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Iso-Spectral Deformations of General Matrix and Their Reductions on Lie Algebras

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Abstract: We study an iso-spectral deformation of the general matrix which is a natural generalization of the nonperiodic Toda lattice equation. This deformation is equivalent to the Cholesky flow, a continuous version of the Cholesky algorithm, introduced by Watkins. We prove the integrability of the deformation and give an explicit formula for the solution to the initial value problem. The formula is obtained by generalizing the orthogonalization procedure of Szegö. Using the formula, the solution to the LU matrix factorization can be constructed explicitly. Based on the root spaces for simple Lie algebras, we consider several reductions of the equation. This leads to generalized Toda equations related to other classical semi-simple Lie algebras which include the integrable systems studied by Bogoyavlensky and Kostant. We show these systems can be solved explicitly in a unified way. The behaviors of the solutions are also studied. Generically, there are two types of solutions, having either sorting property or blowing up to infinity in finite time.

1. Introduction

In this paper we consider an iso-spectral deformation of an arbitrary diagonalizable matrix $L \in \mathfrak{M}(N, \mathbb{R})$. With the deformation parameter $t \in \mathbb{R}$, this is defined by

$$\frac{d}{dt}L = [P, L] \ , \tag{1.1}$$

where P is the generating matrix of the deformation and is given by a projection of L,

$$P = \Pi(L) := (L)_{>0} - (L)_{<0} \ . \tag{1.2}$$

Here $(L)_{>0\ (<0)}$ denotes the strictly upper (lower) triangular part of L. In terms of the standard basis of $\mathfrak{M}(N, \mathbb{R})$, i.e.,

$$E_{ij} = (\delta_{ik}\delta_{jl})_{1 \leqq k, l \leqq N} \ , \tag{1.3}$$

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the matrices L and P are expressed as

$$L = \sum_{1 \leqq i,j \leqq N} a_{ij} E_{ij} , \tag{1.4}$$

$$P = \sum_{1 \leqq i<j \leqq N} a_{ij} E_{ij} - \sum_{1 \leqq j<i \leqq N} a_{ij} E_{ij} . \tag{1.5}$$

Using the commutation relations for E_{ij}, i.e.,

$$[E_{ij}, E_{kl}] = E_{il} \delta_{jk} - E_{jk} \delta_{il} , \tag{1.6}$$

the equations for the components $a_{ij} = a_{ij}(t)$ are written in the form,

$$\frac{da_{ij}}{dt} = 2 \left(\sum_{k=I+1}^{N} - \sum_{k=1}^{J-1} \right) a_{ik} a_{kj} + (a_{II} - a_{JJ}) a_{ij} , \tag{1.7}$$

where $I := \max(i,j)$ and $J := \min(i,j)$. Equation (1.1) is also defined as the compatibility of the following linear equations with iso-spectral property of L:

$$L\Phi = \Phi \Lambda , \tag{1.8}$$

$$\frac{d}{dt} \Phi = P \Phi , \tag{1.9}$$

where Φ is the eigenmatrix, and Λ is the diagonal matrix of eigenvalues,

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) . \tag{1.10}$$

The set of equations (1.8) and (1.9) is also referred as the inverse scattering transform for the system (1.1). The main purpose of this paper is to show the complete integrability of (1.1) with (1.4) and (1.5) by means of explicit integration of the initial value problem using the inverse scattering transform.

One of the most famous and important examples of (1.1) is of course the non-periodic Toda lattice equation, where L is given by a symmetric tridiagonal matrix [15]. The matrices L and P for this equation are commonly written as

$$L_T = \sum_{i=1}^{N} a_i E_{ii} + \sum_{i=1}^{N-1} b_i (E_{i,i+1} + E_{i+1,i}) , \tag{1.11}$$

$$P_T = \sum_{i=1}^{N-1} b_i (E_{i,i+1} - E_{i+1,i}) . \tag{1.12}$$

The integrability of the Toda lattice equation has been shown by the inverse scattering method [8, 14, 15]. In this paper, we call (1.1) with (1.2) the "generalized Toda equation."

Several generalizations of the Toda lattice equation have been considered. In [2], Bogoyavlensky extended the equation based on the simple roots of semi-simple Lie algebra $\mathfrak{g}$, where L and P were given by

$$L_B = \sum_{i=1}^{r} a_i h_i + \sum_{\alpha \in \Pi} b_\alpha (e_\alpha + e_{-\alpha}) , \tag{1.13}$$

$$P_B = \sum_{\alpha \in \Pi} b_\alpha (e_\alpha - e_{-\alpha}) . \tag{1.14}$$

Here the elements $h_i, e_\alpha, e_{-\alpha}$ are Cartan–Weyl bases in $\mathfrak{g}$ with $r = \mathrm{rank}(\mathfrak{g})$ and $\Pi =$ the set of the simple roots. All of these equations associated with semi-simple Lie algebras are shown to be completely integrable hamiltonian systems. In [13] Kostant solved these by using the representation theory of semi-simple Lie algebras. In [1], Bloch et al. showed that these systems can be also written as gradient flow equations on an adjoint orbit of compact Lie group. They then showed that the generic flow assumes the "sorting property" (or convexity). Here the sorting property means that $L(t) \to \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ as $t \to \infty$, with the eigenvalues being ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_N$.

There are also other types of extensions: One of them is to extend L_T in (1.11) to a full symmetric matrix. The corresponding system, which we call the "full symmetric Toda equation," was shown by Deift et al. [5] to be also a complete integrable hamiltonian system. In [11] Kodama and McLaughlin solved the initial value problem of the corresponding inverse scattering problem (1.8) and (1.9), and obtained an explicit formula of the solution in a determinant form. They also showed the sorting property in the generic solution. The full symmetric Toda equation gives a QR-flow defined in [16], and the solution is obtained by the QR factorization method [17]. As a slight generalization of the full symmetric Toda equation, Kodama and Ye [12] considered a system with symmetrizable matrix L, which is expressed as $L_{KY} = L_S S$ with a full symmetric matrix L_S and a diagonal matrix S. A key feature of this system is that the eigenmatrix of L_{KY} can be taken as an element of a noncompact group of matrices, such as $O(p,q)$, and defines an indefinite metric in the eigenspace. The integrability was also shown in a similar way as in [11], and the general solution now assumes either sorting property or blowing up to infinity in finite time as a result of the indefinite metric. This system is equivalent to the HR-flow, a continuous version of the HR algorithm, introduced by Watkins [19].

In [7], Ercolani et al. proposed Eq. (1.1) with matrices,

$$L_H = \sum_{i=1}^{N-1} E_{i,i+1} + \sum_{1 \leqq j \leqq i \leqq N} b_{ij} E_{ij} \,, \tag{1.15}$$

$$P_H = -2(L_H)_{<0} = -2 \sum_{1 \leqq j < i \leqq N} b_{ij} E_{ij} \,, \tag{1.16}$$

which was called the "full Kostant–Toda equation." This is also an extension of the Toda equation (1.11) which can be written in the form,

$$\tilde{L}_T = \sum_{i=1}^{N-1} E_{i,i+1} + \sum_{i=1}^{N} a_i E_{ii} + \sum_{i=1}^{N-1} b_i^2 E_{i+1,i} \,. \tag{1.17}$$

As we will show in this paper, the transformation from (1.11) to (1.17) is given by a rescaling of the eigenvectors of L_T. Several group theoretical structures of the extended system were found. However the question whether the system is completely integrable still remains open in the sense of explicit integration.

It is immediate but important to observe that all of these extensions are special reductions of the generalized Toda equation (1.1). In fact, we show that these reductions are obtained more systematically as certain decompositions of the root spaces of semi-simple Lie algebras.

In terms of the matrix factorization, the generalized Toda equation (1.1) with (1.2) is equivalent to the Cholesky flow introduced by Watkins in [19]. The

Cholesky flow for a general matrix L has the same form as (1.1) with the generating matrix P defined by $-(L)_{<0} - (1/2)\,\mathrm{diag}\,(L)$. Writing P in (1.2) as $P = L - 2(L)_{<0} - \mathrm{diag}(L)$, we see that Eq. (1.1) is the same as the Cholesky flow except a scale change of t by $2t$. Deift et al. showed [6] that the Cholesky flow is a completely integrable system. It can be solved by the following LU-type of matrix factorization:

$$e^{tL(0)} = V(t)W(t)\,, \tag{1.18}$$

where $V(t)$ and $W(t)$ are lower and upper matrices respectively with $\mathrm{diag}(V(t)) = \mathrm{diag}(W(t))$. Then the solution $L(t)$ is given by

$$L(t) = V^{-1}(t)L(0)V(t) = W(t)L(0)W^{-1}(t)\,. \tag{1.19}$$

The above solution is not explicit in the sense of (1.18). The explicit formula of the matrix factorization is a direct consequence of our results.

In this paper we first show the complete integrability of (1.1) with (1.4) and (1.5) by means of the method of inverse scattering transform and give an explicit solution to the initial value problem. Here by complete integrability we mean the solvability of Eq. (1.1). Then we prove the complete integrability of any reductions of (1.1). The content of this paper is as follows: We start with a preliminary in Sect. 2 to give some background information necessary for analysis of the system (1.1) and the inverse scattering scheme (1.8) and (1.9).

In Sect. 3, we solve the initial value problem of (1.9) for the general system (1.1) by generalizing the method developed in [11] and [12]. A key in the method is to use the orthonormalization procedure of Szegö, which is equivalent to the Gram–Schmidt orthogonalization method. This shows the complete integrability of the generalized Toda equation in the sense of the inverse scattering transformation method. Based on our explicit solution, we then give an explicit solution to the Cholesky factorization (1.18).

