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A Matrix Integral Solution to $[P, Q] = P$ and Matrix Laplace Transforms

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Abstract: In this paper we solve the following problems: (i) find two differential operators P and Q satisfying $[P, Q] = P$, where P flows according to the KP hierarchy $\partial P/\partial t_n = [(P^{n/p})_+, P]$, with $p := \operatorname{ord} P \geq 2$; (ii) find a matrix integral representation for the associated τ-function. First we construct an infinite dimensional space $\mathcal{W} = \operatorname{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \dots\}$ of functions of $z \in \mathbb{C}$ invariant under the action of two operators, multiplication by z^p and $A_c := z\,\partial/\partial z - z + c$. This requirement is satisfied, for arbitrary p, if ψ_0 is a certain function generalizing the classical Hänkel function (for $p = 2$); our representation of the generalized Hänkel function as a double Laplace transform of a simple function, which was unknown even for the $p = 2$ case, enables us to represent the τ-function associated with the KP time evolution of the space $\mathcal{W}$ as a "double matrix Laplace transform" in two different ways. One representation involves an integration over the space of matrices whose spectrum belongs to a wedge-shaped contour $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ defined by $\gamma^\pm = \mathbb{R}_+ e^{\pm\pi i/p}$. The new integrals above relate to matrix Laplace transforms, in contrast with matrix Fourier transforms, which generalize the Kontsevich integrals and solve the operator equation $[P, Q] = 1$.

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Introduction

It is a long-standing puzzle in the theory of $2d$-gravity to find an adequate description of gravitational coupling of (p, q) minimal models. One part of it is to find two differential operators P and Q of order p and q respectively, such that $[P, Q] = f(Q)$ for some function f. In the simplest case of $q = 1$ and $f \equiv 1$, such description is provided by 1-matrix models, especially by the Kontsevich integral and their generalizations; see [1, 19, 25]. Going along the chain, $2d$-gravity $\rightarrow$ equilateral triangles $\rightarrow$ discrete matrix models $\rightarrow$ Kontsevich models, this approach has lead to the discovery of integrable structures for non-perturbative partition functions, which take the form of τ-functions of the KP hierarchy (see [7, 25, 31] for review and references). While similar results are believed to be true in the general (p, q)-case, the Kontsevich integral counterparts are still unknown. Note that a minor modification of the generalized Kontsevich integral can be interpreted as a duality transformation between (p, q) and (q, p)-models [18].

So far the most promising approach for finding integrable structures in the general (p, q)-case seems to be the one initiated by Kac–Schwarz in the case $q = 1$ and $f = 1$. So, the general problem comes in two stages: (i) Find a point in Sato's Grassmannian invariant under two symmetry operators, satisfying some commutation relation; the existence of such a plane leads to a system of differential equations specifying the wave function Ψ and thus to an algebra of constraints for the τ-function. (ii) Find a matrix integral representation for this τ-function. Note a matrix representation, beyond the case $q = 1$ and $f = 1$, if it exists at all, was unknown.

The purpose of this paper is to find a τ-function and a matrix integral representation for the equation $[P, Q] = P$ for $q = 1$ and arbitrary p . Remarkably, the matrix integral representation can still be found, but it is far less straightforward and considerably more involved, than the ordinary Kontsevich integral.

The message is the following: whereas the case $[P, Q] = 1$ is described by general *matrix Fourier transforms*, a solution to $[P, Q] = P$ is related to *double Laplace transforms*. While it is not known whether this solution has immediate physical relevance, it may help to shed some light on the (p, q)-case and on the matrix representations of the corresponding τ-functions. In particular, what are the proper multimatrix generalizations of the Kontsevich integrals?

Note this problem has come up in the physical literature, in various different contexts: unitary matrix models have been written down, leading to equations $[P, Q] = P$ for differential operators P and Q in the double scaling limit; see the studies of Dalley, Johnson, Periwal, Minahan, Morris, Shevitz, and Wätterstam [4, 5, 28, 22, 23]). In the mathematical context (inverse scattering and monodromy preserving transformations), see Ablowitz, Flaschka, Fokas and Newell [11, 9, 10]). The solution provided in our paper is new and does *not* require any scaling limit.

Consider the problem of finding a differential operator P of order p and another differential operator Q satisfying

$$[P,Q]=f(P)\,,\quad \text{with } 0\neq f(z)\in\mathbb{C}[z]\,. \tag{1}$$

When P is (formally) deformed with respect to the KP flows, i.e., $\partial P/\partial t_n = [(P^{n/p})_+, P]$, one can introduce the corresponding deformation of Q which preserves Eq. (1). Hence (1) can be considered as a condition on a solution of the p-reduced KP hierarchy.

The basic ingredients of this construction are[1]

- $\psi_0 \in 1+z^{-1}\mathbb{C}[[z^{-1}]]$,
- $A\colon \mathbb{C}((z^{-1}))\to\mathbb{C}((z^{-1}))$, a differential operator in z, which increases the order of an element of $\mathbb{C}((z^{-1}))$ in z exactly by one,

so that $\mathcal{W} := \mathrm{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0,\ldots\}$ belongs to the big stratum of the Sato Grassmannian and satisfies $A\mathcal{W}\subset\mathcal{W}$, such that

- ψ_0 satisfies the differential equation $v(z)\psi_0 = F(A)\psi_0$ for some $v(z)\in\mathbb{C}((z^{-1}))$ and $F(Z)\in\mathbb{C}[Z]$, so that $v(z)\mathcal{W}\subset\mathcal{W}$ also holds.

Let Ψ be the KP wave function corresponding to $\mathcal{W}$. The above conditions lead to the existence of differential operators Q and P in x such that $Q\Psi = A\Psi$ and $P\Psi = v(z)\Psi$. If A coincides with $\partial/\partial v = (1/v')\partial/\partial z$ up to the conjugation by a function, then we have $[P,Q]=1$. And if ψ_0 is defined by a Fourier transform and the action of A on it can be expressed in a suitable way, then the corresponding Hermitian *matrix* Fourier transform, properly normalized, is the corresponding τ-function. See Sect. 3 for details.

The matrix integral approach to (1) has so far needed $\mathrm{ord}\,Q = 1$ at the initial point of the formal KP time flows, requiring $\deg_z f(z)\le 1$. The degree 0 case can be reduced to $[P,Q]=1$. In this paper, we provide a solution to the degree 1 case, or the next simplest instance of (1), which can clearly be reduced to

$$[P,Q]=P\,, \tag{2}$$

with differential operators P and Q. As in the case of $[P,Q]=1$, we write the τ-function of its formal KP deformation explicitly in terms of a matrix integral.

Definition 1. *Let $-1<c<0$, $p\in\mathbb{Z}$, $p\ge 2$. Let $\mathcal{W}$ be the linear span*

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}}\{\psi_0(z),\psi_1(z),\psi_2(z),\ldots\}\,,$$

of generalized Hänkel functions,

$$\psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)}\int_1^\infty \frac{z^{-c}(uz)^k \mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\,,\quad k=0,1,2,\ldots\,, \tag{3}$$

also representable as double Laplace transforms

$$\psi_k(z) = \frac{p^{c+1}}{2\pi \mathrm{i}} z^{(p-1)(c+1)}\mathrm{e}^z \int_0^\infty dx\, x^c \mathrm{e}^{-xz^p}\int_0^\infty dy\, f_k(y)\mathrm{e}^{-xy^p} \tag{4}$$

of the functions

$$f_k(y) = (\zeta^{k+1}\mathrm{e}^{-\zeta y} - \zeta^{-k-1}\mathrm{e}^{-\zeta^{-1}y})y^k,\quad k=0,1,2,\ldots,\ \textit{with}\ \zeta := \mathrm{e}^{\pi\mathrm{i}/p}. \tag{5}$$

[1] $\mathbb{C}[[x]] := \{\sum_{n=0}^\infty a_n x^n \mid a_n\in\mathbb{C}\}$ is the ring of formal power series in x, and $\mathbb{C}((x)) := \{\sum_{-\infty\ll n<\infty} a_n x^n \mid a_n\in\mathbb{C}\}$ is the ring of formal Laurent series in x

Using the asymptotic expansion $\psi_k(z) = z^k(1 + O(1/z)) \in \mathbb{C}((z^{-1}))$ as $\Re z \to \infty$, $\mathscr{W}$ defines a point of the Sato Grassmannian Gr. Let Ψ and τ be the wave (formal Baker–Akhiezer) function and τ-function, respectively, associated with the KP time evolution $\mathscr{W}^t = \mathrm{e}^{-\sum t_i z^i}\mathscr{W}$; see Sects. 1 and 2. Then we have

Theorem 1.

$$\Psi(x,0,z) = \mathrm{e}^{xz}\psi_0\big((1-x)z\big)\,, \tag{6}$$

and it satisfies

$$\left(L\left(x-1,\frac{\partial}{\partial x}\right)-z^p\right)\Psi(x,0,z)=0 \text{ and } \left(L\left(z,\frac{\partial}{\partial z}-1\right)-(x-1)^p\right)\Psi(x,0,z)=0\,, \tag{7}$$

where $L(z,\partial/\partial z)$ is the monic differential operator

$$L\left(z,\frac{\partial}{\partial z}\right) := \frac{1}{z^p}\left(\prod_{i=0}^{p-1}\left(z\frac{\partial}{\partial z}+c-i\right) - cp\prod_{i=0}^{p-2}\left(z\frac{\partial}{\partial z}+c-i\right)\right) = \left(\frac{\partial}{\partial z}\right)^p + \cdots\,.$$

Note that for $p=2$, $L(z,\partial/\partial z) = (\partial/\partial z)^2 - (c^2+c)/z^2$.

Theorem 2. *Let $\mathscr{H}_N$ be the space of $N\times N$ Hermitian matrices, and $\mathscr{H}_N^+$ the subspace of $\mathscr{H}_N$ of positive definite Hermitian matrices. The corresponding τ-function evaluated at*

$$t_n := -\frac{1}{n}\operatorname{tr} Z^{-n}\,, \quad \textit{for } n=1,2,\ldots, \textit{ and with an } N\times N \textit{ diagonal } Z,$$

is given by the following (normalized) double matrix *Laplace transform:*

$$\tau(t) = S_1(t)\frac{\int_{\mathscr{H}_N^+} dX \det X^c \mathrm{e}^{-\operatorname{tr} Z^p X}\int_{\mathscr{H}_N^+} dY\, S_0(Y)\mathrm{e}^{-\operatorname{tr} XY^p}}{\int_{\mathscr{H}_N} dX \exp\operatorname{tr}\left(-\frac{(X+Z)^{p+1}}{p+1}\right)_2}\,,$$

where $(\,)_2$ denotes the terms quadratic in X,

$$S_0(Y) := \frac{\Delta(y^p)}{\Delta(y)^2}\det\big(f_{k-1}(y_i)\big)_{1\le i,k\le N} \quad \textit{and} \quad S_1(t) := \det(Z^{(p-1)(c+1/2)})\mathrm{e}^{\operatorname{tr} Z}\,,$$

where $y=(y_1,\ldots,y_N)$ are the eigenvalues of Y, $y^p = (y_1^p,\ldots,y_N^p)$, and $\Delta(y) := \prod_{i>j}(y_i-y_j) = \det(y_i^{j-1})_{i,j}$, and f_{k-1} are as in (5).

