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Graph Invariants of Vassiliev Type and Application to 4D Quantum Gravity

Nobuharu Hayashi

Institute of Physics, University of Tokyo, Komaba, Tokyo 153, Japan†
E-mail: hayashin@kekvax.kek.jp

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Abstract: We consider graph invariants of Vassiliev type extended by the quantum group link invariants. When they are expanded by x where $q = e^x$, the expansion coefficients are known as the Vassiliev invariants of finite type. In the present paper, we define tangle operators of graphs given by a functor from a category of colored and oriented graphs embedded into a 3-space to a category of representations of the quasi-triangular ribbon Hopf algebra extended by $U_q(sl(2, C))$, which are subject to a quantum group analog of the spinor identity. In terms of them, we obtain the graph invariants of Vassiliev type expressed to be identified with Chern–Simons vacuum expectation values of Wilson loops including intersection points. We also consider the 4d canonical quantum gravity of Ashtekar. It is verified that the graph invariants of Vassiliev type satisfy constraints of the quantum gravity in the loop space representation of Rovelli and Smolin.

1. Introduction

The concept of Vassiliev invariants was introduced in the theory of knot spaces [30]. Let $\mathscr{M}$ be a space of all smooth maps $S^1 \to S^3$ running through a base point with a fixed tangent vector. The knot space is given by $\mathscr{M} \backslash \Sigma$, where Σ is called the *discriminant*, i.e., a set of all singular maps which have multiple points or vanishing tangent vectors. Equivalence classes of knot embeddings by the ambient isotopy of S^3 are in one-to-one correspondence with connected components of $\mathscr{M} \backslash \Sigma$. Each connected component of the knot space is separated by walls which constitute the discriminant Σ.

Vassiliev introduced a system of subgroups of $\tilde{H}^0(\mathscr{M} \backslash \Sigma)$ based on the Alexander duality theorem applied to a space of polynomial maps which approximates $\mathscr{M}$: $\tilde{H}^0(\mathscr{M} \backslash \Sigma) \supset \cdots \supset F_j \supset F_{j-1} \supset \cdots \supset F_1 = 0$. Elements in a quotient group F_j/F_{j-1} are called the Vassiliev invariants of order j. Every element in F_j/F_{j-1} vanishes whenever there are more than j transverse double points. The Vassiliev

† This is not the author's present address.

invariants take into account topological information about a space of embedded graphs. After his pioneering work, Birman and Lin [6] calculated the Vassiliev invariants in a combinatorial way based on a set of axioms and initial data. Elements in F_j/F_{j-1} are given by functionals defined over a space of all configurations of l pairs of $2l$ points on S^1 connected by l cords for $l \leqq j$. Kontsevich provided integral expressions of the functionals of Birman and Lin [18].

A relation of the Vassiliev invariants to the quantum group link invariants such as the HOMFLY polynomials and the Kauffman polynomials was investigated by Birman, Lin [6, 19, 20] and Puinikhin [23, 24]. It was verified that the Vassiliev invariants of finite type appear as expansion coefficients of the quantum group link invariants expanded by x, where $q = e^x$. It implies an intimate relation of the Vassiliev invariants to the CS (Chern–Simons) perturbative expansions. Works of Axelrod, Singer [3, 4] and Bar-Natan [5] revealed it. The Vassiliev invariants can be given by the Feynman integrals in the CS quantum field theory which physicists are familiar with.

Kauffman considered the Vassiliev invariants from a little different point of view. They are perceived as expansion coefficients of a special case of graph invariants of the rigid vertex isotopy extended by the quantum group link invariants [13, 14]. They are called the graph invariants of Vassiliev type. In the perturbative CS analysis, taking the lowest order in the inverse of the CS coupling constant, he showed that the graph invariant of Vassiliev type is given by a CS vacuum expectation value of Wilson loops including intersection points [15]. Such a point of view is expected to provide a neat perspective on the work of Bar-Natan [5] in regard to the local integrability condition. It is a consistency condition for the Vassiliev invariants given by transverse triple points to exist.

In the present paper, we find a non-perturbative generalization of Kauffman's formula in an axiomatic way and consider the local integrability condition in terms of it. We also show that the non-perturbative formula plays a crucial role in investigation of non-perturbative aspects of the 4d quantum gravity of Ashtekar [1]. It is shown that the graph invariants of Vassiliev type provide physical wave-functions of the quantum gravity. At the transverse triple points in graphs, Riemann metrics are non-degenerate.

This paper is organized as follows. Section 2 contains a brief review on Kauffman's graph invariants [13, 14, 16]. The graph invariants of Vassiliev type [15] are introduced as a special case. In Sect. 3, we introduce CS vacuum expectation values of Wilson loops including intersection points and the spinor identity which they are subject to. In Sect. 4, we consider the q-analog (quantum group analog) of the spinor identity in the context of the quasi-triangular ribbon Hopf algebras extended by $U_q(sl(2, C))$. It leads us to tangle operators of graphs naturally identified with the CS vacuum expectation values of Wilson loops including intersection points. The graph invariants of Vassiliev type can be expressed in terms of them. It is consistent with Kauffman's formula. The last section is devoted to a physical application to the 4d quantum gravity of Ashtekar. We apply the canonical quantization and employ the loop space representation [12, 28]. We verify that wave-functions in the loop space representation given by the graph invariants of Vassiliev type satisfy all constraints of the quantum gravity with vanishing cosmological constant.

2. The Rigid Vertex Isotopy and Graph Invariants of Vassiliev Type

This section is devoted to a brief review on Kauffman's graph invariants which are extended by link invariants. The graph G that we consider is closed and composed of rigid vertices, edges and loops. The rigid vertex is a 2-disk from which more than two strings emanate. In addition, the following concept is necessary for the definition of graph invariants.

Definition 2.1. *The rigid vertex embedding is an embedding $\phi: G \to M^3$ by which the image of a neighborhood of each vertex v in G is contained in a proper 2-disk in a ball neighborhood of $\phi(v)$. An isotopy $h_t: M^3 \to M^3$ between two rigid vertex embeddings ϕ_0 and ϕ_1 is called the "rigid vertex isotopy" if it carries through the ball-disk pair for each vertex of G.*

For simplicity, in the present section, we deal with only oriented graphs whose constituents are 4-valent rigid vertices, edges and loops [13, 14]. The 4-valent rigid vertex is a 2-disk with four strings emanating from it, two of which are outgoing and the others are incoming. The graph G is a disconnected sum of a finite number of components, i.e., $G = \amalg_i G_i$.

