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Theory of Tensor Invariants of Integrable Hamiltonian Systems. I. Incompatible Poisson Structures

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Abstract: This paper develops a new theory of tensor invariants of a completely integrable non-degenerate Hamiltonian system on a smooth manifold M^n. The central objects in this theory are supplementary invariant Poisson structures P_c which are incompatible with the original Poisson structure P_1 for this Hamiltonian system. A complete classification of invariant Poisson structures is derived in a neighbourhood of an invariant toroidal domain. This classification resolves the well-known Inverse Problem that was brought into prominence by Magri's 1978 paper devoted to the theory of compatible Poisson structures. Applications connected with the KAM theory, with the Kepler problem, with the basic integrable problem of celestial mechanics, and with the harmonic oscillator are pointed out. A cohomology is defined for dynamical systems on smooth manifolds. The physically motivated concepts of dynamical compatibility and strong dynamical compatibility of pairs of Poisson structures are introduced to study the diversity of pairs of Poisson structures incompatible in Magri's sense. It is proved that if a dynamical system V preserves two strongly dynamically compatible Poisson structures P_1 and P_2 in a general position then this system is completely integrable. Such a system V generates a hierarchy of integrable dynamical systems which in general are not Hamiltonian neither with respect to P_1 nor with respect to P_2. Necessary conditions for dynamical compatibility and for strong dynamical compatibility are derived which connect these global properties with new local invariants of an arbitrary pair of incompatible Poisson structures.

Contents

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1. Introduction

I. In his 1978 paper [34] Magri proved, using the Lenard scheme presented in [23], a general theorem that states that a dynamical system or system of partial differential equations that preserves two compatible non-degenerate Poisson structures (the bi-Hamiltonian system) possesses a sequence of first integrals in involution. If this sequence contains sufficiently many functionally independent first integrals then Liouville's Classical Theorem [32] implies the complete integrability of the bi-Hamiltonian system. Since then, more than one hundred papers and several books have been published devoted to the investigation of the diverse properties of compatible pairs of Poisson structures and bi-Hamiltonian systems. Reviews of these papers and their extended bibliographies are contained in Dorfman's monograph [15] and in Olver's monograph [48].

One of the well-known unsolved problems in this area is

The Inverse Problem. *To classify all invariant Poisson structures for a completely integrable non-degenerate Hamiltonian system on a manifold M^n, $n = 2k$, with a non-degenerate Poisson structure P_1. Are these invariant Poisson structures necessarily compatible with P_1?*

In the present paper we solve this problem. We derive the general and previously unknown formula

$$\omega_c = \mathrm{d}\left(\frac{\partial B(J)}{\partial J_\alpha}\right) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha(I) \wedge \mathrm{d}I_\alpha\,,$$

$$J_\alpha = \frac{\partial H(I)}{\partial I_\alpha}, \quad \alpha = 1,\dots,k \tag{1.1}$$

in the action-angle coordinates $I_1,\dots,I_k$, $\varphi_1,\dots,\varphi_k$. Here $B(J)$ and $f_\alpha(I)$ are arbitrary smooth functions of k variables. The formula (1.1) presents a complete classification of all invariant closed 2-forms ω_c and invariant non-degenerate Poisson structures $P_c = \omega_c^{-1}$ for an arbitrary completely integrable non-degenerate Hamiltonian system with the Hamiltonian function $H(I)$, provided that its invariant submanifolds are compact.

In general the constructed Poisson structures P_c are incompatible with the original Poisson structure $P_1 = \omega_1^{-1}$, $\omega_1 = \mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\alpha$. The original definition of compatibility by Magri [34] states: Two Poisson structures P_1 and P_2 are compatible if their sum $P_1 + P_2$ is also a Poisson structure. This definition is equivalent to the condition that the Schouten bracket $[P_1, P_2]$ vanishes. The theory of compatible Poisson structures is closely connected with Fuchssteiner's theory of hereditary operators [20, 21]. Later, Gelfand and Dorfman in [24, 25] and Magri and Morosi in [36] proved that the compatibility of the non-degenerate Poisson structures is equivalent to the condition that the Nijenhuis tensor N_A [44] vanishes where A is the $(1,1)$ tensor $A = P_1 P_2^{-1}$. Therefore, we prove the incompatibility of the Poisson structures P_c and P_1 by a direct calculation of the non-zero components N^i_{Ajl} of the Nijenhuis tensor in the action-angle coordinates.

Olver in [47] and Turiel in [53] investigated canonical forms of compatible pairs of Poisson structures and integrable systems which preserve them. Ten Eikelder [55], Brouzet [8, 9], Brozet, Molino and Turiel [10], and Fernandes [19] studied the necessary and sufficient conditions for the existence of a compatible invariant Poisson structure for a given completely integrable Hamiltonian system. These compatible invariant Poisson structures correspond to solutions of an overdetermined third-order system of partial differential equations for functions $B(J)$ and $H(I)$ (1.1). This overdetermined system is equivalent to the compatibility condition $N_A = 0$, see Sect. 11.

The solution of the Inverse Problem depends upon whether the invariant submanifolds of the integrable Hamiltonian system are compact or non-compact. In view of the Liouville Theorem [3, 32] almost all of these submanifolds are tori $\mathbb{T}^k$ or toroidal cylinders $\mathbb{T}^m \times \mathbb{R}^{k-m}$, $0 \leqq m < k$, respectively. For the compact case we prove in Theorem 1 that the formula (1.1) presents a complete classification of the non-degenerate Poisson structures P_c which are invariant with respect to the completely integrable non-degenerate Hamiltonian system with Hamiltonian H.

In Theorem 14 a second proof of the complete classification of the non-degenerate invariant Poisson structures is presented along with the classification of degenerate invariant Poisson structures.

For the non-compact case ($\mathbb{T}^m \times \mathbb{R}^{k-m}$) we present in Theorem 2 larger families of invariant closed 2-forms ω_c which include all 2-forms (1.1) and depend upon additional arbitrary functions.

In Sect. 3 we introduce a cohomology for dynamical systems on smooth manifolds. This cohomology $H^*(V, M^n)$ is an invariant that characterizes the global properties of the dynamical system V on the manifold M^n. We prove that the infinite-dimensionality of the cohomologies $H^2(V, M^{2k})$ and $H^4(V, M^{2k})$ is the necessary condition for the non-degenerate integrability of the dynamical system V on the manifold M^{2k}.

II. The second well-known unsolved problem in the theory of compatible Poisson structures is

The Stability Problem. *Let a Hamiltonian system be completely integrable and non-degenerate with respect to a Poisson structure P_1. Let us assume that this system also preserves a second Poisson structure P_2 that is compatible with P_1. Is the property of compatibility of P_2 with P_1 stable?*

In Theorem 7, we prove that the compatibility property is unstable. Using the key formula (1.1) and a method of "toroidal surgeries" we construct a continuum of invariant Poisson structures P_C in any neighbourhood of P_2 which are incompatible with P_1.

In Sects. 5 and 6 we point out applications connected with the Kepler problem, with the basic integrable problem of celestial mechanics, and with the harmonic oscillator. We present explicit formulae for a continuum of invariant symplectic and Poisson structures for these problems. In general, these Poisson structures are incompatible with the original Poisson structure P_1. However, the same formulae contain a continuum of compatible Poisson structures as well. The latter are unstable in a sense that they become incompatible with P_1 after arbitrarily small perturbations inside the general family of invariant Poisson structures.

III. These results show that the notion of the compatibility of Poisson structures and its counterpart, incompatibility, are not conceptionally adequate for a good insight into the diversity of pairs of Poisson structure. Therefore we introduce the following

Definition 1. *Two Poisson structures P_1 and P_2 on a manifold M^n are called dynamically compatible (D.C.) if there exists a dynamical system V on M^n that preserves both of them and such that the set $S \subset M^n$ of its critical points has* $\dim S \leqq n-1$.

In general, two dynamically compatible Poisson structures P_1 and P_2 are not compatible in Magri's sense. This is the case if the corresponding Schouten bracket $[P_1, P_2]$ is not equal to zero.

Definition 2. *Two Poisson structures P_1 and P_2 on a manifold M^n, $n = 2k$, are called strongly dynamically compatible (S.D.C.) if there exists a dynamical system V that preserves both of them and is completely integrable in the Liouville sense non-degenerate Hamiltonian system with respect to some non-degenerate Poisson structure P on the manifold M^n, and such that its invariant submanifolds are compact.*

In this case, Theorem 1 proves that if the invariant Poisson structure P_1 (or P_2) is non-degenerate then the completely integrable dynamical system V is also completely integrable and non-degenerate with respect to the Poisson structure P_1 (or P_2). Theorem 1 also implies that all constructed invariant non-degenerate Poisson structures $P_c = \omega_c^{-1}$ (1.1) are mutually dynamically compatible in the strong sense.

IV. Until now all applications of bi-Hamiltonian systems were limited to the theory of integrable systems. In this paper we develop new applications connected with the Kolmogorov–Arnold–Moser (KAM) theory. This theory studies Hamiltonian perturbations of integrable Hamiltonian systems of the form

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon P_1^{i\alpha} H_{,\alpha} \, . \tag{1.2}$$

For $\varepsilon = 0$, system (1.2) is assumed to be completely integrable and non-degenerate, with compact invariant submanifolds. The classical KAM results [2, 26, 27, 41] on dynamics of the Hamiltonian systems (1.2) for $|\varepsilon| \ll 1$ lead to the following problem.

Admissible Perturbations Problem. *What non-Hamiltonian perturbations*

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon V^i(x) \tag{1.3}$$

possess the same dynamical properties for $|\varepsilon| \ll 1$ *as small Hamiltonian perturbations in* KAM *theory*?

We call such perturbations admissible. In Theorem 5, we prove that all perturbations

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon P_c^{i\alpha} H_{,\alpha} \tag{1.4}$$

are admissible. Here $H(x)$ is an arbitrary smooth function and P_c is an arbitrary non-degenerate Poisson structure that is strongly dynamically compatible with P_1 and is invariant with respect to the unperturbed integrable Hamiltonian system (1.2) for $\varepsilon = 0$. In general, perturbations (1.4) are not Hamiltonian with respect to the original Poisson structure P_1. If the Poisson structure P_c is compatible with P_1 then all perturbations

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon \left(\sum_{m=-l}^{l} a_m A_c^m \right)_\alpha^i P_1^{\alpha\beta} H_{,\beta} \tag{1.5}$$

are admissible for $a_m = \text{const}$, where $A_c = P_1 P_c^{-1}$ is the recursion operator.

These results prove that KAM theory is applicable not only to the small Hamiltonian perturbations (1.2) but also to the rich families of non-Hamiltonian perturbations (1.4) and (1.5). The family of admissible non-Hamiltonian perturbations (1.4) depends upon the $k+1$ arbitrary functions $B(J), f_1(I), \ldots, f_k(I)$ of k variables and one arbitrary function $H(x)$ of $2k$ variables.

V. Any dynamical system that preserves two non-degenerate Poisson structures P_1 and P_2 also preserves the Schouten bracket $[P_1, P_2]$, the $(1,1)$ tensor $A = P_1 P_2^{-1}$, the Nijenhuis tensor N_A, and all tensors which can be constructed from P_1, P_2, $[P_1, P_2]$, A and N_A. Therefore, if P_1 and P_2 are incompatible, the Schouten bracket $[P_1, P_2]$ and the Nijenhuis tensor N_A are not equal to zero, and hence the family of geometric objects which have to be preserved by the dynamical system is greater than that for the compatible case. Thus, one could expect that the family of dynamical systems which preserve two incompatible Poisson structures is smaller than that for the compatible ones.

In general this is true. Indeed, two incompatible Poisson structures will not generally admit any dynamical system preserving both of them. Two arbitrary compatible non-degenerate Poisson structures admit infinitely many bi-Hamiltonian systems which preserve them. However, between these two extreme cases there exists a rich diversity of dynamically compatible and strongly dynamically compatible Poisson structures with utterly different properties. Thus we arrive at the following problem.

The Integrability Problem. *Assume that two non-degenerate Poisson structures* P_1 *and* P_2 *are strongly dynamically compatible on a manifold* M^{2k} *and that the recursion operator* $A = P_1 P_2^{-1}$ *has k functionally independent eigenvalues. Let V be an arbitrary dynamical system on* M^{2k} *that preserves* P_1 *and* P_2. *Is system V integrable*?

In Theorem 10 we prove that dynamical system V is completely integrable with respect to both Poisson structures P_1 and P_2. The proof of Theorem 8 does not use the Lenard scheme [23, 34] that is not applicable for two incompatible Poisson structures. The well-known Lenard recursion relations [23] are not true for the two incompatible Poisson structures considered and therefore the Lenard scheme cannot be applied. The proof is based on the complete classification of all invariant non-degenerate Poisson structures obtained in Theorem 1.

VI. For two compatible non-degenerate Poisson structures P_1 and P_2 in the general position any bi-Hamiltonian system

$$\dot{x}^i = P_1^{i\alpha} H_{1,\alpha} = P_2^{i\alpha} H_{2,\alpha} = V^i \tag{1.6}$$

generates a hierarchy of completely integrable bi-Hamiltonian systems which have the form [35]

$$\dot{x}^i = (A^m V)^i , \tag{1.7}$$

where $A = P_1 P_2^{-1}$ and m is an arbitrary integer. The reasonable question about the role of the compatibility condition for this construction leads to the following.

Integrable Hierarchies Problem. *Let V be an arbitrary dynamical system* (1.6) *that preserves two strongly dynamically compatible Poisson structures P_1 and P_2 and let $A = P_1 P_2^{-1}$. Are the dynamical systems* (1.7) *integrable?*

We show that the compatibility condition is not necessary here and that the problem has positive solution if the recursion operator $A = P_1 P_2^{-1}$ has k functionally independent eigenvalues. In Theorem 11 we prove that even more general dynamical systems

$$\dot{x}^i = \left(\sum_{m=-l}^{l} a_m(x) A^m V \right)^i \tag{1.8}$$

are completely integrable. Here $a_m(x)$ are arbitrary smooth functions of the eigenvalues of the recursion operator A.

For the incompatible case, systems (1.7) for $|m| > 1$ in general do not preserve the Poisson structures P_1 and P_2, in contrast with the compatible case. The (2,0) skew tensors $A^m P_1$ and $A^{-m} P_2$ in general are not Poisson structures for the incompatible P_1 and P_2. The proof of Theorem 11 is entirely different from that for the compatible case [35] and is based on the proof of Theorem 10.

VII. The following problem naturally arises in the course of the investigation of the geometric and algebraic properties of pairs of dynamically compatible and strongly dynamically compatible Poisson structures.

Necessary Conditions Problem. *What are the necessary conditions for dynamical compatibility and for strong dynamical compatibility of two incompatible Poisson structures?*

In Sects. 11 and 12 we present several necessary conditions in terms of the Nijenhuis tensor N_A and other geometric objects assuming the Poisson structures P_1 and P_2 are non-degenerate. The necessary conditions derived are effective in studying concrete problems because they can be verified by direct calculations for arbitrary pairs of Poisson structures.

In Sect. 15 we introduce a distribution $\mathscr{B} \subset T(M^n)$ that is uniquely determined by two arbitrary Poisson structures P_1 and P_2. We derive the following necessary condition for strong dynamical compatibility of the two Poisson structures:

$$\dim \mathscr{B}_x \geqq \frac{n}{2} \tag{1.9}$$

for all points $x \in M^n$, $n = 2k$.

In Sect. 16 we define new invariants of two arbitrary Poisson structures P_1 and P_2. These Poisson structures are determined on a manifold M^n of an arbitrary dimension $n = 2k$ or $n = 2k + 1$ and can both be degenerate. These invariants are the smooth maps

$$f : M^n \longrightarrow RP^{N(n)} \tag{1.10}$$

of the manifold M^n into the real projective spaces RP^N. The maps f are first integrals of any dynamical system that preserves the two Poisson structures P_1 and P_2. The necessary condition for dynamical compatibility has the simple form

$$\operatorname{rank} df(x) \leqq n - 1 \tag{1.11}$$

at all points $x \in M^n$ where the maps f are defined.

2. Complete Classification of Invariant Non-Degenerate Poisson Structures

I. Let P_1^{ij} be a non-degenerate Poisson structure on a manifold M^n, $n = 2k$. A Hamiltonian system

$$\dot{x}^i = P_1^{i\alpha} H_{,\alpha}, \qquad H_{,\alpha} = \partial H / \partial x^\alpha \tag{2.1}$$

is called completely integrable in Liouville's sense if it has $k = n/2$ independent involutive first integrals $F_1(x), \ldots, F_k(x)$:

$$\{F_j, F_l\} = P_1^{\alpha\beta} F_{j,\alpha} F_{l,\beta} = 0 \,. \tag{2.2}$$

The summation with respect to the repeated indices is understood everywhere in this paper.

The Liouville Theorem [1,3,32] implies that almost all points of the manifold M^n (excluding a set $S \subset M^n$, $\dim S \leqq n - 1$) are covered by a system of open toroidal domains $\mathcal{O}_m \subset M^n$ with the action-angle coordinates $I_1, \ldots, I_k$, $\varphi_1, \ldots, \varphi_k$. In these coordinates the completely integrable system (2.1) has the form

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = \frac{\partial H(I)}{\partial I_j} \,. \tag{2.3}$$

The symplectic structure ω_1 has the canonical form $\omega_1 = \mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\alpha$. The Hamiltonian system (2.3) preserves the symplectic structure ω_1 and the Poisson structure $P_1 = \omega_1^{-1} : L_V \omega_1 = 0$, $L_V P_1 = 0$, where L_V is the Lie derivative with respect to the flow (2.3).

The action coordinates $I_1, \ldots, I_k$ are defined in a ball

$$B_r : \sum_{j=1}^{k} (I_j - I_{j0})^2 < r^2 \,. \tag{2.4}$$

The angle coordinates $\varphi_1, \ldots, \varphi_k$ run over a torus $\mathbb{T}^k$, $0 \leqq \varphi_j \leqq 2\pi$, in the compact case or over a toroidal cylinder $\mathbb{T}^m \times \mathbb{R}^{k-m}$, $0 \leqq m < k$ if the manifold $I_j(x) = I_{j0}$ is non-compact.

The set $S \subset M^n$ that is not covered by the system of open toroidal domains $\mathcal{O}_m$ is invariant with respect to the Hamiltonian system (2.1). This set contains all critical points of (2.1) and all homoclinic and heteroclinic trajectories.

II. First we consider a completely integrable Hamiltonian system (2.1) such that the submanifolds of constant level of the k involutive first integrals are compact. Almost all these invariant submanifolds are tori $\mathbb{T}^k$:

$$\mathbb{T}^k : I_1 = c_1, \ldots, \qquad I_k = c_k, \qquad 0 \leqq \varphi_i \leqq 2\pi \,. \tag{2.5}$$

The trajectories of dynamical system (2.3) are everywhere dense on the torus (2.5) if and only if the k numbers $J_i = \partial H / \partial I_i$ are incommensurable over the integers. This means that for arbitrary integers $m_1, \ldots, m_k$, we have

$$m_1 \frac{\partial H(I)}{\partial I_1} + \cdots + m_k \frac{\partial H(I)}{\partial I_k} \neq 0 \,. \tag{2.6}$$

We call a completely integrable system (2.1) $\mathbb{T}^k$-dense if condition (2.6) is met for almost all tori (2.5); or in other words, if the trajectories of system (2.1) are everywhere dense on almost all tori (2.5). This property is invariant and therefore it does not depend upon a choice of concrete action-angle coordinates.

The completely integrable Hamiltonian system (2.1), (2.3) is called non-degenerate if the Kolmogorov condition [26, 27] for the Hessian matrix

$$\det \left\| \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta} \right\| \neq 0 \tag{2.7}$$

is met almost everywhere in the action-angle coordinates

$$I_1, \ldots, I_k, \qquad \varphi_1, \ldots, \varphi_k, \qquad \varphi_i = \varphi_i \operatorname{mod}(2\pi) \,. \tag{2.8}$$

Obviously, any non-degenerate system (2.1), (2.3) is $\mathbb{T}^k$-dense.

Trajectories of the completely integrable non-degenerate Hamiltonian system (2.1) are everywhere dense on almost all tori (2.5). Therefore any smooth first integral $F(I_j, \varphi_j)$ of the system (2.1) is constant on all tori $\mathbb{T}^k$ and hence any first integral F is a function of the action variables only:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 0 \implies F = F(I_1, \ldots, I_k) \,. \tag{2.9}$$

To solve the Inverse Problem we investigate all closed 2-forms ω which are invariant with respect to the system (2.1).

Theorem 1. 1) *In the toroidal domain $\mathcal{O} \subset M^n$ defined by conditions* (2.4) *and* (2.5) *a closed 2-form ω is invariant with respect to the completely integrable non-degenerate Hamiltonian system* (2.1), (2.3) *having compact invariant submanifolds* (2.5) *if and only if it has the form*

$$\omega_c = \mathrm{d} \left(\frac{\partial B(J)}{\partial J_\alpha} \right) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d} f_\alpha(I) \wedge \mathrm{d} I_\alpha \,, \tag{2.10}$$

where $B(J_1, \ldots, J_k)$ and $f_\alpha(I_1, \ldots, I_k)$ are arbitrary functions of k arguments and $J_\alpha = J_\alpha(I_1, \ldots, I_k)$ are functions

$$J_\alpha(I) = \frac{\partial H(I)}{\partial I_\alpha}, \quad \alpha = 1, \ldots, k \,. \tag{2.11}$$

2) *The 2-form* ω_c *is non-degenerate if and only if the two non-degeneracy conditions*

$$\det\left\|\frac{\partial^2 B(J)}{\partial J_\alpha \partial J_\beta}\right\| \neq 0, \qquad \det\left\|\frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta}\right\| \neq 0 \tag{2.12}$$

are met. Then the system (2.1) *has the supplementary Hamiltonian structure*

$$\dot{x}^i = P_c^{i\alpha} H_{c,\alpha}, \qquad P_c = \omega_c^{-1}, \tag{2.13}$$

where the Hamiltonian function $H_c(\tilde{J})$ *is the Legendre transform of the function* $B(J)$:

$$H_c(\tilde{J}) = J_\alpha \frac{\partial B(J)}{\partial J_\alpha} - B(J), \qquad \tilde{J}_\alpha = \frac{\partial B(J)}{\partial J_\alpha}. \tag{2.14}$$

3) *The symplectic structure* (2.10) *has canonical form*

$$\omega_c = \mathrm{d}\tilde{J}_\alpha \wedge \mathrm{d}\tilde{\varphi}_\alpha, \tag{2.15}$$

where functions $\tilde{J}_\alpha$ *and* $\tilde{\varphi}_\alpha$ *are defined by the formulae*

$$\tilde{J}_\alpha = \frac{\partial B(J)}{\partial J_\alpha}, \tag{2.16}$$

$$\tilde{\varphi}_\alpha = \varphi_\alpha - f_\beta(I)\frac{\partial I_\beta}{\partial \tilde{J}_\alpha}, \qquad \tilde{\varphi}_\alpha = \tilde{\varphi}_\alpha \bmod(2\pi). \tag{2.17}$$

The new variables $\tilde{J}_\alpha, \tilde{\varphi}_\alpha$ *are the action-angle coordinates for the Hamiltonian system* (2.13) *with respect to the symplectic structure* ω_c (2.10). *In the action-angle coordinates* $\tilde{J}_\alpha, \tilde{\varphi}_\alpha$, *the Hamiltonian system* (2.13) *is non-degenerate*:

$$\det\left\|\frac{\partial^2 H_c(\tilde{J})}{\partial \tilde{J}_\alpha \partial \tilde{J}_\beta}\right\| = \left(\det\left\|\frac{\partial^2 B(J)}{\partial J_\alpha \partial J_\beta}\right\|\right)^{-1} \neq 0. \tag{2.18}$$

4) *The action variables* $I_1, \dots, I_k$ *are in involution with respect to the Poisson structure* P_c:

$$\{I_\alpha, I_\beta\} = P_c^{jl} I_{\alpha,j} I_{\beta,l} = 0. \tag{2.19}$$

The Hamiltonian system (2.1), (2.13) *is completely integrable with respect to all invariant non-degenerate Poisson structures* P_c.

