Commun. Math. Phys. 178, 653 – 678 (1996)

Communications in Mathematical Physics

Quantum $\mathcal{W}$-Algebras and Elliptic Algebras

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Received: 6 September 1995 / Accepted: 4 October 1995

Abstract: We define a quantum $\mathcal{W}$-algebra associated to $\mathfrak{sl}_N$ as an associative algebra depending on two parameters. For special values of the parameters, this algebra becomes the ordinary $\mathcal{W}$-algebra of $\mathfrak{sl}_N$, or the q-deformed classical $\mathcal{W}$-algebra of $\mathfrak{sl}_N$. We construct free field realizations of the quantum $\mathcal{W}$-algebras and the screening currents. We also point out some interesting elliptic structures arising in these algebras. In particular, we show that the screening currents satisfy elliptic analogues of the Drinfeld relations in $U_q(\widehat{\mathfrak{n}})$.

1. Introduction

1.1. In [1] N. Reshetikhin and the second author introduced new Poisson algebras $\mathcal{W}_q(\mathfrak{g})$, which are q-deformations of the classical $\mathcal{W}$-algebras. The Poisson algebra $\mathcal{W}_q(\mathfrak{g})$ is by definition the center of the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{g}}^L)$ at the critical level, where $\mathfrak{g}^L$ is the Langlands dual Lie algebra to $\mathfrak{g}$. It was shown in [1] that the Wakimoto realization of $U_q(\widehat{\mathfrak{sl}}_N)$ constructed in [2] provides a homomorphism from the center of $U_q(\widehat{\mathfrak{sl}}_N)$ to a Heisenberg–Poisson algebra $\mathcal{H}_q(\mathfrak{sl}_N)$. This homomorphism can be viewed as a free field realization of $\mathcal{W}_q(\mathfrak{sl}_N)$. When $q = 1$, it becomes the well-known Miura transformation [3]. In [1] explicit formulas for this free field realization were given. The structure of these formulas is the same as that of the formulas for the spectra of transfer-matrices in integrable quantum spin chains obtained by the Bethe ansatz method [4]. This is not surprising given that these spectra can actually be computed using the center at the critical level and the Wakimoto realization. For the Gaudin models, which correspond to the $q = 1$ case, this was explained in detail in [5].

1.2. The Poisson algebra $\mathcal{W}_q(\mathfrak{sl}_2)$ is a q-deformation of the classical Virasoro algebra. It has generators t_n, $n \in \mathbb{Z}$. The relations in $\mathcal{W}_q(\mathfrak{sl}_2)$ were computed in [1] using the q-deformed Miura transformation, which is a homomorphism from $\mathcal{W}_q(\mathfrak{sl}_2)$ to

The research of the second author was partially supported by NSF grant DMS-9501414.

a Heisenberg–Poisson algebra $\mathscr{H}_q(\mathfrak{sl}_2)$ with generators λ_n, $n \in \mathbb{Z}$, and relations

$$\{\lambda_n, \lambda_m\} = -h\frac{1-q^n}{1+q^n}\delta_{n,-m} \,. \tag{1.1}$$

Let us form the generating series

$$\Lambda(z) = q^{-1/2} \exp\left(- \sum_{n\in\mathbb{Z}} \lambda_n z^{-n} \right), \qquad t(z) = \sum_{n\in\mathbb{Z}} t_n z^{-n} \,.$$

The q-deformation of the Miura transformation is given by [1]

$$t(z) \to \Lambda(zq^{1/2}) + \Lambda(zq^{-1/2})^{-1} \,. \tag{1.2}$$

Using formulas (1.1) and (1.2) we find the relations in $\mathscr{W}_q(\mathfrak{sl}_2)$ [1]:

$$\{t_n, t_m\} = -h \sum_{l\in\mathbb{Z}} \frac{1-q^l}{1+q^l} t_{n-l}t_{m+l} - h(q^n - q^{-n})\delta_{n,-m} \,, \tag{1.3}$$

where $h = \log q$. As shown in [1], in the limit $q \to 1$, the algebra $\mathscr{W}_q(\mathfrak{sl}_2)$ becomes isomorphic to the classical Virasoro algebra while formula (1.2) becomes the Miura transformation.

In [6] J. Shiraishi, H. Kubo, H. Awata, and S. Odake quantized formulas (1.1), (1.2) and (1.3). This led them to the construction of a non-commutative algebra depending on two parameters q and p, such that when $q = p$ it becomes commutative, and is isomorphic to the Poisson algebra $\mathscr{W}_q(\mathfrak{sl}_2)$. Let us denote this algebra by $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$.

In [6] the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ was defined via its free field realization, i.e. a homomorphism from $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ to a Heisenberg algebra $\mathscr{H}_{q,p}(\mathfrak{sl}_2)$. The formula defining this homomorphism (see (3.2) below) is a normally ordered version of the q-deformed Miura transformation (1.2), just as the free field realization of the Virasoro algebra is a normally ordered version of the ordinary Miura transformation. Shiraishi, e.a., also constructed the screening currents, i.e. operators acting on the Fock representations of $\mathscr{H}_{q,p}(\mathfrak{sl}_2)$, which commute with the action of $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ up to a total difference.[1] It is shown in [6] that if one fixes $\beta \in \mathbb{C}$ and sets $p = q^{1-\beta}$, then in the limit $q \to 1$ the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ becomes isomorphic to the Virasoro algebra with central charge $1 - 6(1-\beta)^2/\beta$.

The work of Shiraishi, e.a. [6] was motivated by their bosonization formula for the Macdonald symmetric functions [7]. The paper [6] reveals a remarkable connection between the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ and Macdonald's functions corresponding to rectangular Young diagrams: those turn out to coincide with singular vectors of $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ in its bosonic Fock representations.

1.3. The goal of the present work is to construct quantum $\mathscr{W}$-algebras generalizing the results of [1] and [6], and to point out some intriguing elliptic structures arising in these algebras. Namely, we construct an algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ depending on q and p, such that when $q = p$ it becomes isomorphic to the q-deformed classical $\mathscr{W}$-algebra $\mathscr{W}_q(\mathfrak{sl}_N)$ from [1]. We construct, along the lines of [1] and [6], a free field

[1] In fact, these screening currents were proposed earlier by S. Lukyanov and Ya. Pugay [19].

realization of $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ by normally ordering the q-deformed Miura transformation from [1], and the screening currents. One can observe many similarities between the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ and the ordinary $\mathscr{W}$-algebra of $\mathfrak{sl}_N$ constructed by V. Fateev and S. Lukyanov [8] (see also [9]), which can be recovered from $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ in the limit $q \to 1$.

The algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ is topologically generated by Fourier coefficients of currents $T_1(z), \dots, T_{N-1}(z)$. The free field realization of $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ is defined by the formula

$$\begin{aligned} &D_{p^{-1}}^N - T_1(z)D_{p^{-1}}^{N-1} + T_2(z)D_{p^{-1}}^{N-2} - \cdots + (-1)^{N-1}T_{N-1}(z)D_{p^{-1}} + (-1)^N \\ &\quad =: (D_{p^{-1}} - \Lambda_1(z))(D_{p^{-1}} - \Lambda_2(zp)) \cdots (D_{p^{-1}} - \Lambda_N(zp^{N-1})) : , \end{aligned} \tag{1.4}$$

where $\Lambda_i(z)$, $i=1,\dots,N$, are generating series of a Heisenberg algebra, and $[D_{p^{-1}} \cdot f](x) = f(xp^{-1})$. In the limit $q \to 1$ this formula becomes the normally ordered Miura transformation from [8].

The screening currents $S_i^{\pm}(z)$ are solutions of the difference equations:

$$D_q S_i^+(z) = p^{-1} : \Lambda_{i+1}(zp^{i/2})\Lambda_i(zp^{i/2})^{-1} S_i^+(z) : ,$$

$$D_{p/q} S_i^-(z) = p^{-1} : \Lambda_{i+1}(zp^{i/2})\Lambda_i(zp^{i/2})^{-1} S_i^-(z) : .$$

Using formula (1.4) one can check that they commute with the currents $T_i(z)$ up to a total difference. This implies that their residues acting between bosonic Fock representations commute with the action of $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$.

Using these operators one can construct singular vectors in the Fock representations of $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$. These singular vectors should give the Macdonald symmetric functions corresponding to Young diagrams with $N-1$ rectangles as was pointed out in [6].

1.4. An interesting aspect of the algebras $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ is the appearance of elliptic functions in their definition and free field realization.

In particular, we show that the series $\Lambda_i(z)$, $i = 1, \dots, N$, satisfy, in the analytic continuation sense, the following relations:

$$\Lambda_i(z)\Lambda_j(w) = \varphi_N\left(\frac{w}{z}\right) \Lambda_j(w)\Lambda_i(z) , \tag{1.5}$$

where

$$\varphi_N(x) = \frac{\theta_{p^N}(xp)\theta_{p^N}(xq^{-1})\theta_{p^N}(xp^{-1}q)}{\theta_{p^N}(xp^{-1})\theta_{p^N}(xq)\theta_{p^N}(xpq^{-1})} ,$$

and $\theta_a(x)$ stands for the θ-function with the multiplicative period a. These relations entail similar relations for the currents $T_i(z)$.

The function $\varphi_N(x)$ can be characterized by the properties that it is an elliptic function, which has three zeroes u_1, u_2, u_3, three poles $-u_1, -u_2, -u_3$, and one of the poles is equal to $1/N$ of the period. These properties imply that the function $\varphi_N(x)$ satisfies the functional equation

$$\varphi_N(x)\varphi_N(xp) \cdots \varphi_N(xp^{N-1}) = 1 .$$

We also show that the screening currents $S_i^+(z)$, $i = 1,\dots,N-1$ satisfy, in the analytic continuation sense, the following relations:

$$S_i^+(z)S_j^+(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{A_{ij}-A_{ij}\beta-1}\frac{\theta_q(\frac{w}{z}p^{A_{ij}/2})}{\theta_q(\frac{z}{w}p^{A_{ij}/2})}S_j^+(w)S_i^+(z)\,, \tag{1.6}$$

where (A_{ij}) is the Cartan matrix. The screening currents $S_i^-(z)$, $i = 1,\dots,N-1$, satisfy the same relations with q replaced by p/q and β replaced by $1/\beta$. Moreover, we show that the screening currents involved in the Wakimoto realization of $U_q(\widehat{\mathfrak{sl}}_N)$ [2] also obey similar relations.

These elliptic relations define algebras, which are closely related to the elliptic algebras introduced by A. Odesskii and the first author in [10]. Such an algebra $U_{q,p}(\widehat{\mathfrak{n}})$ can be viewed as an elliptic deformation of the quantized universal enveloping algebra $U_q(\widehat{\mathfrak{n}})$ (where $\widehat{\mathfrak{n}}$ is the loop algebra of the nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$), introduced by V. Drinfeld [11]. According to [10], the elliptic relations of the type (1.6) imply that the screening currents satisfy certain elliptic analogues of the quantum Serre relations from $U_q(\widehat{\mathfrak{n}})$. We hope to study these relations in more detail in the next paper. We recall that the ordinary screening charges satisfy the ordinary quantum Serre relations [12], see also [18].

In this work we concentrate on the $\mathscr{W}$-algebras associated to $\mathfrak{sl}_N$. In [1] it was shown how to construct the Poisson algebra $\mathscr{W}_q(\mathfrak{g})$ and its free field realization for the general simple Lie algebra $\mathfrak{g}$. We expect that our results on the quantization of $\mathscr{W}_q(\mathfrak{sl}_N)$ can be similarly generalized. At the end of the paper we define the Heisenberg algebra $\mathscr{H}_{q,p}(\mathfrak{g})$ and the screening currents corresponding to the general simply-laced simple Lie algebra $\mathfrak{g}$. We then define the algebra $\mathscr{W}_{q,p}(\mathfrak{g})$ as the commutant of the screening charges in $\mathscr{H}_{q,p}(\mathfrak{g})$. We hope that the homological methods that we used in the study of the ordinary $\mathscr{W}$-algebras [18] can be applied to these quantum $\mathscr{W}$-algebras.

The ordinary $\mathscr{W}$-algebras can be obtained by the quantum Drinfeld–Sokolov reduction from the affine algebras. We expect that the quantum $\mathscr{W}$-algebras can be obtained by an analogous reduction from the quantum affine algebras.

1.5. The paper is organized as follows. In Sect. 2 we recall the results of [1] on the Poisson algebras $\mathscr{W}_q(\mathfrak{sl}_N)$. In Sect. 3 we recall the results of [6] on the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$. We define the algebras $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ in Sect. 4, and their screening currents in Sect. 5. In Sect. 6 we derive relations in the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$. In Sect. 7 we present these relations in elliptic form. Finally, in Sect. 8 we derive the elliptic relations obeyed by the screening currents of $\mathscr{W}_{q,p}(\mathfrak{g})$ and $U_q(\widehat{\mathfrak{g}})$.

2. Poisson Algebras $\mathscr{W}_q(\mathfrak{sl}_N)$

In this section we recall results of [1]. Let us first introduce the Heisenberg–Poisson algebra $\mathscr{H}_q(\mathfrak{sl}_N)$. It has generators $a_i[n]$, $i = 1,\dots,N-1$; $n \in \mathbb{Z}$, and relations

$$\{a_i[n], a_j[m]\} = h(q^{nA_{ij}/2} - q^{-nA_{ij}/2})\delta_{n,-m}\,, \tag{2.1}$$

where (A_{ij}) is the Cartan matrix of $\mathfrak{sl}_N$.

