Commun Math Phys 180, 587–603 (1996)

Communications in Mathematical Physics

Holomorphic Bundles and Many-Body Systems

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Received: 17 May 1995/Accepted: 4 October 1995

Abstract: We show that spin generalization of elliptic Calogero–Moser system, elliptic extension of Gaudin model and their cousins are the degenerations of Hitchin systems. Applications to the constructions of integrals of motion, angle-action variables and quantum systems are discussed. The constructions of classical systems are motivated by Conformal Field Theory, and their quantum counterparts can be thought of as being the degenerations of the critical level Knizhnik–Zamolodchikov–Bernard equations.

1. Introduction

Integrable many-body systems attract attention for the following reasons: they are important in condensed matter physics and they appear quite often in two dimensional gauge theories as well as in conformal field theory. Recently they have been recognized in four dimensional gauge theories.

Among these systems the following ones will be of special interest for us:

1. *Spin generalization of Elliptic Calogero–Moser model* – it describes the system of particles in one (complex) dimension, interacting through the pair-wise potential. The explicit form of the Hamiltonian is:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i \neq j} \mathrm{Tr}\,(S_i S_j) \wp(z_i - z_j)\,,$$

where z_i are the positions of the particles, p_i – corresponding momenta and S_i are the "spins" – $l \times l$ matrices, acting in some auxiliary space. The conditions on S_i will be specified later. The only point to be mentioned is that the Poisson brackets between p, z, S are the following:

$$\{p_i, z_j\} = \delta_{ij}\,,$$

$$\{(S_i)_{ab}, (S_j)_{cd}\} = \delta_{ij}(\delta_{ad}(S_i)_{bc} - \delta_{bc}(S_i)_{ad})\,.$$

2. *Gaudin model and its elliptic counterpart.* We describe first the rational case. Consider a collection of L points on $\mathbb{P}^1$ in generic position: $w_1, \dots, w_L$, assign to each w_a a spin T_a (an $N \times N$ matrix) and define the Hamiltonians [G]:

$$H_a = \sum_{b \neq a} \frac{\mathrm{Tr}(T_a T_b)}{(w_a - w_b)} .$$

The main goal of this note is to include these two (seemingly) unrelated models in the universal family of integrable models, naturally related to the moduli spaces of holomorphic bundles over the curves. It will turn out that the appropriate objects to study are Hitchin systems. As a by-product we shall invent an elliptic Gaudin model, which includes both cases as special limits. We shall also obtain a prescription for construction of integrals of motion and action-angle variables. The paper is organized as follows. In Sect. 3 we recall the construction of Hitchin systems. Section 4 is devoted to the explanation of the mapping between the Hitchin systems and the models, just described. Section 5 deals with action-angle variables and integrals of motion. We conclude with the remarks on the quantization of our constructions.

2. Acknowledgements. I would like to thank Sasha Gorsky for collaboration in [GN1, GN2 and GN3], A. Losev and D. Ivanov for useful discussions, V. Fock, I. Frenkel, G. Moore, A. Polyakov, M. Olshanetsky, A. Rosly, V. Rubtsov and S. Shatashvili for kind advice.

3. Construction of the Systems

We must confess that all the models we are discussing are motivated by the studies of Knizhnik–Zamolodchikov–Bernard equations [KZ, Be, Lo, FV, I]. Our paper is a development of [GN1, GN2]. One of the outcomes of our work might be an insight in the $\mathcal{W}$-generalizations of them.

3.1. Hitchin Systems. Hitchin has introduced in [H] a family of integrable systems. The phase space of these systems can be identified with the cotangent bundle $T^*\mathcal{N}$ to the moduli space $\mathcal{N}$ of stable holomorphic vector bundles of rank N (for the $GL_N(\mathbb{C})$ case) over the compact smooth Riemann surface Σ of genus $g > 1$. His construction can be briefly described as follows. Fix the topological class of the bundles (i.e. let us consider the bundles $\mathscr{E}$ with $c_1(\mathscr{E}) = k$, with k-fixed). Consider the space $\mathscr{A}^s$ of stable complex structures in a given smooth vector bundle V, whose fiber is isomorphic to $\mathbb{C}^N$. The notion of stable bundle comes from geometric invariant theory and implies in this context, that for any proper subbundle U:

$$\frac{deg(U)}{rk(U)} < \frac{deg(V)}{rk(V)} .$$

The quotient of $\mathscr{A}^s/\mathscr{G}$ of the space of all stable complex structures by the gauge group is the moduli space $\mathcal{N}$. Its dimension is given by the Riemann–Roch theorem

$$dim(\mathcal{N}) = N^2(g-1) + 1 .$$

Now consider a cotangent bundle to $\mathscr{A}^s$. It is the space of pairs:

$$\phi, d''_A ,$$

where ϕ is a $Mat_N(\mathbb{C})$ valued $(1,0)$-differential on Σ, d''_A is an operator, defining the complex structure on V:

$$d''_A : \Omega^0(\Sigma, V) \to \Omega^{0,1}(\Sigma, V) .$$

The field ϕ is called a Higgs field and the pair d''_A, ϕ defines what is called a Higgs bundle. In the framework of conformal field theory the Higgs field is usually referred to as the holomorphic current, while holomorphic bundle defines a background gauge field.

