{n_i})}(\tau_1<_l)\mathscr{R}^{(\overset{i}{z}_{n_i})}\quad\cdot\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1<_l)B)$$

This last equation still holds true even if $\overset{i}{y}_1$ is connected by a chord to $\overset{j}{y}_m$. We will show it if $m = n_j$, the other cases are treated with the same method

Let us assume first that $\varphi(s(\overset{i}{y}_1))$ is above $\varphi(s(\overset{j}{y}_m))$ and apply Lemma 1:

$$\mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{U}^{(\overset{i}{y}_1)}\mathscr{U}^{(\overset{j}{y}_{n_j})}\mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}B)$$

$$= \sum_r \mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{R}^{(\overset{i}{z}_2)}\beta^r_{\overset{j}{y}_{n_j}+}\mathscr{U}^{(\overset{j}{y}_{n_j})}(\tau_1 <_l)\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1 <_l)$$

$$\times R^{(\overset{i}{y}_1-\overset{j}{y}_{n_j}+)}_{\overset{i}{y}_1-\overset{j}{y}_{n_j}-}S^2(\alpha^r_{\overset{j}{y}_{n_j}-})\mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}B)$$

$$= \sum_r \mathrm{Tr}_q(\beta^r_{\overset{j}{y}_{n_j}+}A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{R}^{(\overset{i}{z}_2)}\mathscr{U}^{(\overset{j}{y}_{n_j})}(\tau_1 <_l)\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1 <_l)$$

$$\times \mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}BR^{(\overset{i}{y}_1-\overset{j}{y}_{n_j}+)}_{\overset{i}{y}_1-\overset{j}{y}_{n_j}-}S^2(\alpha^r_{\overset{j}{y}_{n_j}-}))$$

$$= \mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{R}^{(\overset{i}{z}_2)}\mathscr{U}^{(\overset{j}{y}_{n_j})}(\tau_1 <_l)\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1 <_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}B)$$

$$= \mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{R}^{(\overset{i}{z}_2)}\mathscr{U}^{(\overset{j}{y}_{n_j})}(\tau_1 <_l)\mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}(\tau_1 <_l)\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1 <_l)B)$$

$$= \mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}(\tau_1 <_l)\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1 <_l)\mathscr{U}^{(\overset{i}{y}_{n_i})}(\tau_1 <_l)\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{U}^{(\overset{i}{y}_1)}(\tau_1 <_l)B) \, .$$

If $\varphi(s(\overset{i}{y}_1))$ is under $\varphi(s(\overset{j}{y}_m))$ we use the same type of proof but use instead relation (48).

Up to now we have shown that we can move $\mathscr{U}^{(\overset{i}{y}_1)}$ to the right with an exchange of $<_l$ into $\tau_1 <_l$. The same arguments apply as well to $\mathscr{U}^{(\overset{i}{y}_2)}$ up to $\mathscr{U}^{(\overset{i}{y}_{n_i})}$. We finally end up with the following equation:

$$\mathrm{Tr}_q(A\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{U}^{(\overset{i}{y}_1)}\mathscr{U}^{(\overset{j}{y}_{n_j})}\mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}B)$$

$$= \mathrm{Tr}_q\left(A\mathscr{U}^{(\overset{j}{y}_{n_j})}\left(\prod_{p=n_i}^{1}\tau_p <\right)\mathscr{R}^{(\overset{j}{z}_{n_j})}\ldots\mathscr{U}^{(\overset{j}{y}_1)}\left(\prod_{p=n_i}^{1}\tau_p <\right)\right.$$

$$\left.\times \mathscr{U}^{(\overset{i}{y}_{n_i})}\left(\prod_{p=n_i}^{1}\tau_p <\right)\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{U}^{(\overset{i}{y}_1)}\left(\prod_{p=n_i}^{1}\tau_p <\right)B\right) \, ,$$

which establishes the result because $\overset{(ij)}{<}_l = \prod_{p=n_i}^{1}\tau_p <$. This ends up the proof that W_L is invariant under relabelling of the connected components of L.

Let us prove now that W_L is invariant under the choice of departure points of the curve $\overset{i}{P}$ used to define W_L. From the first part of the theorem that we just have proved, we can write:

$$W_L = \mathrm{Tr}_q(\mathscr{U}^{(\overset{i}{y}_{n_i})}\mathscr{R}^{(\overset{i}{z}_{n_i})}\ldots\mathscr{U}^{(\overset{i}{y}_1)}A) \, . \tag{56}$$

Once again the idea of the proof is to use Lemmas 1 and 2 to move $\mathcal{U}^{(\overset{i}{y}_{n_i})}$ to the right.

Let us define $c_i <_l$ the strict order deduced from $<_l$ by applying a cyclic permutation to $\overset{i}{y}_{n_i}, \ .., \overset{i}{y}_1$ such that $\overset{i}{y}_{n_{i-1}} c_i <_l \ \cdot <_l c_i <_l \overset{i}{y}_1 c_i <_l \overset{i}{y}_{n_i}$

Assume first that $\overset{i}{y}_{n_i}$ is not connected to any of the points $\overset{i}{y}_p$, we can therefore write, from Lemma 2

$$\mathcal{U}^{(\overset{i}{v}_{n_i})}\mathcal{R}^{(\overset{i}{z}_{n_i})}\mathcal{U}^{(\overset{i}{v}_{n_i-1})} \quad \mathcal{U}^{(\overset{i}{y}_1)} = \mathcal{U}^{(\overset{i}{y}_{n_i-1})}\mathcal{U}^{(\overset{i}{y}_{n_i})}\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \ . \ \mathcal{U}^{(\overset{i}{y}_1)}$$

$$= \mathcal{U}^{(\overset{i}{y}_{n_i-1})}\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \quad \mathcal{U}^{(\overset{i}{y}_{n_i})}\mathcal{U}^{(\overset{i}{y}_1)}$$

$$= \mathcal{U}^{(\overset{i}{y}_{n_i-1})}\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \ . \ \mathcal{U}^{(\overset{i}{y}_1)}\mathcal{R}^{(\overset{i}{z}_1)}\mathcal{U}^{(\overset{i}{v}_{n_i})} \ ,$$

which is equal to·

$$\mathcal{U}^{(\overset{i}{v}_{n_i-1})}(c_i <_l) \cdot \quad \mathcal{U}^{(\overset{i}{y}_1)}(c_i <_l)\mathcal{R}^{(\overset{i}{z}_1)}\mathcal{U}^{(\overset{i}{y}_{n_i})}(c_i <_l) \ . \tag{57}$$

This proof does not work, but the result is still true, with the exception of $n_i = 2$ In that case one has to generalize Lemma 2 to the case where $s(y_1) \cap s(y_2) = \{z, z'\}, z \neq z'$. We leave the details to the reader

Assume now that $\overset{i}{y}_{n_i}$ is connected to one of the points $\overset{i}{y}_p$, and let us choose $p = n_{i-1}$ to show the structure of the proof:

$$W_L = \text{Tr}_q(\mathcal{U}^{(\overset{i}{y}_{n_i})}\mathcal{R}^{(\overset{i}{z}_{n_i})} \ . \ \mathcal{U}^{(\overset{i}{y}_1)}A)$$

$$= \text{Tr}_q(\mathcal{U}^{(\overset{i}{y}_{n_i-1})}(\tau_{\overset{i}{y}_{n_i}\overset{i}{y}_{n_i-1}} <_l)\mathcal{U}^{(\overset{i}{y}_{n_i})}(\tau_{\overset{i}{v}_{n_i}\overset{i}{v}_{n_i-1}} <_l)\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \ . \ \mathcal{U}^{(\overset{i}{v}_1)}A) \ .$$

using Lemma 1 and the property of the quantum trace

Finally we obtain.

$$W_L = \text{Tr}_q(\mathcal{U}^{(\overset{i}{v}_{n_i-1})}(\tau_{\overset{i}{v}_{n_i}\overset{i}{v}_{n_i-1}} <_l)\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \quad \mathcal{U}^{(\overset{i}{y}_{n_i})}(\tau_{\overset{i}{v}_{n_i}\overset{i}{y}_{n_i-1}} <_l)\mathcal{U}^{(\overset{i}{y}_1)}(\tau_{\overset{i}{v}_{n_i}\overset{i}{v}_{n_i-1}} <_l)A)$$

$$= \text{Tr}_q(\mathcal{U}^{(\overset{i}{y}_{n_i-1})}(\tau_{\overset{i}{y}_{n_i}\overset{i}{v}_{n_i-1}} <_l)\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \ . \ \mathcal{U}^{(\overset{i}{y}_1)}(\tau_{\overset{i}{y}_{n_i}\overset{i}{v}_{n_i-1}} <_l)\mathcal{R}^{(\overset{i}{z}_1)}\mathcal{U}^{(\overset{i}{v}_{n_i})}$$

$$\times (\tau_{\overset{i}{v}_{n_i}\overset{i}{y}_{n_i-1}} <_l)A)$$

$$= \text{Tr}_q(\mathcal{U}^{(\overset{i}{y}_{n_i-1})}(c_i <_l)\mathcal{R}^{(\overset{i}{z}_{n_i-1})} \quad \mathcal{U}^{(\overset{i}{y}_1)}(c_i <_l)\mathcal{R}^{(\overset{i}{z}_1)}\mathcal{U}^{(\overset{i}{y}_{n_i})}(c_i <_l)A) \ .$$

It is easy to show using the same method that this result holds as well whatever the value of p can be

We have shown that we can always replace the departure point $\overset{i}{z}_1$ by its neighbour $\overset{i}{z}_{n_i}$. As a result, by a trivial induction, it proves that W_L is invariant under the change of departure point of each connected component.