The rigid vertex isotopy is generated by extended Reidemeister moves as in Fig. 1. The former three moves $R1$, $R2$ and $R3$ are taken into account in the theory of links. The link invariants are defined to be invariant under $R1$, $R2$ and $R3$. The latter two moves $R4$ and $R5$ are additional ones which appear in the theory of graphs composed of the rigid vertices. Graph invariants of the rigid vertex isotopy are defined to be invariant under the extended Reidemeister moves.

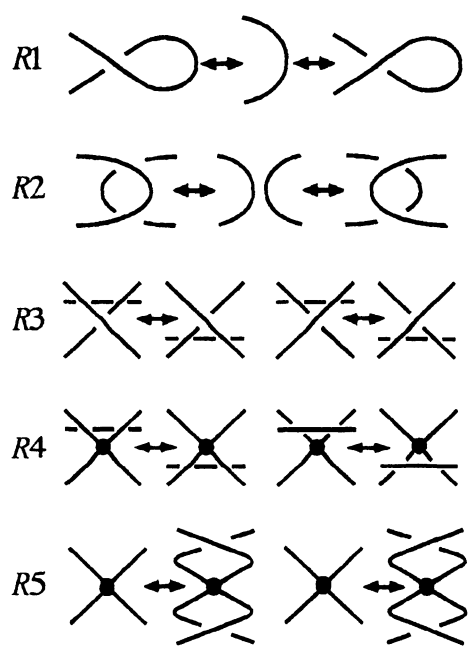


Fig. 1. Extended Reidemeister moves generating the rigid vertex isotopy

Let us introduce Kauffman's construction of the graph invariants [13, 14]. Suppose that we are given the link invariants denoted by $P(L)$. They can always be extended to the graph invariants of the rigid vertex isotopy. To be concrete, they are given by resolutions of any one of the rigid vertices:

$$P(L^{(j)}{}_{\times_\bullet}) = aP(L^{(j-1)}{}_{\times_+}) + bP(L^{(j-1)}{}_{\times_-}) + cP(L^{(j-1)}{}_{\asymp}), \tag{2.1}$$

where $L^{(j)}$ represents a graph G composed of j 4-valent rigid vertices. According to (2.1), it is obvious that $P(G) = \sum_{L \in S} a^{p(L)} b^{n(L)} c^{u(L)} P(L)$, where S represents a set of all links obtained after all the rigid vertices in G are resolved. $p(L)$, $n(L)$ and $u(L)$ stand for a number of positive crossing resolutions, that of negative crossing resolutions and that of unfolding resolutions respectively. We can summarize as follows.

Theorem 2.1 (Kauffman [13, 14]). *Let G be a graph composed of 4-valent rigid vertices. Then $P(G)$ given by* (2.1) *is invariant under the extended Reidemeister moves.*

We are ready to define *graph invariants of Vassiliev type* [15]. They are given by putting $a = 1, b = -1$, and $c = 0$, i.e., $P(L^{(j)}{}_{\times_\bullet}) \equiv P(L^{(j-1)}{}_{\times_+}) - P(L^{(j-1)}{}_{\times_-})$. They are perceived as a special case of Kauffman's graph invariants.

3. CS Vacuum Expectation Values of Wilson Loops Including Intersection Points

Let us introduce Wilson loops given by a singular link. The Wilson loop is a trace of a holonomy along any one of the singular link components L_i^s of the singular link $L^s = \amalg_i L_i^s$, which is not necessarily a disconnected sum. We denote it by $W(A:L_i^s) = \mathrm{Tr}(\mathscr{P}\exp(i\oint_{\gamma_i} A))$. $\mathscr{P}$ represents a path-ordered product along a closed path $\gamma_i(t)$ $(0 \leqq t \leqq 1)$ which describes L_i^s. $\mathscr{P}\exp(i\oint_{\gamma_i} A)$ is the holonomy along L_i^s given by a connection 1-form of a $SU(2)$ -principle bundle over M^3. For later convenience, we denote a holonomy along a line segment from $\gamma_i(s)$ to $\gamma_i(t)$ by $U_i(s, t)$. It is given by $\mathscr{P}\exp(i\int_s^t du \dot{\gamma}_i^\mu(u) A_\mu)$.

For simplicity, we assume that the singular link is composed of N closed paths and includes only one transverse double point. Then let us introduce two types of CS vacuum expectation values of Wilson loops. One type is given by

$$Z(L^{(1)}{}_{\times}) = \int DA\,\mathrm{Tr}(U_j(s,t)\,U_j(t,s)) \cdot \prod_{i \neq j}^{N} W(A:L_i^s) e^{ikS_{cs}(A)}, \tag{3.1}$$

in a case in which the j^{th} singular link component transversely self-intersects at a point where $\gamma_j(s) = \gamma_j(t)$ $(0 \leqq s < t \leqq 1)$, and

$$Z(L^{(1)}{}_{\times}) = \int DA\,\mathrm{Tr}(U_j(s,s)) \cdot \mathrm{Tr}(U_k(t,t)) \cdot \prod_{i \neq j,k}^{N} W(A:L_i^s) e^{ikS_{cs}(A)}, \tag{3.2}$$

in a case in which the j^{th} and k^{th} singular link components transversely intersect each other. $\gamma_j(s)$ and $\gamma_k(t)$ describe L_j^s and L_k^s respectively and satisfy $\gamma_j(s) = \gamma_k(t)$ at the intersection point. The CS action is given by $S_{cs}(A) = \frac{1}{4\pi}\int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$.

The other type is given by

$$Z(L^{(1)}\,\otimes) = \int DA \sum_{a=1}^{\dim SU(2)} \mathrm{Tr}(U_j(s,t)\,T_a U_j(t,s)\,T_a)\cdot \prod_{i \neq j}^{N} W(A:L_i^s)\,e^{ikS_{cs}(A)}\,, \tag{3.3}$$

in the former case, and

$$Z(L^{(1)}\,\otimes) = \int DA \sum_{a=1}^{\dim SU(2)} \mathrm{Tr}(U_j(s,s)\,T_a)\cdot \mathrm{Tr}(T_a U_k(t,t))$$

$$\cdot \prod_{i \neq j,k}^{N} W(A:L_i^s)\,e^{ikS_{cs}(A)}\,, \tag{3.4}$$

in the latter case. We shall call the operator $\sum_{a=1}^{\dim SU(2)}(T_a)_{ij}(T_a)_{kl}$ the *Casimir-like operator*.