Proof. 1) Let us first prove that any 2-form ω_c (2.10) is preserved by the Hamiltonian system (2.1), (2.3). Using classical properties [49] of the Lie derivative $L_V\omega_c = \dot{\omega}_c$ with respect to the dynamical system (2.3) and substituting (2.11) we obtain

$$\dot{\omega}_c = \mathrm{d}\left(\frac{\partial B(J)}{\partial J_\alpha}\right) \wedge \mathrm{d}\left(\frac{\partial H}{\partial I_\alpha}\right) = \frac{\partial^2 B(J)}{\partial J_\alpha \partial J_\beta} \mathrm{d}J_\beta \wedge \mathrm{d}J_\alpha = 0. \tag{2.20}$$

Therefore all 2-forms ω_c (2.10) are invariant with respect to the completely integrable Hamiltonian system (2.1), (2.3).

Now we prove that any closed 2-form ω that is invariant with respect to the dynamical system (2.1), (2.3) has the form (2.10). In the action-angle coordinates

(2.8) any differential 2-form ω is defined by the expression

$$\omega = a_{\alpha\beta}(I,\varphi)\mathrm{d}I_\alpha \wedge \mathrm{d}I_\beta + b_{\alpha\beta}(I,\varphi)\mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\beta + c_{\alpha\beta}(I,\varphi)\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta\,,$$
$$a_{\alpha\beta}(I,\varphi) = -a_{\beta\alpha}(I,\varphi), \qquad c_{\alpha\beta}(I,\varphi) = -c_{\beta\alpha}(I,\varphi)\,. \tag{2.21}$$

Here $a_{\alpha\beta}(I,\varphi), b_{\alpha\beta}(I,\varphi)$ and $c_{\alpha\beta}(I,\varphi)$ are some smooth functions defined in the toroidal domain $\mathcal{O} \subset M^n$ corresponding to the action-angle coordinates (2.8).

The invariant closed 2-form (2.21) has to satisfy the two equations $\dot\omega = 0$, $\mathrm{d}\omega = 0$. The time derivative of the 2-form ω with respect to the system (2.3) has the form

$$\dot\omega = \dot a_{\alpha\beta}\mathrm{d}I_\alpha \wedge \mathrm{d}I_\beta + \dot b_{\alpha\beta}\mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\beta + \dot c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta$$
$$+ b_{\alpha\beta}\mathrm{d}I_\alpha \wedge \mathrm{d}\left(\frac{\partial H}{\partial I_\beta}\right) + c_{\alpha\beta}\mathrm{d}\left(\frac{\partial H}{\partial I_\alpha}\right) \wedge \mathrm{d}\varphi_\beta + c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\left(\frac{\partial H}{\partial I_\beta}\right)\,. \tag{2.22}$$

Therefore, the equation $\dot\omega = 0$ is equivalent to the system of equations

$$2\dot a_{\alpha\beta} = b_{\beta\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\alpha} - b_{\alpha\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\beta}, \qquad \dot b_{\alpha\beta} = 2c_{\beta\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\alpha}, \qquad \dot c_{\alpha\beta} = 0\,. \tag{2.23}$$

In view of the key property of first integrals (2.9) solutions to the linear triangular system (2.23) have the form

$$2a_{\alpha\beta}(t) = 2\tilde c_{\gamma\tau}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\alpha}\frac{\partial^2 H(I)}{\partial I_\tau \partial I_\beta}t^2 + \left(\tilde b_{\beta\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\alpha} - \tilde b_{\alpha\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\beta}\right)t + 2\tilde a_{\alpha\beta}(I)\,,$$
$$b_{\alpha\beta}(t) = 2\tilde c_{\beta\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\alpha}t + \tilde b_{\alpha\beta}(I), \qquad c_{\alpha\beta}(t) = \tilde c_{\alpha\beta}(I_1,\ldots,I_k)\,. \tag{2.24}$$

The components $a_{\alpha\beta}(I,\varphi)$, $b_{\alpha\beta}(I,\varphi)$ and $c_{\alpha\beta}(I,\varphi)$ of the smooth invariant differential 2-form (2.21) are bounded on any torus $\mathbb{T}^k$ (2.5). In view of the non-degeneracy condition (2.7) the exact solutions (2.24) are bounded for all t only if all functions $\tilde c_{\alpha\beta}(I_1,\ldots,I_k) = 0$. Hence using (2.24) and the fact that general trajectories of the system (2.3) are everywhere dense on the tori $\mathbb{T}^k$ we obtain that any invariant 2-form ω (2.21) has the form

$$\omega = \tilde a_{\alpha\beta}(I)\mathrm{d}I_\alpha \wedge \mathrm{d}I_\beta + \tilde b_{\alpha\beta}(I)\mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\beta\,. \tag{2.25}$$

For the 2-form (2.25) the equation $\mathrm{d}\omega = 0$ splits into the $k+1$ independent equations

$$\mathrm{d}(\tilde a_{\alpha\beta}(I)\mathrm{d}I_\alpha \wedge \mathrm{d}I_\beta) = 0, \qquad \mathrm{d}(\tilde b_{\alpha\beta}(I)\mathrm{d}I_\alpha) = 0, \qquad \alpha,\beta = 1,\ldots,k\,. \tag{2.26}$$

In view of Poincaré's Lemma these equations are equivalent to the equations

$$\tilde a_{\alpha\beta}(I)\mathrm{d}I_\alpha \wedge \mathrm{d}I_\beta = \mathrm{d}(f_\gamma(I)\mathrm{d}I_\gamma), \qquad \tilde b_{\alpha\beta}(I)\mathrm{d}I_\alpha = \mathrm{d}F_\beta(I)\,, \tag{2.27}$$

where $f_\gamma(I_1,\ldots,I_k)$ and $F_\beta(I_1,\ldots,I_k)$ are some smooth functions. Substituting formulae (2.27) into (2.25) one gets

$$\omega = \mathrm{d}F_\alpha(I) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha(I) \wedge \mathrm{d}I_\alpha\,. \tag{2.28}$$

Once more considering the equation $\dot{\omega} = 0$ we obtain

$$\dot{\omega} = \mathrm{d}F_\alpha(I) \wedge \mathrm{d}\left(\frac{\partial H}{\partial I_\alpha}\right) = \mathrm{d}\left(F_\alpha(I)\mathrm{d}\frac{\partial H}{\partial I_\alpha}\right) = 0\,. \tag{2.29}$$

Therefore, the Poincaré Lemma implies

$$F_\alpha(I)\mathrm{d}\left(\frac{\partial H}{\partial I_\alpha}\right) = \mathrm{d}B(I)\,, \tag{2.30}$$

where $B(I_1,\dots,I_k)$ is some smooth function. The Kolmogorov condition (2.7) ensures that functions $J_\alpha(I)$ (2.11) form a new system of coordinates in the space of action variables $I_1,\dots,I_k$. In these coordinates Eq. (2.30) yields

$$F_\alpha(I(J)) = \frac{\partial B(I(J))}{\partial J_\alpha}\,. \tag{2.31}$$

Therefore the 2-form (2.28) takes the form (2.10).

Thus we have proved that any invariant closed 2-form ω has form (2.10).

2) In the action-angle coordinates $I_1,\dots,I_k$, $\varphi_1,\dots,\varphi_k$ the closed 2-forms ω_1 and ω_c have the block structure

$$\omega_1 = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}, \qquad P_1 = \omega_1^{-1} = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}, \qquad \omega_c = \begin{pmatrix} \sigma & B \\ -B^t & 0 \end{pmatrix}, \tag{2.32}$$

where e, σ and B are $k \times k$ matrices with entries

$$e_{\alpha\beta} = \delta^\alpha_\beta, \qquad \sigma_{\alpha\beta} = f_{\beta,\alpha} - f_{\alpha,\beta}, \qquad B^\alpha_\beta = \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\gamma}\frac{\partial^2 B(J)}{\partial J_\gamma \partial J_\beta}\,. \tag{2.33}$$

The formulae (2.32) and (2.33) imply

$$\det \| \omega_c \| = \left(\det\left\|\frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta}\right\|\right)^2 \left(\det\left\|\frac{\partial^2 B(J)}{\partial J_\alpha \partial J_\beta}\right\|\right)^2\,. \tag{2.34}$$

This formula proves that the non-degeneracy condition

$$\det \| \omega_c \| \neq 0 \tag{2.35}$$

is equivalent to the two conditions (2.12).

The Poisson structure $P_c = \omega_c^{-1}$ has the block form

$$P_c = \begin{pmatrix} 0 & -(B^t)^{-1} \\ B^{-1} & B^{-1}\sigma(B^t)^{-1} \end{pmatrix}\,. \tag{2.36}$$

Partial derivatives of the function $H_c(I)$ (2.14) have the form

$$\frac{\partial H_c(I)}{\partial I_\alpha} = \frac{\partial H_c}{\partial J_\gamma}\frac{\partial J_\gamma}{\partial I_\alpha} = J_\beta \frac{\partial^2 B(J)}{\partial J_\beta \partial J_\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\alpha} = B^\alpha_\beta \frac{\partial H}{\partial I_\beta}\,. \tag{2.37}$$

Using the block structure (2.36) we obtain

$$(P_c \mathrm{d}H_c)^\alpha = 0, \qquad (P_c \mathrm{d}H_c)^{\alpha+k} = (B^{-1})^\alpha_\gamma B^\gamma_\beta \frac{\partial H}{\partial I_\beta} = \frac{\partial H}{\partial I_\alpha}\,, \tag{2.38}$$

where $\alpha, \beta, \gamma = 1, \dots, k$. Hence we get the equality

$$P_c \mathrm{d}H_c = P_1 \mathrm{d}H \,. \tag{2.39}$$

This equality implies formula (2.13) and presents the supplementary Hamiltonian structure for the Hamiltonian system (2.1).

3) Formulae (2.17) imply that functions $\tilde{\varphi}_\alpha - \varphi_\alpha$ are single-valued. Therefore, the variables $\tilde{\varphi}_\alpha$ are defined mod(2π) along with variables φ_α. Hence $\tilde{\varphi}_\alpha$ play the role of new angle coordinates. In new coordinates (2.16) and (2.17), the symplectic structure (2.10) takes the canonical form

$$\begin{aligned} \omega_c &= \mathrm{d}\tilde{J}_\alpha \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}\left(f_\beta(I) \frac{\partial I_\beta}{\partial \tilde{J}_\alpha} \mathrm{d}\tilde{J}_\alpha \right) \\ &= \mathrm{d}\tilde{J}_\alpha \wedge \mathrm{d}\left(\varphi_\alpha - f_\beta(I) \frac{\partial I_\beta}{\partial \tilde{J}_\alpha} \right) = \mathrm{d}\tilde{J}_\alpha \wedge \mathrm{d}\tilde{\varphi}_\alpha \,. \end{aligned} \tag{2.40}$$

This formula implies that functions $\tilde{J}_\alpha, \tilde{\varphi}_\alpha$ have canonical Poisson brackets with respect to the Poisson structure $P_c = \omega_c^{-1}$. In view of the formula (2.14), the Hamiltonian function H_c (2.13) depends upon the variables $\tilde{J}_\alpha$ only. Therefore, the coordinates $\tilde{J}_\alpha, \tilde{\varphi}_\alpha$ are the action-angle coordinates for the Hamiltonian system (2.13) with respect to the symplectic structure ω_c.

For the Legendre transform (2.14) one has

$$\delta^\alpha_\beta = \frac{\partial J_\alpha}{\partial \tilde{J}_\gamma} \frac{\partial \tilde{J}_\gamma}{\partial J_\beta} = \frac{\partial^2 H_c(\tilde{J})}{\partial \tilde{J}_\alpha \partial \tilde{J}_\gamma} \frac{\partial^2 B(J)}{\partial J_\gamma \partial J_\beta} \,. \tag{2.41}$$

Hence Eq. (2.18) follows.

4) Obviously, the involution of the action variables $I_1, \dots, I_k$ with respect to the Poisson structure P_c (2.19) is an immediate consequence of the block form (2.36). Applying the Liouville Theorem [32] we obtain that system (2.1), (2.13) is completely integrable with respect to the Poisson structure P_c as well. □

III.

Remark 1. Theorem 1 implies that any two of the constructed Poisson structures $P_c = \omega_c^{-1}$ are strongly dynamically compatible. The first part of Theorem 1 was proved in our paper [5].

Remark 2. The original symplectic structure ω_1 has the form (2.10), where $f_\alpha(I) = 0$ and the function $B(J)$ is the Legendre transform $\tilde{B}(J)$ of the Hamiltonian function $H(I)$:

$$\tilde{B}(J) = I_\alpha \frac{\partial H(I)}{\partial I_\alpha} - H(I), \qquad J_\alpha = \frac{\partial H(I)}{\partial I_\alpha} \,. \tag{2.42}$$

Indeed, for this case formula (2.14) presents the inverse Legendre transform. The classical equalities

$$\frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta} = \frac{\partial J_\alpha}{\partial I_\beta}, \qquad \frac{\partial \tilde{B}(J)}{\partial J_\alpha} = I_\alpha, \qquad \frac{\partial^2 \tilde{B}(J)}{\partial J_\alpha \partial J_\beta} = \frac{\partial I_\alpha}{\partial J_\beta} \tag{2.43}$$

imply that the corresponding matrix B (2.33) is the unit matrix

$$B^{\alpha}_{\beta} = \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\gamma} \frac{\partial^2 \tilde{B}(J)}{\partial J_\gamma \partial J_\beta} = \frac{\partial J_\alpha}{\partial I_\gamma} \frac{\partial I_\gamma}{\partial J_\beta} = \frac{\partial J_\alpha}{\partial J_\beta} = \delta^{\alpha}_{\beta} \, . \tag{2.44}$$

Therefore, if $f_\alpha(I) = 0$ and the function $B(J)$ in (2.10) is the Legendre transform (2.42) of the Hamiltonian function $H(I)$ then the symplectic 2-form ω_c (2.10), (2.32) coincides with the original symplectic form ω_1.

Remark 3. A second Hamiltonian structure is known for just a few completely integrable Hamiltonian systems. Theorem 1 reduces the rather difficult search of the second Hamiltonian structure to the classical problem of construction of the action-angle coordinates [18, 42, 43]. When these coordinates are found the formula (2.10) presents a continuous family of symplectic and Poisson structures which are invariant with respect to the system (2.1). Theorem 1 ensures that in this way one obtains all invariant symplectic structures and even all invariant degenerate closed 2-forms if the Kolmogorov condition (2.7) is met.

Remark 4. The recursion operator $A = P_1 P_c^{-1} = P_1 \omega_c$ has the block form

$$A = \begin{pmatrix} B^t & 0 \\ \sigma & B \end{pmatrix} \tag{2.45}$$

in the action-angle coordinates (2.8). In Sect. 11 we prove that the corresponding Nijenhuis tensor $N_A(u, v)$ is not equal to zero in general. Therefore, the Poisson structures P_1 and P_c are incompatible in general.

IV. If a completely integrable non-degenerate Hamiltonian system (2.1) has non-compact invariant submanifolds

$$\mathbb{T}^m \times \mathbb{R}^{k-m} : I_\gamma = c_\gamma, \qquad 0 \leqq \varphi_\alpha \leqq 2\pi, \qquad \rho_i \in \mathbb{R}^1 \, ,$$

$$1 \leqq \gamma \leqq k, \qquad 1 \leqq \alpha \leqq m, \qquad m+1 \leqq i \leqq k \, , \tag{2.46}$$

then the following is true.

Proposition 1. *For any $k+1$ functions $f_\alpha(I_1, \ldots, I_k)$ and $B(J_1, \ldots, J_k)$, where variables J_i are determined by Eqs.* (2.11) *the closed 2-form ω_c* (2.10) *is invariant with respect to the system* (2.1), (2.3). *If the non-degeneracy condition* (2.35) *is met, then the system* (2.1) *has a continuum of supplementary Hamiltonian structures* (2.13). *The Hamiltonian system* (2.1), (2.13) *is completely integrable with respect to all Poisson structures $P_c = \omega_c^{-1}$.*

The proof of Proposition 1 is the same as that for Theorem 1.

V. Completely integrable Hamiltonian systems (2.1) with non-compact invariant submanifolds (2.46) possess greater families of invariant incompatible Poisson structures and invariant closed 2-forms. These Hamiltonian systems have the form

$$\dot{I}_\gamma = 0, \qquad \dot{\varphi}_\alpha = \frac{\partial H(I)}{\partial I_\alpha} = J_\alpha(I), \qquad \dot{\rho}_i = \frac{\partial H(I)}{\partial I_i} = J_i(I) \tag{2.47}$$

in the action-angle coordinates $I_\gamma, \varphi_\alpha, \rho_i$, where $\varphi_\alpha \in S^1$ and $\rho_i \in \mathbb{R}^1$. Let

$$F_\alpha(I), \quad f_\alpha(I), \quad a_{ij}(I), \quad b_{\alpha i}(I), \quad 1 \leqq \alpha, \beta \leqq m, \quad m+1 \leqq i,j,l \leqq k \tag{2.48}$$

be arbitrary functions of the action variables $I_1, \ldots, I_k$. Let $\theta^i(I)$ and $\xi^{ij}(I) = \xi^{ji}(I)$ be arbitrary 1-forms in the domain of the action variables and $c_{\alpha\beta} = -c_{\beta\alpha}$ be arbitrary constants.

Theorem 2. *In the non-compact toroidal domain $\mathcal{O} \subset M^n$ defined by conditions* (2.4) *and* (2.46) *a closed 2-form ω_c represented by the formula*

$$\begin{aligned}\omega_c &= \mathrm{d}F_\alpha(I) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha(I) \wedge \mathrm{d}I_\alpha + c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta \\ &\quad + \mathrm{d}(\rho_i\theta^i) + \mathrm{d}(b_{\alpha i}(I)\rho_i) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}(a_{il}(I)\rho_i\mathrm{d}\rho_l + \rho_i\rho_l\xi^{il})\end{aligned} \tag{2.49}$$

is invariant with respect to the completely integrable Hamiltonian system (2.1), (2.47) *if and only if the functions* (2.48) *and the* 1*-forms θ^i and $\xi^{ij} = \xi^{ji}$ satisfy the equations*

$$\begin{gathered}F_\alpha\mathrm{d}J_\alpha + J_i\theta^i = \mathrm{d}B(I), \qquad b_{\alpha i}J_i - 2c_{\alpha\beta}J_\beta = c_\alpha\,, \\ a_{il}dJ_l + b_{\alpha i}\mathrm{d}J_\alpha + 2J_l\xi^{il} = \mathrm{d}(a_{li}J_l)\,, \\ J_\alpha = \frac{\partial H(I)}{\partial I_\alpha}, \qquad J_l = \frac{\partial H(I)}{\partial I_l}\,,\end{gathered} \tag{2.50}$$

where $B(I)$ is an arbitrary function of the action variables $I_1, \ldots, I_k$ and c_α are arbitrary constants.

Proof. Differentiating the 2-form (2.49) with respect to the dynamical system (2.47) we obtain

$$\begin{aligned}\dot{\omega}_c &= \mathrm{d}(F_\alpha\mathrm{d}J_\alpha + J_i\theta^i) + \mathrm{d}(b_{\alpha i}J_i - 2c_{\alpha\beta}J_\beta) \wedge \mathrm{d}\varphi_\alpha \\ &\quad + \mathrm{d}(\rho_i(a_{il}\mathrm{d}J_l + b_{\alpha i}\mathrm{d}J_\alpha + 2J_l\xi^{il}) + a_{li}J_l\mathrm{d}\rho_i)\,.\end{aligned} \tag{2.51}$$

Three summands in (2.51) depend upon the different variables. Therefore, applying the Poincaré Lemma we get that equation $\dot{\omega}_c = 0$ is equivalent to the system of equations (2.50). □

Equations (2.50) form a linear and triangular system with respect to the unknown functions (2.48) and 1-forms θ^i and $\xi^{ij} = \xi^{ji}$. These equations can be solved as follows.

For $m = k$ system (2.50) reduces to one equation (2.30). Solutions of this equation have the form (2.31).

For $m = k - 1$ and $J_k(I) \neq 0$ system (2.50) implies

$$\begin{gathered}\theta^k = J_k^{-1}(\mathrm{d}B - F_\alpha\mathrm{d}J_\alpha), \qquad 2\xi^{kk} = \mathrm{d}a_{kk} - J_k^{-2}b_\alpha\mathrm{d}J_\alpha\,, \\ b_{\alpha k} = J_k^{-1}b_\alpha, \qquad b_\alpha = 2c_{\alpha\beta}J_\beta + c_\alpha\,.\end{gathered} \tag{2.52}$$

Therefore, for $m = k - 1$ the 2-form (2.49) takes the form

$$\begin{aligned}\omega_c &= \mathrm{d}(F_\alpha + \rho_kJ_k^{-1}b_\alpha) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha \wedge \mathrm{d}I_\alpha + c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta \\ &\quad + \mathrm{d}\left(\rho_kJ_k^{-1}(\mathrm{d}B - F_\alpha\mathrm{d}J_\alpha) - \frac{1}{2}\rho_k^2J_k^{-2}b_\alpha\mathrm{d}J_\alpha\right)\,.\end{aligned} \tag{2.53}$$

This invariant closed 2-form depends upon arbitrary constants $c_\alpha, c_{\alpha\beta} = -c_{\beta\alpha}$ and $2k+1$ arbitrary functions $B(I)$, $F_\alpha(I)$ and $f_\alpha(I)$. For general values of these parameters the 2-form ω_c (2.53) is non-degenerate and the corresponding Poisson structure $P_c = \omega_c^{-1}$ is incompatible with the original Poisson structure P_1.

For $m \leqq k-2$ and $f_k(I) \neq 0$ the following parameters are arbitrary. 1) Functions $a_{il}(I)$ for $m+1 \leqq i,l \leqq k$. 2) Functions $b_{\alpha i}(I)$ and 1-forms θ^i and $\xi^{ij} = \xi^{ji}$ for $m+1 \leqq i,j \leqq k-1$. 3) Functions $B(I)$, $F_\alpha(I)$ and $f_\alpha(I)$. 4) Constants c_α and $c_{\alpha\beta} = -c_{\beta\alpha}$. System of equations (2.50) implies that the unknown 1-forms θ^k and $\xi^{ik} = \xi^{ki}$ and functions $b_{\alpha k}(I)$ have the form

$$\theta^k = J_k^{-1}(\mathrm{d}B - F_\alpha \mathrm{d}J_\alpha - J_i\theta^i)\,,$$

$$\xi^{ki} = \xi^{ik} = (2J_k)^{-1}(\mathrm{d}(a_{li}J_l) - a_{il}\mathrm{d}J_l - b_{\alpha i}\mathrm{d}J_\alpha - 2J_j\xi^{ij})\,,$$

$$\xi^{kk} = (2J_k)^{-1}(\mathrm{d}(a_{lk}J_l) - a_{kl}\mathrm{d}J_l - b_{\alpha k}\mathrm{d}J_\alpha - 2J_i\xi^{ki})\,,$$

$$b_{\alpha k} = J_k^{-1}(2c_{\alpha\beta}J_\beta + c_\alpha - b_{\alpha i}J_i)\,,$$

$$1 \leqq \alpha,\beta \leqq m, \qquad m+1 \leqq i,j \leqq k-1, \qquad m+1 \leqq l \leqq k\,. \tag{2.54}$$

The formulae (2.49) and (2.54) define invariant closed 2-forms ω_c which are non-degenerate for general values of their independent parameters. The corresponding Poisson structures $P_c = \omega_c^{-1}$ are incompatible with the Poisson structure P_1 for the general values of these parameters.

Remark 5. The derived formulae (2.53) and (2.54) show that if the completely integrable Hamiltonian system (2.1) has non-compact invariant submanifolds (2.46) then the family of invariant closed 2-forms ω_c is considerably greater than the complete family (2.10) for the compact and non-degenerate case. The family of invariant closed 2-forms (2.10) depends upon $k+1$ arbitrary functions $B(J)$ and $f_\alpha(I)$. Family (2.53) for $m = k-1$ depends upon $2k+1$ arbitrary functions $B(I)$, $F_\alpha(I)$ and $f_\alpha(I)$. Family (2.54) for $m \leqq k-2$ depends upon a greater number of arbitrary functions and also upon $d(d+3)/2$ arbitrary 1-forms $\theta^i(I)$ and $\xi^{ij}(I) = \xi^{ji}(I)$, where $d = k-m-1$.

Remark 6. Theorem 2 does not depend upon the Kolmogorov condition (2.7). The formulae (2.49)–(2.50), (2.53) and (2.54) present invariant closed 2-forms ω_c independently of whether the completely integrable Hamiltonian system (2.1) is non-degenerate or not.

