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Yangians, Integrable Quantum Systems and Dorey's Rule

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Abstract: It was pointed out by P. Dorey that the three-point couplings between the quantum particles in affine Toda field theories have a remarkable Lie-theoretic interpretation. It is also well known that such theories admit quantum affine algebras as "quantum symmetry groups," and widely believed that the quantum particles correspond to the so-called fundamental representations of these algebras. This led to the conjecture that Dorey's rule should describe when a fundamental representation occurs with non-zero multiplicity in a tensor product of two other fundamental representations. The purpose of this paper is to prove this conjecture, both for quantum affine algebras and for Yangians. The result reveals a hitherto unsuspected role played by Coxeter elements (and their twisted analogues) in the representation theory of these algebras.

1. Introduction

Quantum groups arose from the quantum inverse scattering method, developed by the Leningrad school [13] to solve integrable quantum systems. They provide, in particular, a way to understand the solutions of the quantum Yang–Baxter equation (R-matrices) associated to such systems, and a general framework for producing new solutions. Of special importance are the solutions which depend on a complex ("spectral") parameter; those which are rational, or trigonometric, functions of this parameter arise from the quantum groups called Yangians, or quantum affine algebras, respectively (see [11, 12], and Chapter 12 in [8] for background information).

More recently, quantum groups have arisen in another guise in connection with $1 + 1$ dimensional integrable quantum field theories, namely as the algebras satisfied by certain non-local conserved currents. For example, Yangians appear as "quantum symmetry algebras" in G-invariant Wess–Zumino–Witten models [1], while quantum affine algebras appear in affine Toda field theories (ATFTs) [2]. In [10], Dorey gave a remarkable Lie-theoretic description of the classical three-point couplings (or "fusings") in certain integrable field theories, including ATFTs. It is the purpose of this paper to interpret Dorey's rule in terms of the representation theory of Yangians and quantum affine algebras.

To describe our results in more detail, recall that an ATFT is a theory of scalar fields with exponential interactions determined by the roots of a (possibly twisted) affine Lie algebra. If $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra, and $\hat{\mathfrak{g}}$ is the associated (untwisted) affine Lie algebra, the quantum affine algebra $U_\varepsilon(\hat{\mathfrak{g}})$ is a "quantum symmetry algebra" of the ATFT based on the dual affine algebra $\hat{\mathfrak{g}}^*$, whose Dynkin diagram is obtained from that of $\hat{\mathfrak{g}}$ by reversing the arrows (the deformation parameter ε is related to the coupling constant of the theory, which should be purely imaginary for the quantum affine symmetry to exist – see Sect. 10). Note that $\hat{\mathfrak{g}}$ is self-dual if $\mathfrak{g}$ is simply-laced, but otherwise $\hat{\mathfrak{g}}^*$ is a twisted affine algebra $\hat{\mathfrak{k}}^\sigma$, where $\mathfrak{k}$ is simply-laced and σ is a diagram automorphism of $\mathfrak{k}$.

The manifestation of this quantum affine symmetry of interest to us is the relation, conjectured by physicists, between the so-called "fundamental representations" of $U_\varepsilon(\hat{\mathfrak{g}})$ and the "fusings" of the classical and quantum particles of the ATFT based on $\hat{\mathfrak{g}}^*$ It is well known (see [5, 10 and 14], for example) that the masses of the particles in the theory form the components of the eigenvector with lowest eigenvalue of the Cartan matrix of $\mathfrak{g}$; in particular, there is a natural one-to-one correspondence between these particles and the nodes of the Dynkin diagram of $\mathfrak{g}$. One says that there is a fusing between the particles labelled i, j and k if a certain term in the lagrangian of the theory is non-vanishing (see Sect. 10) Choose a colouring of the nodes of the Dynkin diagram of $\mathfrak{g}$ black or white in such a way that linked nodes have different colour, and let γ be the Coxeter element of the Weyl group of $\mathfrak{g}$ obtained by taking the product of the simple reflections associated to the black nodes, followed by those associated to the white nodes. Let R_i be the γ-orbit of the simple root α_i if i is black, and of $-\alpha_i$ if i is white. Then, Dorey's rule asserts that there is a non-trivial coupling between the particles labelled i, j and k if and only if

$$\text{(D)} \qquad 0 \in R_i + R_j + R_k$$

A little later, it was shown in [23] that (D) also gives the fusing rule for the solitons in the classical theory.

For the theory based on a twisted affine algebra $\hat{\mathfrak{k}}^\sigma$, the particles are in one-to-one correspondence with the orbits of σ on the nodes of the Dynkin diagram of $\mathfrak{k}$, and a twisted version of (D) is required to describe their fusings. One defines a "twisted Coxeter element" $\tilde{\gamma}$ for the pair $(\mathfrak{k}, \sigma)$, with the property that the orbits of $\tilde{\gamma}$ on the set of roots of $\mathfrak{k}$ are in one-to-one correspondence with the orbits of σ on the nodes of the Dynkin diagram of $\mathfrak{k}$ If $\mathfrak{g}$ is the (non-simply-laced) algebra such that $\hat{\mathfrak{g}}^* \cong \hat{\mathfrak{k}}^\sigma$, these orbits are naturally in one-to-one correspondence with the nodes of the Dynkin diagram of $\mathfrak{g}$ Proceeding as above, one obtains an analogue (TD) of (D), in which the indices i, j and k may be viewed as nodes of the Dynkin diagram of $\mathfrak{g}$, although the analogues of the R_i are sets of roots of $\mathfrak{k}$. Then the classical fusings of the ATFT based on $\hat{\mathfrak{g}}^*$, where $\mathfrak{g}$ is non-simply-laced, are given by (TD).

The situation in the quantum theory turns out to be slightly different. This time, the fusings of the ATFT based on $\hat{\mathfrak{g}}^*$ are given by (D) if $\mathfrak{g}$ is simply-laced, but by (D) $\cap$ (TD) otherwise (this can be verified case-by-case using the results in [9], at least when $\mathfrak{g}$ is not of type E or F).

Even without this physical motivation, (D) strongly suggests a connection to representation theory because of its similarity to the condition occurring in the Parthasarathy–Ranga Rao–Varadarajan (PRV) conjecture [24]. This conjecture, proved by Kumar [17] and Mathieu [21], asserts that, if μ_1, μ_2 and μ_3 are dominant weights of $\mathfrak{g}$, $W\mu_1$ the Weyl group orbit of μ_1, and $W(\mu_1)$ the irreducible $\mathfrak{g}$-module with highest weight μ_1, etc., then

$$\text{(PRV)} \qquad 0 \in W\mu_1 + W\mu_2 + W\mu_3$$

implies

$$\mathrm{Hom}_{\mathfrak{g}}(W(\mu_1) \otimes W(\mu_2) \otimes W(\mu_3), \mathbb{C}) \neq 0$$

($\mathbb{C}$ denotes the one-dimensional trivial $\mathfrak{g}$-module). Now, as Braden [4] pointed out, (D) is equivalent to

$$0 \in \Gamma\lambda_i + \Gamma\lambda_j + \Gamma\lambda_k ,$$

where Γ is the cyclic subgroup of W generated by γ, so (D) is obtained from (PRV) by replacing W by Γ (and restricting to fundamental weights).

The fundamental representations of $U_\varepsilon(\hat{\mathfrak{g}})$ to which (D) is related can be characterised as the finite-dimensional irreducible representations of $U_\varepsilon(\hat{\mathfrak{g}})$ which contain a fundamental representation $W_\varepsilon(\lambda_i)$ of $U_\varepsilon(\mathfrak{g})$, and are such that all other irreducible $U_\varepsilon(\mathfrak{g})$-subrepresentations have highest weight strictly less than λ_i (see [7 and 8]). There is, in fact, a family of such representations $V(\lambda_i, a)$ of $U_\varepsilon(\hat{\mathfrak{g}})$, depending on a parameter $a \in \mathbb{C}^\times$. The representations $V(\lambda_i, a)$ and $V(\lambda_i, b)$ are related by twisting by an automorphism of $U_\varepsilon(\hat{\mathfrak{g}})$ which fixes $U_\varepsilon(\mathfrak{g})$ and corresponds, at the classical level, to the automorphism of the loop algebra $\mathfrak{g}[t, t^{-1}]$ which sends t to at/b (the central extension by which $\hat{\mathfrak{g}}$ is obtained from the loop algebra plays no role here, since it acts trivially on all the representations of interest). Now recall that, whether or not $\mathfrak{g}$ is simply-laced, the particles of the ATFT based on $\hat{\mathfrak{g}}^*$ are in one-to-one correspondence with the nodes of the Dynkin diagram of $\mathfrak{g}$. If i, j and k are three such nodes, we would therefore expect a fusing between the quantum particles labelled i, j and k if and only if

$$(\otimes) \qquad \mathrm{Hom}_{U_\varepsilon(\hat{\mathfrak{g}})}(V(\lambda_i, a) \otimes V(\lambda_j, b) \otimes V(\lambda_k, c), \mathbb{C}) \neq 0 ,$$

for some $a, b, c \in \mathbb{C}^\times$. Thus, $(\otimes)$ should hold if and only if i, j and k satisfy (D) when $\mathfrak{g}$ is simply-laced, or (D) $\cap$ (TD) otherwise. This conjecture was first made explicit by MacKay [19, 20].

In this paper, we prove this conjecture when $\mathfrak{g}$ is not of exceptional type. We also prove an analogous result for Yangians (it was actually in the context of Yangians that MacKay originally made his conjecture). In fact, in the body of the paper, we concentrate on the Yangian case, and describe at the end how to translate the main results from the context of Yangians to that of quantum affine algebras. As MacKay has emphasized [20], the truth of the conjecture indicates that there is some beautiful structure in the representation theory of $U_\varepsilon(\hat{\mathfrak{g}})$ which is not evident at our present state of knowledge. It also suggests that it would be interesting to study the representation theory of twisted quantum affine algebras, but this does not seem to have been attempted yet.

One approach to the conjecture is through R-matrices. There is a canonical map $R(a, b) \in \mathrm{End}(V(\lambda_i, a) \otimes V(\lambda_j, b))$ which is a rational function of the spectral parameter a/b, and is such that $\tau R(a, b)$ commutes with the action of $U_\varepsilon(\hat{\mathfrak{g}})$ (τ denotes the flip of the two factors in the tensor product). In some cases, explicit formulas for $R(a, b)$ (or rather its Yangian analogue) were given in [7] (and earlier in [22], but without proper mathematical justification). There is a finite set of values of a/b for which $R(a, b)$ is well defined, but not invertible, and then its kernel is a subrepresentation of $V(\lambda_i, a) \otimes V(\lambda_j, b)$. If one can choose a/b so that this subrepresentation is fundamental, one deduces that $(\otimes)$ holds for some k, c. To use this method to prove the implication (D) (or (D) $\cap$ (TD)) $\Rightarrow$ $(\otimes)$, one would need to compute the R-matrix associated to every pair of fundamental representations; in addition, one would have to prove that every fundamental subrepresentation of $V(\lambda_i, a) \otimes V(\lambda_j, b)$ arises from the R-matrix as above. Because of these difficulties, we employ a different and simpler method, which makes no use of R-matrices, and which establishes the reverse implication at the same time.

2. Yangians

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Cartan matrix $A = (a_{ij})_{i,j \in I}$. Fix coprime positive integers $(d_i)_{i \in I}$ such that the matrix $(d_i a_{ij})$ is symmetric. Let R be the set of roots, R^+ a set of positive roots, and $R^- = -R^+$ The roots can be regarded as functions $I \to \mathbb{Z}$; in particular, the simple roots $\alpha_i \in R^+$ are given by

$$\alpha_i(j) = a_{ji}, \quad (i, j \in I)\,.$$

Let $Q = \bigoplus_{i \in I} \mathbb{Z} \cdot \alpha_i \subset \mathfrak{h}^*$ be the root lattice, and set $Q^+ = \sum_{i \in I} \mathbb{N} \cdot \alpha_i$

A weight is an arbitrary function $\lambda : I \to \mathbb{Z}$; denote the set of weights by P, and let

$$P^+ = \{\lambda \in P \,.\, \lambda(i) \geqq 0 \text{ for all } i \in I\}$$

be the set of dominant weights Define a partial order $\geqq$ on P by

$$\lambda \geqq \mu \quad \text{if and only if } \lambda - \mu \in Q^+$$

Let θ be the unique highest root with respect to $\geqq$

Let $(\,,)$ be the non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ such that the induced form on $\mathfrak{h}^*$ is given by

$$(\alpha_i, \alpha_j) = d_i a_{ij}\,.$$

If $\beta \in R$, set $d_\beta = \frac{1}{2}(\beta, \beta)$. Let W be the Weyl group of $\mathfrak{g}$, let $\{s_i\}_{i \in I}$ be the simple reflections which generate it, and let w_0 be the longest element of W. The dual Coxeter number $\check{h}$ of $\mathfrak{g}$ is

$$\check{h} = 1 + 2\frac{(\rho, \theta)}{(\theta, \theta)}\,,$$

where ρ is half the sum of the positive roots of $\mathfrak{g}$.

Fix a basis $\{H_i\}_{i \in I} \cup \{X_\alpha^\pm\}_{\alpha \in R^+}$ of $\mathfrak{g}$ such that, for all $i \in I$, $\alpha, \beta \in R^+$,

$$[H_i, X_\alpha^\pm] = \pm\alpha(i) X_\alpha^\pm, \qquad [X_\alpha^+, X_\beta^-] = \delta_{\alpha,\beta} H_\alpha\,,$$

$$(H_i, H_j) = d_j^{-1} a_{ij}, \qquad (X_\alpha^+, X_\beta^-) = \delta_{\alpha,\beta}, \qquad (X_\alpha^\pm, X_\beta^\pm) = 0\,,$$

where $H_\alpha = \sum_i n_i H_i$ if $\alpha = \sum_i n_i \alpha_i$. Let $X_i^\pm = X_{\alpha_i}^\pm$

If $\{I_p\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $(\,,)$, let

$$\Omega = \sum_p I_p^2$$

be the Casimir element of the universal enveloping algebra $U(\mathfrak{g})$. We also denote by Ω the element

$$\Omega = \sum_p I_p \otimes I_p \in \mathfrak{g} \otimes \mathfrak{g} .$$

Let κ be 1/4 of the value of Ω acting in the adjoint representation of $\mathfrak{g}$ (the value of κ is given in Sect. 3).

Definition 2.1 ([11]). *The Yangian $Y(\mathfrak{g})$ is the algebra over $\mathbb{C}$ generated by elements $x, J(x)$, for $x \in \mathfrak{g}$, with the following defining relations:*

$$[x, y] \ (in\ Y(\mathfrak{g})) = [x, y] \ (in\ \mathfrak{g}) , \tag{1}$$

$$J(ax + by) = aJ(x) + bJ(y) , \tag{2}$$

$$[x, J(y)] = J([x, y]) , \tag{3}$$

$$[J(x), J([y,z])] + [J(z), J([x, y])] + [J(y), J([z,x])]$$

$$= \sum_{p,q,r} ([x, I_p], [[y, I_q], [z, I_r]])\{I_p, I_q, I_r\} , \tag{4}$$

$$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]]$$

$$= \sum_{p,q,r} ([x, I_p], [[y, I_q], [[z, w], I_r]])\{I_p, I_q, J(I_r)\} , \tag{5}$$

for all $x, y, z \in \mathfrak{g}, a, b \in \mathbb{C}$. Here, for any elements $z_1, z_2, z_3 \in Y(\mathfrak{g})$, we set

$$\{z_1, z_2, z_3\} = \frac{1}{24} \sum_\pi z_{\pi(1)} z_{\pi(2)} z_{\pi(3)} ,$$

the sum being over all permutations π of $\{1, 2, 3\}$.