In Sect. 4, we present reductions of (1.1) according to the classification of semi-simple Lie algebras. The matrix L here then contains "all" the root vectors, and it gives a generalization of the system formulated by Bogoyavlensky [2]. A key ingredient here is to find a matrix representation of the algebra in a decomposition consisting of diagonal, strictly upper and lower matrices (Lie's Theorem [10]). Then the integrability of these systems associated with semi-simple Lie algebras is a direct consequence of the result in Sect. 3.

Section 5 provides other reductions which include the full Kostant–Toda equation and the system with a matrix L having band structure in the elements.

In Sect. 6, we discuss the behavior of the solutions. Generically, in addition to the sorting property, there are solutions blowing up to infinity in finite time, as in the case discussed in [12].

Finally we illustrate the results obtained in this paper with explicit examples in Sect. 7.

2. Preliminary

Here we give some background information necessary for the inverse scattering method (1.8) and (1.9). As we will see in the next section, a key idea for solving these equations is to use an orthogonality of the eigenfunctions of (1.8). This is simply to consider a dual system of (1.8) and (1.9), which are written by

$$\Psi L = \Lambda \Psi \,, \tag{2.1}$$

$$\frac{d}{dt}\Psi = -\Psi P \,, \tag{2.2}$$

where the matrix Ψ is taken to be Φ^{-1}, and of course

$$\Psi\Phi = I, \qquad \Phi\Psi = I \,. \tag{2.3}$$

In terms of the eigenvectors, these matrices can be expressed as

$$\Phi \equiv [\phi(\lambda_1), \dots, \phi(\lambda_N)] = [\phi_i(\lambda_j)]_{1 \leqq i,j \leqq N} \,, \tag{2.4}$$

$$\Psi \equiv [\psi^T(\lambda_1), \dots, \psi^T(\lambda_N)]^T = [\psi_j(\lambda_i)]_{1 \leqq i,j \leqq N} \,. \tag{2.5}$$

Note here that $\phi(\lambda_i)$ and $\psi(\lambda_i)$ are the column and row eigenvectors, respectively. Then Eqs. (2.3) give

$$\sum_{k=1}^{N} \psi_k(\lambda_i)\phi_k(\lambda_j) = \delta_{ij} \,, \tag{2.6}$$

$$\sum_{k=1}^{N} \phi_i(\lambda_k)\psi_j(\lambda_k) = \delta_{ij} \,, \tag{2.7}$$

which are called the "first and second" orthogonality conditions. With (2.7), one can define an inner product $\langle \cdot, \cdot \rangle$ for two functions f and g of λ as

$$\langle f, g \rangle := \sum_{k=1}^{N} f(\lambda_k) g(\lambda_k) \,, \tag{2.8}$$

which we hereafter write as $\langle fg \rangle$. From $L = \Phi\Lambda\Psi$, the entries of L are then expressed by

$$a_{ij} := (L)_{ij} = \langle \lambda \phi_i \psi_j \rangle \,. \tag{2.9}$$

This gives a key equation for the inverse problem where we compute L from the eigenmatrix Φ (and Ψ) with the eigenvalues λ_i. So the eigenmatrix plays the role of the scattering data in the inverse scattering method. Then the method for solving the initial value problem of Eq. (1.1) can be formulated as follows: First we solve the eigenvalue (or scattering) problem (1.8) at $t = 0$, and find the scattering data, $\Phi^0 := \Phi(0)$. Then we solve the time evolution of the eigenmatrix from (1.9), and with the solution $\Phi(t)$ we obtain $L(t)$ through Eq. (2.9).

3. Inverse Scattering Method

In this section, we construct an explicit solution formula for the initial value problem of the generalized Toda equation (1.1) by using the inverse scattering method. A key of this method is to generalize the orthogonalization procedure of Szegö with respect to the inner product (2.8). This is essentially an extension of the method developed in [11].

Following [11] we first "gauge" transform Φ and Ψ by

$$\Phi = G\tilde{\Phi}, \qquad \Psi = \tilde{\Psi} G \,, \tag{3.1}$$

where the matrix G is given by

$$G = \mathrm{diag}[\langle\tilde{\phi}_1\tilde{\psi}_1\rangle^{-1/2}, \dots, \langle\tilde{\phi}_N\tilde{\psi}_N\rangle^{-1/2}] .$$

Note that the gauge transform (3.1) includes freedom in the choice of $\tilde{\phi}$ and $\tilde{\psi}$, that is, (3.1) is invariant under the scaling $\tilde{\phi}_i, \tilde{\psi}_i \to f_i(t)\tilde{\phi}_i,\ f_i(t)\tilde{\psi}_i$, with $\{f_i\}_{i=1}^N$ arbitrary functions of t. With (3.1), Eq. (1.8) and (1.9), as well as (2.1) and (2.2), become

$$(G^{-1}LG)\tilde{\Phi} = \tilde{\Phi}\Lambda, \qquad \tilde{\Psi}(GLG^{-1}) = \Lambda\tilde{\Psi} , \tag{3.2}$$

$$\frac{d}{dt}\tilde{\Phi} = (G^{-1}PG)\tilde{\Phi} - \left(\frac{d}{dt}\log G\right)\tilde{\Phi} , \tag{3.3}$$

$$\frac{d}{dt}\tilde{\Psi} = -\tilde{\Psi}(GPG^{-1}) - \tilde{\Psi}\left(\frac{d}{dt}\log G\right) . \tag{3.4}$$

Noting $G^{-1}(L)_{<0}G = (G^{-1}LG)_{<0}$, etc., we write

$$G^{-1}PG = -2(G^{-1}LG)_{<0} + G^{-1}LG - \mathrm{diag}\,(L) ,$$

$$GPG^{-1} = 2(GLG^{-1})_{>0} - GLG^{-1} + \mathrm{diag}\,(L) ,$$

from which we obtain the equations for the column vectors $\tilde{\phi}(\lambda, t)$ in $\tilde{\Phi}$ and the row vectors $\tilde{\psi}(\lambda, t)$ in $\tilde{\Psi}$,

$$\frac{d\tilde{\phi}}{dt} = -2(G^{-1}LG)_{<0}\tilde{\phi} + \lambda\tilde{\phi} - \left(\mathrm{diag}\,(L) + \frac{d}{dt}\log G\right)\tilde{\phi} , \tag{3.5}$$

$$\frac{d\tilde{\psi}}{dt} = -2\tilde{\psi}(GLG^{-1})_{>0} + \lambda\tilde{\psi} - \tilde{\psi}\left(\mathrm{diag}\,(L) + \frac{d}{dt}\log G\right) . \tag{3.6}$$

We here observe that (3.5) and (3.6) can be split into the following sets of equations by fixing the gauge freedom in the determination of ϕ and ψ. In the components, these are

$$\frac{d\tilde{\phi}_i}{dt} = -2\sum_{j=1}^{i-1}\frac{\langle\lambda\tilde{\phi}_i\tilde{\psi}_j\rangle}{\langle\tilde{\phi}_j\tilde{\psi}_j\rangle}\tilde{\phi}_j + \lambda\tilde{\phi}_i , \tag{3.7}$$

$$\frac{d\tilde{\psi}_j}{dt} = -2\sum_{i=1}^{j-1}\tilde{\psi}_i\frac{\langle\lambda\tilde{\phi}_i\tilde{\psi}_j\rangle}{\langle\tilde{\phi}_i\tilde{\psi}_i\rangle} + \lambda\tilde{\psi}_j , \tag{3.8}$$

$$\frac{1}{2}\frac{d}{dt}\log\langle\tilde{\phi}_i\tilde{\psi}_i\rangle = a_{ii} . \tag{3.9}$$

It is easy to check that (3.7) or (3.8) implies (3.9). It is also immediate from (3.7) and (3.8) that we have:

Proposition 3.1. *The solutions of* (3.7) *and* (3.8) *can be written in the following forms of separation of variables*:

$$\tilde{\phi}(\lambda, t) = M(t)\phi^0(\lambda)e^{\lambda t} , \tag{3.10}$$

$$\tilde{\psi}(\lambda, t) = \psi^0(\lambda)N(t)e^{\lambda t} , \tag{3.11}$$

where $M(t)$ and $N(t)$ are, respectively, lower and upper triangular matrices with $\mathrm{diag}[M(t)] = \mathrm{diag}[N(t)] = I$, *the identity matrix.*

Note here that the initial data for $\tilde{\phi}$ and $\tilde{\psi}$ are chosen as those of ϕ and ψ, i.e. $\tilde{\phi}(\lambda,0) = \phi(\lambda,0) := \phi^0(\lambda)$ and $\tilde{\psi}(\lambda,0) = \psi(\lambda,0) := \psi^0(\lambda)$. As a direct consequence of this proposition, and the orthogonality of the eigenvectors, (2.7), i.e., $\langle \tilde{\phi}_i \tilde{\psi}_j \rangle = 0$ for $i \neq j$, we have:

Corollary 3.1 (Orthogonality). *For each $i,j \in \{2,\dots,N\}$, we have for all $t \in \mathbb{R}$,*

$$\langle \tilde{\phi}_i \psi_l^0 e^{\lambda t} \rangle = 0, \quad \textit{for } l = 1,2,\dots,i-1 , \tag{3.12}$$

$$\langle \phi_k^0 \tilde{\psi}_j e^{\lambda t} \rangle = 0, \quad \textit{for } k = 1,2,\dots,j-1 . \tag{3.13}$$