3. A Cohomology for Dynamical Systems

I. Let $V(x)$ be a smooth vector field on a manifold M^n and

$$\dot{x}^i = V^i(x^1, \ldots, x^n) \tag{3.1}$$

be the corresponding dynamical system. We denote Λ_V^m the space of differential m-forms ω_m on M^n which are invariant with respect to system (3.1).

Let us consider the complex of V-invariant differential forms on M^n,

$$0 \to \Lambda_V^0 \xrightarrow{d} \Lambda_V^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda_V^{n-1} \xrightarrow{d} \Lambda_V^n \to 0\,. \tag{3.2}$$

Definition 3. *The quotient space*

$$H^*(V,M^n) = \operatorname{Ker} \mathrm{d} / \operatorname{Im} \mathrm{d} \tag{3.3}$$

is called the cohomology of the dynamical system V (3.1)*. The wedge product of differential forms induces a ring structure in* $H^*(V,M^n)$*.*

We have the ring homomorphism

$$\alpha : H^*(V,M^n) \to H^*(M^n) \tag{3.4}$$

that transforms a cohomology class of the invariant closed q-forms into the corresponding de Rham's cohomology class [14] of the general closed q-forms.

For any constant $c \neq 0$, we have the isomorphism

$$H^*(cV,M^n) = H^*(V,M^n)\,. \tag{3.5}$$

For $c = 0$ the cohomology $H^*(0,M^n)$ is isomorphic to the de Rham cohomology [14] $H^*(M^n)$.

Remark 7. Using Duff's results [16] it is possible to generalize the constructions of this section for dynamical systems on manifolds M^n with boundary.

II. The homomorphism α has an inverse and therefore is an isomorphism for the following dynamical systems:

1) Assume that all trajectories of the dynamical system (3.1) are closed curves and have the same period T. Let ψ_τ

$$\psi_\tau : M^n \to M^n, \qquad \psi_T = \mathrm{id} \tag{3.6}$$

be the corresponding action of the circle S^1. For any closed q-form ω_q we construct the q-form

$$\alpha^{-1}\omega_q = \frac{1}{T} \oint \psi_\tau^*(\omega_q) \mathrm{d}\tau\,. \tag{3.7}$$

Obviously, the q-form $\alpha^{-1}\omega_q$ is closed and invariant with respect to all diffeomorphisms (3.6). The q-form $\alpha^{-1}\omega_q$ belongs to the same de Rham's cohomology class in $H^q(M^n)$ as the closed q-form ω_q because the q-forms $\psi_\tau^*(\omega_q)$ are homotopic to ω_q for all τ. Therefore $\alpha \circ \alpha^{-1} = \mathrm{id}$ in $H^q(M^n)$ and hence the map α is an isomorphism.

2) Let $M^n = X^{n-k} \times \mathbb{T}^k$, where X^{n-k} is a smooth $(n-k)$-dimensional manifold with a system of local coordinates $x^1, \ldots, x^n$ and $\mathbb{T}^k$ is the k-dimensional torus with angle coordinates $\varphi_1, \ldots, \varphi_k$.

Let us consider the dynamical system

$$\dot{x}^i = 0, \qquad \dot{\varphi}_j = b_j \tag{3.8}$$

on the manifold M^n. Here b_j are arbitrary constants which are incommensurable over the integers in the sense of (2.6). Dynamical system (3.8) generates the following group of diffeomorphisms:

$$\phi_t : (x^i, \varphi_j) \to (x^i, \varphi_j + tb_j)\,. \tag{3.9}$$

Let $f(x,\varphi)$ be an arbitrary smooth function on M^n. Applying the Ergodic Theorem [4] for the subsystem (3.8) on the torus $\mathbb{T}^k$, we obtain

$$\overline{f(x)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x^i, \varphi_j + tb_j)\mathrm{d}t = \int_{\mathbb{T}^k} f(x,\varphi)\mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_k \,. \tag{3.10}$$

Let ω_q be an arbitrary q-form

$$\omega_q = \sum_{l+m=q} \sum_{i,j} a_{i\cdot j}(x,\varphi)\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_l} \wedge \mathrm{d}\varphi_{j_1} \wedge \cdots \wedge \mathrm{d}\varphi_{j_m} \tag{3.11}$$

on the manifold M^n. For all diffeomorphisms ϕ_t (3.9), the differential $\mathrm{d}\phi_t$ is the identity map of the tangent spaces $T_{(x,\varphi)}(X^{n-k} \times \mathbb{T}^k)$. Using this fact, we derive

$$\overline{\omega_q} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi_t^*(\omega_q)\mathrm{d}t = \sum_{l+m=q} \sum_{i,j} \overline{a_{i\cdot j}(x)}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_l} \wedge \mathrm{d}\varphi_{j_1} \wedge \cdots \wedge \mathrm{d}\varphi_{j_m} \,. \tag{3.12}$$

Obviously, the q-form $\overline{\omega_q}$ is closed and invariant with respect to the dynamical system (3.8). Therefore we define

$$\alpha^{-1}\omega_q = \overline{\omega_q} \,. \tag{3.13}$$

This invariant q-form belongs to the same de Rham's cohomology class in $H^q(M^n)$ as the closed q-form ω_q because the q-forms $\phi_\tau^*(\omega_q)$ are homotopic to ω_q for all t. Therefore $\alpha \circ \alpha^{-1} = \mathrm{id}$ in $H^q(M^n)$ and hence the map α is an isomorphism. Hence we obtain the isomorphism of the two cohomologies

$$\alpha : H^*(V, M^n) = H^*(M^n) \tag{3.14}$$

for the dynamical system (3.8) on the manifold $M^n = X^{n-k} \times \mathbb{T}^k$.

The classical harmonic oscillator provides an example of system (3.8), see system (6.7) in Sect. 6 below. Therefore for the harmonic oscillator the cohomology

$$H^*(V, \mathbb{R}^{2k}) = H^*(\mathbb{R}^{2k}) = H^0(\mathbb{R}^{2k}) = \mathbb{R}^1 \tag{3.15}$$

is isomorphic to the ring of reals.

III. Let dynamical system (3.1) be a generic non-integrable Hamiltonian system. Then $V^i = P_1^{ij} H_{,j}$, where P_1 is a non-degenerate Poisson structure on M^{2k}. The corresponding cohomology is isomorphic to the sum

$$H^*(V, M^{2k}) = \mathbb{R}[u]/u^{k+1}\mathbb{R}[u] + H^{2k}(V, M^{2k}) \tag{3.16}$$

of the quotient-ring of polynomials of a single variable u and the infinite-dimensional group $H^{2k}(V, M^{2k})$ that has a trivial law of multiplication. The generator $u \in H^2(V, M^{2k})$ corresponds to the invariant symplectic structure $\omega_1 = P_1^{-1}$. The linear independent elements of the infinite-dimensional group $H^{2k}(V, M^{2k})$ are represented by the invariant closed $2k$-forms

$$\omega_F = F(H)\omega_1 \wedge \cdots \wedge \omega_1 \,. \tag{3.17}$$

There are k factors ω_1 in the wedge product (3.17), $F(H)$ is an arbitrary smooth function of the single variable and $H(x)$ is the Hamiltonian function.

Remark 8. The isomorphism (3.14) for the integrable dynamical system (3.8) and the isomorphism (3.16) for the general non-integrable Hamiltonian system V prove that the cohomology ring $H^*(V, M^n)$ is a new invariant that characterizes simultaneously the topological properties of the manifold M^n and the global properties of the dynamical system V on M^n.

IV. Let P_1^{ij} be a non-degenerate Poisson structure on a manifold M^{2k}. Let us consider a completely integrable in Liouville's sense Hamiltonian system

$$\dot{x}^i = V^i(x) = P_1^{i\alpha} H_{,\alpha}, \qquad H_{,\alpha} = \partial H / \partial x^\alpha \,. \tag{3.18}$$

Definition 4. *Hamiltonian system* (3.18) *is called C-integrable in a domain $\mathcal{O} \subset M^n$ if it is completely integrable in the Liouville sense and in the domain $\mathcal{O}$ all invariant submanifolds of constant level of the k involutive first integrals are compact.*

These invariant submanifolds are tori $\mathbb{T}^k$ (2.5). The Liouville Theorem [32] implies that the C-integrable Hamiltonian system (3.18) has form (2.3) in the action-angle coordinates $I_1, \ldots, I_k$, $\varphi_1, \ldots, \varphi_k$ (2.8).

Definition 5. *A (p, q) tensor T on the manifold M^n is called C-invariant if it is invariant with respect to a C-integrable non-degenerate Hamiltonian system* (3.18).

We consider the C-integrable non-degenerate Hamiltonian system (2.3) in the toroidal coordinates $J_l(I)$ (2.11), φ_l. In these coordinates, the Hamiltonian system (2.3) has the form

$$\dot{J}_l = 0, \qquad \dot{\varphi}_l = J_l \,. \tag{3.19}$$

Let θ be an arbitrary smooth differential 1-form

$$\theta = \theta_i(J, \varphi) \mathrm{d}J_i + \theta_{i+k}(J, \varphi) \mathrm{d}\varphi_i \,. \tag{3.20}$$

Theorem 3. 1) *Differential* 1*-form* θ (3.20) *is C-invariant if and only if*

$$\theta = \theta_i(J) \mathrm{d}J_i \,. \tag{3.21}$$

2) *Any closed C-invariant* 1*-form* θ (3.20) *is exact in the toroidal domain* $\mathcal{O} = B_r \times \mathbb{T}^k$.

Proof. 1) For the 1-form θ (3.20), the invariance equation has the form

$$(L_V \theta)_\beta = \dot{\theta}_\beta + V^\alpha_{,\beta} \theta_\alpha = 0 \,, \tag{3.22}$$

where L_V is the Lie derivative. After substituting formulae (3.19) and (3.20), Eqs. (3.22) imply

$$\dot{\theta}_i = -\theta_{i+k}, \qquad \dot{\theta}_{i+k} = 0 \,. \tag{3.23}$$

In view of (2.9), solutions to (3.23) have the form

$$\theta_i(t) = -\tilde{\theta}_{i+k}(J) t + \tilde{\theta}_i(J), \qquad \theta_{i+k}(t) = \tilde{\theta}_{i+k}(J) \,, \tag{3.24}$$

where $\tilde{\theta}_i(J)$ and $\tilde{\theta}_{i+k}(J)$ are some smooth functions of coordinates $J_1, \ldots, J_k$. Components $\theta_\alpha(J, \varphi)$ of any smooth 1-form (3.20) are bounded on any torus $\mathbb{T}^k$ (2.5). Solutions (3.24) are bounded for all t if and only if $\tilde{\theta}_{i+k}(J) = 0$ for $i = 1, \ldots, k$.

Therefore using (3.24) and the fact that general trajectories of the C-integrable non-degenerate Hamiltonian system (2.3), (3.19) are dense everywhere on the tori $\mathbb{T}^k$ we obtain that the 1-form θ is invariant if and only if it has the form (3.21).

2) If the C-invariant 1-form θ (3.21) is closed then applying the Poincaré Lemma we obtain $\theta = \mathrm{d}F(I)$. □

Proposition 2. *Any C-invariant differential 3-form ω_3 has the form*

$$\omega_3 = b_{ilm}(J)\mathrm{d}J_i \wedge \mathrm{d}J_l \wedge \mathrm{d}\varphi_m + c_{ilm}(J)\mathrm{d}J_i \wedge \mathrm{d}J_l \wedge \mathrm{d}J_m\,, \tag{3.25}$$

where coefficients $c_{ilm}(J)$ are alternating and $b_{ilm}(J)$ satisfy the equations

$$b_{ilm}(J) + b_{lmi}(J) + b_{mil}(J) = 0, \qquad b_{ilm}(J) = -b_{lim}(J)\,. \tag{3.26}$$

Theorem 4. 1) *A closed differential 3-form ω_3 is invariant with respect to the C-integrable non-degenerate Hamiltonian system* (2.3), (3.19) *if and only if it has the form*

$$\omega_3 = \mathrm{d}\left(\frac{\partial B_i(J)}{\partial J_m} + b_{im}(J)\right) \wedge \mathrm{d}J_i \wedge \mathrm{d}\varphi_m + \mathrm{d}(a_{il}(J)\mathrm{d}J_i \wedge \mathrm{d}J_l) \tag{3.27}$$

in the toroidal coordinates J_i, φ_i. Here $B_i(J)$ are arbitrary smooth functions of $J_1, \ldots, J_k$, and coefficients $a_{il}(J)$ and $b_{im}(J)$ satisfy the equations

$$a_{il}(J) = -a_{li}(J), \qquad b_{im}(J) = b_{mi}(J)\,. \tag{3.28}$$

2) *Any closed C-invariant differential 3-form ω_3 is exact. The equation $\omega_3 = d\tilde{\omega}_2$ holds where the C-invariant 2-form $\tilde{\omega}_2$ has the form*

$$\tilde{\omega}_2 = \left(\frac{\partial B_i(J)}{\partial J_m} + \frac{\partial B_m(J)}{\partial J_i} + b_{im}(J)\right) \mathrm{d}J_i \wedge \mathrm{d}\varphi_m + a_{il}(J)\mathrm{d}J_i \wedge \mathrm{d}J_l\,. \tag{3.29}$$

The proof of Proposition 2 and Theorem 4 is based on the same ideas as in the proofs of Theorems 1 and 3 and will be published elsewhere.

Corollary 1. *Assume that a C-integrable non-degenerate Hamiltonian system V* (2.3), (3.19) *is defined in an open toroidal domain $\mathcal{O} = B_r \times \mathbb{T}^k$. Then the first five cohomologies have the form*

$$H^0(V,\mathcal{O}) = \mathbb{R}^1, \qquad H^1(V,\mathcal{O}) = 0, \qquad H^2(V,\mathcal{O}) = \mathbb{R}^\infty\,,$$
$$H^3(V,\mathcal{O}) = 0, \qquad H^4(V,\mathcal{O}) = \mathbb{R}^\infty\,. \tag{3.30}$$

Proof. Theorem 3 implies that each C-invariant closed 1-form is the exterior derivative of some first integral. That means $H^1(V,\mathcal{O}) = 0$. Theorem 4 implies that each C-invariant closed 3-form is the exterior derivative of some C-invariant 2-form. That means $H^3(V,\mathcal{O}) = 0$. Theorem 1 and Theorem 3 imply that $H^2(V,\mathcal{O}) = \mathbb{R}^\infty$. The Proposition 2 implies that the wedge product $\omega_1 \wedge \omega_2$ of two generic C-invariant closed 2-forms ω_1 and ω_2 (2.10) is not the exterior derivative of any C-invariant 3-form ω_3 that necessarily has the form (3.25). Hence the cohomology $H^4(V,\mathcal{O})$ is infinite-dimensional. □

V. The "toroidal surgeries" method presented in Sect. 7 below provides a smooth extension of any C-invariant closed 2-form ω_c (2.10) on the whole manifold M^{2k}.

Therefore, the second and the fourth cohomologies $H^2(V, M^{2k})$ and $H^4(V, M^{2k})$ are infinite-dimensional. Hence we obtain the following consequence.

Corollary 2. *The infinite-dimensionality of the cohomologies* $H^2(V, M^{2k})$ *and* $H^4(V, M^{2k})$ *is the necessary condition for the non-degenerate integrability of the dynamical system* V *on the manifold* M^{2k}.

4. Applications Connected with the KAM Theory

I. Theorem 1 deals with integrable Hamiltonian systems (2.1) which have compact invariant submanifolds (2.5) and are non-degenerate in the Kolmogorov sense (2.7). This class of integrable systems is exactly the starting point for the Kolmogorov–Arnold–Moser theory [2, 26, 27, 41] that studies Hamiltonian perturbations of integrable Hamiltonian systems

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon P_1^{i\alpha} H_{,\alpha} \,. \tag{4.1}$$

Kolmogorov's Theorem [2, 26] assumes that for $\varepsilon = 0$ system (4.1)

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} \tag{4.2}$$

is completely integrable, non-degenerate and has compact invariant submanifolds. In (4.1), the Hamiltonian function $H(x^1, \ldots, x^n)$ is arbitrary smooth and $|\varepsilon|$ is sufficiently small.

It is well-known [3, 27] that generic non-Hamiltonian perturbations can destroy all invariant tori (2.5) and that the dynamics of trajectories of the general perturbed system is not quasi-periodic.

Definition 6. *For an integrable system* (4.2) *a perturbation* $\varepsilon V^i(x^1, \ldots, x^n)$ *is called admissible if dynamical system*

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon V^i(x) \tag{4.3}$$

possesses the same dynamical properties for sufficiently small $|\varepsilon|$ *as the Hamiltonian perturbations* (4.1) *in* KAM *theory.*

Theorem 1 implies the existence of a rich family of admissible perturbations which depend upon $k+1$ arbitrary functions of k variables and one arbitrary function of $2k$ variables and which are non-Hamiltonian with respect to the Poisson structure P_1.

Theorem 5. 1) *For any completely integrable non-degenerate Hamiltonian system* (4.2) *with compact invariant submanifolds the dynamical system*

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon P_c^{i\alpha} H_{,\alpha} \tag{4.4}$$

possesses the same dynamical properties as the Hamiltonian perturbations (4.1) *in* KAM *theory. Here* $H(x^1, \ldots, x^n)$ *is an arbitrary smooth function and* P_c *is an arbitrary non-degenerate Poisson structure that is invariant with respect to the integrable Hamiltonian system* (4.2). *In a neighbourhood of an invariant torus* (2.5) *the admissible perturbations* (4.4) *depend upon* $k+1$ *arbitrary functions* (2.10),

$$B(J_1, \ldots, J_k), \qquad f_\alpha(I_1, \ldots, I_k), \qquad \alpha = 1, \ldots, k \,. \tag{4.5}$$

2) *If a supplementary invariant non-degenerate Poisson structure* P_2 *is compatible with* P_1 *then all perturbations*

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon \left(\sum_{m=-l}^{l} a_m A_2^m \right)_\alpha^i P_1^{\alpha\beta} H_{,\beta} \tag{4.6}$$

are admissible. Here $A_2 = P_1 P_2^{-1}$, $a_m = \text{const}$ *and the* (1, 1) *tensor*

$$\sum_{m=-l}^{l} a_m A_2^m \tag{4.7}$$

is assumed to be non-degenerate.

Proof. 1) Let us consider dynamical system (4.4) in the action-angle coordinates I_α, φ_α (2.8) associated with the integrable Hamiltonian system (4.2). In view of Theorem 1 any invariant non-degenerate Poisson structure P_c has the form $P_c = \omega_c^{-1}$, where the symplectic structure ω_c is defined by (2.10). The integrable system (4.2) has also form (2.13)

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} = P_c^{i\alpha} H_{c,\alpha} \tag{4.8}$$

with new Hamiltonian function

$$H_c(\tilde{J}) = J_\alpha \frac{\partial B(J)}{\partial J_\alpha} - B(J), \qquad \tilde{J}_\alpha = \frac{\partial B(J)}{\partial J_\alpha}\,. \tag{4.9}$$

Using formulae (4.8) we present dynamical system (4.4) in the form

$$\dot{x}^i = P_c^{i\alpha} H_{c,\alpha} + \varepsilon P_c^{i\alpha} H_{,\alpha}\,. \tag{4.10}$$

Obviously, this system is Hamiltonian with respect to the Poisson structure P_c or symplectic structure ω_c. Theorem 1 implies that the unperturbed completely integrable Hamiltonian system ($\varepsilon = 0$)

$$\dot{x}^i = P_c^{i\alpha} H_{c,\alpha} \tag{4.11}$$

is non-degenerate with respect to the Poisson structure P_c. Indeed, the symplectic form $\omega_c = P_c^{-1}$ has the canonical form (2.15) in the action-angle coordinates $\tilde{J}_\alpha, \tilde{\varphi}_\alpha$ (2.16), (2.17). In these coordinates the non-degeneracy condition

$$\det \left\| \frac{\partial^2 H_c(\tilde{J})}{\partial \tilde{J}_\alpha \partial \tilde{J}_\beta} \right\| \neq 0 \tag{4.12}$$

is met in view of (2.18).

Therefore all conditions of KAM theory are satisfied for the system (4.10). Hence the dynamical system (4.4) is an admissible perturbation of (4.2).

Dynamical system (4.4) is not Hamiltonian with respect to the Poisson structure P_1 if at least one eigenvalue of the recursion operator $A_c = P_1 P_c^{-1}$ is not constant. Indeed, this system preserves the Poisson structure P_c. If it also preserved the Poisson structure P_1 then all eigenvalues of the recursion operator A_c would be first integrals of this system. But system (4.4), (4.10) does not have any additional first integrals in general because the function $H(x)$ is arbitrary.

2) If a supplementary invariant Poisson structure P_2 is compatible with P_1 then the $(2,0)$ tensor

$$P_a^{ij} = \left(\sum_{m=-l}^{l} a_m A_2^m \right)_\alpha^i P_1^{\alpha j} \tag{4.13}$$

is a Poisson structure in view of Magri's Theorem [35]. Obviously, the Poisson structure P_a^{ij} is invariant with respect to the integrable system (4.2) and non-degenerate because the (1,1) tensor (4.7) is non-degenerate. Applying Theorem 1 we obtain that the unperturbed ($\varepsilon = 0$) system (4.6) can be presented in the form

$$\dot{x}^i = P_a^{i\alpha} H_{a,\alpha} \tag{4.14}$$

with a Hamiltonian function $H_a(\tilde{J})$ of the type (4.9). Therefore dynamical system (4.6) takes the form

$$\dot{x}^i = P_a^{i\alpha} H_{a,\alpha} + \varepsilon P_a^{i\alpha} H_{,\alpha} \tag{4.15}$$

that is Hamiltonian with respect to the Poisson structure P_a^{ij}. Hamiltonian system (4.15) satisfies all conditions of KAM theory because system (4.14) is completely integrable and non-degenerate in view of Theorem 1. □

Theorem 5 implies the following consequence.

Corollary 3. *The* KAM *theory is applicable not only for small Hamiltonian perturbations* (4.1) *but also for the rich family of non-Hamiltonian perturbations* (4.4). *The family of admissible non-Hamiltonian perturbations* (4.4) *depends upon the* $k+1$ *arbitrary functions of* k *variables* $B(J), f_1(I), \ldots, f_k(I)$ *and upon one arbitrary function of* $2k$ *variables* $H(x)$.

Remark 9. Formulae (4.4) and (4.6) imply that any invariant non-degenerate Poisson structure P_2 that is compatible with P_1 leads to a larger family of admissible perturbations than an incompatible Poisson structure P_c. The family (4.6) depends upon an arbitrary Laurent polynomial (4.7) or an arbitrary analytic function for $|l| \to \infty$. Nevertheless, the whole family of admissible perturbations (4.4) is more general because the incompatible invariant Poisson structures P_c depend upon $k+1$ arbitrary smooth functions of k variables (4.5) and compatible Poisson structures P_2 are exceptional cases among them, see Sect. 11.

II. Let $P_0, \ldots, P_N$ be arbitrary non-degenerate invariant Poisson structures for integrable Hamiltonian system (4.2), and $A_i(x) = P_1 P_i^{-1}$ be the corresponding recursion operators. For an integer multi-index $\tau = (\tau_0, \ldots, \tau_N)$ and $N+1$ (1,1)-tensors $A_i(x)$ we define a (1,1) tensor

$$A^\tau(x) = A_0^{\tau_0} \cdots A_N^{\tau_N} \,. \tag{4.16}$$

Let $f_\tau(x)$ and $H_\tau(x)$ be arbitrary first integrals of system (4.2).