The Yangian $Y(\mathfrak{g})$ has a Hopf algebra structure with counit ε, comultiplication Δ and antipode S given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x , \tag{6}$$

$$\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, \Omega] , \tag{7}$$

$$S(x) = -x, \qquad S(J(x)) = -J(x) + \kappa x , \tag{8}$$

$$\varepsilon(x) = \varepsilon(J(x)) = 0 . \tag{9}$$

We shall also need the following presentation of $Y(\mathfrak{g})$, given in [12]:

Theorem 2.2. *The Yangian $Y(\mathfrak{g})$ is isomorphic to the associative algebra with generators $X^{\pm}_{i,r}, H_{i,r}, i \in I, r \in \mathbb{N}$, and the following defining relations:*

$$[H_{i,r}, H_{j,s}] = 0 , \tag{10}$$

$$[H_{i,0}, X^{\pm}_{j,s}] = \pm d_i a_{ij} X^{\pm}_{j,s} , \tag{11}$$

$$[H_{i,r+1}, X_{j,s}^{\pm}] - [H_{i,r}, X_{j,s+1}^{\pm}] = \pm \frac{1}{2} d_i a_{ij} (H_{i,r} X_{j,s}^{\pm} + X_{j,s}^{\pm} H_{i,r}) , \tag{12}$$

$$[X_{i,r}^{+}, X_{j,s}^{-}] = \delta_{i,j} H_{i,r+s} , \tag{13}$$

$$[X_{i,r+1}^{\pm}, X_{j,s}^{\pm}] - [X_{i,r}^{\pm}, X_{j,s+1}^{\pm}] = \pm \frac{1}{2} d_i a_{ij} (X_{i,r}^{\pm} X_{j,s}^{\pm} + X_{j,s}^{\pm} X_{i,r}^{\pm}) , \tag{14}$$

$$\sum_{\pi} [X_{i,r_{\pi(1)}}^{\pm}, [X_{i,r_{\pi(2)}}^{\pm}, \ldots, [X_{i,r_{\pi(m)}}^{\pm}, X_{j,s}^{\pm}] \cdots]] = 0 , \tag{15}$$

for all sequences of non-negative integers $r_1, \ldots, r_m$, *where* $m = 1 - a_{ij}$ *and the sum is over all permutations* π *of* $\{1, \ldots, m\}$

The isomorphism f *between the two realizations of* $Y(\mathfrak{g})$ *is given by*

$$f(H_i) = d_i^{-1} H_{i,0}, \qquad f(J(H_i)) = d_i^{-1} H_{i,1} + f(v_i) ,$$

$$f(X_i^{\pm}) = X_{i,0}^{\pm}, \qquad f(J(X_i^{\pm})) = X_{i,1}^{\pm} + f(w_i^{\pm}) , \tag{16}$$

where

$$v_i = \frac{1}{4} \sum_{\beta \in \Delta^+} \frac{d_\beta}{d_i} (\beta, \alpha_i)(X_\beta^+ X_\beta^- + X_\beta^- X_\beta^+) - \frac{d_i}{2} H_i^2 ,$$

$$w_i^{\pm} = \pm \frac{1}{4} \sum_{\beta \in \Delta^+} d_\beta ([X_i^{\pm}, X_\beta^{\pm}] X_\beta^{\mp} + X_\beta^{\mp} [X_i^{\pm}, X_\beta^{\pm}])$$

$$- \frac{1}{4} d_i (X_i^{\pm} H_i + H_i X_i^{\pm}) . \qquad \square$$

Remarks. 1. The presentation (2.1) of $Y(\mathfrak{g})$ shows that there is a canonical map $\mathfrak{g} \to Y(\mathfrak{g})$ (it is known that this map is injective). Thus, any $Y(\mathfrak{g})$-module may be regarded as a $\mathfrak{g}$-module.

2. If π is a permutation of I such that

$$H_i \mapsto H_{\pi(i)}, \qquad X_i^{\pm} \mapsto X_{\pi(i)}^{\pm}$$

defines a Lie algebra automorphism of $\mathfrak{g}$, the assignment

$$H_{i,k} \mapsto H_{\pi(i),k}, \qquad X_{i,k}^{\pm} \mapsto X_{\pi(i),k}^{\pm}$$

defines a Hopf algebra automorphism of $Y(\mathfrak{g})$. We denote both of these automorphisms simply by π.

We shall make use of two further types of automorphism of $Y(\mathfrak{g})$.

Proposition 2.3. *There is a one-parameter group* $\{\tau_a\}_{a \in \mathbb{C}}$ *of Hopf algebra automorphisms of* $Y(\mathfrak{g})$ *given in terms of presentation* (2.1) *by*

$$\tau_a(x) = x, \qquad \tau_a(J(x)) = J(x) + ax ,$$

for $x \in \mathfrak{g}$, *and in terms of presentation* (2.2) *by*

$$\tau_a(H_{i,k}) = \sum_{r=0}^{k} \binom{k}{r} a^{k-r} H_{i,r}, \qquad \tau_a(X_{i,k}^{\pm}) = \sum_{r=0}^{k} \binom{k}{r} a^{k-r} X_{i,r}^{\pm} \qquad \square$$

This is Proposition 2.6 in [7].

The second automorphism is an extension of the Cartan involution

$$\varphi_0(H_i) = -H_i, \qquad \varphi_0(X_i^{\pm}) = X_i^{\mp} \tag{17}$$

of $\mathfrak{g}$ to $Y(\mathfrak{g})$.

Proposition 2.4. *There exists a unique algebra automorphism φ of $Y(\mathfrak{g})$ such that*

$$\varphi(H_{i,k}) = (-1)^{k+1}H_{i,k}, \qquad \varphi(X_{i,k}^{\pm}) = (-1)^k X_{i,k}^{\mp},$$

for all $i \in I$, $k \in \mathbb{N}$. Moreover, φ is a coalgebra anti-automorphism of $Y(\mathfrak{g})$.

Proof. It is easy to check that applying φ to one of the defining relations in (2.2) gives another of the defining relations. Hence, the assignment in the statement of the proposition extends uniquely to an algebra homomorphism $Y(\mathfrak{g}) \to Y(\mathfrak{g})$, and it is obvious that φ is an involution.

Using the isomorphism f in (2.2), it is clear that $\varphi|_{\mathfrak{g}} = \varphi_0$ and that

$$\varphi(J(H_i)) = J(H_i).$$

Hence,

$$\varphi(J(X_i^{\pm})) = \mp\frac{1}{2}\varphi([X_i^{\pm}, J(H_i)]) = \mp\frac{1}{2}[X_i^{\mp}, J(H_i)] = -J(X_i^{\mp}).$$

To prove that

$$(\varphi \otimes \varphi) \circ \Delta = \Delta^{\mathrm{op}} \circ \varphi,$$

where Δ^{op} denotes the opposite comultiplication of $Y(\mathfrak{g})$, it suffices to show that both sides agree when applied to a set of generators of $Y(\mathfrak{g})$, such as $\{H_i, X_i^{\pm}, J(H_i), J(X_i^{\pm})\}_{i\in I}$. This is now straightforward, making use of the formula for Δ in (2.1) and the observation that $(\varphi_0 \otimes \varphi_0)(\Omega) = \Omega$. □

We shall also need the following weak version of the Poincaré–Birkhoff–Witt theorem for $Y(\mathfrak{g})$.

Proposition 2.5. *Let Y^+, Y^- and Y^0 be the subalgebras of $Y(\mathfrak{g})$ generated by the $X_{i,k}^+$, the $X_{i,k}^-$ and the $H_{i,k}$, respectively ($i \in I$, $k \in \mathbb{N}$). Then,*

$$Y(\mathfrak{g}) = Y^- \cdot Y^0 \cdot Y^+. \qquad \square$$

The proof is straightforward.

3. Finite-Dimensional Representations

If W is a $\mathfrak{g}$-module and $\lambda \in P$, the weight space

$$W_\lambda = \{w \in W | H_i \cdot w = \lambda(i)w \text{ for all } i \in I\}.$$

If $W_\lambda \neq 0$, λ is called a weight of W, and the set of such weights is denoted by $P(W)$.

A non-zero vector $w \in W$ is called a $\mathfrak{g}$-highest weight vector if $w \in W_\lambda$ for some $\lambda \in P(W)$ and $X_i^+ \cdot w = 0$ for all $i \in I$. Let W^+ be the set of $\mathfrak{g}$-highest weight vectors of W, and set $W_\lambda^+ = W^+ \cap W_\lambda$. If $W = U(\mathfrak{g}) \cdot w$, then W is called a highest weight $\mathfrak{g}$-module with highest weight λ. Lowest weight vectors and $\mathfrak{g}$-modules are defined similarly. For any $\lambda \in P^+$ denote by $W(\lambda)$ the unique irreducible highest weight $\mathfrak{g}$-module with highest weight λ. If W is any finite-dimensional $\mathfrak{g}$-module, we have

$$W \cong \bigoplus_{\lambda \in P^+} W(\lambda)^{\oplus m_\lambda(W)},$$

where the multiplicities $m_\lambda(W)$ are given by

$$m_\lambda(W) = \dim(W^+ \cap W_\lambda).$$

We recall that the Casimir operator $\Omega \in U(\mathfrak{g})$ acts on $W(\lambda)$ by the scalar $(\lambda + 2\rho, \lambda)$ In particular, $\kappa = \frac{1}{2} d_\theta \check{h}$.

Let W^* be the dual $\mathfrak{g}$-module of W, and let W^{φ_0} be the $\mathfrak{g}$-module obtained by twisting W with the Cartan involution φ_0 of $\mathfrak{g}$. For $\lambda \in P$, let $\overline{\lambda} = -w_0(\lambda)$. It is well known that

$$m_\lambda(W) = m_{\overline{\lambda}}(W^*) = m_{\overline{\lambda}}(W^{\varphi_0})$$

Suppose now that V is a $Y(\mathfrak{g})$-module. Set

$$V^{++} = \{v \in V^+ | X_{i,k}^+ \cdot v = 0 \text{ for all } i \in I, k \in \mathbb{N}\},$$

and for any $\lambda \in P^+$, set $V_\lambda^{++} = V^{++} \cap V_\lambda$. Note that, by (2 2), V^{++} is preserved by the action of Y^0, and so, if $V^{++} \neq 0$, it contains a non-zero Y^0-eigenvector v (say), so that

$$H_{i,k} \cdot v = d_{i,k} v,$$

for some $d_{i,k} \in \mathbb{C}$ Such a vector v is called a $Y(\mathfrak{g})$-highest weight vector, V is called $Y(\mathfrak{g})$-highest weight if $V = Y(\mathfrak{g}) \cdot v$ for some $Y(\mathfrak{g})$-highest weight vector $v \in V$, and the collection of scalars $\mathbf{d} = \{d_{i,k}\}_{i \in I, k \in \mathbb{N}}$ is called its highest weight. It is not difficult to show that, for every $\mathbf{d} = (d_{i,k})_{i \in I, k \in \mathbb{N}}$, there is an irreducible $Y(\mathfrak{g})$-module $V(\mathbf{d})$, unique up to isomorphism, such that $V(\mathbf{d})$ has highest weight $\mathbf{d}$. Lowest weight vectors and modules for $Y(\mathfrak{g})$ are defined similarly.

The following theorem of Drinfel'd [12] classifies the finite-dimensional irreducible $Y(\mathfrak{g})$-modules

Theorem 3.1. (i) *Every finite-dimensional irreducible $Y(\mathfrak{g})$-module is both highest weight and lowest weight*

(ii) *If $\mathbf{d} = (d_{i,k})_{i \in I, k \in \mathbb{N}}$, the $Y(\mathfrak{g})$-module $V(\mathbf{d})$ is finite-dimensional if and only if there exist monic polynomials $P_i \in \mathbb{C}[u]$ such that*

$$\frac{P_i(u + d_i)}{P_i(u)} = 1 + \sum_{k=0}^{\infty} d_{i,k} u^{-k-1}, \tag{18}$$

in the sense that the right-hand side is the Laurent expansion of the left-hand side about $u = \infty$ □

If V is a finite-dimensional irreducible $Y(\mathfrak{g})$-module, we call the associated I-tuple of polynomials $(P_i)_{i \in I}$ the Drinfel'd polynomials of V.

In general, if V is any finite-dimensional $Y(\mathfrak{g})$-module and $v \in V$ is a $Y(\mathfrak{g})$-highest weight vector, with

$$H_{i,k} \cdot v = d_{i,k}^v v$$

for some $d^v_{i,k} \in \mathbb{C}$, it follows from (3.1) that there exist monic polynomials P^v_i such that

$$\frac{P^v_i(u+d_i)}{P^v_i(u)} = 1 + \sum_{k=0}^{\infty} d^v_{i,k} u^{-k-1} .$$

Proposition 3.2. *Let V_1, V_2 be finite-dimensional $Y(\mathfrak{g})$-modules, and let $v_1 \in V_1$, $v_2 \in V_2$ be $Y(\mathfrak{g})$-highest weight vectors. Then,*

$$P^{v_1 \otimes v_2}_i = P^{v_1}_i P^{v_2}_i . \qquad \square$$

This is Proposition 2.15 in [7].

The $Y(\mathfrak{g})$-modules of interest in this paper are defined as follows.

Definition 3.3. *If $i \in I$, $a \in \mathbb{C}$, then $V_a(\lambda_i)$ is the finite-dimensional irreducible $Y(\mathfrak{g})$-module with Drinfel'd polynomials*

$$P_j(u) = \begin{cases} u - a & \text{if } j = i , \\ 1 & \text{if } j \neq i . \end{cases}$$

We call $V_a(\lambda_i)$ a fundamental $Y(\mathfrak{g})$-module.

Given a finite-dimensional $Y(\mathfrak{g})$-module V, we can define the following associated $Y(\mathfrak{g})$-modules:

(i) $V(a)$: this is obtained pulling back V through τ_a;
(ii) V^φ: this is obtained pulling back V through φ;
(iii) the left dual ${}^tV$ and right dual V^t: these are given by the following actions of $Y(\mathfrak{g})$ on the vector space dual of V:

$$(y \cdot f)(v) = f(S(y) \cdot v), \quad y \in Y(\mathfrak{g}), f \in {}^tV, v \in V ,$$

$$(y \cdot f)(v) = f(S^{-1}(y)) \cdot v, \quad y \in Y(\mathfrak{g}), f \in V^t, v \in V .$$

Clearly, if V is irreducible, so are all the representations defined above.

Proposition 3.4. *Let U, V and W be finite-dimensional $Y(\mathfrak{g})$-modules, and let $a \in \mathbb{C}$. Then,*

(i) $(U \otimes V)^\varphi \cong V^\varphi \otimes U^\varphi$;
(ii) $V(a)^\varphi \cong V^\varphi(-a)$;
(iii) $\mathrm{Hom}_{Y(\mathfrak{g})}(U, V \otimes W) \cong \mathrm{Hom}_{Y(\mathfrak{g})}({}^tV \otimes U, W)$;
(iv) $\mathrm{Hom}_{Y(\mathfrak{g})}(U, W \otimes V) \cong \mathrm{Hom}_{Y(\mathfrak{g})}(U \otimes V^t, W)$;
(v) ${}^{tt}V \cong V(-2\kappa)$, $V^{tt} \cong V(2\kappa)$, ${}^t(V^t) \cong ({}^tV)^t \cong V$;
(vi) $(V \otimes W)^t \cong (W^t \otimes V^t)$, ${}^t(V \otimes W) \cong {}^tW \otimes {}^tV$;
(vii) $(V(a))^t \cong V^t(a)$, ${}^t(V(a)) \cong ({}^tV)(a)$.

Proof. Part (i) follows from the fact that φ is a coalgebra anti-automorphism of $Y(\mathfrak{g})$, and part (ii) from the identity

$$\varphi \cdot \tau_a = \tau_{-a} \cdot \varphi ,$$

which is proved by checking that the two sides agree when applied to any of the generators $H_{i,k}, X^{\pm}_{i,k}$. Parts (iii)–(vii) are straightforward. $\square$

The following result describes the Drinfel'd polynomials of the modules defined above. If $i \in I$, define $\bar{i} \in I$ by $\lambda_{\bar{i}} = \overline{\lambda_i}$.

Proposition 3.5. *Let V be a finite-dimensional irreducible $Y(\mathfrak{g})$-module with Drinfel'd polynomials P_i $(i \in I)$, and let $a \in \mathbb{C}$. Then:*

(i) *The Drinfel'd polynomials P_i^a of $V(a)$ are given by*

$$P_i^a(u) = P_i(u-a)$$

(ii) *The Drinfel'd polynomials ${}^tP_i$ and P_i^t of ${}^tV$ and V^t, respectively, are given by*

$${}^tP_i(u) = P_{\bar{i}}(u+\kappa), \qquad P_i^t(u) = P_{\bar{i}}(u-\kappa) .$$

(iii) *The Drinfel'd polynomials P_i^φ of V^φ are given by*

$$P_i^\varphi(u) = (-1)^{\deg(P_i)} P_{\bar{i}}(\kappa + d_i - u)$$

Proof Parts (i) and (ii) were proved in [7]. We now prove part (iii) Let $0 \neq v \in V$ be a $Y(\mathfrak{g})$-lowest weight vector, and let

$$H_{i,k} \cdot v = \overline{d_{i,k}} v, \qquad (\overline{d_{i,k}} \in \mathbb{C}) .$$

Then, v is a $Y(\mathfrak{g})$-highest weight vector in V^φ and, in V^φ, we have, by (2.4),

$$H_{i,k} \cdot v = (-1)^{k+1} \overline{d_{i,k}} v .$$

Hence, the Drinfel'd polynomials P_i^φ of V^φ satisfy

$$\frac{P_i^\varphi(u+d_i)}{P_i^\varphi(u)} = 1 + \sum_{k=0}^{\infty} (-1)^{k+1} \overline{d_{i,k}} u^{-k-1} . \tag{19}$$

On the other hand, by Propositions 3 1 and 3.2 in [7],

$$\frac{P_{\bar{i}}(u-\kappa)}{P_{\bar{i}}(u+d_i-\kappa)} = 1 + \sum_{k=0}^{\infty} \overline{d_{i,k}} u^{-k-1} . \tag{20}$$

The result follows on comparing (19) and (20). □

Corollary 3.6. *Let $i \in I$, $a \in \mathbb{C}$ Then:*

(i) ${}^t(V_a(\lambda_i)) \cong V_{a-\kappa}(\lambda_{\bar{i}})$, $(V_a(\lambda_i))^t \cong V_{a+\kappa}(\lambda_{\bar{i}})$;
(ii) $(V_a(\lambda_i))^\varphi \cong V_{\kappa+d_i-a}(\lambda_{\bar{i}})$ □

We shall also need the following result.