A generalization to more complicated cases in which there exist more than one transverse double point is trivial. In such cases, it is convenient to introduce the following notation. Let $Z(L^{(k,j-k)})$ $(0 \leqq k \leqq j)$ be a CS vacuum expectation value of Wilson loops including only j transverse double points. $j-k$ Casimir-like operators are inserted at $j-k$ transverse double points one by one. In the notation, $Z(L^{(0,0)})$ is given by a link, and $Z(L^{(1,0)})$ and $Z(L^{(0,1)})$ correspond to $Z(L^{(1)}\times)$ and $Z(L^{(1)}\,\otimes)$ respectively. In general, Casimir-like operators inserted at intersection points can be eliminated by the Fierz identity satisfied by the CS vacuum expectation values:

$$Z(L^{(k,j-k)}\,\otimes) = \frac{1}{2}\,Z(L^{(k,j-k-1)}\times) - \frac{1}{4}\,Z(L^{(k+1,j-k-1)}\times)\,. \tag{3.5}$$

It is derived from the Fierz identity given by $\sum_a (T_a)_{ij}(T_a)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N}\delta_{ij}\delta_{kl}$, where $N = 2$ for $SU(2)$.

Let us introduce the spinor identity satisfied by CS vacuum expectation values $Z(L^{(j,0)})$ $(j \geqq 1)$. Let $U(\alpha)$ and $U(\beta)$ be elements in $SU(2)$, i.e., invertible 2×2 matrices, which represent holonomies along closed paths α and β. From the spinor identity given by $\varepsilon_{ab}\varepsilon^{cd} = \delta_a^c\delta_b^d - \delta_a^d\delta_b^c$, it follows that the Wilson loops are subject to Mandelstam's identity [10] for $SU(2)$:

$$\begin{aligned}\mathrm{Tr}(U(\alpha))\mathrm{Tr}(U(\beta)) &= \mathrm{Tr}(U(\alpha)\,U(\beta)) + \mathrm{Tr}(U(\alpha)U(\beta)^{-1})\\ &= \mathrm{Tr}(U(\alpha\circ\beta)) + \mathrm{Tr}(U(\alpha\circ\beta^{-1}))\,,\end{aligned} \tag{3.6}$$

where $\alpha\circ\beta$ and $\alpha\circ\beta^{-1}$ are composite and closed paths. Thus the spinor identity satisfied by the CS vacuum expectation values takes the following form:

$$Z(\alpha\cup\beta) = Z(\alpha\circ\beta) + Z(\alpha\circ\beta^{-1})\,. \tag{3.7}$$

On the left-hand side, $Z(\alpha\cup\beta)$ is given by $\mathrm{Tr}(U(\alpha))\mathrm{Tr}(U(\beta))$. On the right-hand side, $Z(\alpha\circ\beta)$ and $Z(\alpha\circ\beta^{-1})$ are given by $\mathrm{Tr}(U(\alpha\circ\beta))$ and $\mathrm{Tr}(U(\alpha\circ\beta^{-1}))$ respectively.

The spinor identity (3.7) plays a role of resolving transverse double points. It is enough to consider two classes, i.e., the 1*st and* 2*nd classes.* A transverse double point of the 1st class is a self-intersection point of a closed and composite path $\alpha\circ\beta$, as in Fig. 2-G_1. On the other hand, a transverse double point of the 2nd class is an

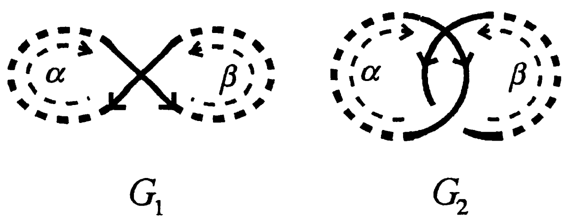
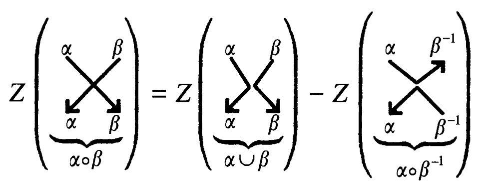


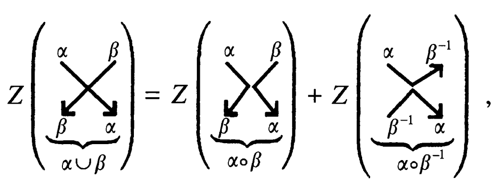
Fig. 2. In G_1 (G_2), a transverse double point of the 1st class (the 2nd class) is illustrated. They are also identified with graphs composed of the 4-valent rigid vertices.

intersection point of two closed paths α and β, as in Fig. 2-G_2. The spinor identity takes the following form:



(3.8)

when applied to resolve a transverse double point of the 1st class. On the other hand, it takes



(3.9)

when applied to resolve a transverse double point of the 2nd class.

The q-analog of the spinor identity corresponding to (3.7) is our main theme. Before proceeding to the part based on the Hopf algebras, we consider Kauffman's extension theorem (Theorem 2.1) from the point of view of the perturbative CS quantum field theory [11]. It is crucial to notice that a transverse double point of the 1st class and that of the 2nd class can be identified with 4-valent rigid vertices.

Let us take the lowest order in $1/k$ (the inverse of the CS coupling constant). We can show that a CS vacuum expectation value of Wilson loops including only j transverse double points, where no Casimir-like operators are inserted, is expressed by a sum of two ones after one of the transverse double points is resolved in two possible ways. To see it, we need to use a manipulation, attaching a small loop to the intersection point to be resolved. In one way, we resolve it to be a positive crossing. In the other way, we resolve it to be a negative crossing. Contributions of the small loop can be evaluated by the following three facts. The first is that the holonomy can be expanded by an area tensor $\sigma^{\mu\nu}$ given by the small loop γ as $U(\gamma) = Id. + F(A)_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}(\sigma^2)$. The second is that the curvature tensor satisfies $F^i(A)_{\mu\nu} = \varepsilon_{\mu\nu\lambda}\delta S_{cs}/\delta A^i_\lambda$. The third is that it is possible to carry out integration by parts in the CS path-integral. (We should say that the last one is an assumption rather than a fact.) After all, one can get $Z(L^{(j,0)}{}_{\times}) = \frac{1}{2}(Z(L^{(j-1,0)}{}_{\times}) + Z(L^{(j-1,0)}{}_{\times}))$ taking the lowest order in $1/k$. More general cases in which Casimir-like operators are inserted at intersection points can be considered in the same way by eliminating them by the Fierz identity (3.5). Contributions of higher orders remain to be investigated as a future problem.

4. Reshetikhin-Turaev Construction of Graph Invariants and Correspondence with the CS Vacuum Expectation Values of Wilson Loops

The present section is devoted to the q-analog of the spinor identity and construction of graph invariants with a natural correspondence with the CS vacuum expectation values of Wilson loops including intersection points. The consideration is based on the quasi-traingular ribbon Hopf algebra extended by $U_q(sl(2, C))$ [17, 25, 26, 29]. We should comment that the q-analog of the spinor identity leads us to the q-analog of Penrose's spin-network [21, 22]. We mean by Penrose's spin-network a planar graph colored by $SU(2)$ representations and further specified by the spinor identity.