Theorem 6. *For a completely integrable non-degenerate Hamiltonian system* (4.2) *all perturbations* (*of an arbitrary scale*)

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \sum_{|\tau| \leqq m} f_\tau A_\alpha^{\tau i} P_1^{\alpha\beta} H_{\tau,\beta} \tag{4.17}$$

are admissible. All invariant tori of system (4.2) *are also invariant with respect to the dynamical system* (4.17). *General trajectories of* (4.17) *are quasi-periodic.*

Proof. The completely integrable Hamiltonian system (4.2) has form (2.3) in the action-angle coordinates I_l, φ_l. The Kolmogorov condition (2.7) implies that all first integrals of this system are functions of the action variables I_l only:

$$f_\tau = f_\tau(I_l), \qquad H_\tau = H_\tau(I_l)\,. \tag{4.18}$$

Therefore vector fields $V_\tau^\alpha = P_1^{\alpha\beta} H_{\tau,\beta}$ have coordinates

$$V_\tau^j = 0, \qquad V_\tau^{j+k} = H_\tau(I)_{,j}\,, \tag{4.19}$$

where $j = 1,\ldots,k$. In the action-angle coordinates any recursion operator $A_i = P_1 P_i^{-1}$ has the lower triangular block form (2.45) where all entries depend upon the action variables I_l only. Therefore, the (1,1) tensors A^τ (4.16) also have the lower triangular block form (2.45). Hence using the key property of first integrals (2.9) and the block structure (2.32) we obtain that the dynamical system (4.17) has the form

$$\dot I_j = 0, \qquad \dot\varphi_j = p^j(I) \tag{4.20}$$

in the action-angle coordinates I_l, φ_l. Here $p^j(I_1,\ldots,I_k)$ are functions of the action variables and $j = 1,\ldots,k$. Obviously, formulae (4.20) complete the proof of Theorem 6. □

5. Applications Connected with the Kepler Problem

I. The classical Kepler problem is described by the Hamiltonian system in the phase space $\mathbb{R}^6$

$$\dot p_i = -\frac{\partial H(p,q)}{\partial q_i}, \qquad \dot q_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \omega_1 = \mathrm{d}p_i \wedge \mathrm{d}q_i, \qquad i = 1,2,3 \tag{5.1}$$

with the Hamiltonian

$$H(p,q) = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - \frac{GM_0 m}{r}\,. \tag{5.2}$$

Here m is the mass of the moving particle, M_0 is the mass of the attracting centre, G is the gravitational constant and $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$. As it is well known, the Kepler problem has three first integrals of angular momentum

$$M_i = \varepsilon_{ijk}\, p_j q_k \tag{5.3}$$

and three Lenz and Runge first integrals

$$R_i = \frac{1}{m}\varepsilon_{ijk}\, p_j M_k + \frac{GM_0 m}{r} q_i\,. \tag{5.4}$$

Here ε_{ijk} is an alternating tensor, $\varepsilon_{123} = 1$, $i,j,k = 1,2,3$.

The existence of first integrals (5.3) and (5.4) implies that the Kepler problem (5.1) possesses a continuum of invariant closed 2-forms

$$\omega_c = \omega_1 + \mathrm{d}F_i(M,R) \wedge \mathrm{d}G_i(M,R)\,. \tag{5.5}$$

Here $F_i(M,R)$ and $G_i(M,R)$ are arbitrary smooth functions of first integrals M_j and R_l and $i = 1,2,3$.

The invariant 2-forms (5.5) are non-degenerate for the generic functions $F_i(M,R)$ and $G_i(M,R)$. The corresponding invariant Poisson structures $P_c = \omega_c^{-1}$ are incompatible with $P_1 = \omega_1^{-1}$ in general. This is obvious because the rank of the system of differential 1-forms $\mathrm{d}M_j$, $\mathrm{d}R_l$ is equal to 5 and functions $F_i(M,R)$ and $G_i(M,R)$ are arbitrary.

An additional compatible Poisson structure was constructed for the Kepler problem in [37] by another method.

II. The basic problem of celestial mechanics is the problem of dynamics of n planets with masses m_α, $\alpha = 1,\ldots,n$, around the Sun that is assumed to be in the origin of the Euclidean space $\mathbb{R}^3$ and has mass $M_0 \gg m_\alpha$. The Hamiltonian of this problem has the form

$$H(p,r) = \sum_{\alpha=1}^{n} \left(\frac{p_\alpha^2}{2m_\alpha} - \frac{GM_0 m_\alpha}{|r_\alpha|} \right) - \sum_{\alpha \neq \beta}^{n} \frac{Gm_\alpha m_\beta}{|r_\alpha - r_\beta|}\,, \tag{5.6}$$

where vectors r_α and p_α define position and momentum of the α-th planet.

In view of $m_\alpha/M_0 \ll 1$ dynamics of the Solar system is studied as a small perturbation of the basic integrable problem that is described by the Hamiltonian

$$H(p,r) = \sum_{\alpha=1}^{n} \left(\frac{p_\alpha^2}{2m_\alpha} - \frac{GM_0 m_\alpha}{|r_\alpha|} \right), \tag{5.7}$$

where the gravitational interaction between planets is neglected.

The basic integrable problem with Hamiltonian (5.7) is the direct product of n Kepler problems. Let $M_{\alpha i}$ and $R_{\alpha i}$ be the angular momentum first integrals (5.3) and the Lenz and Runge first integrals (5.4) for the α-th planet Kepler Problem. Let $F_{\alpha i}(M_{\beta j}, R_{\gamma l})$ and $G_{\alpha i}(M_{\beta j}, R_{\gamma l})$ be arbitrary smooth functions of the $6n$ arguments, where $\alpha, \beta, \gamma = 1,\ldots,n$ and $i,j,l = 1,2,3$. Obviously, the closed 2-forms

$$\omega_1 = \sum_{\alpha,\, i} \mathrm{d}p_{\alpha i} \wedge \mathrm{d}r_{\alpha i}, \qquad \omega_c = \omega_1 + \sum_{\alpha,\, i} \mathrm{d}F_{\alpha i}(M,R) \wedge \mathrm{d}G_{\alpha i}(M,R) \tag{5.8}$$

are invariant with respect to the flow of the direct product of n Kepler problems (5.7).

The same arguments as for the Kepler problem (5.2) prove that the invariant closed 2-forms ω_c (5.8) are non-degenerate for the generic functions $F_{\alpha i}(M,R)$ and $G_{\alpha i}(M,R)$ and that the invariant Poisson structures $P_c = \omega_c^{-1}$ (5.8) in general are incompatible with the original Poisson structure $P_1 = \omega_1^{-1}$.

Remark 10. Using the classical Poincare canonical elements [50] and methods of Sect. 2, it is possible to present the invariant symplectic structures (5.5) and (5.8) by the explicit formulae in the corresponding action-angle coordinates. The analogous formulae for the harmonic oscillator are presented in the next section.

6. Invariant Poisson Structures for the Harmonic Oscillator

I. The classical harmonic oscillator is described by the Hamiltonian system with the Hamiltonian

$$H(p,q) = \sum_{j=1}^{k} \left(\frac{1}{2m_j} p_j^2 + \frac{1}{2} a_j q_j^2 \right), \quad a_j > 0, \quad m_j > 0\,, \tag{6.1}$$

in the phase space $\mathbb{R}^{2k}$ with the standard symplectic structure $\omega = dp_j \wedge dq_j$.

The corresponding action-angle variables I_l, φ_l are connected with p_l, q_l by the formulae

$$I_l = \frac{1}{2c_l} p_l^2 + \frac{1}{2} c_l q_l^2, \qquad \varphi_l = \arctan\left(c_l \frac{q_l}{p_l} \right), \qquad c_l = \sqrt{a_l m_l}\,, \tag{6.2}$$

$$p_l = \sqrt{2c_l I_l} \cos \varphi_l, \qquad q_l = \sqrt{\frac{2}{c_l} I_l} \sin \varphi_l\,. \tag{6.3}$$

Indeed, using (6.2) we find

$$\mathrm{d}I_l = \frac{1}{c_l} p_l \mathrm{d}p_l + c_l q_l \mathrm{d}q_l, \qquad \mathrm{d}\varphi_l = \frac{1}{2I_l}(p_l \mathrm{d}q_l - q_l \mathrm{d}p_l)\,. \tag{6.4}$$

These formulae imply that the symplectic form ω has the form

$$\omega = \sum_{j=1}^{k} \mathrm{d}p_j \wedge \mathrm{d}q_j = \sum_{j=1}^{k} \mathrm{d}I_j \wedge \mathrm{d}\varphi_j\,. \tag{6.5}$$

Formulae (6.3) yield the following expression for the Hamiltonian function (6.1):

$$H(p,q) = H(I) = \omega_1 I_1 + \cdots + \omega_k I_k, \qquad \omega_l = \sqrt{\frac{a_l}{m_l}}\,. \tag{6.6}$$

Hence the dynamics of the harmonic oscillator is defined by the simplest integrable Hamiltonian system

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = \omega_j \tag{6.7}$$

in the action-angle coordinates. The Hamiltonian system (6.7) is degenerate as much as possible since the corresponding Hessian matrix (2.7) is identically equal to zero for the linear Hamiltonian function (6.6).

Let $f_\alpha(I_1,\dots,I_k)$ and $g_\alpha(I_1,\dots,I_k)$ be arbitrary smooth functions of the action variables, and $c_{\alpha\beta}$ be arbitrary constants, $\alpha, \beta = 1,\dots,k$. Any closed differential 2-form

$$\omega_c = \mathrm{d}f_\alpha(I) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}g_\alpha(I) \wedge \mathrm{d}I_\alpha + c_{\alpha\beta} \mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta \tag{6.8}$$

is invariant with respect to the flow (6.7). Indeed, the Lie derivative $L_V \omega_c$ with respect to dynamical system (6.7) vanishes: $L_V \omega_c = 0$.

Using formulae (6.4) one can easily obtain the expression for the invariant 2-form (6.8) in the Cartesian coordinates p_l, q_l.

II. If frequencies ω_l (6.6) are incommensurable over the integers (see (2.6)) then all trajectories of dynamical system (6.7) are everywhere dense on the invariant tori

$\mathbb{T}^k$. Applying Theorem 9 from Sect. 8 below to the $\mathbb{T}^k$-dense Hamiltonian system (6.7) we obtain that formula (6.8) represents all invariant closed 2-forms.

If frequencies ω_l (6.6) are commensurable over the integers then they satisfy some $m < k$ linear independent equations

$$c_{i1}\omega_1 + \cdots + c_{ik}\omega_k = 0, \quad i = 1, \ldots, m \tag{6.9}$$

with integer coefficients $c_{ij} \in \mathbb{Z}$. For this case system (6.7) has m additional first integrals

$$\psi_i(\varphi) = c_{i1}\varphi_1 + \cdots + c_{ik}\varphi_k, \qquad \frac{\mathrm{d}\psi_i(\varphi)}{\mathrm{d}t} = 0\,, \tag{6.10}$$

which are defined mod(2π). Let

$$f_\alpha(I_1, \ldots I_k, \psi_1, \ldots, \psi_m), \qquad g_\alpha(I_1, \ldots, I_k, \psi_1, \ldots \psi_m), \quad \alpha = 1, \ldots k \tag{6.11}$$

be arbitrary smooth functions of $k + m$ variables which are 2π-periodic with respect to the variables ψ_i. Formulae (6.7) and (6.10) imply that system (6.7) preserves the following closed differential 2-forms:

$$\omega_c = \mathrm{d}f_\alpha(I, \psi(\varphi)) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}g_\alpha(I, \psi(\varphi)) \wedge \mathrm{d}I_\alpha + c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta\,, \tag{6.12}$$

where $c_{\alpha\beta}$ are arbitrary constants.

Invariant closed 2-forms ω_c (6.8) and (6.12) are non-degenerate for the generic functions $f_\alpha(I, \psi)$. The corresponding generic Poisson structures $P_c = \omega_c^{-1}$ are incompatible with the original Poisson structure $P = \omega^{-1}$ because the Nijenhuis tensor N_A for the recursion operator $A = P\omega_c$ does not vanish in general, see Sect. 11.

However, formula (6.8) contains a continuum of Poisson structures $\tilde{P}_c = \tilde{\omega}_c^{-1}$ which are compatible with the original Poisson structure $P = \omega^{-1}$ (6.5). Indeed, let ω_c^{-1} have the form

$$\tilde{\omega}_c = f_\alpha(I_\alpha)\mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\alpha\,, \tag{6.13}$$

where $f_\alpha(I_\alpha)$ are arbitrary smooth functions of the single variable. Then the formula

$$(\omega_1^{-1} + \tilde{\omega}_c^{-1})^{-1} = \frac{f_\alpha(I_\alpha)}{1 + f_\alpha(I_\alpha)}\mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\alpha \tag{6.14}$$

is true and therefore the compatibility condition $\mathrm{d}(\omega_1^{-1} + \omega_c^{-1})^{-1} = 0$ (6.15) is satisfied. The corresponding recursion operator $A = P_1\tilde{\omega}_c$ has doubly degenerate spectrum $f_\alpha(I_\alpha)$, $\alpha = 1, \ldots, n$.

The invariant Poisson structures $\tilde{P}_c = \tilde{\omega}_c^{-1}$ (6.13) which are compatible with $P_1 = \omega_1^{-1}$ are unstable in a sense that they become incompatible with P_1 after arbitrarily small perturbations inside the general families (6.8) and (6.12) of invariant Poisson structures $P_c = \omega_c^{-1}$.

7. Instability of the Property of Compatibility of Invariant Poisson Structures

I. Assume that a completely integrable non-degenerate Hamiltonian system (2.1) is given on a symplectic manifold M^n, $n = 2k$, with a symplectic form ω_1 and Poisson structure $P_1 = \omega_1^{-1}$. Assume there exists a second non-degenerate Poisson structure P_2 that is invariant with respect to the dynamical system (2.1) and is compatible with the original Poisson structure $P_1 = \omega_1^{-1}$. The following theorem proves that for the supplementary invariant Poisson structure P_2 the property of compatibility with P_1 is unstable.

Theorem 7. *In any neighbourhood of the Poisson structure P_2 there exists a non-degenerate Poisson structure P_C that is incompatible with the Poisson structure P_1 and invariant with respect to the Hamiltonian system* (2.1).

Proof. In view of the Liouville Theorem for any point $p \in M^n \backslash S$ there exists some toroidal domain $\mathcal{O}_p \subset M^n$, $p \in \mathcal{O}_p$, with action-angle coordinates $I_1, \ldots, I_k$, $\varphi_1, \ldots, \varphi_k$ in $\mathcal{O}_p$, where the symplectic structure ω_1 has canonical form $\omega_1 = \mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\alpha$ and the completely integrable Hamiltonian system (2.1) has form (2.3). Assume that the Kolmogorov condition (2.7) is satisfied at some point $I_0 \in \mathcal{O}_p$ with action coordinates $I_{01}, \ldots, I_{0k}$. Then the map

$$\phi : (I_1, \ldots, I_k) \to (J_1, \ldots, J_k), \qquad J_i = \partial H(I)/\partial I_i \tag{7.1}$$

is a diffeomorphism in a neighbourhood of the point I_0. Therefore, for some $r > 0$ the ball B_r (2.4) is transformed into an open set $V_r \subset R^k$ that contains two balls $B_1 \subset B_2$:

$$B_\alpha : \sum_{i=1}^{k} (J_i - J_{i0})^2 \leqq \delta_\alpha^2, \qquad \delta_1 < \delta_2, \qquad (J_0) = \phi(I_0)\,. \tag{7.2}$$

Let $B_p(J_1, \ldots, J_k)$ be an arbitrary smooth function on $\mathbb{R}^k$ that is constant outside of the ball B_2 and is not constant inside the ball $B_1 \subset B_2$.

We construct a global invariant symplectic structure ω_p on the manifold M^n from the second invariant symplectic structure $\omega_2 = P_2^{-1}$ by the following "toroidal surgery" in the action-angle coordinates $I_1, \ldots, I_k$, $\varphi_1, \ldots, \varphi_k$. Let $\mathcal{O}_r \subset M^n$ be the set diffeomorphic to the direct product $B_r \times \mathbb{T}^k$ or $B_r \times \mathbb{T}^m \times \mathbb{R}^{k-m}$, where the action coordinates $I_1, \ldots, I_k$ satisfy the inequality (2.4) and the angle coordinates $\varphi_1, \ldots, \varphi_k$ are arbitrary. The symplectic structure ω_p coincides with the original 2-form ω_2 in $M^n \backslash \mathcal{O}_r$. Inside the set $\mathcal{O}_r$ the 2-form ω_p is defined by the formula

$$\omega_p = \omega_2 + \varepsilon \mathrm{d}\left(\frac{\partial B_p(J)}{\partial J_\alpha}\right) \wedge \mathrm{d}\varphi_\alpha\,. \tag{7.3}$$

The constructed 2-form ω_p is defined globally on the manifold M^n and is smooth and closed. It coincides with ω_2 outside of the set

$$\phi^{-1}(B_2) \times \mathbb{T}^m \times \mathbb{R}^{k-m} \supset \phi^{-1}(B_1) \times \mathbb{T}^m \times \mathbb{R}^{k-m}\,, \tag{7.4}$$

and is different from ω_2 inside the set $\phi^{-1}(B_1) \times \mathbb{T}^m \times \mathbb{R}^{k-m}$. The 2-form ω_p is preserved by the Hamiltonian system (2.1) in view of Theorem 1 and is non-degenerate for sufficiently small ε because the 2-form ω_2 is non-degenerate. Therefore, the invariant Poisson structure $P_C = \omega_p^{-1}$ is a small perturbation of the original Poisson structure P_2.

For the recursion operator $A_p = P_1 \omega_p$ the Nijenhuis tensor N_{A_p} is not equal to zero in a neighbourhood of the point I_0. Indeed, function $B_p(J)$ (7.3) is arbitrary in the neighbourhood of the point $J_0 = \phi(I_0)$ and therefore it does not satisfy the overdetermined third-order nonlinear system of partial differential equations (11.9) that follows from the compatibility condition $N_{A_p} = 0$. Hence we obtain that the invariant Poisson structure $P_C = \omega_p^{-1}$ is incompatible with P_1 in a neighbourhood of the point I_0. The instability is proved. $\square$

II. The method of "toroidal surgeries" that has been used in Theorem 7 can be applied for any completely integrable non-degenerate Hamiltonian system (2.1). This method provides the globalization of the invariant closed 2-forms (2.10) and (2.49) and the corresponding incompatible Poisson structures from the action-angle coordinates to the whole manifold M^n.

8. Invariant Poisson Structures for the Degenerate Integrable Hamiltonian Systems

I. The Kepler problem, the basic integrable problem of celestial mechanics and the harmonic oscillator problem provide the classical examples of degenerate completely integrable Hamiltonian systems. The following theorem is a generalization of the concrete constructions of Sect. 6. We assume that m angle coordinates $\varphi_1, \dots, \varphi_m$, $0 \leqq m \leqq k$ run over the torus $\mathbb{T}^m$ and $k - m$ coordinates $\rho_{m+1} = \varphi_{m+1}, \dots, \rho_k = \varphi_k$ run over the Euclidean space $\mathbb{R}^{k-m}$ and $1 \leqq \alpha, \beta, \gamma \leqq k$.

Theorem 8. 1) *The closed 2-form*

$$\omega_c = \mathrm{d}F_\alpha(I) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha(I) \wedge \mathrm{d}I_\alpha + c_{\alpha\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}\varphi_\beta \tag{8.1}$$

is invariant with respect to the degenerate Hamiltonian system (2.3) *if the functions $f_\alpha(I)$ are arbitrary, the functions $F_\alpha(I)$ satisfy the equation*

$$F_\alpha(I)\mathrm{d}\left(\frac{\partial H}{\partial I_\alpha}\right) = \mathrm{d}B(I) \tag{8.2}$$

with some smooth function $B(I)$ and the skew-symmetric matrix $c_{\alpha\beta}$ is constant and satisfies the algebraic equation

$$c_{\alpha\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\beta} = 0\,. \tag{8.3}$$

2) *If the 2-form ω_c (8.1) is non-degenerate then system* (2.1), (2.3) *has the supplementary Hamiltonian form*

$$\dot{x}^i = P_c^{i\alpha}\psi_{c\alpha}\,, \tag{8.4}$$

where $P_c = \omega_c^{-1}$ and ψ_c is a closed 1-form

$$\psi_c = \mathrm{d}H_c(I) + g_\alpha \mathrm{d}\varphi_\alpha\,, \tag{8.5}$$

and function $H_c(I)$ and constants g_α have the form

$$H_c(I) = \frac{\partial H(I)}{\partial I_\beta}F_\beta(I) - B(I), \qquad g_\alpha = c_{\alpha\gamma}\frac{\partial H(I)}{\partial I_\gamma}\,. \tag{8.6}$$

3) *If matrix B with the components*

$$B_\beta^\alpha = \frac{\partial F_\beta(I)}{\partial I_\alpha} \tag{8.7}$$

is non-degenerate and all constants $c_{\alpha\beta} = 0$ then system (2.3) *has the supplementary Hamiltonian form*

$$\dot{x}^i = P_c^{i\alpha}H_{c,\alpha}\,. \tag{8.8}$$

System (8.8) *is completely integrable with respect to the Poisson structure* P_c.

Proof. 1) Differentiating the closed 2-form ω_c (8.1) with respect to the dynamical system (2.3) we obtain

$$\dot{\omega}_c = \mathrm{d}\left(F_\alpha(I)\mathrm{d}\left(\frac{\partial H}{\partial I_\alpha}\right)\right) + 2c_{\alpha\gamma}\frac{\partial^2 H(I)}{\partial I_\gamma \partial I_\beta}\mathrm{d}\varphi_\alpha \wedge \mathrm{d}I_\beta\,. \tag{8.9}$$

Substituting equalities (8.2) and (8.3) we find

$$\dot{\omega}_c = \mathrm{d}(\mathrm{d}B(I)) = 0\,. \tag{8.10}$$

Therefore the 2-form ω_c (8.1) is invariant with respect to the Hamiltonian system (2.3).

2) In view of Eqs. (8.2) and (8.7) we obtain for the function H_c (8.6),

$$\frac{\partial H_c(I)}{\partial I_\alpha} = \frac{\partial H}{\partial I_\beta}\frac{\partial F_\beta}{\partial I_\alpha} = B^\alpha_\beta \frac{\partial H}{\partial I_\beta}\,. \tag{8.11}$$

In the action-angle coordinates (2.8) the symplectic form ω_c (8.1) has the block structure

$$\omega_c = \begin{pmatrix} \sigma & B \\ -B^t & c \end{pmatrix}, \tag{8.12}$$

where matrix B has entries (8.7) and matrix σ has entries (2.33). The original Hamiltonian system (2.3) is defined by the vector field $\dot{x} = V$ that has the components

$$V^\alpha = 0, \qquad V^{k+\alpha} = \frac{\partial H}{\partial I_\alpha}, \quad \alpha = 1,\ldots,k\,. \tag{8.13}$$

Formulae (8.11)–(8.13) imply that the 1-form $\omega_c V$ has components

$$(\omega_c V)_\alpha = (\omega_c)_{\alpha j}V^j = B^\alpha_\beta\frac{\partial H}{\partial I_\beta} = \frac{\partial H_c(I)}{\partial I_\alpha}\,,$$

$$(\omega_c V)_{\alpha+k} = (\omega_c)_{\alpha+k\cdot j}V^j = c_{\alpha\gamma}\frac{\partial H}{\partial I_\gamma} = g_\alpha\,. \tag{8.14}$$

Equations (8.3) imply that $g_\alpha = \text{const}$. Formulae (8.5) and (8.14) yield the equality

$$\omega_c V = \psi_c\,, \tag{8.15}$$

where ψ_c is the closed 1-form (8.5). Hence we obtain

$$\dot{x} = V = P_c\psi_c\,, \tag{8.16}$$

where $P_c = \omega_c^{-1}$. Therefore the representation (8.4) is proved.

3) If in (8.12) matrix B is non-degenerate and matrix $c = 0$ then

$$\det \|\omega_c\| = (\det \|B\|)^2 \neq 0\,. \tag{8.17}$$

In this case the Poisson structure P_c has the block form (2.36) and hence the Poisson brackets (2.19) vanish. The condition $c_{\alpha\beta} = 0$ implies the equality $\psi_c = \mathrm{d}H_c(I)$. Therefore, applying the Liouville Theorem [32] we obtain that system (2.1), (8.8) is completely integrable with respect to the Poisson structure P_c as well. □

Obviously, Eq. (8.2) has solutions of the form

$$B(I) = B(J(I)), \qquad F_\alpha(I) = \frac{\partial B(J(I))}{\partial J_\alpha}, \qquad J_\alpha(I) = \frac{\partial H(I)}{\partial I_\alpha}, \tag{8.18}$$

where $B(J_1, \ldots, J_k)$ is an arbitrary function. For this case equality (8.2) follows from the definition of the differential $\mathrm{d}B(I)$.

The closed 1-form ψ_c (8.5) is not exact if one of the constants $g_\alpha \neq 0$ and the corresponding coordinate φ_α is periodic. For this case the Hamiltonian system (8.4) has no single-valued Hamiltonian function that would be defined in a neighbourhood of the invariant submanifold $\mathbb{T}^m \times \mathbb{R}^{k-m}$ (2.46).

II. Recall that a completely integrable system (2.1) is called $\mathbb{T}^k$-dense if the trajectories of system (2.1) are everywhere dense on almost all tori (2.5).