Proposition 3.7. *Let V be a finite-dimensional highest weight $Y(\mathfrak{g})$-module Then, V^φ is also a highest weight $Y(\mathfrak{g})$-module*

Proof Let $0 \neq v \in V_\lambda$ $(\lambda \in P^+)$ be a $Y(\mathfrak{g})$-highest weight vector. By (2.5), $m_\lambda(V)=1$ and $m_\mu(V) = 0$ unless $\mu \leqq \lambda$. Let W be the $\mathfrak{g}$-submodule of V of type $W(\lambda)$; then $v \in W$ Let v^- be a lowest weight vector (for $\mathfrak{g}$) in W. Then, v^- is a $Y(\mathfrak{g})$-highest weight vector in V^φ and

$$Y(\mathfrak{g}) \cdot v^- \supset U(\mathfrak{g}) \cdot v^- = W ,$$

so $v \in Y(\mathfrak{g}) \cdot v^-$, and hence

$$V^\varphi = Y(\mathfrak{g}) \cdot v \subset Y(\mathfrak{g}) \cdot v^- . \qquad \square$$

We conclude this section with the following results.

Proposition 3.8. *Let V be a finite-dimensional $Y(\mathfrak{g})$-module. Then, V is irreducible if and only if V and ${}^tV$ (resp. V and V^t) are both highest weight $Y(\mathfrak{g})$-modules.*

Proof. The "only if" part follows from (3.1) (i). For the converse, suppose that V and ${}^tV$ are highest weight (the other case is identical). Let $v \in V_\lambda$ ($\lambda \in P^+$) be a $Y(\mathfrak{g})$-highest weight vector. Let $0 \neq W$ be an irreducible $Y(\mathfrak{g})$-submodule of V, and let μ (say) be the highest weight of W as a $\mathfrak{g}$-module; thus, $\mu \leqq \lambda$. Then, ${}^tW$ is a quotient of ${}^tV$, and these $\mathfrak{g}$-modules have maximal weights $\bar{\mu}$ and $\bar{\lambda}$, respectively (cf. the proof of (3.7)). Since ${}^tV$ is a $Y(\mathfrak{g})$-highest weight module, its highest weight vector must map to a non-zero element of ${}^tW$. Hence, $\bar{\lambda} \leqq \bar{\mu}$, so $\lambda \leqq \mu$. Thus, $\lambda = \mu$ and $W = V$. □

Along similar lines, we have the following result whose simple proof we omit.

Proposition 3.9. *Let V be a finite-dimensional $Y(\mathfrak{g})$-module, and assume that, as a $\mathfrak{g}$-module, V has a unique maximal weight $\lambda \in P^+$. Then, $Y(\mathfrak{g}) \cdot v$ is a proper submodule of V if and only if V^t (resp. ${}^tV$) contains a $Y(\mathfrak{g})$-highest weight vector of weight strictly less than $\bar{\lambda}$.* □

4. Dorey's Rule

Let $s_1, s_2, \ldots, s_n$ be the simple reflections in the Weyl group W of $\mathfrak{g}$ (in some order), and let $\gamma = s_1 s_2 \cdots s_n$ be the associated Coxeter element of W. Define roots

$$\phi_i = s_n s_{n-1} \cdots s_{i+1}(\alpha_i)\,,$$

and let R_i be the γ-orbit of ϕ_i. It is known that the ϕ_i are precisely the positive roots which become negative under the action of γ, and that each R_i contains precisely h roots, where the Coxeter number h is the order of γ (see [16, 25, 26]).

Definition 4.1. *If $p \geqq 2$, we say that indices $i_1, i_2, \ldots, i_p \in I$ satisfy condition* (D_p) *if and only if $0 \in R_{i_1} + R_{i_2} + \cdots + R_{i_p}$.*

Note that the condition (D_p) appears to depend on a number of arbitrary choices: we had to pick a Cartan subalgebra $\mathfrak{h}$, a set of positive roots R^+, and an ordering of the set of simple reflections. However, we have

Proposition 4.2. *For any $p \geqq 2$, the condition (D_p) is independent of the choices made.*

Proof. Let G be a (connected, complex) Lie group with Lie algebra $\mathfrak{g}$. If $\bar{\mathfrak{h}}$ is another Cartan subalgebra, and $\bar{R}^+$ a set of positive roots with respect to $\bar{\mathfrak{h}}$, there exists $g \in G$ such that $\bar{\mathfrak{h}} = \mathrm{Ad}(g)(\mathfrak{h})$ and $\bar{R}^+ = \mathrm{Ad}(g)^*(R^+)$. Then, the $\bar{\alpha}_i = \mathrm{Ad}(g)^*(\alpha_i)$ are the simple roots in $\bar{R}^+$, and the $\bar{s}_i = \mathrm{Ad}(g) \circ s_i \circ \mathrm{Ad}(g^{-1})$ are the corresponding simple reflections. Using the Coxeter element $\bar{\gamma} = \bar{s}_1 \bar{s}_2 \cdots \bar{s}_n$, it is easy to see that, in an obvious notation, $\bar{R}_i = \mathrm{Ad}(g)^*(R_i)$, and it follows immediately that

$$0 \in R_{i_1} + \cdots + R_{i_p} \quad \text{iff } 0 \in \bar{R}_{i_1} + \cdots + \bar{R}_{i_p}\,.$$

Thus, we may work with a fixed Cartan subalgebra and set of positive roots, and need only consider the effect of re-ordering the set of simple reflections. It is well known (see [26], Lemma 2.3) that any such re-ordering can be achieved by a sequence of moves of the following two types:

(i) $s_1 s_2 \cdots s_{n-1} s_n \mapsto s_n s_1 s_2 \cdots s_{n-1}$;
(ii) $s_1 \cdots s_{i-1} s_i s_{i+1} s_{i+2} \cdots s_n \mapsto s_1 \cdots s_{i-1} s_{i+1} s_i s_{i+2} \cdots s_n$, where $s_i s_{i+1} = s_{i+1} s_i$.

Thus, it suffices to prove that, if $\bar{\gamma}$ is the Coxeter element obtained from γ by performing one of these moves, the condition (D_p) obtained by using $\bar{\gamma}$ is equivalent to that obtained using γ. Define $\bar{\phi}_i$ and $\bar{R}_i$ in the obvious way.

For a move of type (i), it is easy to see that $\bar{\phi}_j = s_n(\phi_j)$ if $j \neq n$, and $\bar{\phi}_n = s_n\gamma^{-1}(\phi_n)$ Since $\bar{\gamma} = s_n \gamma s_n$, it follows that $\bar{R}_j = s_n(R_j)$ for all j. It follows as before that the condition (D_p) is unchanged.

For type (ii), $\bar{\gamma} = \gamma$ and it is clear that $\bar{\phi}_j = \phi_j$ except possibly when $j = i$ or $i+1$. But

$$\bar{\phi}_i = s_n \cdots s_{i+2}(\alpha_i) = s_n \cdots s_{i+2} s_{i+1}(\alpha_i) = \phi_i\ ,$$

since $s_{i+1}(\alpha_i) = \alpha_i$, and

$$\bar{\phi}_{i+1} = s_n \cdots s_{i+2} s_i(\alpha_{i+1}) = s_n \cdots s_{i+2}(\alpha_{i+1}) = \phi_{i+1}\ ,$$

since $s_i(\alpha_{i+1}) = \alpha_{i+1}$. Thus, $\bar{R}_j = R_j$ for all j. □

Remark. Despite this result, it is sometimes convenient to make a particular choice of γ, as follows (see [4, 5 and 10], for example). Choose a partition

$$I = I_\circ \amalg I_\bullet \tag{21}$$

such that

$$a_{ij} = 0 \quad \text{if } i,j \in I_\circ \quad \text{or if } i,j \in I_\bullet\ .$$

It is clear that such a partition exists and is unique up to interchanging $I_\circ$ and $I_\bullet$. Since s_i and s_j commute if $i,j \in I_\circ$ or if $i,j \in I_\bullet$, the Weyl group elements

$$\gamma_\circ = \prod_{i\in I_\circ} s_i, \qquad \gamma_\bullet = \prod_{i\in I_\bullet} s_i$$

are well defined. Then we take $\gamma = \gamma_\circ\gamma_\bullet$. Note that $\gamma_\circ^2 = \gamma_\bullet^2 = 1$, so that $\gamma^{-1} = \gamma_\bullet\gamma_\circ$.

With this choice, it is easy to see that

$$\phi_i = \begin{cases} \gamma_\bullet\alpha_i & \text{if } i \in I_\circ\ , \\ \alpha_i & \text{if } i \in I_\bullet\ . \end{cases}$$

Note that $\gamma\phi_i = \gamma_\circ\alpha_i = -\alpha_i$ if $i \in I_\circ$; on the other hand, if $i \in I_\bullet$, it is clear that α_i occurs with coefficient -1 in the root $\gamma\phi_i = -\gamma_\circ\alpha_i$, so $\gamma\phi_i \in R^-$. It follows that $\alpha_i \in R_i$ if $i \in I_\bullet$, and $-\alpha_i \in R_i$ if $i \in I_\circ$.

We observe next that, for an arbitrary choice of γ,

$$\phi_i = \lambda_i - \gamma^{-1}\lambda_i\ .$$

Indeed, recalling that $s_i\lambda_j = \lambda_j$ if $i \neq j$, and $= \lambda_i - \alpha_i$ if $i = j$, we have

$$\begin{aligned}\gamma^{-1}(\lambda_i) &= s_n s_{n-1}\cdots s_1(\lambda_i) = s_n s_{n-1}\cdots s_i(\lambda_i)\\ &= s_n s_{n-1}\cdots s_{i+1}(\lambda_i - \alpha_i) = \lambda_i - s_n s_{n-1}\cdots s_{i+1}(\alpha_i) = \lambda_i - \phi_i\,.\end{aligned}$$

Hence, (D_p) is equivalent to the condition

$$\gamma(\gamma^{r_1}\lambda_{i_1} + \gamma^{r_2}\lambda_{i_2} + \cdots + \gamma^{r_p}\lambda_{i_p}) = \gamma^{r_1}\lambda_{i_1} + \gamma^{r_2}\lambda_{i_2} + \cdots + \gamma^{r_p}\lambda_{i_p}$$

for some $r_1, r_2, \ldots, r_p \in \mathbb{Z}$. Since one is not an eigenvalue of γ on $\mathfrak{h}^*$ (see [16], Lemma 8.1, for example), this last equation is equivalent to

$$(\mathrm{D}'_p) \qquad \gamma^{r_1}\lambda_{i_1} + \gamma^{r_2}\lambda_{i_2} + \cdots + \gamma^{r_p}\lambda_{i_p} = 0$$

for some $r_1, r_2, \ldots, r_p \in \mathbb{Z}$. Now, the PRV conjecture [24], now a theorem [17,21], asserts that, if $\mu_1, \mu_2, \ldots, \mu_p \in P^+$ and if $0 \in W\mu_1 + W\mu_2 + \cdots + W\mu_p$ (where $W\mu$ is the Weyl group orbit of $\mu \in P^+$), then

$$\mathrm{Hom}_{\mathfrak{g}}(W(\mu_1) \otimes W(\mu_2) \otimes \cdots \otimes W(\mu_p), \mathbb{C}) \neq 0\,.$$

Hence, we have

Proposition 4.3. *If $p \geqq 2$ and $i_1, i_2, \ldots, i_p \in I$ satisfy condition* (D_p), *then*

$$(\mathrm{CG}_p) \qquad \mathrm{Hom}_{\mathfrak{g}}(W(\lambda_{i_1}) \otimes W(\lambda_{i_2}) \otimes \cdots \otimes W(\lambda_{i_p}), \mathbb{C}) \neq 0\,. \qquad \square$$

This result (and its proof) are due to Braden [4]. A generalisation of it can also be deduced from the main results of this paper, without using the PRV conjecture (see the remark at the end of Sect. 8).

In the case where $\mathfrak{g}$ is not simply-laced, we shall need a twisted version of condition (D_p). For this, we recall that the dual affine Lie algebra $\hat{\mathfrak{g}}^*$, whose Dynkin diagram is obtained by reversing the arrows in that of the affine Lie algebra $\hat{\mathfrak{g}}$, is the twisted affine Lie algebra associated to a diagram automorphism σ of a simply-laced algebra $\tilde{\mathfrak{g}}$ (see [15]). Following [25], choose nodes $\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_n$ of the Dynkin diagram of $\tilde{\mathfrak{g}}$, one from each orbit of σ, and define the *twisted Coxeter element* $\tilde{\gamma}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\gamma} = \tilde{s}_{\tilde{i}_1}\tilde{s}_{\tilde{i}_2}\cdots\tilde{s}_{\tilde{i}_n}\sigma$$

(here, and elsewhere in this section, a ˜ is used to denote objects associated with $\tilde{\mathfrak{g}}$). Define roots $\tilde{\phi}_{\tilde{i}_r}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\phi}_{\tilde{i}_r} = \sigma^{-1}\tilde{s}_{\tilde{i}_n}\tilde{s}_{\tilde{i}_{n-1}}\cdots\tilde{s}_{\tilde{i}_{r+1}}(\tilde{\alpha}_{\tilde{i}_r})\,,$$

and let $\tilde{R}_{\tilde{i}_r}$ be the $\tilde{\gamma}$-orbit of $\tilde{\phi}_{\tilde{i}_r}$. Note that there is a natural one-to-one correspondence between the set I of nodes of the Dynkin diagram of $\mathfrak{g}$ and the set of orbits of σ on the nodes of the Dynkin diagram of $\tilde{\mathfrak{g}}$. Thus, if $p \geqq 2$ and $i_1, i_2, \ldots, i_p \in I$, the following condition makes sense:

$$(\mathrm{TD}_p) \qquad 0 \in \tilde{R}_{i_1} + \tilde{R}_{i_2} + \cdots + \tilde{R}_{i_p}\,.$$

This is the twisted analogue of condition (D_p) that we shall need. As in (4 3), (TD_p) is independent of the choices made in defining it. It suffices to prove independence of the choice of node from each σ-orbit, for a given choice of total ordering of these orbits Unfortunately, we have only been able to verify this by a case-by-case check

We now make the conditions (D_p) and (TD_p) explicit, beginning with the case $p = 2$

Proposition 4.4. *(D_2) and (TD_2) are both equivalent to (CG_2)*

Proof For condition (D_2), we take γ to be the "white-black" Coxeter element defined in the previous remark, and distinguish two cases:

(a) $\mathfrak{g}$ *is not of type A_n with n even* In this case, h is even and it is well known that $w_0 = \gamma^{h/2}$; moreover, it is easy to check that, for all $i \in I$, i and $\bar{i}$ have the same colour Suppose that i and $\bar{i}$ are both black Then, $\alpha_i \in R_i$ and $\alpha_{\bar{i}} \in R_{\bar{i}}$ so the result follows from

$$\alpha_{\bar{i}} = -w_0(\alpha_i) = -\gamma^{h/2}(\alpha_i) \in -R_i \ .$$

If, on the other hand, i and $\bar{i}$ are both white, then $\gamma_\bullet(\alpha_i) \in R_i$ and $\gamma_\bullet(\alpha_{\bar{i}}) \in R_{\bar{i}}$, and the result follows from

$$\gamma_\bullet(\alpha_{\bar{i}}) = -\gamma_\bullet\gamma^{h/2}(\alpha_i) = -\gamma^{-h/2}\gamma_\bullet(\alpha_i) \in -R_i$$

(b) $\mathfrak{g}$ *is of type A_n with n even* In this case, h is odd, i and $\bar{i}$ always have different colour, and $w_0 = \gamma_\bullet\gamma^{(h-1)/2} = \gamma^{(h+1)/2}\gamma_\bullet$. The argument now proceeds essentially as in case (a)

This proves that, for all $\mathfrak{g}$, (CG_2) implies (D_2) The converse is the case $p = 2$ of (4 3).