Now we obtain the formulae for the eigenvectors of L in terms of the initial data $\{\phi_i^0(\lambda)\}_{1 \leqq i \leqq N}$ and $\{\psi_j^0(\lambda)\}_{1 \leqq j \leqq N}$:

Theorem 3.1. *The solutions $\tilde{\phi}_i(\lambda,t)$ and $\tilde{\psi}_j(\lambda,t)$ of* (3.7) *and* (3.8) *are given by*

$$\tilde{\phi}_i(\lambda,t) = \frac{e^{\lambda t}}{D_{i-1}(t)} \begin{vmatrix} c_{11} & \dots & c_{1,i-1} & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1} & \dots & c_{i,i-1} & \phi_i^0(\lambda) \end{vmatrix} , \tag{3.14}$$

$$\tilde{\psi}_j(\lambda,t) = \frac{e^{\lambda t}}{D_{j-1}(t)} \begin{vmatrix} c_{11} & \dots & c_{1j} \\ \vdots & \ddots & \vdots \\ c_{j-1,1} & \dots & c_{j-1,j} \\ \psi_1^0(\lambda) & \dots & \psi_j^0(\lambda) \end{vmatrix} , \tag{3.15}$$

where $c_{ij}(t) = \langle \phi_i^0 \psi_j^0 e^{2\lambda t} \rangle$, and $D_k(t)$ is the determinant of the $k \times k$ matrix with entries $c_{ij}(t)$, i.e.,

$$D_k(t) = \det[(c_{ij}(t))_{1 \leqq i,j \leqq k}] . \tag{3.16}$$

(*Note here that $c_{ij}(0) = \delta_{ij}$ and $D_k(0) = 1$.*)

Proof. From Eqs. (3.12) and (3.13) with (3.10) and (3.11), we have

$$\sum_{l=1}^{i} M_{il}(t) \langle \phi_l^0 \psi_k^0 e^{2\lambda t} \rangle = 0, \quad \text{for } 1 \leqq k \leqq i-1 , \tag{3.17}$$

$$\sum_{k=1}^{j} \langle \phi_l^0 \psi_k^0 e^{2\lambda t} \rangle N_{kj}(t) = 0, \quad \text{for } 1 \leqq l \leqq j-1 . \tag{3.18}$$

Solving (3.17) and (3.18) for M_{il} and N_{kj} with $M_{ii} = N_{jj} = 1$, we obtain

$$M_{il}(t) = \frac{(-1)^{i+l}}{D_{i-1}(t)} D_i \begin{bmatrix} l \\ i \end{bmatrix} (t) := \frac{\Delta_{l,i}(t)}{D_{i-1}(t)}, \quad 1 \leqq l \leqq i , \tag{3.19}$$

$$N_{kj}(t) = \frac{(-1)^{k+j}}{D_{j-1}(t)} D_j \begin{bmatrix} j \\ k \end{bmatrix} (t) := \frac{\Delta_{j,k}(t)}{D_{j-1}(t)}, \quad 1 \leqq k \leqq j , \tag{3.20}$$

where $D_i\left[{l \atop i}\right]$ is the determinant of D_i after removing the l^{th} row and i^{th} column. From (3.10) and (3.11), we then have

$$\tilde{\phi}_i(\lambda,t) = e^{\lambda t}\sum_{l=1}^{i} M_{il}\phi_l^0 = \frac{e^{\lambda t}}{D_{i-1}(t)}\sum_{l=1}^{i}\phi_l^0(\lambda)\Delta_{l,i}(t)\,, \tag{3.21}$$

$$\tilde{\psi}_j(\lambda,t) = e^{\lambda t}\sum_{k=1}^{j}\psi_k^0 N_{kj} = \frac{e^{\lambda t}}{D_{j-1}(t)}\sum_{k=1}^{j}\Delta_{j,k}(t)\psi_k^0(\lambda)\,. \tag{3.22}$$

Noticing that $\Delta_{l,i}$ is the cofactor of the element c_{li} of the matrix $(c_{mn})_{1\leqq m,n\leqq i}$ and $\Delta_{j,k}$ is for c_{jk} of $(c_{mn})_{1\leqq m,n\leqq j}$ we confirm (3.21) and (3.22) are just (3.14) and (3.15). □

We then note:

Corollary 3.2. *The gauge factors* $\langle\tilde{\phi}_i\tilde{\psi}_i\rangle$ *can be expressed by*

$$\langle\tilde{\phi}_i\tilde{\psi}_i\rangle(t) = \frac{D_i(t)}{D_{i-1}(t)}\,. \tag{3.23}$$

Proof. From (3.21) and (3.22), we have

$$\tilde{\phi}_i(\lambda)\tilde{\psi}_i(\lambda) = \frac{1}{D_{i-1}^2}\sum_{l,k=1}^{i}\Delta_{l,i}\Delta_{i,k}\phi_l^0(\lambda)\psi_k^0(\lambda)e^{2\lambda t}\,.$$

Then taking the bracket $\langle\,,\rangle$ in (2.8) leads to

$$\langle\tilde{\phi}_i\tilde{\psi}_i\rangle = \frac{1}{D_{i-1}^2}\sum_{l,k=1}^{i}\Delta_{l,i}\Delta_{i,k}c_{lk}\,.$$

Using the relation $\sum_{l=1}^{i}\Delta_{l,i}c_{lk} = D_i\delta_{ik}$ with $\Delta_{i,i} = D_{i-1}$ complete the proof. □

This yields the formulae for the normalized eigenfunctions

$$\phi_i(\lambda,t) = \frac{e^{\lambda t}}{\sqrt{D_i(t)D_{i-1}(t)}}\begin{vmatrix} c_{11} & \dots & c_{1,i-1} & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1} & \dots & c_{i,i-1} & \phi_i^0(\lambda)\end{vmatrix}\,, \tag{3.24}$$

$$\psi_j(\lambda,t) = \frac{e^{\lambda t}}{\sqrt{D_j(t)D_{j-1}(t)}}\begin{vmatrix} c_{11} & \dots & c_{1j} \\ \vdots & \ddots & \vdots \\ c_{j-1,1} & \dots & c_{j-1,j} \\ \psi_1^0(\lambda) & \dots & \psi_j^0(\lambda)\end{vmatrix}\,. \tag{3.25}$$

With the formula (3.24) and (3.25), we now have the solution (2.9) of the inverse scattering problem (1.8) and (1.9).

The above derivation of the eigenvectors is the same as the orthogonalization procedure of Szegö [18], which is equivalent to the Gram–Schmidt orthogonalization as observed in [11].

To see the connection with the Cholesky factorization method (1.18), we have:

Corollary 3.3. *The matrices* $V(t)$ *and* $W(t)$ *in the Cholesky factorization* (1.18) *can be explicitly represented by*

$$V(t) = M^{-1}(t)G^{-1}(t), \quad \text{and} \quad W(t) = G^{-1}(t)N^{-1}(t), \tag{3.26}$$

where $G(t)$ *is the gauge matrix in* (3.1), $M(t)$ *is the lower triangular matrix in* (3.10) *and* $N(t)$ *is the upper triangular matrix in* (3.11).

Proof. From (3.1), (3.10) and (3.11), we can write the solutions $\Phi(t)$ and $\Psi(t)$ as

$$\Phi(t) = GM\Phi^0 e^{\Lambda t} = GMe^{L(0)t}\Phi^0 ,$$

$$\Psi(t) = e^{\Lambda t}\Psi^0 NG = \Psi^0 e^{L(0)t} NG .$$

Using the relation $\Phi(t)\Psi(t) = I$, we have

$$GMe^{2L(0)t}NG = I ,$$

which leads to $e^{2L(0)t} = (GM)^{-1}(NG)^{-1}$. Since $\mathrm{diag}(M(t)) = \mathrm{diag}(N(t)) = I$, thus $\mathrm{diag}((GM)^{-1}) = \mathrm{diag}((NG)^{-1})$. This completes the proof. □

Remark 1. The generalized Toda equation (1.1) with (1.2) possesses a hierarchy defined by

$$\frac{\partial}{\partial t_n}L = [P_n, L], \quad n = 1, 2, \dots , \tag{3.27}$$

where P_n is given by

$$P_n = \Pi(L^n) \equiv (L^n)_{>0} - (L^n)_{<0} . \tag{3.28}$$

The commutativity of these flows can be shown by the "zero" curvature conditions of P_n, i.e.,

$$\frac{\partial P_m}{\partial t_n} - \frac{\partial P_n}{\partial t_m} + [P_m, P_n] = 0 , \tag{3.29}$$

which is a direct consequence of the choice of (3.28) [12]. The solution for the hierarchy is then obtained by extending the argument λt in the solution formula to $\xi(\lambda, t) := \sum_{n=1}^{\infty} \lambda^n t_n$ [12].