Theorem 9. *Assume that a completely integrable Hamiltonian system* (2.1) *is* $\mathbb{T}^k$-*dense in the compact toroidal domain* $\mathcal{O} \subset M^n$ *defined by conditions* (2.4) *and* (2.5). *Then the following is true*:

1) *A closed* 2-*form* ω_c *is invariant with respect to system* (2.3) *if and only if it has the form* (8.1) *and Eqs.* (8.2) *and* (8.3) *are satisfied.*

2) *If the degenerate Hessian matrix satisfies the condition*

$$\operatorname{rank} \left\| \frac{\partial^2 H(I)}{\partial I_\alpha \partial I_\beta} \right\| = k - 1 \tag{8.19}$$

then all invariant closed 2-*forms* ω_c *have the form*

$$\omega_c = \mathrm{d}F_\alpha(I) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha(I) \wedge \mathrm{d}I_\alpha \,. \tag{8.20}$$

Proof. 1) The sufficiency of Eqs. (8.1)–(8.3) follows from Theorem 8. Their necessity is proved by the same arguments as in the proof of Theorem 1.

2) In view of (8.19) Eq. (8.3) yields

$$\operatorname{rank} \|c_{\alpha\beta}\| \leqq 1 \,. \tag{8.21}$$

The rank of the skew matrix $c_{\alpha\beta}$ is even. Hence the inequality (8.21) yields $c_{\alpha\beta} = 0$. Therefore Eq. (8.20) follows from (8.1).

9. The Integrability Problem

I. In this section we present a solution of the Integrability Problem that is formulated in Sect. 1.

Theorem 10. *Assume that two non-degenerate Poisson structures* P_1 *and* P_2 *on a manifold* M^{2k} *are strongly dynamically compatible and their recursion operator* $A = P_1 P_2^{-1}$ *has k functionally independent eigenvalues. Then the following is true*:

1) *Any dynamical system*

$$\dot{x}^i = V^i(x^1, \ldots, x^{2k}) = P_1^{i\alpha} \theta_{1 \cdot \alpha} = P_2^{i\alpha} \theta_{2 \cdot \alpha} \tag{9.1}$$

that preserves the two Poisson structures P_1 *and* P_2 *is completely integrable in the Liouville sense with respect to* P_1 *and* P_2. *Here* θ_1 *and* θ_2 *are closed* 1-*forms.*

2) *There exist the action-angle coordinates* I_j, φ_j *where the Poisson structure* P_1 *has the canonical form*

$$P_1 = \sum_{j=1}^{k} dI_j \wedge d\varphi_j \tag{9.2}$$

and where all closed 1*-forms* θ_1 (9.1) *have the form* $\theta_1 = \mathrm{d}H_1(I)$, *where* $H_1(I)$ *are the corresponding Hamiltonian functions. All dynamical systems* (9.1) *have simultaneously the form*

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = \frac{\partial H_1(I)}{\partial I_j} \tag{9.3}$$

in the same system of the action-angle coordinates I_j, φ_j.

3) *Flows of all dynamical systems which preserve the two strongly dynamically compatible Poisson structures* P_1 *and* P_2 *commute with each other.*

Proof. 1) Definition 2 implies that if two Poisson structures P_1 and P_2 are strongly dynamically compatible then there exists a dynamical system

$$\dot{x}^\alpha = \tilde{V}^\alpha(x^1, \ldots, x^{2k}) \tag{9.4}$$

that preserves P_1 and P_2 and that is an integrable and non-degenerate Hamiltonian system with respect to some non-degenerate Poisson structure P, and such that its invariant submanifolds are compact. The Liouville Theorem implies that these submanifolds are tori $\mathbb{T}^k$. Applying Theorem 1 we obtain that system (9.4) is completely integrable and non-degenerate with respect to both Poisson structures P_1 and P_2. Let I_j, φ_j be the action-angle coordinates with respect to the Poisson structure P_1 (9.2) where system (9.4) has the form

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = \frac{\partial \tilde{H}(I)}{\partial I_j} \tag{9.5}$$

with a non-degenerate Hamiltonian function $\tilde{H}(I_1, \ldots, I_k)$. Applying Statement 1 of Theorem 1, we obtain that the second invariant non-degenerate Poisson structure P_2 is equal to ω_2^{-1}, where ω_2 is a closed differential 2-form (2.10). In the action-angle coordinates I_j, φ_j this 2-form has block structure (2.32). Therefore, the corresponding (1,1) tensor $A^i_j = P_1^{i\alpha} P^{-1}_{2\alpha j}$ has the block structure

$$A = P_1 P_2^{-1} = \begin{pmatrix} B^t & 0 \\ \sigma & B \end{pmatrix}, \tag{9.6}$$

where matrices $B = B(I)$ and $\sigma = \sigma(I)$ depend upon the action coordinates I_j only.

Let $C(\lambda, x)$ be the characteristic polynomial of the $k \times k$ matrix $B(x)$, $x \in M^{2k}$:

$$C(\lambda, x) = \det(B(x) - \lambda) = \sum_{m=0}^{k} c_m(x) \lambda^m . \tag{9.7}$$

The block structure (9.6) implies that the characteristic polynomial $P(\lambda, x) = \det(A(x) - \lambda)$ of the operator $A(x)$ is the square of the polynomial $C(\lambda, x)$

$$P(\lambda, x) = C^2(\lambda, x) . \tag{9.8}$$

Hence we obtain that every eigenvalue $\lambda_i(x)$ of the recursion operator $A(x) = P_1 \omega_c$ has an even multiplicity $m_i = 2k_i$.

The formula (9.6) implies that functions

$$H_m(x) = \operatorname{Tr} A^m(x) \tag{9.9}$$

depend upon the action variables I_j only. That follows from the key property (2.9) as well because functions (9.9) are first integrals of the non-degenerate integrable system (9.5). Therefore, the Hamiltonian flows

$$\dot{x}^\alpha = V_m^\alpha(x) = P_1^{\alpha\beta}(\operatorname{Tr} A^m(x))_{,\beta} \tag{9.10}$$

have the form

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = \frac{\partial H_m(I)}{\partial I_j} \tag{9.11}$$

in the action-angle coordinates I_j, φ_j.

The family of functions $H_m(x)$ (9.9) and the k distinct eigenvalues $\lambda_i(x)$ are functionally equivalent. All these functions are in involution with respect to the Poisson structure P_1 because they depend upon the action variables I_j only.

We have assumed that the recursion operator A (9.6) has k functionally independent eigenvalues. Therefore, the submanifold ($d_i = \text{const}$)

$$M^k : \lambda_i(I) = d_1, \dots, \lambda_k(I) = d_k \tag{9.12}$$

is a torus $\mathbb{T}^k(I_j = c_j)$ or a union of several tori $\mathbb{T}^k$.

Any dynamical system V (9.1) has first integrals (9.9). Therefore, vector field V (9.1) is tangent to the tori $\mathbb{T}^k$ and hence dynamical system (9.1) has the form

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = V^{j+k}\,. \tag{9.13}$$

Any Hamiltonian system V (9.1) is completely integrable with respect to the Poisson structure P_1 because functions (9.9) are involutive first integrals of this system and there are k functionally independent first integrals (9.9).

2) The closed 1-form θ_1 has the form

$$\theta_1 = \theta_{1\cdot j}(I,\varphi)\mathrm{d}I_j + \theta_{1\cdot j+k}(I,\varphi)\mathrm{d}\varphi_j \tag{9.14}$$

in the action-angle coordinates I_j, φ_j (9.2). Therefore, the Hamiltonian system V (9.1) has the form

$$\dot{I}_j = -\theta_{1\cdot j+k}(I,\varphi), \qquad \dot{\varphi}_j = \theta_{1\cdot j}(I,\varphi)\,. \tag{9.15}$$

The two formulae (9.13) and (9.15) for the same dynamical system (9.1) imply the equalities $\theta_{1\cdot j+k} = 0$. These equalities and condition $\mathrm{d}\theta_1 = 0$ yield

$$\theta_1 = \theta_{1\cdot j}(I)\mathrm{d}I_j\,. \tag{9.16}$$

Applying the Poincaré Lemma for the closed 1-form (9.16) we obtain

$$\theta_1 = \mathrm{d}H_1(I), \qquad \theta_{1\cdot j} = \frac{\partial H_1(I)}{\partial I_j}\,. \tag{9.17}$$

Therefore the representation (9.3) is proved for any system (9.1).

3) The commutativity of all flows (9.1) follows from their simultaneous form (9.3) in the same system of action-angle coordinates I_j, φ_j. □

II. Magri's Theorem [34] states that if a dynamical system V (9.1) preserves two compatible Poisson structures P_1 and P_2 then functions $H_m(x) = \operatorname{Tr} A^m(x)$ are in involution with respect to both Poisson structures P_1 and P_2. The proof of involutiveness is based on the identities

$$P_1^{\alpha\beta}\left(\frac{1}{m}\operatorname{Tr} A^m\right)_{,\beta} = P_2^{\alpha\gamma}\left(\frac{1}{m+1}\operatorname{Tr} A^{m+1}\right)_{,\gamma} \tag{9.18}$$

and follows from the Lenard scheme [23]. For the non-degenerate Poisson structures P_1 and P_2, the identities (9.18) are equivalent to the compatibility of P_1 and P_2. Therefore, identities (9.18) are not true for any pair of incompatible non-degenerate Poisson structures. However we have proved in Theorem 10 by another method that all first integrals $H_m = \operatorname{Tr} A^m(x)$ are in involution. Our proof is independent upon the Lenard scheme that is not applicable for two general strongly dynamically compatible Poisson structures.

The involutiveness of first integrals $H_m = \operatorname{Tr} A^m(x)$ with respect to both Poisson structures P_1 and P_2 implies that all flows $V_m : \dot{x} = P_1 dH_m$ commute and all flows $\tilde{V}_m : \dot{x} = P_2 dH_m$ commute. The identity (9.18) implies an excessive information that all Hamiltonian flows V_m preserve not only the Poisson structure P_1 but also P_2 and that all Hamiltonian flows $\tilde{V}_m$ preserve not only the Poisson structure P_2 but also P_1.

Remark 11. These properties are not necessary for the Liouville integrability. Therefore, the dynamical systems which preserve two compatible Poisson structures undergo the more rigid mechanism of integrability that those preserving two strongly dynamically compatible Poisson structures. For the incompatible case Theorem 10 implies that the commuting flows V_m preserve the Poisson structure P_1 and do not preserve the Poisson structure P_2 for two general strongly dynamically compatible Poisson structures P_1 and P_2. Analogously the commuting flows $\tilde{V}_m$ preserve P_2 and do not preserve P_1. These facts do not confirm Olver's prediction that "*it would appear that incompatible bi-Hamiltonian systems are, in a sense, even more integrable than compatible ones*" [47, p. 187].

The incompatible bi-Hamiltonian systems V preserve the non-zero Nijenhuis tensor N^i_{Ajk} and all invariants which can be constructed from tensors P_1, P_2, A and N_A. However, we prove in Theorem 12 (see Sect. 12) that all arising scalar invariants (12.7) are equal to zero. Therefore, these invariants do not provide additional first integrals which could lead to an excessive integrability of the system V under investigation.

Remark 12. Let C_1 be a class of C-integrable non-degenerate Hamiltonian systems on a manifold M^{2k} with a non-degenerate Poisson structure P_1, see Definition 4 in Sect. 3. Let C_2 be a class of dynamical systems which preserve two strongly dynamically compatible non-degenerate Poisson structures P_1 and P_2 provided that the recursion operator $A = P_1 P_2^{-1}$ has k functionally independent eigenvalues. Let C_3 be the class of all C-integrable Hamiltonian systems on the Poisson manifold M^{2k}. The inclusions

$$C_1 \subset C_2 \subset C_3 \tag{9.19}$$

hold. Indeed, inclusion $C_1 \subset C_2$ is proved in Theorem 1. The inclusion $C_2 \subset C_3$ is proved in Theorem 10.

Remark 13. Brouset in [8, 9] and Fernandes in [19] proved that class C_1 is not included into the class $\tilde{C}_2$ of dynamical systems which preserve two compatible in

Magri's sense Poisson structures provided that the recursion operator $A = P_1 P_2^{-1}$ has k functionally independent eigenvalues. Magri proved in [34] that $\tilde{C}_2 \subset \tilde{C}_3$, where $\tilde{C}_3$ is the class of all integrable Hamiltonian systems on M^{2k}.

10. Hierarchy of Integrable Dynamical Systems

I. Let $L(z)$ be a Laurent polynomial

$$L(z) = \sum_{m=-l}^{l} a_m(x) z^m , \tag{10.1}$$

where coefficients $a_m(x)$ are arbitrary smooth functions of the eigenvalues of the recursion operator $A = P_1 P_2^{-1}$. We define a function $H_L(x)$ on the manifold M^{2k}

$$H_L(x) = \operatorname{Tr}(L(A(x))), \quad A = P_1 P_2^{-1} \tag{10.2}$$

and the Hamiltonian system

$$\dot{x}^i = P_1^{i\alpha} H_{L,\alpha} . \tag{10.3}$$

Theorem 11. *Assume that two non-degenerate Poisson structures P_1 and P_2 on a manifold M^{2k} are strongly dynamically compatible and their recursion operator $A = P_1 P_2^{-1}$ has k functionally independent eigenvalues. Then the following is true:*

1) *Any dynamical system* (9.1) *that preserves the two Poisson structures P_1 and P_2 generates a hierarchy of integrable dynamical systems*

$$\dot{x}^i = (A^m V)^i , \tag{10.4}$$

where m is an arbitrary integer.

2) *Invariants $H_m(x) = \operatorname{Tr} A^m(X)$ of the recursion operator A are first integrals for all dynamical systems* (10.4).

3) *All flows* (10.3) *and* (10.4) *commute*

$$[A^m V, A^l V] = 0, \qquad [A^m V, P_1 \mathrm{d} H_L] = 0, \qquad [P_1 \mathrm{d} H_{L_1}, P_1 \mathrm{d} H_{L_2}] = 0 . \tag{10.5}$$

4) *All dynamical systems* (9.5) *as well as the more general dynamical systems*

$$\dot{x}^i = (L(A)V)^i \tag{10.6}$$

are completely integrable. All flows (10.6) *for different $L(A)$ commute with each other. Here $L(A)$ is an arbitrary Laurent polynomial* (10.1).

Proof. 1) Theorem 10 implies that dynamical system (9.1) is completely integrable with respect to both Poisson structures P_1 and P_2. This system has the form (9.3) in the action-angle coordinates I_j, φ_j which are constructed in Theorem 10. The recursion operator A has form (9.6) in the coordinates I_j, φ_j.

Formula (9.6) implies that the $(1, 1)$ tensors A^l and A^{-l}, $l > 0$, have the block structure

$$A^l = \begin{pmatrix} (B^t)^l & 0 \\ \sigma_l & B^l \end{pmatrix}, \qquad A^{-l} = \begin{pmatrix} (B^t)^{-l} & 0 \\ \sigma_{-l} & B^{-l} \end{pmatrix} . \tag{10.7}$$

Matrices σ_l and σ_{-l} are defined by the formulae

$$\sigma_l = \sum_{p+q=l-1} B^p \sigma (B^t)^q, \qquad \sigma_{-l} = -B^{-l} \sigma_l (B^t)^{-l}, \tag{10.8}$$

which follow readily by induction.

In the action-angle coordinates I_j, φ_j vector field V (9.3) has components

$$V^j = 0, \qquad V^{j+k} = \frac{\partial H_1(I)}{\partial I_j}. \tag{10.9}$$

Therefore, formulae (10.7) imply that components of the vector fields $A^m V$ have the form

$$(A^m V)^j = 0, \qquad (A^m V)^{k+j} = (B^m(I))^j_l \frac{\partial H}{\partial I_l}, \tag{10.10}$$

where $j, l = 1, \ldots, k$. These vector fields are tangent to the tori $\mathbb{T}^k$ (2.5). Therefore, the tori $\mathbb{T}^k$ are invariant submanifolds for all dynamical systems (10.4). All these systems are integrable in view of (10.10).

The last is true as well for any dynamical system (10.6) corresponding to an arbitrary Laurent polynomial (10.1) because components of the vector field $L(A)V$ have the form

$$(L(A)V)^j = 0, \qquad (L(A)V)^{k+j} = p^j(I) = L(B(I))^j_l \frac{\partial H}{\partial I_l}. \tag{10.11}$$

2) Dynamical system (9.1) preserves the two Poisson structures P_1 and P_c. Therefore it preserves the $(1,1)$ tensor $A = P_1 P_c^{-1}$ and all its invariants as well. Hence the functions $H_m(x) = \operatorname{Tr} A^m(x)$ (9.9) and $H_L(x)$ (10.2) are first integrals of system (9.1). We have proved in Theorem 10 that all eigenvalues $\lambda_j(I)$ of the recursion operator A are constant on the tori $\mathbb{T}^k$ ($I_j = c_j$), see (9.12). Hence all functions (9.9) and (10.2) are constant on the tori $\mathbb{T}^k$ as well.

Therefore, in view of (10.10) we obtain that functions (9.9) and (10.2) are first integrals of the dynamical systems (10.4) as well.

3) Vector fields $A^m V$ are tangent to the tori $\mathbb{T}^k$ ($I_j = c_j$) and their components in the action-angle coordinates I_j, φ_j depend upon the action variables I_j only. Obviously the same is true for the vector fields $P_1 \mathrm{d} H_L(I)$ in view of the canonical form (9.2). Therefore, all these vector fields and the corresponding flows (10.3) and (10.4) commute.

4) In the action-angle coordinates I_j, φ_j the dynamical system (10.6), (10.11) has the form

$$\dot{I}_j = 0, \qquad \dot{\varphi}_j = p^j(I_1, \ldots, I_k). \tag{10.12}$$

Obviously, this system is integrable.

System (10.12) preserves the closed 2-form

$$\omega_2 = \sum_{j=1}^{k} \mathrm{d}p^j(I) \wedge \mathrm{d}\varphi_j. \tag{10.13}$$

If the k functions $p^1(I), \ldots, p^k(I)$ are functionally independent then dynamical system (10.6), (10.12) has the Hamiltonian form

$$\dot{p}^j = 0, \qquad \dot{\varphi}_j = \frac{\partial H_0(p)}{\partial p^j}, \qquad H_0(p) = \frac{1}{2}((p^1)^2 + \cdots + (p^k)^2). \tag{10.14}$$

Hence we obtain that in the non-degenerate case dynamical system (10.6) is completely integrable in the Liouville sense with respect to the symplectic structure (10.13) in a toroidal neighbourhood $\mathcal{O} = B_r \times \mathbb{T}^k$ of any invariant torus $\mathbb{T}^k$.

Commutativity of all flows (10.6) follows from the formulae (10.11). $\square$

II. The dynamical systems (10.4) have the form

$$\dot{x}^i = P^{i\alpha}_{m+1} H_{1,\alpha} = P^{i\alpha}_m H_{2,\alpha}, \quad m > 0\,, \tag{10.15}$$

$$\dot{x}^i = \tilde{P}^{i\alpha}_{|m|} H_{1,\alpha} = \tilde{P}^{i\alpha}_{|m|+1} H_{2,\alpha}, \quad m < 0\,. \tag{10.16}$$

Here the $(2,0)$ tensors

$$P_l = A^{l-1} P_1, \qquad \tilde{P}_l = A^{1-l} P_2, \quad l > 0 \tag{10.17}$$

are skew.

For $l \geqq 2$ the $(2,0)$ tensors P_l and $\tilde{P}_l$ are not Poisson structures if the original Poisson structures P_1 and P_2 are incompatible. In the proof of Theorem 11 we have constructed the symplectic structures for systems (10.4) and (10.6) by the explicit formulae (10.13) in the action-angle coordinates.

The methods of the present paper differ substantially from the methods used in papers and monographs [8–13, 15, 19–22, 24, 25, 33–37, 40, 45–48, 53, 54] for pairs of compatible Poisson structures where all $(2,0)$ tensors P_l and $\tilde{P}_l$ (10.17) are themselves Poisson structures and therefore systems (10.15) and (10.16) are bi-Hamiltonian.

For $l = 1$ formulae (10.17) yield the tensors P_1 and $\tilde{P}_1 = P_2$. Systems (10.15), (10.16) take the form

$$\dot{x}^i = (AV)^i = (P_1 P_2^{-1} P_1)^{i\alpha} H_{1,\alpha} = P_1^{i\alpha} H_{2,\alpha}, \quad m = 1\,, \tag{10.18}$$

$$\dot{x}^i = (A^{-1}V)^i = P_2^{i\alpha} H_{1,\alpha} = (P_2 P_1^{-1} P_2)^{i\alpha} H_{2,\alpha}, \quad m = -1\,. \tag{10.19}$$

These systems preserve the Poisson structure P_1 or P_2 respectively. They are completely integrable with respect to P_1 or P_2 by the same arguments as in Theorem 10.

11. The Nijenhuis Tensor for the Recursion Operator

I. Assume that two incompatible non-degenerate Poisson structures P_1 and P_c on a manifold M^n, $n = 2k$ are strongly dynamically compatible. In view of Definition 2, Sect. 1, there exists a dynamical system V on the manifold M^n that preserves both of them and is completely integrable and non-degenerate with respect to some non-degenerate Poisson structure P, and such that its invariant submanifolds are compact. Using Statement 4 of Theorem 1, we obtain that the dynamical system V is completely integrable and non-degenerate with respect to both Poisson structures P_1 and P_c.

Let us use the action-angle coordinates I_j, φ_j (2.8) where the Poisson structure P_1 has the canonical form $P_1 = \omega_1^{-1}$, $\omega_1 = \mathrm{d}I_j \wedge \mathrm{d}\varphi_j$. Applying Statement 1 of Theorem 1 for the invariant Poisson structure P_c, we obtain that in the action-angle coordinates I_j, φ_j the recursion operator $A = P_1 P_c^{-1}$ has block structure (2.45) with the following entries:

$$A^i_j = B^j_i(I), \qquad A^i_{j+k} = 0, \qquad A^{i+k}_j = \sigma_{ij}(I), \qquad A^{i+k}_{j+k} = B^i_j(I)\,, \tag{11.1}$$

where $1 \leqq i, j \leqq k$. Formulae (2.33) imply

$$B^i_l(I) = \frac{\partial B_l(I)}{\partial I_i}, \qquad B_l(I) = \frac{\partial B(J(I))}{\partial J_l}, \qquad \sigma_{ij}(I) = f_{j,i} - f_{i,j}\,. \tag{11.2}$$

For any $(1,1)$ tensor A^α_β the Nijenhuis tensor $N^\alpha_{\beta\gamma}$ is defined by the formula [44]

$$N^\alpha_{\beta\gamma} = A^\alpha_{\gamma,\tau}A^\tau_\beta - A^\alpha_{\beta,\tau}A^\tau_\gamma + (A^\tau_{\beta,\gamma} - A^\tau_{\gamma,\beta})A^\alpha_\tau\,. \tag{11.3}$$

Substituting formulae (11.1) we obtain that the following components of the Nijenhuis tensor (11.3) vanish:

$$N^{i+k}_{j+k\cdot l+k} = N^i_{j+k\cdot l+k} = N^i_{j+k\cdot l} = 0\,. \tag{11.4}$$

Here and below we assume $1 \leqq i,j,l,m \leqq k$. Components N^i_{jl} coincide with those for the Nijenhuis tensor of the $(1,1)$ tensor $B^t(I)$ in the domain of the action variables:

$$N^i_{jl} = B^l_{i,m}B^j_m - B^j_{i,m}B^l_m + (B^j_{m,l} - B^l_{m,j})B^m_i\,. \tag{11.5}$$

The other components have the form

$$N^{i+k}_{j+k\cdot l} = B^m_{j,l}B^i_m - B^i_{j,m}B^l_m, \qquad N^{i+k}_{j\cdot l+k} = B^i_{l,m}B^j_m - B^m_{l,j}B^i_m\,, \tag{11.6}$$

$$N^{i+k}_{jl} = \sigma_{il,m}B^j_m - B^i_m\sigma_{ml,j} - \sigma_{ij,m}B^l_m + B^i_m\sigma_{mj,l} + \sigma_{im}(B^j_{m,l} - B^l_{m,j})\,. \tag{11.7}$$

Proposition 3. *The k-dimensional linear subspace $\mathscr{L}_x = T_x(\mathbb{T}^k)$ is a commutative ideal with respect to the algebraic structure defined by the Nijenhuis tensor $N(u,v)$ in the tangent space $T_x(M^n)$.*

Indeed, equalities (11.4) mean that

$$N(\mathscr{L}_x, \mathscr{L}_x) = 0, \qquad N(T_x(M^n), \mathscr{L}_x) \subset \mathscr{L}_x\,. \tag{11.8}$$

Therefore subspace $\mathscr{L}_x$ is a commutative ideal.