Let us now show that W_L is a gauge invariant element It is easy to show that W_L is a co-invariant element under the coaction Ω_x, where $x \in \mathcal{V} \setminus \varphi(Y)$: this is the

same standard proof of Prop. 2 in [10]. The only non-trivial part comes from the points of $\varphi(Y)$, i.e. the intersection points of the loops $\overset{i}{P}$. Let $y \in Y$, according to the previous part of the proof. We can always write $W_L = \text{tr}_q(\mathcal{U}^{(y)}A\mathcal{U}^{(y')}B)$, where $y \sim y'$, $\varphi(y)$ above $\varphi(y')$ and A and B are elements of the algebra Λ containing no variable $\overset{\alpha}{u}(l)$ with l incident to $\varphi(y)$.

We have:

$$\Omega_{\varphi(y)}(W_L)$$

$$= \text{Tr}_q(U_{[zy]}S(g_{y^-})R^{(y^- y^+)-1}g_{y^+}U_{[yt]}AU_{[z'y']}S(g_{y'-})R^{(y'^- y'^+)-1}g_{y'+}U_{[y't']}B)$$

$$= \text{Tr}_q(U_{[zy]}g_{y^+}R^{(y^- y^+)-1}S(g_{y^-})U_{[yt]}AU_{[z'y']}g_{y'+}R^{(y'^- y'^+)-1}S(g_{y'-})U_{[y't']}B)$$

$$= \text{Tr}_q(g_{y^+}U_{[zy]}R^{(y^- y^+)-1}S(g_{y^-})g_{y'+}U_{[yt]}AU_{[z'y']}R^{(y'^- y'^+)-1}S(g_{y'-})U_{[y't']}B)$$

$$= \sum_i \text{Tr}_q(g_{y^+}U_{[zy]}R^{(y^- y^+)-1}S(g_{y^-})\beta^i_{y'+}R^{(y^- y'^+)-1}S^2(\alpha^i_{y^-})g_{y'+}U_{[yt]}A$$

$$\times U_{[z'y']}R^{(y'^- y'^+)-1}S(g_{y'-})U_{[y't']}B)$$

$$= \sum_i \text{Tr}_q(\beta^i_{y'+}g_{y^+}U_{[zy]}R^{(y^- y^+)-1}S(g_{y^-})R^{(y^- y'^+)-1}g_{y'+}U_{[yt]}A$$

$$\times U_{[z'y']}R^{(y'^- y'^+)-1}S(g_{y'-})U_{[y't']}BS^2(\alpha^i_{y^-}))$$

$$= \sum_i \text{Tr}_q(\alpha^i_{y^+}\beta^i_{y'+}g_{y^+}U_{[zy]}R^{(y^- y^+)-1}g_{y'+}R^{(y^- y'^+)-1}S(g_{y^-})U_{[yt]}A$$

$$\times U_{[z'y']}R^{(y'^- y'^+)-1}S(g_{y'-})U_{[y't']}B)$$

$$= \text{tr}_q(R^{(y^- y'^+)-1}_{y^+ y'+}g_{y^+}g_{y'+}U_{[zy]}R^{(y^- y^+)-1}R^{(y^- y'^+)-1}U_{[yt]}A$$

$$\times U_{[z'y']}R^{(y'^- y'^+)-1}U_{[y't']}BS(g_{y^-})S(g_{y'-}))$$

$$= \text{Tr}_q(g_{y'+}g_{y^+}R^{(y^- y'^+)-1}_{y^+ y'+}U_{[zy]}R^{(y^- y^+)-1}R^{(y^- y'^+)-1}U_{[yt]}A$$

$$\times U_{[z'y']}R^{(y'^- y'^+)-1}U_{[y't']}BS(g_{y^-})S(g_{y'-}))$$

$$= \text{Tr}_q(R^{(y^- y'^+)-1}_{y^+ y'+}U_{[zy]}R^{(y^- y^+)-1}R^{(y^- y'^+)-1}U_{[yt]}A$$

$$\times U_{[z'y']}R^{(y'^- y'^+)-1}U_{[y't']}B) .$$

As a result we get $\Omega_{\varphi(y)}(W_L) \in \Lambda \otimes 1$. Applying $(id \otimes \varepsilon)$ and using the comodule definition, it follows that: $\Omega_{\varphi(y)}(W_L) = W_L \otimes 1$, i.e. W_L is an element of Λ^{inv}.

Let L, L' be two links of $\Sigma \times [0,1]$, and $\{\overset{i}{S_L} \, i = 1, \ldots, p_L\}$, (resp. $\{\overset{i}{S_{L'}} \, i = 1, \ldots, p_{L'}\}$) be the circles associated to the chord diagram of L (resp. L'). The family of circles of the chord diagram of $L * L'$ is the union $\{\overset{i}{S_L} \, i = 1, \ldots, p_L\} \cup \{\overset{i}{S_{L'}} \, i = 1, \ldots, p_{L'}\}$. We can choose a labelling of the circles of the chord diagram of $L * L'$ such that those of L appear before those of L'. Because $\varphi(L)$ is above $\varphi(L')$ the expression of $\mathcal{R}^{(y)}$ such that $\varphi(y) \in \varphi(L) \cap \varphi(L')$ does not connect L and L', i.e.

$\mathscr{R}^{(y)} = R^{(y^- y^+)-1}$. From the definition of $W_{L*L'}$ we immediately get $W_{LL'} = W_L W_{L'}$. This fact is the final step in the proof of the theorem. □

As already explained in Sect. 2, when L is a simple loop in $\Sigma \times [0, 1]$, W_L has two equivalent expressions: the first one, called "expanded form" (Eq. 28), can be written as a partial quantum trace over the space $\Lambda \otimes \bigotimes_{s \in \mathscr{S}} \mathrm{Hom}(V_{d(s)^-}, V_{e(s)^+})$, whereas the second one (Eq. 27), called "contracted form," is expressed as a partial quantum trace over the space $\Lambda \otimes \bigotimes_{t=1}^{p} \mathrm{End}(\overset{\alpha_t}{V})$. The relation between them is a direct consequence of the following result:

$$\text{if } A \in \mathrm{Hom}(X, Y) \text{ and } B \in \mathrm{Hom}(Y, Z) \text{ then } BA = \mathrm{tr}_Y(P_{Y,Z} B \otimes A).$$

The definition of W_L for arbitrary links is naturally defined in terms of the U_s, it can of course be written in terms of a quantum partial trace over $\Lambda \otimes \bigotimes_{t=1}^{p} \mathrm{End}(\overset{\alpha_t}{V})$. The expression is quite messy, the only essential point being that we can switch from one expression to the other. This contracted form enjoys the following property:

Lemma 3. *Assume that there exists a connected open path* $P \subset \overset{i}{P}$ *such that*

1. $e(P)$ *and* $d(P)$ *are not intersection points of the projected link*
2. $\forall y \in \varphi^{-1}(P)$ *and* $y' \in Y \setminus \varphi^{-1}(P)$ *with* $y \sim y'$, $\phi(s(y))$ *is above* $\phi(s(y'))$,
then the Wilson loop associated to L *can be uniquely written as*

$$W_L = \mathrm{tr}_{\overset{\alpha_t}{V}}(\overset{\alpha_t}{\mu} v_{\alpha_t}^{-\frac{1}{2}\varepsilon(e(P), \overset{i}{P})} u_P v_{\overset{\alpha_t}{2}}^{\frac{1}{2}\varepsilon(d(P), \overset{i}{P})} A), \tag{58}$$

where A *is an element of* $\Lambda \otimes \mathrm{End}(\overset{\alpha_t}{V}, \overset{\alpha_t}{V})$ *which components are linear combinations of the matrix element of* $(\overset{\alpha}{u}(l))_{l \in \phi(L)\setminus P}$ *with coefficients depending only on the set of links in* $\phi(L)\setminus P$ *and the ciliation at each vertex of* $\phi(L)\setminus P$. *A path satisfying properties* 1) *and* 2) *will be said to be "on top" of the link* L

Proof. From Theorem (1), we can always assume that $P \subset \overset{1}{P}$, and from the independence under the choice of the $\phi(z)$ for $z \in Z$, we can always assume that $e(P) = \phi(\overset{1}{z_1})$ As a result we can write: $W_L = \mathrm{Tr}(\mu_{\mathscr{S}} \prod_{t=1}^{p} \overset{i}{\sigma} \mathscr{U}^{(1)} \prod_{j=2}^{p} \mathscr{U}^{(j)})$. Because P is on the top of L we have $\mathscr{R}^{(y)} = R^{(y^- y^+)-1}$ if $y \in Y$ and $\varphi(y) \in P$ Using the formula

$$\mathrm{tr}_{V_{e(\overset{+}{s_2})}}(\mu_{e(s_2^+)} u_{s_1} R^{(d(s_1)^- e(s_2)^+)-1} u_{s_2}) = v_{\alpha_1}^{\varepsilon(\phi(e(s_2)), \overset{1}{P})} u_{s_1} u_{s_2}, \tag{59}$$

we obtain the desired result. □

Proposition 2 (Regular isotopy). *The element* $\widehat{W}_L = W_L \prod_{F \in \mathscr{F}} \delta_{\partial F}$ *of* Λ_{CS} *depends only on the regular isotopy class of the link* L, *i e it satisfies the Reidemeister moves of type* $0, 2, 3$.

Proof. Let us first show an already interesting result in itself: if L, L' are two links In $\Sigma \times \mathbf{R}$, L_1 (resp $L'1$) are connected curves included in L (resp L') and P

(resp. P') are the projections of L_1 (resp. L_1') such that:

- $L\backslash L_1 = L'\backslash L_1'$
- $\phi(L)\backslash P = \phi(L')\backslash P'$
- $e(P) = e(P')$, $d(P) = d(P')$ and none of these points are intersection point
- P (resp. P') is on top of L (resp. L')
- the curve $C = PP'^{-1}$ is a closed simple homologically trivial curve, then the following equality is true:

$$\widehat{W}_L = \widehat{W}_{L'} \ . \tag{60}$$

In order to show this easy result let us denote $e = e(P) = e(P')$ and $d = d(P) = d(P')$, and assume that P and P' are colored by α_i.