4.1. The Q-Analog of the Spinor Identity and Tangle Operators of Graphs. It is known that the Jones polynomial $P(L)$ given by a link L defers by a phase factor from a tangle operator $F(L)$ of L (which is also called Kauffman's bracket of L). Suppose that $M^3 = S^3$ and all link components of L are in the 2-dimensional representation of $U_q(sl(2, C))$. Then it follows that $P(L) = \alpha^{-\omega(L)}F(L)$[1]. The tangle operator $F(L)$ is given by a functor F from a category of colored and framed links to a category of representations of the quasi-triangular Hopf algebra. The tangle operators of links can be extended to those of graphs composed of the rigid vertices by possible resolutions based on Kauffman's extension theorem. Tangle operators of graphs are related to the graph invariants via $P(G) = \alpha^{-\omega(G)}F(G)$[2].

Let us begin with defining tangle operators of graphs composed of 4-valent rigid vertices which can be identified with certain objects in the CS quantum field theory such as $Z(L^{(k,j-k)})$.

Definition 4.1. *Let $L^{(k,j-k)}$ ($j \geqq 1, 0 \leqq k \leqq j$) be a graph composed of j 4-valent rigid vertices. j-k vertices are marked by ©. The rest of the k vertices have no marks. A tangle operator given by $L^{(k,j-k)}$, which is denoted by $F(L^{(k,j-k)})$, is given by resolutions of the two kinds of 4-valent rigid vertices*[3]. *We define it to satisfy*

$$F(L^{(k,j-k)}{}_{\times}) \equiv \frac{1}{e(\frac{1}{2}) + e(-\frac{1}{2})}(F(L^{(k-1,j-k)}{}_{\times}) + F(L^{(k-1,j-k)}{}_{\times})) , \qquad (4.1)$$

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[1] According to Kauffman's convention [15], $e(x) = \exp(\frac{-i\pi}{k}x)$ and $\alpha = e(\frac{3}{2})$

[2] $\omega(G) = \sum_{p \in \Gamma(G)} \varepsilon(p)$. $\Gamma(G)$ represents a set of all positive and negative crossings and rigid vertices in G. The signature for the positive crossing, the negative crossing and the rigid vertex takes 1, -1 and 0 respectively.

[3] All link and graph diagrams have the blackboard framings.

when any one of the rigid vertices with no marks is resolved, and

$$F(L^{(k,j-k)}\,\textcircled{\times}) \equiv \frac{1}{4(e(\frac{1}{2}) - e(-\frac{1}{2}))}(F(L^{(k,j-k-1)}\,\asymp) - F(L^{(k,j-k-1)}\,\times))\,, \quad (4.2)$$

when any one of the rigid vertices marked by © is resolved. In addition, we formally define

$$F(L^{(k,j-k)}\,\times) \equiv F(L^{(k,j-k)}\,\asymp)\,, \quad (4.3)$$

$$F(L^{(k,j-k)}\,\times) \equiv F(L^{(k,j-k)}\,\asymp)\,. \quad (4.4)$$

Since any pair of corners on a 2-disk contacting at a single point can also be regarded as a 4-valent rigid vertex, Kauffman's extension theorem (Theorem 2.1) was applied to the definitions of (4.3) and (4.4). We also notice from (3.5) that (4.3) leads us to the q-analog of the Fierz identity. Equation (4.2) was defined in accordance with it. It can be easily checked that every tangle operator $F(L^{(j,0)})$ $(j \geqq 1)$ behaves as $Z(L^{(j,0)})$ under simultaneous orientation reverses of any singular link components and mirror reflections of the whole graph diagram.

We are ready to describe the *q-analog of the spinor identity*. Let us remember that in the CS field theory, there are two classes of transverse double points of Wilson loops as in Fig. 2. The spinor identity (3.7) is of different forms depending on how it is applied to resolve a transverse double point of either the 1st or 2nd class. Since every 4-valent rigid vertex can be identified with either a transverse double point of the 1st class or a transverse double point of the 2nd class, it is necessary to take account of the two classes of rigid vertices to describe the q-analog of the spinor identity. It is described as follows.

Theorem 4.1. *Suppose that we are given tangle operators $F(L^{(j,0)})$ $(j \geqq 1)$ by Definition* 4.1. *They satisfy the q-analog of the spinor identity:*

$$F(L^{(j,0)}\,\times) = F(L^{(j-1,0)}\,\asymp) - F(L^{(j-1,0)}\,\times)\,, \quad (4.5)$$

when a rigid vertex of the 1*st class is resolved, and*

$$F(L^{(j,0)}\,\times) = F(L^{(j-1,0)}\,\asymp) + F(L^{(j-1,0)}\,\times)\,, \quad (4.6)$$

when a rigid vertex of the 2*nd class is resolved.*

Before proving Theorem 4.1, we prove the following lemma.

Lemma 4.1. *Let all link components of links be in the* 2*-dimensional representation* V^2 *and* $k \in Z$ *be larger than* 2 *or equal to* 2. *Then the tangle operator of links satisfy*[4]

$$F(\asymp) = \frac{1}{e(1) - e(-1)}\left(e\left(\frac{1}{2}\right)F(\times) - e\left(-\frac{1}{2}\right)F(\times)\right), \quad (4.7)$$

and

$$F(\asymp) = \frac{1}{e(1) - e(-1)}\left(e\left(-\frac{1}{2}\right)F(\times) - e\left(\frac{1}{2}\right)F(\times)\right). \quad (4.8)$$

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[4] For simplicity, we employ the notation in which $F(L_D)$ is denoted by $F(D)$. D represents an elementary subdiagram of L.