The cotangent bundle $T^*\mathscr{A}^s$ can be endowed with a holomorphic symplectic form:

$$\omega = \int_\Sigma \mathrm{Tr}\delta\phi \wedge \delta d''_A ,$$

where $\delta d''_A$ can be identified with a $(0,1)$-form with values in N by N matrices. The gauge group $\mathscr{G}$ acts on $T^*\mathscr{A}^s$ by the transformations:

$$\phi \to g^{-1}\phi g ,$$

$$d''_A \to g^{-1} d''_A g ,$$

and preserves the form ω. Therefore, a moment map is defined:

$$\mu = [d''_A, \phi] .$$

Taking the zero level of the moment map and factorizing it along the orbits of $\mathscr{G}$ we get the symplectic quotient, which can be identified with $T^*\mathscr{N}$. Now the Hitchin Hamiltonians are constructed with the help of holomorphic $(1-j, 1)$-differentials v_{j,i_j}, where i_j labels a basis in the linear space

$$H^1(\Sigma, \mathscr{K} \otimes \mathscr{T}^j) = \mathbb{C}^{(2j-1)(g-1)}$$

for $j > 1$ and $\mathbb{C}^g$ for $j = 1$. Take a gauge invariant $(j,0)$-differential $\mathrm{Tr}\phi^j$ and integrate it over Σ with the weight v_{j,i_j}:

$$H_{j,i_j} = \int_\Sigma v_{j,i_j} \mathrm{Tr}\, \phi^j .$$

Obviously, on $T^*\mathscr{A}^s$ these functions Poisson-commute. Since they are gauge invariant, they will Poisson-commute after reduction. Also it is obvious that they are functionally independent and their total number is equal to

$$g + \sum_{j=2}^{N} (2j-1)(g-1) = N^2(g-1) + 1 = dim(\mathscr{N}) .$$

Therefore, we have an integrable system.

3.2. Holomorphic Bundles over Degenerate Curves. Now let us consider a degeneration of the curve. Recall, that the normalization of the stable curve Σ is a collection of a smooth curves Σ_α with possible marked points, such that any component of genus zero has at least three marked points and every component of genus one has at least one such point. For each component Σ_α we have a subset $X_\alpha = \{x^1_\alpha, \dots, x^{L_\alpha}_\alpha\}$ of points. Let us denote the pair $(\Sigma_\alpha, X_\alpha)$ as C_α. The disjoint union of C_α's is mapped onto Σ by the normalization map π. Let us denote by $X_{\alpha\beta}$ the set of double points

$\pi(X_\alpha) \cap \pi(X_\beta)$ for $\alpha \neq \beta$ and as $X_{\alpha\alpha}$ the set of double points in $\pi(X_\alpha)$ (these appear due to pinching the handles). The union of all $X_{\alpha\beta}$ we shall denote by $X \subset \Sigma$. We define $x^{ij}_{\alpha\beta} \in X_{\alpha\beta}$ as $\pi(x^i_\alpha) \cap \pi(x^j_\beta)$. Notice, that it may be empty. *A stable bundle $\mathscr{E}$ over Σ* is a collection of holomorphic bundles $\mathscr{E}_\alpha$ over Σ_α of rank N (there might be some generalizations with different ranks of the bundle over different components – these are unnatural as a degeneration of the bundle over a smooth curve) with the identifications $g^{ij}_{\alpha\beta}$ of the fibers

$$g^{ij}_{\alpha\beta} : \mathscr{E}_\alpha|_{x^i_\alpha} \longrightarrow \mathscr{E}_\beta|_{x^j_\beta}$$

with the obvious condition: $g^{ij}_{\alpha\beta} g^{ji}_{\beta\alpha} = 1$.

The gauge group acts on the complex structure of the bundle $\mathscr{E}_\alpha$ for each α as in the smooth curve case. The new ingredient is the action on $g^{ij}_{\alpha\beta}$. Fix a gauge transformation g_α for each component of Σ. Then $g^{ij}_{\alpha\beta}$ are acted on by g_α as follows:

$$g^{ij}_{\alpha\beta} \longrightarrow g_\beta(x^j_\beta)^{-1} g^{ij}_{\alpha\beta} g_\alpha(x^i_\alpha) \,.$$

Now we have to introduce a notion of stable bundle. The condition of stability is:

For each collection of proper subbundles $\mathscr{F}_\alpha \subset \mathscr{E}_\alpha$, such that

$$g^{ij}_{\alpha\beta}(\mathscr{F}_\alpha|_{x^i_\alpha}) = \mathscr{F}_\beta|_{x^j_\beta}$$

and

$$rk(\mathscr{F}_\alpha) = N' < N$$

for each α the following inequality holds:

$$deg(\mathscr{F}_\alpha) < \frac{N'}{N} deg(\mathscr{E}_\alpha)$$

for any α.

Let $\mathscr{A}$ denote the space of collections of $d''_{A,\alpha}$ operators in each $\mathscr{E}_\alpha$ together with $g^{ij}_{\alpha\beta}$ for each α and β. Let $\mathscr{A}^s$ denote the subspace of $\mathscr{A}$, consisting of *stable* objects. The cotangent bundle $T^*\mathscr{A}^s$ can be identified with the space of collections of pairs

$$(\mathscr{E}_\alpha, \phi_\alpha), \phi_\alpha \in \Omega^{1,0}(\Sigma_\alpha) \otimes End(\mathscr{E}_\alpha)$$

and

$$(g^{ij}_{\alpha\beta}, p^{ij}_{\alpha,\beta}), p^{ij}_{\alpha\beta} \in T^*_{g^{ij}_{\alpha\beta}} Hom(\mathscr{E}_\alpha|_{x^i_\alpha}, \mathscr{E}_\beta|_{x^j_\beta}) \,.$$

We normalize $p^{ij}_{\alpha\beta}$: $p^{ij}_{\alpha\beta} = -Ad^*(g^{ij}_{\alpha\beta}) p^{ji}_{\beta\alpha}$. The Higgs fields ϕ_α are allowed to have singularities at the marked points. As we will see, they could have poles there. Now we shall proceed as in the previous section. Consider the gauge group action on $T^*\mathscr{A}^s$. Since the gauge group $\mathscr{G}$ is essentially the product of gauge groups $\mathscr{G}_\alpha$, the moment map is a collection of the moment maps for each component Σ_α:

$$\mu_\alpha = [d''_{A,\alpha}, \phi_\alpha] + \sum_{\beta, i, j} p^{ij}_{\alpha\beta} \delta^2(x^i_\alpha) \,,$$

where the sum over i runs from 1 up to L_α while β and j are determined from the condition, that $\pi(x^j_\beta) = \pi(x^i_\alpha)$. Let us now repeat the procedure of reduction. At the first step we should restrict ourselves onto the zero level of the moment map. It means that ϕ_α becomes a meromorphic section of the bundle $End(\mathscr{E}_\alpha) \otimes \Omega^{1,0}(\Sigma_\alpha)$ with the first order poles at the double points. The residue of ϕ_α at the point x_α^i equals $p_{\alpha\beta}^{ij}$ for appropriate β, j. This condition is compatible with the definition of the canonical bundle over the stable curve. On the next step we take a quotient with respect to the gauge group action and get the reduced space $T^*\mathscr{N}$. The space $\mathscr{N}$ is the quotient of $\mathscr{A}^s$ by $\mathscr{G}$. The symplectic form on $T^*\mathscr{N}$ can be written as:

$$\omega = \sum_\alpha \omega_\alpha + \sum_{(\alpha,i),(\beta,j)} \mathrm{Tr}\delta(g_{\beta\alpha}^{ji} p_{\alpha\beta}^{ij}) \wedge \delta g_{\alpha\beta}^{ij} .$$