Remark 1. The parameter q that we use in this paper corresponds to q^2 in [1]. The algebra $\mathscr{W}_q(\mathfrak{g})$ corresponds to $\mathscr{W}_{h/2}(\mathfrak{g})$ in the notation of [1]. □

Define now new generators $\lambda_i[n]$, $i = 1,\dots,N$; $n \in \mathbb{Z}$, according to the formula

$$\lambda_i[n] - \lambda_{i+1}[n] = q^{ni/2} a_i[n], \quad i = 1,\dots,N-1; n \in \mathbb{Z}\,,$$

$$\sum_{i=1}^{N} q^{(1-i)n} \lambda_i[n] = 0\,. \tag{2.2}$$

From these formulas we derive the Poisson brackets (see [1])

$$\{\lambda_i[n], \lambda_i[m]\} = -h \frac{(1-q^n)(1-q^{n(N-1)})}{1-q^{nN}} \delta_{n,-m}\,, \tag{2.3}$$

$$\{\lambda_i[n], \lambda_j[m]\} = h \frac{(1-q^n)^2}{1-q^{nN}} q^{-n} \delta_{n,-m}, \quad i < j\,, \tag{2.4}$$

where $n \neq 0$.

Introduce the generating functions

$$\Lambda_i(z) = q^{i-(N+1)/2} \exp\left(-\sum_{m \in \mathbb{Z}} \lambda_i[m] z^{-m}\right)\,. \tag{2.5}$$

From (2.3) and (2.4) we find:

$$\{\Lambda_i(z), \Lambda_i(w)\} = -h \left\{ \sum_{m \in \mathbb{Z}} \left(\frac{w}{z}\right)^m \frac{(1-q^m)(1-q^{m(N-1)})}{1-q^{mN}} \right\} \Lambda_i(z)\Lambda_i(w)\,, \tag{2.6}$$

$$\{\Lambda_i(z), \Lambda_j(w)\} = h \left\{ \sum_{m \in \mathbb{Z}} \left(\frac{w}{zq}\right)^m \frac{(1-q^m)^2}{1-q^{mN}} \right\} \Lambda_i(z)\Lambda_j(w)\,, \tag{2.7}$$

if $i < j$.

Now let us define generating functions $t_i(z)$, $i = 0,\dots,N$, whose coefficients lie in $\mathscr{H}_q(\mathfrak{sl}_N)$: $t_0(z) = 1$, and

$$t_i(z) = \sum_{1 \leqq j_1 < \dots < j_i \leqq N} \Lambda_{j_1}(z)\Lambda_{j_2}(zq) \cdots \Lambda_{j_{i-1}}(zq^{i-2})\Lambda_{j_i}(zq^{i-1})\,, \tag{2.8}$$

$i = 1,\dots,N$. Formula (2.2) implies that

$$t_N(z) = \Lambda_1(z)\Lambda_2(zq) \cdots \Lambda_N(zq^{N-1}) = 1\,.$$

Formula (2.8) can be rewritten succinctly as follows:

$$\begin{aligned} &D_{q^{-1}}^N - t_1(z) D_{q^{-1}}^{N-1} + t_2(z) D_{q^{-1}}^{N-2} - \dots + (-1)^{N-1} t_{N-1}(z) D_{q^{-1}} + (-1)^N \\ &= (D_{q^{-1}} - \Lambda_1(z))(D_{q^{-1}} - \Lambda_2(zq)) \cdots (D_{q^{-1}} - \Lambda_N(zq^{N-1}))\,, \end{aligned} \tag{2.9}$$

where D_a stands for the a-difference operator:

$$(D_a \cdot f)(x) = f(xa)\,.$$

In the limit $q \to 1$ we have: $\Lambda_i(z) = 1 - h\chi_i(z) + o(h)$ and $D_{q^{-1}} = 1 - h\partial_z + o(h)$, where $h = \log q$. Hence the right-hand side of (2.9) becomes in this limit

$$(-1)^N h^N (\partial_z - \chi_1(z))(\partial_z - \chi_2(z)) \cdots (\partial_z - \chi_N(z)) + o(h^N)\,,$$

and we obtain the standard Miura transformation corresponding to the classical $\mathscr{W}$-algebra $\mathscr{W}(\mathfrak{sl}_N)$, see e.g. [3]. This shows that the generators of $\mathscr{W}(\mathfrak{sl}_N)$ can be recovered as certain linear combinations of $t_0(z),\dots,t_{N-1}(z)$ and their derivatives in the limit $q \to 1$.

The coefficients of the series $t_i(z)$, $i=1,\dots,N-1$, generate a Poisson subalgebra $\mathscr{W}_q(\mathfrak{sl}_N)$ of $\mathscr{H}_q(\mathfrak{sl}_N)$. The relations between them are as follows (see [1]):

$$\begin{aligned}\{t_i(z),t_j(w)\} = &-h\left\{\sum_{m\in\mathbb{Z}}\left(\frac{wq^{j-i}}{z}\right)^m \frac{(1-q^{im})(1-q^{m(N-j)})}{1-q^{mN}}\right\} t_i(z)t_j(w)\\ &+h\sum_{r=1}^{i}\delta\left(\frac{w}{zq^r}\right) t_{i-r}(w)t_{j+r}(z)\\ &-h\sum_{r=1}^{i}\delta\left(\frac{wq^{j-i+r}}{z}\right) t_{i-r}(z)t_{j+r}(w)\,,\end{aligned}$$

if $i \leqq j$ and $i+j \leqq N$; and

$$\begin{aligned}\{t_i(z),t_j(w)\} = &-h\left\{\sum_{m\in\mathbb{Z}}\left(\frac{wq^{j-i}}{z}\right) \frac{(1-q^{im})(1-q^{m(N-j)})}{1-q^{mN}}\right\} t_i(z)t_j(w)\\ &+h\sum_{r=1}^{N-j}\delta\left(\frac{w}{zq^r}\right) t_{i-r}(w)t_{j+r}(z)\\ &-h\sum_{r=1}^{N-j}\delta\left(\frac{wq^{j-i+r}}{z}\right) t_{i-r}(z)t_{j+r}(w)\,,\end{aligned}$$

if $i \leqq j$ and $i+j > N$.

Remark 2. It is natural to define the Poisson algebra $\mathscr{W}_q(\mathfrak{sl}_\infty)$ with generators $t_i(z)$, $i \geqq 1$, and relations

$$\begin{aligned}\{t_i(z),t_j(w)\} = &\,h\sum_{r=0}^{i}\delta\left(\frac{w}{zq^r}\right) t_{i-r}(w)t_{j+r}(z)\\ &-h\sum_{r=0}^{i}\delta\left(\frac{wq^{j-i+r}}{z}\right) t_{i-r}(z)t_{j+r}(w), \quad i \leqq j\,. \quad \square\end{aligned}$$

3. The Algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$

In this section we recall the results of Shiraishi–Kubo–Awata–Odake [6] on the quantum deformation of $\mathscr{W}_q(\mathfrak{sl}_2)$. However, some of our notation will be different from theirs.

Let h,β be two complex numbers, such that neither h nor $h\beta$ belongs to $2\pi i\mathbb{Q}$. Set $q=e^h$ and $p=e^{h(1-\beta)}$. We will use this notation throughout the paper.