Proof. Let us introduce a coupon D [17], where D is an isomorphism: $V^{2*} \to V^2$ provided that $k \geqq 2$. V^{2*} is dual to V^2. Let G_c represent a colored framed graph obtained by inserting the coupon D and its inverse D^{-1} on a link component K of a link L so that p extrema on K are separated. Then there is the following fact that

$$F(G_c) = (-1)^p F(L)\,. \tag{4.9}$$

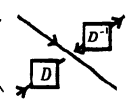
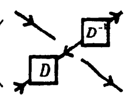
Equation (4.7) is proved as follows. From the nature of the quantum group R matrix, tangle operators of elementary subdiagrams satisfy the skein relation:

$$F()() = \frac{1}{e(1) - e(-1)}\left(e\left(\frac{1}{2}\right)F(\times) - e\left(-\frac{1}{2}\right)F(\times)\right). \tag{4.10}$$

When rotated by 90 degrees, it looks like

$$F(\asymp) = \frac{1}{e(1) - e(-1)}\left(e\left(\frac{1}{2}\right)F(\times) - e\left(-\frac{1}{2}\right)F(\times)\right)\cdot \tag{4.11}$$

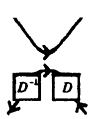
We can find that according to (4.11), the tangle operators of links satisfy

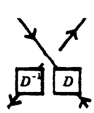
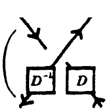
$F(\asymp) = \frac{1}{e(1)-e(-1)}\left(e\left(\frac{1}{2}\right)F\left(\right.\right.$  $\left.\left.\right) - e\left(-\frac{1}{2}\right)F\left(\right.\right.$  $\left.\left.\right)\right)$

$$= \frac{1}{e(1) - e(-1)}\left(e\left(\frac{1}{2}\right)F(\times) - e\left(-\frac{1}{2}\right)F(\times)\right). \tag{4.12}$$

Thus we arrived at (4.7).

Equation (4.8) is proved as follows. We can find that the tangle operators of links satisfy

$F(\asymp) = -F\left(\right.$  $\left.\right)$

$= \frac{-1}{e(1)-e(-1)}\left(e\left(\frac{1}{2}\right)F\left(\right.\right.$  $\left.\left.\right) - e\left(-\frac{1}{2}\right)F\left(\right.\right.$  $\left.\left.\right)\right)$

$$= \frac{1}{e(1) - e(-1)}\left(e\left(-\frac{1}{2}\right)F(\times) - e\left(\frac{1}{2}\right)F(\times)\right)\cdot \tag{4.13}$$

The first equality follows from (4.9). The second equality follows from the skein relation (4.11). The last equality also follows from (4.9), because there exists an even number of extrema between the two coupons D and D^{-1} in both the diagrams in the second line. Thus we accomplished the proof of the lemma. □

We are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let us prove the theorem when $j = 1$ first. We use (4.3), (4.4) and (4.3) rotated by 90 degrees on the right-hand sides of (4.5) and (4.6) and substitute (4.7), (4.8) and (4.10). We arrive at the definition (4.1) for $j = k = 1$.

In the following, let us assume that $j \geqq 2$. It is enough to prove (4.5), because we can prove (4.6) in the same way. Suppose that all rigid vertices in $L^{(j,0)}$ except the depicted one as a subdiagram are resolved on the left-hand side of (4.5). According to the definition (4.1), we get

$$F(L^{(j,0)}{}_{\times}) = \frac{1}{(e(\frac{1}{2}) + e(-\frac{1}{2}))^{j-1}} \sum_{s=1}^{2^{j-1}} F(L_s^{(1,0)}{}_{\times}) . \tag{4.14}$$

On the other hand, we can resolve all the $j-1$ rigid vertices on the right-hand side of (4.5). It follows that

$$F(L^{(j-1,0)}{}_{\times}) - F(L^{(j-1,0)}{}_{\times}) = \frac{1}{(e(\frac{1}{2}) + e(-\frac{1}{2}))^{j-1}} \times \sum_{s=1}^{2^{j-1}} F(L_s^{(0,0)}{}_{\times}) - F(L_s^{(0,0)}{}_{\times}) . \tag{4.15}$$

It is enough to show that for every s,

$$F(L_s^{(1,0)}{}_{\times}) = F(L_s^{(0,0)}{}_{\times}) - F(L_s^{(0,0)}{}_{\times}) . \tag{4.16}$$

The diagrams $L_s^{(1,0)}{}_{\times}$, $L_s^{(0,0)}{}_{\times}$ and $L_s^{(0,0)}{}_{\times}$ for each s are identical outside the subdiagrams depicted, if the orientation of the diagrams is ignored. Proof of (4.16) is accomplished by repeating the proof when $j = 1$. Thus Theorem 4.1 was proved. □

We have shown that the tangle operators of graphs given by Definition 4.1 satisfy the q-analog of the spinor identity. But we wonder if there exist other definitions with the same properties as the tangle operators of graphs given by Definition 4.1. In regard to it, it can be verified that Definition 4.1 is unique except for a normalization factor.

Proposition 4.1. *Let $F(L^{(j,0)})$ $(j \geqq 1)$ be a tangle operator of a graph composed of j 4-valent rigid vertices which behave as $Z(L^{(j,0)})$. We further suppose that $F(L^{(j,0)}{}_{\times}) = \alpha F(L^{(j-1,0)}{}_{\times}) + \beta F(L^{(j-1,0)}{}_{\times})$, where α and β are complex numbers. Then it follows that $\alpha = \beta \in R$, and $F(L^{(j,0)}{}_{\times})$ satisfies the q-analog of the spinor identity, i.e.,* (4.5) *and* (4.6), *if and only if*

$$F(L^{(j-1,0)}{}_{\times}) = \alpha\left(e\left(\frac{1}{2}\right) + e\left(-\frac{1}{2}\right)\right)F(L^{(j-1,0)}{}_{\times}) , \tag{4.17}$$

and

$$F(L^{(j-1,0)}{}_{\times}) = \alpha\left(e\left(\frac{1}{2}\right) + e\left(-\frac{1}{2}\right)\right)F(L^{(j-1,0)}{}_{\times}) . \tag{4.18}$$

One can easily prove it using Lemma 4.1 and the skein relation. The reason why $\alpha = \beta \in R$ is that every tangle operator $F(L^{(j,0)})$ must have the same properties as a CS vacuum expectation value of Wilson loops including only j transverse double points under simultaneous orientation reverses of any singular link components and mirror reflections of the whole graph diagram. α plays a role of a normalization factor. For instance, let α be a parameter such that $F(\otimes) = (e(1) + e(-1) - 1)F(\frown)$. Then α is determined to be $\alpha = (e(\frac{1}{2}) + e(-\frac{1}{2}))^{-1}$. It was used in Definition 4.1.