The function $\tau(t)$ also has the following matrix integral representation

$$\tau(t) = S_1(t)\frac{\int_{\mathscr{H}_N^\gamma} m(dW)\int_{\mathscr{H}_N^+} dX \det X^c\big(\Delta(w^p)/\Delta(w)\big)\mathrm{e}^{-\operatorname{tr} W}\mathrm{e}^{\operatorname{tr} X(W^p-Z^p)}}{\int_{\mathscr{H}_N} dX \exp\operatorname{tr}\left(-\frac{(X+Z)^{p+1}}{p+1}\right)_2}\,,$$

integrated over the space of matrices

$$\mathscr{H}_N^\gamma = \{W = UD_\gamma U^{-1} \mid U \in \mathbf{U}(N), D_\gamma := \operatorname{diag}(w_1,\ldots,w_N) \in (\gamma)^N\},$$

where γ denotes a wedge-shaped contour in $\mathbb{C}$, defined in Sect. 4 (see Fig. 1), in terms of a complex-valued measure

$$m(dW) = dU\,dw \prod_{1\le i<j\le N}(w_i-w_j)^2.$$

Theorem 3. *(i) The algebra of stabilizers of* $\mathscr{W}$,

$$S_{\mathscr{W}} := \{\phi(z, \partial/\partial z) \in \mathbb{C}((z^{-1}))[\partial/\partial z] \text{ such that } \phi\mathscr{W} \subset \mathscr{W}\},$$

is generated by $A_c := z\frac{\partial}{\partial z} - z + c$, z^p *and* $\xi := z^{-p}F(A_c)$, *where* $F(u) = \prod_0^{p-1}(u - i) - cp\prod_0^{p-2}(u - i)$:

$$S_{\mathscr{W}} = \mathbb{C}[A_c, z^p, \xi] \subset \mathbb{C}((z^{-1}))[\partial/\partial z].$$

Moreover, $\mathscr{W} = \mathbb{C}[A_c]\psi_0$, *and* ψ_0 *satisfies the differential equation*

$$F(A_c)\psi_0 = (-z)^p\psi_0(z)\,. \tag{8}$$

(ii) A family of solutions to the operator equation $[P, Q] = P$ *is given by the differential operators* P *and* Q *in* x, *defined equivalently by*

$$P\Psi = z^p\Psi\,, \quad Q\Psi = \frac{1}{p}A_c\Psi\,, \tag{9}$$

or by

$$P = S\left(\frac{d}{dx}\right)^p S^{-1} \quad \text{and} \quad Q = \frac{1}{p}(MP^{1/p} - P^{1/p} + c)\,,$$

where $M = S\left(\sum_1^\infty k\bar{t}_k(d/dx)^{k-1}\right)S^{-1}$, $\bar{t}_k = t_k + \delta_{k,1}x$, *with wave operator*[2]

$$S = \frac{\tau(\bar{t} - [(d/dx)^{-1}])}{\tau(\bar{t})}\,.$$

(iii) The function $\tau(t)$ *satisfies, in terms of the* W*-generators in Eq. (20), the following constraints*

$$\sum_{\substack{0\le i\le m\\ 0\le j\le i}} \alpha_{m,i}\binom{i}{j}\frac{(-1)^{i-j}}{j+1}W^{(j+1)}_{i+np-j}\tau(t) = a_{m,n,c}\tau(t)\,, \quad m, n = 0, 1, 2, \ldots, \tag{10}$$

for some constants $a_{m,n,c}$, *where the constants* $\alpha_{n,i}$ *are defined by the formula*[3] $(x \cdot d/dx)^n = \sum_{i=0}^n \alpha_{n,i}x^i(d/dx)^i$. *In particular, setting* $m = 1$, $\tau(t)$ *satisfies Virasoro constraints of the form (with* $W^{(2)}_{np} = \sum_{i+j=np}{:}J^{(1)}_i J^{(1)}_j{:}$*)*

$$\left(\frac{1}{2}W^{(2)}_{np} - \frac{\partial}{\partial t_{np+1}} - a_{1,n,c}\right)\tau = 0\,, \quad n = 0, 1, 2, \ldots. \tag{11}$$

Remark 1. The constants $a_{m,n,c}$ in (10) can all be calculated; in particular, the Virasoro constraint (11) for $n = 0$ becomes:

$$\left(\sum_1^\infty it_i\frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_1} - \frac{c(1+c)(p-1)}{2}\right)\tau = 0\,.$$

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[2] For $\alpha \in \mathbb{C}$, define $[\alpha] := (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \quad) \in \mathbb{C}^\infty$

[3] More explicitly, $\alpha_{n,i} = \frac{1}{i!}\sum_{j=0}^i \binom{i}{j}(-1)^{i-j}j^n$ Note that it vanishes if $n > 0$ and $i = 0$

1. The KP Hierarchy

Throughout, x is a formal scalar variable near 0, and z is a formal scalar variable near ∞. If $g(z) = cz^q\big(1 + O(z^{-1})\big)$, $c \neq 0$, then $\operatorname{ord}_z g(z) := q$ is the *order* of $g(z)$.

Throughout, we denote $\partial/\partial x$ by D. The algebra of ordinary pseudodifferential operators in x is denoted by $\mathscr{D}$ (the word "in x" may be dropped if there is no fear of confusion), with its splitting $\mathscr{D} = \mathscr{D}_+ + \mathscr{D}_-$ into the subalgebras of ordinary differential operators and of ordinary pseudodifferential operators of negative order:

$$\mathscr{D} = \left\{ \sum_{-\infty < i \leq n} a_i D^i \;\middle|\; n \in \mathbb{Z} \text{ arbitrary}, \; a_i = a_i(x) \right\},$$

$$A = \sum a_i D^i \in \mathscr{D} \quad \Rightarrow \quad A_+ = \sum_{i \geq 0} a_i D^i \in \mathscr{D}_+ \;\text{ and }\; A_- = A - A_+ \in \mathscr{D}_- .$$

The ring $\mathscr{D}$ acts on the space of functions of the form $\sum_{-\infty < i \ll \infty} a_i(x) z^i e^{xz}$ simply by extending the formulas $D^n e^{xz} = z^n e^{xz}$ and $A(Be^{xz}) = (A \circ B)e^{xz}$, A, $B \in \mathscr{D}$. When $A \in \mathscr{D}_+$, this definition of $A(Be^{xz})$ coincides with the usual action of A, as a differential operator, on Be^{xz} as a formal series in x with z-dependent coefficients.

A pseudodifferential operator in x may depend on the KP time variables $t = (t_1, t_2, \ldots)$ introduced below, but not on z unless otherwise noted. We are not specific about the regularity of the coefficients of pseudodifferential operators. The operators S, L, M etc., associated to a point $\mathscr{W}$ of the big stratum Gr^0 of the Sato Grassmannian (see below) have regular (i.e., formal power series) coefficients; otherwise, the singularities of those operators can be controled by the Schubert stratum to which $\mathscr{W} \in \mathrm{Gr}$ belongs. In particular, there exist n, $m \geq 0$ such that $x^n S$ and $S^{-1} x^m$ at $t = 0$ have regular coefficients. See [29] for details.

As in [2], we set $\bar{t} = (x + t_1, t_2, t_3, \ldots)$, and

$$\tilde{\partial} = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots \right).$$

The elementary Schur functions p_n are defined by $\exp\left(\sum_1^\infty t_n z^n\right) = \sum_0^\infty p_n(t) z^n$.

1.1. KP hierarchy. The operator $L = L(t) = D + \sum_{j=-\infty}^{-1} a_j(x,t) D^j \in \mathscr{D}$, with $t = (t_1, t_2, \ldots)$, subjected to the KP equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, \ldots,$$

is known to have the following representation in terms of an operator $S \in 1 + \mathscr{D}_-$ called the wave operator, and the associated, formally infinite order pseudodifferential operator

$$W := S e^{\sum_{i=1}^\infty t_i D^i},$$

as follows:

$$L = SDS^{-1} = WDW^{-1}, \tag{12}$$

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad \text{and} \quad \frac{\partial W}{\partial t_n} = (L^n)_+ W.$$

The wave function

$$\Psi(t,z) := \Psi(x,t,z) := We^{xz} = Se^{\sum_{i=1}^{\infty} \bar{t}_i z^i}, \tag{13}$$

where $\bar{t}_i = t_i + \delta_{i,1}x$, satisfies

$$L\Psi = z\Psi \quad \text{and} \quad \frac{\partial\Psi}{\partial t_n} = (L^n)_+\Psi, \tag{14}$$

and has the following representation in terms of a scalar-valued function associated to S called the tau function τ:

$$\begin{aligned}\Psi(t,z) &= \frac{\tau(\bar{t}-[z^{-1}])}{\tau(\bar{t})} e^{\sum_1^{\infty} \bar{t}_i z^i} \\ &= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} z^{-n} e^{\sum_1^{\infty} \bar{t}_i z^i} \\ &= \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} D^{-n} e^{\sum_1^{\infty} \bar{t}_i z^i},\end{aligned}$$

implying in view of (13)

$$S = \frac{\tau(\bar{t}-[D^{-1}])}{\tau(\bar{t})} := \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} D^{-n}. \tag{15}$$

Moreover, using (13), we have

$$\frac{\partial}{\partial z}\Psi = \frac{\partial}{\partial z} We^{xz} = W\frac{\partial}{\partial z}e^{xz} = Wxe^{xz} = WxW^{-1}\Psi,$$

thus leading to the operator

$$\begin{aligned}M &:= WxW^{-1} = Se^{\sum t_k D^k} xe^{-\sum t_k D^k} S^{-1} = S\left(x + \sum_1^{\infty} kt_k D^{k-1}\right) S^{-1} \\ &= S\left(\sum_1^{\infty} k\bar{t}_k D^{k-1}\right) S^{-1}\end{aligned} \tag{16}$$

satisfying

$$M\Psi = (\partial/\partial z)\Psi \quad \text{and} \quad [L, M] = W[D, x]W^{-1} = 1,$$

and for any formal series $f = f(x, \xi)$,

$$f(M, L) = Wf(x, D)W^{-1}. \tag{17}$$

1.2. Symmetries. Consider the Lie algebra w_∞ of operators

$$w_\infty := \mathbb{C}[z, z^{-1}][d/dz] = \mathrm{span}_{\mathbb{C}}\left\{ z^\alpha \left(\frac{\partial}{\partial z}\right)^\beta \,\middle|\, \alpha, \beta \in \mathbb{Z}, \beta \geq 0 \right\},$$

and its completion $\overline{w}_\infty := \mathbb{C}((z^{-1}))[\partial/\partial z]$ in the z^{-1}-adic topology, for the customary commutation relation [,]. Acting on Ψ, we have

$$z^\alpha (\partial/\partial z)^\beta \Psi = M^\beta L^\alpha \Psi\,, \tag{18}$$

motivating the definition of the following vector fields, called symmetries, on Ψ:

$$\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}\Psi := (M^\beta L^\alpha)_- \Psi\,.$$

We require that these flows act trivially on parameters x, t, and hence on $S^{-1}MS = \sum k\bar{t}_k D^{k-1}$, for instance.

Lemma 1. *There is an injective homomorphism of Lie algebras*

$$\overline{w}_\infty/\mathbb{C} \longrightarrow \left\{ \begin{array}{l} \textit{Lie algebra of vector fields} \\ \textit{on the manifold of wave functions } \Psi \\ \textit{commuting with the KP flows } \partial/\partial t_n \end{array} \right\}$$

$$z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta \longmapsto \mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}\Psi = (M^\beta L^\alpha)_- \Psi\,,$$

i.e.,

$$\left[\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}, \mathbb{Y}_{z^{\alpha'}(\partial/\partial z)^{\beta'}}\right] = \mathbb{Y}_{[z^\alpha(\partial/\partial z)^\beta, z^{\alpha'}(\partial/\partial z)^{\beta'}]}\,.$$

This definition differs from the one in [2] by the sign. Here this definition is chosen to make it consistent with the natural action of $\overline{w}_\infty$ on the Grassmannian discussed in the next section, rather than its negative. These vector fields induce vector fields on S and $L = SDS^{-1}$, as

$$\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}(S) = (M^\beta L^\alpha)_- S$$

and

$$\mathbb{Y}_{z^\alpha(\partial/\partial z)^\beta}(L) = [(M^\beta L^\alpha)_-, L]\,.$$

Proposition 1 ([2]). *We have*

$$-\frac{(M^n L^{n+\ell})_- \Psi}{\Psi} = (\mathrm{e}^{-\eta} - 1)\left.\frac{\frac{1}{n+1}W_\ell^{(n+1)}(\tau)}{\tau}\right|_{t_1 \to t_1 + x}, \quad n, \ell \in \mathbb{Z},\ n \geq 0, \tag{19}$$

where the $W_\ell^{(n+1)}$, the generators of the W_∞-algebra, are the coefficients in the expansion of the vertex operator

$$\begin{aligned} X(t, \lambda, \mu) &:= \exp\left(\sum_{i=1}^\infty (\mu^i - \lambda^i)t_i\right) \exp\left(\sum_{i=1}^\infty \frac{\lambda^{-i} - \mu^{-i}}{i}\frac{\partial}{\partial t_i}\right) \\ &= \sum_{k=0}^\infty \frac{(\mu - \lambda)^k}{k!} \sum_{\ell=-\infty}^\infty \lambda^{-\ell-k} W_\ell^{(k)}, \quad \textit{with } W_\ell^{(0)} = \delta_{\ell,0}. \end{aligned} \tag{20}$$

2. Grassmannian

Let $H := \mathbb{C}((z^{-1}))$, $H_+ := \mathbb{C}[z]$, and $H_- := z^{-1}\mathbb{C}[[z^{-1}]]$, so that $H = H_+ \oplus H_-$. We denote by Gr the Grassmannian manifold of linear subspaces $\mathcal{W}$ of H of relative dimension 0 with respect to H_+, i.e., the natural map

$$\pi_{\mathcal{W}} \colon \mathcal{W} \hookrightarrow H \xrightarrow{\pi} H/H_- \simeq H_+$$

being Fredholm of index 0. $\mathrm{Gr}^0 := \{\mathcal{W} \in \mathrm{Gr} \mid \pi_{\mathcal{W}} \text{ is isomorphism}\}$ is the big (open) Schubert stratum of Gr.