Let us calculate the dimension of $T^*\mathscr{N}$. We shall calculate the (complex) dimension of $\mathscr{N}$ by means of the following trick. The moduli space $\mathscr{N}$ can be projected onto the direct product of moduli spaces $\mathscr{N}_\alpha$ of the stable bundles over Σ_α's. Actually, the map is to the product of the moduli of holomorphic bundles, but the open dense subset, consisting of the stable bundles is covered. The projection simply takes the collection of $\mathscr{E}_\alpha$'s to the product of equivalence classes in $\mathscr{N}_\alpha$'s. The fiber of this map can be identified with the quotient G/H, where G is the set of collections of $g_{\alpha\beta}^{ij}$, while H is the group of automorphisms of $\times_\alpha \mathscr{E}_\alpha$. This group is a product over all components Σ_α of the groups H_α. For the genus zero component H_α is $GL_N(\mathbb{C})$, while the genus one component provides a maximal torus – $(\mathbb{C}^*)^N$. Other components contribute $\mathbb{C}^*$. Therefore, at a generic point, we conclude that the dimension of $\mathscr{N}$ is

$$\begin{aligned} dim(\mathscr{N}) &= \sum_\alpha dim(\mathscr{N}_\alpha) + dim(G/H) \\ &= 1 + N^2 E(\Sigma) + \sum_\alpha N^2(g(\Sigma_\alpha) - 1) = N^2(g-1) + 1 , \end{aligned} \tag{3.1}$$

where we have used the Riemann–Roch theorem in the form

$$dim(\mathscr{N}_\alpha) - dim(H_\alpha) = N^2(g(\Sigma_\alpha) - 1) ;$$

$E(\Sigma)$ is the total number of double points.

3.3. Hamiltonian Systems on $T^*\mathscr{N}$. Now we shall define the Hamiltonians. For each α we take $\nu_{\alpha,l,k}$ – the k^{th} holomorphic $(1-l, 1)$ differential on $\Sigma_\alpha - X_\alpha$ and construct a holomorphic function on $T^*\mathscr{A}^s$:

$$H_{\alpha,l,k} = \int_{\Sigma_\alpha} \nu_{\alpha,l,k} \mathrm{Tr}(\phi_\alpha^l) .$$

Obviously, all $H_{\alpha,l,k}$ descend to $T^*\mathscr{N}$ and Poisson-commute there.

4. Gaudin Model, Spin Elliptic Calogero–Moser System and so on ...

4.1. Genus Zero Models. Consider a component of genus zero. Let us describe explicitly the part of $T^*\mathscr{N}$ related to this component as well as the Hamiltonians. We shall omit the label α in the subsequent formulas to save space. Thus, we have $C = (\mathbb{P}^1, x_1, \ldots, x_L)$.

Gaudin model. Assume first, that for $a \neq b$: $\pi(x_a) \neq \pi(x_b)$. Then $p^{ab}_{\alpha\beta}$ can be denoted simply as T_a without any confusion. There are no continuous moduli of stable holomorphic bundles over the sphere. For simplicity we consider the case when $\mathscr{E}$ is trivial and therefore, we can assume that d''_A is just the $\bar{\partial}$ operator. The moment map condition:

$$0 = \bar{\partial}\phi + \sum_a T_a \delta^2(x_a)$$

is easily solved:

$$\phi(x) = -\frac{1}{2\pi\sqrt{-1}} \sum_a \frac{T_a}{(x - x_a)} .$$

Notice, however, that every trivial holomorphic bundle over $\mathbb{P}^1$ has an automorphism group $GL_N(\mathbb{C})$, which acts nontrivially on ϕ as well as on T_i. In fact, the reduction with respect to this subgroup is forced by our equation: the sum of residues of ϕ must vanish, giving rise to the constraint: $\sum_a T_a = 0$, which is nothing but the moment map for the $GL_N(\mathbb{C})$ action.

Our Hamiltonians in this case boil down to

$$H_{l,a,b} = Res_{x_a}(x - x_a)^{b-1}\mathrm{Tr}\phi^l(x) ,$$

where $1 \leqq b \leqq l, 1 \leqq a \leqq L$. These Hamiltonians (essentially $H_{2,a,2}$) are called *Gaudin Hamiltonians*[1]

$$H_{1,a,1} = \mathrm{Tr}(T_a), H_{2,a,1} = \mathrm{Tr}(T_a^2) ,$$

$$H_{2,a,2} = \sum_{b \neq a} \frac{\mathrm{Tr}(T_b T_a)}{(x_b - x_a)} ,$$

etc.

Spin Calogero–Moser and Rational Ruijsenaars Systems. Now consider the genus zero component Σ_α with double points. Let us decompose the set of marked points X_α as

$$X_\alpha = S \cup T \cup \sigma(T) , \tag{4.1}$$

where $t \in T$ and $\sigma(t) \in \sigma(T)$ are mapped to $x^{t\sigma(t)}_{\alpha\alpha}$, while the restriction of π on S is surjective. We denote $p^{t\sigma(t)}_{\alpha\alpha} = p_t$, $g^{t\sigma(t)}_{\alpha\alpha} = g_t$. We have: $p^{\sigma(t)t}_{\alpha\alpha} = -Ad^*(g_t)p_t$. Solving the moment map condition as before, we get:

$$\phi_\alpha(x) = \sum_s \frac{p_s}{(x - x_s)} + \sum_t \frac{p_t}{(x - x_t)} - \frac{Ad^*(g_t)p_t}{(x - x_{\sigma(t)})} .$$

The sum of the residues vanishes, giving rise to the moment equation:

$$\sum_s p_s + \sum_t (p_t - Ad^*(g_t)p_t) = 0 . \tag{4.2}$$

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[1] Historically, these Hamiltonians in the classical context were introduced by R. Garnier [Ga], and L Schlesinger [Sch] Nowadays, by *Gaudin* system one understands the quantum counterpart of this model, when T_a's are replaced by the generators of the Lie algebra, acting in representation, attached to the point x_a.