From the definition of a path being on top of L, we have:

$$\prod_{F\in\mathscr{F}} \delta_{\partial F} W_L = \prod_{F\in\mathscr{F}} \delta_{\partial F}\,\underset{V}{\mathrm{tr}}_{\alpha_i}(\overset{\alpha_i}{\mu}\,v_{\alpha_i}^{-\frac{1}{2}\varepsilon(e,L)}u_P v_{\alpha_i}^{\frac{1}{2}\varepsilon(d,L)}A)$$

$$= \prod_{F\in\mathscr{F}} \delta_{\partial F} v_{\alpha_i}^{-\frac{1}{2}\varepsilon(e,L)+\frac{1}{2}\varepsilon(d,L)+\frac{1}{2}\varepsilon(e,C)-\frac{1}{2}\varepsilon(d,C)}$$

$$\times \underset{V}{\mathrm{tr}}_{\alpha_i}(\overset{\alpha_i}{\mu}\,(v_{\alpha_i}^{-\frac{1}{2}\varepsilon(e,C)}u_P v_{\alpha_i}^{\frac{1}{2}\varepsilon(d,C)}u_{P'-1})u_{P'}A)$$

$$= \prod_{F\in\mathscr{F}} \delta_{\partial F} v_{\alpha_i}^{-\frac{1}{2}\varepsilon(e,L)+\frac{1}{2}\varepsilon(d,L)+\frac{1}{2}\varepsilon(e,C)-\frac{1}{2}\varepsilon(d,C)}$$

$$\times \underset{\otimes_j V}{\mathrm{tr}}_{\alpha_i}(\overset{\alpha_i}{\mu}\,u_C u_{P'}A)$$

$$= \prod_{F\in\mathscr{F}} \delta_{\partial F} v_{\alpha_i}^{-\frac{1}{2}\varepsilon(e,L)+\frac{1}{2}\varepsilon(d,L)+\frac{1}{2}\varepsilon(e,C)-\frac{1}{2}\varepsilon(d,C)+\frac{1}{2}\varepsilon(e,L')-\frac{1}{2}\varepsilon(d,L')}$$

$$\times \underset{\otimes_j V}{\mathrm{tr}}_{\alpha_i}(\overset{\alpha_i}{\mu}\,v_{\alpha_i}^{-\frac{1}{2}\varepsilon(e,L')}u_{P'}\,v_{\alpha_i}^{\frac{1}{2}\varepsilon(d,L')}A)$$

$$= \prod_{F\in\mathscr{F}} \delta_{\partial F} W_{L'} \ .$$

The last line comes from the identity:

$$\varepsilon(e,C) + \varepsilon(e,L') - \varepsilon(e,L) = 1 \ . \tag{61}$$

This ends the proof of the intermediate result.

The proof of Reidemeister moves of type $0, 2, 3$ is a direct byproduct of the previous result, the proof of all these moves being easily reduced to move an open strand on the top of the link. $\square$

In [4] it was shown that if $\mathscr{T}$ and $\mathscr{T}'$ are two ciliated triangulations of Σ, the algebra $\Lambda_{CS}(\mathscr{T})$ and $\Lambda_{CS}(\mathscr{T}')$ are isomorphic. Let us denote $i_{\mathscr{T},\mathscr{T}'} : \Lambda_{CS}(\mathscr{T}) \rightarrow \Lambda_{CS}(\mathscr{T}')$ as this isomorphism. It was shown that this isomorphism preserves the linear form h: if $a \in \Lambda_{CS}(\mathscr{T})$, then $h(a) = h'(i_{\mathscr{T},\mathscr{T}'}(a))$, where h (resp. h') is the linear invariant form on $\Lambda(\mathscr{T})$ (resp. on $\Lambda(\mathscr{T}')$).

Proposition 3 (Invariance under the change of triangulation). *Let $\mathscr{T}$ and $\mathscr{T}'$ be two triangulations of Σ, and let $R(\mathscr{T},\mathscr{T}')$ be any triangulation which is a common refinement of $\mathscr{T}$ and $\mathscr{T}'$. Let L (resp. L') be links in Σ such that L (resp. L')*

Fig. 2.

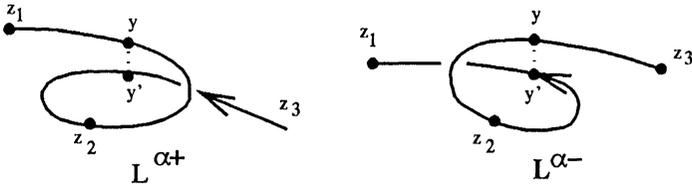*has all its projected components composed with links of $\mathcal{T}$ (resp $\mathcal{T}'$). If L and L' are related by a finite set of moves of type $0, 2, 3$ involving the triangulation $R(\mathcal{T}, \mathcal{T}')$ then: $i_{\mathcal{T},\mathcal{T}'}(\widehat{W}_L) = \widehat{W}_{L'}$. A direct consequence of this result is the equality of expectation values, i e.: $h(\widehat{W}_L) = h'(\widehat{W}_{L'})$.*

Proof. This is a direct consequence of Proposition 2 and the expression of $i_{\mathcal{T},\mathcal{T}'}$. $\square$

The only move which cannot be deduced from Proposition 2 is the first move. Although being reduced to the move of strand on top of the link, the strand P which is moved to P' is such that $e(P), d(P)$ are elements of $\phi(Y)$ or PP'^{-1} is not a simple curve So we cannot apply directly the intermediate result.

In the next theorem we study the behaviour of $\widehat{W}_L$ under the move of type 1.

Proposition 4 (Type I moves). *Let L be as usual a link in $\Sigma \times [0,1]$ and $\overset{i}{P}$ the set of projected curves on Σ and let $L^{\propto\pm}$ be another link whose projection $\overset{i}{P}{}^{\propto\pm}$ differs from $\overset{i}{P}$ by a move of type I (see Fig 2) applied to a curve colored by α_i, we have the following relation.*

$$\widehat{W}_{L^{\propto\pm}} = v_{\alpha_i}^{\pm 1} \widehat{W}_L \tag{62}$$

Proof. Let C denote the closed curve $C = \phi([y'z_2]) \cup \phi([z_2 y])$, we have·

$$\delta_C W_{L^{\propto+}} = \delta_C v_{\alpha_i}^{-\frac{1}{2}\varepsilon(z_1{}^-z_1{}^+)+\frac{1}{2}(\varepsilon(y^-y^+)+\varepsilon(z_2{}^-z_2{}^+)+\varepsilon(y'^-y'^+))+\frac{1}{2}\varepsilon(z_3{}^-z_3{}^+)}$$

$$\times \mathrm{tr}_{\underset{V}{\alpha_i}}(\overset{\alpha_i}{\mu} u_{[z_1 y]} u_{[yz_2]} u_{[z_2 y']} u_{[y'z_3]} A)$$

$$= \delta_C v_{\alpha_i}^{-\frac{1}{2}\varepsilon(z_1{}^-z_1{}^+)-\frac{1}{2}(\varepsilon(y^-v^+)+\varepsilon(z_2{}^-z_2{}^+)+\varepsilon(y'^-y'^+))+\frac{1}{2}\varepsilon(z_3{}^-z_3{}^+)}$$

$$\times \mathrm{tr}_{V_{z_2}\otimes V_v \otimes V_{y'}\otimes V}^{\alpha_i}(P_{z_1^+ y'^+}+P_{z_1^+ z_2}+P_{z_1^+ y'}+\overset{\alpha}{\mu}_{y^+}\overset{\alpha}{\mu}_{z_2^+}\overset{\alpha}{\mu}_{y'^+}\overset{\alpha_i}{\mu}$$

$$\times u_{[z_1 y]} R^{(y^- v^+)-1} u_{[yz_2]} R^{(z_2{}^- z_2{}^+)-1} u_{[z_2 y']} R^{(y'^- y'^+)-1} u_{[y'z_3]} A)$$

$$= \delta_C v_{\alpha_i}^{-\frac{1}{2}\varepsilon(z_1{}^-z_1{}^+)-\frac{1}{2}(\varepsilon(y^-v^+)+\varepsilon(z_2{}^-z_2{}^+)+\varepsilon(y'^-y'^+))+\frac{1}{2}\varepsilon(z_3{}^-z_3{}^+)}$$

$$\times \mathrm{tr}_{V_{z_2}\otimes V_v \otimes V_{y'}\otimes V}^{\alpha_i}(P_{z_1^+ y'^+}+P_{z_1^+ z_2}+P_{z_1^+ y'^+}\overset{\alpha}{\mu}_{y^+}\overset{\alpha}{\mu}_{z_2^+}\overset{\alpha}{\mu}_{y'^+}\overset{\alpha_i}{\mu}$$

$$\times u_{[yz_2]} R^{(z_2{}^- z_2{}^+)-1} u_{[z_2 y']} u_{[z_1 y]} R^{(y'^- v^-)} R^{(y'^- y'^+)-1} u_{[y'z_3]} A)$$

$$= \delta_C v_{\alpha_i}^{-\frac{1}{2}\varepsilon(z_1^- z_1^+) - \frac{1}{2}(\varepsilon(y^- y^+) + \varepsilon(y'^- y'^+)) + \frac{1}{2}\varepsilon(z_3^- z_3^+) + \frac{1}{2}\varepsilon(y'^- y^+)}$$

$$\times \mathrm{tr}_{V_y \otimes V_{y'} \otimes V}^{\alpha_i} (P_{z_1^+ y'^+} + P_{z_1^+ y^+} \overset{\alpha}{\mu}_{y^+} \overset{\alpha}{\mu}_{y'^+} \overset{\alpha_i}{\mu}$$

$$\times v_{\alpha_i}^{-\frac{1}{2}\varepsilon(y'^- y^+)} u_{[yz_2]} v_{\alpha_i}^{\frac{1}{2}\varepsilon(z_2^- z_2^+)} u_{[z_2 y']} u_{[z_1 y]} R^{(y'^- y^-)} R^{(y'^- y'^+)} u_{[y' z_3]} A)$$