Given a wave function $\Psi = \Psi(x, t, z)$, let $\mathcal{W}$ be the point of Gr defined by[4]

$$\begin{aligned} \mathcal{W} &= \mathrm{span}_{\mathbb{C}}\left\{ \frac{\partial^j}{\partial x^j}\Psi(0,0,z) \,\middle|\, j = 0,1,2,\dots \right\} \\ &= \mathrm{span}_{\mathbb{C}}\left\{ \frac{\partial^{j_1+\quad +j_N}}{\partial t_1^{j_1}\dots\partial t_N^{j_N}}\Psi(0,0,z) \,\middle|\, N \geq 0,\ j_1,\dots,j_N \geq 0 \right\}. \end{aligned}$$

The first line guarantees $\mathcal{W} \in \mathrm{Gr}$, and the second line follows from the first by using the second equation in (14), i.e., the KP time evolutions of Ψ. Hence up to the t-adic completion we have

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}}\left\{ \left(\frac{\partial}{\partial x}\right)^j \Psi(0,t,z) \,\middle|\, j = 0,1,2,\dots \right\},$$

so that, letting $\psi = \mathrm{e}^{-\sum t_i z^i}\Psi$ and

$$\mathcal{W}^t := \mathrm{e}^{-\sum t_i z^i}\mathcal{W} = \mathrm{span}_{\mathbb{C}}\{(\partial/\partial x)^j \psi(0,t,z) \mid j = 0,1,2,\dots\},$$

we have $\psi = (\pi_{\mathcal{W}^t})^{-1}(1)$, i.e., ψ is the preimage of 1 by the map $\pi_{\mathcal{W}^t} \colon \mathcal{W}^t \to H_+$.

The corresponding τ-function $\tau(t)$ is the determinant of the composite map

$$\mathcal{W} \xrightarrow{g} \mathcal{W}^t \xrightarrow{\pi_{\mathcal{W}^t}} H/H_- \simeq H_+, \tag{21}$$

where g denotes the multiplication by $\mathrm{e}^{-\sum t_i z^i}$. Given $\mathcal{W}$, the determinant is well-defined up to a constant which is determined by the choice of a basis $\{\psi_k\}_{k=0}^{\infty}$, $\psi_k = z^k\big(1 + O(z^{-1})\big)$ for $k \gg 0$, of $\mathcal{W}$. We take $\{z^k\}_{k=0}^{\infty}$ as the basis of H_+. More specifically, $\tau(t)$ is defined as the limit as $n \to \infty$ of the determinant of

$$\mathcal{W}_n \hookrightarrow \mathcal{W} \to H_+ \to H_+/z^n H_+, \tag{22}$$

where the middle arrow is the composite map in (21), $\mathcal{W}_n = \mathrm{span}_{\mathbb{C}}\{\psi_k\}_{k=0}^{n-1}$, and the determinant is computed with respect to the bases $\{\psi_k\}_{k=0}^{n-1}$ of $\mathcal{W}_n$ and $\{z^k\}_{k=0}^{n-1}$ of $H_+/z^n H_+$. The limit exists in the t-adic topology of $\mathbb{C}[[t]]$, i.e., for any multi-index α, there exists a positive integer n_α such that, if $n \geq n_\alpha$, then the coefficient of t^α in the determinant of (22) is independent of n, and gives the coefficient of t^α in $\tau(t)$. This finiteness property is an immediate consequence of the fact that, expanding $\tau(t)$ in terms of Schur functions, the coefficients give the Plücker coordinates of $\mathcal{W}$. See [29] for details.

The $\overline{w}_\infty$-action on Ψ becomes the natural action of $\overline{w}_\infty$ on Gr: As an ordinary differential operator in z, each $A \in \overline{w}_\infty$ acts on H, which defines a vector field on Gr.

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[4] If Ψ is singular at $(x,t) = 0$, we need to replace $(\partial/\partial x)^j\Psi(0,0,z)$ in the first line by $(\partial/\partial x)^j\left(x^n\Psi(x,0,z)\right)|_{x=0}$ for some $n > 0$, and make a similar replacement in the second line (see [29] for details) We chose to write the formulas for $\mathcal{W} \in \mathrm{Gr}^0$ for simplicity

2.1. Stabilizers. Given $\mathscr{W} \in \mathrm{Gr}$, we shall call

$$S_{\mathscr{W}} := \{Q := Q(z, \partial/\partial z) \in \overline{w}_\infty \mid Q\mathscr{W} \subset \mathscr{W}\}$$

the *stabilizer* of $\mathscr{W}$. In this subsection we shall observe basic properties of the stabilizer which can be obtained without referring to matrix integrals.

Lemma 2. *Let* $\mathscr{W} \in \mathrm{Gr}$ *and* $A := \sum_{-\infty<i\ll\infty, 0\le j\ll\infty} c_{ij} z^i (\partial/\partial z)^j \in \overline{w}_\infty$. *If*

$$A\mathscr{W} \subset \mathscr{W}, \tag{23}$$

then

$$Q_A := \sum_{\substack{-\infty<i\ll\infty \\ 0\le j\ll\infty}} c_{ij} M^j L^i \in \mathscr{D}_+ .$$

Conversely, if $Q \in \mathscr{D}_+$ *is of this form, i.e.,* $Q = Q_A$ *for some* $A \in \overline{w}_\infty$, *then this* A *satisfies (23).*

Proof. We have

$$A\Psi(t, z) = Q_A \Psi(t, z) \tag{24}$$

by definition. Since $A\mathscr{W} \subset \mathscr{W}$, and since the Taylor coefficients (or Laurent coefficients if $\mathscr{W} \notin \mathrm{Gr}^0$) in x of Ψ generates $\mathscr{W}$, $A\Psi$ is a $\mathbb{C}[[x, t]]$-linear combination of Ψ, $D\Psi$, $D^2\Psi$, ..., i.e., $A\Psi = Q\Psi$ for some $Q \in \mathscr{D}_+$. Hence, since (24) determines Q_A uniquely, Q_A itself must be in $\mathscr{D}_+$. Conversely, suppose $Q_A \in \mathscr{D}_+$, and let $\Psi(x, 0, z) = \sum f_n(z) x^n$ be the Taylor (or Laurent) expansion of $\Psi(x, 0, z)$ at $x = 0$. Then each Taylor coefficient in x of $Q_A\Psi$ is a linear combination of $\{f_n(z)\}$, and hence it belongs to $\mathscr{W}$, so that by (24) $Af_n \in \mathscr{W}$ for every n (the action of A on f_n is well-defined since A is a differential operator in z). Since $\{f_n\}$ is a basis of $\mathscr{W}$, we have $A\mathscr{W} \subset \mathscr{W}$.

Corollary 1. *Let* $p \neq 0$ *be an integer, and let* $Q \in \mathscr{D}_+$ *such that* $\mathrm{ad}(L^p)^N Q = 0$ *for* $N \gg 0$. *Then* $Q = Q_A$ *for some* $A \in \overline{w}_\infty$ *such that* $A\mathscr{W} \subset \mathscr{W}$ *holds. In particular, a solution to the string equation (1) always comes from a pair of* $A \in \overline{w}_\infty$ *and* $\mathscr{W} \in \mathrm{Gr}$, *such that* $A\mathscr{W} \subset \mathscr{W}$ *(and* $z^p\mathscr{W} \subset \mathscr{W}$ *due to the extra assumption* $P = L^p \in \mathscr{D}_+$*).*

Proof. Writing $Q = \sum_{ij} c_{ij} M^j L^i$, let $A = \sum_{ij} c_{ij} z^i (\partial/\partial z)^j$. Since $\mathrm{ad}(L^p)^N Q = 0$ we have $\mathrm{ad}(z^p)^N A = 0$, which implies that A is a differential operator in z. Hence the "converse" part of Lemma 2 applies.

Lemma 3. *Let* A, $B \in \overline{w}_\infty$, $\psi_0 = 1 + O(z^{-1}) \in 1 + H_-$ *and* $\mathscr{W} \in \mathrm{Gr}$. *Suppose* A *acts on the monomials* z^k, $k \in \mathbb{Z}$, *as*

$$Az^k = z^{k+1}\left(c_k + O(z^{-1})\right),$$

and $c_k \neq 0$ *if* $k \ge 0$. *Then the following conditions are equivalent:*

(i) $\psi_0 \in \mathscr{W}$, $A\mathscr{W} \subset \mathscr{W}$ *and* $B\mathscr{W} \subset \mathscr{W}$;
(ii) $\mathscr{W} = \mathrm{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \dots\}$, *and* ψ_0 *satisfies the differential equations*

$$BA^n\psi_0 = F_n(A)\psi_0, \quad n = 0, 1, \dots \tag{25}$$

for some $F_n(s) \in \mathbb{C}[s]$.

In particular, under these conditions $\mathcal{W}$ belongs to the big stratum Gr^0 of Gr. If, moreover, A and B satisfy a commutation relation of the form

$$[A, B] = a(A)B + b(A) \tag{26}$$

for some $a(s)$, $b(s) \in \mathbb{C}[s]$, then in (25) it suffices to assume only the $n = 0$ case, i.e.,

$$B\psi_0 = F(A)\psi_0 \tag{27}$$

for some $F(s) \in \mathbb{C}[s]$.

Proof. Since $\psi_0 \in \mathcal{W}$, $A\mathcal{W} \subset \mathcal{W}$ implies $\mathcal{W}' := \mathrm{span}_{\mathbb{C}}\{\psi_0, A\psi_0, A^2\psi_0, \ldots\} \subset \mathcal{W}$. Since $\psi_0 = 1 + O(z^{-1})$ and A raises the order of a function in z by 1, the map $\mathcal{W}' \to H_+$ is bijective, and $\mathcal{W}' \in \mathrm{Gr}^0$. In particular, both $\mathcal{W}$ and $\mathcal{W}'$ are of relative dimension 0, so that $\mathcal{W} = \mathcal{W}'$. Conversely, $\mathcal{W} = \mathcal{W}'$ clearly implies $\psi_0 \in \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$. Assume these equivalent conditions. Then $B\mathcal{W} \subset \mathcal{W}$ if and only if $B\mathcal{W}' \subset \mathcal{W}'$ if and only if the differential equations of the form (25) are satisfied. Finally, when A and B satisfy a commutation relation of the form (26), the n^{th} equation in (25) implies the $(n + 1)^{\text{st}}$ one, so that (27) suffices.

The following propositions take a closer look at the $[P, Q] = 1$ case and $[P, Q] = P$ case, to show that essentially those elements in $\overline{w}_\infty$ which give rise to P and Q in the sense of Lemma 2, and their polynomials, are the only elements of the stabilizer.

Proposition 2. *Let $p \in \mathbb{Z}$, $p > 0$. Let $A \in \overline{w}_\infty$ be such that $[A, z^p] = 1$. If $\mathcal{W} \in$ Gr satisfies $z^p\mathcal{W} \subset \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$, then the stabilizer of $\mathcal{W}$ is generated by z^p and A, i.e., $S_{\mathcal{W}} = \mathbb{C}[A, z^p]$.*

Proof. Since $[A, z^p] = 1$, A is a first order differential operator in z, so that any $C \in S_{\mathcal{W}}$ can be written as $C = \sum_{-\infty < i \ll \infty, 0 \le j \le N} a_{ij} z^i A^j$ for some $N \geq 0$. It suffices to prove that $a_{ij} = 0$ if $i < 0$ or if $i \not\equiv 0 \bmod p$. Suppose A raises the order of a function in z by k: $\mathrm{ord}_z\, Az^\ell = \ell + k$. Let I be the set of pairs (i, j) such that $i < 0$ or $i \not\equiv 0 \bmod p$, $a_{ij} \neq 0$, and $i + kj$ is maximum among all such a_{ij}'s. We have $|I| < \infty$, and we only need to prove $|I| = 0$. Suppose this is not true. Let $C_0 := \sum_{(i,j)\in I} a_{ij} z^i A^j$. Noting

$$[A, z^i A^j] = [A, (z^p)^{i/p}]A^j = (i/p)z^{i-p}A^j,$$

so that $\mathrm{ad}(A)^n(z^i A^j) = 0$ for $n \gg 0$ if and only if $i \geq 0$ and $i \equiv 0 \bmod p$, we see that for $n \gg 0$ the leading terms of $\mathrm{ad}(A)^n C$ are $\mathrm{ad}(A)^n C_0$, which lowers the order of a function in z, and does not annihilate the function for a general n. This cannot happen since $\mathrm{ad}(A)^n C\mathcal{W} \subset \mathcal{W}$, and since in $\mathcal{W}$ the order of functions in z are bounded from below.