Remark 2. The well known QR flow for a general matrix $L \in \mathfrak{M}(N, \mathbb{R})$ is in the same form as (1.1) with the following (skew-symmetric) generating matrix P:

$$P = (L)_{>0} - (L^T)_{<0} = (L)_{>0} - [(L)_{>0}]^T . \tag{3.30}$$

It has been studied extensively in [4, 6, 16, 17] and [19]. They showed that this equation is a completely integrable hamiltonian system and can be solved in the sense of a matrix factorization of QR type, and the solution converges to a matrix in the triangular form. Our method developed in this section can be also applied to this problem as follows: First we note that the product $\Phi^*\Phi$ of the eigenmatrix Φ and its adjoint $\Phi^* := \overline{\Phi}^T$ is invariant under this flow (1.1). Then we define a hermitian matrix $S = (s_{ij})_{1 \leqq i,j \leqq N}$ as the inverse of $\Phi^*\Phi$, i.e.,

$$\Phi S \Phi^* = I . \tag{3.31}$$

The matrix S is determined from the initial eigenmatrix Φ^0, and $S\Phi^*$ gives the inverse of Φ, that is, we have $S\Phi^*$ for Ψ in our method. Note that if L is symmetric,

S is an identity matrix I and $\Phi \in O(N)$. In general, we see from the Binet–Cauchy theorem that S is positive definite. Equation (3.31) now gives the orthogonality relation,

$$\sum_{1 \leqq k, l \leqq N} \phi_i(\lambda_k) s_{kl} \overline{\phi_j(\lambda_l)} = \delta_{ij} , \tag{3.32}$$

from which we define the following inner product as in (2.8):

$$\langle\langle f, g \rangle\rangle := \sum_{1 \leqq k, l \leqq N} f(\lambda_k) s_{kl} \overline{g(\lambda_l)} = \overline{\langle\langle g, f \rangle\rangle} . \tag{3.33}$$

This leads to a positive definite metric. Then following the procedure in this section with some modifications based on $\Psi = S\Phi^*$, we obtain the same result for the eigenvectors (3.24) except the quantities c_{ij} which are now given by

$$c_{ij} = \langle\langle \phi_i^0 e^{\lambda t}, \phi_j^0 e^{\lambda t} \rangle\rangle = \bar{c}_{ji} . \tag{3.34}$$

The solution $L(t)$ is then given by $L(t) = \Phi \Lambda S \Phi^*$, i.e.,

$$a_{ij}(t) = \langle\langle \lambda \phi_i, \phi_j \rangle\rangle(t) . \tag{3.35}$$

Thus, we can show explicitly the integrability of Eq. (1.1) with the generator P given by (3.30) for arbitrary diagonal matrix L, and as a result of the positivity in the metric, the solution converges to an upper triangular matrix. The solution Φ can be also given in the same form as in Proposition 3.1, i.e.,

$$\Phi(t) = G(t) M(t) \Phi^0 e^{t\Lambda} .$$

Then writing $\Phi = W(t)\Phi^0$, one can show from (1.9) with (3.30) that $W(t)$ is an orthogonal matrix, and we have an explicit formula of the QR factorization,

$$e^{tL(0)} = [G(t)M(t)]^{-1} W(t) .$$

The detail will be discussed elsewhere.

Remark 3. In [19], Watkins introduced the LU flow as a continuous version of the LU algorithm. Deift et al. [6] showed that it is a completely integrable hamiltonian system. The flow on a general matrix $L \in \mathfrak{M}(N, \mathbb{R})$ is in the same form as (1.1) with the following generating matrix P:

$$P = -2(L)_{<0} . \tag{3.36}$$

It can be shown that the LU flow is related to (1.1) through a similarity transform (same as the transform in Proposition 5.2) which immediately implies its integrability through our scheme.

4. Isospectral Flows on Simple Lie Algebras

In this section, we consider the generalized Toda equations (1.1) associated with simple Lie algebras $\mathfrak{g}$, and show their integrability. The matrices L and P here are given by a generalization of (1.13) and (1.14), i.e.,

$$L_{\mathfrak{g}} = \sum_{i=1}^{r} a_i h_i + \sum_{\alpha\in\Delta^+} b_\alpha e_\alpha + \sum_{\beta\in\Delta^-} c_\beta e_\beta \,, \tag{4.1}$$

$$P_{\mathfrak{g}} = \sum_{\alpha\in\Delta^+} b_\alpha e_\alpha - \sum_{\beta\in\Delta^-} c_\beta e_\beta \,. \tag{4.2}$$

Here h_i are the bases for the Cartan subalgebra with $r = \mathrm{rank}(\mathfrak{g})$, Δ^+ and Δ^- are the sets of positive and negative roots with the corresponding root vectors e_α and $e_\beta(= e_{-\alpha})$. These vectors $\{h_i, e_\alpha\}$ form the Cartan–Weyl bases of the simple Lie algebra $\mathfrak{g}$ which satisfy for $i,j \in \{1,\dots,r\}$ and $\alpha, \beta \in \Delta := \Delta^+ \cup \Delta^-$,

$$\begin{aligned}
&[h_i, h_j] = 0, \qquad [h_i, e_\alpha] = \alpha(h_i) e_\alpha \,, \\
&[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}, \quad \text{if } \alpha+\beta \in \Delta \,, \\
&[e_\alpha, e_{-\alpha}] = h_\alpha, \quad \text{for } \alpha \in \Delta^+ \,.
\end{aligned} \tag{4.3}$$

Using representations of the Cartan–Weyl bases, we now express (4.1) and (4.2) in matrix form for each simple Lie algebra. Then we prove that Eq. (1.1) with those $L_{\mathfrak{g}}$ and $P_{\mathfrak{g}}$ associated with the Lie algebra $\mathfrak{g}$ is completely integrable by the inverse scattering method developed in Sect. 3. The key ingredient in the proof is to show that for each simple Lie algebra $\mathfrak{g}$ there exists a "permutation" matrix $O_{\mathfrak{g}}$ such that the matrices $L_{\mathfrak{g}}$ and $P_{\mathfrak{g}}$ are similar to L and P in (1.1) with P defined by (1.2), i.e.,

$$L = O_{\mathfrak{g}} L_{\mathfrak{g}} O_{\mathfrak{g}}^T \,, \tag{4.4}$$

$$P = O_{\mathfrak{g}} P_{\mathfrak{g}} O_{\mathfrak{g}}^T = \Pi(L) \,. \tag{4.5}$$

In other words, we look for a similarity transform such that the matrix representations of e_α for $\alpha \in \Delta^+$ and e_β for $\beta \in \Delta^-$ are transformed to strictly upper and lower triangular matrices, respectively. The existence of such representations is due to Lie's theorem [10]. Then the result in Sect. 3 implies the integrability of the system (1.1) with $L_{\mathfrak{g}}$ and $P_{\mathfrak{g}}$. Note here that the generalized Toda equation is invariant under the similarity transform. Here we consider all the classical simple Lie algebras A_n, B_n, C_n and D_n. The system associated with the exceptional algebra can be treated in the same way. For convenient matrix representations of the Cartan–Weyl bases, we follow the notations in [3] and [10]:

A_n: Let E_{ij} be the $(n+1)\times(n+1)$ matrix defined in (1.3). We then take an element of the Cartan subalgebra as $h = \sum_{i=1}^{n+1} \lambda_i E_{ii}$ with $\sum_{i=1}^{n+1} \lambda_i = 0$. Using (1.6) for E_{ij}, we have

$$[h, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \,. \tag{4.6}$$

Thus E_{ij} gives a root vector corresponding to the root $\alpha(h) = \lambda_i - \lambda_j$. The simple roots are defined as

$$\alpha_k(h) = \lambda_k - \lambda_{k+1}, \quad \text{for } k = 1,\dots,n \,. \tag{4.7}$$

Then the positive (negative) roots are given by $\lambda_i - \lambda_j$ with $i < j$ $(i > j)$. This implies that the choice of the P_{A_n} is the same as that in (1.2). Note also that adding some constant to the Cartan subalgebra, one can choose h_i of the basis to be E_{ii}.

Namely, the generalized Toda equation (1.1) with (1.4) and (1.5) can be considered as an iso-spectral flow on the simple Lie algebra A_n.

C_m: The element of this algebra is given by a $2m \times 2m$ matrix X satisfying $X^T J + JX = 0$, where J is defined by

$$J = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix} . \tag{4.8}$$

Here 0_m is the $m \times m$ 0-matrix, and I_m is the $m \times m$ identity matrix. We then choose the following bases with the $2m \times 2m$ matrix E_{ij} defined in (1.3),

$$\begin{aligned} e^1_{ij} &= E_{ij} - E_{j+m,i+m}, \quad 1 \leqq i,j \leqq m , \\ e^2_{ij} &= E_{i,j+m} + E_{j,i+m}, \quad 1 \leqq i \leqq j \leqq m , \\ e^3_{ij} &= E_{i+m,j} + E_{j+m,i}, \quad 1 \leqq i \leqq j \leqq m . \end{aligned} \tag{4.9}$$

Writing $h = \sum_{i=1}^m \lambda_i e^1_{ii}$ as a general element of the Cartan subalgebra, we have

$$\begin{aligned} [h, e^1_{ij}] &= (\lambda_i - \lambda_j) e^1_{ij}, \quad i \neq j , \\ [h, e^2_{ij}] &= (\lambda_i + \lambda_j) e^2_{ij}, \quad i \leqq j , \\ [h, e^3_{ij}] &= -(\lambda_i + \lambda_j) e^3_{ij}, \quad i \leqq j . \end{aligned} \tag{4.10}$$

The simple roots are taken by

$$\begin{aligned} \alpha_k(h) &= \lambda_k - \lambda_{k+1}, \quad \text{for } 1 \leqq k \leqq m-1 , \\ \alpha_m(h) &= 2\lambda_m , \end{aligned} \tag{4.11}$$

from which the sets of positive and negative root vectors $\Sigma^+_{C_m}$ and $\Sigma^-_{C_m}$ are given by

$$\Sigma^+_{C_m} = \{ e^1_{ij}, e^2_{kl} \mid 1 \leqq i < j \leqq m, \ 1 \leqq k \leqq l \leqq m \} , \tag{4.12}$$

$$\Sigma^-_{C_m} = \{ e^1_{ij}, e^3_{kl} \mid 1 \leqq j < i \leqq m, \ 1 \leqq k \leqq l \leqq m \} . \tag{4.13}$$