4.2. The Graph Invariants of Vassiliev type in Chern–Simons Representation. It is time to introduce a new representation of the graph invariants of Vassiliev type $P(L_{\times_\bullet}) = P(L_{\times_+}) - P(L_{\times_-})$, in which they can be identified with the CS vacuum expectation values of Wilson loops including intersection points. We shall call it the *CS (Chern–Simons) representation.* One can notice that the graph invariants of Vassiliev type are composed of tangle operators of another kind of 4-valent rigid vertices. They are given by

$$F(\otimes_V) \equiv \alpha^{-1}F(\times_+) - \alpha F(\times_-)$$
$$= (e(1) + e(-1) - 1)(e(1) - e(-1))$$
$$\times\left(2F(\otimes_C) - \frac{1}{2}\frac{e(1) + e(-1) + 1}{e(1) + e(-1) - 1}F(\times_\bullet)\right), \tag{4.19}$$

where $F(\otimes_C)$ and $F(\times_\bullet)$ are tangle operators of the two kinds of 4-valent rigid vertices. They are given by Definition 4.1:

$$F(\times_\bullet) \equiv \frac{1}{e(\frac{1}{2}) + e(-\frac{1}{2})}(F(\times_+) + F(\times_-)), \tag{4.20}$$

$$F(\otimes_C) \equiv \frac{1}{4(e(\frac{1}{2}) - e(-\frac{1}{2}))}(F(\times_+) - F(\times_-)). \tag{4.21}$$

It is crucial that $F(\otimes_V)$ is a linear combination of $F(\otimes_C)$ and $F(\times_\bullet)$. The former (latter) elementary subdiagram corresponds to a (no) Casimir-like operator insertion at the transverse double point.

The graph invariant of Vassiliev type in the CS representation is given as follows. Suppose that we are given a graph composed of j 4-valent rigid vertices. Then it is expressed by

$$P(L^{(j)}{}_{\times_\bullet}) = \alpha^{-\omega(L^{(j)})}F(L^c)\circ\bigotimes_{t=1}^{j} F^t(\otimes_V), \tag{4.22}$$

where $F(L^c)$ represents a tangle operator of the complement L^c obtained by cutting neighborhoods of the j rigid vertices out of $L^{(j)}$. $F^t(\otimes_V)$ represents a tangle operator of the t^{th} rigid vertex.

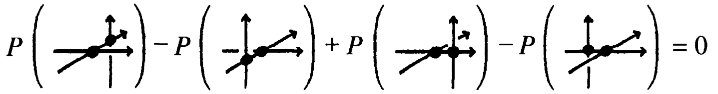
It is interesting to compare our formula with Kauffman's based on the perturbative CS quantum field theory [15]. Taking the lowest order in $1/k$, one can find that (4.19) becomes

$$F(\otimes_V) = \frac{4\pi i}{k}\left(F(\times_\bullet) - \frac{3}{4}F(\otimes_C)\right). \tag{4.23}$$

This coincides with Kauffman's formula expressed by the CS vacuum expectation values of Wilson loops. From this fact, (4.22) can be regarded as a non-perturbative generalization.

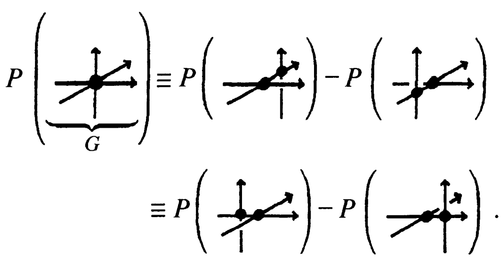
It should be mentioned that Kauffman's formula can exist for any Lie algebra. A generalization of our argument restricted to $U_q(sl(2, C))$ to other quasi-traingular Hopf algebras remains to be investigated.

4.3. Invariants of Graphs Including Transverse Triple Points and the Local Integrability Condition. In the theory of Vassiliev invariants [30], they are defined to be subject to the local integrability condition (that is also called the 4-term relation) [6, 19], which is a consistency condition for the Vassiliev invariants given by transverse triple points to exist. Let us recall that the Vassiliev invariants are expansion coefficients of the graph invariants of Vassiliev type expanded by x. The local integrability condition satisfied by the graph invariants of Vassiliev type is of the following form:[5]



(4.24)

The invariants of graphs including transverse triple points are defined by

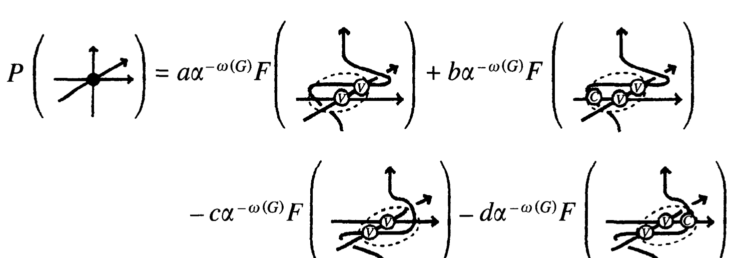


(4.25)

(4.26)

In the present section, we clarify how the local integrability condition is satisfied in the CS representation.

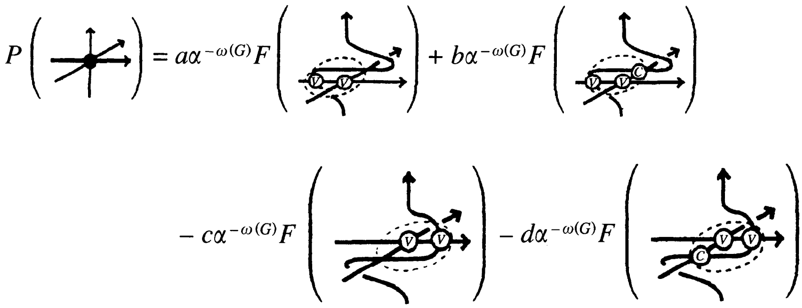
It can be shown that the invariants of graphs including transverse double and triple points are given by tangle operators of 4-valent and 6-valent rigid vertices. The 6-valent vertex is a 2-disk from which six strings emanate. Using (4.20), (4.21) and (4.22), one can find that from the definition (4.25),



(4.27)

[5] All graph diagrams coincide outside the elementary subdiagrams

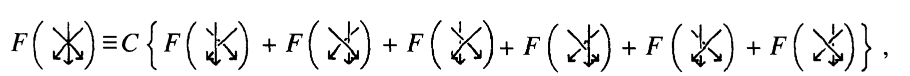
and from the definition (4.26),



(4.28)

In each diagram on the right-hand sides of (4.27) and (4.28), the vertical axis was deformed to form a 6-valent rigid vertex. One can find the coefficients a, b, c and d by rewriting (4.20) and (4.21).