Proposition 3. *Let $p \in \mathbb{Z}$, $p > 0$. Let $A = z\partial/\partial z - a(z)$, where $a(z) \in z + \mathbb{C}[[z^{-1}]]$, and $\psi_0 = 1 + O(z^{-1}) \in 1 + H_-$. Let $\mathcal{W} \in$ Gr be the point of the Grassmannian determined by the conditions $\psi_0 \in \mathcal{W}$ and $A\mathcal{W} \subset \mathcal{W}$. Suppose $\mathcal{W}$ also satisfies $z^p\mathcal{W} \subset \mathcal{W}$. Let $F(s) = c\prod_{i=1}^{p}(s - c_i) \in \mathbb{C}[s]$, where $c_i \in \mathbb{C}$, $c \in \mathbb{C}^*$, be the polynomial of degree p as in (27) with $B = z^p$, i.e., ψ_0 satisfies the equation*

$$F(A)\psi_0 = z^p\psi_0\,. \tag{28}$$

Then if F satisfies the following genericity condition:

(G) *For any* $n \not\equiv 0 \bmod p$, *we have* $(F)+n := \sum(c_i+n) \not\equiv (F) \bmod p$, *i.e.*, $\pi_p\big((F)+n\big) \neq \pi_p(F)$, *where* $(F) = \sum_{i=1}^{p}(c_i)$ *is the divisor of* F, *and* $\pi_p\colon \mathbb{C} \to \mathbb{C}/p\mathbb{Z}$ *is the natural projection,*

then the stabilizer of $\mathscr{W}$ *is generated by* A, z^p *and* $\xi := z^{-p}F(A)$, *i.e.*,

$$S_{\mathscr{W}} = \mathbb{C}[A, z^p, \xi]\,. \tag{29}$$

Remark 2. Condition (G) is equivalent to

(G′) There does *not* exist $n \mid p$, $0 < n < p$, and $H(s) \in \mathbb{C}[s]$ of degree n such that $F(s) = \prod_{i=0}^{p/n} H(s-in)$;

and if it is not satisfied, i.e., if $F(s) = \prod_{i=0}^{p/n} H(s-in)$ for some $n \mid p$ and H, then taking such (n, H) of the smallest n, we observe from our proof below that $\mathbb{C}[A, z^p, \xi] \subset S_{\mathscr{W}} \subset \mathbb{C}[A, z^n, \xi']$, where $\xi' = z^{-n}H(A)$.

Remark 3. The right-hand side of (29) equals $\sum_{i,j,k\geq 0} \mathbb{C}a^i b^j c^k$, where (a, b, c) is any permutation of (A, z^p, ξ); the order does not matter because

$$[A, z^p] = pz^p\,, \quad [A, \xi] = -pF(A) \quad \text{and} \quad [z^p, \xi] = F(A) - F(A-p)\,. \tag{30}$$

Remark 4. Condition (G) is satisfied by the F in Theorem 3: Since

$$F(s) = \left(\prod_{i=0}^{p-2}(s-i)\right)\big(s-(p-1+cp)\big)$$

and $-1 < c < 0$, there is no period less than p in the divisor of F modulo p.

Proof of Prop. 3. Using the commutation relations (30), the definition of $\mathscr{W}$, and Eq. (28), we observe easily that $S_{\mathscr{W}} \supset \mathbb{C}[A, z^p, \xi]$. We prove the converse inclusion in two steps. Only Step 2 needs Condition (G).

Step 1. We observe that $S_{\mathscr{W}}$ is spanned by the z-homogeneous elements in $S_{\mathscr{W}}$, i.e., the elements of $S_{\mathscr{W}}$ of the form $z^n f(A)$, where $n \in \mathbb{Z}$ and $f(s) \in \mathbb{C}[s]$.

Indeed, let $S' \subset S_{\mathscr{W}}$ be the subspace of $S_{\mathscr{W}}$ spanned by the z-homogeneous elements, and suppose that $S'' := S_{\mathscr{W}} \setminus S' \neq \emptyset$. Let N be a nonnegative integer such that

$$S''^{(N)} := \{C \in S'' \mid \operatorname{ord}_{\partial/\partial z} C \leq N\}$$

is nonempty. Let $C \in S''^{(N)}$ be such that, writing

$$C = \sum z^n f_n(A)\,, \tag{31}$$

$n_0(C) := \max\{n \mid f_n \not\equiv 0\}$ is the smallest in $S''^{(N)}$. Such a C exists because

Claim: $\{n_0(C) \mid C \in S''^{(N)}\}$ is bounded below.

Proof. Indeed it is bounded from below by $-2N+1$: since $C \in S''^{(N)}$ is an ordinary differential operator of order $\leq N$, and since $\psi_0, A\psi_0, \ldots, A^{N-1}\psi_0$ are linearly independent, we have $CA^i\psi_0 \not\equiv 0$ for some i, $0 \leq i < N$. Since $A^i\psi_0 = (-1)^i z^i\big(1 + O(z^{-1})\big)$, since $C\mathscr{W} \subset \mathscr{W}$, and since $\mathscr{W}$ is a span of $A^j\psi_0$ for $j \geq 0$, we observe that C does not decrease the order of $A^i\psi_0$ in z by more than $N-1$. This implies, using the notation of (31), that $n + \deg f_n \geq -(N-1)$ for some n. Hence $n_0(C) \geq n \geq -\deg f_n - (N-1) \geq -(2N-1)$.

Now let

$$C' := [A, C] - n_0(C)C = \sum \big(n - n_0(C)\big) z^n f_n(A). \tag{32}$$

Clearly $C' \in S_{\mathscr{W}}$. We have $\operatorname{ord}_{\partial/\partial z} C' \le \operatorname{ord}_{\partial/\partial z} C \le N$, and $n_0(C') \le n_0(C) - 1$. Hence by the minimality of $n_0(C)$, we must have $C' \notin S''^{(N)}$, so that $C' \in S'$. Thus each term $\big(n - n_0(C)\big) z^n f_n(A)$ in (32) belongs to S', and only finitely many f_n are non-zero. As a finite linear combination of such, we have $C'' := C - z^{n_0(C)} f_{n_0(C)}(A) \in S'$, so that $z^{n_0(C)} f_{n_0(C)}(A) = C - C''$ must also belong to $S_{\mathscr{W}}$, and hence to S', since it is z-homogeneous. This implies $C = C'' + (C - C'') \in S'$, which is a contradiction.

Step 2. Let $f(s) \not\equiv 0$ be any constant coefficient polynomial, and let n be an integer. We prove that

$$z^n f(A) \in S_{\mathscr{W}} \quad \text{implies} \quad p \mid n\,,$$

and that, when $n < 0$, $z^n f(A) \in S_{\mathscr{W}}$ must have the form $\xi^k h(A)$ for $k := -n/p > 0$ and some $h(s) \in \mathbb{C}[s]$.

Suppose $z^n f(A) \in S_{\mathscr{W}}$. We assume $n \neq 0$ without loss of generality. Since $z^n f(A)\psi_0 \in \mathscr{W}$, by Lemma 3 there exists another polynomial $g(s) \in \mathbb{C}[s]$, such that

$$z^n f(A)\psi_0 = g(A)\psi_0\,. \tag{33}$$

First assume $n > 0$. Let $\ell > 0$ be the least common multiple of p and n. Noting

$$z^{2p}\psi_0 = z^p F(A)\psi_0 = F(A-p) z^p \psi_0 = F(A-p)F(A)\psi_0$$

etc., we have

$$\left(\prod_{i=0}^{\ell/p-1} F(A - ip)\right)\psi_0 = z^\ell \psi_0 \tag{34}$$

from (28), and

$$\left(\prod_{j=0}^{\ell/n-1} G(A - jn)\right)\psi_0 = z^\ell \psi_0 \tag{35}$$

from (33), where $G(s) = g(s)/f(s-n)$ is a rational function in s, and $G(A - jn)$ in (35) is understood as an element of the field of fractions of $\mathbb{C}[A]$; this makes sense because, since $\{A^n\psi_0\}_{n=0,1,}$ is linearly independent, the representation

$$\mathbb{C}[s] \ni f(s) \mapsto f(A)\psi_0 \in \mathscr{W}$$

is faithful.

Comparing the left-hand sides of (34) and (35), we thus have the equality

$$\prod_{i=0}^{\ell/p-1} F(s - ip) = \prod_{j=0}^{\ell/n-1} G(s - jn) \tag{36}$$

of rational functions in s. Since the left-hand side of it is a polynomial of s, so is the right-hand side. Let D be the divisor of this polynomial, and let π_ℓ be the natural map $\mathbb{C} \to \mathbb{C}/\ell\mathbb{Z}$. From the left-(resp. right-)hand side of (36) the image $\pi_\ell(D)$ of divisor D on the cylinder $\mathbb{C}/\ell\mathbb{Z}$ is invariant under the translation by p (resp. n). But the genericity condition (G) implies that if $\pi_\ell(D)$ is invariant under the translation by $k \in \mathbb{Z}$, then $p \mid k$. Hence $p \mid n$.

Note here that, since ℓ is the least common multiple of p and n, this implies $\ell = n$, so that the right-hand side of (36) is $G(s)$ itself. Hence

$$g(s)/f(s-n) = G(s) = \prod_{i=0}^{n/p-1} F(s-ip)\,.$$

In particular, $g(s)/f(s-n)$ is a polynomial.

In the case where $n < 0$, after rewriting (33) as

$$z^{-n} g(A)\psi_0 = f(A)\psi_0\,,$$

we switch the roles of f and g, and n and $-n$, to proceed exactly the same way to prove $p \mid n$ and

$$f(s)/g(s+n) = \prod_{i=0}^{-n/p-1} F(s-ip)\,.$$

Thus we have

$$\begin{aligned} z^n f(A) &= z^n \left(\prod_{i=0}^{-n/p-1} F(A-ip) \right) g(A+n) \\ &= \left(z^{-p} F(A)\right)^{-n/p} g(A+n) \\ &= \xi^k g(A+n) =: \xi^k h(A)\,, \end{aligned}$$

proving the last assertion of Step 2, and hence completing the proof of Prop. 3.

2.2. Symmetric functions and matrix integrals. In this subsection, we prove a number of lemmas regarding symmetric functions.

Lemma 4. *Let s and N be positive integers. Let $F\left(x^{(1)}, \ldots, x^{(s)}\right)$ be a function which is symmetric in each $x^{(r)} := \left(x_1^{(r)}, \ldots, x_N^{(r)}\right) \in \mathbb{C}^N$, $r = 1, \ldots, s$; let $f_1, \ldots, f_s$ be functions of two variables, and let $B\left(x^{(s)}\right)$ be a skew-symmetric function of $x^{(s)}$. If $C_1, \ldots, C_s$ denote s fixed contours in $\mathbb{C}$, then the integral*

$$\begin{aligned} \Phi\left(x^{(0)}\right) &:= \int \cdots \int_{(C_1)^N \times \quad \times (C_s)^N} \prod_{r=1}^{s} \prod_{i=1}^{N} dx_i^{(r)} \cdot \\ &\quad \cdot F\left(x^{(1)}, \ldots, x^{(s)}\right) B\left(x^{(s)}\right) \prod_{r=1}^{s} \det\left(f_r\left(x_i^{(r-1)}, x_j^{(r)}\right)\right)_{1 \le i,j \le N}\,, \end{aligned}$$

where $x^{(0)} \in \mathbb{C}^N$ comes in as the first argument of f_1, is skew-symmetric in $x^{(0)}$, and

$$\begin{aligned} \Phi\left(x^{(0)}\right) &= (N!)^s \int \cdots \int_{(C_1)^N \times \quad \times (C_s)^N} \prod_{r,i} dx_i^{(r)} \cdot \\ &\quad \cdot F\left(x^{(1)}, \ldots, x^{(s)}\right) B\left(x^{(s)}\right) \prod_{r=1}^{s} \prod_{i=1}^{N} f_r\left(x_i^{(r-1)}, x_i^{(r)}\right)\,. \end{aligned}$$

Proof. For any (good) functions $A = A(x^{(1)},\dots,x^{(s)})$ and $h = h(x^{(1)},\dots,x^{(s)})$, let

$$\langle Ah\rangle := \int\cdots\int_{(C_1)^N\times\ \ \times(C_s)^N}\prod_{r,i} dx_i^{(r)}\cdot A(x^{(1)},\dots,x^{(s)})h(x^{(1)},\dots,x^{(s)})\,.$$

For any $\sigma_r\in\mathfrak{S}_N$, let $x_{\sigma_r}^{(r)} := (x_{\sigma_r 1}^{(r)},\dots,x_{\sigma_r N}^{(r)})$, and $h^{(\sigma_1,\ \ ,\sigma_s)}(x^{(1)},\dots,x^{(s)}) := h(x_{\sigma_1}^{(1)},\dots,x_{\sigma_s}^{(s)})$. Clearly $\langle Ah\rangle = \langle A^{(\sigma_1,\ \ ,\sigma_s)}h^{(\sigma_1,\ \ ,\sigma_s)}\rangle$. If, moreover, A is symmetric in each of $x^{(1)},\dots,x^{(s-1)}$, and skew-symmetric in $x^{(s)}$, i.e., $A^{(\sigma_1,\ \ ,\sigma_s)} = (-1)^{\varepsilon(\sigma_s)}A$, then we have

$$\langle Ah\rangle = \langle A^{(\sigma_1,\ \ ,\sigma_s)}h^{(\sigma_1,\ \ ,\sigma_s)}\rangle = (-1)^{\varepsilon(\sigma_s)}\langle Ah^{(\sigma_1,\ \ ,\sigma_s)}\rangle \quad \forall\sigma_r\in\mathfrak{S}_N\,.$$

Applying this to $h(x^{(1)},\dots,x^{(s)}) := \prod_r\prod_i f_r(x_i^{(r-1)},x_i^{(r)})$, and summing it up over $(\sigma_1,\dots,\sigma_s)\in(\mathfrak{S}_N)^s$, we obtain

$$\begin{aligned}
&(N!)^s\Big\langle A\prod_r\prod_i f_r(x_i^{(r-1)},x_i^{(r)})\Big\rangle\\
&= \Big\langle A\sum_{\sigma_1,\ \ ,\sigma_s}(-1)^{\varepsilon(\sigma_s)}\prod_r\prod_i f_r(x_{\sigma_{r-1}i}^{(r-1)},x_{\sigma_r i}^{(r)})\Big\rangle, \quad \text{with } \sigma_0=\mathrm{id},\\
&= \Big\langle A\sum_{\sigma_1,\ \ ,\sigma_s}\prod_r(-1)^{\varepsilon(\sigma_r)-\varepsilon(\sigma_{r-1})}\prod_i f_r(x_{\sigma_{r-1}i}^{(r-1)},x_{\sigma_r i}^{(r)})\Big\rangle\\
&= \Big\langle A\prod_r\sum_{\sigma\in\mathfrak{S}_N}(-1)^{\varepsilon(\sigma)}\prod_i f_r(x_i^{(r-1)},x_{\sigma i}^{(r)})\Big\rangle\\
&= \Big\langle A\prod_r\det\Big(f_r(x_i^{(r-1)},x_j^{(r)})\Big)_{i,j}\Big\rangle\,.
\end{aligned}$$

Setting here $A = F(x^{(1)},\dots,x^{(s)})B(x^{(s)})$ proves the identity in Lemma 4. Finally, $\Phi(x^{(0)})$ is skew-symmetric in $x^{(0)}$ since $\det\Big(f_1(x_i^{(0)},x_j^{(1)})\Big)$ is.