This is the moment with respect to the group $GL_N(\mathbb{C})$ of automorphisms which acts on the data (p_s, p_t, g_t) as follows:

$$g_t \to g^{-1} g_t g, p_t \to g^{-1} p_t g, p_s \to g^{-1} p_s g .$$

Now let us specialize to the case when $\#T = 1$. The moment map condition will give

$$p_t - Ad^*(g_t) p_t = -\sum_s p_s .$$

The phase space is the quotient of the manifold of p_t, p_s, g_t, satisfying this condition by the action of $GL_N(\mathbb{C})$. We have two options: either we parameterize the quotient by the conjugacy class of g_t, or by the one of p_t.

Consider the first option. Generically one can diagonalize g_t, and there will be left a group of diagonal matrices, which will act nontrivially on p_t and p_s's. Let us denote the eigenvalues of g_t by

$$g_t \sim diag(e^{z_1}, \ldots, e^{z_N}) .$$

Then in the basis, where g_t is diagonal, p_t has a form:

$$(p_t)_{ij} = p_i \delta_{ij} + \frac{\sum_s (p_s)_{ij}}{1 - e^{z_i - z_j}}$$

with the further condition

$$\sum_s (p_s)_{ii} = 0 \tag{4.3}$$

for any i. This condition has an elliptic nature, as we will see, and has a very natural origin: the double point on the sphere comes from the pinching the handle.

Explicitly calculating $H_{2,t,2}$ we get:

$$H_{2,t,2} = \sum_i \frac{p_i^2}{2} + \sum_{i,j} \frac{\Sigma_{ij}}{sinh^2(\frac{z_i - z_j}{2})}$$

with

$$\Sigma_{ij} = \sum_{s,s'} (p_s)_{ij} (p_{s'})_{ji} .$$

This Hamiltonian describes the particles with spin interaction. In Sect. 4.3 we shall represent the coupling Σ_{ij} as $\mathrm{Tr}\, S_i S_j$, which is the form of spin interaction we advertised in introduction. This model for $\#S = 1$ is the *Spin generalization of Sutherland model* [S].

Now let us investigate another option – namely, we diagonalize p_t. For simplicity we shall treat the case $\#S = 1$. We have:

$$p_t = diag(\theta_1, \ldots, \theta_N)$$

and

$$(g_t)_{ij} (\theta_i - \theta_j) = (\tilde{p}_s)_{ij} ,$$

where $\tilde{p}_s = g_t p_s$. Now we make a further assumption. Suppose, that for some $v \in \mathbb{C}^*$ the matrix $p_s - v\mathbf{Id}$ has rank one.

Then,

$$p_s = v\mathbf{Id} + u \otimes v ,$$

where $v \in (\mathbb{C}^N)^*, u \in \mathbb{C}^N$.

Therefore, we can solve for g_t:

$$(g_t)_{ij} = \frac{\tilde{u}_i v_j}{\theta_i - \theta_j - \nu}$$

and $\tilde{u}_i = (g_t u)_i$. The consistency requires a linear equation:

$$\sum_j \frac{u_j v_j}{\theta_i - \theta_j - \nu} = 1$$

for any i. The solution is

$$u_i v_i = \frac{P(\theta_i + \nu)}{P'(\theta_i)},$$

where $P(\theta) = \prod(\theta - \theta_i)$.

Finally we introduce the coordinates z_i, defined by

$$e^{z_i} u_i = \tilde{u}_i .$$

The Hamiltonians we can consider in this approach are the characters of g_t. We have:

$$\mathrm{Tr}\, g_t^k = \sum_{I \subset \{1, \ ,N\}, \#I = k} e^{z_I} \prod_{i \in I, j \not\subset I} \frac{\theta_{ij} + \nu}{\theta_{ij}},$$

$$z_I = \sum_{i \in I} z_i, \theta_{ij} = \theta_i - \theta_j, k = 1, \ldots, N . \tag{4.4}$$

The system we get is the classical limit of *Rational Macdonald* system. If, instead of taking $G = GL_N(\mathbb{C})$ we would consider $SU(N)$, we will get what is called a rational Ruijsenaars–Schneider model [R, RS, GN2].

It is clear, that without the simplifying assumption that p_s has the form of the scalar matrix plus the matrix of rank one, we would get the *Spin Generalization of Ruijsenaars–Schneider Model.* Conjecturally, our reduction provides a Hamiltonian description for the spin generalization of Ruijsenaars–Schneider model, considered in [KrZ] (for its rational limit).

4.2. Elliptic Models. The next interesting example is the genus one component. Again we omit label α and $p^{ij}_{\alpha\beta}$ gets replaced by p_i. Generic holomorphic bundles over the torus are decomposable into the direct sum of the line bundles:

$$\mathscr{E} = \bigoplus_{i=1}^{N} \mathscr{L}_i .$$

Therefore, the moduli space $\mathscr{N}_\alpha$ can be identified with the N^{th} power of the Jacobian of the curve, divided by the action of the permutation group. Let us introduce the coordinates $z_1, \ldots, z_N$ on $\mathscr{N}_\alpha$. They are defined up to the elliptic affine Weyl group action. Let τ be the modular parameter of the elliptic curve. Then there are shifts

$$z_i \to z_i + 2\pi\sqrt{-1}\frac{m_i \tau + n_i}{\tau - \bar{\tau}}$$

with $m_i, n_i \in \mathbb{Z}$, induced by the gauge transformations

$$\exp\left(diag\left(2\pi\sqrt{-1}\frac{n_i(x-\bar{x})+m_i(x\bar{\tau}-\bar{x}\tau)}{\tau-\bar{\tau}}\right)\right)$$

as well as permutations of z_i's. Up to these equivalencies z_i's are the honest coordinates.