Then the matrix L_{C_m} can be represented by

$$L_{C_m} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} , \tag{4.14}$$

where $A_1, \ldots, A_4$ are the $m \times m$ matrices satisfying the relations

$$A_1^T = -A_4, \qquad A_2 = A_2^T, \qquad A_3 = A_3^T . \tag{4.15}$$

The matrix P_{C_m} is now given by

$$P_{C_m} = \begin{pmatrix} \Pi(A_1) & A_2 \\ -A_3 & -\Pi(A_4) \end{pmatrix} . \tag{4.16}$$

We then obtain:

Proposition 4.1. *With the permutation matrix O_{C_m}, we have the generalized Toda equation* (1.1) *on C_m with L-P pair given by*

$$L = O_{C_m} L_{C_m} O_{C_m}^T , \tag{4.17}$$

$$P = O_{C_m} P_{C_m} O_{C_m}^T = \Pi(L) , \tag{4.18}$$

where O_{C_m} is given by

$$O_{C_m} = \begin{pmatrix} I_m & 0_m \\ 0_m & Q_m \end{pmatrix} , \tag{4.19}$$

with the $m \times m$ matrix Q_m,

$$Q_m = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ 0 & \vdots & \vdots & 0 \\ 1 & \dots & 0 & 0 \end{pmatrix} = Q_m^T . \tag{4.20}$$

Proof. From (4.14) and (4.16), it suffices to show

$$-Q_m \Pi(A_4) Q_m^T = \Pi(Q_m A_4 Q_m^T) . \tag{4.21}$$

Recall that the multiplication of Q_m from the left (right) implies an exchange of rows (columns). Then we see

$$Q_m (A_4)_{>0(<0)} Q_m^T = (Q_m A_4 Q_m^T)_{<0(>0)} , \tag{4.22}$$

which implies the assertion. □

Note that Eq. (1.1) with L_{C_m} and P_{C_m} is just a reduction of the generalized Toda equation on A_{2m-1}.

Example 1. We take the simplest case C_2. The matrices L_{C_2} and P_{C_2} are represented as

$$L_{C_2} = \begin{pmatrix} a_1 & b_1 & b_2 & b_4 \\ c_1 & a_2 & b_4 & b_3 \\ c_2 & c_4 & -a_1 & -c_1 \\ c_4 & c_3 & -b_1 & -a_2 \end{pmatrix} , \tag{4.23}$$

and

$$P_{C_2} = \begin{pmatrix} 0 & b_1 & b_2 & b_4 \\ -c_1 & 0 & b_4 & b_3 \\ -c_2 & -c_4 & 0 & c_1 \\ -c_4 & -c_3 & -b_1 & 0 \end{pmatrix} . \tag{4.24}$$

Under the similarity transformation with O_{C_2} defined in (4.19), L_{C_2} and P_{C_2} becomes

$$L = O_{C_2} L_{C_2} O_{C_2}^T = \begin{pmatrix} a_1 & b_1 & b_4 & b_2 \\ c_1 & a_2 & b_3 & b_4 \\ c_4 & c_3 & -a_2 & -b_1 \\ c_2 & c_4 & -c_1 & -a_1 \end{pmatrix} , \tag{4.25}$$

and

$$P = O_{C_2} P_{C_2} O_{C_2}^T = \begin{pmatrix} 0 & b_1 & b_4 & b_2 \\ -c_1 & 0 & b_3 & b_4 \\ -c_4 & -c_3 & 0 & -b_1 \\ -c_2 & -c_4 & c_1 & 0 \end{pmatrix} . \tag{4.26}$$

Note here that under the similarity transformation the root space is decomposed into the diagonal, upper and lower triangular parts of the matrix (Lie's theorem).

D_m: The matrix representation of this algebra is given by a $2m \times 2m$ matrix X satisfying $X^T K + KX = 0$, where K is defined by

$$K = \begin{pmatrix} 0_m & I_m \\ I_m & 0_m \end{pmatrix} . \tag{4.27}$$

Then the bases can be chosen as

$$\begin{aligned} e_{ij}^1 &= E_{ij} - E_{j+m,i+m}, \quad 1 \leqq i,j \leqq m , \\ e_{ij}^2 &= E_{i,j+m} - E_{j,i+m}, \quad 1 \leqq i < j \leqq m , \\ e_{ij}^3 &= E_{i+m,j} - E_{j+m,i}, \quad 1 \leqq i < j \leqq m . \end{aligned} \tag{4.28}$$

With a general element $h = \sum_{i=1}^m \lambda_i e_{ii}^1$ of the Cartan subalgebra, we have

$$\begin{aligned} [h, e_{ij}^1] &= (\lambda_i - \lambda_j) e_{ij}^1, \quad j \neq k , \\ [h, e_{ij}^2] &= (\lambda_i + \lambda_j) e_{ij}^2, \quad i < j , \\ [h, e_{ij}^3] &= -(\lambda_i + \lambda_j) e_{ij}^3, \quad i < j , \end{aligned} \tag{4.29}$$

from which the simple roots may be taken as

$$\begin{aligned} \alpha_k(h) &= \lambda_k - \lambda_{k+1}, \quad \text{for } 1 \leqq k \leqq m-1 , \\ \alpha_m(h) &= \lambda_{m-1} + \lambda_m . \end{aligned} \tag{4.30}$$

The sets of positive and negative root vectors $\Sigma_{D_m}^+$ and $\Sigma_{D_m}^-$ are then given by

$$\Sigma_{D_m}^+ = \{ e_{ij}^1, e_{ij}^2 \mid 1 \leqq i < j \leqq m \} , \tag{4.31}$$

$$\Sigma_{D_m}^- = \{ e_{ij}^1, e_{ji}^3 \mid 1 \leqq j < i \leqq m \} . \tag{4.32}$$

The matrix L_{D_m} is then expressed as

$$L_{D_m} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} , \tag{4.33}$$

where the $m \times m$ matrices $A_1, \ldots, A_4$ satisfy

$$A_1^T = -A_4, \qquad A_2 = -A_2^T, \qquad A_3 = -A_3^T . \tag{4.34}$$

The matrix P_{D_m} is

$$P_{D_m} = \begin{pmatrix} \Pi(A_1) & A_2 \\ -A_3 & -\Pi(A_4) \end{pmatrix} . \tag{4.35}$$

It is then immediate to see from Proposition 4.1 that the permutation matrix O_{D_m} is the same as in the case of C_m. Namely we have:

Proposition 4.2. *With the permutation matrix* $O_{D_m} = O_{C_m}$ *given in* (4.19), *we have*

$$L = O_{D_m} L_{D_m} O_{D_m}^T , \tag{4.36}$$

$$P = O_{D_m} P_{D_m} O_{D_m}^T = \Pi(L) . \tag{4.37}$$

B_m: The element of this algebra satisfies the same relation as in D_m, $X^T K + KX = 0$, except now K is the $(2m+1) \times (2m+1)$ matrix defined by

$$K = \begin{pmatrix} 1 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & 0_m & I_m \\ \mathbf{0} & I_m & 0_m \end{pmatrix} , \tag{4.38}$$

where $\mathbf{0}$ is the m-column vector with 0 entries. This algebra is referred to as the orthogonal algebra $so(2m+1)$, while the algebra D_m is as $so(2m)$, and has the same bases as (4.28) with additional elements,

$$\begin{aligned} e_i^4 &= E_{0i} - E_{i+m,0}, \quad 1 \leqq i \leqq m , \\ e_i^5 &= E_{i0} - E_{0,i+m}, \quad 1 \leqq i \leqq m , \end{aligned} \tag{4.39}$$

where we have labeled the indices of the matrix E_{ij} as $0 \leqq i, j \leqq 2m$. With the expression $h = \sum_{i=1}^m \lambda_i e_{ii}^1$ as in the case of D_m, we have

$$[h, e_i^4] = -\lambda_i e_i^4, \qquad [h, e_i^5] = \lambda_i e_i^5 . \tag{4.40}$$

The simple roots are then chosen as

$$\begin{aligned} \alpha_k(h) &= \lambda_k - \lambda_{k+1}, \quad \text{for } 1 \leqq k \leqq m-1 , \\ \alpha_m(h) &= \lambda_m . \end{aligned} \tag{4.41}$$

The sets of positive and negative root vectors are now

$$\Sigma_{B_m}^+ = \{e_{ij}^1, e_{ij}^2, e_k^5 \mid 1 \leqq i < j \leqq m, \ 1 \leqq k \leqq m\} , \tag{4.42}$$

$$\Sigma_{B_m}^- = \{e_{ij}^1, e_{ji}^3, e_k^4 \mid 1 \leqq j < i \leqq m, \ 1 \leqq k \leqq m\} . \tag{4.43}$$