Lemma 5. *(See [19, Lemma 4.2], [17, Eq. (2.21)], [26, Theorem 8.18].) Let*

$$\mathscr{W} = \mathrm{span}_{\mathbb{C}}\{\psi_0(z),\psi_1(z),\psi_2(z),\dots\}\in\mathrm{Gr}$$

with functions

$$\psi_k(z) = \sum_{-\infty<j\le k} a_{j,k}z^j, \qquad k=0,1,2,\dots,$$

such that $a_{kk}=1$ for $k\gg 0$, i.e., $\mathrm{ord}_z\,\psi_k(z)\le k$, and $\psi_k(z)=z^k(1+O(z^{-1}))$ for $k\gg 0$. Let $N>0$ be any integer such that this condition holds for $k\ge N$. Let $z_1,\dots,z_N$ be formal scalar variables near ∞. Then the τ-function $\tau(t)$ at

$$t_n := -\frac{1}{n}\sum_{i=1}^N z_i^{-n}, \quad n=1,2,\dots, \tag{37}$$

is given by

$$\tau(t) = \frac{\det\big(\psi_{j-1}(z_i)\big)_{1\le i,j\le N}}{\det(z_i^{j-1})_{1\le i,j\le N}}\,. \tag{38}$$

Proof. This lemma is stated by Kontsevich in [19], essentially without proofs; see [17, Sect. 2.3] for a proof using free fermions. To keep the notation simple, let us denote by $(1-z)^{-1}$ and $(-z+1)^{-1}$ the geometric series $\sum_0^\infty z^n$ and $-\sum_{-\infty}^{-1} z^n$, respectively. Let $\delta(z) := (1-z)^{-1} - (-z+1)^{-1} = \sum_{-\infty}^{\infty} z^n$, which plays the role of delta function, in the sense that

$$\delta(z/y)f(z) = \delta(z/y)f(y)\,, \tag{39}$$

as is obvious by taking $f(z) = z^m$ (see [6]). Let $\sigma := \prod_{i=1}^N (-z_i) = (-1)^N z_1 \ldots z_N$. Let $\sigma_i := 1\big/\prod_{j(\neq i)}(1 - z_i/z_j)$, $i = 1, \ldots, N$, understood as rational functions of z_j's, so that we have the following identity of formal power series in z:

$$\prod_{i=1}^N \left(1 - \frac{z}{z_i}\right)^{-1} = \sum_{i=1}^N \sigma_i \left(1 - \frac{z}{z_i}\right)^{-1}.$$

From (37) we have

$$\begin{aligned} g := \exp\left(-\sum_{n=1}^\infty t_n z^n\right) &= \prod_{i=1}^N \left(1 - \frac{z}{z_i}\right)^{-1} = \sum_{i=1}^N \sigma_i \left(1 - \frac{z}{z_i}\right)^{-1} \\ &= \sum_{i=1}^N \sigma_i \delta(z/z_i) + \sum_{i=1}^N \sigma_i \left(-\frac{z}{z_i} + 1\right)^{-1} \\ &= \sum_{i=1}^N \sigma_i \delta(z/z_i) + \prod_{i=1}^N \left(-\frac{z}{z_i} + 1\right)^{-1}, \end{aligned}$$

so that by using (39), we have

$$\begin{aligned} g\psi_j(z) &= \sum_{i=1}^N \sigma_i \delta(z/z_i)\psi_j(z) + \left(\prod_{i=1}^N \left(-\frac{z}{z_i} + 1\right)^{-1}\right)\psi_j(z) \\ &= \sum_{i=1}^N \sigma_i \delta(z/z_i)\psi_j(z_i) + z^{-N}\big(\sigma + O(z^{-1})\big)\psi_j(z)\,. \end{aligned}$$

Denoting by B the matrix of the composite map in (21) with respect to the bases $\{\psi_j\}_{j=0}^\infty$ and $\{z^k\}_{k=0}^\infty$, we have thus $B = B^0 + B^1$, where

$$B^0 = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ z_1^{-2} & \cdots & z_N^{-2} \\ \vdots & \cdots & \vdots \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \psi_1(z_1) & \cdots \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \psi_1(z_N) & \cdots \end{pmatrix},$$

$$B^1 = \left(\begin{array}{ccc|cccccc} \cdots & 0 & 0 & \overbrace{0 \cdots 0}^{N} & \sigma & & * \\ \cdots & 0 & 0 & 0 \cdots 0 & 0 & \sigma & \\ \cdots & \vdots & \vdots & \vdots \cdots \vdots & \vdots & \ddots & \ddots \\ \cdots & \vdots & \vdots & \vdots \cdots \vdots & \vdots & \cdots & \ddots \end{array}\right) \left(\begin{array}{ccc} \vdots & \vdots & \cdots \\ a_{-2,0} & a_{-2,1} & \cdots \\ a_{-1,0} & a_{-1,1} & \cdots \\ \hline a_{00} & a_{01} & \cdots \\ 0 & a_{11} & \cdots \\ 0 & 0 & \ddots \\ \vdots & \vdots & \ddots \end{array}\right),$$

S_N is the diagonal matrix diag$(\sigma_1, \dots, \sigma_N)$, and a_{kj}, $-\infty < k < \infty$, $0 \le j < \infty$, are the Laurent coefficients of $\psi_j = \sum_k a_{kj} z^k$.

Let us apply some column operations on B. Adding an appropriate linear combination of first N columns to the $(N+i)^{\text{th}}$ column ($i > 0$), we can eliminate the column ${}^t\big(\psi_{N+i}(z_1), \dots, \psi_{N+i}(z_N)\big)$, $i > 0$, from B^0. Since N is large enough so that $a_{jj} = 1$ for $j \ge N$, B^1 has the form

$$\left(\begin{array}{c|cccc} & \sigma & & * & \\ & & \sigma & & \\ O_{\infty\times N} & & & \ddots & \\ & \mathbf{0} & & & \end{array}\right),$$

so that the "$*$" part can be eliminated by further column operations on columns $N+1$, $N+2$, ..., which do not alter the B^0-part. Here $O_{m\times n}$ is the $m \times n$ zero matrix. The matrix B can thus be reduced to $B' = B'^0 + B'^1$, where

$$B'^0 = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ z_1^{-2} & \cdots & z_N^{-2} \\ \vdots & \cdots & \vdots \end{pmatrix} S_N \left(\begin{array}{ccc|c} \psi_0(z_1) & \cdots & \psi_N(z_1) & \\ \vdots & \vdots & \vdots & O_{N\times\infty} \\ \psi_0(z_N) & \cdots & \psi_N(z_N) & \end{array}\right),$$

$$B'^1 = \left(O_{\infty\times N} \,\middle|\, \sigma I_\infty\right).$$

Let n, $n \ge N$, be an integer. Note that the column operations needed to bring B into B' only adds linear combinations of lower numbered columns to higher ones. Hence, denoting by B_n, B'_n, B'^0_n and B'^1_n the matrices of the first n rows and columns in B, B', B'^0 and B'^1, respectively, we have $\det B_n = \det B'_n = \det(B'^0_n + B'^1_n)$, with

$$B'^0_n = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{-1} & \cdots & z_N^{-1} \\ \vdots & \cdots & \vdots \\ z_1^{-n+1} & \cdots & z_N^{-n+1} \end{pmatrix} S_N \left(\begin{array}{ccc|c} \psi_0(z_1) & \cdots & \psi_N(z_1) & \\ \vdots & \vdots & \vdots & O_{N\times(n-N)} \\ \psi_0(z_N) & \cdots & \psi_N(z_N) & \end{array}\right),$$

and

$$B'^1_n = \left(\begin{array}{c|c} O_{(n-N)\times N} & \sigma I_{n-N} \\ \hline O_{N\times N} & O_{N\times(n-N)} \end{array}\right).$$

Since the last $n - N$ columns of B'^0_n are 0, we have

$$B'_n = \left(\begin{array}{c|c} * & \sigma I_{n-N} \\ \hline Z & O_{N\times(n-N)} \end{array}\right),$$

where Z consists of the last N rows and the first N columns of B'^0_n:

$$Z = \begin{pmatrix} z_1^{-n+N} & \cdots & z_N^{-n+N} \\ \vdots & \cdots & \vdots \\ z_1^{-n+1} & \cdots & z_N^{-n+1} \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \cdots & \psi_N(z_1) \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \cdots & \psi_N(z_N) \end{pmatrix}.$$

Hence we have, using $\sigma = (-1)^N z_1 \dots z_N$,

$$\det B_n = \det B'_n = (-1)^{N(n-N)} \det Z \det(\sigma I_{n-N}) = (z_1 \ldots z_N)^{n-N} \det Z = (z_1 \ldots z_N)^{1-N} \det Z',$$

where

$$Z' = \begin{pmatrix} z_1^{N-1} & \cdots & z_N^{N-1} \\ \vdots & \cdots & \vdots \\ z_1^1 & \cdots & z_N^1 \\ 1 & \cdots & 1 \end{pmatrix} S_N \begin{pmatrix} \psi_0(z_1) & \cdots & \psi_N(z_1) \\ \vdots & \vdots & \vdots \\ \psi_0(z_N) & \cdots & \psi_N(z_N) \end{pmatrix}.$$

Noticing

$$\det(z_j^{N-i})_{1\le i,j\le N} = (-1)^{N(N-1)/2} \det(z_j^{i-1})_{1\le i,j\le N},$$

and

$$\det S_N = \prod_1^N \sigma_i = \frac{\left(\prod_{j=1}^N z_j\right)^{N-1}}{\prod_{i,j\neq i}(z_j - z_i)} = \frac{(z_1 \ldots z_N)^{N-1}}{(-1)^{N(N-1)/2} \det(z_j^{i-1})^2_{1\le i,j\le N}},$$

we observe that $\det B_n$ coincides with the right-hand side of (38). Since $n \geq N$ is arbitrary, this completes the proof of Lemma 5.

Lemma 6. *Let* $Z := \operatorname{diag}(z_1, \ldots, z_N)$. *Let* $\lambda := \big((p-1)(N-1), (p-1)(N-2), \ldots, p-1\big)$. *For a polynomial* $f(y,z)$, *let us denote by* $\big(f(y,z)\big)_2$ *the terms in* $f(y,z)$ *which are quadratic in* y. *Then we have*[5]

$$\frac{\Delta(z^p)}{\Delta(z)} = F_\lambda\left(-\operatorname{tr} Z, -\frac{1}{2}\operatorname{tr} Z^2, -\frac{1}{3}\operatorname{tr} Z^3, \ldots\right) = c\prod z_i^{-\frac{p-1}{2}} \left(\int_{\mathscr{H}_N} dY \exp \operatorname{tr}\left(-\frac{(Y+Z)^{p+1}}{p+1}\right)_2\right)^{-1},$$

where c is a non-zero constant which depends only on N and p.