No double points. First, we dispose of the case when π doesn't map two points to one. Now the moment map condition is

$$\bar{\partial}\phi_{ij} + (z_i - z_j)\phi_{ij} + \sum_a (T_a)_{ij}\delta^2(x_a) = 0\,.$$

The solution of this equation yields a Lax operator ϕ:

$$i \neq j: \quad \phi_{ij} = \frac{\exp(z_{ij}\frac{x-\bar{x}}{\tau-\bar{\tau}})}{2\pi\sqrt{-1}}\sum_a (T_a)_{ij}\frac{\sigma(z_{ij}+x-x_a)}{\sigma(z_{ij})\sigma(x-x_a)}e^{z_{ij}\frac{\bar{x}_a-x_a}{\tau-\bar{\tau}}}\,.$$

$$i = j: \quad \phi_{ii}(x) = w_i + \frac{1}{2\pi\sqrt{-1}}\sum_a (T_a)_{ii}\zeta(x-x_a)\,, \tag{4.5}$$

where we have denoted for brevity: $z_{ij} = z_i - z_j$. In these formulas σ and ζ are Weierstrass elliptic functions for the curve with periods 1 and τ. Since the sum of residues of meromorphic form ϕ_{ii} vanishes, we get the following equation:

$$\sum_a (p_a)_{ii} = 0\,, \quad i = 1,\ldots,N, \tag{4.6}$$

which coincides with the moment for the maximal torus action. When the elliptic curve degenerates down to the rational one with the double point this equation becomes just (4.3).

Now we can compute our Hamiltonians. We have:

$$-4\pi^2\mathrm{Tr}\,\phi^2(x) = \sum_i\left(w_i + \sum_a (T_a)_{ii}\zeta(x-x_a)\right)^2$$
$$- \sum_{i\neq j;\,a,b}(T_a)_{ij}(T_b)_{ji}\frac{\sigma(z_{ij}+x-x_a)\sigma(z_{ji}+x-x_b)}{\sigma(z_{ij})^2\sigma(x-x_a)\sigma(x-x_b)}e^{z_{ij}\frac{\bar{x}_{ab}-x_{ab}}{\tau-\bar{\tau}}}\,, \tag{4.7}$$

where $x_{ab} = x_a - x_b$.

Expanding this expression as:

$$-4\pi^2\mathrm{Tr}\,\phi^2(x) = \left(\sum_a \wp(x-x_a)H_{2,2,a} + \zeta(x-x_a)H_{2,1,a}\right) + H_{2,0}\,,$$

we obtain:

$$H_{2,2,a} = \mathrm{Tr}T_a^2\,, \tag{4.8}$$

as it could be guessed,

$$H_{2,1,a} = \sum_i w_i(T_a)_{ii} + \sum_{b\neq a;i}(T_a)_{ii}(T_b)_{ii}\zeta(x_{ab})$$
$$+ \sum_{b\neq a;i\neq j} e^{z_{ij}(x_a-x_b)}(T_b)_{ij}(T_a)_{ji}\frac{\sigma(z_{ij}+x_{ab})}{\sigma(z_{ij})\sigma(x_{ab})}e^{z_{ij}\frac{x_{ab}-\bar{x}_{ab}}{\tau-\bar{\tau}}}\,. \tag{4.9}$$

These Hamiltonians will be called *Elliptic Gaudin Hamiltonians*. The next interesting Hamiltonian is:

$$H_{2,0} = \sum_i w_i^2 + \sum_{i \neq j; a} (T_a)_{ij}(T_a)_{ji}\wp(z_{ij}) + \sum_{i; a \neq b} (T_a)_{ii}(T_b)_{ii}(\wp(x_{ab}) - \zeta^2(x_{ab}))$$
$$+ \sum_{i \neq j; a \neq b} e^{z_{ij}x_{ab}}(T_a)_{ij}(T_b)_{ji}e^{z_{ij}\frac{x_{ab}-\bar{x}_{ab}}{\tau-\bar{\tau}}} \frac{\sigma(z_{ij}+x_{ab})}{\sigma(z_{ij})\sigma(x_{ab})}(\zeta(x_{ab}+z_{ij}) - \zeta(z_{ij}))\,. \tag{4.10}$$

Higher Hamiltonians provide us with the rest of the integrals of motion of this model.

Double points. In the case, when there are double points, the formulas for the Lax operator and Hamiltonians are nearly the same, the only difference is in the condition on the T_a's. It is easy to see that the condition is: let us decompose the set of marked points as in (4.1) and introduce the notations p_t, p_s as before. Then the elliptic analogue of (4.2) is

$$\left[\sum_s p_s + \sum_t (p_t - Ad^*(g_t)p_t)\right]_{ii} = 0, \quad i = 1, \dots, N\,. \tag{4.11}$$

4.3. Parabolic Structures, Spins and Coadjoint Orbits. In this section we shall explain the relation of our moduli spaces to the moduli spaces of bundles with parabolic structures (this is what one usually expects to be considered on the punctured curve). Then we shall map the notations p_a for the Lie algebra elements to the spin notations S_i, which were used in the beginning of the paper.

First of all, the integrable systems we have defined have an obvious invariant subvariety: the conjugacy classes of all T_a's are conserved. Indeed, since ϕ has a pole at x_a with the residue T_a, then near x_a

$$\mathrm{Tr}\,\phi^n(x) \sim \frac{\rho_{a,n}}{(x-x_a)^n}\,, \tag{4.12}$$

where $\rho_{a,n}$ equals:

$$\rho_{a,n} = \mathrm{Tr}\,T_a^n,$$

hence the trace $\mathrm{Tr}\,T_a^n$ is a constant of motion. Therefore, each T_a will represent a point on the coadjoint orbit O_a of $SL_N(\mathbb{C})$. This orbit (which is generically diffeomorphic to the cotangent bundle to the compact flag variety) defines a parabolic structure at the point x_a (which is a fixed flag in the fiber over it).