Let $\mathcal{H}'_{q,p}(\mathfrak{sl}_2)$ be the Heisenberg algebra with generators $\lambda[n]$, $n \in \mathbb{Z}$, and relations:

$$[\lambda[n], \lambda[m]] = -\frac{1}{n}\frac{(1-q^n)(1-(p/q)^n)}{1+p^n}\delta_{n,-m}\,, \quad n \neq 0\,. \tag{3.1}$$

In the limit $\beta \to 0$, in which $p \to q$ we can recover the Poisson bracket (1.1) as the β-linear term of the bracket (3.1).

For $\mu \in \mathbb{C}$, let π_μ be the Fock representation of the algebra $\mathcal{H}'_{q,p}(\mathfrak{sl}_2)$, which is generated by a vector v_μ, such that $\lambda[n]v_\mu = 0$, $n > 0$, and $\lambda[0]v_\mu = \mu v_\mu$. Let

$$\mathcal{H}_{q,p}(\mathfrak{sl}_2) = \lim_{\leftarrow} \mathcal{H}'_{q,p}(\mathfrak{sl}_2)/I_n, \quad n > 0\,,$$

where I_n is the left ideal of $\mathcal{H}'_{q,p}(\mathfrak{sl}_2)$ generated by all polynomials in $\lambda[m], m > 0$, of degrees greater than or equal to n ($\deg \lambda[m] = m$). By definition, the action of $\mathcal{H}_{q,p}(\mathfrak{sl}_2)$ on the modules π_μ is well-defined.

Introduce the generating function

$$\Lambda(z) = p^{-1/2}q^{-\lambda[0]} : \exp\left(-\sum_{m\neq 0}\lambda[m]z^{-m}\right):,$$

where columns stand for the standard normal ordering. Now define the power series $T(z) = \sum_{m\in\mathbb{Z}} T[m]z^{-m}$ by the formula

$$T(z) =: \Lambda(zp^{1/2}) : + : \Lambda(zp^{-1/2})^{-1} : \,. \tag{3.2}$$

The coefficients $T[n]$ of the power series $T(z)$ belong to $\mathcal{H}_{q,p}(\mathfrak{sl}_2)$. They satisfy the following relations [6]:

$$\sum_{l=0}^{\infty} f_l(T[n-l]T[m+l] - T[m-l]T[n+l]) = \frac{(1-q)(1-p/q)}{1-p}(p^{-n}-p^n)\delta_{n,-m}\,, \tag{3.3}$$

where f_l's are given by the generating function

$$f(x) = \sum_{l=0}^{\infty} f_l x^l = \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\frac{(1-q^m)(1-(p/q)^m)}{1+p^m}x^m\right)\,. \tag{3.4}$$

In the limit $q \to 1$, formulas (3.2) and (3.3) become formulas (1.2) and (1.3), respectively.

Introduce an additional operator Q, such that $[\lambda[n], Q] = \beta\delta_{n,0}$. The operator $e^{\alpha Q}, \alpha \in \mathbb{C}$, acts from π_μ to $\pi_{\mu+\alpha\beta}$ by sending v_μ to $v_{\mu+\alpha\beta}$. In [6] two screening currents were constructed:

$$S^+(z) = e^{Q}z^{s^+[0]} : \exp\left(\sum_{m\neq 0} s^+[m]z^{-m}\right):, \tag{3.5}$$

$$S^-(z) = e^{-Q/\beta}z^{-s^-[0]} : \exp\left(-\sum_{m\neq 0} s^-[m]z^{-m}\right):, \tag{3.6}$$

where

$$s^+[m] = \frac{1+p^m}{q^{-m}-1}\lambda[m], \quad m\neq 0, \qquad s^+[0] = 2\lambda[0]\,,$$

$$s^-[m] = \frac{1+p^m}{(q/p)^m-1}\lambda[m], \quad m\neq 0, \qquad s^-[0] = 2\lambda[0]/\beta\,.$$

The Fourier coefficients of $S^+(z)$ act from π_μ to $\pi_{\mu+\beta}$, and the Fourier coefficients of $S^-(z)$ act from π_μ to $\pi_{\mu-1}$.

They satisfy [6]:

$$[T[n], S^+(w)] = \mathscr{D}_q C_n^+(w), \qquad [T[n], S^-(w)] = \mathscr{D}_{p/q} C_n^-(w),$$

where $C_n^\pm(w)$ are certain operator-valued power series, and

$$[\mathscr{D}_a \cdot f](x) = \frac{f(x) - f(xa)}{x(1-a)}.$$

This implies that $T[n]$, $n \in \mathbb{Z}$, commute with the screening charges $\int S^\pm(z)dz$, whenever they are well-defined [6].[2] In the limit $q \to 1$, those become the two screening charges of the Virasoro algebra.

4. The Algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$

4.1. Heisenberg Algebra. Let $\mathscr{H}'_{q,p}(\mathfrak{sl}_N)$ be the Heisenberg algebra with generators $a_i[n]$, $i = 1, \dots, N-1$; $n \in \mathbb{Z}$, and relations

$$[a_i[n], a_j[m]] = \frac{1}{n} \frac{(1-q^n)(p^{A_{ij}n/2} - p^{-A_{ij}n/2})(1-(p/q)^n)}{1-p^n} \delta_{n,-m}, \tag{4.1}$$

where $n \neq 0$. This formula was derived from the commutation relations (3.1) in the case of $\mathfrak{sl}_2$, which follow from [6], and from the condition that in the limit $\beta \to 0$ the β-linear term should give us the Poisson bracket (2.1).

For each weight μ of the Cartan subalgebra of $\mathfrak{sl}_N$, let π_μ be the Fock representation of $\mathscr{H}'_{q,p}(\mathfrak{sl}_N)$ generated by a vector v_μ, such that $a_i[n]v_\mu = 0$, $n > 0$, and $a_i[0]v_\mu = \mu(\alpha_i^\vee)v_\mu$, where $\alpha_i^\vee$ is the i^{th} coroot of $\mathfrak{sl}_N$.

Let $\mathscr{H}_{q,p}(\mathfrak{sl}_N)$ be the completion of $\mathscr{H}'(\mathfrak{sl}_N)$ defined in the same way as in the case of $\mathfrak{sl}_2$, see Sect. 3. The algebra $\mathscr{H}_{q,p}(\mathfrak{sl}_N)$ acts on the modules π_μ.

Introduce new generators $\lambda_i[n]$ of $\mathscr{H}_{q,p}(\mathfrak{sl}_N)$ by the formulas

$$\lambda_i[n] - \lambda_{i+1}[n] = p^{ni/2} a_i[n], \quad i = 1, \dots, N-1; n \in \mathbb{Z}, \tag{4.2}$$

$$\sum_{i=1}^{N} p^{(1-i)n} \lambda_i[n] = 0. \tag{4.3}$$

From these formulas and (4.1) we derive the commutation relations between them:

$$[\lambda_i[n], \lambda_i[m]] = -\frac{1}{n} \frac{(1-q^n)(1-p^{n(N-1)})(1-(p/q)^n)}{1-p^{nN}} \delta_{n,-m}, \tag{4.4}$$

$$[\lambda_i[n], \lambda_j[m]] = \frac{1}{n} \frac{(1-q^n)(1-p^n)(1-(p/q)^n)}{1-p^{nN}} p^{-n} \delta_{n,-m}, \quad i < j, \tag{4.5}$$

where $n \neq 0$.