Proof. The Schur function associated with the partition λ is given by (see [21])

$$F_\lambda\left(-\sum_1^N y_i, -\frac{1}{2}\sum_1^N y_i^2, -\frac{1}{3}\sum_1^N y_i^3, \ldots\right) := \frac{\Delta_{\lambda+\delta}(y)}{\Delta_\delta(y)},$$

where $\delta = (N-1 > N-2 > \cdots > 1 > 0)$ and $\Delta_\mu(y) = \det(y_i^{\mu_j})_{1\le i,j\le N}$. Therefore we have, with $\lambda + \delta = \big(p(N-1) > p(N-2) > \cdots > p > 0\big)$,

$$\frac{\Delta(z^p)}{\Delta(z)} = \frac{\Delta_{\lambda+\delta}(z)}{\Delta_\delta(z)} = F_\lambda\left(-\sum_1^N z_i, -\frac{1}{2}\sum_1^N z_i^2, -\frac{1}{3}\sum_1^N z_i^3, \ldots\right),$$

establishing the first equality of Lemma 6. In order to establish the second one, note

[5] F_λ is the Schur function for the partition λ

$$
\begin{aligned}
\operatorname{tr}\left(\frac{(Y+Z)^{p+1}}{p+1}\right)_2 &= \frac{1}{2}\operatorname{tr}(Y^2Z^{p-1} + YZYZ^{p-2} + \cdots + YZ^{p-1}Y) \\
&= \frac{1}{2}\sum_{i,j} Y_{ij}Y_{ji}(z_i^{p-1} + z_i^{p-2}z_j + \cdots + z_j^{p-1}) \\
&= \frac{1}{2}\sum_{i,j} Y_{ij}Y_{ji}\left(\frac{z_i^p - z_j^p}{z_i - z_j}\right).
\end{aligned}
$$

Hence, performing a Gaussian integration, we find

$$
\begin{aligned}
\int dY \exp\operatorname{tr}\left(-\frac{(Y+Z)^{p+1}}{p+1}\right)_2 &= \int dY \exp\left(-\frac{1}{2}\sum_{i,j} Y_{ij}Y_{ji}\frac{z_i^p - z_j^p}{z_i - z_j}\right) \\
&= (2\pi)^{N^2/2}\left(\prod_{1\le i,j\le N}\frac{z_i - z_j}{z_i^p - z_j^p}\right)^{1/2} \\
&= \frac{(2\pi)^{N^2/2}}{p^{N/2}}\prod_{1\le i<j\le N}\frac{z_i - z_j}{z_i^p - z_j^p}\prod_1^p z_i^{-\frac{p-1}{2}} \\
&= \frac{(2\pi)^{N^2/2}}{p^{N/2}}\frac{\Delta(z)}{\Delta(z^p)}\prod_1^N z_i^{-\frac{p-1}{2}},
\end{aligned}
$$

establishing Lemma 6.

Remark 5. In general we have

$$
\int_{\mathscr{H}} dY\, e^{-\operatorname{tr}(V(Y+Z))_2} = (2\pi)^{N^2/2}\frac{\Delta(z)}{\Delta(V'(z))}\frac{1}{\sqrt{\prod_1^N V''(z_i)}}.
$$

The following lemma is due to Harish Chandra, Bessis–Itzykson–Zuber and Duistermaat–Heckman among others:

Lemma 7. *Given $N \times N$-diagonal matrices X and Y, we have*

$$
\int_{\mathrm{U}(N)} e^{\operatorname{tr} XUYU^\dagger}\, dU = (2\pi)^{\frac{N(N-1)}{2}}\frac{\det(e^{x_i y_j})_{1\le i,j\le N}}{\Delta(X)\Delta(Y)}.
$$

A proof can be found in [13].

3. Matrix Fourier Transforms

In this section we explain how generalized Kontsevich integrals (see [19, 1, 24]) are closely related to the theory of Fourier transforms. Indeed, if $V(x)$ grows sufficiently at infinity, any *Fourier transform*

$$
a(y) = \int_{-\infty}^{\infty} e^{-V(x)+xy}\, dx \tag{40}
$$

leads to a linear space of functions $\mathscr{W}$ *invariant under two operators A and $V'(z)$* satisfying $[A, V'(z)] = 1$.

(i) The point is that $a(y)$ satisfies the differential equation

$$V'\left(\frac{\partial}{\partial y}\right)a(y) = ya(y)\,, \tag{41}$$

as seen from

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} e^{-V(x)+xy} dx = \int_{-\infty}^{\infty} \big(-V'(x) + y\big) e^{-V(x)+xy} dx \\ &= \left(-V'\left(\frac{\partial}{\partial y}\right) + y\right) a(y). \end{aligned}$$

Thus setting $y = V'(z)$ in (41) and $A_0 := V''(z)^{-1}\partial/\partial z = \partial/\partial y|_{y=V'(z)}$, the function $a\big(V'(z)\big)$ satisfies the differential equation

$$V'(A_0)a\big(V'(z)\big) = V'(z)a\big(V'(z)\big)\,. \tag{42}$$

(ii) The method of stationary phase applied to integrals (40) and their derivatives leads to the following estimate, upon Taylor expanding $V(x)$ around $x = z$,

$$\begin{aligned} &\left(\frac{\partial}{\partial y}\right)^n a(y)\bigg|_{y=V'(z)} \\ &= \int_{-\infty}^{\infty} x^n e^{-V(x)+xV'(z)} dx \\ &= \int_{-\infty}^{\infty} x^n e^{-\left(V(z)+(x-z)V'(z)+(1/2)(x-z)^2V''(z)+O(x-z)^3\right)+xV'(z)} dx \\ &= e^{-V(z)+zV'(z)} \int_{-\infty}^{\infty} x^n e^{-(1/2)(x-z)^2V''(z)\left(1+(V'''/V'')O(x-z)\right)} dx \\ &= e^{-V(z)+zV'(z)} \frac{1}{\sqrt{V''}} \left(\int_{-\infty}^{\infty}\left(\frac{y}{\sqrt{V''}} + z\right)^n e^{-y^2/2} dy + O(1/z)\right) \\ &= \rho(z)^{-1} z^n \big(1 + O(1/z)\big)\,, \end{aligned} \tag{43}$$

with

$$\rho(z) = \frac{1}{\sqrt{2\pi}} e^{V(z)-zV'(z)} \sqrt{V''(z)}\,.$$

Therefore defining

$$A := \rho(z) \frac{\partial}{\partial y}\bigg|_{y=V'(z)} \circ \rho(z)^{-1}$$

and

$$\psi_n(z) := A^n \psi_0(z) := \rho(z) \frac{\partial^n}{\partial y^n} a(y)\bigg|_{y=V'(z)}\,, \qquad n = 0, 1, \ldots,$$

the differential equation (42) implies

$$V'(A)\psi_0(z) = V'(z)\psi_0(z)\,.$$

This, combined with (43), proves that the linear span

$$\mathscr{W} := \operatorname{span}_{\mathbb{C}}\{\psi_k(z) = z^k\big(1 + O(1/z)\big) \mid k = 0, 1, 2, \ldots\}$$

is invariant under the operators A and $V'(z)$, i.e.,

$$A\mathscr{W} \subset \mathscr{W} \quad \text{and} \quad V'(z)\mathscr{W} \subset \mathscr{W}, \quad \text{with} \quad [A, V'(z)] = 1\,.$$

(iii) By Lemma 5 and the fact that $\psi_k(z) = A^k\psi_0(z)$, the τ-function corresponding to $\mathscr{W}$, at time t as in (37), is given by

$$\begin{aligned}
\tau(t) &= \frac{\det\big(A^{j-1}\psi_0(z_i)\big)_{1\le i,j\le N}}{\det(z_i^{j-1})_{1\le i,j\le N}} \\
&= \frac{1}{\Delta(z)} \det\left(\rho(z_i)\left(\frac{\partial}{\partial y}\right)^{j-1} \int_{-\infty}^{\infty} \mathrm{e}^{-V(x)+xy}\,dx\Bigg|_{y=V'(z_i)}\right)_{1\le i,j\le N} \\
&= \frac{\prod_1^N \rho(z_i)}{\Delta(z)} \int_{\mathbb{R}^N} dx\, \mathrm{e}^{-\sum_1^N V(x_i)} \Delta(x) \prod_1^N \mathrm{e}^{x_\alpha V'(z_\alpha)} \\
&= \frac{\prod_1^N \rho(z_i)}{N!\Delta(z)} \int_{\mathbb{R}^N} dx\, \mathrm{e}^{-\sum_1^N V(x_i)} \Delta(x) \det\big(\mathrm{e}^{x_\alpha V'(z_\beta)}\big)_{1\le\alpha,\beta\le N}, \\
&\qquad \text{using Lemma 4 with } s = 1 \text{ and the skew-symmetry of } \Delta(x), \\
&= \frac{\prod_1^N \rho(z_i)}{N!\Delta(z)/\Delta\big(V'(z)\big)} \int_{\mathbb{R}^N} dx\, \mathrm{e}^{-\sum_1^N V(x_i)} \Delta^2(x) \frac{\det\big(\mathrm{e}^{x_\alpha V'(z_\beta)}\big)_{1\le\alpha,\beta\le N}}{\Delta(x)\Delta\big(V'(z)\big)} \\
&= c\frac{\prod_1^N \rho(z_i)}{\Delta(z)/\Delta\big(V'(z)\big)} \int_{\mathbb{R}^N} dx\, \mathrm{e}^{-\sum_1^N V(x_i)} \Delta^2(x) \int_{\mathrm{U}(N)} dU\, \mathrm{e}^{\operatorname{tr} UXU^{-1}V'(Z)}, \\
&\qquad \text{using Lemma 7, with } X = \operatorname{diag}(x), \\
&= c'\mathrm{e}^{\operatorname{tr}(V(Z)-ZV'(Z))} \frac{\int_{\mathscr{H}_N} dX\, \mathrm{e}^{-\operatorname{tr} V(X)} \mathrm{e}^{\operatorname{tr} XV'(Z)}}{\int_{\mathscr{H}_N} dY\, \mathrm{e}^{-\operatorname{tr}(V(Y+Z))_2}}, \quad \text{using Lemma 6,} \\
&= c''\frac{\int_{\mathscr{H}_N} dY\, \mathrm{e}^{-\operatorname{tr}(V(Y+Z))_{\ge 2}}}{\int_{\mathscr{H}_N} dY\, \mathrm{e}^{-\operatorname{tr}(V(Y+Z))_2}}, \quad \text{upon setting } X = Y + Z,
\end{aligned}$$

for some constants c, c' and c'' depending on N.

4. Generalized Hänkel Functions, Differential Equations and Laplace Transforms

This section deals with the properties of Hänkel functions and their generalizations.

Lemma 8. *The family of integrals*

$$\psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^{\infty} \frac{z^{-c}(uz)^k \mathrm{e}^{-(u-1)z}}{(u^p - 1)^{c+1}}\,du\,, \quad \begin{matrix} -1 < c < 0, \\ k = 0, 1, \ldots,\ p = 2, 3, \ldots \end{matrix} \tag{44}$$

admits, for large $z > 0$, an asymptotic expansion in $\mathbb{C}((z^{-1}))$ of the form

$$\psi_k(z) = z^k\big(1 + O(1/z)\big)\,, \tag{45}$$

with $\psi_0(z)$ satisfying the differential equation

$$e^z z^{-c}\left(\prod_{i=0}^{p-1}\left(z\frac{\partial}{\partial z}-i\right)-cp\prod_{i=0}^{p-2}\left(z\frac{\partial}{\partial z}-i\right)\right)z^c e^{-z}\psi_0(z)=(-z)^p\psi_0(z)\,, \qquad (8)$$

or equivalently

$$e^z z^{-c}\left(z^p\left(\frac{\partial}{\partial z}\right)^p - cp\,z^{p-1}\left(\frac{\partial}{\partial z}\right)^{p-1}\right)z^c e^{-z}\psi_0(z)=(-z)^p\psi_0(z)\,. \qquad (8')$$

Moreover $\psi_k(z)$ admits the following representation in terms of a double integral[6]

$$\begin{aligned}\psi_k(z) &= \frac{p^{c+1}}{2\pi i}z^{(p-1)(c+1)}\int_\gamma dw\int_0^\infty dx\, e^{z-w}w^k x^c e^{x(w^p-z^p)}\\ &= \frac{p^{c+1}}{2\pi i}z^{(p-1)(c+1)}e^z\int_0^\infty dx\, x^c e^{-xz^p}\int_0^\infty dy\, f_k(y)e^{-xy^p}\,, \qquad (46)\end{aligned}$$

where, in the first integral, $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ denotes the contour consisting of two half-lines $\gamma^\pm = \mathbb{R}_+\zeta^{\pm 1}$, $\zeta := e^{\pi i/p}$, through the origin making an angle $\pm\pi/p$ with the positive real axis, with the orientation given as to go from $\zeta^{-1}\cdot\infty$ to 0 to $\zeta\cdot\infty$ (see Fig. 1 (a)), and where in the second integral,

$$f_k(y)=(\zeta^{k+1}e^{-\zeta y}-\zeta^{-k-1}e^{-\zeta^{-1}y})y^k=\sum_{j=0}^\infty\frac{(-1)^j}{j!}a_{j+k+1}y^{j+k},$$

where $a_n=\zeta^n-\zeta^{-n}=2i\sin(n\pi/p)$.