II. The compatibility of the two non-degenerate Poisson structures P_1 and P_c is equivalent to the vanishing of the Nijenhuis tensor $N_A(u,v) = 0$ [24,25,36]. This condition implies

$$N^{i+k}_{j+k\cdot l} = \frac{\partial^2 B_j(I)}{\partial I_l \partial I_m}\frac{\partial B_m(I)}{\partial I_i} - \frac{\partial^2 B_j(I)}{\partial I_i \partial I_m}\frac{\partial B_m(I)}{\partial I_l} = 0\,,$$

$$B_l(I) = \frac{\partial B(J(I))}{\partial J_l}, \qquad J_l(I) = \frac{\partial H(I)}{\partial I_l}\,. \tag{11.9}$$

The overdetermined third-order nonlinear system of partial differential equations (11.9) has solutions only for exceptional pairs of functions $B(J)$ and $H(I)$.

For example if function $\tilde{B}(J)$ is the Legendre transform (2.42) of the function $H(I)$ then we have

$$\tilde{B}_l(I) = \frac{\partial \tilde{B}(J(I))}{\partial J_l} = I_l, \qquad B^i_l(I) = \frac{\partial \tilde{B}_l(I)}{\partial I_i} = \delta^i_l \,. \tag{11.10}$$

Hence Eqs. (11.9) are satisfied and components of the Nijenhuis tensor (11.5) and (11.6) vanish. For the case (11.10) components N^{i+k}_{jl} (11.7) vanish for any matrix σ_{ij}. Therefore, any symplectic form

$$\tilde{\omega} = \omega_1 + (f_{\beta,\alpha} - f_{\alpha,\beta})\mathrm{d}I_\alpha \wedge \mathrm{d}I_\beta \tag{11.11}$$

defines a Poisson structure $\tilde{P} = \tilde{\omega}^{-1}$ that is compatible with the Poisson structure P_1. The corresponding recursion operator $A = P_1\tilde{\omega}$ (2.45) satisfies an algebraic equation $(A-1)^2 = 0$ and has non-diagonal 2×2 Jordan blocks.

The function $B(J)$ is the most important element of the invariant symplectic form (2.10) because this function determines the incompatibility of the two Poisson structures P_c and P_1.

If two functions $H(I)$ and $B(J)$ are in general position then Eqs. (11.9) are not satisfied and therefore the invariant Poisson structure P_c (2.13) is incompatible with the original Poisson structure P_1.

III. For any tangent vector $u \in T_x(M^n)$ we define an operator N_u:

$$N_u w = N(u, w) \,. \tag{11.12}$$

Tangent vector u has the following coordinates:

$$u = u^1 e_1 + \cdots + u^k e_k + v^1 e_{1+k} + \cdots + v^k e_{2k} \,,$$

$$e_j = \frac{\partial}{\partial I_j}, \qquad e_{j+k} = \frac{\partial}{\partial \varphi_j}, \quad j = 1, \ldots, k \,. \tag{11.13}$$

The formulae (11.4) imply that in the action-angle coordinates operators N_{e_j} have the following block structure:

$$N_{e_j} = \begin{pmatrix} V_j & 0 \\ U_j & W_j \end{pmatrix}, \qquad N_{e_{j+k}} = \begin{pmatrix} 0 & 0 \\ Q_j & 0 \end{pmatrix}, \tag{11.14}$$

where V_j, U_j, W_j and Q_j are $k \times k$ matrices which depend upon the action variables $I_1, \ldots, I_k$ only.

Let us define the following polynomial-valued function on the tangent bundle $T(M^n)$:

$$P_N(u, \lambda) = \det(N_u - \lambda) \,. \tag{11.15}$$

For the Nijenhuis tensor (11.14), polynomial (11.15) has the form

$$P_N(u, \lambda) = \det(V_j u^j - \lambda) \det(W_j u^j - \lambda) \,. \tag{11.16}$$

Obviously, this polynomial is a product of two polynomials of degree k. Polynomial (11.16) does not depend upon the coordinates $v^1, \ldots, v^k$ (11.13). Hence the identity holds

$$P_N(u + v, \lambda) = P_N(u, \lambda) \,, \tag{11.17}$$

where $u \in T_x(M^n)$ and $v \in \mathscr{L}_x$. This identity implies $P_N(v, \lambda) = (-\lambda)^n$ for all tangent vectors $v \in \mathscr{L}_x$.

12. Necessary Conditions Problem

I. Let P_1 and P_c be two incompatible non-degenerate Poisson structures on a manifold M^n, $n = 2k$. Let $A = P_1 P_c^{-1}$ be the recursion operator and $N_A(u, v)$ be the corresponding Nijenhuis tensor. The $(1,1)$ tensor A^i_j defines a family of differential forms

$$\mathrm{d}H_m, \quad H_m = \frac{1}{m} \operatorname{Tr} A^m \tag{12.1}$$

and a family of vector fields

$$X_\alpha = A^l P_1 \mathrm{d}H_m, \quad \alpha = (l, m)\,. \tag{12.2}$$

Using the operators N_u (11.12) we obtain a family of differential 1-forms

$$\varphi_m(u) = \operatorname{Tr}(A^m N_u)\,, \tag{12.3}$$

a family of bilinear forms

$$g_m(u, \tilde{u}) = \operatorname{Tr}(N_u A^m N_{\tilde{u}})\,, \tag{12.4}$$

and a family of polynomial-valued functions on the tangent bundle $T(M^n)$,

$$P_{lN}(u, \lambda) = \det(A^l N_u - \lambda) = \sum_{m=0}^{n} p_{lm}(u) \lambda^m\,. \tag{12.5}$$

These geometric objects lead to a family of vector fields

$$Y_\alpha = A^l P_1 \varphi_m, \quad \alpha = (l, m)\,, \tag{12.6}$$

and families of functions on the manifold M^n

$$f_\gamma = g_m(Z_\alpha, Z_\beta), \qquad h_\delta = P_1(A^p \psi_l, A^q \psi_m), \qquad r_\delta = P_2(A^p \psi_l, A^q \psi_m)\,, \tag{12.7}$$

where

$$Z_\alpha = X_\alpha \quad \text{or } Y_\alpha, \qquad \psi_m = \varphi_m \quad \text{or } \mathrm{d}H_m, \qquad \gamma = (m, \alpha, \beta), \qquad \delta = (l, m, p, q)\,. \tag{12.8}$$

II.

Theorem 12. *The following properties of the geometric objects* (12.1)–(12.7) *constitute the necessary conditions for two incompatible non-degenerate Poisson structures P_1 and P_c to be strongly dynamically compatible*:

1) *The algebraic structure defined by the Nijenhuis tensor $N_A(u, v)$ in the tangent space $T_x(M^n)$ possesses a k-dimensional commutative ideal $\mathscr{L}_x \subset T_x(M^n)$. The linear subspace $\mathscr{L}_x$ is Lagrangian with respect to both symplectic structures $\omega_1 = P_1^{-1}$ and $\omega_c = P_c^{-1}$. The algebraic structures which are defined by the Nijenhuis tensor are isomorphic along any curve $x(t) \in M^n$ that is tangent to the distribution $\mathscr{L}_{x(t)}$.*

2) *The differential 2-forms* $\mathrm{d}\varphi_m$ *and bilinear forms* g_m *have*

$$\operatorname{rank} \mathrm{d}\varphi_m \leqq k, \qquad \operatorname{rank} g_m \leqq k . \tag{12.9}$$

3) *All vector fields* $Z_\alpha = X_\alpha$ *or* Y_α *are mutually commutative*

$$[Z_\alpha, Z_\beta] = 0 . \tag{12.10}$$

4) *The functions* f_γ, h_δ *and* r_δ *are identically equal to zero. All functions* $H_m(x)$ *are in involution with respect to the Poisson structures* P_1 *and* P_c,

$$P_1(\mathrm{d}H_l, \mathrm{d}H_m) = P_c(\mathrm{d}H_l, \mathrm{d}H_m) = 0 . \tag{12.11}$$

5) *Polynomial* $P_{IN}(u, \lambda)$ (12.5) *is reducible and is a product of two polynomials of degree* $k = n/2$.

Proof. 1) Assume that two Poisson structures P_1 and P_c are strongly dynamically compatible. Then we define the k-dimensional distribution $\mathscr{L}_x$ in the form $\mathscr{L}_x = T_x(\mathbb{T}^k)$, where tori $\mathbb{T}^k$ are defined by the integrable system V, see Definition 2 in Sect. 1. By this definition $\mathscr{L}_x$ is Lagrangian with respect to the symplectic structure ω_1. Theorem 1 implies that any invariant closed 2-form ω_c has form (2.10) in the action-angle coordinates (2.8) corresponding to the completely integrable Hamiltonian system V (2.3). Therefore the distribution $\mathscr{L}_x$ is Lagrangian with respect to any invariant symplectic structure $\omega_c = P_c^{-1}$. Formulae (11.8) prove that the subspace $\mathscr{L}_x$ is a commutative ideal.

The phase flow corresponding to the system (2.3) preserves the two Poisson structures P_1 and P_c. Therefore, it preserves the recursion operator $A = P_1 P_c^{-1}$ and the corresponding Nijenhuis tensor $N_A(u, v)$ and all geometric objects (12.1)–(12.7). Hence these tensors are isomorphic along any trajectory of the system (2.3). In view of the Kolmogorov condition (2.7) the general trajectories of this system are everywhere dense on the tori (2.5). Therefore, the algebraic structures defined by the Nijenhuis tensor $N_A(u, v)$ on the tangent spaces $T_{x_1}(M^n)$ and $T_{x_2}(M^n)$ are isomorphic if the points x_1 and x_2 belong to the same torus $\mathbb{T}^k$ (2.5). Hence this is true along any curve $x(t)$ tangent to the distribution $\mathscr{L}_{x(t)}$ because such a curve lies on a torus (2.5).

2) Formulae (10.7), (10.8) and (11.14) imply that operators $A^m N_u$ have the form

$$A^m N_u = \begin{pmatrix} (B^t)^m V_\alpha u^\alpha & 0 \\ U_{mu} & B^m W_\alpha u^\alpha \end{pmatrix}, \quad U_{mu} = \sigma_m V_\alpha u^\alpha + B^m (U_\alpha u^\alpha + Q_\alpha v^\alpha) \tag{12.12}$$

in coordinates (11.13). Hence we obtain

$$\varphi_m(u) = \operatorname{Tr}(A^m N_u) = \operatorname{Tr}((B^t)^m V_\alpha + B^m W_\alpha) u^\alpha , \tag{12.13}$$

$$g_m(u, \tilde{u}) = \operatorname{Tr}(N_u A^m N_{\tilde{u}}) = \operatorname{Tr}(V_\alpha (B^t)^m V_\beta + W_\alpha B^m W_\beta) u^\alpha \tilde{u}^\beta , \tag{12.14}$$

where matrices $B(I)$, $V_\alpha(I)$, $W_\alpha(I)$ depend upon the action variables $I_1, \dots, I_k$ only. Obviously, formulae (12.13)–(12.14) yield the relations (12.9).

3) Formulae (2.32), (10.7) and (12.13) imply that coordinates of the vector fields $Z_\alpha = X_\alpha$ (12.2) or Y_α (12.6) have the form

$$Z_\alpha = (0, \dots, 0,\ z_\alpha^1(I), \dots, z_\alpha^k(I)) . \tag{12.15}$$

Hence the commutativity relations (12.10) follow.

4) Substituting expression (12.15) into (12.14) we obtain that all functions f_γ vanish.

The Kolmogorov condition (2.7) implies that any first integral $F(x)$ of the system is a function of the action variables $I_1,\ldots,I_k$. Therefore the differential form $\mathrm{d}F$ has components

$$\mathrm{d}F = (F_{,1},\ldots,F_{,k},\ 0,\ldots,0)\,. \tag{12.16}$$

Applying (12.16) for first integrals $H_m(x)$ (12.1) and using the block forms of the Poisson structures P_1 (2.32) and P_c (2.36) we obtain the equalities (12.11).

Using the block forms of the operators A^l (10.7) and formulae (12.13),(12.16) we find that the 1-forms $A^l\psi_m$ have components

$$A^l\psi_m = (\psi_{\alpha 1},\ldots,\psi_{\alpha k},\ 0,\ldots,0), \quad \alpha = (l,m)\,. \tag{12.17}$$

These expressions and the block forms of the Poisson structures P_1 (2.32) and P_c (2.36) imply that all functions h_δ and r_δ (12.7) vanish.

5) In view of (12.12) polynomial (12.5) has the form

$$P_{lN}(u,\lambda) = \det((B^t)^l V_\alpha u^\alpha - \lambda)\det(B^l W_\alpha u^\alpha - \lambda)\,. \tag{12.18}$$

Obviously, formula (12.18) presents polynomial $P_{lN}(u,\lambda)$ as the product of two polynomials of degree k. □

III. Theorem 12 can be applied for many incompatible Poisson structures P_1 and P_c. For example if the hypersurfaces of constant level $H_m(x) = \text{const}$ are compact for one of the functions $H_m(x)$ (12.1) and one of the necessary conditions 1)–5) is not met then no integrable non-degenerate Hamiltonian system exists that would preserve the two Poisson structures P_1 and P_c.

13. Canonical Forms for Non-Degenerate Completely Integrable Hamiltonian Systems

I. In this section we study canonical forms for the non-degenerate completely integrable Hamiltonian systems in the domains (see (2.4))

$$\mathcal{O} = B_r \times \mathbb{T}^m \times \mathbb{R}^{k-m}\,. \tag{13.1}$$

Theorem 13. *Any non-degenerate completely integrable Hamiltonian system (2.1)–(2.3) in a toroidal domain $\mathcal{O} \subset M^n$ is diffeomorphically equivalent to one of the $k+1$ universal Hamiltonian systems in the Cartesian coordinates p_i, q_i:*

$$\dot p_i = -\frac{\partial H_m}{\partial q_i}, \qquad \dot q_i = \frac{\partial H_m}{\partial p_i}\,. \tag{13.2}$$

The Hamiltonian functions H_m have $k+1$ canonical forms

$$H_m = \frac{1}{8}\sum_{i=1}^{m}(p_i^2+q_i^2)^2 + \frac{1}{2}\sum_{i=m+1}^{k} p_i^2\,, \tag{13.3}$$

where $m = 0,1,\ldots,k$.

Proof. The Kolmogorov condition (2.7) ensures that the k functions

$$J_l(I) = \frac{\partial H(I)}{\partial I_l}, \quad l = 1,\dots,k, \tag{13.4}$$

form a system of local coordinates in the space of action variables $I_1,\dots,I_k$. In these coordinates the dynamical system (2.3) takes the canonical form

$$\dot{J}_l = 0, \qquad \dot{\varphi}_l = J_l = \frac{\partial H_0(J)}{\partial J_l}, \tag{13.5}$$

$$H_0(J) = \frac{1}{2}(J_1^2 + \cdots + J_k^2). \tag{13.6}$$

Obviously system (13.5) is Hamiltonian and preserves the symplectic structure

$$\omega_2 = \mathrm{d}J_l \wedge \mathrm{d}\varphi_l. \tag{13.7}$$

In the original action-angle coordinates I_j, φ_l this structure has the form

$$\omega_2 = \frac{\partial^2 H(I)}{\partial I_j \partial I_l} \mathrm{d}I_j \wedge \mathrm{d}\varphi_l. \tag{13.8}$$

Hamiltonian system (13.5) is the universal canonical form for any non-degenerate completely integrable system (2.3) in the coordinates $J_1,\dots,J_k$ (13.4) and the original angle coordinates $\varphi_1,\dots,\varphi_k$.

The canonical forms in the Cartesian coordinates depend upon the topology of the invariant submanifolds $I_j = \text{const}$. In view of the Liouville Theorem [32] we assume that m coordinates $\varphi_1,\dots,\varphi_m$ are defined $\mathrm{mod}(2\pi)$ and run over the torus $\mathbb{T}^m$. The other $k-m$ coordinates $\rho_{m+1},\dots,\rho_k$ run over the Euclidean space $\mathbb{R}^{k-m}$. We define the Cartesian coordinates

$$\begin{aligned} &p_i = \sqrt{2J_i}\cos\varphi_i, \qquad q_i = \sqrt{2J_i}\sin\varphi_i, \quad i = 1,\dots,m \leqq k, \\ &p_j = J_j, \qquad q_j = \rho_j, \quad j = m+1,\dots,k, \qquad J_\alpha = \frac{\partial H(I)}{\partial I_\alpha}. \end{aligned} \tag{13.9}$$

In these coordinates symplectic structure (13.7) takes the canonical form

$$\omega_2 = \sum_{i=1}^{k} \mathrm{d}p_i \wedge \mathrm{d}q_i. \tag{13.10}$$

The Hamiltonian function $H_0(J)$ (13.6) takes the form (13.3). □

Remark 14. The symplectic form (13.7) is a special case of the invariant closed 2-forms ω_c (2.10). The corresponding functions $f_\alpha(I) = 0$ and function $B(J)$ has the form

$$B(J) = \frac{1}{2}(J_1^2 + \cdots + J_k^2). \tag{13.11}$$

The symplectic structure (13.7) implies the supplementary Hamiltonian representation (2.13) for the system (2.3). The corresponding Hamiltonian function $H_c(J)$ (2.14) coincides with $H_0(J)$ (13.6). In coordinates $I_1,\dots,I_k$ it is defined by the formula

$$H_c(I) = J_\alpha \frac{\partial B(J)}{\partial J_\alpha} - B(J) = \frac{1}{2}\left(\left(\frac{\partial H(I)}{\partial I_1}\right)^2 + \cdots + \left(\frac{\partial H(I)}{\partial I_k}\right)^2\right) . \qquad (13.12)$$

In coordinates J_α, φ_β the original symplectic structure $\omega_1 = \mathrm{d}I_\alpha \wedge \mathrm{d}\varphi_\alpha$ has the form

$$\omega_1 = \mathrm{d}\left(\frac{\partial \tilde{B}(J)}{\partial J_\alpha}\right) \wedge \mathrm{d}\varphi_\alpha = \frac{\partial^2 \tilde{B}(J)}{\partial J_\alpha \partial J_\beta} \mathrm{d}J_\alpha \wedge \mathrm{d}\varphi_\beta , \qquad (13.13)$$

where function $\tilde{B}(J)$ is the Legendre transform (2.42) of the Hamiltonian function $H(I)$.

Remark 15. Theorem 1 proves that any completely integrable Hamiltonian system (2.1) has a continuum of invariant Poisson structures in a toroidal domain $\mathcal{O} \subset M^n$, which are incompatible with the original Poisson structure P_1 (2.4). But among these structures there exist a continuum of compatible pairs. For example the Poisson structure $P_2 = \omega_2^{-1}$ (13.8) is compatible with all invariant Poisson structures $P_G = \omega_G^{-1}$, where the symplectic structure ω_G has the form

$$\omega_G = \mathrm{d}G_\alpha(J_\alpha) \wedge \mathrm{d}\varphi_\alpha, \qquad J_\alpha(I) = \frac{\partial H(I)}{\partial I_\alpha} . \qquad (13.14)$$

Here $G_\alpha(x)$ are arbitrary smooth functions of the single variable x. This is obvious because the corresponding (1,1) tensor $\tilde{A} = P_2 P_G^{-1}$ is diagonal in the coordinates J_i, φ_j and has the diagonal entries

$$\tilde{A}_\alpha^\alpha = \tilde{A}_{k+\alpha}^{k+\alpha} = G'_\alpha(J_\alpha), \quad \alpha = 1, \dots, k . \qquad (13.15)$$

For example if $G_\alpha(x) = G(x) = \frac{1}{2}x^2$, then the eigenvalues of the (1,1) tensor $\tilde{A}$ are equal to $J_\alpha(I)$ and have multiplicity 2. Hence we obtain that for any non-degenerate completely integrable Hamiltonian system (2.1) first integrals $J_\alpha(I_1, \dots, I_k)$ (13.14) can be presented as eigenvalues of the recursion operator $\tilde{A} = P_2 P_G^{-1}$ for two invariant compatible Poisson structures P_2 and P_G.

II. Let us consider in the toroidal domain $\mathcal{O} \subset M^n$ the original Poisson structure $P_1 = \omega_1^{-1}$ and the Poisson structure $P_2 = \omega_2^{-1}$ (13.7)–(13.8). The corresponding recursion operator $A = P_1 P_2^{-1}$ has the block structure

$$A = P_1 P_2^{-1} = \begin{pmatrix} \frac{\partial^2 H(I)}{\partial I_i \partial I_j} & 0 \\ 0 & \frac{\partial^2 H(I)}{\partial I_i \partial I_j} \end{pmatrix} \qquad (13.16)$$

in the action-angle coordinates $I_1, \dots, I_k$, $\varphi_1, \dots, \varphi_k$. For this (1,1) tensor the $k \times k$ matrices V_j, U_j, W_j and Q_j (11.14) have the following entries:

$$(V_j)_{il} = H_{,ilm} H_{,jm} - H_{,jim} H_{,ml}, \qquad (U_j)_{il} = 0 ,$$

$$(W_j)_{il} = H_{,ilm} H_{,jm} - H_{,jlm} H_{,mi}, \qquad (Q_j)_{il} = H_{,jlm} H_{,mi} - H_{,jim} H_{,ml} . \qquad (13.17)$$

Obviously the equality $W_j = V_j^t$ holds. Therefore, the formulae (11.12)–(11.14) imply that operator N_u has the following block structure:

$$N_u = \begin{pmatrix} V_j u^j & 0 \\ Q_j v^j & V_j^t u^j \end{pmatrix} , \tag{13.18}$$

where $j = 1, \ldots, k$. Hence we obtain that polynomial (11.15) is a perfect square

$$P_N(u, \lambda) = (\det(V_j u^j - \lambda))^2 . \tag{13.19}$$

This remarkable algebraic fact is a manifestation of a general theorem that will be published in our next paper. This theorem states that if a (1,1) tensor A is the recursion operator for two incompatible Poisson structures, $A = P_1 P_2^{-1}$, then the corresponding polynomial (11.15) is a perfect square.

14. General Invariant Poisson Structures

I. Olver in [47] and Turiel in [53] studied canonical forms of compatible pairs of Poisson structures and integrable systems which preserve them.

In this section, we present a classification of all Poisson structures $P^{\alpha\beta}$ which are invariant with respect to the integrable non-degenerate Hamiltonian system (2.3) provided that all its invariant submanifolds are compact. For the non-degenerate case $\det \| P^{\alpha\beta} \| \neq 0$ we give a second proof of the main results of Theorem 1.

The Kolmogorov condition (2.7) implies that the k functions $J_\alpha(I)$ (2.11) form a system of local coordinates in a ball B_r (2.4). Hamiltonian system (2.3) takes the equivalent form

$$\dot{J}_i = 0, \qquad \dot{\varphi}_i = J_i \tag{14.1}$$

in coordinates

$$J_1, \ldots, J_k, \qquad \varphi_1, \ldots, \varphi_k, \qquad \varphi_i = \varphi_i \bmod(2\pi) . \tag{14.2}$$

Trajectories of system (14.1) are everywhere dense on almost all tori (2.5).

In the toroidal coordinates J_i, φ_i a (2,0) tensor $P^{\alpha\beta}(J_i, \varphi_i)$ has the block structure

$$P^{\alpha\beta}(J_i, \varphi_i) = \begin{pmatrix} p_1 & p_3 \\ p_2 & p_0 \end{pmatrix} , \tag{14.3}$$

where $p_0, p_1, p_2, p_3(J_i, \varphi_i)$ are $k \times k$ matrices.

Theorem 14. 1) *A Poisson structure $P^{\alpha\beta}$ is invariant with respect to the dynamical system* (14.1) *if and only if it has the form*

$$P^{\alpha\beta} = \begin{pmatrix} 0 & -p(J) \\ p(J) & p_0(J) \end{pmatrix} , \tag{14.4}$$

where matrices $p(J)$ and $p_0(J)$ satisfy the equations

$$p^t(J) = p(J), \qquad p_0^t(J) = -p_0(J) , \tag{14.5}$$

$$p^{ij}_{,m} p^{ml} = p^{il}_{,m} p^{mj} , \tag{14.6}$$

$$p^{ij}_{0,m} p^{ml} + p^{jl}_{0,m} p^{mi} + p^{li}_{0,m} p^{mj} = 0 . \tag{14.7}$$

2) *Any first integrals F and G of an integrable non-degenerate Hamiltonian system* (2.3) *are in involution with respect to any invariant Poisson structure*

$$P^{\alpha\beta}\frac{\partial F}{\partial u^\alpha}\frac{\partial G}{\partial u^\beta}=0\,. \tag{14.8}$$

3) *If the invariant Poisson structure* (14.4) *is non-degenerate then*

$$p=B^{-1},\qquad p_0=p\sigma p\,,$$

$$B_{il}(J)=\frac{\partial^2 B(J)}{\partial J_i\partial J_l},\qquad 2\sigma_{il}(J)=\frac{\partial f_l(J)}{\partial J_i}-\frac{\partial f_i(J)}{\partial J_l} \tag{14.9}$$

for some functions $B(J)$ and $f_i(J)$.