In fact, the flag structure can be decoded from the T_a with the help of the following construction: For simplicity, we consider the case without double points on the component C_α. Again for simplicity we assume, that this coadjoint orbit is of the generic type.

Fix a point x_a and denote T_a simply as p. Introduce a sequence of vector spaces

$$\mathscr{E}^r, \dots, \mathscr{E}^0\,.$$

Let $d^i = dim\mathscr{E}^i$, we assume that $d^i > d^{i+1}$. Consider the space of operators

$$U^i : \mathscr{E}^i \to \mathscr{E}^{i+1}, \qquad V^i : \mathscr{E}^{i+1} \to \mathscr{E}^i$$

with the canonical symplectic form:

$$\sum_i \operatorname{Tr} \delta U^i \wedge \delta V^i \ .$$

This form is invariant under the action of the group

$$H = \times_{i=1}^{r} GL(\mathscr{E}^i)$$

by the changes of bases. Therefore, one can make a Hamiltonian reduction at some central level of the moment map. Formally it amounts to imposing the constraints:

$$U^{r-1}V^{r-1} = \zeta^r \mathbf{Id}_{\mathscr{E}^r} \ ,$$

$$U^{i-1}V^{i-1} - V^i U^i = \zeta^i \mathbf{Id}_{\mathscr{E}^i}$$

for $i = 1,\ldots,r-1$. Here the complex numbers ζ^i are related to the eigenvalues λ_i of the matrix p via:

$$\lambda_i - \lambda_{i-1} = \zeta^i, \lambda^{-1} = 0 \ ,$$

the multiplicity of the eigenvalue λ^i equals $d^i - d^{i+1}$.

Finally, $p = V^0 U^0 + \zeta^0 \mathbf{Id}_{\mathscr{E}^0}$.

The flag

$$\mathscr{F}^r \subset \cdots \subset \mathscr{F}^0 = \mathscr{E}^0$$

is constructed as:

$$\mathscr{F}^i = Im(V^0 \cdots V^{i-1}) : \mathscr{E}^i \to \mathscr{E}^0 \ .$$

Now for each point x_a the orbit of p_a can be represented with the help of the set of matrices U_a^i, V_a^i and the numbers ζ_a^i such that $p_a = V_a^0 U_a^0 + \zeta_a^0 \mathbf{Id}$. Here a is a multiindex, including the numbers of the components of curves and the numbers of the points there.

Then for $i \neq j$ we could replace

$$(T_a)_{ij}(T_a)_{ji} \to Tr_{\mathscr{E}_a^1} S_i^a S_j^a \ ,$$

$$(S_i^a)_{\gamma\delta} = (U^a)_i^{\gamma} (V^a)_{\delta}^i \ . \tag{4.13}$$

Hence, for the sphere with one double point and one extra puncture and for the elliptic curve with one puncture we get precisely the *Spin Generalization of Calogero–Moser–Sutherland Model.*

If the number of punctures ($\#S$) is greater than one, then the representation of the coupling as the product of spins of the type, just described, is not convenient. For the products $p_a p_b$ with $a \neq b$ there is no such interpretation. Therefore, S_I operators in the Gaudin model as we described it in the Introduction are just the matrices p_a.

The global consequence of this presentation is that one can view the moduli space of bundles over the degenerate curve as a family of products over the set of components of the curve of the moduli of bundles with parabolic structures. The restriction on the parabolic structures at the glued points is that the weights ζ^i and dimensions d^i should coincide for both components, glued at the point.

5. Action-Angle Variables

We first recall the construction of Hitchin in the case of the compact curve of genus $g > 2$.

Given a point in the moduli space of Higgs bundles one can construct a *spectral curve* $S \subset \mathbb{P}(T^*\Sigma \oplus 1)$:

$$\mathcal{R}(x, \lambda) = det(\phi(x) - \lambda)\,,$$

where λ is a linear coordinate on the fiber of the cotangent bundle $T^*\Sigma$. This curve is well-defined, since the equation, which defines it, is gauge invariant.

The curve S is an N-sheeted ramified covering of Σ, its genus can be computed by the adjunction formula or using Riemann–Hurwitz theorem,

$$g(S) = N^2(g-1) + 1\,,$$

which agrees with the dimension of the moduli space of stable bundles over Σ. Denote by p the projection $S \to \Sigma$.

Given a stable bundle $\mathcal{E}$ over Σ we can pull it back onto S. There is a line subbundle $\mathcal{L} \subset p^*\mathcal{E}$, whose fiber at a generic point (x, λ) is an eigenspace of $\phi(x)$ with the eigenvalue λ. Conversely, given a line bundle $\mathcal{L}$ on Σ, one can take its direct image, which (again at a generic point) is defined as

$$\mathcal{E}_x = \bigoplus_{y \in p^{-1}(x)} \mathcal{L}_y\,.$$

Therefore, under the flow, generated by the Hitchin Hamiltonians, the $\mathcal{L}$ changes. It can be shown that these flows extend to the linear commuting vector fields on the Jacobian $Jac(S)$ of S.

Thus, the linear coordinates on $Jac(S)$ are the coordinates of the angle-type, whereas the integrals of λ over the corresponding cycles in S give the action variables. The construction of the covering spectral curve and abelianization of the problem resembles Knizhnik's idea [K] of replacing the correlators of the analytic fields on the covering of the Riemann surface by the correlators on the underlying Riemann surface with the insertions of additional vertex operators. Let us also remark that a quite analogous construction was invented by Krichever in [Kr] in connection to the elliptic Calogero–Moser System.

5.1. Degeneration of the Spectral Curve. We shall adopt the same definition of the spectral curve in the case of degenerate Σ.