For (TD_2), we have only been able to verify the result by using a case-by-case check. □

The crucial case for us is $p = 3$.

Proposition 4.5. *Let the nodes of the Dynkin diagram of* $\mathfrak{g}$ *be numbered as in* [3], *and let* $1 \leqq i \leqq j \leqq k \leqq n$

(a) $\mathfrak{g} = A_n$. *(i,j,k) satisfies (D_3) if and only if*
  (i) $i + j \leqq n$, $k = n + 1 - (i + j)$, *or*
  (ii) $i + j > n + 1$, $k = 2n + 2 - i - j$
(b) $\mathfrak{g} = B_n$ $(n \geqq 3)$: *(i,j,k) satisfies (D_3) if and only if*
  (i) $i + j \leqq n - 1$, $k = i + j$, *or*
  (ii) $i + j \geqq n + 1$, $k = 2n - i - j$; *or*
  (iii) $i < n$, $j = k = n$,
and satisfies (TD_3) if and only if one of the conditions (i), (iii) *or*
  (ii)′ $i + j \geqq n$, $k = 2n - 1 - i - j$
holds
(c) $\mathfrak{g} = C_n$ $(n \geqq 2)$. *(i,j,k) satisfies (D_3) if and only if*
  (i) $i + j \leqq n$, $k = i + j$, *or*
  (ii) $i + j \geqq n$, $k = 2n - i - j$,
and satisfies (TD_3) if and only if one of the conditions (i) *or*
  (ii)′ $i + j \geqq n + 2$, $k = 2n + 2 - i - j$
holds

(d) $\mathfrak{g} = D_n$ ($n \geqq 4$): *(i,j,k) satisfies* (D_3) *if and only if*
(i) $i + j \leqq n-2$, $k = i+j$; *or*
(ii) $i + j \geqq n$, $k = 2n - i - j - 2$; *or*
(iii) $i \leqq n-2$, $n-i$ *is even*, $j = k = n-1$ *or* $j = k = n$; *or*
(iv) $i \leqq n-2$, $n-i$ *is odd*, $j = n-1$, $k = n$.

(e_6) $\mathfrak{g} = E_6$: *the triples satisfying* (D_3) *are*

$$(1,1,1),(1,1,5),(1,2,3),(1,2,6),(1,3,4),(1,5,5),(2,2,2),(2,2,4),(2,4,4),$$
$$(2,5,6),(3,3,3),(3,3,6),(3,4,5),(3,6,6),(4,4,4),(4,5,6),(5,5,5),(6,6,6).$$

(e_7) $\mathfrak{g} = E_7$: *the triples satisfying* (D_3) *are*

$$(1,1,1),(1,1,3),(1,1,6),(1,2,2),(1,2,5),(1,2,7),(1,3,6),(1,4,4),(1,7,7),$$
$$(2,2,4),(2,3,5),(2,3,7),(2,4,5),(2,6,7),(3,3,3),(3,4,4),(3,4,6),(3,5,7),$$
$$(3,6,6),(4,4,4),(4,5,5),(4,5,7),(4,6,6),(5,5,6),(5,6,7),(6,6,6),(6,7,7).$$

(e_8) $\mathfrak{g} = E_8$: *the triples satisfying* (D_3) *are*

$$(1,1,1),(1,1,2),(1,1,3),(1,1,6),(1,2,5),(1,2,8),(1,3,4),(1,3,5),(1,3,6),$$
$$(1,3,7),(1,4,5),(1,5,5),(1,7,7),(1,7,8),(1,8,8),(2,2,2),(2,2,3),(2,2,4),$$
$$(2,2,5),(2,2,8),(2,3,8),(2,4,5),(2,4,6),(2,5,5),(2,6,6),(2,6,7),(2,6,8),$$
$$(2,7,8),(3,3,3),(3,3,4),(3,3,7),(3,4,5),(3,4,7),(3,5,6),(3,5,8),(3,6,8),$$
$$(3,7,7),(4,4,4),(4,4,5),(4,4,6),(4,4,7),(4,4,8),(4,5,8),(4,6,6),(4,6,7),$$
$$(5,5,5),(5,6,7),(5,6,8),(5,7,7),(6,6,6),(6,7,7),(6,7,8),(7,7,7),(7,8,8),$$
$$(8,8,8).$$

(f) $\mathfrak{g} = F_4$ (α_3 *short*): *the triples satisfying* (D_3) *are*

$$(1,1,1),(1,1,2),(1,2,2),(1,3,4),(1,4,4),(2,2,2),(2,3,3),(2,3,4),(3,3,3),$$
$$(3,3,4),(3,4,4),(4,4,4),$$

and those satisfying (TD_3) *are*

$$(1,1,1),(1,1,2),(1,1,3),(1,2,3),(1,3,4),(1,4,4),(2,2,2),$$
$$(2,2,3),(2,2,4),(2,3,4),(3,4,4),(4,4,4).$$

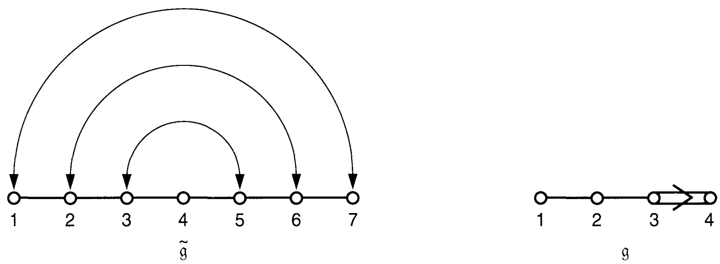
(g) $\mathfrak{g} = G_2$ (α_2 *short*): *the triples satisfying* (D_3) (*resp.* (TD_3)) *are*

$(1,1,1),(1,2,2),(2,2,2)$ (*resp.* $(1,1,1),(1,1,2),(1,2,2),(2,2,2)$). □

Remarks. 1. It is interesting to note that, in each of (a)–(d), case (ii) of condition (D_3) can be written $k = h - i - j$, where h is the Coxeter number of $\mathfrak{g}$, and that in case (b) (resp. (c)), condition (ii)$'$ can be written $k = \check{h} - i - j$ (resp. $2\check{h} - i - j$), where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$. (This mysterious factor of 2 in the C_n case is apparently well known to physicists.)

2. If $\mathfrak{g}$ is of type D_5, the triple $2,2,2$ satisfies (CG_3) (because $W(\lambda_2)$ is the adjoint $\mathfrak{g}$-module), but does not satisfy (D_3). Thus, the converse of (4.3) is false when $p = 3$. This result was first noted in [18].

The proof of (4.6) is a straightforward, if tedious, computation. We discuss the example of $\mathfrak{g} = B_4$ to show what is involved. From [15], we see that $\tilde{\mathfrak{g}}$ is of type A_7 and σ is the obvious involution:



We take

$$\gamma = s_1 s_2 s_3 s_4, \qquad \tilde{\gamma} = \tilde{s}_1 \tilde{s}_2 \tilde{s}_3 \tilde{s}_4 \sigma .$$

Then,

$$\phi_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4, \qquad \phi_2 = \alpha_2 + \alpha_3 + 2\alpha_4, \qquad \phi_3 = \alpha_3 + 2\alpha_4, \qquad \phi_4 = \alpha_4 ;$$

$$\tilde{\phi}_1 = \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{\alpha}_7, \qquad \tilde{\phi}_2 = \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6, \qquad \tilde{\phi}_3 = \tilde{\alpha}_4 + \tilde{\alpha}_5, \qquad \tilde{\phi}_4 = \tilde{\alpha}_4$$

The orbits are as follows·

$$\begin{aligned}
R_1 &= \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4)\} ,\\
R_2 &= \{\pm(\alpha_1 + \alpha_2), \pm(\alpha_2 + \alpha_3), \pm(\alpha_2 + \alpha_3 + 2\alpha_4)\} ,\\
R_3 &= \{\pm(\alpha_3 + 2\alpha_4), \pm(\alpha_1 + \alpha_2 + \alpha_3), \pm(\alpha_2 + 2\alpha_3 + 2\alpha_4), \pm(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)\} ,\\
R_4 &= \{\pm\alpha_4, \pm(\alpha_3 + \alpha_4), \pm(\alpha_2 + \alpha_3 + \alpha_4), \pm(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\} ;\\
\tilde{R}_1 &= \{\pm\tilde{\alpha}_1, \pm\tilde{\alpha}_2, \pm\tilde{\alpha}_3, \pm\tilde{\alpha}_6, \pm\tilde{\alpha}_7, \pm(\tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{\alpha}_7), \pm(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5)\} ,\\
\tilde{R}_2 &= \{\pm(\tilde{\alpha}_1 + \tilde{\alpha}_2), \pm(\tilde{\alpha}_2 + \tilde{\alpha}_3), \pm(\tilde{\alpha}_6 + \tilde{\alpha}_7), \pm(\tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6), \pm(\tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5) ,\\
&\qquad \pm(\tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{\alpha}_7), \pm(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6)\} ,\\
\tilde{R}_3 &= \{\pm(\tilde{\alpha}_4 + \tilde{\alpha}_5), \pm(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3), \pm(\tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5), \pm(\tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6),\\
&\qquad \pm(\tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6), \pm(\tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{\alpha}_7),\\
&\qquad \pm(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{\alpha}_7)\} ,\\
\tilde{R}_4 &= \{\pm\tilde{\alpha}_4, \pm\tilde{\alpha}_5, \pm(\tilde{\alpha}_3 + \tilde{\alpha}_4), \pm(\tilde{\alpha}_5 + \tilde{\alpha}_6), \pm(\tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4), \pm(\tilde{\alpha}_5 + \tilde{\alpha}_6 + \tilde{\alpha}_7),\\
&\qquad \pm(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4)\} .
\end{aligned}$$

By inspection, one sees that i, j, k satisfies (D_3) or (TD_3) exactly in the following cases:

$$(D_3).\ (1,1,2),\ (1,2,3),\ (1,4,4),\ (2,3,3),\ (2,4,4),\ (3,4,4) ,$$

$$(TD_3):\ (1,1,2),\ (1,2,3),\ (1,3,3),\ (1,4,4),\ (2,2,3),\ (2,4,4),\ (3,4,4) .$$

These results are in accordance with (4.6).

For $p \geqq 4$, the following result determines (D_p) inductively in terms of (D_2) and (D_3).

Proposition 4.6. *Fix $p \geqq 4$ and $i_1, i_2, \ldots, i_p \in I$. Then, $i_1, i_2, \ldots, i_p$ satisfy* (D_p) *if and only if there exists $2 \leqq r \leqq p$ such that either*

(i) *i_1, i_r satisfy* (D_2) *and $i_2, \ldots, \hat{i}_r, \ldots, i_p$ satisfy* (D_{p-2}) (*the* ˆ *indicates that the index is to be omitted*), *or*

(ii) *there exists $j \in I$ such that i_1, i_r, j satisfy* (D_3) *and $\bar{j}, i_2, \ldots, \hat{i}_r, \ldots, i_p$ satisfy* (D_{p-1}).

The same result holds with (D_p) *replaced by* (TD_p) *throughout.*

Proof. The "if" part is trivial, and the "only if" part follows immediately from the following simple fact about root systems. Let $\beta_1, \beta_2, \ldots, \beta_p$ be roots ($p \geqq 3$) and suppose that $\beta_1 + \beta_2 + \cdots + \beta_p = 0$; then, there exists $2 \leqq r \leqq p$ such that $\beta_1 + \beta_r$ is either zero or a root. To prove this, just observe that $-2 = -(\beta_1, \check{\beta}_1) = (\beta_2 + \cdots + \beta_p, \check{\beta}_1)$, so, for some $2 \leqq r \leqq p$, we must have $(\beta_r, \check{\beta}_1) < 0$. □

We conjecture, based on extensive computer calculations for small values of p and algebras of low rank (including the exceptional algebras), that one can always take $r = 2$ in this result.

The main purpose of this paper is to study the following conjecture, first made explicit by MacKay [20] (when $p = 3$), but implicit in the work of several authors on affine Toda field theories (see [5, 10 and 23], for example).

Conjecture 4.7. *Let $p \geqq 2$ and let $i_1, i_2, \ldots, i_p \in I$. Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a_1}(\lambda_{i_1}) \otimes V_{a_2}(\lambda_{i_2}) \otimes \cdots \otimes V_{a_p}(\lambda_{i_p}), \mathbb{C}) \neq 0 \,, \tag{23}$$

for some $a_1, a_2, \ldots, a_p \in \mathbb{C}$, if and only if $i_1, i_2, \ldots, i_p$ satisfy

$$\begin{cases} (D_p) & \textit{when } \mathfrak{g} \textit{ is simply-laced}\,, \\ \textit{both } (D_p) \textit{ and } (TD_p) & \textit{when } \mathfrak{g} \textit{ is not simply-laced}\,. \end{cases}$$

It follows immediately from (3.6) and (4.4) that this conjecture is true when $p = 2$. The main result of this paper is to prove the conjecture when $p = 3$ and $\mathfrak{g}$ is not of exceptional type (this is the content of Sects. 5–8). In dealing with the $p = 3$ case, it is useful to observe that, if i, j, k satisfies (23), so does any permutation of i, j, k (the same is obviously true of condition (D_3)). To see this, note first that, by (3.4) and (3.6) (i), (23) holds if and only if

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_{c-\kappa}(\lambda_{\bar{k}})) \neq 0 \,,$$

which in turn holds if and only if

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c-2\kappa}(\lambda_k) \otimes V_a(\lambda_i) \otimes V_b(\lambda_j), \mathbb{C}) \neq 0 \,.$$

Thus, (23) is preserved by cyclic permutations of (i, j, k). On the other hand, by 3.6 (ii), (23) is equivalent to

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{\bar{c}}(\lambda_{\bar{k}}) \otimes V_{\bar{b}}(\lambda_{\bar{j}}) \otimes V_{\bar{a}}(\lambda_{\bar{i}}), \mathbb{C}) \neq 0 \,, \tag{24}$$

where $\bar{a} = \kappa + d_i - a$, etc. But, it is known that there exists a diagram automorphism π of $\mathfrak{g}$ such that $\pi(i) = \bar{i}$ for all $i \in I$. Twisting by the corresponding automorphism of $Y(\mathfrak{g})$ shows that (24) is equivalent to

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{\bar{c}}(\lambda_k)\otimes V_{\bar{b}}(\lambda_j)\otimes V_{\bar{a}}(\lambda_i),\mathbb{C})\neq 0\,.$$

Hence, (23) is also preserved by the permutation $(i,j,k)\mapsto(k,j,i)$ Since this, together with the cyclic permutations, generates the whole symmetric group on three letters, (23) is preserved by all permutations of (i,j,k). It follows that, in proving (4.5), we may always assume that i,j and k are in some fixed order.

As to the case $p\geqq 4$, we note that Proposition 4 6, together with the $p=2$ and $p=3$ cases of Conjecture 4 7, implies the following weak version of the "if" part of (4 7) for arbitrary p.