$$= \delta_C v_{\alpha_i}^{-\frac{1}{2}(\varepsilon(z_1^- z_1^+) - \frac{1}{2}(\varepsilon(y^- y^+) + \varepsilon(y'^- y'^+)) + \frac{1}{2}\varepsilon(z_3^- z_3^+) + \frac{1}{2}\varepsilon(y'^- y^+)}$$

$$\times \mathrm{tr}_{V_y \otimes V_{y'} \otimes V}^{\alpha_i} (P_{z_1^+ y'^+} + P_{z_1^+ y^+} \overset{\alpha}{\mu}_{y^+} \overset{\alpha}{\mu}_{y'^+} \overset{\alpha_i}{\mu}$$

$$\times \mathrm{id}_{(V_{y^+}, V_{y'^-})} u_{[z_1 y]} R^{(y'^- y^-)} R^{(y'^- y'^+)-1} u_{[y' z_3]} A)$$

$$= (\text{after the use of } \delta_C v_{\alpha_i}^{-\frac{1}{2}\varepsilon(y'^- y^+)} u_{[yz_2]} v_{\alpha_i}^{\frac{1}{2}\varepsilon(z_2^- z_2^+)} u_{[z_2 y']} = \delta_C \mathrm{id}_{(V_{y^+}, V_{y'^-})}$$

where $\mathrm{id}_{(V_{y^+}, V_{y'^-})}$ denote the identity endomorphism)

$$= \delta_C v_{\alpha_i}^{-\frac{1}{2}(\varepsilon(y^- y^+) + \varepsilon(y'^- y'^+)) - \varepsilon(y'^- y^-) + \varepsilon(y'^- y'^+) - \frac{1}{2}\varepsilon(y^- y'^+) + \frac{1}{2}\varepsilon(y'^- y^+)}$$

$$\times \mathrm{tr}_{V}^{\alpha_i} (\overset{\alpha_i}{\mu} v_{\alpha_i}^{-\frac{1}{2}\varepsilon(z_1^- z_1^+)} u_{[z_1 y]} v_{\alpha_i}^{\frac{1}{2}\varepsilon(y^- y'^+)} u_{[yz_3]} v_{\alpha_i}^{\frac{1}{2}\varepsilon(z_3^- z_3^+)} A)$$

$$= v_{\alpha_i}^{-\frac{1}{2}(\varepsilon(y^- y^+) + \varepsilon(y'^- y'^+)) - \varepsilon(y'^- y^-) + \varepsilon(y'^- y'^+) - \frac{1}{2}\varepsilon(y^- y'^+) + \frac{1}{2}\varepsilon(y'^- y^+)} \delta_C W_L .$$

From the relation:

$$\varepsilon(y'^- y'^+) + \varepsilon(y^- y'^-) = \varepsilon(y^- y'^+) + 1 ,$$

we can write:

$$-\frac{1}{2}(\varepsilon(y^- y^+) + \varepsilon(y'^- y'^+)) - \varepsilon(y'^- y^-) + \varepsilon(y'^- y'^+) - \frac{1}{2}\varepsilon(y^- y'^+) + \frac{1}{2}\varepsilon(y'^- y^+)$$

$$= \frac{1}{2}(\varepsilon(y'^- y^+) + \varepsilon(y^+ y^-) + \varepsilon(y^- y'^+) + \varepsilon(y'^+ y'^-)) + 1 = 1 ,$$

which leads to $\delta_C W_L^{\propto +} = v_{\alpha_i} \delta_C W_L$. We finally obtain the relation:

$$\widehat{W}_{L^{\propto +}} = v_{\alpha_i} \widehat{W}_L . \tag{63}$$

The proof in the other case is very similar and it can be shown that:

$$\widehat{W}_{L^{\propto -}} = v_{\alpha_i}^{-1} \widehat{W}_L . \qquad \square \tag{64}$$

From Propositions 2 and 3 we immediately obtain that $\widehat{W}_L$ is a ribbon invariant, the link being endowed with the "blackboard framing" associated to the projection $\Sigma \times [0,1] \to \Sigma$. Let n_i be the writhe of the ribbon defined by L_i, Propositions 2 and 3 imply that

$$I(L) = \prod_{i=1}^{p} v_{\alpha_i}^{-n_i} \widehat{W}_L \tag{65}$$

is a link invariant element of the algebra Λ_{CS}.

A particularly simple situation appears when $\Sigma = S^2$, because in that case W_L is essentially a number.

Indeed let $\Sigma = S^2$, then from the independence of the algebra structure of Λ_{CS} under the choice of graph [4] we obtain that $\Lambda_{CS}(S^2)$ is an algebra of dimension 1, (this comes from the fact that S^2 is homeomorphic to a disk whose boundary has been identified to a point). As a result we get $\Lambda_{CS}(S^2) = \mathbf{C}\prod_{f \in \mathscr{F}} \delta_{\partial f}$. We then

deduce that.

$$\widehat{W}_L = w_L \prod_{f \in \mathcal{T}} \delta_{\partial f} \, , \tag{66}$$

with $w_L \in \mathbf{C}$.

This number w_L satisfies the following equation:

$$w_{L*L'} = w_L w_{L'} \, , \tag{67}$$

which is a trivial consequence of the identity $W_L W_{L'} = W_{L \star L'}$ and $\dim \Lambda_{CS}(S^2) = 1$.

In the next section we will show that w_L is the Reshetikhin–Turaev invariant (denoted $RT(L)$) of the framed link L. We will have therefore a new description of these invariants in term of traces of holonomies of flat connections in the spirit of the work of E. Witten.

When Σ is not homeomorphic to S^2, Λ_{CS} is no more a trivial algebra. It is then a quantization of the space $\text{Fun}(\text{Hom}(\pi_1(\Sigma), G)/G, \mathbf{C})$, a presentation by generators and relations being given in [4]. This presentation is simply obtained by generalizing the definition of Λ to arbitrary cell decomposition of the surface. By taking the simplest one, i.e. one 2-cell, 2g edges and one point, these authors have obtained a nice presentation of the Moduli algebra. This description is one of the steps to obtain the complete set of irreducible unitary representation of Λ_{CS} (even with punctures) as described in [5]. We also have to mention the work [19, 2] for other interesting constructions related to quantization of the moduli space of flat connections.

4. Relation Between the Observables W_L with $L \subset S^2 \times [0, 1]$ and Reshetikhin–Turaev Invariants

The aim of this section is to show that in the case of the sphere the invariant of link w_L is equal to the Reshetikhin–Turaev invariant of the ribbon (endowed with the blackboard framing) associated to L. The idea of the method is very simple: because $\widehat{W}_L = w_L a_{YM}$, we obtain $w_L = h(w_L a_{YM})/h(a_{YM})$. It remains to integrate $w_L a_{YM}$ over all links of the triangulation. This is quite technical, the final answer being just the Reshetikhin–Turaev invariant of the link L in the shadow world

Before explaining this proof we will first show that when $A = \mathscr{U}_q(sl_2)$ and L is a link colored with the fundamental representation, then $w_L = q^{\frac{3}{2} \sum_i n_i} P(L, q^2)$, where $P(L, z)$ is the Jones polynomial of the link L.

Proposition 5 (Skein relations-Jones Polynomial). *Let* L_+, L_-, L_0 *be links in* $\Sigma \times [0, 1]$ *which coincide outside a ball and look as in* (*Fig 3*) *inside the ball.*

In addition assume that $A = \mathscr{U}_q(sl_2)$ *and that the links* L_+, L_-, L_0 *have all their components colored with the fundamental representation of A. We have the skein relations·*

$$q^{\frac{1}{2}} \widehat{W}_{L_+} - q^{-\frac{1}{2}} \widehat{W}_{L_-} = (q - q^{-1}) \widehat{W}_{L_0} \, . \tag{68}$$

Moreover.

$$w_L = q^{\frac{3}{2} \sum_i n_i} P(L, q^2) \, , \tag{69}$$

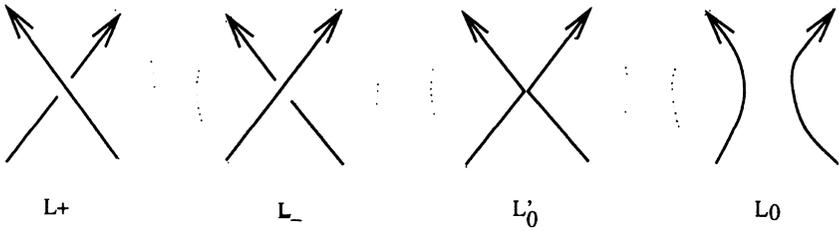*where* $P(L, z)$ *is the Jones polynomial of the link L in the variable z.*

$$L+ \qquad\qquad L_- \qquad\qquad L_0' \qquad\qquad L_0$$

Fig. 3.