Obviously, the normalization of S can be also decomposed as the disjoint union of the components S_α, labeled as the components Σ_α and S_α covers Σ_α with some fixed branching at the points x^i_α. Indeed, the behavior of ϕ_α near the point x^i_α is known, since the residue is known. Let us fix the conjugacy class of $p^{ij}_{\alpha\beta}$. Suppose that it has k^i_1 eigenvalues of multiplicity 1, k^i_2 eigenvalues of multiplicity 2, and so on. Since near the point x^i_α ϕ_α behaves like:

$$\phi_\alpha(x) \sim \frac{p^{ij}_{\alpha\beta}}{x - x^i_\alpha}$$

for appropriate j and β, then λ behaves like

$$\lambda_m(x) \sim \frac{p_m}{x - x^i_\alpha},$$

where p_m is the m^{th} eigenvalue of $p^{ij}_{\alpha\beta}$. Following [BBKT], we can find

$$N^{ij}_{\alpha\beta} = \sum_{m=1}^{\infty} k^i_m$$

points $P_1, \dots, P_{N^{ij}_{\alpha\beta}}$ above x^i_α, such that the local parameters $Z_1, \dots, Z_{N^{ij}_{\alpha\beta}}$ near them are defined:

$$Z_m = \lambda_m(x) - \frac{p_m}{x - x^i_\alpha}.$$

The discriminant Δ_α of ϕ_α is a meromorphic $N(N-1)$-differential on Σ_α. At each point x^i_α it has a pole of the order

$$o^i_\alpha = N^2 - \sum_m k^i_m m^2.$$

The zeroes of Δ_α determine the branching points of the covering

$$S_\alpha \to \Sigma_\alpha.$$

The number of the branching points equals, therefore,

$$2N(N-1)(g(\Sigma_\alpha) - 1) + \sum_i o^i_\alpha.$$

The genus of S_α can be computed with the help of the Riemann–Hurwitz formula, which gives:

$$g(S_\alpha) = 1 + N^2(g-1) + \frac{1}{2}\sum_i o^i_\alpha.$$

Now the Hamiltonian flow due to our Hamiltonians produces a motion of the line bundle over S_α and it covers the Jacobian of the completed curve $\bar{S}_\alpha$, therefore, the coordinates of the particles will be determined by the same equation:

$$\theta\left(\sum_i U^i t_i + Z_0\right),$$

as in the simplest one-punctured case. Here θ is a θ-function on the Jacobian of $\bar{S}_\alpha$, and $\vec{U}$ defines an embedding of the moduli space of the holomorphic bundles over $\bar{\Sigma}_\alpha$ into the Jacobian, as we have described it.

The details of the reconstruction of all angle-type variables will be published elsewhere [GN3]. Remark that this problem was solved for a one-punctured elliptic curve for a specific orbit in [BBKT].

6. Formulas for General Case-Genus g Curve with L Punctures

In this section we consider only one component Σ of the stable curve. We assume that it has genus g and L punctures. We also assume, that Σ has no double points.

Using the formulas for the twisted meromorphic forms on the curve, quoted in [I], we can easily write down the formula for the solution of the main equation

$$[d''_A, \phi] + \sum_i p_i \delta^2(x_i) = 0 .$$

In order to do this, we choose the following coordinatization of the moduli space $\mathcal{N}$ of holomorphic bundles over Σ. Namely, over the open dense subset of $\mathcal{N}$ one can parameterize the holomorphic bundle by choosing a set of g twists: elements of the complex group G, assigned to the A-cycles of Σ. More precisely, let us fix the representatives a_k, $k = 1, \dots, g$, of the A-cycles and let $\tilde{\Sigma}$ be the surface Σ with the small neighborhoods of a_k removed. Topologically $\tilde{\Sigma}$ is a sphere with $2g$ holes.

The boundary of the neighborhood of a_k consists of two circles $a_k^{\pm}$. In order to glue back the surface Σ one has to attach the projective transformations γ_k, which map $a_k^+ \to a_k^-$. These transformations generate the Shottky group. On the sphere one can find such a gauge transformation h that

$$d''_A = h^{-1} \bar{\partial} h .$$

Obviously,

$$h(g_k(x))|_{a_k^-} = H_k(x) h(x)|_{a_k^+} ,$$

where H_k is a holomorphic G-valued function, defined in the vicinity of a_k^+. Generically one can find a constant representative of H_k (this is a Riemann–Hilbert problem).

Once such a gauge h transformation is chosen, the equation for ϕ can be restated in words as the following: find a meromorphic form on $\tilde{\Sigma}$, which satisfy the following requirements:

in the vicinity of x_i: $\phi \sim \frac{p_i}{x - x_i}$,
twisting: $\phi(\gamma_k(x)) d\gamma_k(x) = Ad_{H_k} \phi(x) dx$.

The answer can be conveniently written in terms of the Poincaré series ([I]): Introduce

$$\omega_k[x_0] \in \Omega^1(\mathbb{P}^1) \otimes End(Lie\, G) ,$$

$$\theta[x, x_0] \in \Omega^1(\mathbb{P}^1) \otimes End(Lie\, G) ,$$

where $x, x_0 \in \mathbb{P}^1$,

$$\omega_k[x_0]_z dz = \sum_{\gamma \in \Gamma} Ad(H_\gamma^{-1}) dlog \frac{\gamma(z) - \gamma_k(x_0)}{\gamma(z) - x_0} ,$$

$$\theta[x, x_0]_z dz = \sum_{\gamma \in \Gamma} Ad(H_\gamma^{-1}) dlog \frac{\gamma(z) - x}{\gamma(z) - x_0} ,$$

where the sum runs over all elements of the Shottky group (it is a free group with g generators – loops around B-cycles. One assigns to the "word"

$$\gamma = b_{i_1}^{n_1} b_{i_2}^{n_2} \cdots b_{i_k}^{n_k}$$

an element of G:

$$H_\gamma = H_{i_1}^{n_1} H_{i_2}^{n_2} \cdots H_{i_k}^{n_k}) .$$

In these formulas x_0 is an auxiliary point on the sphere.

Finally, the solution has the form:

$$\phi[x_0] = \sum_{k=1}^{g} \omega_k(w_k)[x_0] + \sum_{i=1}^{L} \theta(p_i)[x_i, x_0] .$$

The momenta w_k are defined from the condition:

$$\int_{a_k} \phi = w_k .$$

The final remark concerns the vanishing of the residue at the point x_0:

$$\sum_k Ad(H_k^{-1})(w_k) - w_k + \sum_i p_i = 0 . \tag{6.1}$$

This equation we met in the degenerate form in Sect. 4.1. It has the meaning of the moment map for the action of G on the product of g copies of T^*G and the coadjoint orbits of p_i.