Proof. Setting $v=(u-1)z$, and using

$$\Gamma(-c)=\int_0^\infty\frac{e^{-v}}{v^{c+1}}\,dv \quad \text{for } c<0,$$

we first observe that for each $n\ge 0$,

$$\begin{aligned}\psi_k(z) &= \frac{p^{c+1}z^k}{\Gamma(-c)}\int_0^\infty\frac{(1+v/z)^k e^{-v}}{v^{c+1}p^{c+1}\left(1+\frac{1}{p}\left(\sum_{i=2}^p\binom{p}{i}(v/z)^{i-1}\right)\right)^{c+1}}\,dv\\ &= z^k\left(1+\tilde b_{k,1}z^{-1}+\cdots+\tilde b_{k,n}z^{-n}+O(1/z^{n+1})\right)\end{aligned}$$

as $z\to\infty$, where the $\tilde b_{k,i} := \big(\Gamma(-c+i)/\Gamma(-c)\big)b_{k,i} = \left(\prod_{j=0}^{i-1}(-c+j)\right)b_{k,i}$ are obtained from the coefficients $b_{k,i}$ of the expansion[7]

---

[6] If $p=2$, so that γ becomes the imaginary axis, these integrals should be interpreted by replacing ζ by $\zeta_\varepsilon = e^{(\pi i/2)-\varepsilon}$, and γ by $\mathbb{R}_+\zeta_\varepsilon+\mathbb{R}_+\zeta_\varepsilon^{-1}$, and then taking the limit as $\varepsilon\downarrow 0$

[7] Noting that the radius of convergence of this power series is $|\zeta-1|$, one can get a precise growth estimate of the coefficients of $\psi_k(z)$ which implies that, in particular, as always with the string equation, $\mathscr{W}$ does not belong to the L^2-Grassmannian of Segal–Wilson [30].

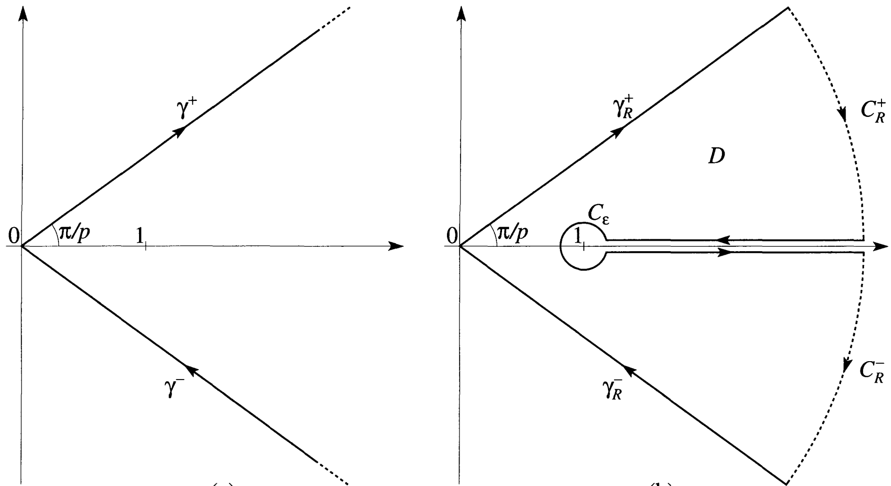

Fig. 1. Contours of integration: (a) contour γ for (46); (b) closed contour for (48)

$$\frac{(1+s)^k}{\left(1+\frac{1}{p}\left(\sum_{i=2}^{p}\binom{p}{i}s^{i-1}\right)\right)^{c+1}} = 1 + \sum_{i=1}^{\infty} b_{k,i}s^i\,, \tag{47}$$

confirming the asymptotic expansion (45).

Moreover, setting

$$\varphi_0(z) = \int_1^\infty \frac{z^{-c}e^{-uz}}{(u^p-1)^{c+1}}\,du\,,$$

we have for $c<0$ and $\mathrm{Re}\,z>0$,

$$\begin{aligned}
0 &= -z^{p-1-c}\frac{e^{-uz}}{(u^p-1)^c}\Bigg|_{u=1}^{u=\infty} \\
&= -z^{p-1}\int_1^\infty \frac{\partial}{\partial u}\left((u^p-1)\frac{z^{-c}e^{-uz}}{(u^p-1)^{c+1}}\right)du \\
&= (-1)^p\int_1^\infty \left((-zu)^p - cp(-zu)^{p-1} - (-z)^p\right)\frac{z^{-c}e^{-uz}}{(u^p-1)^{c+1}}\,du \\
&= (-1)^p z^{-c}\left(z^p\left(\frac{\partial}{\partial z}\right)^p - cp\,z^{p-1}\left(\frac{\partial}{\partial z}\right)^{p-1} - (-z)^p\right)z^c\varphi_0(z) \\
&= (-1)^p z^{-c}\left(\prod_{i=0}^{p-1}\left(z\frac{\partial}{\partial z}-i\right) - cp\prod_{i=0}^{p-2}\left(z\frac{\partial}{\partial z}-i\right) - (-z)^p\right)z^c\varphi_0(z)\,,
\end{aligned}$$

using in the last line the operator identity

$$\prod_{i=0}^{p-1}\left(z\frac{\partial}{\partial z}-i\right) = z^p\left(\frac{\partial}{\partial z}\right)^p,$$

thus showing that $\psi_0(z)$ satisfies the differential equation (8) or (8′).

Consider a bounded domain $D \subset \mathbb{C}$, whose boundary consists of the lines $\gamma_R^{\pm}$, making an angle $\pm\pi/p$ with the positive real axis, two circle segments $C_R^{\pm}$, about the origin, of large enough radius R and a small circle about 1 of radius ε connected to $C_R^{\pm}$, as in Fig. 1 (b). The function $\mathrm{e}^{-uz}/(u^p-1)^{c+1}$ is univalued in D and all its singularities lie outside D. By Cauchy's theorem we have

$$\left(\int_{\gamma_R^-}+\int_{\gamma_R^+}+\int_{C_R^+}+\int_R^{1+\varepsilon}+\int_{C_\varepsilon}+\int_{1+\varepsilon}^R+\int_{C_R^-}\right)\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^c}\,du=0\,. \tag{48}$$

Observe that, for $z>0$ and $p>2$, we have $z\cos\theta \geq z\cos(\pi/p)>0$ for $0\leq\theta\leq\pi/p$, implying

$$\int_{C_R^{\pm}}\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du=O\big(R^{k-(c+1)p+1}\mathrm{e}^{-Rz\cos(\pi/p)}\big)\to 0$$

as $R\uparrow\infty$. Since $c<0$, we also have

$$\int_{C_\varepsilon}\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du=O(\varepsilon^{-c})\to 0$$

as $\varepsilon\downarrow 0$. So, taking limits as $\varepsilon\downarrow 0$ and $R\uparrow\infty$ leads to

$$\begin{aligned}\int_\gamma\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du &= -\left(\int_\infty^1+\int_{1-\mathrm{i}0}^{\infty-\mathrm{i}0}\right)\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\\ &= (1-\mathrm{e}^{-2\pi\mathrm{i}(c+1)})\int_1^\infty\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\\ &= 2\mathrm{i}\mathrm{e}^{-\pi\mathrm{i}c}\sin\pi c\int_1^\infty\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\,.\end{aligned}$$

Note that, since $u^p-1<0$ along $[1,\infty)$, we have the following Γ-function representation

$$\frac{1}{(u^p-1)^{c+1}}=-\frac{\mathrm{e}^{-\pi\mathrm{i}c}}{\Gamma(c+1)}\int_0^\infty dx\,x^c\mathrm{e}^{x(u^p-1)}\,,$$

and thus

$$\begin{aligned}\psi_k(z) &= \frac{p^{c+1}}{\Gamma(-c)}\int_1^\infty\frac{z^{-c}(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\\ &= \frac{p^{c+1}\mathrm{e}^{\pi\mathrm{i}c}}{2\mathrm{i}\sin\pi c\,\Gamma(-c)}z^{-c}\int_\gamma\frac{(uz)^k\mathrm{e}^{-(u-1)z}}{(u^p-1)^{c+1}}\,du\\ &= -\frac{p^{c+1}z^{-c}}{2\mathrm{i}\sin\pi c\Gamma(-c)\Gamma(c+1)}\int_\gamma du\,(uz)^k\mathrm{e}^{-(u-1)z}\int_0^\infty dx\,x^c\mathrm{e}^{x(u^p-1)}\\ &= \frac{p^{c+1}}{2\pi\mathrm{i}}z^{-c}\int_\gamma du\,(uz)^k\mathrm{e}^{-(u-1)z}\int_0^\infty dx\,x^c\mathrm{e}^{x(u^p-1)}\,,\\ &= \frac{p^{c+1}}{2\pi\mathrm{i}}z^{(p-1)(c+1)}\int_\gamma dw\int_0^\infty dx\,\mathrm{e}^{z-w}w^k x^c\mathrm{e}^{xw^p}\mathrm{e}^{-xz^p}\,,\end{aligned}$$

upon setting $w = uz$. Here we used the Γ-function duplication, $\Gamma(-c)\Gamma(c+1) = -\pi/\sin\pi c$, $-1 < c < 0$. Working out the integral over γ, interchanging the integrations and using $\zeta^{\pm p} = -1$, we find

$$\begin{aligned}\psi_k(z) &= \frac{p^{c+1}}{2\pi \mathrm{i}} z^{(p-1)(c+1)} \mathrm{e}^z \int_0^\infty dx\, x^c \mathrm{e}^{-xz^p} \cdot \\ &\quad \cdot \left(\zeta^{-k-1} \int_\infty^0 dy\, \mathrm{e}^{-\zeta^{-1} y} y^k \mathrm{e}^{-xy^p} + \zeta^{k+1} \int_0^\infty dy\, \mathrm{e}^{-\zeta y} y^k \mathrm{e}^{-xy^p} \right) \\ &= \frac{p^{c+1}}{2\pi \mathrm{i}} z^{(p-1)(c+1)} \mathrm{e}^z \int_0^\infty dx\, x^c \mathrm{e}^{-xz^p} \int_0^\infty dy\, f_k(y) \mathrm{e}^{-xy^p}\end{aligned}$$

with

$$\begin{aligned}f_k(y) &= (\zeta^{k+1} \mathrm{e}^{-\zeta y} - \zeta^{-k-1} \mathrm{e}^{-\zeta^{-1} y}) y^k \\ &= \sum_{j=0}^\infty \frac{(-1)^j}{j!} (\zeta^{j+k+1} - \zeta^{-j-k-1}) y^{j+k},\end{aligned}$$

as announced in (46), thus ending the proof of Lemma 8.

Lemma 9. *The linear space spanned by the generalized Hänkel functions,*

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}} \left\{ \psi_k(z) = \frac{p^{c+1}}{\Gamma(-c)} \int_1^\infty \frac{z^{-c} (uz)^k \mathrm{e}^{-(u-1)z}}{(u^p - 1)^{c+1}}\, du \;\middle|\; k = 0, 1, 2, \ldots \right\}$$

is invariant under

$$z^p \quad \text{and} \quad A_c := z^{-c} \mathrm{e}^z z \frac{\partial}{\partial z} \circ \mathrm{e}^{-z} z^c = z \frac{\partial}{\partial z} - z + c$$

(so that $[(1/p)A, z^p] = z^p$*), with* ψ_0 *satisfying the differential equation (8).*

Proof. The space $\mathcal{W}$ is invariant under A_c, because

$$\begin{aligned}A_c \psi_k(z) &= \frac{p^{c+1}}{\Gamma(-c)} z^{-c} \mathrm{e}^z z \frac{\partial}{\partial z} z^c \mathrm{e}^{-z} \int_1^\infty \frac{z^{-c} (uz)^k \mathrm{e}^{-(u-1)z}}{(u^p - 1)^{c+1}}\, du \\ &= \frac{p^{c+1}}{\Gamma(-c)} z^{-c} \mathrm{e}^z z \frac{\partial}{\partial z} \int_1^\infty \frac{(uz)^k \mathrm{e}^{-uz}}{(u^p - 1)^{c+1}}\, du \\ &= k\psi_k(z) - \psi_{k+1}.\end{aligned}$$

Moreover, the operator

$$\prod_{i=0}^{p-1} (A_c - i) - cp \prod_{i=0}^{p-2} (A_c - i)$$

has the form $\sum_0^p \alpha_j A_c^j$, with $\alpha_p = 1$. From Lemma 8, the solution to the differential equation

$$\left(\prod_{i=0}^{p-1} (A_c - i) - cp \prod_{i=0}^{p-2} (A_c - i) \right) \psi_0(z) = (-z)^p \psi_0(z)$$

is given by the function in (44) or (46) for $k = 0$. An asymptotic expansion of the form

$$\psi_0(z) = 1 + O(z^{-1})$$

follows from (45).

5. Proof of the Main Statements

5.1. Proof of Theorems 3 and 1 and Remark 1. In Lemma 9, we have constructed a space $\mathcal{W}$ and an operator $A = A_c$ such that

$$A\mathcal{W} \subset \mathcal{W} \quad \text{and} \quad z^p\mathcal{W} \subset \mathcal{W},$$

with the lowest order element $\psi_0 \in \mathcal{W}$ satisfying Eq. (8). Proposition 3 and Remark 4 imply that the stabilizer of $\mathcal{W}$ is $\mathbb{C}[A, z^p, z^{-p}F(A)]$, proving Theorem 3, Part (i).