[2] The same screening charges were constructed earlier in [19], (see also [20]). Moreover, it was proposed in [19] that the deformed Virasoro algebra should be the subalgebra of the Heisenberg algebra, which commutes with these screening charges.

Let us introduce the power series

$$\Lambda_i(z) = p^{i-(N+1)/2} q^{-\lambda_i[0]} : \exp\left(- \sum_{m \neq 0} \lambda_i[m] z^{-m} \right) : . \tag{4.6}$$

We can compute the operator product expansions (OPEs) of these power series using the following lemma. Introduce the notation

$$(x|\alpha_1, \dots, \alpha_k; t)_\infty = \prod_{i=1}^{k} \prod_{n=0}^{\infty} (1 - \alpha_i x t^n) .$$

Lemma 1. *Let $b[n]$, $n \in \mathbb{Z}$, and $c[n]$, $n \in \mathbb{Z}$, satisfy commutation relations*:

$$[b[n], c[m]] = -\frac{1}{n(1-t^n)} \left(\sum_{i=1}^{k} \alpha_i^n - \sum_{j=1}^{l} \beta_j^n \right) \delta_{n,-m} ,$$

where $n \neq 0$ and $|t| < 1$. Then for $|z| > \max_{i,j}\{|\alpha_i|, |\beta_j|\}\,|w|$ the composition

$$: \exp\left(\sum_{n \neq 0} b[n] z^{-n} \right) :: \exp\left(\sum_{n \neq 0} c[n] z^{-n} \right) :$$

acting on each module π_μ exists and is equal to

$$\frac{(\frac{w}{z}|\alpha_1, \dots, \alpha_k)_\infty}{(\frac{w}{z}|\beta_1, \dots, \beta_l)_\infty} : \exp\left(\sum_{n \in \mathbb{Z}} b[n] z^{-n} \right) \exp\left(\sum_{n \in \mathbb{Z}} c[n] z^{-n} \right) : .$$

Proof. Direct computation based on formula

$$\exp\left(-\sum_{n>0} \frac{x^n}{n} \right) = 1 - x . \qquad \square$$

Let us assume that $|p| < 1$ and $|z| \gg |w|$; more precisely, it suffices that $|z| > |w| pq^{-1}$, and $|z| > |w|q$. Then we find from formula (4.4) and Lemma 1:

$$\Lambda_i(z)\Lambda_i(w) = \frac{(\frac{w}{z}|1, p, p^{N-1}q, p^N q^{-1}; p^N)_\infty}{(\frac{w}{z}|q, pq^{-1}, p^{N-1}, p^N; p^N)_\infty} : \Lambda_i(z)\Lambda_i(w) : . \tag{4.7}$$

In the same way we obtain:

$$\Lambda_i(z)\Lambda_j(w) = \frac{(\frac{w}{z}|qp^{-1}, q^{-1}, p; p^N)_\infty}{(\frac{w}{z}|p^{-1}, q, pq^{-1}; p^N)_\infty} : \Lambda_i(z)\Lambda_j(w) : , \quad i < j , \tag{4.8}$$

$$\Lambda_i(z)\Lambda_j(w) = \frac{(\frac{w}{z}|p^{N-1}q, p^N q^{-1}, p^{N+1}; p^N)_\infty}{(\frac{w}{z}|p^{N-1}, p^N q, p^{N+1} q^{-1}; p^N)_\infty} : \Lambda_i(z)\Lambda_j(w) : , \quad i > j , \tag{4.9}$$

where $|z| \gg |w|$.

Remark 3. When $|p| < 1$, the functions appearing in the right-hand side of formulas (4.7)–(4.9) are power series in w/z, whose coefficients are rational functions in p. $\square$

4.2. Definition of the Quantum $\mathscr{W}$-Algebra. Now we define generating functions $T_i(z)$, $i = 0,\dots,N$, whose coefficients lie in $\mathscr{H}_q(\mathfrak{sl}_N) : T_0(z) = 1$, and

$$T_i(z) = \sum_{1 \leqq j_1 < \cdots < j_i \leqq N} : \Lambda_{j_1}(z)\Lambda_{j_2}(zp)\cdots\Lambda_{j_{i-1}}(zp^{i-2})\Lambda_{j_i}(zp^{i-1}) : , \tag{4.10}$$

$i = 1,\dots,N$. Formula (4.3) implies that

$$T_N(z) =: \Lambda_1(z)\Lambda_2(zp)\cdots\Lambda_N(zp^{N-1}) := 1 .$$

Formula (4.10) can be rewritten as follows:

$$\begin{aligned} &D^N_{p^{-1}} - T_1(z)D^{N-1}_{p^{-1}} + T_2(z)D^{N-2}_{p^{-1}} - \cdots + (-1)^{N-1}T_{N-1}(z)D_{p^{-1}} + (-1)^N \\ &=: (D_{p^{-1}} - \Lambda_1(z))(D_{p^{-1}} - \Lambda_2(zp))\cdots(D_{p^{-1}} - \Lambda_N(zp^{N-1})) : . \end{aligned} \tag{4.11}$$

Formulas (4.10) and (4.11) are quantum deformations of formulas (2.8) and (2.9) –(2.9).

We define the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ as the subalgebra of $\mathscr{H}_q(\mathfrak{sl}_N)$ generated by the Fourier coefficients of the power series $T_i(z)$, $i = 1,\dots,N-1$, given by formula (4.10). It is clear from the definition that in the limit $\beta \to 0$, i.e. $p \to q$, the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ becomes the Poisson algebra $\mathscr{W}_q(\mathfrak{sl}_N)$ defined in [1], see Sect. 2.

Remark 4. The currents $\Lambda(z)$ and $T(z)$ that were used in the definition of $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ in Sect. 3 correspond to $\Lambda_1(z)$ and $T_1(zp^{1/2})$, respectively. $\square$

Let us fix β and consider the limit $q \to 1$ with $p = q^{1-\beta}$. Then we have: $\Lambda_i(z) = 1 - h\chi_i(z) + o(h)$ and $D_{p^{-1}} = 1 - h(1-\beta)\partial_z + o(h)$, where $h = \log q$. Hence the right-hand side of (4.11) becomes in this limit

$$(-1)^N h^N((1-\beta)\partial_z - \chi_1(z))((1-\beta)\partial_z - \chi_2(z))\cdots((1-\beta)\partial_z - \chi_N(z)) + o(h^N) ,$$

and we obtain the normally ordered Miura transformation corresponding to the $\mathscr{W}$-algebra of $\mathfrak{sl}_N$, introduced by Fateev and Lukyanov [8]. In the notation of [18], this algebra is $\mathscr{W}_{\sqrt{\beta}}(\mathfrak{sl}_N)$ with central charge $(N-1) - N(N+1)(1-\beta)^2/\beta$. Thus, in the limit $q \to 1$, the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ becomes isomorphic to $\mathscr{W}_{\sqrt{\beta}}(\mathfrak{sl}_N)$. The generating currents of $\mathscr{W}_{\sqrt{\beta}}(\mathfrak{sl}_N)$ can be recovered as certain linear combinations of $T_0(z),\dots,T_{N-1}(z)$ and their derivatives in the limit $q \to 1$.