Proof. Let us assume first that y_p and y_q belongs to the same circle of the chord diagram. We can write:

$$\mathscr{U}^{(1)} = U_{[z_{n+1}y_n]}R^{(y_n^- y_n^+)-1}U_{[y_nz_n]}AU_{[z_{p+1}y_p]}R^{(y_p^- y_p^+)-1}U_{[y_pz_p]}B . \qquad (70)$$

Using this notation we obtain:

$$W_{L_+} = \mathrm{Tr}_q\left(\prod_i \mathscr{U}^{(i)}\right)$$

$$= \mathrm{Tr}_q(U_{[z_{n+1}y_n]}R^{(y_n^- y_n^+)-1}U_{[y_nz_n]}AU_{[z_{p+1}y_p]}R^{(y_p^- y_p^+)-1}U_{[y_pz_p]}BC)$$

$$= \mathrm{Tr}_q(U_{[y_nz_n]}U_{[z_{n+1}y_n]}AU_{[y_pz_p]}U_{[z_{p+1}y_p]}BC)$$

$$= \mathrm{Tr}_q(U_{[y_nz_n]}AU_{[z_{n+1}y_n]}\beta_{y_p^+}^{(y_n^- y_p^+)}R^{(y_n^- y_p^+)-1}S^2(\alpha_{y_n^-}^{(y_n^- y_p^+)})U_{[y_pz_p]}U_{[z_{p+1}y_p]}BC)$$

$$= \mathrm{Tr}_q(\beta_{y_p^+}^{(y_n^- y_p^+)}U_{[y_nz_n]}AU_{[y_pz_p]}U_{[z_{n+1}y_n]}U_{[z_{p+1}y_p]}S^2(\alpha_{y_n^-}^{(y_n^- y_p^+)})BC)$$

$$= \mathrm{Tr}_q(U_{[y_nz_n]}AU_{[y_pz_p]}U_{[z_{n+1}y_n]}U_{[z_{p+1}y_p]}R_{y_n^- y_p^-}^{(y_n^- y_p^+)}BC)$$

$$= \mathrm{Tr}_q(U_{[y_nz_n]}AU_{[z_{p+1}y_p]}R^{(y_p^- y_p^+)-1}U_{[y_pz_p]}U_{[z_{n+1}y_n]}R^{(y_p^- y_n^-)}R_{y_n^- y_p^-}^{(y_n^- y_p^+)}BC)$$

$$= \mathrm{Tr}_q(U_{[y_nz_n]}AU_{[z_{p+1}y_p]}R^{(y_p^- y_p^+)-1}U_{[y_pz_p]}BR^{(z_{n+1}^- z_{n+1}^+)-1}$$

$$\times U_{[z_{n+1}y_n]}R^{(y_p^- y_n^-)}R_{y_n^- y_p^-}^{(y_n^- y_p^+)}C) .$$

Using the same method, we can check that:

$$W_{L_-} = \mathrm{Tr}_q(U_{[z_{n+1}y_n]}R^{(y_n^+ y_p^+)}R^{(y_n^- y_n^+)-1}R^{(y_n^- y_p^+)-1}U_{[y_nz_n]}A$$

$$\times U_{[z_{p+1}y_p]}R^{(y_p^- y_p^+)-1}R^{(y_p^- y_n^-)}R_{y_p^- y_n^-}^{(y_p^- y_n^+)}U_{[y_pz_p]}BC)$$

$$= \mathrm{Tr}_q(U_{[y_nz_n]}AU_{[z_{p+1}y_p]}R^{(y_p^- y_p^+)-1}U_{[y_pz_p]}BR^{(z_{n+1}^- z_{n+1}^+)-1}$$

$$\times U_{[z_{n+1}y_n]}R_{y_p^- y_n^-}^{(y_p^- y_n^+)}R_{y_n^- y_p^-}^{(y_n^+ y_p^+)}C) .$$

From the standard expression of R in the fundamental representation

$$R = q^{-\frac{1}{2}}\left(q\sum_{i=1}^{2}E_{ii}\otimes E_{ii} + \sum_{i\neq j}^{2}E_{ij}\otimes E_{ji} + (q-q^{-1})E_{12}\otimes E_{21}\right), \qquad (71)$$

we obtain:

$$q^{\frac{1}{2}}R^{(y_p^- y_n^-)}R^{(y_n^- v_p^+)}_{v_n^- y_p^-} - q^{-\frac{1}{2}}R^{(y_p^- y_n^-)}_{y_p^- v_n^-}R^{(y_n^+ y_p^+)}_{y_n^- y_p^-} = (q - q^{-1})R^{(y_p^- y_p^-)}_{y_p^- y_n^-}P_{v_n^- y_p^-}$$

As a result we get

$$q^{\frac{1}{2}}W_{L_+} - q^{-\frac{1}{2}}W_{L_-} = (q - q^{-1})\text{Tr}_q(P_{y_n^+ v_p^+}U_{[y_n z_n]}AU_{[z_{p+1}v_p]}R^{(y_p^- y_p^+)-1}U_{[v_p z_p]}$$

$$\times BR^{(z_{n+1}^- z_{n+1}^+)}U_{[z_{n+1}y_n]}R^{(y_p^- y_p^+)}_{y_p^- y_n^-}C)$$

$$= (q - q^{-1})\text{Tr}_q(P_{y_n^+ y_p^+}U_{[y_n z_n]}AU_{[z_{p+1}y_p]}U_{[v_p z_p]}BR^{(z_{n+1}^- z_{n+1}^+)}U_{[z_{n+1}y_n]}C)$$

The effect of the permutation operator after taking the trace is to identify the points y_n^- with y_p^+ and y_p^- with y_n^+, therefore,

$$q^{\frac{1}{2}}W_{L_+} - q^{-\frac{1}{2}}W_{L_-} = (q - q^{-1})W_{L_0'} \, ,$$

where $W_{L_0'}$ is the obvious generalization of the construction of the Wilson loop to the link L_0' (ambient isotopic to L_0) whose projection has a non-transverse intersection at $\phi(y_p)$ (see Fig. 3).

We can now conclude that·

$$q^{\frac{1}{2}}\widehat{W}_{L_+} - q^{-\frac{1}{2}}\widehat{W}_{L_-} = (q - q^{-1})\widehat{W}_{L_0} . \tag{72}$$

Similar arguments would also lead to the same result when y_p and y_q do not belong to the same circle.

One can easily compute the factor v_f in the fundamental representation, we get $v_f = q^{-\frac{3}{2}}$ As a result we obtain that $i(L) = q^{\frac{3}{2}\sum_i n_i}w_L$ is a link invariant, satisfying the skein relations $q^2 i(L_+) - q^{-2}i(L_-) = (q - q^{-1})i(L_0)$. From uniqueness of link invariants satisfying these axioms we get $i(L) = P(L, q^2)$. $\square$

We can now sketch a proof of the equality $w_L = RT(L)$ in the case where $A = \mathscr{U}_q(sl2)$ for any colour of the components of L. We will assume for simplicity that L has only one component. Let $\overset{\alpha}{L} \# \overset{\beta}{L}$ be the cabling of the framed knot L with two components coloured by α and β. It can be shown (it is not completely obvious) that we have.

$$\widehat{W}(\overset{\alpha}{L} \# \overset{\beta}{L}) = \sum_{\gamma} N^{\gamma}_{\alpha\beta}\widehat{W}(\overset{\gamma}{L}) \tag{73}$$

This last equation generalizes the fusion equation (Eq. 30) to the case of links. From the structure of the fusion ring of $sl(2)$ (it is a polynomial algebra in one variable), we immediately obtain that we can write:

$$\widehat{W}(\overset{\alpha}{L}) = \sum_{n=0}^{m} A_{\alpha}(n)\widehat{W}(\overset{f}{L}^{\#n}) \tag{74}$$

with $A_{\gamma}(n) \in \mathbf{C}$. But we also have the same fusion equation for the Reshetikhin–Turaev invariant (this is trivial from the quasitriangularity property). As a result we

also get:

$$RT(\overset{\alpha}{L}) = \sum_{n=0}^{m} A_\alpha(n) RT(\overset{f}{L}{}^{\#n}) \tag{75}$$

with the same $A_\alpha(n)$ as in (Eq. 74). It remains to use the equality $RT(L') = w_{L'} = P(L', q^2)$ for any link with arbitrary components coloured with the fundamental representation to obtain the equality $RT(L) = w_L$ for any knot L arbitrarily colored. When L has more than one component, it is easy to see that a generalization of this proof works as well.

In the rest of this section we will give the proof of the theorem announced at the beginning, i.e. in the case of the sphere the invariant of link w_L is equal to the Reshetikhin–Turaev invariant. This result is quite technical and the details are not particularly interesting in themselves.

Let L be a link in $D \times [0,1]$, where $D = [-1,1]^2$ and choose a braid with n colored strands which closure gives L. We will denote by $\overset{i}{Q}_{i=1,\,..,n}$ the projections of the strands on D and assume that they are in generic positions. Let β_i be the colour of the strand $\overset{i}{Q}$. These β_i take their values in the set $\{\alpha_j\}$. Let $\Delta = [0,1] \times [0,\frac{1}{2}]$ be the domain depicted in the picture where all the crossings of the $\overset{i}{Q}_{i=1,\,..,n}$ are located, and divide Δ in r strips $\Delta_j = D \cap ([0,1] \times [\frac{r-j}{2r}, \frac{r-j+1}{2r}])$, $1 \leq j \leq r$ such that in each of these strips there is only one crossing. Let us denote by C_j and C'_j the upper horizontal part and lower horizontal part of the boundary of Δ_j and by L_j and R_j the left vertical part and right vertical part of the boundary of Δ_j. Let us also define $\Delta_0 = (D \backslash \Delta) \cap \{(x,y), y \geq 0\}$ and $\Delta_{r+1} = (D \backslash \Delta) \cap \{(x,y), y \leq 0\}$.

It will be convenient to use the paths $\overset{i}{Q}_j = \overset{i}{Q} \cap \Delta_j$, for $j \in \{0, \ldots, r+1\}$. In each of the domains Δ_j the piece of the link is an elementary braid composed of n strands with at most one crossing.

We now construct a cell decomposition of D as follows:

- the set $\mathscr{V}$ of vertices is defined to be

$$\mathscr{V} = \{x \in \partial(\Delta_i \cap \Delta_j) \mid 0 \leq i \leq j \leq p+1\} \cup \left(\bigcup_i \overset{i}{Q}\right) \cap \left(\bigcup_{j=0}^{p+1} \partial\Delta_j\right) \cup_{i,i'} (\overset{i}{Q} \cap \overset{i'}{Q}) \tag{76}$$

- the set $\mathscr{F}$ of faces of the cell decomposition is defined to be the set of connected components of $D \backslash (\bigcup_i \overset{i}{Q} \bigcup_{j=0}^{p+1} \partial\Delta_j)$.

The ciliation at each vertex is shown on the following figure:

Let us denote by $\beta_{i_n}, \ldots, \beta_{i_1}$, in this order (from left to right), the incoming colors of the curves $\{\overset{i}{Q}_j, i = 1, \ldots, n\}$ according to Fig. 5 below, and denote by the couple $(m_j, m_j - 1)$ the location of the elementary braid (in order to simplify the notations we have put $m = m_j$ in some of the formulas below).