Proposition 4.8. *Let $\mathfrak{g}$ be of type A,B,C or D, let $p\geqq 2$ and let $i_1,i_2,\dots,i_p\in I$ satisfy* (D_p) *if $\mathfrak{g}$ is simply-laced, or* (D_p) *and* (TD_p) *if $\mathfrak{g}$ is not simply-laced Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a_1}(\lambda_{i_1})\otimes V_{a_r}(\lambda_{i_r})\otimes V_{a_2}(\lambda_{i_2})\otimes\cdots\otimes V_{a_{r-1}}(\lambda_{i_{r-1}})$$
$$\otimes V_{a_{r+1}}(\lambda_{i_{r+1}})\otimes\cdots\otimes V_{a_p}(\lambda_{i_p}),\mathbb{C})\neq 0\,,$$

for some $a_1,a_2,\dots,a_p\in\mathbb{C}$

Proof We assume that $i_1,i_2,\dots,i_p$ satisfy (ii) in (4 6) (the other case is easier) By the $p=3$ case of (4.7),

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a_1}(\lambda_{i_1})\otimes V_{a_r}(\lambda_{i_r})\otimes V_b(\lambda_j),\mathbb{C})\neq 0\,,$$

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_c(\lambda_{\bar{j}})\otimes V_{a_2}(\lambda_{i_2})\otimes\cdots\otimes V_{a_{r+1}}(\lambda_{i_{r+1}})\otimes V_{a_{r-1}}(\lambda_{i_{r-1}})$$
$$\otimes\cdots\otimes V_{a_p}(\lambda_{i_p}),\mathbb{C})\neq 0\,,$$

for some $a_1,a_2,\dots,a_p,b,c\in\mathbb{C}$. By (3.4) and (3.6), there are non-zero (hence injective) $Y(\mathfrak{g})$-module homomorphisms

$$V_b(\lambda_j)\to V_{a_r+\kappa}(\lambda_{\bar{i}_r})\otimes V_{a_1+\kappa}(\lambda_{\bar{i}_1})\,,$$

$$V_c(\lambda_{\bar{j}})\to V_{a_p-\kappa}(\lambda_{\bar{i}_p})\otimes\cdots\otimes V_{a_{r+1}-\kappa}(\lambda_{i_{r+1}})\otimes V_{a_{r-1}-\kappa}(\lambda_{i_{r-1}})\otimes\cdots\otimes V_{a_2-\kappa}(\lambda_{\bar{i}_2})\,.$$

By twisting with a suitable automorphism τ_a (see (2.3)), we may assume that $c=b+\kappa$ Taking the tensor product of the last two homomorphisms, we then obtain an injective homomorphism

$$V_{b+\kappa}(\lambda_{\bar{j}})\otimes V_b(\lambda_j)$$
$$\downarrow$$
$$V_{a_p-\kappa}(\lambda_{\bar{i}_p})\otimes\cdots\otimes V_{a_{r+1}-\kappa}(\lambda_{i_{r+1}})\otimes V_{a_{r-1}-\kappa}(\lambda_{i_{r-1}})\otimes\cdots\otimes V_{a_2-\kappa}(\lambda_{\bar{i}_2})$$
$$\otimes V_{a_r+\kappa}(\lambda_{\bar{i}_r})\otimes V_{a_1+\kappa}(\lambda_{\bar{i}_1})\,.$$

Finally, composing with the injective homomorphism

$$\mathbb{C}\to V_{b+\kappa}(\lambda_{\bar{j}})\otimes V_b(\lambda_j)$$

given by (3.4) and (3.6), we obtain an injective (hence non-zero) homomorphism

$$\begin{array}{c}\mathbb{C}\\ \downarrow\\ V_{a_p-\kappa}(\lambda_{\bar{i}_p})\otimes\cdots\otimes V_{a_{r+1}-\kappa}(\lambda_{i_{r+1}})\otimes V_{a_{r-1}-\kappa}(\lambda_{i_{r-1}})\otimes\cdots\otimes V_{a_2-\kappa}(\lambda_{\bar{i}_2})\\ \otimes V_{a_r+\kappa}(\lambda_{\bar{i}_r})\otimes V_{a_1+\kappa}(\lambda_{\bar{i}_1})\,.\end{array}$$

Equivalently,

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a_1}(\lambda_{i_1})\otimes V_{a_r}(\lambda_{i_r})\otimes V_{a_2-2\kappa}(\lambda_{i_2})\otimes\cdots\otimes V_{a_{r-1}-2\kappa}(\lambda_{i_{r-1}})$$
$$\otimes V_{a_{r+1}-2\kappa}(\lambda_{i_{r+1}})\otimes\cdots\otimes V_{a_p-2\kappa}(\lambda_{i_p}),\mathbb{C})\neq 0\,. \qquad \square$$

The same argument shows that the conjecture stated after the proof of (4.6) implies the "if" part of (4.7) in full generality (when $\mathfrak{g}$ is not of exceptional type).

5. Some Preliminary Lemmas

In this section we collect some results which describe the restriction of $Y(\mathfrak{g})$-modules to "diagram subalgebras" of $Y(\mathfrak{g})$.

Definition 5.1. *Let* $\emptyset\neq J\subseteq I$.

(i) $\mathfrak{g}_J$ *is the Lie subalgebra of* $\mathfrak{g}$ *generated by the* H_i *and the* $X_i^{\pm}$ *for* $i\in J$;

(ii) Y_J *is the subalgebra of* $Y(\mathfrak{g})$ *generated by the* $H_{i,k}$ *and the* $X_{i,k}^{\pm}$, *for* $i\in J$, $k\in\mathbb{N}$.

(iii) $Q_J=\sum_{i\in J}\mathbb{Z}\cdot\alpha_i$, $Q_J^+=\sum_{i\in J}\mathbb{N}\cdot\alpha_i$.

It is clear from (2.2) that there is an algebra homomorphism $Y(\mathfrak{g}_J)\to Y_J$ which maps $H_{i,k}\mapsto H_{i,k}$ and $X_{i,k}^{\pm}\mapsto X_{i,k}^{\pm}$, for all $i\in J$, $k\in\mathbb{N}$. In particular, every $Y(\mathfrak{g})$-module may be regarded as a $Y(\mathfrak{g}_J)$-module. If V is a highest weight $Y(\mathfrak{g})$-module with highest weight vect or v, set

$$V_J=Y(\mathfrak{g}_J)\cdot v\,.$$

Note that V_J is preserved by the action of $\mathfrak{h}$, since $[\mathfrak{h},Y_J]\subseteq Y_J$.

Lemma 5.2. *Let* $\emptyset\neq J\subseteq I$.

(i) *Let* V *be a highest weight* $Y(\mathfrak{g})$*-module with highest weight* $\lambda\in P^+$ *(as a* $\mathfrak{g}$*-module). Then,*

$$V_J=\bigoplus_{\eta\in Q_J^+}V_{\lambda-\eta}\,.$$

(ii) *If* V *is an irreducible* $Y(\mathfrak{g})$*-module, then* V_J *is an irreducible* $Y(\mathfrak{g}_J)$*-module.*

(iii) *If* V *and* W *are irreducible* $Y(\mathfrak{g})$*-modules with highest weights* λ *and* μ, *then,*

$$V_J\otimes W_J=\bigoplus_{\eta\in Q_J^+}(V\otimes W)_{\lambda+\mu-\eta}\,. \qquad \square$$

The proof is straightforward (see Lemma 4.3 in [7] for part (ii)).

The canonical map $Y(\mathfrak{g}_J)\to Y(\mathfrak{g})$ is not a homomorphism of Hopf algebras. Nevertheless, we have

Lemma 5.3. *Let V and W be finite-dimensional irreducible $Y(\mathfrak{g})$-modules and let $\emptyset \neq J \subseteq I$ Then, $V_J \otimes W_J$ is a $Y(\mathfrak{g}_J)$-submodule of $V \otimes W$.* □

This is Lemma 2.15 from [7]. The following is a more precise result.

Lemma 5.4. *Let U, V and W be finite-dimensional irreducible $Y(\mathfrak{g})$-modules with highest weights (as $\mathfrak{g}$-modules) λ, μ and ν, respectively, and let $\emptyset \neq J \subseteq I$*

(i) *Assume that $\lambda + \mu - \nu \in Q_J^+$ Then, any non-zero $Y(\mathfrak{g})$-module homomorphism $U \otimes V \to W$ maps $U_J \otimes V_J$ onto W_J In fact, this restriction defines an injective linear map*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(U \otimes V, W) \to \mathrm{Hom}_{Y(\mathfrak{g}_J)}(U_J \otimes V_J, W_J)\,. \tag{25}$$

(ii) *Assume that $U_J \otimes V_J$ is a highest weight $Y(\mathfrak{g}_J)$-module and that $U \otimes V$ has an irreducible quotient $Y(\mathfrak{g})$-module with highest weight $\nu < \lambda + \mu$ Then, $\lambda + \mu - \nu \in Q^+ \backslash Q_J^+$*

Proof The fact that any $Y(\mathfrak{g})$-module homomorphism $f \cdot U \otimes V \to W$ maps $U_J \otimes V_J$ into W_J follows from (5.2) (i) and (iii). If $f \neq 0$, the image of f contains a $Y(\mathfrak{g})$-highest weight vector $w \in W$. By (5.2) (i) and (iii) again, w is in the image of the restriction of f to $U_J \otimes W_J$. By (5.2) (ii), f is surjective, and the linear map (25) is injective.

Part (ii) follows immediately from part (i). □

Lemma 5.5. *Let V and W be finite-dimensional irreducible $Y(\mathfrak{g})$-modules, and let $\emptyset \neq J \subseteq I$ Assume that $V_J \otimes W_J$ contains a non-zero $Y(\mathfrak{g}_J)$-highest weight vector u which is also an $\mathfrak{h}$-eigenvector of weight $\nu \in P^+$. Then, $(V \otimes W)_\nu$ contains a $Y(\mathfrak{g})$-highest weight vector*

Proof Clearly, $\lambda + \mu - \nu \in Q_J^+$. It follows that $u \in (V \otimes W)_\nu^{++}$. The result now follows from the discussion preceeding (3.1) □

6. The A_n Case

In this section $\mathfrak{g}$ is of type A_n ($n \geqq 1$) The Coxeter number h of $\mathfrak{g}$ is $n+1$. Proposition 4.6 implies that Conjecture 4.7 is a special case of

Theorem 6.1. *Let $1 \leqq i, j, k \leqq n$, $a, b, c \in \mathbb{C}$ Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0 \tag{26}$$

if and only if one of the following holds:

(i) $i + j < n + 1$, $k = i + j$, $b - a = \frac{1}{2}(i + j)$, $c - a = \frac{1}{2}j$;

(ii) $i + j > n + 1$, $k = i + j - n - 1$, $b - a = n + 1 - \frac{1}{2}(i + j)$, $c - a = \frac{1}{2}(n + 1 - j)$

Remark It follows from 6.3 (i) below that the space

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k))$$

is one-dimensional when it is non-zero.

We shall also prove the following:

Theorem 6.2. *Let* $1 \leqq i \leqq j \leqq n$, $a, b \in \mathbb{C}$. *Then,* $V_a(\lambda_i) \otimes V_b(\lambda_j)$ *is not a highest weight* $Y(\mathfrak{g})$*-module if and only if*

$$b - a = \frac{1}{2}(j - i) + r \quad \textit{for some } 0 < r \leqq \min(i, n + 1 - j)\,.$$

Hence, $V_a(\lambda_i) \otimes V_b(\lambda_j)$ *is reducible as a* $Y(\mathfrak{g})$*-module if and only if*

$$b - a = \pm\left(\frac{1}{2}(j - i) + r\right) \quad \textit{for some } 0 < r \leqq \min(i, n + 1 - j)\,.$$

Remark. One can show further that, when $b - a = \frac{1}{2}(j - i) + r$ for some $0 < r \leqq \min(i, n + 1 - j)$, the $Y(\mathfrak{g})$-module $V_a(\lambda_i) \otimes V_b(\lambda_j)$ has a Jordan–Hölder series of length two:

$$0 \to V \to V_a(\lambda_i) \otimes V_b(\lambda_j) \to V_{a+\frac{1}{2}}(\lambda_{i-r}) \otimes V_{a+\frac{1}{2}(j-i+r)}(\lambda_{j+r}) \to 0\,,$$

where V is an irreducible $Y(\mathfrak{g})$-module such that

$$V \cong \bigoplus_{s=0}^{r-1} W(\lambda_{i-s} + \lambda_{j+s})$$

as $\mathfrak{g}$-modules.

We begin with the following.

Proposition 6.3. *Let* $1 \leqq i, j, k \leqq n$.

(i) *We have*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda_i) \otimes W(\lambda_j), W(\lambda_k)) = \begin{cases} \mathbb{C} & \textit{if } k = i + j \textit{ or } k = i + j - n - 1\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

(ii) *As* $\mathfrak{g}$*-modules, we have*

$$V_a(\lambda_i) \cong W(\lambda_i)\,.$$

(iii) *Let* $a, b, c \in \mathbb{C}$. *Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) = 0$$

if $k \neq i + j$ *or* $i + j - n - 1$. *If* $k = i + j$ *or* $i + j - n - 1$, *and* a *and* b *are fixed, the space*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0$$

for at most one value of c, *in which case it is one-dimensional.*

Proof. Part (i) is easy, part (ii) is well known (see [11 and 8]), and part (iii) is immediate from parts (i) and (ii). □

Proof of 6.1. By induction on n. The case $n = 1$ is proved in [6]. Twisting by φ and using (3.5), we see that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0$$

if and only if

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{\frac{1}{2}(n+3)-b}(\lambda_{n+1-j}) \otimes V_{\frac{1}{2}(n+3)-a}(\lambda_{n+1-i}), V_{\frac{1}{2}(n+3)-c}(\lambda_{n+1-k})) \neq 0$$

Hence it suffices to prove the theorem when $i + j < n + 1$.

Assume that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0 \tag{27}$$

Since $i + j < n$, 6.3 implies that $k = i + j$ Noting that

$$\lambda_i + \lambda_j - \lambda_{i+j} \in Q_J^+ ,$$

where $J = \{1, 2, \ldots, i + j - 1\}$, (5.4) gives

$$\mathrm{Hom}_{Y(\mathfrak{g}_J)}(V_a(\lambda_i) \otimes V_b(\lambda_j), \mathbb{C}) \neq 0 ,$$

whence $b - a = \frac{1}{2}(i + j)$ by (3.6).

The value of c can be computed as follows Using (3.5) and (3.6), (27) implies that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}(i+j)}(\lambda_j), V_{a+\frac{1}{2}(n+1)}(\lambda_{n+1-i}) \otimes V_c(\lambda_{i+j})) \neq 0 ,$$

and hence, by taking left duals, that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c-\frac{1}{2}(n+1)}(\lambda_{n+1-i-j}) \otimes V_a(\lambda_i), V_{a-\frac{1}{2}(n+1-i-j)}(\lambda_{n+1-j})) \neq 0 .$$

The first part of the proof now shows that

$$a - \left(c - \frac{1}{2}(n + 1)\right) = \frac{1}{2}(i + n + 1 - i - j) ,$$

i e

$$c - a = \frac{1}{2}j$$

For the "if" part, suppose that $k = i + j$, $b - a = \frac{1}{2}(i + j)$ and $c - a = \frac{1}{2}j$. Using (5 4) with $J = \{n - i - j - 2, \ldots, n\}$ and (5.5), we see that

$$(V_{b-\frac{1}{2}(n+1)}(\lambda_{n+1-j}) \otimes V_{a-\frac{1}{2}(n+1)}(\lambda_{n+1-i}))^{++}_{\lambda_{n+1-i-j}} \neq 0 .$$

Since there is no non-zero dominant weight strictly less than $\lambda_{n+1-i-j}$, it follows that for some $c' \in \mathbb{C}$,

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c'-\frac{1}{2}(n+1)}(\lambda_{n+1-i-j}), V_{b-\frac{1}{2}(n+1)}(\lambda_{n+1-j}) \otimes V_{a-\frac{1}{2}(n+1)}(\lambda_{n+1})) \neq 0$$

Applying (3 5) and (3 6) shows that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_{c'}(\lambda_k)) \neq 0 .$$

But then, by 6.3 (iii), c' is uniquely determined, and by the "only if" part, $c' = c$

The proof of 6.1 is now complete □

Proof of 6 2 By induction on n. If $n = 1$, the result is contained in [6]. Assuming the result is known when $\mathfrak{g}$ is of type A_m for $m < n$, we prove it when $\mathfrak{g}$ is of type A_n by induction on $\min(i, n + 1 - j)$. If $i = 1$ or $j = n$, the result follows from

(6.1) and (6.3), since

$$W(\lambda_1)\otimes W(\lambda_j)\cong W(\lambda_1+\lambda_j)\oplus W(\lambda_{j+1}),$$

$$W(\lambda_n)\otimes W(\lambda_j)\cong W(\lambda_n+\lambda_j)\oplus W(\lambda_{j-1}).$$

Assume now that $\min(i,n+1-j)>1$. To prove the "only if" part of (6.2), consider the case $i+j<n+1$ (resp. the case $i+j>n+1$).

Since $V_a(\lambda_i)\otimes V_b(\lambda_j)$ is not a highest weight $Y(\mathfrak{g})$-module, there exists an irreducible $Y(\mathfrak{g})$-module V with $\mathfrak{g}$-highest weight $\lambda=\lambda_i+\lambda_j-\eta$, for some $0\neq\eta\in Q^+$, such that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i)\otimes V_b(\lambda_j),V)\neq 0.$$

By (3.5), (3.6) and (6.1), we have

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}}(\lambda_{i-1})\otimes V_b(\lambda_j),V_{a+\frac{1}{2}(n-i)}(\lambda_n)\otimes V)\neq 0,$$

(resp.