5. Screening Currents for $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$

Introduce operators $Q_i, i = 1,\dots,N-1$, which satisfy commutation relations $[a_i[n], Q_j] = A_{ij}\beta\delta_{n,0}$. The operators e^{Q_i} act from π_μ to $\pi_{\mu+\beta\alpha_i}$.

Now we can define the screening currents as the generating functions

$$S_i^+(z) = e^{Q_i} z^{s_i^+[0]} : \exp\left(\sum_{m \neq 0} s_i^+(m) z^{-m}\right) : , \tag{5.1}$$

$$S_i^-(z) = e^{-Q_i/\beta} z^{-s_i^-[0]} : \exp\left(-\sum_{m \neq 0} s_i^-(m) z^{-m}\right) : , \tag{5.2}$$

where

$$s_i^+[m] = \frac{a_i[m]}{q^{-m}-1}, \quad m \neq 0, \qquad s_i^+[0] = a_i[0]\,, \tag{5.3}$$

$$s_i^-[m] = -\frac{a_i[m]}{(q/p)^m - 1}, \quad m \neq 0, \qquad s_i^-[0] = a_i[0]/\beta \tag{5.4}$$

(compare with (3.5) and (3.6)).

Let

$$A_i(z) = q^{-a[0]} : \exp\left(-\sum_{m\neq 0} a_i[m] z^{-m}\right) : \,. \tag{5.5}$$

Then we have

$$A_i(z) =: S_i^+(z) S_i^+(zq)^{-1} :\,, \tag{5.6}$$

and

$$A_i(z) =: S_i^-(z) S_i^-(zp/q)^{-1} : \,. \tag{5.7}$$

Formulas (5.6) and (5.7) show that the screening currents are solutions of the following difference equations:

$$D_q S_i^+(z) =: A_i(z)^{-1} S_i^+(z) :\,,$$

$$D_{p/q} S_i^-(z) =: A_i(z)^{-1} S_i^-(z) : \,.$$

In the limit $q \to 1$ they become the differential equations defining the ordinary screening currents.

We also have from (4.2):

$$A_i(z) = p : \Lambda_i(zp^{i/2}) \Lambda_{i+1}(zp^{i/2})^{-1} : \,. \tag{5.8}$$

Theorem 1. *The screening currents commute with the algebra $\mathcal{W}_{q,p}(\mathfrak{sl}_N)$ up to a total difference. More precisely, for any $A \in \mathcal{W}_{q,p}(\mathfrak{sl}_N)$ we have*:

$$[A, S_i^+(w)] = \mathcal{D}_q C_i^+(w), \qquad [A, S_i^-(w)] = \mathcal{D}_{p/q} C_i^-(w)\,,$$

where $C^\pm(w)$ are certain operator-valued power series, and

$$[\mathcal{D}_a f](x) = \frac{f(x) - f(xa)}{x(1-a)}\,.$$

Proof. Let us consider the case of the screening currents $S_i^+(z)$; the case of $S_i^-(z)$ can be treated in the same way.

Consider the difference operator (4.11). We want to prove that each term of this operator has the property that all of its Fourier coefficients commute with $S_i^+(z)$ up to a total $\mathcal{D}_q$-difference.

From formulas (5.3), (4.2), (4.4), (4.5) we obtain the following commutation relations:

$$[\lambda_i[n], s_i^+[m]] = -\frac{1}{m} p^{n(i/2-1)} (1 - (p/q)^n) \delta_{n,-m}\,,$$

$$[\lambda_{i+1}[n], s_i^+[m]] = \frac{1}{m} p^{ni/2} (1 - (p/q)^n) \delta_{n,-m}\,,$$

where $m \neq 0$, and

$$[\lambda_j[n], s_i^+[m]] = 0, \quad j \neq i, \ i+1 \, .$$

From these commutation relations we derive the following OPEs (cf. Lemma 1):

$$\Lambda_i(z)S_i^+(w) = \frac{p(z - wp^{i/2-1})}{q(z - wp^{i/2}q^{-1})} : \Lambda_i(z)S_i^+(w) : , \quad |z| \gg |w| \, ,$$

$$S_i^+(w)\Lambda_i(z) = \frac{p(z - wp^{i/2-1})}{q(z - wp^{i/2}q^{-1})} : \Lambda_i(z)S_i^+(w) : , \quad |w| \gg |z| \, ,$$

$$\Lambda_{i+1}(z)S_i^+(w) = \frac{q(z - wp^{i/2+1}q^{-1})}{p(z - wp^{i/2})} : \Lambda_{i+1}(z)S_i^+(w) : , \quad |z| \gg |w| \, ,$$

$$S_i^+(w)\Lambda_{i+1}(z) = \frac{q(z - wp^{i/2+1}q^{-1})}{p(z - wp^{i/2})} : \Lambda_{i+1}(z)S_i^+(w) : , \quad |w| \gg |z| \, ,$$

and

$$\Lambda_j(z)S_i^+(w) =: \Lambda_j(z)S_i^+(w) : , \quad \forall z, w \, ,$$

if $j \neq i, \ i+1$.

The last formula means that $S_i^+(w)$ commutes with all Fourier coefficients of $\Lambda_j(z)$ if $j \neq i, \ i+1$. Therefore it is sufficient to consider the OPE between the factor

$$\begin{aligned} &: (D_{p^{-1}} - \Lambda_i(zp^{i-1}))(D_{p^{-1}} - \Lambda_{i+1}(zp^i)) : \\ &\quad = D_{p^{-1}}^2 - (\Lambda_i(zp^{i-1}) + \Lambda_{i+1}(zp^{i-1}))D_{p^{-1}} + : \Lambda_i(zp^{i-1})\Lambda_{i+1}(zp^i) : \end{aligned}$$

in formula (4.11) and $S_i^+(w)$. We have to show that all Fourier coefficients of each of the terms commute with $S_i^+(w)$ up to a total difference.

For the term $: \Lambda_i(zp^{i-1})\Lambda_{i+1}(zp^i) :$ we have according to the OPEs above:

$$: \Lambda_i(zp^{i-1})\Lambda_{i+1}(zp^i) : S_i^+(w) =: \Lambda_i(zp^{i-1})\Lambda_{i+1}(zp^i)S_i^+(w) : ,$$

which means that all Fourier coefficients of $: \Lambda_i(zp^{i-1})\Lambda_{i+1}(zp^i) :$ commute with $S_i^+(w)$.