We can define an element B_j for $0 \leq j \leq r+1$

$$B_j \in \Lambda \otimes \text{End}(\overset{\beta_{i_1}}{V} \otimes \cdots \otimes \overset{\beta_{i_n}}{V}, \overset{\beta_{i_1}}{V} \otimes \cdots \overset{\beta_{i_m}}{V} \otimes \overset{\beta_{i_{m-1}}}{V} \otimes \cdots \overset{\beta_{i_n}}{V}) \tag{77}$$
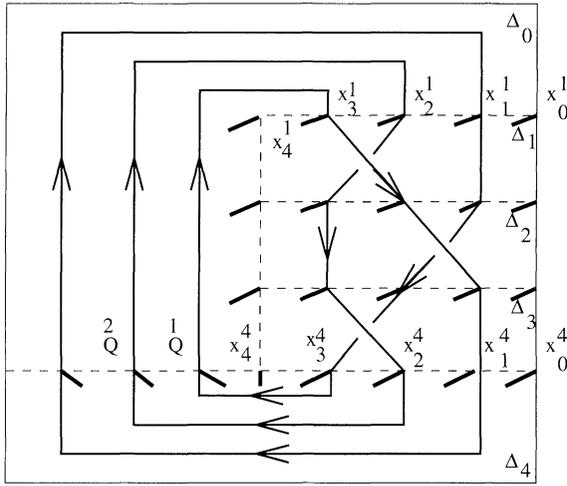
Fig. 4.
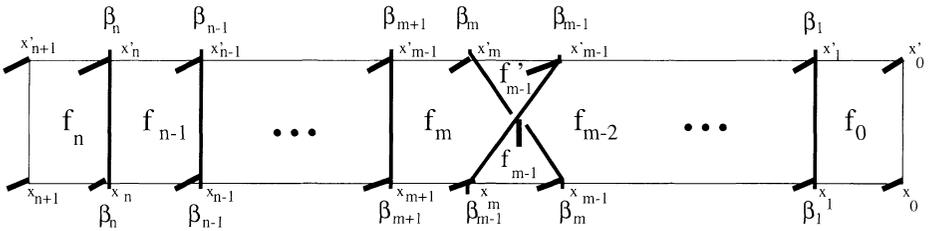
Fig. 5.

defined by:

$$B_j = v(B_j)P_{mm-1}u(\overset{i_n}{Q}_j)_n \cdot \quad u(\overset{i_m}{Q}_j)_m u(\overset{i_{m-1}}{Q}_j)_{m-1} \cdots u(\overset{i_1}{Q}_j)_1$$

$$\text{if } \overset{i_m}{Q}_j \text{ is above } \overset{i_{m-1}}{Q}_j \text{ and } 1 \leqq j \leqq r, \tag{78}$$

$$B_j = v(B_j)P_{mm-1}u(\overset{i_n}{Q}_j)_n \quad \cdots u(\overset{i_{m-1}}{Q}_j)_{m-1}u(\overset{i_m}{Q}_j)_m \cdots u(\overset{i_1}{Q}_j)_1$$

$$\text{if } \overset{i_m}{Q}_j \text{ is under } \overset{i_{m-1}}{Q}_j \text{ and } 1 \leqq j \leqq r, \tag{79}$$

$$B_j = v(B_j)u(\overset{i_n}{Q}_j)_n \quad \cdot \quad u(\overset{i_m}{Q}_j)_m u(\overset{i_{m-1}}{Q}_j)_{m-1} \cdot \quad u(\overset{i_1}{Q}_j)_1$$

$$\text{if } j = 0 \text{ or } r+1 \tag{80}$$

with

$$v(B_j) = \prod_{k=1}^{n} v_{\beta_{i_k}}^{\frac{1}{2}\varepsilon(d(\overset{i_k}{Q}_j),\overset{i_k}{Q})} \text{ if } j \neq 0 \quad \text{and} \quad v(B_j) = \prod_{k=1}^{n} v_{\beta_{i_k}}^{-\frac{1}{2}\varepsilon(d(\overset{i_k}{Q}_0),\overset{i_k}{Q})} \text{ if } j = 0$$

It is not difficult to see that·

Lemma 4. W_L *can be expressed as*:

$$W_L = \mathrm{tr}_{(\beta) \atop V} \left(\overset{(\beta)}{\mu} \prod_{j=r+1}^{0} B_j \right), \tag{81}$$

where $(\beta) = (\beta_n, \ldots, \beta_1)$. $\widehat{W}_L$ *can also be written*:

$$\widehat{W}_L = \left(\prod_{f \in \mathscr{F}} \delta_{\partial f} \right) W_L = \mathrm{tr}_{(\beta) \atop V} \left(\overset{(\beta)}{\mu} \prod_{j=r+1}^{0} \widehat{B}_j \right), \tag{82}$$

where we have defined $\widehat{B}_j = (\prod_{f \in \mathscr{F} \cap \varDelta_j} \delta_{\partial f}) B_j$.

Proof. The reader is invited to prove it. It is not difficult and uses only the exchange relations between the edge variables $u(\overset{i}{Q}_j)$. $\square$

Our strategy will consist in showing that $\widehat{W}_L = (\prod_{f \in \mathscr{F}} \delta_{\partial f}) \mathrm{tr}_{(\beta) \atop V}(\overset{(\beta)}{\mu} \prod_{j=r+1}^{0} B_j)$ can also be written as:

$$\widehat{W}_L = \left(\prod_{f \in \mathscr{F}} \delta_{\partial f} \right) \mathrm{tr}_{(\beta) \atop V} \left(\overset{(\beta)}{\mu} \prod_{j=r+1}^{0} \check{R}_j \right), \tag{83}$$

where $\check{R}_j$ is the matrix $P_{mm-1}(\beta_m \otimes \beta_{m-1})(R^{\pm})$ associated to the colored braid generator defined by the strip $\varDelta_j$. This will be sufficient to show that $w_L = RT(L)$. However this is not at all a trivial result and we will use integration over the links of the cell decomposition to obtain it.

Using again the commutation properties of Proposition 1 and the specific choice of ciliation, we have:

$$\begin{aligned}
\widehat{B}_j &= \prod_{f \in \mathscr{F} \cap \varDelta_j} \delta_{\partial f} B_j \\
&= v(B_j) P_{m,m-1} \left(\prod_{k=n}^{m+1} \delta_{\partial f_k} u(\overset{i_k}{Q}_j)_k \right) \delta_{\partial f_m} \delta_{\partial f'_{m-1}} \delta_{\partial f_{m-1}} \\
&\quad \times u(\overset{i_{m-1}}{Q}_j)_{m-1} u(\overset{i_m}{Q}_j)_m \prod_{k=m-2}^{1} \delta_{\partial f_k} u(\overset{i_k}{Q}_j)_k.
\end{aligned}$$

We will first establish a formula giving the expression of

$$\int \widehat{B}_j \prod_{l \in \mathscr{L} \cap \{\overset{i}{Q}_j, i=1, ,n\}} dh(u(l)) \tag{84}$$

in term of specific elements of the algebra $\varLambda$ (which will be described in the sequel) and in terms of the R matrix expressed in the shadow world.

Let us denote by $R_{pq}^{(\pm)} \begin{pmatrix} v_3 & b' & v_4 \\ c & & a \\ v_2 & b & v_1 \end{pmatrix}$, where p, q, a, b, c, b' are irreducible representations of A and v_1, v_2, v_3, v_4 are integers labelling multiplicities, the value of

$\overset{pq}{\check{R}}{}^{\pm 1}$ in the shadow world, i.e

$$\psi_{a,v_4}^{b'p}(\psi_{b',v_3}^{cq} \otimes \underset{V}{id}_{b'})(\underset{V}{id}_c \otimes \overset{pq}{\check{R}}{}^{\pm 1})(\phi_{cp}^{b,v_2} \otimes \underset{V}{id}_q)\phi_{bq}^{a,v_1} = R_{pq}^{(\pm)}\begin{pmatrix} v_3 & b' & v_4 \\ c & & a \\ v_2 & b & v_1 \end{pmatrix}\underset{V}{id}_a, \quad (85)$$

Lemma 5. *The result of integration over gauge fields associated to interior links of a plaquette $P = A \cup B \cup C \cup B'$ describing an overcrossing:*
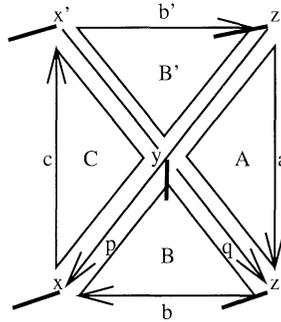
Fig. 6.

has the following expression:

$$\int dh(u(xy))dh(u(yz'))dh(u(zy))dh(u(yx'))\delta_{\partial A}\delta_{\partial B}\delta_{\partial C}\delta_{\partial B'} \overset{p}{u}(xyz')_2 \overset{q}{u}(zyx')_1$$

$$= (v_p v_q)^{-\frac{1}{2}} \sum_{\substack{a,b,c,b' \in \mathrm{Irr}(A) \\ \imath_1,v_2,\imath_3,\imath_4}} [d_c]\left(\frac{v_a v_{b'}}{v_b v_c}\right)^{\frac{1}{2}} R_{pq}^{(+)}\begin{pmatrix} v_3 & b' & v_4 \\ c & & a \\ v_2 & b & v_1 \end{pmatrix}\mathscr{T}_P\begin{pmatrix} v_3 & b' & v_4 \\ c & & a \\ v_2 & b & v_1 \end{pmatrix},$$

where $\mathscr{T}_P\begin{pmatrix} v_3 & b' & v_4 \\ c & & a \\ v_2 & b & v_1 \end{pmatrix}$ is an element of $\Lambda \otimes \mathrm{End}(\overset{q}{V} \otimes \overset{p}{V}, \overset{q}{V} \otimes \overset{p}{V})$ defined by.

$$\mathscr{T}_P\begin{pmatrix} v_3 & b' & v_4 \\ c & & a \\ v_2 & b & v_1 \end{pmatrix} = \underset{V}{\mathrm{tr}}_c(\overset{c}{\mu} \overset{c}{u}(x'x)\phi_{cp}^{b,v_2} \overset{b}{u}(xz)\phi_{bq}^{a,v_1} \overset{a}{u}$$

$$\times (zz')\psi_{a,v_4}^{b'p} \overset{b'}{u}(z'x')\psi_{b',v_1}^{cq} \overset{cq}{R'}). \quad (86)$$

In the case of an undercrossing the same formula is valid after the exchange of $\overset{p}{u}(xyz')_2 \overset{q}{u}(zyx')_1$ in $\overset{q}{u}(zyx')_1 \overset{p}{u}(xyz')_2$ and $R^{(+)}$ in $R^{(-)}$.