Let Ψ and τ be the wave function and the τ-function, respectively, associated with the KP time evolution $\mathcal{W}^t = e^{-\sum t_i z^i}\mathcal{W}$ of $\mathcal{W}$. We now define the operators P and Q in the x-variable, via the operators A and z^p in the z-variable, by means of

$$z^p\Psi(t,z) = P\Psi(t,z) \quad \text{and} \quad (1/p)A\Psi(t,z) = Q\Psi(t,z).$$

According to Lemma 2, P and Q are differential operators. They satisfy $[P,Q] = P$ since $[(1/p)A, z^p] = z^p$. Note that P and Q can also be written:

$$P = L^p = SD^pS^{-1}$$

and

$$Q = \frac{1}{p}(ML - L + c) = \frac{1}{p}S\left(\sum_1^\infty k\bar{t}_k D^k - D + c\right)S^{-1},$$

where

$$S = \frac{\tau(t - [D^{-1}])}{\tau(t)}$$

in terms of the τ-function above, and L and M are as in (12) and (16), proving Theorem 3, Part (ii).

Since $(M-1)L = pQ - c$ is a differential operator, we also have, using the notation α_{ij} as in the statement of Theorem 3,

$$\begin{aligned}\big((M-1)L\big)^m L^{np} &= \sum_{i=1}^m \alpha_{m,i}(M-1)^i L^{i+np} \\ &= \sum_{\substack{0\le i\le m\\ 0\le j\le i}} \alpha_{m,i}\binom{i}{j}(-1)^{i-j}M^jL^{i+np}\end{aligned}$$

is a differential operator. Thus

$$\sum_{\substack{0\le i\le m\\ 0\le j\le i}} \alpha_{m,i}\binom{i}{j}(-1)^{i-j}(M^jL^{i+np})_-\Psi = 0,$$

implying (10), upon using (19), completing the proof of Theorem 3.

To prove Remark 1, we evaluate

$$\left(\sum it_i \frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_1} - a\right)\tau = 0$$

at $t = 0$ to find $-\big((\partial\tau/\partial t_1)/\tau\big)|_{t=0} = a$. Remember, on the one hand,

$$\Psi(0,0,z) = \psi_0(z) = (1 + \tilde{b}_{01}z^{-1} + \cdots),$$

and on the other hand

$$\begin{aligned}\Psi(0,0,z) &= \left.\frac{\tau(t_1 + x - z^{-1}, \ldots)}{\tau(t_1 + x, \ldots)}\right|_{x=0,t=0} \\ &= \left.\left(1 - \tau^{-1}\frac{\partial\tau}{\partial x}z^{-1} + \cdots\right)\right|_{t=0}.\end{aligned}$$

Therefore $a = \tilde{b}_{01} = (-c)b_{0,1} = c(1+c)(p-1)/2$ as stated in Remark 1, as implied by (47).

To prove Theorem 1, note that at $t = 0$,

$$\begin{aligned}Q|_{t=0} &= (1/p)S((x-1)(\partial/\partial x) + c)S^{-1} \\ &= (1/p)(x-1)(\partial/\partial x) + c + (\text{negative order terms}).\end{aligned}$$

Since Q must be a differential operator, the negative order terms vanish, and $Q|_{t=0} = (1/p)(x-1)(\partial/\partial x) + c$. Thus, from the second equation in (9), we have

$$0 = \big(Q|_{t=0} - A_c\big)\Psi(x,0,z) = \big((x-1)(\partial/\partial x) - z(\partial/\partial z - 1)\big)\Psi(x,0,z). \tag{49}$$

Since this is a first order equation and the line $x = 0$ is noncharacteristic, $\Psi(x,0,z)$ is determined by (49) together with the initial condition $\Psi(0,0,z) = \psi_0(z)$. It is easy to check that the right-hand side of (6) satisfies these conditions. Finally, (7) follows from (8): Writing (8) as

$$F\left(z\frac{\partial}{\partial z} + c\right)\big(\mathrm{e}^{-z}\psi_0(z)\big) = (-z)^p\mathrm{e}^{-z}\psi_0(z),$$

substituting $(1-x)z$ for z, using the scaling invariance of $z\,\partial/\partial z$, and dividing both sides by z^p, we get

$$\frac{1}{z^p}F\left(z\frac{\partial}{\partial z} + c\right)\big(\mathrm{e}^{(x-1)z}\psi_0((1-x)z)\big) = (x-1)^p\mathrm{e}^{(x-1)z}\psi_0((1-x)z). \tag{50}$$

Multiplying both sides of this formula by e^z, and using the identity $\mathrm{e}^z(z\,\partial/\partial z + c) = (z\,\partial/\partial z - z + c)\circ\mathrm{e}^z$, we get the second formula in (7). Next, switching the roles of z and $1 - x$ in (50), we get

$$\frac{1}{(x-1)^p}F\left((x-1)\frac{\partial}{\partial x} + c\right)\big(\mathrm{e}^{(x-1)z}\psi_0((1-x)z)\big) = z^p\mathrm{e}^{(x-1)z}\psi_0((1-x)z).$$

Multiplying both sides of this formula by e^z, and using the fact that e^z commutes with $(x-1)\partial/\partial x + c$, we get the first formula in (7), completing the proof of Theorem 1.

5.2. Proof of Theorem 2. Setting $t_n = -\frac{1}{n}\sum_{i=1}^{n} z_i^{-n}$, $n = 1, 2, \ldots$, and using Lemma 5, and Lemma 4 with $s = 2$, we have

$$
\begin{aligned}
\tau(t) &= \frac{\det\big(\psi_{k-1}(z_i)\big)_{1\le k,i\le N}}{\Delta(z)} \\
&= \frac{a^N}{\Delta(z)} \det\left(z_i^{(p-1)(c+1)} e^{z_i} \int_0^\infty dx \int_0^\infty dy\, x^c e^{-xz_i^p} f_{k-1}(y) e^{-xy^p}\right)_{k,i} \\
&= \frac{a^N S_2(t)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx \int_{\mathbb{R}_+^N} dy \left(\prod_1^N x_i^c\right) \cdot \\
&\quad \cdot \det\big(f_{k-1}(y_i)\big)_{k,i}\, e^{-\sum_1^N x_i z_i^p} e^{-\sum_1^N x_i y_i^p} \\
&= \frac{a^N S_2(t)}{(N!)^2 \Delta(z)} \int_{\mathbb{R}_+^N} dx \int_{\mathbb{R}_+^N} dy \left(\prod_1^N x_i^c\right) \cdot \\
&\quad \cdot \det\big(f_{k-1}(y_i)\big)_{k,i} \det\left(e^{-x_i z_j^p}\right)_{i,j} \det\left(e^{-x_i y_j^p}\right)_{i,j} \\
&= \frac{a^N S_2(t)\Delta(z^p)}{(N!)^2 \Delta(z)} \int_{\mathbb{R}_+^N} dx \int_{\mathbb{R}_+^N} dy \left(\prod_1^N x_i^c\right) \Delta(x)^2 \Delta(y)^2 \cdot \\
&\quad \cdot S_0(y) \frac{\det\left(e^{-x_i z_j^p}\right)_{i,j}}{\Delta(x)\Delta(z^p)} \frac{\det\left(e^{-x_i y_j^p}\right)_{i,j}}{\Delta(x)\Delta(y^p)},
\end{aligned}
$$

where $a = p^{c+1}/2\pi i$,

$$S_2(t) = \prod_1^N \left(z_i^{(p-1)(c+1)} e^{z_i}\right),$$

and

$$S_0(y_1, y_2, \ldots, y_N) = \frac{\Delta(y^p)}{\Delta(y)} \frac{\det\big(f_{k-1}(y_i)\big)_{1\le i,k\le N}}{\Delta(y)}.$$

So we have, for some constants C, C' and C'' depending on N, p and c,

$$
\begin{aligned}
\tau(t) &= C \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx\, \Delta(x)^2 \left(\prod_1^N x_i^c\right) \int_{\mathbb{R}_+^N} dy\, \Delta(y)^2 S_0(y) \cdot \\
&\quad \cdot \int_{\mathrm{U}(N)} dU_X\, e^{-\operatorname{tr} Z^p U_X^{-1} x U_X} \int_{\mathrm{U}(N)} dV_Y\, e^{-\operatorname{tr} x V_Y^{-1} y^p V_Y} \\
&\qquad \text{using Lemma 7} \\
&= C \frac{S_2(t)\Delta(z^p)}{\Delta(z)} \int_{\mathbb{R}_+^N} dx\, \Delta(x)^2 \left(\prod_1^N x_i^c\right) \int_{\mathbb{R}_+^N} dy\, \Delta(y)^2 S_0(y) \cdot \\
&\quad \cdot \int_{\mathrm{U}(N)} dU_X\, e^{-\operatorname{tr} Z^p U_X^{-1} x U_X} \int_{\mathrm{U}(N)} dU_Y\, e^{-\operatorname{tr} U_X^{-1} x U_X U_Y^{-1} y^p U_Y} \\
&\qquad \text{setting } U_Y = V_Y U_X \text{ for fixed } U_X \text{ in the last} \\
&\qquad \text{integral and noting that } dU_X\, dU_Y = dU_X\, dV_Y
\end{aligned}
$$

$$
\begin{aligned}
&= C\frac{S_2(t)\Delta(z^p)}{\Delta(z)}\int_{\mathbb{R}_+^N} dx\,\Delta(x)^2\left(\prod_1^N x_i^c\right)\int_{\mathrm{U}(N)} dU_X\,\mathrm{e}^{-\operatorname{tr} Z^pU_X^{-1}xU_X}\,.\\
&\quad\cdot\int_{\mathbb{R}_+^N} dy\,\Delta^2(y)S_0(y)\int_{\mathrm{U}(N)} dU_Y\,\mathrm{e}^{-\operatorname{tr} U_X^{-1}xU_XU_Y^{-1}y^pU_Y}\\
&= C'\frac{S_2(t)\Delta(z^p)}{\Delta(z)}\int_{\mathscr{H}_N^+} dX\,\det(X^c)\mathrm{e}^{-\operatorname{tr} Z^pX}\int_{\mathscr{H}_N^+} dY\,S_0(y)\mathrm{e}^{-\operatorname{tr} XY^p}\\
&= C''S_1(t)\frac{\int_{\mathscr{H}_N^+} dX\,\det(X^c)\mathrm{e}^{-\operatorname{tr} Z^pX}\int_{\mathscr{H}_N^+} dY\,S_0(y)\mathrm{e}^{-\operatorname{tr} XY^p}}{\int_{\mathscr{H}_N} dX\,\exp\operatorname{tr}\left(-\frac{((X+Z)^{p+1})_2}{p+1}\right)}\,,
\end{aligned}
$$

where we used Lemma 6 in the last equality, and the definition of $S_1(t)$ in Theorem 2. A similar calculation, outlined below, implies the second formula for τ, upon using the first representation of $\psi_k(z)$ in (46):

$$
\begin{aligned}
\tau(t) &= \frac{\det\left(\psi_{k-1}(z_i)\right)_{1\le k,i\le N}}{\Delta(z)}\,,\quad \text{with } t_n = -\tfrac{1}{n}\textstyle\sum_{i=1}^{\infty} z_i^{-n}\,,\\
&= \frac{1}{\Delta(z)}\det\left(a\,\mathrm{e}^{z_i}z_i^{(p-1)(c+1)}\int_\gamma dw\int_0^\infty dx\,\mathrm{e}^{-w}w^{k-1}x^c\mathrm{e}^{xw^p}\mathrm{e}^{-xz_i^p}\right)_{k,i}\\
&= \frac{a^N}{\Delta(z)}\mathrm{e}^{\sum z_i}\prod z_i^{(p-1)(c+1)}\,.\\
&\quad\cdot\int_\gamma\cdots\int_\gamma dw\int_0^\infty\cdots\int_0^\infty dx\,\mathrm{e}^{-\sum w_i}\prod x_i^c\Delta(w)\prod_{i=1}^N\mathrm{e}^{-z_i^px_i}\prod_{i=1}^N\mathrm{e}^{x_iw_i^p}\\
&= \frac{a^N}{(N!)^2}\mathrm{e}^{\sum z_i}\prod z_i^{(p-1)(c+1)}\frac{1}{\Delta(z)}\int_{\gamma^N} dw\int_{\mathbb{R}_+^N} dx\,\mathrm{e}^{-\sum w_i}\prod x_i^c\Delta(w)\,\cdot\\
&\quad\cdot\det\left(\mathrm{e}^{-z_i^px_j}\right)_{1\le i,j\le N}\det\left(\mathrm{e}^{x_iw_j^p}\right)_{1\le i,j\le N}\\
&= \cdots\\
&= C'''S_1(t)\frac{\int_{\mathscr{H}_N^\gamma} m(dW)\int_{\mathscr{H}_N^+} dX\,\det X^c\left(\Delta(w^p)/\Delta(w)\right)\mathrm{e}^{-\operatorname{tr} W}\mathrm{e}^{\operatorname{tr} X(W^p-Z^p)}}{\int_{\mathscr{H}_N} dX\,\exp\operatorname{tr}\left(-\frac{(X+Z)^{p+1}}{p+1}\right)_2}\,,
\end{aligned}
$$

ending the proof of Theorem 2.

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