Now consider the linear term $\Lambda_i(zp^{i-1}) + \Lambda_{i+1}(zp^{i-1})$. We have according to the OPEs above:

$$\begin{aligned} (\Lambda_i(zp^{i-1}) + \Lambda_{i+1}(zp^{i-1}))S_i^+(w) &= \frac{p(z - wp^{-i/2})}{q(z - wp^{-i/2+1}q^{-1})} : \Lambda_i(zp^{i-1})S_i^+(w) : \\ &+ \frac{q(z - wp^{i/2+2}q^{-1})}{p(z - wp^{-i/2+1})} : \Lambda_{i+1}(zp^{i-1})S_i^+(w) : , \end{aligned} \tag{5.9}$$

for $|z| \gg |w|$, and the same formula for the product in the opposite order for $|w| \gg |z|$. Therefore we can compute the commutator

$$[\int (\Lambda_i(zp^{i-1}) + \Lambda_{i+1}(zp^{i-1}))z^n dz, S_i^+(w)]$$

by evaluating the residues in the right-hand side of (5.9). We find that this commutator is equal to

$$
\begin{aligned}
&(p/q-1)[:\Lambda_i(wp^{i/2}q^{-1})S_i^+(w):(wp^{-i/2+1}q^{-1})^{n+1}\\
&\quad -p^{-1}q:\Lambda_{i+1}(wp^{i/2})S_i^+(w):(wp^{-i/2+1})^{n+1}]\\
&=\mathscr{D}_q[p^{i/2-1}(1-q)(p-q):\Lambda_i(wp^{i/2}q^{-1})S_i^+(w):(wp^{-i/2+1}q^{-1})^{n+2}]\,,
\end{aligned}
$$

because by formulas (5.6) and (5.8)

$$:\Lambda_i(wp^{i/2})S_i^+(wq):=p^{-1}:\Lambda_{i+1}(wp^{i/2})S_i^+(w):\ .$$

This completes the proof. □

Corollary 1. *Any element of the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ commutes with the operators $\int S_i^{\pm}(z)dz$ acting on the Fock representations, whenever they are well-defined.*

In the limit $q\to 1$, the operators $\int S_i^{\pm}(z)dz$ become the ordinary screening charges $\int e^{\alpha_\pm\phi_i(z)}dz$, where $\alpha_\pm=\pm\beta^{\pm 1/2}$.

Corollary 1 implies that one can construct intertwining operators between the Fock representations π_μ of $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$, and hence singular vectors, by integrating products of the screening currents over suitable cycles. For the ordinary $\mathscr{W}$-algebra of $\mathfrak{sl}_N$, the screening charges satisfy quantum Serre relations, and the integration cycles correspond to the singular vectors in the Verma modules over the quantum group $U_q(\mathfrak{sl}_N)$, see [12–14, 18]. We expect an analogous structure for the $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ screening charges.

6. Relations in $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$

6.1. Relations between $T_1(z)$ and $T_m(w)$. Let us again assume that $|p|<1$. Introduce the formal power series $f_{m,N}(x)$ by the formula

$$f_{m,N}(x)=\frac{(x|p^{m-1}q,p^mq^{-1},p^{N-1},p^N;p^N)_\infty}{(x|p^{m-1},p^m,p^{N-1}q,p^Nq^{-1};p^N)_\infty}\,. \tag{6.1}$$

The function $f_{m,N}(x)$ is a very-well-poised basic hypergeometric series

$${}_6\phi_5\left[\begin{matrix}x, & x^{1/2}p^N, & -x^{1/2}p^N, & p^{N-m}, & pq^{-1}, & q\\ & x^{1/2}, & -x^{1/2}, & xp^m, & xp^{N-1}q, & xp^Nq^{-1}\end{matrix};p^N,x\right],$$

see formula (2.7.1) of [15].

In what follows we use the notation

$$\delta(x)=\sum_{n\in\mathbb{Z}}x^n\,.$$

Theorem 2. *The formal power series* $T_1(z)$ *and* $T_m(w)$ *satisfy the following relations*:

$$f_{m,N}\left(\frac{w}{z}\right) T_1(z)T_m(w) - f_{m,N}\left(\frac{z}{w}\right) T_m(w)T_1(z)$$
$$= \frac{(1-q)(1-p/q)}{1-p}\left(\delta\left(\frac{w}{zp}\right) T_{m+1}(z) - \delta\left(\frac{wp^m}{z}\right) T_{m+1}(w)\right) . \qquad (6.2)$$

Proof. Using the OPEs (4.7)–(4.9), we obtain that when $|z| \gg |w|$

$$\Lambda_i(z) : \Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1}) :$$

is equal to

$$f_{m,N}\left(\frac{w}{z}\right)^{-1} : \Lambda_i(z)\Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1}) : ,$$

if $i = j_k$ for some $k \in \{1,\ldots,m\}$; and

$$f_{m,N}\left(\frac{w}{z}\right)^{-1} \frac{(z-wp^{k-1}q)(z-wp^kq^{-1})}{(z-wp^{k-1})(z-wp^k)} : \Lambda_i(z)\Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1}) : ,$$

if $j_k < i < j_{k+1}$. Here and below the case $i < j_1$ corresponds to $k = 0$ and the case $i > j_m$ corresponds to $k = m$.

On the other hand, when $|w| \gg |z|$,

$$: \Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1}) : \Lambda_i(z)$$

is equal to

$$f_{m,N}\left(\frac{z}{w}\right)^{-1} : \Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1})\Lambda_i(z) : ,$$

if $i = j_k$ for some $k \in \{1,\ldots,m\}$; and

$$f_{m,N}\left(\frac{z}{w}\right)^{-1} \frac{(z-wp^{k-1}q)(z-wp^kq^{-1})}{(z-wp^{k-1})(z-wp^k)} : \Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1})\Lambda_i(z) : ,$$

if $j_k < i < j_{k+1}$.

Since the normally ordered product does not depend on the order of the factors, we conclude that the analytic continuations of

$$f_{m,N}\left(\frac{w}{z}\right) \Lambda_i(z) : \Lambda_{j_1}(w)\cdots\Lambda_{j_m}(wp^{m-1}) :$$

and

$$f_{m,N}\left(\frac{z}{w}\right) : \Lambda_{j_1}(w)\cdots\Lambda_{j_m}(wp^{m-1}) : \Lambda_i(z)$$

coincide.

Therefore

$$\int_{C_R} f_{m,N}\left(\frac{w}{z}\right) \Lambda_i(z) : \Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1}) : z^n dz$$
$$- \int_{C_r} f_{m,N}\left(\frac{z}{w}\right) : \Lambda_{j_1}(w)\Lambda_{j_2}(wp)\cdots\Lambda_{j_m}(wp^{m-1}) : \Lambda_i(z) z^n dz ,$$

where C_R and C_r are circles on the z plane of radii $R \gg |w|$ and $r \ll |w|$, respectively, is equal to 0 if $i = j_k$ for some $k \in \{1,\dots,m\}$; and the sum of the residues of

$$\frac{(z - wp^{k-1}q)(z - wp^kq^{-1})}{(z - wp^{k-1})(z - wp^k)} : \Lambda_i(z)\Lambda_{j_1}(w)\cdots\Lambda_{j_m}(wp^{m-1}) : ,$$

if $j_k < i < j_{k+1}$.