Proof. We first show the identity:

$$\delta_{\partial A}\delta_{\partial B}\delta_{\partial C}\delta_{\partial B'}(\overset{p}{u}(xyz')_1 \overset{q}{u}(zyx')_2 - \overset{p}{u}(xzz')_1 \overset{q}{u}(zz'x')_2) = 0 \quad (87)$$

which is a simple consequence of the flatness condition and the particular choice of ciliation

Indeed,

$$\delta_{\partial A}\delta_{\partial B}\delta_{\partial C}\delta_{\partial B'}\overset{p}{u}(xyz')_1\overset{q}{u}(zyx')_2$$

$$= \delta_{\partial B}\delta_{\partial C}\delta_{\partial B'}\overset{p}{u}(xy)_1 v_P^{\frac{1}{2}\varepsilon((xy)(yz))}\delta_{\partial A}\overset{p}{u}(yzz')_1\overset{q}{u}(zyx')_2$$

$$= \delta_{\partial A}\delta_{\partial B}\delta_{\partial C}\delta_{\partial B'}\overset{p}{u}(xyzz')_1\overset{q}{u}(zyx')_2$$

$$= \delta_{\partial A}\delta_{\partial B}\delta_{\partial C}\delta_{\partial B'}\overset{p}{u}(xzz')_1\overset{q}{u}(zyx')_2$$

$$= \delta_{\partial B}\delta_{\partial C}\overset{p}{u}(xzz')_1\delta_{\partial A}\delta_{\partial B'}\overset{q}{u}(zyx')_2$$

$$= \delta_{\partial A}\delta_{\partial B'}\delta_{\partial B}\delta_{\partial C}\overset{p}{u}(xzz')_1\overset{q}{u}(zz'x')_2 .$$

Therefore we obtain:

$$\int dh(u(xy))dh(u(yz'))dh(u(zy))dh(u(yx'))\delta_{\partial A}\delta_{\partial B}\delta_{\partial C}\delta_{\partial B'}\times\overset{p}{u}(xyz')_1\overset{q}{u}(zyx')_2$$

$$= \delta_{\partial P}\overset{p}{u}(xzz')_1\overset{q}{u}(zz'x')_2 ,$$

where P denotes the plaquette $A\cup B\cup B'\cup C$.

After the use of the decomposition rules of gauge field elements we obtain:

$$\delta_{[x'xzz']}\overset{p}{u}_{[xzz']}\overset{q}{u}_{[zz'x']}(v_p v_q)^{\frac{1}{2}}$$

$$= \sum_c[d_c]\mathrm{tr}_{V_c}(\mu\overset{c}{u}_{[x'x]}\overset{c}{u}_{[xz]}\overset{c}{u}_{[zz']}\overset{c}{u}_{[z'x']})(\overset{p}{u}_{[xz]}\overset{p}{u}_{[zz']})(\overset{q}{u}_{[zz']}v_q\overset{q}{u}_{[z'x']})$$

$$= \sum_c[d_c]v_q\mathrm{tr}_{V_c}(\mu\overset{c}{u}_{[x'x]}\overset{c}{u}_{[xz]}\overset{p}{u}_{[xz]}\overset{pc}{R}{}'\overset{c}{u}_{[zz']}\overset{p}{u}_{[zz']}\overset{q}{u}_{[zz']}\overset{pc}{R}{}^{-1}\overset{qc}{R}{}^{-1}\overset{c}{u}_{[z'x']}\overset{q}{u}_{[z'x']})$$

$$= \sum_{cbab'}[d_c]v_q\mathrm{tr}_{V_c}(\mu\overset{c}{u}_{[x'x]}\phi^b_{cp}\overset{b}{u}_{[xz]}\psi^{pc}_b\overset{pc}{\check{R}}\phi^b_{cp}\phi^a_{bq}\overset{a}{u}_{[zz']}\psi^{qb}_a\overset{bq}{\check{R}}{}^{-1}\psi^{pc}_b$$

$$\times\overset{cp}{\check{R}}{}^{-1}\overset{pq}{R}{}'\phi^{b'}_{cq}\overset{b'}{u}_{[z'x']}\psi^{qc}_{b'}P_{cq})$$

$$= \sum_{c,b,a,b'}[d_c]\mathrm{tr}_{V_c}(\mu\overset{c}{u}_{[x'x]}\phi^b_{cp}\overset{b}{u}_{[xz]}\phi^a_{bq}\overset{a}{u}_{[zz']}\psi^{b'}_a\overset{p}{u}_{[z'x']}\psi^{cq}_{b'}\overset{cq}{R}{}')$$

$$\times R^{(+)}_{pq}\begin{pmatrix}&b'&\\c&&a\\&b&\end{pmatrix}\left(\frac{v_a v_{b'}}{v_b v_c}\right)^{\frac{1}{2}},$$

which establishes the result previously announced (in order to improve the check of this proof we have deliberately omitted multiplicities). $\square$

Let $\Delta_{j_2,j_1}=\bigcup_{j=j_1}^{j_2}\Delta_j$ for $1\leqq j_1\leqq j_2\leqq p$, we can write $\partial\Delta_{j_2,j_1}=C_{j_1}\cup R_{j_2,j_1}\cup C'_{j_2}\cup L_{j_2,j_1}$, where as usual R_{j_2,j_1} and L_{j_2,j_1} denote the right vertical part and left vertical part of the boundary of Δ_{j_2,j_1}. It will be convenient to choose for each $(\beta)\in\mathrm{Irr}(A)^{\times n},\gamma_0,\gamma_n\in\mathrm{Irr}(A)$ a particular basis, denoted $\{\psi_\rho,\rho\in I^{(\beta)}(\gamma_n,\gamma_0)\}$ of the spaces $\mathrm{Hom}_A(\overset{\gamma_n}{V}\otimes\overset{(\beta)}{V},\overset{\gamma_0}{V})$, recursively defined by: $\{\psi_\sigma(\psi^{\gamma_{n+1}\beta_{n+1}}_{\gamma_n,m}\otimes id_{(\beta)}),\sigma\in I^{(\beta)}(\gamma_n,\gamma_0),\gamma_n\in\mathrm{Irr}(A),m=1,\ldots,N^{\gamma_n}_{\gamma_{n+1}\beta_{n+1}}\}$ is the basis $\{\psi_\rho,\rho\in I^{(\beta_{n+1},\,\beta_1)}(\gamma_{n+1},\gamma_0)\}$.

We will also define a basis $\{\phi_\rho, \rho \in O^{(\beta)}(\gamma_0, \gamma_n)\}$), of the spaces $\mathrm{Hom}_A(\overset{\gamma_0}{V},$ $\overset{\gamma_n}{V} \otimes \overset{(\beta)}{V})$ recursively defined by: $\{(\phi_{\gamma_{n+1}\beta_{n+1}}^{\gamma_n, m} \otimes id_{(\beta)}) \phi_\sigma, \sigma \in O^{(\beta)}(\gamma_n, \gamma_0), \gamma_n \in \mathrm{Irr}(A),$ $m = 1, \ ., N_{\gamma_{n+1}\beta_{n+1}}^{\gamma_n}\}$ is the basis $\{\phi_\rho, \rho \in O^{(\beta_{n+1}, \overset{V}{,\beta_1})}(\gamma_{n+1}, \gamma_0)\}$

If ψ is an element of $\mathrm{Hom}_A(\overset{\gamma_n}{V} \otimes V^{(\beta)}, \overset{\gamma_0}{V})$ and ϕ is an element of $\mathrm{Hom}_A(\overset{\gamma_n}{V} \otimes \overset{(\beta)}{V}, \overset{\gamma_0}{V})$ we have

$$\psi \circ \phi = a_{\psi,\phi} id_{\overset{\gamma_0}{V}}, \tag{88}$$

where $a_{\psi,\phi} \in \mathbf{C}$. We will use the notation $\langle \psi, \phi \rangle$ to denote the number $a_{\psi,\phi}$.

The previous choices of basis assure that $\langle \psi_\rho, \phi_\sigma \rangle \in \{0, 1\}$ if $\rho \in I^{(\beta)}(\gamma_n, \gamma_0)$ and $\sigma \in O^{(\beta)}(\gamma_0, \gamma_n)$.