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i)\otimes V_{b-\frac{1}{2}}(\lambda_{j+1}),V\otimes V_{b-\frac{1}{2}(j+1)}(\lambda_1))\neq 0).$$

If $b-a\neq\frac{1}{2}(j-i)+r$ for any $1<r\leqq i$, then by the induction hypothesis on $\min(i,n+1-j)$, $V_{a+\frac{1}{2}}(\lambda_{i-1})\otimes V_b(\lambda_j)$ (resp. $V_a(\lambda_i)\otimes V_{b-\frac{1}{2}}(\lambda_{j+1})$) is a highest weight $Y(\mathfrak{g})$-module, so

$$\lambda_{i-1}+\lambda_j\leqq\lambda_n+\lambda_i+\lambda_j-\eta,$$

i.e.

$$\eta\leqq\alpha_i+\cdots+\alpha_n,$$

(resp.

$$\lambda_i+\lambda_{j+1}\leqq\lambda_1+\lambda_i+\lambda_j-\eta,$$

i.e.

$$\eta\leqq\alpha_1+\cdots+\alpha_j).$$

This, together with the requirement that $\lambda\in P^+$, forces $\eta=\alpha_i+\cdots+\alpha_j$, so $\lambda=\lambda_{i-1}+\lambda_{j+1}$. Noting that $\lambda_i+\lambda_j-\lambda_{i-1}-\lambda_{j+1}\in Q_J^+$, where $J=\{i,i+1,\dots,j\}$, it follows from (5.4) that

$$\mathrm{Hom}_{Y(\mathfrak{g}_J)}(V_a(\lambda_1)\otimes V_b(\lambda_{j-i+1}),\mathbb{C})\neq 0.$$

By (3.6), we see that $b-a=\frac{1}{2}(j-i)+1$, as required.

We now prove the "if" part of (6.2), assuming it when $\mathfrak{g}$ is of type A_m for $m<n$, and for smaller values of $\min(i,n-j+1)$ when $\mathfrak{g}$ is of type A_n. We consider three cases.

Suppose first that $i+j<n+1$ (resp. $i+j>n+1$). Let

$$b-a=\frac{1}{2}(j-i)+r\quad\text{for some } 0<r\leqq i, \tag{28}$$

and assume for a contradiction that $V_a(\lambda_i)\otimes V_b(\lambda_j)$ is a highest weight $Y(\mathfrak{g})$-module. Let $J'=\{1,2,\dots,n-1\}$ (resp. $J'=\{2,\dots,n\}$). Since $V_a(\lambda_i)\otimes V_b(\lambda_j)$ is assumed to be $Y(\mathfrak{g})$-highest weight,

$$V_a(\lambda_i)_{J'}\otimes V_b(\lambda_j)_{J'}\subset Y(\mathfrak{g})\cdot(v_i\otimes v_j),$$

where v_i and v_j are $Y(\mathfrak{g})$-highest weight vectors in $V_a(\lambda_i)$ and $V_b(\lambda_j)$. But then (5.2) implies that

$$V_a(\lambda_i)_{J'} \otimes V_b(\lambda_j)_{J'} \subset Y(\mathfrak{g}_{J'}) \cdot (v_i \otimes v_j),$$

and hence that $V_a(\lambda_i)_{J'} \otimes V_b(\lambda_j)_{J'}$ is $Y(\mathfrak{g}_{J'})$-highest weight. By the induction hypothesis on n, $b-a$ cannot take any of the values in (28). This is the desired contradiction.

If $i+j=n+1$, the argument used above fails when $b-a=\frac{1}{2}(n+1)$, since in that case $V_a(\lambda_i)_{J'} \otimes V_b(\lambda_j)_{J'}$ is $Y(\mathfrak{g}_{J'})$-highest weight. But for this value of $b-a$, the contradiction is immediate from (3.6).

We have now completely proved Theorem 6.2, except for the final statement, which follows immediately from (3.6) and (3.8). □

7. The D_n Case

In this section $\mathfrak{g}$ is of type D_n, ($n \geqq 4$). The Coxeter number h is $2n-2$. Conjecture 4.7 is a special case of

Theorem 7.1. *Let* $1 \leqq i \leqq j \leqq k \leqq n$, $a,b,c \in \mathbb{C}$ *Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0$$

if and only if one of the following holds:

(i) $i+j \leqq n-2$, $k=i+j$, $b-a=\frac{1}{2}(i+j)$, $c-a=\frac{1}{2}j$;

(ii) $i+j \geqq n$, $j \leqq n-2$, $k=2n-i-j-2$, $b-a=\frac{1}{2}(i+j)$, $c-a=\frac{1}{2}j$;

(iii) $i \leqq n-2$, $j=n-1$, $b-a=\frac{1}{2}(n+i-1)$, $c-a=\frac{1}{2}(n-i-1)$,

$$k = \begin{cases} \overline{n-1} & \text{if } n-i \text{ is even}, \\ \overline{n} & \text{if } n-i \text{ is odd}, \end{cases}$$

(iv) $i \leqq n-2$, $j=n$, $b-a=\frac{1}{2}(n+i-1)$, $c-a=\frac{1}{2}(n-i-1)$,

$$k = \begin{cases} \overline{n} & \text{if } n-i \text{ is even}, \\ \overline{n-1} & \text{if } n-i \text{ is odd}. \end{cases}$$

Moreover, the space of homomorphisms in (26) *is one-dimensional when it is non-zero*

We shall also prove

Theorem 7.2. *Let* $1 \leqq i \leqq j \leqq n$ *Then,* $V_a(\lambda_i) \otimes V_b(\lambda_j)$ *is not a highest weight* $Y(\mathfrak{g})$*-module if and only if one of the following holds*:

(i) $j \leqq n-2$,

$$b-a = \begin{cases} \frac{1}{2}(j-i)+r & \text{for some } 0 < r \leqq \min(i,n-j), \text{ or} \\ n-1-r-\frac{1}{2}(j-i) & \text{for some } 0 \leqq r < \min(i,n-j), \end{cases}$$

(ii) $i \leqq n-2$, $j=n-1$ *or* n,

$$b-a = \frac{1}{2}(n-1-i)+r \quad \text{for some } 0 < r \leqq i\,;$$

(iii) $i = j = n - 1$ *or* n,

$$b - a = n - r - 1 \quad \textit{for some } 0 \leqq r \leqq n - 2 \textit{ with } n - r \textit{ even};$$

(iv) $i = n - 1$, $j = n$,

$$b - a = n - r - 1 \quad \textit{for some } 0 \leqq r \leqq n - 2 \textit{ with } n - r \textit{ odd}.$$

Hence, $V_a(\lambda_i) \otimes V_b(\lambda_j)$ *is reducible as a* $Y(\mathfrak{g})$*-module if and only if* $\pm(b - a)$ *takes one of the above values.*

We first recall from [7], Theorem 6.2, the $\mathfrak{g}$-module structure of the fundamental $Y(\mathfrak{g})$-modules.

Proposition 7.3. *Let* $a \in \mathbb{C}$*. Then as a* $\mathfrak{g}$*-module,*

$$V_a(\lambda_i) \cong \begin{cases} \bigoplus_{k=0}^{[\frac{i}{2}]} W(\lambda_{i-2k}) & \textit{if } i \leqq n-2, \\ W(\lambda_i) & \textit{if } i = n-1 \textit{ or } n, \end{cases}$$

where $\lambda_0 = 0$. □

We first prove

Proposition 7.4. *Let* $1 \leqq j \leqq n - 2$ *and let* v_1 *and* v_j *be* $Y(\mathfrak{g})$*-highest weight vectors in* $V(\lambda_1)$ *and* $V(\lambda_j)$*, respectively.*

(i) $V_a(\lambda_1) \otimes V_b(\lambda_j)$ *is not a* $Y(\mathfrak{g})$*-highest weight module if and only if* $b - a = \frac{1}{2}(j + 1)$ *or* $n - \frac{1}{2}(j + 1)$.

(ii) *If* $b - a = \frac{1}{2}(j + 1)$ *or* $n - \frac{1}{2}(j + 1))$*, then* $V_a(\lambda_1) \otimes V_b(\lambda_j)$ *has a Jordan–Hölder series of length two, namely*

$$0 \to Y(\mathfrak{g}) \cdot (v_1 \otimes v_j) \to V_a(\lambda_1) \otimes V_b(\lambda_j) \to V_{a+\frac{1}{2}j}(\lambda_{j+1}) \to 0,$$

or

$$0 \to Y(\mathfrak{g}) \cdot (v_1 \otimes v_j) \to V_a(\lambda_1) \otimes V_b(\lambda_j) \to V_{a+n-1-\frac{1}{2}j}(\lambda_{j-1}) \to 0),$$

respectively (if $j = n - 2$*, the first short exact sequence should be replaced by*

$$0 \to Y(\mathfrak{g}) \cdot (v_1 \otimes v_j) \to V_a(\lambda_1) \otimes V_b(\lambda_j) \to V_{a+\frac{1}{2}(n-2)}(\lambda_{n-1}) \otimes V_{a+\frac{1}{2}(n-2)}(\lambda_n) \to 0).$$

Proof. Let $b - a = \frac{1}{2}(j + 1)$ and $J = \{1, 2, \ldots, n - 2\}$. By (6.1), the $Y(\mathfrak{g})$-submodule $V_b(\lambda_j)_J \otimes V_a(\lambda_1)_J$ of $V_b(\lambda_j) \otimes V_a(\lambda_1)$ has a $Y(\mathfrak{g}_J)$-highest weight vector of weight $\lambda_{j+1} = \lambda_1 + \lambda_j - \alpha_1 - \cdots - \alpha_j$ for $\mathfrak{g}$. Hence, by (5.5), $V_b(\lambda_j) \otimes V_a(\lambda_1)$ has a $Y(\mathfrak{g})$-highest weight vector of weight λ_{j+1}. But then, by (3.14), $V_a(\lambda_1) \otimes V_b(\lambda_j)$ cannot be $Y(\mathfrak{g})$-highest weight.

For the converse, assume that $V_a(\lambda_1) \otimes V_b(\lambda_j)$ is not $Y(\mathfrak{g})$-highest weight, let $M = V_a(\lambda_1) \otimes V_b(\lambda_j)/Y(\mathfrak{g}) \cdot (v_1 \otimes v_j)$, let N be an irreducible quotient of M, and let $\lambda \in P^+$ be the maximal weight of N as a $\mathfrak{g}$-module. The dominant weights $\lambda < \lambda_1 + \lambda_j$ are of two types:

(i) $j \geqq 3$, $\lambda = \lambda_1 + \lambda_{j-2k}$, $0 < k \leqq [j/2]$, and
(ii) $\lambda = \lambda_{j+1-2k}$, $0 \leqq k \leqq [(j + 1)/2]$,

with the understanding that $\lambda_0 = 0$ In case (i),

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_1) \otimes V_b(\lambda_j), N) \neq 0 ,$$

which implies that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_b(\lambda_j), V_{a+n-1}(\lambda_1) \otimes N) \neq 0 ,$$

and hence that $\lambda_j \leqq 2\lambda_1 + \lambda_{j-2k}$ This implies that $k = 1$, i.e. $\lambda = \lambda_1 + \lambda_{j-2}$. But then $\lambda_1 + \lambda_j - \lambda \in Q^+_{J'}$, where $J' = \{j-1, j, \dots, n\}$. Hence, (5.5) implies that

$$\mathrm{Hom}_{Y(\mathfrak{g}_{J'})}(V_b(\lambda_2), \mathbb{C}) \neq 0 ,$$

which is absurd. Thus, case (ii) must hold. As above, one sees that $k = 0$ or 1 (and $k = 1$ if $j = n-2$), so we must have either

(iia) $\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_1) \otimes V_b(\lambda_j), V_c(\lambda_{j+1})) \neq 0$ (if $j < n-2$), or
(iib) $\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_1) \otimes V_b(\lambda_j), V_c(\lambda_{j-1})) \neq 0$,

for some $c \in \mathbb{C}$.

In case (iia), note that $\lambda_1 + \lambda_j - \lambda_{j+1} = \alpha_1 + \cdot \ \cdot + \alpha_j \in Q^+_{J''}$, where $J'' = \{1, 2, .. \ , n-2\}$. Hence,

$$\mathrm{Hom}_{Y(\mathfrak{g}_{J''})}(V_a(\lambda_1)_{J''} \otimes V_b(\lambda_j)_{J''}, V_c(\lambda_{j+1})_{J''}) \neq 0 ,$$

which gives $b - a = \frac{1}{2}(j+1)$ and $c - a = \frac{1}{2}j$, by (6.1).

In case (iib), we get

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_b(\lambda_j), V_{a+n-1}(\lambda_1) \otimes V_c(\lambda_{j-1})) \neq 0 ,$$

and hence, taking left duals,

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c-n+1}(\lambda_{j-1}) \otimes V_a(\lambda_1), V_{b-n+1}(\lambda_j)) \neq 0 .$$

Finally, twisting with φ gives

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{n-a}(\lambda_1) \otimes V_{2n-1-c}(\lambda_{j-1}), V_{2n-1-b}(\lambda_j)) \neq 0 .$$

We are now in the situation of (iia). Hence,

$$2n - 1 - c - (n-a) = \frac{1}{2}j \quad \text{and} \quad 2n - 1 - b - (n-a) = \frac{1}{2}(j-1) ,$$

i.e.

$$b - a = n - \frac{1}{2}(j+1) \quad \text{and} \quad c - a = n - 1 - \frac{1}{2}j .$$

We have now proved (i) In fact, the preceding argument shows that, if V is an irreducible quotient of $V_a(\lambda_1) \otimes V_b(\lambda_j)$ with highest weight different from $\lambda_1 + \lambda_j$, then either

$$b - a = \frac{1}{2}(i+j) \quad \text{and} \quad V \cong V_{a+\frac{1}{2}j}(\lambda_{j+1}),$$

or

$$b - a = n - \frac{1}{2}(j+1) \quad \text{and} \quad V \cong V_{a+n-1-\frac{1}{2}j}(\lambda_{j-1}) . \tag{29}$$

We prove part (ii) of (7.4) when $b-a=\frac{1}{2}(j+1)$; the other case is similar. First, if N is any $Y(\mathfrak{g})$-submodule (or quotient module) of $V_a(\lambda_1)\otimes V_b(\lambda_j)$, then, since $\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-n+1}(\lambda_1)\otimes N, V_b(\lambda_j))\neq 0$, (resp. $\mathrm{Hom}_{Y(\mathfrak{g})}({}^tN\otimes V_a(\lambda_1), V_{b-n+1}(\lambda_j))$ $\neq 0$), we see by using (7.3) that

$$m_0(N)\neq 0 \quad \text{if } j \text{ is odd,} \quad \text{and} \quad m_1(N)\neq 0 \quad \text{if } j \text{ is even}\,.$$

If $L=Y(\mathfrak{g})\cdot(v_1\otimes v_j)$ is reducible for $Y(\mathfrak{g})$, let $L'\subset L$ be an irreducible $Y(\mathfrak{g})$-submodule. Then, $L'\cong V_e(\lambda_{j-1})$ for some $e\in\mathbb{C}$, and we get

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_e(\lambda_{j-1}), V_a(\lambda_1)\otimes V_b(\lambda_j))\neq 0\,.$$

But this is impossible when $b-a=\frac{1}{2}(j+1)$, by (29).

Let M be the quotient $V_a(\lambda_1)\otimes V_b(\lambda_j)/L$. Then, $M=Y(\mathfrak{g})\cdot M_{\lambda_{j+1}}$, since otherwise M would have an irreducible quotient which would have to be of highest weight λ_{j-1}, and we have seen above that this is impossible for this value of $b-a$. Since M is non-zero, this shows that $M_{\lambda_{j+1}}\neq 0$. On the other hand, since $M_{\lambda_{j+1}}\subseteq M^+$, we have

$$\dim(M_{\lambda_{j+1}})\leqq m_{\lambda_{j+1}}(V_a(\lambda_1)\otimes V_b(\lambda_j))\,.$$

This multiplicity is one, and so $M_{\lambda_{j+1}}$ is one-dimensional. Thus, M is a highest weight $Y(\mathfrak{g})$-module with $\mathfrak{g}$-highest weight λ_{j+1}. If M is not irreducible for $Y(\mathfrak{g})$, it contains an irreducible $Y(\mathfrak{g})$-submodule, which must be of the form $V_d(\lambda_{j+1-2k})$ for some $1\leqq k\leqq[(j+1)/2]$, $d\in\mathbb{C}$. By (7.3), this means that $m_{\lambda_1}(M)=2$ if j is even, and $m_0(M)=2$ if j is odd. But this would mean that $m_0(L)=0$ or $m_{\lambda_1}(L)=0$, and we have seen that this is impossible. □

To prove (7.1), we need

Proposition 7.5. *Let* $1\leqq i\leqq n-2$, $a\in\mathbb{C}$.