The matrix L_{B_m} is then expressed as

$$L_{B_m} = \begin{pmatrix} 0 & \mathbf{b}_1^T & \mathbf{b}_2^T \\ -\mathbf{b}_2 & A_1 & A_2 \\ -\mathbf{b}_1 & A_3 & A_4 \end{pmatrix} , \tag{4.44}$$

where $\mathbf{b}_1, \mathbf{b}_2$ are the m-column vectors, and the $m \times m$ matrices $A_1, \ldots, A_4$ satisfy the same relations as (4.34). The matrix P_{B_m} is now given by

$$P_{B_m} = \begin{pmatrix} 0 & -\mathbf{b}_1^T & \mathbf{b}_2^T \\ -\mathbf{b}_2 & \Pi(A_1) & A_2 \\ \mathbf{b}_1 & -A_3 & -\Pi(A_4) \end{pmatrix} . \tag{4.45}$$

We then have:

Proposition 4.3. *With the* $(2m+1) \times (2m+1)$ *permutation matrix* O_{B_m}, *we have*

$$L = O_{B_m} L_{B_m} O_{B_m}^T , \tag{4.46}$$

$$P = O_{B_m} P_{B_m} O_{B_m}^T = \Pi(L) , \tag{4.47}$$

where O_{B_m} *is given by*

$$O_{B_m} = \begin{pmatrix} \mathbf{0} & I_m & 0_m \\ 1 & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & 0_m & Q_m \end{pmatrix} . \tag{4.48}$$

Proof. Under the similarity transformation with O_{B_m}, L and P in (4.46) and (4.47) are given by

$$L = \begin{pmatrix} A_1 & -\mathbf{b}_2 & A_2 Q_m \\ \mathbf{b}_1^T & 0 & \mathbf{b}_2^T Q_m \\ Q_m A_3 & -Q_m \mathbf{b}_1 & Q_m A_4 Q_m^T \end{pmatrix} , \tag{4.49}$$

and

$$P = \begin{pmatrix} \Pi(A_1) & -\mathbf{b}_2 & A_2 Q_m \\ -\mathbf{b}_1^T & 0 & \mathbf{b}_2^T Q_m \\ -Q_m A_3 & Q_m \mathbf{b}_1 & -Q_m \Pi(A_4) Q_m^T \end{pmatrix} . \tag{4.50}$$

Then Eq. (4.21) immediately leads to the result. □

Example 2. We take the simplest case B_2, where L_{B_2} and P_{B_2} are represented by

$$L_{B_2} = \begin{pmatrix} 0 & c_3 & c_4 & -b_3 & -b_4 \\ b_3 & a_1 & b_1 & 0 & b_2 \\ b_4 & c_1 & a_2 & -b_2 & 0 \\ -c_3 & 0 & -c_2 & -a_1 & -c_1 \\ -c_4 & c_2 & 0 & -b_1 & -a_2 \end{pmatrix} , \tag{4.51}$$

and

$$P_{B_2} = \begin{pmatrix} 0 & -c_3 & -c_4 & -b_3 & -b_4 \\ b_3 & 0 & b_1 & 0 & b_2 \\ b_4 & c_1 & 0 & -b_2 & 0 \\ c_3 & 0 & c_2 & 0 & c_1 \\ c_4 & -c_2 & 0 & -b_1 & 0 \end{pmatrix} . \tag{4.52}$$

Under the similarity transformation with O_{B_2} defined in (4.48), L and P in (4.49) and (4.50) become

$$L = \begin{pmatrix} a_1 & b_1 & b_3 & 0 & -b_2 \\ c_1 & a_2 & b_4 & b_2 & 0 \\ c_3 & c_4 & 0 & -b_4 & -b_3 \\ 0 & c_2 & -c_4 & -a_2 & -b_1 \\ -c_2 & 0 & -c_3 & -c_1 & -a_1 \end{pmatrix}, \tag{4.53}$$

and

$$P = \begin{pmatrix} 0 & b_1 & b_3 & 0 & -b_2 \\ -c_1 & 0 & b_4 & b_2 & 0 \\ -c_3 & -c_4 & 0 & -b_4 & -b_3 \\ 0 & -c_2 & c_4 & 0 & -b_1 \\ c_2 & 0 & c_3 & c_1 & 0 \end{pmatrix}. \tag{4.54}$$

5. Reductions on Root Spaces

As we have explained in the introduction, several generalizations of the Toda equation may be obtained by taking reductions of the generalized Toda equation (1.1) with the general matrix L. We then showed in the previous section that the equations on simple Lie algebras studied in [2] are generalized by taking all the root vectors in the algebras. In this section, we consider reductions of these equations by restricting the set of roots in the sums in (4.1).

Let S^+ and S^- be subsets of positive and negative roots of a simple Lie algebra $\mathfrak{g}$ defined by, for $\forall\alpha_0 \in S^+$ and $\forall\beta_0 \in S^-$,

$$S^+ := \{\alpha \in \Delta^+ \mid \alpha \prec \alpha_0\}, \tag{5.1}$$

$$S^- := \{\beta \in \Delta^- \mid \beta \succ \beta_0\}. \tag{5.2}$$

Here "$\prec$" and "$\succ$" are the standard partial orderings between roots. We then consider Eq. (1.1) with the matrices $\hat{L}$ and $\hat{P}$ given by

$$\hat{L} = \sum_{i=1}^{n} a_i h_i + \sum_{\alpha\in S^+} b_\alpha e_\alpha + \sum_{\beta\in S^-} c_\beta e_\beta, \tag{5.3}$$

$$\hat{P} = \sum_{\alpha\in S^+} b_\alpha e_\alpha - \sum_{\beta\in S^-} c_\beta e_\beta, \tag{5.4}$$

where $n = \mathrm{rank}(\mathfrak{g})$. We here claim:

Proposition 5.1. *Equation* (1.1) *with $\hat{L}$ and $\hat{P}$ is a reduction of the generalized Toda equation on* $\mathfrak{g}$.

Proof. All we need to show is that the commutator $[\hat{P}, \hat{L}]$ is in the span of the root vectors whose roots are in S^+ and S^-. From (5.3) and (5.4), $[\hat{L}, \hat{P}]$ can be written as

$$[\hat{P}, \hat{L}] = -\sum_{i=1}^{n} a_i[h_i, \hat{P}] + 2\sum_{\alpha\in S^+}\sum_{\beta\in S^-} b_\alpha c_\beta [e_\alpha, e_\beta]. \tag{5.5}$$

Using (4.3) we first note that the terms $[h_i, \hat{P}]$ do not produce any new root vectors. The second term, which then gives $[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}$, has a root $\alpha+\beta \in S^+ \cup S^-$ (if $\alpha+\beta \in \Delta$), since $\alpha \in \Delta^+$ and $\beta \in \Delta^-$. This completes the proof. $\square$

Example 3. The generalized Toda equation with band matrix L. This example can be obtained as the following reduction on A_{N-1}: Consider the subsets of the roots S^+ and S^- given by

$$S^+ = \{(i,j) \in \Delta^+ \mid 0 < j-i \leqq M^+ \leqq N-1\}, \tag{5.6}$$

$$S^- = \{(i,j) \in \Delta^- \mid 0 < i-j \leqq M^- \leqq N-1\}, \tag{5.7}$$

where M^+ and M^- are some positive integers. Then the corresponding matrix $\hat{L}$ which we denote $L_{(M^+,M^-)}$ becomes

$$L_{(M^+,M^-)} = \begin{pmatrix} a_{11} & \dots & a_{1,1+M^+} & 0 & \dots & 0 \\ \vdots & \ddots & \dots & \ddots & \dots & \vdots \\ a_{1+M^-,1} & \dots & \dots & \dots & \ddots & 0 \\ 0 & \ddots & \dots & \ddots & \dots & a_{N-M^+,N} \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & a_{N,N-M^-} & \dots & a_{NN} \end{pmatrix}. \tag{5.8}$$

As a special case of this example, we now construct the full Kostant–Toda equation having a L_H-P_H pair given in (1.15) and (1.16). Here we choose S^+ and S^- to be the sets of the simple roots (i.e. $M^+ = 1$) and of all the negative roots (i.e. $M^- = N-1$), respectively. Thus the corresponding matrix is expressed as

$$L_{(1,N-1)} = \begin{pmatrix} a_{11} & b_1 & 0 & \dots & 0 \\ a_{21} & a_{22} & b_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \dots & \dots & a_{N-1,N-1} & b_{N-1} \\ a_{N1} & \dots & \dots & a_{N,N-1} & a_{NN} \end{pmatrix}. \tag{5.9}$$

Then we claim:

Proposition 5.2. *The full Kostant–Toda equation is obtained from the generalized Toda equation* (1.1) *with* $L_{(1,N-1)}$ *and* $P_{(1,N-1)} := \Pi(L_{(1,N-1)})$ *through a similarity transform* $L_H = HL_{(1,N-1)}H^{-1}$, *where* H *is given by*

$$H = \mathrm{diag}\left[1,\ b_1,\ b_1b_2, \dots, \prod_{i=1}^{N-1} b_i\right]. \tag{5.10}$$

Proof. From Eq. (1.7) for $L_{(1,N-1)}$, we have

$$\frac{db_i}{dt} = (a_{i+1,i+1} - a_{ii})b_i, \quad \text{for } i = 1,\dots,N-1, \tag{5.11}$$

from which H satisfies

$$\frac{d}{dt}H = (\mathrm{diag}(L_H) - a_{11}I_N)H. \tag{5.12}$$

Note here that $\mathrm{diag}(L_H) = \mathrm{diag}(L_{(1,N-1)}) = \mathrm{diag}[a_{11},\dots,a_{NN}]$. Then the derivative of L_H is calculated as

$$\frac{d}{dt}L_H = [\mathrm{diag}(L_H), L_H] + [\Pi(L_H), L_H], \tag{5.13}$$

where we have used $dL_{(1,N-1)}/dt = [P_{(1,N-1)}, L_{(1,N-1)}]$, and $HP_{(1,N-1)}H^{-1} = \Pi(L_H)$. Noting the relation

$$\Pi(L_H) + \mathrm{diag}(L_H) = L_H - 2(L_H)_{<0}\,, \tag{5.14}$$

we complete the proof. □

Thus the full Kostant–Toda equation can be solved through the generalized Toda equation with the $L_{(1,N-1)}$-$P_{(1,N-1)}$ pair as the reduction on A_{N-1}, that is, with the solution $L_{(1,N-1)}$, $L_H = HL_{(1,N-1)}H^{-1}$. The similarity transform H in (5.10) was introduced by Kostant [13] to write the original nonperiodic Toda equation in the Hessenberg matrix form.