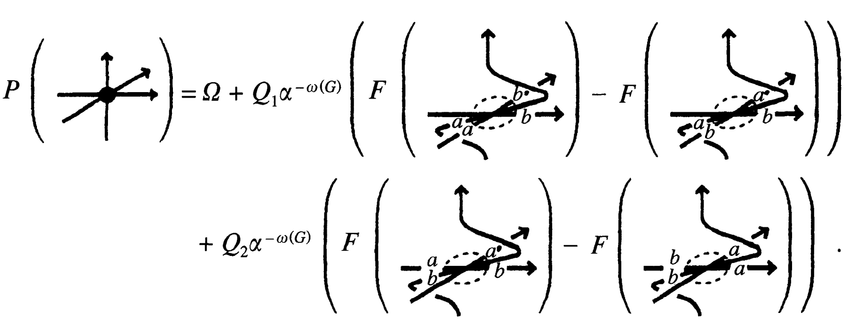
Kauffman's extension theorem (Theorem 2.1) for graphs composed of 4-valent rigid vertices is trivially extended to that for graphs including 6-valent rigid vertices. It means that one can construct invariants of graphs including 6-valent rigid vertices in terms of the link invariants. Then there are tangle operators of various kinds of 6-valent rigid vertices specified by different resolutions. As we did in Sect. 4.1, we can characterize them in accordance with all the properties that the CS vacuum expectation values of Wilson loops have under orientation reverses of Wilson loops and the spinor identity (3.7) applied to resolve a planar triple point identified with a 6-valent rigid vertex. For instance, as a result, the tangle operator of a special kind of 6-valent rigid vertices identified with a planar triple point of Wilson loops with no Casimir-like operators inserted in it is given by



(4.29)

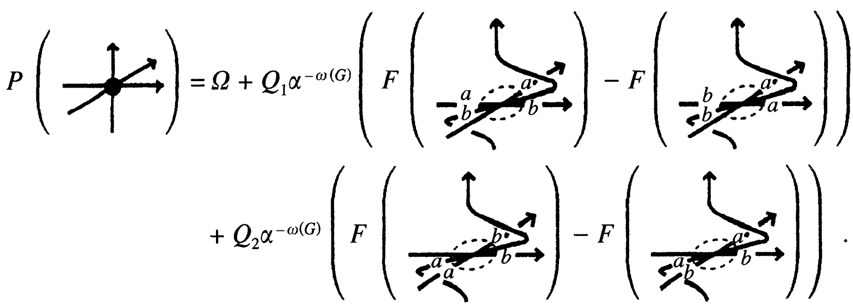
where C is a real normalization factor fixed by a certain diagram. Other tangle operators of 6-valent rigid vertices corresponding to planar triple points where Casimir-like operators are inserted are also defined with the help of the spinor and Fierz identities.

In such a direction, the CS representation can shed light on the local integrability condition. We can get the following expression from (4.27):

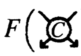
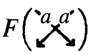


(4.30)

On the other hand, from (4.28), we can get



(4.31)

The convention F() = F() was employed. It is easy, but tedious to find the explicit expressions of Q_1 and Q_2 where $Q_1 \neq Q_2$. Ω represents common terms not including the Lie commutation relations. We can see, from the coincidence of (4.30) and (4.31), how the local integrability condition is satisfied by the Lie commutation relations in the CS representation. It was shown that our approach provides a non-perturbative perspective on the local integrability condition not based on the perturbative CS quantum field theory.

5. Application to the 4D Quantum Gravity

We considered the CS representation of the graph invariants of Vassiliev type. It is important from mathematical and physical points of view. From the former, it gives a correspondence between the quantum group graph invariants and the CS path-integrals. It also shed new light on the local integrability condition that the graph invariants of Vassiliev type are subject to. On the other hand, from the latter, it enables us to investigate non-perturbative aspects of the 4d quantum gravity of Ashtekar based on the canonical quantization.

This section is devoted to the latter. We employ the loop space representation [12, 28], in which wave-functions are defined over a space of multiple loops in a 3-space. We aim at verifying that the graph invariants of Vassiliev type in the CS representation are just physical wave-functions in the loop space representation. Let us begin with a brief review on the canonical quantization of Ashtekar's gravity and the loop space representation.

5.1. Physical Wave-Functions in the Loop Space Representation. Let M^4 be a (real analytic) 4-manifold with a co-dimension one foliation, and $\Sigma^3(t)$ a leaf. t is the parameter of time, which is given by $t = \tau(\Sigma^3)$ in terms of a smooth map $\tau: \Sigma^3 \to R$. Suppose that we are given complex-valued functionals $\psi(A: \Sigma^3(t))$ defined over an affine space $\mathscr{A}$ of $su(2)$-valued connection 1-forms[6] over $\Sigma^3(t)$. They are sections of

[6] In Ashtekar's gravity, the self-dual connections A^4 ($A^4 = A_0 dt + A_i dx^i$) and the tetrads $\tilde{E}$ defined over M^4 are dynamical variables. In the (3 + 1)-decomposition and in $A_0 = 0$ gauge, $A \equiv A_i dx^i$ is a coordinate of the configuration space on which wave-functions are defined in the canonical quantization.

a line bundle over $\mathscr{A}$ specified by a set of constraints. In the canonical quantization, the constraints of Ashtekar's gravity with vanishing cosmological constant are given by[7]

$$\hat{\mathfrak{G}}[\varepsilon^i]\psi(A:\Sigma) = \int_\Sigma d^3x\varepsilon^i(x)\hat{\mathscr{G}}_i\psi(A:\Sigma)$$

$$= i\int_\Sigma d^3x\varepsilon^i(x)\mathscr{D}_a\frac{\delta}{\delta A_a^i(x)}\psi(A:\Sigma) = 0\,, \tag{5.1}$$

$$\hat{\mathfrak{M}}[N^a]\psi(A:\Sigma) = \int_\Sigma d^3xN_a(x)\hat{\mathscr{M}}^a\psi(A:\Sigma)$$

$$= i\int_\Sigma d^3xN_b(x)\frac{\delta}{\delta A_a^i(x)}F^i_{ab}\psi(A:\Sigma) = 0\,, \tag{5.2}$$

$$\hat{\mathfrak{H}}[N]\psi(A:\Sigma) = \int_\Sigma d^3x\underset{\sim}{N}(x)\hat{\mathscr{H}}\psi(A:\Sigma)$$

$$= \int_\Sigma d^3x\underset{\sim}{N}(x)\varepsilon^{ijk}\frac{\delta}{\delta A_a^i(x)}\frac{\delta}{\delta A_b^j(x)}F^k_{ab}\psi(A:\Sigma) = 0\,. \tag{5.3}$$

where $\varepsilon^i(x)$, the shift functions $N_a(x)$ and the lapse function $\underset{\sim}{N}(x)$ are analytic functions over M^4. The first constraint is called the Gauss law constraint which generates the gauge transformations, the second the momentum constraint which generates diffeomorphisms of Σ, the last one the Hamiltonian constraint (that is also called the Wheeler–DeWitt equation) which generates diffeomorphisms in the time direction.