An easy computation shows that the reduced symplectic form is the result of the Hamiltonian reduction with respect to the natural action of the group G with moment (6.1) of the sum of the symplectic forms on the orbits, attached to the points x_i and of the g copies of the Liouville form on the T^*G, where the momentum for H_k is w_k.

7. Applications to Quantization

It is straightforward to quantize our models. When the conjugacy classes of $p_{\alpha\beta}^{ij}$ are fixed, their quantum counterparts become simply the generators of the group, acting in the corresponding representations of G. The condition on the residues of ϕ_α and ϕ_β at the double point gets translated to the fact that the representations, sitting at the points x_α^i and x_β^j, belonging to one double point, are dual to each other.

The pinched handle corresponds to the regular representation of the group, and the corresponding generators $p_{\alpha\alpha}^{ij}$ and $p_{\alpha\alpha}^{ji}$ are left- and right-invariant vector fields on the group.

Then the Schrödinger equations for the wavefunctions coincide with the critical level Knizhnik–Zamolodchikov–Bernard equations [KZ, Be, Lo, EK1, EFK]. The result of the quantization should follow from the degeneration of the Beilinson–Drinfeld construction [Beil].

Also, it would be nice to realize the meaning of the generalized KZ equations of [Ch] along the lines of our approach. As far as it seems now, these equations are inspired by the occasional fact that the Hamiltonians we have written for the punctured elliptic curve are almost symmetric under the exchange: $z_i \leftrightarrow x_a$.

Finally, note that using the results of [I] one can easily write down the quadratic Hamiltonians for an arbitrary curve (unfortunately, at the moment only in terms of the covering of the open dense subset of the actual phase space), while [FV] allows one to get the expression for the wave-functions of the elliptic Gaudin model in terms of the solutions of the Bethe Ansatz-like equations.

When the paper was completed we were notified about the recent paper by B. Enriquez and V. Rubtsov [ER] on a related subject. We would like to thank the authors of [ER] for their comments.

References

[BBKT] Babelon, O., Billey, E., Krichever, I., Talon, M.: Spin generalization of the Calogero–Moser system and the Matrix KP equation. hepth/9411160

[Beil] Beilinson, A., Drinfeld, V.: Quantization of Hitchin's fibration and Langlands program A. Beilinson's lectures at IAS, Fall 1994

[Be] Bernard, D.: Nucl. Phys. **B303**, 77 (1988), **B309**, 14 (1988)

[C] Calogero, F.: J. Math. Phys. **12**, 419 (1971)

[Ch] Cherednik, I.: Difference-elliptic operators and root systems. hep-th/9410188

[EF] Etingof, P , Frenkel, I.: Central extensions of current groups in two dimensions. hepth/9303047

[EK1] Etingof, P.: Quantum integrable systems and representations of Lie algebras hepth/9311132

[EK2] Etingof, P., Kirillov, A.Jr.: On the affine analogue of Jack's and Macdonald's polynomials. Yale preprint, 1994

[EFK] Etingof, P , Frenkel, I., Kirillov-Jr.A.: Spherical functions on affine Lie groups. Yale preprint, 1994

[FV] Felder, G., Varchenko, A.: Integral formula for the solutions of the elliptic KZB equations Int. Math. Res. Notices **5**, 222–233 (1995)

[Ga] Garnier, R.: Sur une classe de systemès differentiels Abeliéns déduits de la theorie des équations lineáres. Rend. del Circ. Matematice Di Palermo, **43**, **4** (1919)

[GN1] Gorsky, A., Nekrasov, N.: Elliptic Calogero–Moser System from Two Dimensional Current Algebra hepth/9401021

[GN2] Gorsky, A., Nekrasov, N : Relativistic Calogero–Moser model as gauged WZW theory Nucl. Phys **B436**, 582–608 (1995)

[GN3] Nekrasov, N., Gorsky, A.: Integrable systems on moduli spaces. In preparation

[FF] Feigin, B., Frenkel, E., Reshetikhin, N.: Gaudin Model, Critical Level and Bethe Ansatz. Commun. Math. Phys **166**, 27–62 (1995)

[G] Gaudin, M : J Physique **37**, 1087–1098 (1976)

[H] Hitchin, N.: Duke Math. J. **54**, No. 1 (1987)

[I] Ivanov, D.: Knizhnik–Zamolodchikov–Bernard equations on Riemann surfaces. hep-th/9410091

[KKS] Kazhdan, D., Kostant, B., Sternberg, S.: Comm. on Pure and Appl. Math **XXXI**, 481–507 (1978)

[KZ] Knizhnik, V., Zamolodchikov, A.: Nucl. Phys. **B247**, 83 (1984)

[K] Knizhnik, V.: Analytic fields on Riemann surfaces. Commun. Math. Phys. **112**, 567 (1987)

[Lo] Losev, A.: Coset construction and Bernard Equations. CERN-TH.6215/91

[OP] Olshanetsky, M., Perelomov, A.: Phys. Rep. **71**, 313 (1981)

[M] Moser, J.: Adv. Math. **16**, 197–220 (1975)

[Kr] Krichever, I.: Funk. Anal. and Appl. **12**, 1, 76–78 (1978), **14**, 282–290, (1980)

[RS] Ruijsenaars, S N.M., Schneider, H.: Ann. Phys. **170**, 370–405 (1986)

[R] Ruijsenaars, S.: Commun. Math. Phys. **110**, 191–213 (1987)

[S] Sutherland, B.: Phys. Rev. **A5**, 1372–1376 (1972)

[Sch] Schlesinger, L : Über eien Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten Journal für die reine und angewandte Mathematik, Bd. **CXLI**, 96-145 (1912)
[V] Verlinde, E.: Nucl. Phys **B300**, 360 (1988)
[ER] Enriquez, B , Rubtsov, V.: Hitchin systems, higher Gaudin operators and r-matrices. alg-geom/9503010

Communicated by G Felder