6. Behaviors of the Solutions

Here we study the behavior of the solution of the generalized Toda equation obtained in Sect. 3 by following the approach in [12]. Many results obtained in [12] are valid for this more general situation. First we note:

Lemma 6.1. *The determinants D_i for $i = 1, 2, \ldots, N$ in* (3.16) *are real functions.*

Proof. In the construction of the solutions $\Phi(t)$ and $\Psi(t)$, the "gauge" G is fixed by (3.9). In terms of D_i, (3.9) is

$$a_{ii} = \frac{1}{2}\frac{d}{dt}\log\frac{D_i}{D_{i-1}}\,. \tag{6.1}$$

Note that $D_0 \equiv 1$, $D_i(0) = 1$ and a_{ii} are real functions. Then we see by induction that all D_i are real functions. □

In (6.1), suppose $D_i(t_0) = 0$ for some finite t_0 and some i. Then if $L(t_0)$ is a finite matrix, $D_{i-1}(t_0)$ must be also 0. By induction, $D_1(t_0) = 0$, but $D_0(t) \equiv 1$, this forces a_{11} to be infinite, which is a contradiction. So we have:

Lemma 6.2. *Suppose $D_i(t_0) = 0$ for some $t_0 < \infty$ and some i. Then $L(t)$ blows up to infinity at t_0.*

We note that D_i for $i = 1, 2, \ldots, N$ are the i^{th} leading principal minors of the product of matrices $\Phi_e\Psi_e$, where Φ_e and Ψ_e are defined by

$$\Phi_e := \begin{pmatrix} e^{\lambda_1 t}\phi_1^0(\lambda_1) & \ldots & e^{\lambda_N t}\phi_1^0(\lambda_N) \\ \vdots & \ddots & \vdots \\ e^{\lambda_1 t}\phi_N^0(\lambda_1) & \ldots & e^{\lambda_N t}\phi_N^0(\lambda_N) \end{pmatrix},$$

and

$$\Psi_e := \begin{pmatrix} e^{\lambda_1 t}\psi_1^0(\lambda_1) & \ldots & e^{\lambda_1 t}\psi_N^0(\lambda_1) \\ \vdots & \ddots & \vdots \\ e^{\lambda_N t}\psi_1^0(\lambda_N) & \ldots & e^{\lambda_N t}\psi_N^0(\lambda_N) \end{pmatrix}.$$

Then from the Binet–Cauchy theorem, we have:

Lemma 6.3. *The determinants D_i with $i = 1, 2, \ldots, N$ can be expressed as*

$$D_i(t) = \sum_{J_{iN}} e^{2\sum_{k=1}^{i}\lambda_{j_k}t} \begin{vmatrix} \phi_1^0(\lambda_{j_1}) & \ldots & \phi_1^0(\lambda_{j_i}) \\ \vdots & \ddots & \vdots \\ \phi_i^0(\lambda_{j_1}) & \ldots & \phi_i^0(\lambda_{j_i}) \end{vmatrix} \begin{vmatrix} \psi_1^0(\lambda_{j_1}) & \ldots & \psi_i^0(\lambda_{j_1}) \\ \vdots & \ddots & \vdots \\ \psi_1^0(\lambda_{j_i}) & \ldots & \psi_i^0(\lambda_{j_i}) \end{vmatrix}, \tag{6.2}$$

where $J_{iN} = (j_1, \ldots, j_i)$ *represents all possible combinations for* $1 \leqq j_1 < \cdots < j_i \leqq N$. *In particular* $D_0(t) \equiv 1$, *and* $D_N(t) = \exp(2\sum_{i=1}^{N} \lambda_i t)$.

This lemma is very useful to study the asymptotic behavior of D_i for large t. We now obtain:

Theorem 6.1. *Let the eigenvalues of* L *be all real and ordered as* $\lambda_1 > \lambda_2 > \cdots > \lambda_N$. *Suppose that* $\det(\Phi_k^0) \neq 0$ *and* $\det(\Psi_k^0) \neq 0$ *for* $k = 1, \ldots, N$, *where* Φ_k^0 *and* Ψ_k^0 *are the* k^{th} *leading principal submatrices of* Φ^0 *and* Ψ^0, *respectively. Then as* $t \to \infty$, *the eigenfunctions* $\phi_i(\lambda_j, t)$ *and* $\psi_j(\lambda_i, t)$ *satisfy*

$$\phi_i(\lambda_j, t) \to \delta_{ij} \times \frac{\det(\Phi_i^0)\det(\Psi_{i-1}^0)}{\sqrt{\det(\Phi_i^0 \Psi_i^0)\det(\Phi_{i-1}^0 \Psi_{i-1}^0)}}, \tag{6.3}$$

$$\psi_j(\lambda_i, t) \to \delta_{ij} \times \frac{\det(\Phi_{i-1}^0)\det(\Psi_i^0)}{\sqrt{\det(\Phi_i^0 \Psi_i^0)\det(\Phi_{i-1}^0 \Psi_{i-1}^0)}}, \tag{6.4}$$

which implies the sorting property as $t \to \infty$, *that is,* $L(t) = \Phi(t)\Lambda\Psi(t) \to \Lambda$.

Proof. Here we give a proof for (6.3). The case for $\psi_j(\lambda_i, t)$ is obtained in the same way. Using Lemma 6.3, and from the ordering in the eigenvalues we see that the leading order term for D_i is given by

$$D_i(t) \to e^{2\sum_{k=1}^{i}\lambda_k t}\det(\Phi_i^0 \Psi_i^0), \quad \text{as}\, t \to \infty\,. \tag{6.5}$$

From (3.24) and (6.5), the eigenfunctions behave as $t \to \infty$,

$$\phi_i(\lambda; t) \to \frac{e^{(\lambda - 2\sum_{k=1}^{i-1}\lambda_k - \lambda_i)t}}{\sqrt{\det(\Phi_i^0 \Psi_i^0)\det(\Phi_{i-1}^0 \Psi_{i-1}^0)}} \begin{vmatrix} c_{11}(t) & \ldots & c_{1,i-1}(t) & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ c_{i1}(t) & \ldots & c_{i,i-1} & \phi_i^0(\lambda) \end{vmatrix}. \tag{6.6}$$

The dominant term in the determinant gives

$$e^{2\sum_{k=1}^{i-1}\lambda_k t} \sum_{\mathbb{P}_{i-1}} \psi_1^0(\lambda_{l_1}) \cdots \psi_{i-1}^0(\lambda_{l_{i-1}}) \begin{vmatrix} \phi_1^0(\lambda_{l_1}) & \ldots & \phi_1^0(\lambda_{l_{i-1}}) & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ \phi_i^0(\lambda_{l_1}) & \ldots & \phi_i^0(\lambda_{l_{i-1}}) & \phi_i^0(\lambda) \end{vmatrix}$$

$$= e^{2\sum_{k=1}^{i-1}\lambda_k t}\det(\Psi_{i-1}^0) \begin{vmatrix} \phi_1^0(\lambda_1) & \ldots & \phi_1^0(\lambda_{i-1}) & \phi_1^0(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ \phi_i^0(\lambda_1) & \ldots & \phi_i^0(\lambda_{i-1}) & \phi_i^0(\lambda) \end{vmatrix}, \tag{6.7}$$

where $\mathbb{P}_k$ is the permutation $\begin{pmatrix} 1 & 2 & \cdots & k \\ l_1 & l_2 & \cdots & l_k \end{pmatrix}$. Noting that the determinant in (6.7) is zero for $\lambda = \lambda_j$, with $j = 1, \ldots, i-1$, we complete the proof. □

This theorem implies that if all the eigenvalues of L are real, then generic solutions have the "sorting property" in the asymptotic sense. It should be however noted that $D_i(t)$ might be zero for some "finite" times, where the solution blows up (Lemma 6.2). The next theorem provides sufficient conditions for the solutions to blow up to infinity in finite time.