(i) *If* $n-i$ *is even,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_n)\otimes V_b(\lambda_n), V_c(\lambda_i))\neq 0$$

$$\textit{iff } b-a=n-i-1,\quad c-a=\frac{1}{2}(n-i-1)\,,$$

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_{n-1})\otimes V_b(\lambda_{n-1}), V_c(\lambda_i))\neq 0$$

$$\textit{iff } b-a=n-i-1,\quad c-a=\frac{1}{2}(n-i-1)\,.$$

(ii) *If* $n-i$ *is odd,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_{n-1})\otimes V_b(\lambda_n), V_c(\lambda_i))\neq 0$$

$$\textit{iff } b-a=n-i-1,\quad c-a=\frac{1}{2}(n-i-1)\,,$$

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_n)\otimes V_b(\lambda_{n-1}), V_c(\lambda_i))\neq 0$$

$$\textit{iff } b-a=n-i-1,\quad c-a=\frac{1}{2}(n-i-1)\,.$$

Proof. We prove the first statement in part (i); the proofs in the other cases are similar. In [7], Proposition 6.2, we established that, if $b-a=n-i-1$, there exist $c,c' \in \mathbb{C}$ such that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_n)\otimes V_b(\lambda_n), V_c(\lambda_i)) \neq 0$$

and

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_{n-1})\otimes V_b(\lambda_{n-1}), V_{c'}(\lambda_i)) \neq 0 .$$

To see that $c=c'=a+\frac{1}{2}(n-i-1)$, notice that since $m_{\lambda_i}(V(\lambda_n)\otimes V(\lambda_n))=1$, the values of c and c' are uniquely determined by a and b. But now, twisting by φ and applying τ_{a+b} gives

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_n)\otimes V_b(\lambda_n), V_{a+b-c}(\lambda_i)) \neq 0$$

and

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_{n-1})\otimes V_b(\lambda_{n-1}), V_{a+b-c'}(\lambda_i)) \neq 0 ,$$

and hence

$$a+b-c=c \quad \text{and} \quad a+b-c'=c'.$$

Conversely, suppose that $\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_n)\otimes V_b(\lambda_n), V_c(\lambda_i)) \neq 0$. We prove by induction on n that $b-a$ and $c-a$ have the stated values. If $n=4$, the result follows from (7.4) (i) by using a diagram automorphism of order three of $Y(\mathfrak{g})$, so the induction begins Assume the result when $\mathfrak{g}$ is of type D_m with $m<n$. Now, $2\lambda_n-\lambda_i \in Q_J^+$, where $J=\{2,3,\ldots,n\}$, so by the induction hypothesis on n, we get

$$b-a=n-1-(i-1)-1=n-i-1 .$$

The value of $c-a$ is determined as before. □

Proof of Theorem 7 1 We only have to prove the theorem in cases (i) and (ii), since (7.5) establishes cases (iii) and (iv).

The "only if" part is proved by induction on n. The induction actually begins at $n=3$, when $\mathfrak{g}$ is of type A_3, and the result in that case is contained in (6.1). Assume now that $n \geqq 4$ and that the result is known when $\mathfrak{g}$ is of type D_m for $m<n$.

Suppose then that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i)\otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0 . \tag{30}$$

This implies by Proposition 7.4 that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}}(\lambda_{i-1})\otimes V_{a+\frac{1}{2}(i-1)}(\lambda_1)\otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0 ,$$

and hence

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}(i-1)}(\lambda_1)\otimes V_b(\lambda_j), V_{a+n-\frac{3}{2}}(\lambda_{i-1})\otimes V_c(\lambda_k)) \neq 0 .$$

Let F be a non-zero element in $\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}(i-1)}(\lambda_1)\otimes V_b(\lambda_j), V_{a+n-\frac{3}{2}}(\lambda_{i-1})\otimes V_c(\lambda_k))$, and let v_1 and v_j be $Y(\mathfrak{g})$-highest weight vectors in $V_{a+\frac{1}{2}(i-1)}(\lambda_1)$ and $V_b(\lambda_j)$, respectively. We first prove that one of the following must hold:

(α) $b-a=\frac{1}{2}(i+j)$ and $F(v_1\otimes v_j)=0$;
(β) $b-a=n-1-\frac{1}{2}(j-i)$ and $F(v_1\otimes v_j)=0$;

(γ) $F(v_1 \otimes v_j) \neq 0$, $i+j \leqq n$, $k = i+j-2$, $b-a = \frac{1}{2}(i+j)-1$, $c-a = \frac{1}{2}j$;
(δ) $F(v_1 \otimes v_j) \neq 0, i+j \geqq n+2, k = 2n-i-j, b-a = \frac{1}{2}(i+j)-1, c-a = \frac{1}{2}j$.

If $F(v_1 \otimes v_j) = 0$, then, by (7.4), we see that either (α) or (β) must hold. On the other hand, if $F(v_1 \otimes v_j) \neq 0$, then, using the fact that $\lambda_{i-1} + \lambda_k - \lambda_1 - \lambda_j \in Q_J^+$, where $J = \{2, 3, \ldots, n\}$, we see from (5.5) that, if $i > 2$,

$$\mathrm{Hom}_{Y(\mathfrak{g}_J)}(V_{a+\frac{1}{2}}(\lambda_{i-2}) \otimes V_b(\lambda_{j-1}), V_c(\lambda_{k-1})) \neq 0 .$$

The induction hypothesis on n now shows that either (γ) or (δ) must hold. If $i = 1$, the same conclusion follows from (7.4). Finally, if $i = 2$, we get $b = c$ and $j = k$, so

$$F : V_{a+\frac{1}{2}}(\lambda_1) \otimes V_b(\lambda_j) \to V_{a+n-\frac{3}{2}}(\lambda_1) \otimes V_b(\lambda_j) .$$

Since $a + \frac{1}{2} \neq a + n - \frac{3}{2}$, 3.2 implies that $F(v_1 \otimes v_j) = 0$, contradicting our assumption. This completes the proof that one of (α)–(δ) must hold.

Next, we prove that (30) implies that (α) must hold. Observe that if we twist by φ and apply τ_{a+b-n}, then (30) implies that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_j) \otimes V_b(\lambda_i), V_{a+b-c}(\lambda_k)) \neq 0 .$$

Suppose that a, b and c satisfy the conditions in (γ) or (δ) above. Then, it is easy to see that a, b and $a+b-c$ do not satisfy any of the conditions (α)–(δ). Thus, the only possibilities are (α) and (β). We prove by induction on i that (β) is impossible.

If $i = 1$, we know by (7.4) that $k = j+1$ (since $i \leqq j \leqq k$) and that $b - a = \frac{1}{2}(j+1)$, so (β) is impossible in this case. Assume that (α) is the only possibility for $i-1$. If (β) holds for i, we see from (7.4) that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+n-1-\frac{1}{2}(j-i+1)}(\lambda_{j-1}), V_{a+n-\frac{3}{2}}(\lambda_{i-1}) \otimes V_c(\lambda_k)) \neq 0 ,$$

or equivalently that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}}(\lambda_{i-1}) \otimes V_{a+n-1-\frac{1}{2}(j-i+1)}(\lambda_{j-1}), V_c(\lambda_k)) \neq 0 .$$

Since (α) holds for $i-1$, we get

$$a + n - 1 - \frac{1}{2}(j - i + 1) - \left(a - \frac{1}{2}\right) = \frac{1}{2}(i + j - 2) ,$$

i.e. $j = n$, contradicting our assumption that $j < n$. This completes the induction, and proves that (30) implies (α).

We now show, again by induction on i, that (30) implies that either (i) or (ii) in the statement of (7.1) must hold. If $i = 1$, the result follows from (7.4).

Assume the result for $i-1$. To complete the induction we consider four cases:

Case 1. $j < k \leqq n-2$. By (7.4), (α) gives $k = i+j$ or $2n-i-j-2$,

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}}(\lambda_{i-1}) \otimes V_{a+\frac{1}{2}(i+j-1)}(\lambda_{j+1}), V_c(\lambda_k)) \neq 0 ,$$

and

$$c - \left(a - \frac{1}{2}\right) = \frac{1}{2}(j+1) ,$$

i.e. $c - a = \frac{1}{2}j$. Thus, either (i) or (ii) in 7.1 must hold for i.

Case 2 $j = k < n-2$. Again, (α) gives

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}}(\lambda_{i-1}) \otimes V_{a+\frac{1}{2}(i+j-1)}(\lambda_{j+1}), V_c(\lambda_j)) \neq 0$$

Unfortunately, (7 1) does not apply to this because $j+1 \nleq j$ However, the non-vanishing of this last space of homomorphisms is equivalent to

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c-n+1}(\lambda_j) \otimes V_{a-\frac{1}{2}}(\lambda_{i-1}), V_{a+\frac{1}{2}(i+j-1)-n+1}(\lambda_{j+1})) \neq 0\,,$$

and hence, twisting by φ and then applying τ_{n-1}, to

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{-a+\frac{1}{2}}(\lambda_{i-1}) \otimes V_{-c+n-1}(\lambda_j), V_{-a-\frac{1}{2}(i+j-1)+n-1}(\lambda_{j+1})) \neq 0$$

We can apply the induction hypothesis to this inequality, and this gives that $j+1 = i-1+j$ or $2n-(i-1)-j-2$, and

$$-c+n-1-\left(-a+\frac{1}{2}\right) = \frac{1}{2}(i-1+j)$$

and

$$-a-\frac{1}{2}(i+j-1)+n-1-\left(-a+\frac{1}{2}\right) = \frac{1}{2}j$$

In both cases, we get $i+2j = 2n-2$ and $c-a = \frac{1}{2}j$, so (7 1) (ii) is satisfied (we already know that $b-a = \frac{1}{2}(i+j)$)

Case 3 $i < n-2$, $j = k = n-2$ We show that this case is possible only if $i = 2$ (it is obvious that i must be even). We first determine the value of c Observe that (30) implies that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c-n+1}(\lambda_{n-2}) \otimes V_a(\lambda_i), V_{b-n+1}(\lambda_{n-2})) \neq 0\,,$$

and hence, twisting by φ and applying τ_{n-1}, we get

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{-a}(\lambda_i) \otimes V_{-c+n-1}(\lambda_{n-2}), V_{-b+n-1}(\lambda_{n-2})) \neq 0$$

Since (α) must hold for this, we get

$$c = a + \frac{1}{2}(n-i)\,.$$

By (7 4), we see that (30) implies

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-n+\frac{1}{2}(i+3)}(\lambda_1) \otimes V_{a+\frac{1}{2}}(\lambda_{i+1}) \otimes V_{a+\frac{1}{2}(n+i-2)}(\lambda_{n-2}), V_{a+\frac{1}{2}(n-i)}(\lambda_{n-2})) \neq 0\,,$$

which gives

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}}(\lambda_{i+1}) \otimes V_{a+\frac{1}{2}(n+i-2)}(\lambda_{n-2}), V_{a+\frac{1}{2}(i+1)}(\lambda_1) \otimes V_{a+\frac{1}{2}(n-i)}(\lambda_{n-2})) \neq 0$$

By (7 4) again, the module on the right-hand side of this space of homomorphisms is irreducible. Thus, if v_1 and v_{n-2} are $Y(\mathfrak{g})$-highest weight vectors in $V_{a+\frac{1}{2}(i+1)}(\lambda_1)$ and $V_{a+\frac{1}{2}(n-i)}(\lambda_{n-2})$, respectively, $v_1 \otimes v_{n-2}$ must be in the image of any non–zero homomorphism F in

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}}(\lambda_{i+1}) \otimes V_{a+\frac{1}{2}(n+i-2)}(\lambda_{n-2}), V_{a+\frac{1}{2}(i+1)}(\lambda_1) \otimes V_{a+\frac{1}{2}(n-i)}(\lambda_{n-2}))\,.$$

Since $\lambda_{i+1} - \lambda_1 \in Q_J^+$, where $J = \{2,3,\ldots,n\}$, we see from (5.5) that

$$\mathrm{Hom}_{Y(\mathfrak{g}_J)}(V_{a+\frac{1}{2}}(\lambda_i) \otimes V_{a+\frac{1}{2}(n+i-2)}(\lambda_{n-3}), V_{a+\frac{1}{2}(n-i)}(\lambda_{n-3})) \neq 0\,.$$

The induction hypothesis on n now proves that $i = 2$.

Case 4. $i = j = k = n-2$. In this case, we have $b = a + n - 2$ and $c = a + \frac{1}{2}(n-2)$. Equation (30) implies that

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{c-n+1}(\lambda_{n-2}) \otimes V_a(\lambda_{n-2}), V_{b-n+1}(\lambda_{n-2})) \neq 0\,.$$

But (α) does not hold for this if $n \neq 4$, and (7.1) (ii) holds if $n = 4$.

Finally, the inductive step, and with it the proof of the "only if" part of (7.1), is complete.

We now prove the "if" part. Suppose that (i) holds. Taking $J = \{1,2,\ldots, n-2\}$, we see that $V_{a+\frac{1}{2}(i+j)}(\lambda_j) \otimes V_a(\lambda_i)$ has a $Y(\mathfrak{g})$-highest weight vector of weight λ_{i+j}. This vector cannot generate a reducible highest weight $Y(\mathfrak{g})$-submodule, since otherwise it would have an irreducible $Y(\mathfrak{g})$-submodule, which would necessarily be of the form $V_c(\lambda_r)$ for some $r < i + j$, and this is impossible by the "only if" part of (7.1).

Suppose now that (ii) holds. Recall from the discussion in Sect. 4 that we may assume that $i \leqq j \leqq k$.

Consider first the case when n, i and j are all even. By (7.5),

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}(n+i-1)}(\lambda_n) \otimes V_a(\lambda_i), V_{a-\frac{1}{2}(n-i-1)}(\lambda_n)) \neq 0\,,$$

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}(n-i-1)}(\lambda_n), V_{a+j-\frac{1}{2}(n+i-1)}(\lambda_n) \otimes V_{a+\frac{1}{2}(i+j)-n+1}(\lambda_j)) \neq 0\,.$$

Hence,

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a-\frac{1}{2}(n+i-1)}(\lambda_n) \otimes V_a(\lambda_i), V_{a+j-\frac{1}{2}(n-i-1)}(\lambda_n) \otimes V_{a+\frac{1}{2}(i+j)-n+1}(\lambda_j)) \neq 0\,,$$

or equivalently,

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_{a+\frac{1}{2}(i+j)}(\lambda_j), V_{a+\frac{1}{2}(n-i-1)}(\lambda_n) \otimes V_{a+j-\frac{1}{2}(n-i-1)}(\lambda_n)) \neq 0\,.$$

We consider the composite of a non-zero element F of this space of homomorphisms with the non-zero homomorphism

$$V_{a+\frac{1}{2}(n-i-1)}(\lambda_n) \otimes V_{a+j-\frac{1}{2}(n-i-1)}(\lambda_n) \to V_{a+\frac{1}{2}j}(\lambda_{2n-2-i-j})$$

given by (7.5). This composite cannot be zero, otherwise the image of F would be a $Y(\mathfrak{g})$-module N for which

$$\mathrm{Hom}_{Y(\mathfrak{g})}({}^tN \otimes V_a(\lambda_i), V_{a+\frac{1}{2}(i+j)-n+1}(\lambda_j)) \neq 0\,. \tag{31}$$

Moreover, the irreducible $\mathfrak{g}$-modules occurring in N would be among the set

$$\{W(2\lambda_n), W(\lambda_{n-2}), W(\lambda_{n-4}), \ldots, W(\lambda_{2n-i-j})\} .$$

Since $i \leqq k = 2n - i - j - 2$,

$$m_0({}^tN \otimes W(\lambda_i)) = 0, \qquad m_0(W(\lambda_j)) = 1 ,$$

so (31) is impossible

The proofs in the other cases are similar applications of (7.5) and the $\mathfrak{g}$-module decomposition of the fundamental $Y(\mathfrak{g})$-modules. We omit the details.

This completes the proof of the "if" part of (7.1). □

It remains to give the

Proof of 7 2 We proceed by induction on n. As usual, the induction starts at $n = 3$, where the result is known from (6.2). Assume now that (7.2) is known when $\mathfrak{g}$ is of type D_m for $m < n$. To prove the result when $\mathfrak{g}$ is of type D_n, we consider first the case when $i \leqq j \leqq n - 2$, and proceed by induction on $\min(i, n - j)$. The induction starts when $i = 1$, this case being covered by (7.4).