Proof. 1) The invariance of a $(2,0)$ tensor $P^{\alpha\beta}$ with respect to dynamical system (14.1) is equivalent to the vanishing of the Lie derivative $L_V P$, where vector field V (14.1) has components

$$V^1=\cdots=V^k=0,\qquad V^{i+k}=J_i\,. \tag{14.10}$$

For any vector field V the Lie derivative $L_V P$ has the form [49]

$$(L_V P)^{\alpha\beta}=\dot P^{\alpha\beta}-V^\alpha_{,\gamma}P^{\gamma\beta}-V^\beta_{,\gamma}P^{\alpha\gamma}\,. \tag{14.11}$$

After substituting (14.3), (14.10) and (14.11) the invariance equation $L_V P=0$ takes the form of the matrix system

$$\dot p_1=0,\qquad \dot p_2=p_1,\qquad \dot p_3=p_1,\qquad \dot p_0=p_2+p_3\,. \tag{14.12}$$

In view of the key property of first integrals (2.9) solutions to (14.12) have the form

$$p_1(t)=\tilde p_1(J),\qquad p_2(t)=\tilde p_1(J)t+\tilde p_2(J),\qquad p_3(t)=\tilde p_1(J)t+\tilde p_3(J)\,,$$

$$p_0(t)=\tilde p_1(J)t^2+(\tilde p_2(J)+\tilde p_3(J))t+p_0(J)\,. \tag{14.13}$$

All entries of matrices p_a (14.3) are smooth functions of J_i,φ_i. Hence $p_a(t)$ are bounded on any torus $\mathbb{T}^k$ (2.5). Solutions (14.13) are bounded for $-\infty<t<+\infty$ if and only if

$$\tilde p_1(J)=0,\qquad \tilde p_2(J)=p(J),\qquad \tilde p_3(J)=-p(J)\,. \tag{14.14}$$

Therefore any invariant (2,0) tensor P has the block form (14.4).

By definition a Poisson structure $P^{\alpha\beta}$ satisfies equations

$$P^{\alpha\beta}=-P^{\beta\alpha}\,, \tag{14.15}$$

$$P^{\alpha\beta}_{,\tau}P^{\tau\gamma}+P^{\beta\gamma}_{,\tau}P^{\tau\alpha}+P^{\gamma\alpha}_{,\tau}P^{\tau\beta}=0\,. \tag{14.16}$$

Formulae (14.5) follow from (14.15) and (14.4). In view of (14.4) formulae (14.16) have different meaning for different (α,β,γ). Let $1\leqq i,j,l,m\leqq k$. Formulae (14.16) are identically true when

$$(\alpha,\beta,\gamma)=(i,j,l)\quad\text{or}\quad(\alpha,\beta,\gamma)=(i,j,l+k)\,. \tag{14.17}$$

Formulae (14.16) are equivalent to (14.6) when

$$(\alpha, \beta, \gamma) = (i, j+k, l+k)\,, \tag{14.18}$$

and they are equivalent to (14.7) when

$$(\alpha, \beta, \gamma) = (i+k, j+k, l+k)\,. \tag{14.19}$$

2) In view of (2.9) any first integral F of Hamiltonian system (2.3), (14.1) has the form $F = F(J_1, \dots, J_k)$. Therefore the involution relation (14.8) follows from (14.4).

3) Formula (14.4) implies

$$\det \|P(J)\| = (\det \|p(J)\|)^2\,. \tag{14.20}$$

Hence the invariant Poisson structure (14.4) is non-degenerate if and only if the symmetric matrix $p(J)$ is non-degenerate. Let $B(J) = p^{-1}(J)$ be the inverse (symmetric) matrix. Multiplying Eq. (14.6) with $B_{qi}B_{rj}B_{pl}$ and contracting with respect to the indices i, j, l we obtain

$$B_{qr,p} = B_{qp,r}, \quad 1 \leqq p, q, r \leqq k\,. \tag{14.21}$$

These equalities yield

$$B_{qr} = \frac{\partial B_q(J)}{\partial J_r} \tag{14.22}$$

for some functions $B_q(J)$. The symmetry of matrix B and (14.22) imply the equalities

$$B_q(J) = \frac{\partial B(J)}{\partial J_q}, \qquad B_{qr}(J) = \frac{\partial^2 B(J)}{\partial J_q \partial J_r} \tag{14.23}$$

for some function $B(J)$.

Let $\sigma = p^{-1} p_0 p^{-1}$. Then one has

$$p_0 = p\sigma p, \qquad \sigma^t = -\sigma\,. \tag{14.24}$$

In view of (14.5) and (14.6), Eq. (14.7) takes an equivalent form

$$\sigma_{ij,l} + \sigma_{jl,i} + \sigma_{li,j} = 0\,. \tag{14.25}$$

That means that the 2-form

$$\omega_2 = \sigma_{il}(J)\mathrm{d}J_i \wedge \mathrm{d}J_l \tag{14.26}$$

is closed, $d\omega_2 = 0$. The Poincaré Lemma implies that locally the 2-form ω_2 is exact

$$\omega_2 = \mathrm{d}(f_i(J)\mathrm{d}J_i)\,. \tag{14.27}$$

Hence the last of equalities (14.9) follows. □

II. In Theorem 14, we have presented the second proof of the classification of the non-degenerate invariant Poisson structures that has been discovered in Theorem 1. Theorem 14 also implies the existence of families of invariant degenerate Poisson structures (14.4)–(14.7). For example one gets an invariant Poisson structure (14.4) if

$$p^{il}(J) = 0, \qquad p_0^{il}(J) = -p_0^{li}(J)\,, \tag{14.28}$$

where functions $p_0^{il}(J)$ are arbitrary. In this case Eqs. (14.5)–(14.7) are satisfied identically. The family (14.4), (14.28) of degenerate Poisson structures depends upon $k(k-1)/2$ arbitrary functions $p_0^{il}(J)$. Thus this family is larger than the family of all non-degenerate Poisson structures (14.9) that depends upon $k+1$ arbitrary functions $f_l(J)$ and $B(J)$.

One gets another family if for some m,

$$p^{il}(J) = f_0(J)\delta^i_m\delta^l_m, \qquad p_0^{im}(J) = -p_0^{mi}(J) = f_i(J), \quad i \neq m\,,$$

$$p^{il}_{0,m}(J) = 0, \quad i \neq m, \quad l \neq m, \quad 1 \leqq m \leqq k\,. \tag{14.29}$$

Here $f_0(J), f_i(J)$ are k arbitrary functions of k variables $J_1, \ldots, J_k$ and $p_0^{il}(J) = -p_0^{li}(J)$ are $(k-1)(k-2)/2$ arbitrary functions of $k-1$ variables J_i for $i \neq m$. A direct substitution proves that functions (14.29) satisfy all Eqs. (14.5)–(14.7). Therefore, the corresponding matrices $p(J)$ and $p_0(J)$ define a family of invariant degenerate Poisson structures (14.4).

In general, the constructed Poisson structures (14.28) and (14.29) are incompatible with the original Poisson structure P_1 (2.32). That follows from the explicit formulae (15.23) for components of the Schouten bracket, see Sect. 15 below.

15. Necessary Conditions for Strong Dynamical Compatibility

I. Let P and Q be two arbitrary Poisson structures on a manifold M^n, $n = 2k$. Their Schouten bracket $[P, Q]$ is an alternating $(3, 0)$ tensor that has the following components:

$$2[P,Q]^{\alpha\beta\gamma} = P^{\alpha\beta}_{,\tau}Q^{\tau\gamma} + P^{\beta\gamma}_{,\tau}Q^{\tau\alpha} + P^{\gamma\alpha}_{,\tau}Q^{\tau\beta}$$

$$+ Q^{\alpha\beta}_{,\tau}P^{\tau\gamma} + Q^{\beta\gamma}_{,\tau}P^{\tau\alpha} + Q^{\gamma\alpha}_{,\tau}P^{\tau\beta}\,. \tag{15.1}$$

Let $\Lambda(T(M^n))$ be the exterior algebra of the tangent bundle $T(M^n)$

$$\Lambda(T(M^n)) = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_n\,. \tag{15.2}$$

Recall that $\Lambda_1 = T(M^n)$. Any alternating $(k, 0)$ tensor is a section of the fibre bundle Λ_k. The Poisson structures P and Q are sections of the fibre bundle Λ_2. The Schouten bracket $[P, Q]$ is a section of the fibre bundle Λ_3. The fibres of the bundles Λ_0 (scalar functions) and Λ_n (alternating $(n, 0)$ tensors) are one-dimensional.

Let S_m for $m = 0, 1, \ldots, k-2$ be an alternating $(n-1, 0)$ tensor of the form

$$S_m = [P,Q] \wedge P \wedge \cdots \wedge P \wedge Q \wedge \cdots \wedge Q\,, \tag{15.3}$$

where there are m factors P and $k-2-m$ factors Q. Let us denote $\langle W_r, \omega_r\rangle$ the complete contraction of the product of the $(r, 0)$ tensor W_r and the $(0, r)$ tensor ω_r.

II. We define $k-1$ differential 1-forms ξ^m by the formula

$$\xi^m = S_m \rfloor \omega_n\,, \tag{15.4}$$

where ω_n is an arbitrary non-degenerate n-form on M^n. Formula (15.4) means that

$$\xi^m(u) = \langle S_m \wedge u, \omega_n\rangle \tag{15.5}$$

for any tangent vector $u \in T_x(M^n)$.

The Poisson structures P and Q transform the 1-forms ξ^m into the vector fields U_m and V_m by the formulae

$$U_m^\tau = P^{\tau\nu}\xi_\nu^m, \qquad V_m^\tau = Q^{\tau\nu}\xi_\nu^m . \tag{15.6}$$

III. Let $R_l \in \Lambda_{n-2}$ be the wedge product of l factors P and $k-1-l$ factors Q, $l = 0, 1, \ldots, k-1$,

$$R_l = P \wedge \cdots \wedge P \wedge Q \wedge \cdots \wedge Q . \tag{15.7}$$

We define $2k(k-1)$ differential 1-forms ζ^α and ϑ^α,

$$\zeta^\alpha = (R_l \wedge U_m) \rfloor \omega_n, \qquad \vartheta^\alpha = (R_l \wedge V_m) \rfloor \omega_n . \tag{15.8}$$

Here α is the multi-index $\alpha = (l, m)$.

Let us denote the 1-forms ξ^m, ζ^α and ϑ^α by the general symbol θ^α. These 1-forms are defined uniquely up to a functional factor (as well as the n-form ω_n). They generate the distribution $\mathscr{B}^\perp \subset T^*(M^n)$ and assign the orthogonal distribution

$$\mathscr{B} \subset T(M^n), \qquad \langle \mathscr{B}, \mathscr{B}^\perp \rangle = 0 . \tag{15.9}$$

The distribution $\mathscr{B}$ is uniquely determined by the system of Pfaff equations

$$\theta^\alpha(u) = 0 . \tag{15.10}$$

These equations for the tangent vectors $u \in \mathscr{B}_x$ are equivalent to the system of equations in the exterior algebra $\Lambda(T(M^n))$,

$$S_m \wedge u = 0, \qquad R_l \wedge U_m \wedge u = 0, \qquad R_l \wedge V_m \wedge u = 0 . \tag{15.11}$$

Any dynamical system that preserves the two Poisson structures P and Q also preserves the two distributions $\mathscr{B}$ and $\mathscr{B}^\perp$.

Remark 16. The distributions $\mathscr{B}$ and $\mathscr{B}^\perp$ have very simple form if the Poisson structures P and Q are compatible. Indeed, in this case their Schouten bracket vanishes: $[P, Q] = 0$. Therefore the $k-1$ tensors S_m, $m = 0, 1, \ldots, k-2$ (15.3) and the 1-forms ξ^m (15.4) and ζ^α, ϑ^α (15.8) also vanish. Hence we obtain that $\mathscr{B}^\perp = 0$ and therefore distribution $\mathscr{B}$ (15.9) coincides with the tangent bundle $T(M^n)$ and has dimension $n = 2k$.

IV. **Necessary conditions. I.** *Assume that two incompatible Poisson structures P and Q are strongly dynamically compatible. Then the following necessary conditions are satisfied*:

1) *The distribution $\mathscr{B}^\perp \subset T^*(M^n)$ is annihilated by the Poisson structures P and Q and by their Schouten bracket $[P, Q]$. It means that the equations*

$$\langle P, \theta^\alpha \wedge \theta^\beta \rangle = 0, \qquad \langle Q, \theta^\alpha \wedge \theta^\beta \rangle = 0 , \tag{15.12}$$

$$\langle [P, Q], \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \rangle = 0, \qquad \langle [P, Q], \theta^\alpha \wedge \mathrm{d}\theta^\beta \rangle = 0 \tag{15.13}$$

hold for all differential 1-forms θ^α, θ^β and $\theta^\gamma \in \mathscr{B}^\perp$.

2) *The distribution $\mathscr{B} \subset T(M^n)$ satisfies the condition*

$$\dim \mathscr{B}_x \geqq k . \tag{15.14}$$

If $\dim \mathscr{B}_x = k$ *for a dense open set* $\mathcal{O} \subset M^n$, *then the distribution* $\mathscr{B}$ *is integrable and its fibres are tori* $\mathbb{T}^k$.

Proof. 1) Definition 2, Sect. 1, implies that the two Poisson structures P and Q are invariant with respect to some completely integrable Hamiltonian system

$$\dot{x}^i = P_1^{i\alpha} H_{,\alpha} , \tag{15.15}$$

where P_1 is some non-degenerate Poisson structure on M^n. Theorem 14 implies that the invariant Poisson structures P and Q have the following $k \times k$ block forms:

$$P = \begin{pmatrix} 0 & -p(J) \\ p(J) & p_0(J) \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -q(J) \\ q(J) & q_0(J) \end{pmatrix} \tag{15.16}$$

in coordinates J_i, φ_i (14.2). Here $p(J)$, $p_0(J)$, $q(J)$ and $q_0(J)$ are $k \times k$ matrices satisfying the equations

$$p^t = p, \qquad p_0^t = -p_0, \qquad q^t = q, \qquad q_0^t = -q_0 . \tag{15.17}$$

One has the basis of vector fields

$$e_i = \frac{\partial}{\partial J_i}, \qquad e_{i+k} = \frac{\partial}{\partial \varphi_i}, \quad i = 1, \ldots, k \tag{15.18}$$

presented in the local coordinates J_i, φ_i (14.2). The k vector fields $e_{i+k}(x)$ form a basis in the invariant k-dimensional distribution

$$\mathscr{L}_x = T_x(\mathbb{T}^k) \subset T_x(M^n) \tag{15.19}$$

that is tangent to the invariant tori $\mathbb{T}^k$ (2.5). The block structures (15.16) mean that the alternating $(2,0)$ tensors P and Q have the form

$$P = -2p^{ij}(J) e_i \wedge e_{j+k} + p_0^{ij}(J) e_{i+k} \wedge e_{j+k} , \tag{15.20}$$

$$Q = -2q^{ij}(J) e_i \wedge e_{j+k} + q_0^{ij}(J) e_{i+k} \wedge e_{j+k} . \tag{15.21}$$

A direct calculation of the Schouten bracket (15.1) leads to the following expressions for its components:

$$2[P,Q]^{i \cdot j+k \cdot l+k} = p^{ij}_{,m} q^{ml} - p^{li}_{,m} q^{mj} + q^{ij}_{,m} p^{ml} - q^{li}_{,m} p^{mj} , \tag{15.22}$$

$$\begin{aligned} -2[P,Q]^{i+k \cdot j+k \cdot l+k} &= p^{ij}_{0,m} q^{ml} + p^{jl}_{0,m} q^{mi} + p^{li}_{0,m} q^{mj} \\ &\quad + q^{ij}_{0,m} p^{ml} + q^{jl}_{0,m} p^{mi} + q^{li}_{0,m} p^{mj} , \end{aligned} \tag{15.23}$$

$$[P,Q]^{i \cdot j \cdot l+k} = 0, \qquad [P,Q]^{i \cdot j \cdot l} = 0 . \tag{15.24}$$

These formulae mean that the alternating $(3,0)$ tensor $[P,Q]$ has the form

$$[P,Q] = C^{ijl}(J) e_i \wedge e_{j+k} \wedge e_{l+k} + D^{ijl}(J) e_{i+k} \wedge e_{j+k} \wedge e_{l+k} . \tag{15.25}$$

The alternating $(n-1,0)$ tensor S_m (15.3) has the form

$$S_m = W_m \wedge e_{1+k} \wedge e_{2+k} \wedge \cdots \wedge e_{2k-1} \wedge e_{2k} , \tag{15.26}$$

where W_m is some alternating $(k-1,0)$ tensor. Indeed, formulae (15.20), (15.21) and (15.25) imply that in the wedge product (15.3) each monomial has at least k factors e_{i+k}. Therefore every non-zero monomial contains all factors e_{j+k} for $j = 1,\dots,k$.

Formula (15.26) yields that $S_m \wedge e_{i+k} = 0$ for all $i = 1,\dots,k$. Therefore the 1-forms ξ^m (15.5) have the form

$$\xi^m = \xi^m_i(J,\varphi)\mathrm{d}J_i\,. \tag{15.27}$$

Using the block structures (15.16) we obtain that the vector fields U_m and V_m (15.6) have the form

$$U_m = U^i_m(J,\varphi)e_{i+k}, \qquad V_m = V^i_m(J,\varphi)e_{i+k}\,. \tag{15.28}$$

These formulae along with (15.20) and (15.21) imply that the $(n-1,0)$ tensors $R_l \wedge U_m$ and $R_l \wedge V_m$ (15.7) have the same structure as tensor S_m (15.26). Hence we get

$$R_l \wedge U_m \wedge e_{i+k} = 0, \qquad R_l \wedge V_m \wedge e_{i+k} = 0 \tag{15.29}$$

for all $i = 1,\dots,k$. Therefore the 1-forms ζ^α and ϑ^α (15.8) have the same structure as the 1-forms ξ^m (15.27). Thus we have proved that all 1-forms θ^α (or $\xi^m,\zeta^\alpha,\vartheta^\alpha$) have the form

$$\theta^\alpha = \theta^\alpha_i(J,\varphi)\mathrm{d}J_i\,. \tag{15.30}$$

Now Eqs. (15.12) and (15.13) follow readily from the formulae (15.20), (15.21), (15.25) and (15.30).

2) The formulae (15.30) imply that the invariant k-dimensional distribution $\mathscr{L}$ (15.19) satisfies the equations

$$\theta^\alpha(\mathscr{L}) = 0\,. \tag{15.31}$$

Therefore $\mathscr{L}$ is embedded into the distribution $\mathscr{B}$ (15.10). Hence the condition (15.14) follows. If $\dim \mathscr{B}_x = k$ then the embedding $\mathscr{L} \subset \mathscr{B}$ implies that $\mathscr{B} = \mathscr{L}$. □

Corollary 4. *Assume that a dynamical system*

$$\dot{x}^i = V^i(x^1,\dots,x^n) \tag{15.32}$$

preserves the two Poisson structures P and Q and is completely integrable in the Liouville sense non-degenerate Hamiltonian system with respect to some non-degenerate Poisson structure P_1 and its invariant submanifolds are tori $\mathbb{T}^k$. Then the distribution $\mathscr{B}$ (15.9) contains tangent spaces of these invariant tori $\mathbb{T}^k$:

$$\mathscr{B}_x \supset T_x(\mathbb{T}^k)\,, \tag{15.33}$$

or $\theta^\alpha(T(\mathbb{T}^k)) = 0$ for all α.

Proof. The completely integrable non-degenerate Hamiltonian system (15.32) has form (15.15). The tangent spaces $T_x(\mathbb{T}^k)$ satisfy the Pfaff equations $dJ_l(T_x(\mathbb{T}^k))=0$, $l = 1,\dots,k$, in the corresponding coordinates J_l,φ_l (14.2). Therefore Eqs. (15.30) imply $\theta^\alpha(T_x(\mathbb{T}^k)) = 0$. Hence the inclusion (15.33) follows. □

IV. The necessary conditions (15.12)–(15.14) are applicable as well for the distributions $\mathscr{B}^\perp_m \subset T^*(M^n)$ and $\mathscr{B}_m \subset T(M^n)$ which are defined as follows. Let us denote the constructed 1-forms θ^α as $\theta^{1\,\alpha}$. We define a family of 1-forms $\theta^{i+1\,\alpha}$

from the family of 1-forms $\theta^{i\,\alpha}$ by induction. Let vector fields $U_{i\alpha}$ and $V_{i\alpha}$ have the coordinates

$$U^{\tau}_{i\alpha} = P^{\tau\nu}\theta^{i\cdot\alpha}_{\nu}, \qquad V^{\tau}_{i\alpha} = Q^{\tau\nu}\theta^{i\cdot\alpha}_{\nu}. \tag{15.34}$$

The 1-forms $\zeta^{li\alpha}$ and $\vartheta^{li\alpha}$ are defined as in (15.8):

$$\zeta^{li\alpha} = (R_l \wedge U_{i\alpha}) \rfloor \omega_n, \qquad \vartheta^{li\alpha} = (R_l \wedge V_{i\alpha}) \rfloor \omega_n. \tag{15.35}$$

The family $\theta^{i+1\cdot\alpha}$ consists of all 1-forms $\theta^{i\cdot\alpha}$, $\zeta^{li\alpha}$ and $\vartheta^{li\alpha}$, where $(i+1\cdot\alpha)$ is a new multi-index. These 1-forms generate a distribution $\mathscr{B}^{\perp}_{i+1} \subset T^*(M^n)$. The corresponding distribution $\mathscr{B}_{i+1} \subset T(M^n)$ is defined by the system of Pfaff equations

$$\theta^{i+1\cdot\alpha}(u) = 0, \quad u \in \mathscr{B}_{i+1\cdot x}. \tag{15.36}$$

Obviously one has the embeddings

$$\mathscr{B}^{\perp} \subset \cdots \subset \mathscr{B}^{\perp}_{i} \subset \mathscr{B}^{\perp}_{i+1} \subset \cdots \subset \mathscr{B}^{\perp}_{l}, \tag{15.37}$$

$$\mathscr{B} \supset \cdots \supset \mathscr{B}_{i} \supset \mathscr{B}_{i+1} \supset \cdots \supset \mathscr{B}_{l}. \tag{15.38}$$

These inclusions stabilize at some $l \leqq n$ because $\dim \mathscr{B}^{\perp}_{j} \leqq n$. We denote the stabilized distributions as $\mathscr{B}^{\perp}_{l}$ and $\mathscr{B}_{l}$. The necessary conditions (15.12)–(15.14) are equally applicable for the stabilized distributions $\mathscr{B}^{\perp}_{l}$ and $\mathscr{B}_{l}$ with the corresponding 1-forms $\theta^{l\cdot\alpha}$ and for all intermediate distributions (15.37) and (15.38) and have the same proof as above.

16. Necessary Conditions for Dynamical Compatibility

I. In this section we define a series of new invariants of two arbitrary Poisson structures P and Q which are determined on a manifold M^n of an arbitrary dimension $n = 2k$ or $n = 2k + 1$ and can be both degenerate. These invariants are preserved by any dynamical system that preserves the two Poisson structures P and Q. Constructions of this section are based on the one-dimensionality of the linear spaces $\Lambda_n(x)$ and $\Lambda^n(x)$ of alternating $(n,0)$ and $(0,n)$ tensors for each point $x \in M^n$.