Theorem 6.2. *Suppose some eigenvalues of L are not real,* $\det \Phi_n^0 \neq 0$ *and* $\det \Psi_n^0 \neq 0$, *for* $n = 1,\dots,N$. *Then* $L(t)$ *blows up to infinity in finite time.*

Proof. We order the eigenvalues of $\tilde{L}$ by their real parts. We still assume all the eigenvalues to be distinct. Since L is a real matrix, the complex eigenvalues appear as pairs. For convenience, we also assume that there is at most one pair having the same real part. Suppose $\lambda_k + i\beta_k$ and $\lambda_k - i\beta_k$ are the first pair of complex eigenvalues. Then from (6.2), the leading order term in D_k is

$$e^{2\sum_{l=1}^{k}\lambda_l t+2i\beta_k t}\begin{vmatrix}\phi_1^0(\lambda_1) & \dots & \phi_1^0(\lambda_k+i\beta_k)\\ \vdots & \ddots & \vdots\\ \phi_k^0(\lambda_1) & \dots & \phi_k^0(\lambda_k+i\beta_k)\end{vmatrix}\begin{vmatrix}\psi_1^0(\lambda_1) & \dots & \psi_1^0(\lambda_k+i\beta_k)\\ \vdots & \ddots & \vdots\\ \psi_k^0(\lambda_1) & \dots & \psi_k^0(\lambda_k+i\beta_k)\end{vmatrix}$$

$$+\,e^{2\sum_{l=1}^{k}\lambda_l t-2i\beta_k t}\begin{vmatrix}\phi_1^0(\lambda_1) & \dots & \phi_1^0(\lambda_k-i\beta_k)\\ \vdots & \ddots & \vdots\\ \phi_k^0(\lambda_1) & \dots & \phi_k^0(\lambda_k-i\beta_k)\end{vmatrix}\begin{vmatrix}\psi_1^0(\lambda_1) & \dots & \psi_1^0(\lambda_k-i\beta_k)\\ \vdots & \ddots & \vdots\\ \psi_k^0(\lambda_1) & \dots & \psi_k^0(\lambda_k-i\beta_k)\end{vmatrix}.$$

Since D_k is real by Lemma 1, one can write the above as

$$e^{2\sum_{l=1}^{k}\lambda_l t}[A\cos(2\beta_k t) + B\sin(2\beta_k t)]\,,$$

where A and B are two real constants. The above is an oscillating function about zero. Thus by Lemma 6.2, we conclude that $L(t)$ blows up to infinity in finite time. □

Remark 4. All the results in this section remain valid for the full Kostant–Toda equation defined by (1.15) and (1.16). To see this, from Proposition 5.2, we solve $L_{(1,N-1)}$ with $L_{(1,N-1)}(0) = L_H(0)$. Then $L_H(t)$ is related to $L_{(1,N-1)}(t)$ through $L_H = HL_{(1,N-1)}H^{-1}$, where H is defined in (5.10) with $b_i(0) = 1$, $i = 1,\dots,N-1$. In the case $L_{(1,N-1)}(t)$ has the sorting property, since b_is in (5.11) all go to zero, one verifies L_H also has the sorting property. Thus Theorem 6.1 holds. In the blowing up case, since the transform by H (5.10) doesn't change the diagonal element, Lemma 6.2 holds, thus Theorem 6.2 is valid. In [9], the solution behavior of the tridiagonal Kostant–Toda equation is considered. A neccessary and sufficient condition for the blowing up is obtained.

7. Example

In this section, we demonstrate the results obtained in this paper by taking an explicit form of the matrix L. The main purpose here is to solve the generalized Toda equation (1.1) for this explicit matrix, and discuss the behavior of the solution.

Let us consider a 2×2 matrix $L(t) = (a_{ij})_{1\le i,j\le 2}$. The generalized Toda equation then gives

$$\frac{d}{dt}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix}2a_{12}a_{21} & a_{12}(a_{22}-a_{11})\\ a_{21}(a_{22}-a_{11}) & -2a_{21}a_{12}\end{pmatrix}. \tag{7.1}$$

The initial data of $L(t)$ is assumed to be

$$L(0) = \begin{pmatrix}0 & 1\\ a & b\end{pmatrix}, \tag{7.2}$$

where a and b are arbitrary constants. The eigenvalues of $L(0)$, λ_1 and λ_2, are

$$\lambda_{1,2} = \frac{1}{2}(b \pm \sqrt{b^2 + 4a}) . \tag{7.3}$$

Then the initial eigenmatrices Φ^0 and Ψ^0 are expressed by

$$\Phi^0 = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \tag{7.4}$$

$$\Psi^0 = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}. \tag{7.5}$$

In order to compute the solutions $\Phi(t)$ and $\Psi(t)$ from (3.24) and (3.25), we need the quantities $c_{ij} = \langle \phi^0 \psi^0 e^{2\lambda t} \rangle$. From (7.4) and (7.5), they are

$$\begin{aligned} c_{11}(t) &= \frac{1}{\lambda_2 - \lambda_1}(\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}) , \\ c_{12}(t) &= \frac{1}{\lambda_2 - \lambda_1}(-e^{2\lambda_1 t} + e^{2\lambda_2 t}) , \\ c_{21}(t) &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}(e^{2\lambda_1 t} - e^{2\lambda_2 t}) , \\ c_{22}(t) &= \frac{1}{\lambda_2 - \lambda_1}(-\lambda_1 e^{2\lambda_1 t} + \lambda_2 e^{2\lambda_2 t}) , \end{aligned} \tag{7.6}$$

from which the determinants $D_i(t)$ in (3.16) become

$$D_1(t) = c_{11}(t), \qquad D_2(t) = \begin{vmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{vmatrix} = e^{2(\lambda_1 + \lambda_2)t} . \tag{7.7}$$

We now have the solutions (Theorem 3.1),

$$\Phi(t) = \frac{1}{\sqrt{D_1(t)}} \begin{pmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_2 t} & \lambda_2 e^{\lambda_1 t} \end{pmatrix}, \tag{7.8}$$

$$\Psi(t) = \frac{1}{(\lambda_2 - \lambda_1)\sqrt{D_1(t)}} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} & -e^{\lambda_2 t} \\ -\lambda_1 e^{\lambda_2 t} & e^{\lambda_1 t} \end{pmatrix}. \tag{7.9}$$

The solution $L(t)$ of the generalized Toda equation is then obtained from (2.9), $a_{ij}(t) = \langle \lambda \phi_i \psi_j \rangle (t)$,

$$L(t) = \frac{1}{\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}} \begin{pmatrix} \lambda_1 \lambda_2 (e^{2\lambda_1 t} - e^{2\lambda_2 t}) & (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t} \\ -\lambda_1 \lambda_2 (\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)t} & \lambda_2^2 e^{2\lambda_1 t} - \lambda_1^2 e^{2\lambda_2 t} \end{pmatrix}. \tag{7.10}$$

Now let us discuss the solution behavior for $t > 0$. First we assume both eigenvalues λ_1 and λ_2 to be real. With the choice of the eigenvalues in (7.3), we have $\lambda_1 \geqq \lambda_2$. Then if $\lambda_1 \lambda_2 \leqq 0$, then the function $D_1(t)$ does not vanish for all t. This implies the sorting property (Theorem 6.1). For the case of $\lambda_1 > \lambda_2 > 0$, the D_1 vanishes and we have the blowing up in the solution at the time $t = t_B > 0$,

$$t_B = \frac{1}{2(\lambda_1 - \lambda_2)} \log \frac{\lambda_1}{\lambda_2} . \tag{7.11}$$

This formula also implies that for $0 > \lambda_1 > \lambda_2$ we have the sorting result for $t > 0$. Note here that the blowing up occurs at one time $t = t_B$ (7.11), and then the solution $L(t)$ will be sorted as $t \to \infty$, with the asymptotic forms of the eigenmatrices, i.e., (6.3) and (6.4),

$$\Phi(t) \to \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_2}} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{7.12}$$

$$\Psi(t) \to \frac{1}{\sqrt{\lambda_2(\lambda_2 - \lambda_1)}} \begin{pmatrix} \lambda_2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.13}$$

For the case of the complex eigenvalue $\lambda_1 = \bar{\lambda}_2 := \alpha + i\beta$, $D_1(t)$ is expressed as

$$D_1(t) = e^{2\alpha t} \sec\theta \cos(2\beta t + \theta) \tag{7.14}$$

with $\tan\theta = \alpha/\beta$. This indicates the blowing up (Theorem 6.2).

In the case of degenerate eigenvalues $\lambda_1 = \lambda_2$ (i.e., $b^2 + 4a = 0$), we take the limit $\lambda_2 \to \lambda_1 := \lambda_0$ in (7.10), and obtain

$$L(t) = \frac{1}{1 - 2\lambda_0 t} \begin{pmatrix} -2\lambda_0^2 t & 1 \\ -\lambda_0^2 & 2\lambda_0(1 - \lambda_0 t) \end{pmatrix}. \tag{7.15}$$

which shows the "sorting property" as $t \to \infty$, i.e., $L(t) \to \lambda_0 I_2$. It should be noted however that $L(0)$ with the degenerate eigenvalues is not similar to the "diagonal" matrix $\lambda_0 I_2$.

We summarize the above results in the figure, where we classify the behavior of the solution in terms of the parameters a and b in (7.2).

In the figure, the shaded region corresponds to the blowing up solutions for $t > 0$ where the eigenvalues are either complex or real with $\lambda_1 > \lambda_2 > 0$. The other region including the positive b-axis and the lower boundary of the curve $b^2 + 4a = 0$ gives the sorting property.

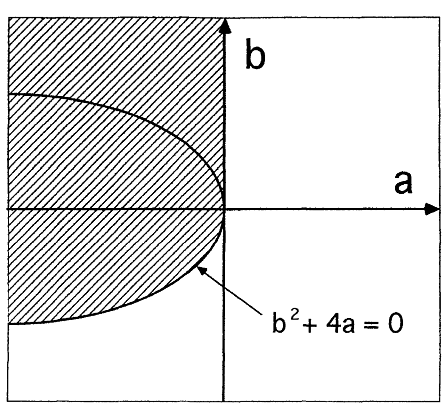

Fig. 1. The bifurcation diagram for the solution (7.10)

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