Let us introduce the loop space representation. Wave-functions in the loop space representation are given by the Wilson loops $W(A:L_i^s)$ $(1 \leqq i \leqq N)$, which are gauge invariant objects. $L^s = \Pi_{i=1}^N L_i^s$ represents a singular link including not only transverse intersection points but also corners. Each L_i^s represents a closed path. We allow a finite number of corners to contact one another at a single point. If Casimir-like operators are inserted at the transverse intersection points or at points where a finite number of corners are contacting one another, they can be eliminated by the Fierz identity. Let us denote a wave-function given by L^s by $\psi(L^s:\Sigma)$. It is given by the functional integral:

$$\psi(L^s:\Sigma) \equiv \int_{\mathscr{A}} DA\prod_{i=1}^N W(A:L_i^s)\psi(A:\Sigma)\,. \tag{5.4}$$

Constraints in the loop space representation are induced by the constraints (5.1), (5.2) and (5.3). For differential operators $\hat{\mathscr{O}} = \hat{\mathscr{G}}\ \hat{\mathscr{M}}, \hat{\mathscr{H}}$, the constraints are given by

$$\hat{\mathscr{O}}_L\psi(L^s:\Sigma) \equiv \int_A DA\prod_{i=1}^N W(A:L_i^s)(\hat{\mathscr{O}}\psi(A:\Sigma))$$

$$= \int_A DA\left(\hat{\mathscr{O}}^\dagger\prod_{i=1}^N W(A:L_i^s)\right)\psi(A:\Sigma) = 0\,. \tag{5.5}$$

[7] For brevity, $\Sigma^3(t)$ is denoted by Σ.

We used integration by parts to obtain the second equality. $\hat{\mathcal{O}}^\dagger$ represents the adjoint operator. The Gauss law constraint is trivial provided that $\psi(A:\Sigma)$ is gauge-invariant.

The graph invariants of Vassiliev type play a crucial role in the canonical quantization of Ashtekar's gravity. They are regarded as physical wave-functions in the loop space representation. It is obvious by definition that the graph invariants of Vassiliev type are subject to the momentum constraint, because the momentum operator $\hat{\mathcal{M}}_L$ generates the rigid vertex isotopy of the 3-space Σ. On the other hand, it is less obvious that they satisfy the Hamiltonian constraint. The following theorem is significant.

Theorem 5.1. *Let $P(G:\Sigma)$*[8] *be a graph invariant of Vassiliev type in the CS representation. G represents a singular link including only transverse double and triple points. Then it satisfies.*

$$\hat{\mathcal{H}}_L P(G:\Sigma) = 0. \tag{5.6}$$

Proof. It suffices to prove $\hat{\mathcal{H}}_L P(L:\Sigma) = 0$, because $P(G:\Sigma)$ is given by the link invariants $P(L:\Sigma)$. It follows from a fact that the action of the Hamiltonian is trivial when acting on functionals of links. □

Non-trivial cases appear only when the Hamiltonian acts on functionals of singular links including transverse triple points or points, at each of which a finite number of corners are contacting one another with tangent vectors spanning a 3-dimensional vicinity [7, 8].

5.2. The Graph Invariants of Vassiliev Type and the Half-Flat Geometry. We showed that the graph invariants of Vassiliev type are physical wave-functions in a sense that they satisfy all the constraints of the quantum gravity of Ashtekar. Let us consider the physical implication. We recall that the graph invariants of Vassiliev type in the CS representation can be identified with the CS vacuum expectation values of Wilson loops including intersection points. According to the perturbative CS quantum field theory, contributions of flat connections over Σ dominate in the classical limit. In the quantum gravity of Ashtekar, it corresponds to vanishing of the self-dual curvature $F(A)$ in the $3+1$ formulation, where A represents Ashtekar's connection. Then the 4-geometry is *half-flat* [2]. In addition to the half-flatness, suppose that we are given three complex structures I, J and K subject to the quaternionic relations, i.e., $I^2 = J^2 = K^2 = -1$ and $IJ = -JI = K$. Then we say that the 4-geometry is *hyperkähler* [27].

Let us consider complex structures on spaces of multiple loops in the 3-space Σ. In a case in which all loops have no intersection points, a complex structure on a space of multiple loops (i.e., a space of links) is defined as follows. Let N be a section of the normal bundle over a loop, then another vector B is given by $B = T \times N = J_p N$ at p. T is the unit tangent vector, and J_p represents the conformal structure on the normal plane. The three vectors T, N and B constitute the *Frénet frame* as indicated in Fig. 3. The complex structure on the space of links is induced

[8] We do not have to require that Σ should be restricted to S^3, because invariants of graphs in Σ are obtained from those of graphs in S^3 via the Dehn surgery.

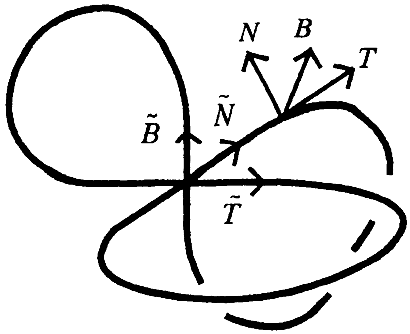


Fig. 3. The Frénet frames.

by J_p over Σ. A generalization to singular links including intersection points is also possible [9].

Singular links including transverse triple points are physically significant. At each transverse triple point, it seems that we can introduce three almost-complex structures because there exist three normal planes given by three independent tangent vectors along loops. However, the three almost-complex structures can not be independent. Let I_p, J_p and K_p be the three almost-complex structures at p, a transverse triple point. Without lack of generality, we can put $\tilde{B} = I_p\tilde{T}$, $\tilde{N} = J_p\tilde{B}$ and $\tilde{T} = K_p\tilde{N}$ as illustrated in Fig. 3. One can easily check that the three almost-complex structures must satisfy a set of compatibility conditions that is just a set of the quaternionic relations. Thus we can get the *almost-hyperkähler structure* [27] on the space of singular links including transverse triple points. After all, we can summarize as follows.

Remark 5.1. In the canonical quantization of Ashtekar's gravity with vanishing cosmological constant, physical wave-functions in the loop space representation given by the graph invariants of Vassiliev type are solutions to the Hamiltonian constraint. Furthermore, when the graphs are including transverse triple points, the graph invariants of Vassiliev type are characterized by the half-flat geometry with the almost-hyperkähler structure in the classical limit.

We end this section with a few comments. The physical wave-functions given by the graph invariants of Vassiliev type can not describe the physically realistic universe. The reason is that non-degenerate metrics evaluated in terms of them can be given only at the transverse triple points, which are discretely located. An idea to construct wave-functions of the realistic universe must be to consider the inductive limit of the Vassiliev invariants, i.e., $\varinjlim F_j$. Finally, it should be mentioned that we aim at constructing the Hilbert space of the quantum gravity. The loop space representation seems most hopeful.

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