We will now associate to each domain Δ_{j_2, j_1}, an element of $\Lambda \otimes \mathrm{Hom}(\overset{\gamma_n}{V} \otimes \overset{(\beta')}{V}, V_{\gamma_0})$ defined by:

$$\mathscr{I}_{\Delta_{j_2,j_1}}(\rho) = \psi_\rho \overset{\gamma_n}{u}(x'_{n+1} \cdot \quad x'_0 x_0)_0 \prod_{i=n}^{1} \overset{\beta'_i}{u}(x_i \cdot \quad x'_0 x_0)_{n-i+1}, \tag{89}$$

and an element of $\Lambda \otimes \mathrm{Hom}(V_{\gamma_0}, \overset{\gamma_n}{V} \otimes \overset{(\beta)}{V})$ defined by·

$$\mathscr{O}_{\Delta_{j_2,j_1}}(\sigma) = \prod_{i=1}^{n} \overset{\beta_i}{u}(x_i \quad x_0)_i \overset{\gamma_n}{u}(x'_{n+1} x_{n+1} \quad x_0)_0 \phi_\sigma. \tag{90}$$

With these elements we can build a generalisation of the element $\mathscr{T}$ introduced in Lemma (5) by:

$$\mathscr{T}_{\Delta_{j_2,j_1}} \left(\gamma_n \quad \begin{matrix} \rho' \\ \\ \rho \end{matrix} \quad \gamma_0 \right) = \mathrm{tr}_{\overset{\gamma_n}{V}}(\overset{\gamma_n}{\mu} \, \mathscr{O}_{\Delta_{j_2,j_1}}(\rho) \mathscr{I}_{\Delta_{j_2,j_1}}(\rho')), \tag{91}$$

where $(\beta), (\beta')$ are elements of $\mathrm{Irr}(A)^{\times n}$, γ_0, γ_n are elements of $\mathrm{Irr}(A)$ and $\rho \in O^{(\beta')}(\gamma_0, \gamma_n)$ and $\rho' \in I^{(\beta)}(\gamma_n, \gamma_0)$ according to the picture below:
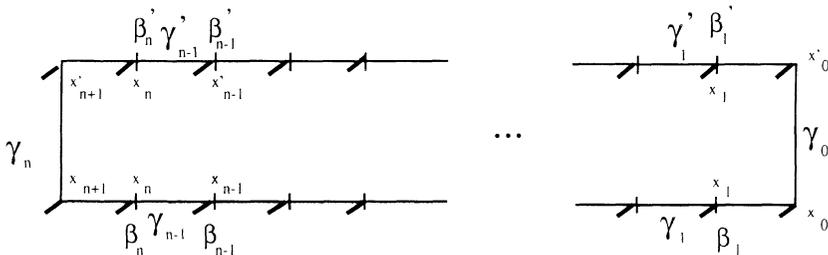

Fig. 7.

These elements are satisfying remarkable properties which are collected in the next three lemmas:

Lemma 6. *Let* j_1, j_2, j_3 *be integers such that* $1 \leq j_1 \leq j_2 \leq j_3 \leq r$ *the following relation holds*:

$$\int \mathscr{T}_{\Delta_{j_3, j_2}} \begin{pmatrix} & \sigma'' & \\ \gamma_n' & & \gamma_0' \\ & \sigma' & \end{pmatrix} \mathscr{T}_{\Delta_{j_2, j_1}} \begin{pmatrix} & \rho' & \\ \gamma_n & & \gamma_0 \\ & \rho & \end{pmatrix} \prod_{l \in \mathscr{L} \cap C_{j_2}} dh(u(l))$$

$$= \frac{1}{[d_{\gamma_n}]} \delta_{\gamma_0, \gamma_0'} \delta_{\gamma_n, \gamma_n'} \mathscr{T}_{\Delta_{j_3, j_1}} \begin{pmatrix} & \sigma'' & \\ \gamma_n & & \gamma_0 \\ & \rho & \end{pmatrix} \langle \psi_{\rho'}, \phi_{\sigma'} \rangle, \qquad (92)$$

where $(\beta), (\beta'), (\beta'')$ *are elements of* $\mathrm{Irr}(A)^{\times n}$, $\gamma_0, \gamma_n, \gamma_0', \gamma_n'$ *are elements of* $\mathrm{Irr}(A)$ *and* $\rho \in O^{(\beta)}(\gamma_0, \gamma_n), \rho' \in I^{(\beta')}(\gamma_n, \gamma_0)$. $\sigma' \in O^{(\beta')}(\gamma_0', \gamma_n')$ *and* $\sigma'' \in I^{(\beta'')}(\gamma_n', \gamma_0')$.

Lemma 7. *Consider the strip* Δ_j *and let us denote* $(\beta) = (\beta_{i_n}, \dots, \beta_{i_1})$ *the incoming representations.*

The formula of Lemma 5 can be extended to the case of a strip Δ_j,

$$\int \prod_{k=1}^{n} dh(u(\overset{i_k}{Q}_j)) \widehat{B}_j = \sum_{\substack{\gamma_0, \gamma_n \in \mathrm{Irr}(A) \\ \rho' \in I^{(\beta)}(\gamma_n, \gamma_0) \\ \rho \in O^{(\beta)}(\gamma_0, \gamma_n)}} [d_{\gamma_n}] \mathscr{T}_{\Delta_j} \begin{pmatrix} & \rho' & \\ \gamma_n & & \gamma_0 \\ & \rho & \end{pmatrix} \langle \psi_{\rho'}, \check{R}_j \phi_\rho \rangle. \qquad (93)$$

Lemma 8 (Closure of the braid).

$$\int \prod_{l \in (\mathscr{L} \backslash \partial D) \cap (\Delta_0 \cup \Delta_{r+1})} dh(u_l) \mathrm{tr}_{(\beta)} \begin{pmatrix} (\beta) \\ \mu \widehat{B}_{r+1} \mathscr{T}_\Delta \begin{pmatrix} & \rho' & \\ \gamma_n & & \gamma_0 \\ & \rho & \end{pmatrix} \widehat{B}_0 \end{pmatrix} = \overset{\gamma_0}{W}_{\partial D} \langle \psi_{\rho'}, \phi_\rho \rangle. \quad (94)$$

The proofs of these lemmas are not very difficult and are left to the reader. They imply the following proposition:

Proposition 6 (Reshetikhin–Turaev invariant). *Let* L *be a link in* $D \times [0, 1]$, *the element* $\widehat{W}_L$ *can be written*

$$\widehat{W}_L = w_L \prod_{f \in \mathscr{F}} \delta_{\partial f}, \qquad (95)$$

where $w_L = RT(L)$.

Proof. From Lemmas (6,7) we obtain:

$$\int \prod_{l \in (\Delta \backslash \partial \Delta) \cap \mathscr{L}} dh(u_l) \prod_{j=r}^{1} \widehat{B}_j = \sum_{\substack{\gamma_0, \gamma_n \in \mathrm{Irr}(A) \\ \rho' \in I^{(\beta)}(\gamma_n, \gamma_0) \\ \rho \in O^{(\beta)}(\gamma_0, \gamma_n)}} \mathscr{T}_\Delta \begin{pmatrix} & \rho' & \\ \gamma_n & & \gamma_0 \\ & \rho & \end{pmatrix} \left\langle \psi_{\rho'}, \prod_{j=r}^{1} \check{R}_j \phi_\rho \right\rangle. \qquad (96)$$

Using Lemma (8) we get:

$$\int \prod_{l \in (D \backslash \partial D) \cap \mathscr{L}} dh(u_l) \mathrm{tr}_{(\beta)} \mathrm{tr} \begin{pmatrix} (\beta) & 0 \\ \mu & \prod_{j=r+1}^{0} \widehat{B}_j \end{pmatrix}$$

$$= \sum_{\substack{\gamma_0, \gamma_n \in \mathrm{Irr}(A) \\ \rho' \in I^{(\beta)}(\gamma_n, \gamma_0) \\ \rho \in O^{(\beta)}(\gamma_0, \gamma_n)}} [d_{\gamma_n}] \overset{\gamma_0}{W}_{\partial D} \langle \psi_{\rho'}, \phi_\rho \rangle \left\langle \psi_{\rho'}, \prod_{j=r}^{1} \check{R}_j \phi_\rho \right\rangle. \qquad (97)$$

From the relation:

$$\sum_{\substack{\gamma_n \in \operatorname{Irr}(A) \\ \rho' \in I^{(\beta)}(\gamma_n, \gamma_0) \\ \rho \in O^{(\beta)}(\gamma_0, \gamma_n)}} [d_{\gamma_n}] \langle \psi_{\rho'}, \phi_\rho \rangle \phi_\rho (id_{\underset{V}{\gamma_n}} \otimes A) \psi_{\rho'} = [d_{\gamma_0}] id_{\underset{V}{\gamma_0}} \operatorname*{tr}_{\underset{V}{(\beta)}} (\overset{(\beta)}{\underset{V}{\mu}} A) , \qquad (98)$$

which holds for every $A \in \operatorname{End}(\overset{(\beta)}{V})$ (this relation is a consequence of relations (13,14) on the Clebsh–Gordan maps (see Proposition 14 of [10])) we obtain:

$$\int \prod_{l \in (D \backslash \partial D) \cap \mathscr{L}} dh(u_l) \operatorname*{tr}_{(\beta)} tr \left(\overset{(\beta)}{\underset{V}{\mu}} \prod_{J=\iota+1}^{0} \widehat{B}_j \right) = \left(\sum_{\gamma_0} [d_{\gamma_0}] \overset{\gamma_0}{\widehat{W}_{\partial D}} \right) \operatorname*{tr}_{(\beta)} \left(\overset{(\beta)}{\underset{V}{\mu}} \prod_{j=\iota}^{1} \check{R}_j \right) , \quad (99)$$

which is the last step in the proof $\square$

5. Conclusion

In this work we continued the analysis of combinatorial quantization of hamiltonian Chern–Simons theory We have defined elements of the observable algebra Λ_{CS} which are associated to any link in $\Sigma \times \mathbf{R}$ These elements are expected to be the precise definition of the Wilson loop elements in E Witten's formalism. This is supported by the fact that they are ribbon invariants and that their expectation value when $\Sigma = S^2$ is precisely the Reshetikhin–Turaev invariant of the link. In the next work [8] we have studied this combinatorial approach when the 3-manifold M is arbitrary It will be much more convenient to work with a Heegaard splitting of M rather than using a surgery presentation. If Σ_g and $f \in \operatorname{Mod}(\Sigma_g)$ is any Heegaard splitting of M we can associate an element a_f of $\Lambda_{CS}(\Sigma_g)$ which expectation value gives an invariant of M, which can be shown to be the Reshetikhin–Turaev invariant of M This invariant can be expressed as the partition function of a non-commutative lattice gauge theory associated to a cellular decomposition of M One important technical point which we will have to develop is the truncation of the spectrum when q is a root of unity.

The expression of the observables associated to any link that we have found is derived from a few principles· gauge invariance, independence under the choice of departure points. We expect that similar arguments will lead to the construction of observables associated to links with arbitrary self crossings. This could shed light on the construction of states in the canonical quantization program of pure gravity [7].

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