First we assume that $n = 2k$. Let $T_m \in \Lambda_{2k}$ be the wedge product of m factors P and $k - m$ factors Q, $m = 0, 1, \ldots, k$:

$$T_m = P \wedge \cdots \wedge P \wedge Q \wedge \cdots \wedge Q. \tag{16.1}$$

Assume that at least one of the tensors $T_m(x)$ is not equal to zero in a neighbourhood of a point $x \in M^{2k}$. Using the fact that $\dim \Lambda_n(x) = 1$, we define a map of the manifold M^{2k} into the real projective space RP^k:

$$f_1 : M^{2k} \to RP^k, \tag{16.2}$$

$$f_1(x) = T_0(x) : T_1(x) : \cdots : T_k(x) \in RP^k. \tag{16.3}$$

This map is not defined in the points x where all $(n,0)$ tensors $T_m(x) = 0$.

Any dynamical system

$$\dot{x}^i = V^i(x^1, \ldots, x^n), \qquad L_V P = 0, \qquad L_V Q = 0 \tag{16.4}$$

that preserves the two Poisson structures P and Q also preserves all $(n,0)$ tensors T_m (16.1), which are proportional one to another because $\dim \Lambda_n(x) = 1$. Therefore the map f_1 (16.2) is first integral of dynamical system (16.4).

This construction defines first integrals of dynamical system (16.4) when both Poisson structures P and Q are degenerate and therefore the recursion operator $A = PQ^{-1}$ does not exist. For degenerate Poisson structures P and Q only $k-2$ coordinates of the map $f_1(x)$ (16.2) can be non-zero because tensors T_0 and T_k vanish.

II. Let us consider the alternating $(n,0)$ tensors

$$T_{ml} = S_m \wedge U_l, \qquad R_{ml} = S_m \wedge V_l\,, \tag{16.5}$$

where the $(2k-1,0)$ alternating tensors S_m have form (15.3) and vector fields U_l and V_l have form (15.6) and $m,l = 0,1,\ldots,k-2$. We assume that at least one of the tensors (16.5) is not equal to zero. These tensors are determined by the formulae (15.3)–(15.6) uniquely up to a common factor because the space $\Lambda^n(x)$ of alternating n-forms ω_n is one-dimensional. Therefore, for any point $x \in M^{2k}$ the $2(k-1)^2$ tensors (16.5) that belong to the one-dimensional space $\Lambda_n(x)$ uniquely define a point of the projective space RP^N, $N = 2(k-1)^2 - 1$. Hence we obtain the map

$$f_2 \,:\, M^{2k} \to RP^N\,, \tag{16.6}$$

$$f_2(x) = T_{00}(x) : R_{00}(x) : \cdots : R_{k-2\cdot k-2}(x) \in RP^N\,. \tag{16.7}$$

This map is first integral of any dynamical system (16.4).

The map (16.7) is not defined in the points x, where all $(n,0)$ tensors (16.5) vanish; for example, in the points x where the distribution $\mathscr{B}_x$ (15.10) has dimension n. Indeed, all 1-forms ζ^m (15.4) and vectors U_l, V_l (15.6) vanish at these points. Hence the tensors $T_{ml}(x)$ and $R_{ml}(x)$ (16.5) vanish and therefore the map f_2 (16.7) is not defined. The formulae (15.26) and (15.28) imply that tensors (16.5) vanish identically if the two Poisson structures P and Q are strongly dynamically compatible.

By proceeding in the same way one can construct more complicated first integrals

$$f_\alpha \,:\, M^{2k} \to RP^{N(\alpha,k)} \tag{16.8}$$

by considering the alternating $(n,0)$ tensors

$$T_{mi\alpha} = S_m \wedge U_{i\alpha}, \qquad R_{mi\alpha} = S_m \wedge V_{i\alpha}\,, \tag{16.9}$$

where vector fields $U_{i\alpha}$ and $V_{i\alpha}$ have form (15.34).

III. Assume that dimension of the manifold M^n is odd $n = 2k+1$. Let ω_n be any non-degenerate n-form on M^n. We define $k+1$ differential 1-forms ζ^m for $m = 0,1,\ldots,k$ by the formula

$$\zeta^m = T_m \rfloor \omega_n\,, \tag{16.10}$$

where the alternating $(2k,0)$ tensors T_m have the form (16.1). The Poisson structures P and Q transform the 1-forms ζ^m into the vector fields $\tilde{U}_m$ and $\tilde{V}_m$:

$$\tilde{U}^\tau_m = P^{\tau\nu}\zeta^m_\nu, \qquad \tilde{V}^\tau_m = Q^{\tau\nu}\zeta^m_\nu\,. \tag{16.11}$$

We define the $2(k+1)^2$ alternating $(n,0)$ tensors

$$\tilde{T}_{ml} = T_m \wedge \tilde{U}_l, \qquad \tilde{R}_{ml} = T_m \wedge \tilde{V}_l \,. \tag{16.12}$$

Assume that at least one of the tensors (16.12) is not equal to zero. These tensors are defined uniquely up to a common factor because the n-form ω_n in (16.10) is defined up to a factor. In view of $\dim \Lambda_n(x) = 1$ tensors (16.12) uniquely define for each point $x \in M^n$ a point of the projective space $RP^{\tilde{N}}$, $\tilde{N} = 2(k+1)^2 - 1$. Hence we obtain the smooth map

$$f_3 \,:\, M^{2k+1} \to RP^{\tilde{N}} \,, \tag{16.13}$$

$$f_3(x) = \tilde{T}_{00}(x) : \tilde{R}_{00}(x) : \cdots : \tilde{R}_{kk}(x) \in RP^{\tilde{N}} \,. \tag{16.14}$$

This map is first integral of any dynamical system (16.4).

IV. Assume that P and Q are arbitrary Poisson structures on a manifold M^n of odd dimension $n = 2k+1$. We define a distribution $\mathscr{B} \subset T(M^{2k+1})$ by the $k+1$ Pfaff equations

$$\zeta^m(u) = 0 \,, \tag{16.15}$$

where ζ^m are the 1-forms (16.10) and $m = 0, 1, \ldots, k$. This distribution $\mathscr{B}$ has dimension k in general. The map f_3 (16.13) is not defined in points x, where $\dim \mathscr{B}_x = 2k+1$. Indeed, all 1-forms ζ^m (16.10) and vectors $\tilde{U}_m$, $\tilde{V}_m$ (16.11) vanish at these points. Hence the tensors $\tilde{T}_{ml}(x)$ and $\tilde{R}_{ml}(x)$ (16.12) also vanish and therefore the map $f_3(x)$ (16.14) is not defined.

V. Let us define the k alternating $(n,0)$ tensors $(n = 2k+1)$

$$\tilde{S}_m = [P,Q] \wedge P \wedge \cdots \wedge P \wedge Q \wedge \cdots \wedge Q \,, \tag{16.16}$$

where there are m factors P and $k-1-m$ factors Q, $m = 0, 1, \ldots, k-1$. Assuming that at least one tensor $\tilde{S}_m(x) \neq 0$ we obtain the map

$$f_4 \,:\, M^{2k+1} \to RP^{k-1} \,, \tag{16.17}$$

$$f_4(x) = \tilde{S}_0(x) : \tilde{S}_1(x) : \cdots : \tilde{S}_{k-1}(x) \in RP^{k-1} \,. \tag{16.18}$$

This map is first integral of any dynamical system (16.4) because it preserves all tensors $\tilde{S}_m$ (16.15) and $\dim \Lambda_n(x) = 1$.

The direct product of the maps (16.13) and (16.17),

$$f_3 \times f_4 \,:\, M^{2k+1} \to RP^{\tilde{N}} \times RP^{k-1} \,. \tag{16.19}$$

also is first integral of any dynamical system (16.4).

Remark 17. First integrals f_1, f_2, f_α, f_3 and f_4 possess the following properties:

1) They are defined in some open domains $\mathcal{O} \subset M^n$.

2) These open domains are invariant with respect to any dynamical system V (16.4) that preserves the two Poisson structures P and Q. Indeed, the non-zero components of maps f_i remain to be non-zero after any diffeomorphism defined by the dynamical system V.

3) First integrals f_1 (16.2) and f_3 (16.13) depend upon the components of the Poisson structures P and Q and do not depend upon their partial derivatives.

4) First integrals f_2 (16.6), f_α (16.8) and f_4 (16.17) depend upon the Schouten bracket $[P, Q]$, and therefore upon the first order partial derivatives of the Poisson structures P and Q.

VI. **Necessary conditions. II.** *If two Poisson structures P and Q on a manifold M^n are dynamically compatible then the necessary condition*

$$\operatorname{rank} df(x) \leqq n-1 \tag{16.20}$$

is satisfied at all points $x \in \mathcal{O} \subset M^n$, where one of the maps

$$f \,:\, f_2, \qquad f_1 \times f_2, \quad f_\alpha, \quad f_3, \quad f_3 \times f_4 \tag{16.21}$$

is determined.

Proof. The constructions of the maps f (16.21) imply that they are determined in some open domains $\mathcal{O} \subset M^n$ which are invariant with respect to any dynamical system that preserves the Poisson structures P and Q. If the two Poisson structures P and Q are dynamically compatible then such dynamical system (16.4) does exist. Every map f (16.21) is first integral of this system. Therefore, every map f (16.21) is constant on each trajectory of system (16.4) in the invariant open domain $\mathcal{O}$, where f is defined. Hence the condition (16.20) follows. □

17. Concluding Remarks on the Role of the Compatibility Condition

(i) In the present paper, we have studied the geometric and algebraic properties of pairs of Poisson structures which are invariant with respect to some integrable dynamical system on a manifold M^n, $n = 2k$. We have proved that the compatibility in Magri's sense [34] of these structures is an exceptional and unstable phenomenon.

(ii) In Theorem 1, we have derived the complete classification of the non-degenerate Poisson structures P_c which are invariant with respect to a given completely integrable non-degenerate Hamiltonian system provided that the invariant submanifolds of this system are compact. The classification is given in a toroidal domain $\mathcal{O}$ in the action-angle coordinates I_α, φ_α, where the original Poisson structure P_1 has the standard form. This classification is presented by the general and previously unknown formula

$$\omega_c = \mathrm{d}\left(\frac{\partial B(J)}{\partial J_\alpha}\right) \wedge \mathrm{d}\varphi_\alpha + \mathrm{d}f_\alpha(I) \wedge \mathrm{d}I_\alpha\,,$$

$$J_\alpha(I) = \frac{\partial H(I)}{\partial I_\alpha}, \quad \alpha = 1, \ldots, k\,, \tag{17.1}$$

that describes all invariant closed 2-forms ω_c. Invariant non-degenerate Poisson structures are $P_c = \omega_c^{-1}$. Here $B(J)$ and the $f_\alpha(I)$ are arbitrary smooth functions of k variables and $H(I)$ is the Hamiltonian of the given integrable system. For a general function $B(J)$ the two Poisson structures P_c and P_1 are incompatible. Only exceptional Poisson structures P_c are compatible with P_1. The corresponding functions $B(J)$ are connected with the Hamiltonian function $H(I)$ by the overdetermined third-order nonlinear system of partial differential equations (11.9).

(iii) In Theorem 7, we have proved that for any invariant non-degenerate Poisson structure P_2 the property of compatibility with P_1 is unstable. By means of the method of "toroidal surgeries," we have constructed the supplementary invariant Poisson structures $P_C = \omega_p^{-1}$ (7.3) which are defined globally on the manifold M^n, are arbitrarily close to P_2 and are incompatible with the original Poisson structure P_1.

(iv) In Sect. 3, we have introduced a cohomology for dynamical systems on smooth manifolds. This cohomology $H^*(V, M^n)$ is a new invariant that characterizes the global properties of the dynamical system V on the manifold M^n. We have proved that the infinite-dimensionality of the cohomologies $H^2(V, M^{2k})$ and $H^4(V, M^{2k})$ is the necessary condition for the non-degenerate integrability of the dynamical system V on the manifold M^{2k}.

(v) In Sects. 5 and 6, we have pointed out applications connected with the Kepler problem, with the basic integrable problem of celestial mechanics, and with the harmonic oscillator. We have presented explicit formulae for a continuum of invariant symplectic and Poisson structures for these problems. In general, these Poisson structures are incompatible with the original Poisson structure P_1. However, these same formulae contain a continuum of compatible Poisson structures as well. The latter are unstable in a sense that they become incompatible with P_1 after arbitrarily small perturbations inside the general family of invariant Poisson structures.

(vi) The results obtained show that Magri's notion of compatibility of two Poisson structures and its counterpart, incompatibility, are not conceptionally adequate for a good insight into the diversity of pairs of Poisson structures. Therefore, we have introduced the new concepts of dynamical compatibility and strong dynamical compatibility of two arbitrary Poisson structures.

(vii) In Theorems 5 and 6, we have demonstrated that strongly dynamically compatible non-degenerate Poisson structures P_1 and P_2 have applications connected with the Kolmogorov–Arnold–Moser theory [2, 26, 27, 41]. Theorem 5 implies that KAM theory is applicable not only to small Hamiltonian perturbations of integrable non-degenerate Hamiltonian systems

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon P_1^{i\alpha} H_{,\alpha} , \tag{17.2}$$

but also to the rich family of non-Hamiltonian perturbations

$$\dot{x}^i = P_1^{i\alpha} H_{0,\alpha} + \varepsilon P_2^{i\alpha} H_{,\alpha} . \tag{17.3}$$

The family (17.3) depends upon the $k+1$ arbitrary smooth functions $B(J), f_1(I), \ldots, f_k(I)$ (17.1) of k variables and one arbitrary smooth function $H(x)$ of $2k$ variables.

(viii) In Theorem 10, we have proved that if on a manifold M^{2k} a dynamical system V preserves two strongly dynamically compatible non-degenerate Poisson structures P_1 and P_2 and the recursion operator $A = P_1 P_2^{-1}$ has k functionally independent eigenvalues then system V is completely integrable with respect to P_1 and P_2. Flows of all such dynamical systems commute with each other. The proof of Theorem 10 is independent upon the Lenard scheme [23, 34] that is not applicable for the two incompatible Poisson structures P_1 and P_2.

(ix) In Theorem 11, we have proved that any dynamical system $\dot{x}^i = V^i(x)$ that preserves two strongly dynamically compatible non-degenerate Poisson structures P_1 and P_2 in the general position generates an infinite hierarchy of completely integrable dynamical systems

$$\dot{x}^i = (A^m V)^i \,, \tag{17.4}$$

where $A = P_1 P_2^{-1}$ and m is an arbitrary integer. In contrast with the compatible case, neither P_1 nor P_2 are preserved in general by dynamical systems (17.4) for $|m| > 1$. Flows of all dynamical systems (17.4) commute with each other.

(x) In Theorem 12, we have presented several necessary conditions for strong dynamical compatibility of two non-degenerate incompatible Poisson structures P_1 and P_2. These necessary conditions are formulated in terms of the Nijenhuis tensor N_A and other geometric objects constructed from P_1, P_2, $A = P_1 P_2^{-1}$ and N_A.

(xi) In Sect. 15, we have introduced a distribution $\mathscr{B} \subset T(M^n)$ that is uniquely determined by two arbitrary Poisson structures P_1 and P_2. Necessary conditions for strong dynamical compatibility of the two Poisson structures are derived which connect the global property of strong dynamical compatibility with the local geometric invariants of the distribution $\mathscr{B}$.

(xii) In Sect. 16, we have introduced new invariants of an arbitrary pair of Poisson structures P_1 and P_2. These Poisson structures are defined on a manifold M^n of an arbitrary dimension $n = 2k$ or $n = 2k+1$ and can both be degenerate. The invariants are the smooth maps f of the manifold M^n into the real projective spaces $RP^{N(n)}$. For the two Poisson structures P_1 and P_2, we have derived the necessary condition for dynamical compatibility that has the form

$$\operatorname{rank}\, df(x) \leqq n-1 \tag{17.5}$$

at all points $x \in M^n$ where the maps f are defined.

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References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics. London: The Benjamin/Cummings Publishing, 1978
2. Arnold, V.I.: Proof of A.N. Kolmogorov's theorem on the preservation of quasi-periodic motions under small perturbation of the Hamiltonian. Uspekhi Mat. Nauk U.S.S.R **18**, Ser. **5**, 13–40 (1963)
3. Arnold, V.I : Mathematical methods of classical mechanics. Berlin, Heidelberg, New York: Springer, 1978
4. Birkhoff, G.D.: Dynamical systems. Providence, Rhode Island: American Mathematical Society, 1966
5. Bogoyavlenskij, O.I.: Invariant incompatible Poisson structures. Proc. R. Soc. Lond. A **450**, 723–730 (1995)
6 Bogoyavlenskij, O.I.: Incompatible Poisson structures and integrable Hamiltonian systems. C. R. Math. Rep. Acad. Sci. Canada **17**, 123–128 (1995)
7. Bogoyavlenskij, O.I.: A cohomology for dynamical systems. C. R. Math. Rep. Acad. Sci. Canada **17**, 253–258 (1995)
8. Brouzet, R.: Systèmes bihamiltoniens et complète intégrabilité en dimension 4. C. R. Acad. Sci. Paris **311**, S. I, 895–898 (1990)
9. Brouzet, R.: About the existence of recursion operators for completely integrable Hamiltonian systems near a Liouville torus. J. Math. Phys. **34**, 1309–1313 (1993)
10. Brouzet, R., Molino, P., Turiel, F.J.: Gèomètrie des systèmes bihamiltoniens. Indag. Mathem., N.S., **4**, 269–296 (1993)

11. Crampin, M., Marmo, G., Rubano, C : Conditions for the complete integrability of a dynamical system admitting alternative Lagrangians. Phys. Lett. A **97**, 88–90 (1983)
12 Das, A., Okubo, S.: A systematic study of the Toda lattice. Ann. Phys. **190**, 215–232 (1989)
13. De Fillipo, S., Vilasi, G., Marmo, G., Salerno, M.: A new characterization on completely integrable systems, Nuovo Cimento B **83**, 97–112 (1984)
14 De Rham, G : Sur l'Analysis situs des variétés a *n* dimensions. J. de Mathem. Pures and Appl. **10**, 115–200 (1931)
15. Dorfman, I.Ya.: Dirac structures and integrability of nonlinear evolution equations. New York: Wiley and Sons, 1993
16 Duff, G F D.: Differential forms in manifolds with boundary. Ann. Math. **56**, 115–127 (1952)
17. Duff, G.F.D., Spencer, D C : Harmonic tensors of Riemannian manifolds with boundary Ann. Math **56**, 128–156 (1952)
18. Duistermaat, J.J : On global action-angle coordinates. Comm. Pure Appl. Math. **33**, 687–706 (1980)
19. Fernandes, R.L.: Completely integrable bi-Hamiltonian systems J. Dyn. Diff Eq. **6**, 53–69 (1994)
20. Fuchssteiner, B.: Applications of hereditary symmetries to nonlinear evolution equations. Nonlinear Analysis, Theory, Methods and Applic. **3**, 849–862 (1979)
21 Fuchssteiner, B.: The Lie algebra structure of degenerate Hamiltonian and bi-Hamiltonian systems. Progr. Theor. Phys. **68**, 1082–1104 (1982)
22. Fuchssteiner, B., Fokas, A.S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries. Physica D **4**, 47–66 (1981)
23. Gardner, C.S , Greene, J.M., Kruskal, M.D., Miura, R M.: Korteweg-de Vries Equation and Generalizations VI. Methods for exact solution. Comm. Pure Appl. Math. **27**, 97–133 (1974)
24 Gelfand, I M., Dorfman, I.Ya.: Hamiltonian operators and algebraic structures related to them. Funct. Anal. Applic. **13**, 248–262 (1979)
25. Gelfand, I.M., Dorfman, I.Ya.: The Schouten bracket and Hamiltonian operators Funct. Anal. Applic. **14**, 223–226 (1980)
26. Kolmogorov, A N.: On conservation of conditionally periodic motions under small variations of the Hamiltonian. Dokl. Akad. Nauk SSSR **98**, 527–530 (1954)
27 Kolmogorov, A.N.: The general theory of dynamical systems and classical mechanics. In: Proc. Intern. Congr. Math. 1954, **1** Amsterdam: North-Holland Publ. Co., 1957, pp. 315–333
28. Kosmann–Schwarzbach, Y.: Géométrie des systèmes bihamiltoniens. Seminaire de Math. Sup., Press. Univers. de Montreal **102**, 185–216 (1986)
29 Kosmann–Schwarzbach, Y., Magri, F : Poisson–Nijenhuis structures. Ann. Inst. Henri Poincaré **53**, 35–81 (1990)
30. Landi, G., Marmo, G., Vilasi, G.: Recursion operators: Meaning and existence for completely integrable systems. J Math. Phys **35**, 808–815 (1994)
31. Lichnerowicz, A.: Les variétés de Poisson et leurs algèbres de Lie associées. J. Diff. Geom **12**, 253–300 (1977)
32 Liouville, J.: Note sur l'intégration des équations différentielles de la Dynamique J. Math. Pures Appl. **20**, 137–138 (1855)
33. Magnano, G., Magri, F : Poisson–Nijenhuis structures, truncated loop algebras and Sato's KP hierarchy. In: Integrable systems and quantum groups Singapore: World Scientific Publishing Company, 1992, pp. 147–172
34 Magri, F : A simple model of an integrable Hamiltonian system. J. Math Phys. **19**, 1156 –1162 (1978)
35 Magri, F : A geometrical approach to the nonlinear solvable equations. In: Nonlinear evolution equations and dynamical systems, M. Boiti, F Pempinelli and G. Soliani (Eds), Lecture Notes in Physics, **120**. Berlin, Heidelberg, New York: Springer, 1980, pp. 233–263
36. Magri, F , Morosi, C.: A geometrical characterization of integrable Hamiltonian systems through the theory of the Poisson–Nijenhuis manifolds. Quaderno S **19**, Universita di Milano (1984)
37 Marmo, G., Vilasi, G.: When do recursion operators generate new conservation laws? Phys. Lett. B **277**, 137–140 (1992)
38. Marsden, J , Weinstein, A : Reduction of symplectic manifolds with symmetry. Rep. Math. Phys. **5**, 121–130 (1974)
39 Marsden, J , Ratiu, T., Weinstein, A.: Semi-direct products and reduction in mechanics. Trans. Amer. Math. Soc **281**, 147–177 (1984)

40. McKean, H.P.: Compatible brackets in Hamiltonian mechanics In: Important developments in soliton theory. A.S. Fokas, V.E. Zakharov (eds.). Berlin, Heidelberg, New York: Springer, 1993, pp. 344–354
41. Moser, J.: Convergent series expansions for quasi-periodic motions. Math. Annalen **169**, 136 –176 (1967)
42 Nekhoroshev, N.N.: Action-angle variables and their generalizations. Trans. Moscow Math. Soc. **26**, 180–198 (1972)
43. Nekhoroshev, N.N.: The Poincaré–Lyapunov–Liouville–Arnol'd Theorem. Funct. Anal. Appl. **28**, 128–129 (1992)
44. Nijenhuis, A.: X_{n-1}-forming sets of eigenvectors. Proc. Kon Ned. Akad. Amsterdam **54**, 200 –212 (1951)
45. Oevel, W., Zhang, H., Fuchssteiner, B.: Mastersymmetries and multi-Hamiltonian formulations for some integrable lattice systems Progr. Theor. Phys. **81**, 294–308 (1989)
46. Olver, P J.: BiHamiltonian systems In: Pitman Research Notes in Mathematics Series **157**. B.D. Sleeman, R.J. Jarvis (eds.). New York: Longman Scientific and Technical, 1987, pp. 176 –193
47. Olver, P.J.: Canonical forms and integrability of bi-Hamiltonian systems. Phys. Lett. A **148**, 177–187 (1990)
48. Olver, P.J.: Applications of Lie groups to differential equations, 2nd edition. Berlin, Heidelberg, New York: Springer, 1993
49. Palais, R.S.: A definition of the exterior derivative in terms of Lie derivatives. Proc. Amer Math. Soc. **5**, 902–908 (1954)
50. Poincaré, H.: Les méthodes nouvelles de la méchanique céleste. T **1**. Paris: Gauthier - Villars, 1892
51. Siegel, C.L., Moser, J.K.: Lectures on celestial mechanics. Berlin, Heidelberg, New York: Springer, 1971
52. Souriau, J.M.: Structure des systèmes dynamiques. Paris: Dunod, 1970
53. Turiel, F.J : Classification locale d'un couple de formes symplectiques Poisson-compatibles. C.R. Acad. Sci. Paris **308**, S. I, 575–578 (1989)
54. Vilasi, G.: New aspects of complete integrability. In: Integrable systems and quantum groups Singapore: World Scientific Publishing Company, 1992, pp. 126–146
55. Ten Eikelder, H.M.M.: On the local structures of recursion operators for symmetries. Proc Kon. Ned. Akad. Amsterdam **89**, 389–403 (1986)

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