Assume that $V_a(\lambda_i) \otimes V_b(\lambda_j)$ is not $Y(\mathfrak{g})$-highest weight. Let v_i and v_j be $Y(\mathfrak{g})$-highest weight vectors in $V(\lambda_i)$ and $V(\lambda_j)$, respectively, and let N be an irreducible $Y(\mathfrak{g})$-quotient of $V(\lambda_i) \otimes V(\lambda_j)/Y(\mathfrak{g}) \cdot (v_i \otimes v_j)$. Then, we have

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), N) \neq 0 ,$$

and $N_{\lambda_i+\lambda_j} = 0$. Assume for a contradiction that $b - a$ takes none of the values

$$\frac{1}{2}(j - i) + r, \qquad 0 < r \leqq \min(i, n - j) ,$$

or

$$n - 1 + \frac{1}{2}(i - j) - r, \qquad 0 \leqq r < \min(i, n - j) .$$

If $i + j \leqq n$, we use (7.4) to get

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}(1-i)}(\lambda_1) \otimes V_{a+\frac{1}{2}}(\lambda_{i-1}) \otimes V_b(\lambda_j), N) \neq 0 ,$$

and hence

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}}(\lambda_{i-1}) \otimes V_b(\lambda_j), V_{a+n-\frac{1}{2}(i+1)}(\lambda_1) \otimes N) \neq 0 . \tag{32}$$

By the assumption on $b - a$ and the induction hypothesis on $\min(i, n - j)$, $V_a(\lambda_{i-1}) \otimes V_b(\lambda_j)$ is $Y(\mathfrak{g})$-highest weight. Hence, $\lambda_1 + \lambda \geqq \lambda_{i-1} + \lambda_j$, so $\lambda = \lambda_i + \lambda_j - \eta$, where $\eta = \sum_{i=1}^{n} r_i \alpha_i \in Q^+$ satisfies

$$\eta \leqq \alpha_1 + 2\alpha_2 + \cdots + \alpha_{i-1} + 2(\alpha_i + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n .$$

If $r_1 = 0$, let $J = \{2, 3, \ldots, n\}$. Then, by (5.4),

$$\mathrm{Hom}_{Y(\mathfrak{g}_J)}(V_a(\lambda_{i-1}) \otimes V_b(\lambda_{j-1}), N_J) \neq 0 .$$

By the induction hypothesis on n, we have

$$b-a=\begin{cases}\frac{1}{2}(j-i)+r & \text{for some } 0<r\leqq i-1, \text{ or}\\ n-2+\frac{1}{2}(i-j)-r & \text{for some } 0\leqq r<i-1\,.\end{cases}$$

But all these values have been excluded.

Using $\lambda\in P^+$, one sees that the case $r_1>0$ is possible only if $i=2$, and then either $\eta=\lambda_2$, or $\eta\in Q^+_{J'}$, where $J'=\{1,2,\ldots,n-2,n-1\}$ or $\{1,2,\ldots,n-2,n\}$. In the first case, $\lambda=\lambda_j$ and (32) becomes

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_{a+\frac{1}{2}}(\lambda_1)\otimes V_b(\lambda_j),V_{a+n-\frac{1}{2}(i+1)}(\lambda_1)\otimes V_c(\lambda_j))\neq 0\,, \tag{33}$$

for some $c\in\mathbb{C}$. By the assumption on $b-a$, $V_{a+\frac{1}{2}}(\lambda_1)\otimes V_b(\lambda_j)$ is $Y(\mathfrak{g})$-highest weight, but then (33) contradicts (3.2). In the second case, (5.4) gives

$$\mathrm{Hom}_{Y(\mathfrak{g}_{J'})}(V_a(\lambda_2)\otimes V_b(\lambda_j),N_{J'})\neq 0\,,$$

and then (6.2) gives

$$b-a=\frac{1}{2}j \quad\text{or}\quad \frac{1}{2}j+1\,.$$

Both of these values have been excluded.

Thus, we have obtained the desired contradiction when $i+j\leqq n$. If $i+j>n$, one uses a similar argument, but using (7.4) to replace (32) by

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i)\otimes V_{b-\frac{1}{2}}(\lambda_{j+1}),N\otimes V_{b+\frac{1}{2}(1-j)}(\lambda_1))\neq 0\,.$$

This proves the "only if" part of (7.2) when $j\leqq n-2$.

For the "if" part, note that the $i=1$ case is contained in (7.4). Suppose that $i>1$. If $b-a=\frac{1}{2}(j-i)+r$, where $0<r\leqq\min(i,n-j)$, let $J''=\{1,2,\ldots,n-1\}$. By (6.2), $V_a(\lambda_i)_{J''}\otimes V_b(\lambda_j)_{J''}$ is not $Y(\mathfrak{g})$-highest weight, so by (5.5), neither is $V_a(\lambda_i)\otimes V_b(\lambda_j)$. If $b-a=n-i+\frac{1}{2}(i-j)-r$, where $0<r<\min(i,n-j)$, one uses the same argument with J'' replaced by $\{2,3,\ldots,n\}$ and uses the induction hypothesis on n instead of (6.2). For the remaining value $b-a=n-1+\frac{1}{2}(i-j)$, note that, if $i\neq j$, we have

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i)\otimes V_{a+n-1+\frac{1}{2}(i-j)}(\lambda_j),V_{a+n-1-\frac{1}{2}(j)}(\lambda_{j-i}))\neq 0\,,$$

by (7.1), while if $i=j$, then by (3.4) and (3.6), we have

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i)\otimes V_{a+n-1}(\lambda_i),\mathbb{C})\neq 0\,.$$

In any case, this implies that $V_a(\lambda_i)\otimes V_b(\lambda_j)$ is not $Y(\mathfrak{g})$-highest weight.

We now consider part (ii). If $j=n$, the result follows by the above argument, using (5.5) and (6.2) (and the same J''). If $j=n-1$, replace J'' with $\{1,2,\ldots,n-2,n\}$.

Finally, parts (iii) and (iv) follow immediately from (7.5). □

8. The B_n and C_n Cases

In this section, we give the analogues of Theorems 6.1 and 7.1 when $\mathfrak{g}$ is of type B_n or C_n.

Theorem 8.1. *Let $\mathfrak{g}$ be of type B_n, and let $1 \leqq i \leqq j \leqq k \leqq n$, $a, b, c \in \mathbb{C}$ Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j), V_c(\lambda_k)) \neq 0$$

if and only if one of the following holds·

(i) $i + j \leqq n - 1$, $k = i + j$, $b - a = i + j$, $c - a = j$,
(ii) $i < n$, $j = k = n$, $b - a = n + i - 1$, $c - a = n - i - 1$ □

The proof of this theorem is very similar to that of (7.1). Although we shall omit the details, we remark that the argument used to prove the existence of a non-zero homomorphism of $Y(\mathfrak{g})$-modules

$$V_a(\lambda_1) \otimes V_b(\lambda_j) \to V_c(\lambda_{2n-2-i-j})$$

(for suitable a, b, c) when $\mathfrak{g}$ is of type D_n fails to produce a non-zero homomorphism

$$V_a(\lambda_i) \otimes V_b(\lambda_j) \to V_c(\lambda_{2n-i-j})$$

when $\mathfrak{g}$ is of type B_n (as would be predicted by condition (D) alone) because of a difference in the way that tensor products of spin modules for $\mathfrak{g}$ behave in the two cases. Namely, in the B_n case, the tensor product of the spin module with itself contains every fundamental $\mathfrak{g}$-module except the spin module, whereas in the D_n case, the tensor product of the two spin modules, or of a spin module with itself, contains only "half" the remaining fundamental $\mathfrak{g}$-modules.

Theorem 8.2. *Let $\mathfrak{g}$ be of type C_n, and let $1 \leqq i \leqq j \leqq k \leqq n, a, b, c \in \mathbb{C}$ Then,*

$$\mathrm{Hom}_{Y(\mathfrak{g})}(V_a(\lambda_i) \otimes V_b(\lambda_j) \otimes V_c(\lambda_k), \mathbb{C}) \neq 0$$

if and only if $i + j \leqq n$, $k = i + j$, $b - a = \frac{1}{2}(i + j)$, $c - a = \frac{1}{2}j$ □

The proof of this theorem is similar to that of (6 1). Note that the fundamental $Y(\mathfrak{g})$-modules are irreducible as $\mathfrak{g}$-modules in both the A_n and C_n cases (see [11 and 8])

Remark We can use (6 1), (7.1), (8 1) and (8.2) to prove (4.2), avoiding the use of the PRV conjecture In fact, we can prove a more general result. Suppose, for example, that $\mathfrak{g}$ is of type D_n and that $i, j \leqq n - 2, a, b \in \mathbb{C}$ Let V be any irreducible quotient $Y(\mathfrak{g})$-module of $V_a(\lambda_i) \otimes V_b(\lambda_j)$, and let λ be the highest weight of V as a $\mathfrak{g}$-module Then,

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda_i) \otimes W(\lambda_j), W(\lambda)) \neq 0 \tag{34}$$

Indeed, since λ is dominant and $\leqq \lambda_i + \lambda_j$, we have either

(i) $\lambda = \lambda_k$ for some k, or
(ii) $\lambda = \lambda_k + \lambda_\ell$ for some k, ℓ.

In case (i), we know that (i, j, k) satisfies the conditions in (4.5), and then (34) is easily checked In case (ii), $\lambda_i + \lambda_j - \lambda_k - \lambda_\ell \in Q_J^+$, where $J = \{2, 3, \quad, n\}$, so by (5 4) and an obvious induction on n, we have

$$\mathrm{Hom}_{\mathfrak{g}_J}(W(\lambda_i)_J \otimes W(\lambda_j)_J, W(\lambda)_J) \neq 0 ,$$

which implies

$$\operatorname{Hom}_{\mathfrak{g}}(W(\lambda_i) \otimes W(\lambda_j), W(\lambda)) \neq 0 .$$

Similar arguments apply in the other cases.

9. The Quantum Affine Case

In this section, we indicate how to translate the preceding results from the context of Yangians to that of quantum affine algebras. We use freely the notation established in [8], Chapter 12. We assume throughout that the deformation parameter ε is not a root of unity.

To find the quantum affine version of (6.1), for example, one replaces $Y(\mathfrak{g})$ by $U_\varepsilon(\hat{\mathfrak{g}})$, $V_a(\lambda_i)$ by $V_\varepsilon(\lambda_i, a)$, etc., and conditions such as

$$b - a = \frac{1}{2}(i + j), \qquad c - a = \frac{1}{2}j$$

in (6.1) (i) by

$$b/a = \varepsilon^{i+j}, \qquad c/a = \varepsilon^j .$$

Similarly, in (6.2), the condition for $V_\varepsilon(\lambda_i, a) \otimes V_\varepsilon(\lambda_j, b)$ not to be a highest weight $U_\varepsilon(\hat{\mathfrak{g}})$-module is

$$b/a = \varepsilon^{j-i+2r} \quad \text{for some } 0 < r \leqq \min(i, n+1-j) .$$

The main results in Sects. 7 and 8 can be translated in the same way. We leave this to the reader, as well as the straightforward problem of appropriately reformulating the proofs.

10. Appendix: Dorey's Rule and Affine Toda Theories

In this section, we sketch how Dorey's condition arises in the context of ATFTs. We shall consider only those ATFTs based on untwisted affine algebras. The exposition is a slightly expanded version of that given in [14].

We begin by summarizing some results related to Coxeter elements, for which we follow [16]. Let $\alpha_0 = -\theta$, $X_0^{\pm} = X_\theta^{\mp}$, and $\hat{I} = I \amalg \{0\}$. Let $k_i\, (i \in I)$ be the coprime positive integers such that

$$\sum_{i \in \hat{I}} k_i \alpha_i = 0$$

(so that $k_0 = 1$), and set

$$X^{\pm} = \sum_{i \in \hat{I}} \sqrt{k_i}\, X_i^{\pm} .$$

Since X^+ is a regular element [16], its centralizer is a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$. Note that $[X^+, X^-] = 0$, so $X^{\pm} \in \mathfrak{h}'$. Recall also that $h = \sum_{i \in \hat{I}} k_i$.

Let $H \in \mathfrak{h}$ be such that $\alpha_i(H) = 1$ for all $i \in I$, and set

$$A = \exp\left(\frac{2\pi\sqrt{-1}}{h} H\right) .$$

Thus, A lies in a connected complex simple Lie group G with Lie algebra $\mathfrak{g}$.

Note that the centralizer of A in $\mathfrak{g}$ is $\mathfrak{h}$ On the other hand, it is clear that

$$\mathrm{Ad}(A)(X^{\pm}) = \omega^{\pm 1} X^{\pm} ,$$

where $\omega = e^{2\pi\sqrt{-1}/h}$ It follows that $\mathrm{Ad}(A)(\mathfrak{h}') = \mathfrak{h}'$ In fact, it is known [16] that $\mathrm{Ad}(A)|_{\mathfrak{h}'}$ is a Coxeter transformation of $\mathfrak{h}'$, i e. one can choose an ordered set of simple roots $\alpha'_1, \alpha'_2, \quad, \alpha'_n$ of $\mathfrak{g}$ with respect to $\mathfrak{h}'$ such that

$$\gamma' \equiv \mathrm{Ad}(A)\,|_{\mathfrak{h}'} = s'_1 s'_2 \quad s'_n ,$$

where s'_i is the i^{th} simple reflection in the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}'$. Define, for $i \in I$,

$$\phi'_i = s'_n s'_{n-1} \quad s'_{i+1}(\alpha'_i) ,$$

and let R'_i be the γ'-orbit of ϕ'_i

Choose root vectors $X_{\alpha'}$, for every root α' of $\mathfrak{g}$ with respect to $\mathfrak{h}'$, such that

$$\mathrm{Ad}(A)(X_{\alpha'}) = X_{\gamma'(\gamma')} ,$$

and set

$$\tilde{H}_i = \sum_{\beta' \in R'_i} X_{\beta'} . \tag{35}$$

It is clear that $\mathrm{Ad}(A)(\tilde{H}_i) = \tilde{H}_i$, so $\tilde{H}_i \in \mathfrak{h}$. Obviously, the $\tilde{H}_i$ are linearly independent, and hence form a basis of $\mathfrak{h}$

We now turn to Toda field theory The ATFT based on the affine Lie algebra $\hat{\mathfrak{g}}$ is defined by the lagrangian

$$\mathscr{L}(\Psi) = \int\int \left\{ \left\| \frac{\partial \Psi}{\partial t} \right\|^2 - \left\| \frac{\partial \Psi}{\partial x} \right\|^2 + \frac{m^2}{\beta^2} (e^{\mathrm{ad}(\Psi)}(X^+), X^-) \right\} dxdt ,$$

where Ψ is a function of the coordinates (x, t) on $1 + 1$ dimensional spacetime with values in $\mathfrak{h}$, $(\ ,\ )$ is the invariant bilinear form on $\mathfrak{g}$, m^2 is a (positive) mass scale, and β is a coupling constant (usually either real or purely imaginary). Since the $\tilde{H}_i$ are a basis of $\mathfrak{h}$, we can write

$$\Psi = \sum_{i \in I} \psi_i \tilde{H}_i , \tag{36}$$

where the ψ_i are scalar-valued functions. The component ψ_i is associated with the i^{th} particle of the theory The potential term

$$V(\Psi) = (e^{\mathrm{ad}(\Psi)}(X^+), X^-)$$

in $\mathscr{L}(\Psi)$ can be expanded formally as a power series in the ψ_i,

$$V(\Psi) = \sum_{p=0}^{\infty} \sum_{i_1, i_2, \quad, i_p \in I} V_{i_1 i_2 \quad i_p} \psi_{i_1} \psi_{i_2} \quad \psi_{i_p} ,$$

and one says that there is a coupling (or fusing) between the particles labelled $i_1, i_2, \ldots, i_p$ if $V_{i_1 i_2 \quad i_p} \neq 0$

From (36),

$$V_{i_1 i_2 \ i_p} = \frac{1}{p!}\sum_{\pi} ([\tilde{H}_{i_{\pi(1)}}, [\tilde{H}_{i_{\pi(2)}}, \dots, [\tilde{H}_{i_{\pi(p)}}, X^+] \cdots]], X^-),$$

where the sum is over all permutations π of $\{1, 2, \dots, p\}$. Next, using (35), we get

$$V_{i_1 i_2 \ i_r} = \frac{1}{p!}\sum_{\pi} \sum_{\beta_1' \in R'_{i_{\pi(1)}}} \sum_{\beta_2' \in R'_{i_{\pi(2)}}} \cdots \sum_{\beta_p' \in R'_{i_{\pi(p)}}} ([X_{\beta_1'}, [X_{\beta_2'}, \dots, [X_{\beta_p'}, X^+] \cdots]], X^-). \tag{37}$$

Since the weight of

$$[X_{\beta_1'}, [X_{\beta_2'}, \dots, [X_{\beta_p'}, X^+] \cdots]]$$

with respect to $\mathfrak{h}'$ is $\beta_1' + \beta_2' + \cdots + \beta_p'$, it is clear that the term on the right-hand side of (37) corresponding to $\beta_1', \beta_2', \dots, \beta_p'$ can be non-zero only if

$$\beta_1' + \beta_2' + \cdots + \beta_p' = 0,$$

and hence that

$$V_{i_1 i_2 \ i_p} \neq 0 \quad \text{only if } 0 \in R'_{i_1} + R'_{i_2} + \cdots + R'_{i_p}.$$

Thus, the p-point coupling $V_{i_1 i_2 \ i_p} \neq 0$ only if $i_1, i_2, \dots, i_p$ satisfies (D_p). The converse statement also holds when $p = 3$, but this requires a case-by-case analysis, which we omit.

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