But the latter is equal to $(1-q)(1-p/q)/(1-p)$ times

$$(: \Lambda_{j_1}(w)\cdots\Lambda_{j_k}(wp^{k-1})\Lambda_i(wp^{k-1})\Lambda_{j_{k+1}}(wp^k)\cdots\Lambda_{j_m}(wp^{m-1}) : w^{n+1}p^{(n+1)(k-1)}$$
$$- : \Lambda_{j_1}(w)\cdots\Lambda_{j_k}(wp^{k-1})\Lambda_i(wp^k)\Lambda_{j_{k+1}}(wp^k)\cdots\Lambda_{j_m}(wp^{m-1}) : w^{n+1}p^{(n+1)k}) .$$

After summation over $j_1 < j_2 < \cdots < j_m$, all of these terms will cancel out except for

$$\frac{(1-q)(1-p/q)}{1-p} : \Lambda_i(wp^{-1})\Lambda_{j_1}(w)\cdots\Lambda_{j_m}(wp^{m-1}) : w^{n+l+1}p^{-(n+1)}$$

with $i < j_1$; and

$$-\frac{(1-q)(1-p/q)}{1-p} : \Lambda_{j_1}(w)\cdots\Lambda_{j_m}(wp^{m-1})\Lambda_i(wp^m) : w^{n+l+1}p^{(n+1)m}$$

with $i > j_m$. This gives us formula (6.2). $\square$

In the limit $p \to q$ formula (6.2) gives the Poisson bracket between $t_1(z)$ and $t_m(w)$ from [1], see Sect. 2.

Formula (6.2) shows that the Fourier coefficients $T_1[n]$ of the power series $T_1(z)$ generate the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$. In particular, $T_i(z)$ can be written as a degree i expression in $T_1[n]$, $n \in \mathbb{Z}$.

One can also derive similar relations between $T_i(z)$ and $T_j(w)$ with $i,j > 1$. These relations are quadratic, and involve products of $T_{i-r}(z)$ and $T_{j+r}(w)$, where $r = 1,\dots,i-1$, if $i+j \leqq N$ and $i \leqq j$; and $r = 1,\dots,N-j$, if $i+j > N$ and $i \leqq j$. In the limit $p \to q$ these relations give the Poisson brackets between $t_i(z)$ and $t_j(w)$ from [1], which are described in Sect. 2.

Let us define analogues of the Verma modules over the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$; in the case of $\mathfrak{sl}_2$ this has been done in [6].

Although the 0^{th} Fourier coefficients $T_i[0]$ of the series $T_i(z)$ do not commute with each other, they commute modulo the left ideal generated by $T_i[n]$, $i = 1,\dots,N-1$; $n > 0$. We can therefore define a Verma module $M_{\gamma_1,\dots,\gamma_{N-1}}$ as a $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$-module generated by a vector $v_{\gamma_1,\dots,\gamma_{N-1}}$, such that $T_i[n]v_{\gamma_1,\dots,\gamma_{N-1}} = 0$, if $n > 0$, and $T_i[0]v_{\gamma_1,\dots,\gamma_{N-1}} = \gamma_i v_{\gamma_1,\dots,\gamma_{N-1}}$. The relations in $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ imply that the module $M_{\gamma_1,\dots,\gamma_{N-1}}$ has a PBW basis which consists of lexicographically ordered monomials in $T_i[n], n < 0$, applied to the highest weight vector $v_{\gamma_1,\dots,\gamma_{N-1}}$.

6.2. Relations in $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$. In this case the relations are:

$$f_{1,2}\left(\frac{w}{z}\right) T_1(z)T_1(w) - f_{1,2}\left(\frac{z}{w}\right) T_1(w)T_1(z)$$
$$= \frac{(1-q)(1-p/q)}{1-p}\left(\delta\left(\frac{w}{zp}\right) - \delta\left(\frac{wp}{z}\right)\right) . \tag{6.3}$$

These relations are equivalent to the relations of Shiraishi, e.a. [6] given by formula (3.3), because their $f(x)$ coincides with $f_{1,2}(x)$ given by (6.1). Formula (6.1) can be simplified in this case:

$$f_{1,2}(x) = \frac{1}{1-x} \frac{(x|q, pq^{-1}; p^2)_\infty}{(x|pq, p^2q^{-1}; p^2)_\infty} .$$

6.3. Relations in $\mathscr{W}_{q,p}(\mathfrak{sl}_3)$. In this case we have:

$$f_{1,3}(x) = \frac{(x|q, pq^{-1}, p^2, p^3; p^3)_\infty}{(x|1, p, p^2q, p^3q^{-1}; p^3)_\infty} ,$$

$$f_{2,3}(x) = \frac{(x|pq, p^2q^{-1}, p^3; p^3)_\infty}{(x|p, p^2q, p^3q^{-1}; p^3)_\infty} .$$

The relations are the following:

$$f_{1,3}\left(\frac{w}{z}\right) T_1(z)T_1(w) - f_{1,3}\left(\frac{z}{w}\right) T_1(w)T_1(z)$$

$$= \frac{(1-q)(1-p/q)}{1-p} \left(\delta\left(\frac{w}{zp}\right) T_2(z) - \delta\left(\frac{wp}{z}\right) T_2(w)\right) ;$$

$$f_{2,3}\left(\frac{w}{z}\right) T_1(z)T_2(w) - f_{2,3}\left(\frac{z}{w}\right) T_2(w)T_1(z)$$

$$= \frac{(1-q)(1-p/q)}{1-p} \left(\delta\left(\frac{w}{zp}\right) - \delta\left(\frac{wp^2}{z}\right)\right) ;$$

$$f_{1,3}\left(\frac{w}{z}\right) T_2(z)T_2(w) - f_{1,3}\left(\frac{z}{w}\right) T_2(w)T_2(z)$$

$$= \frac{(1-q)(1-p/q)}{1-p} \left(\delta\left(\frac{w}{zp}\right) T_1(w) - \delta\left(\frac{wp}{z}\right) T_1(z)\right) .$$

In the limit $p \to q$ they become the relations in $\mathscr{W}_q(\mathfrak{sl}_3)$ described in [1].

7. Relations in Elliptic Form

We recall that q and p are assumed to be generic with $|p| < 1$.

7.1. The case of $\mathfrak{sl}_2$. Consider the OPE given by formula (4.7):

$$\Lambda(z)\Lambda(w) = f_{1,2}\left(\frac{w}{z}\right)^{-1} : \Lambda(z)\Lambda(w) : , \tag{7.1}$$

where

$$f_{1,2}\left(\frac{w}{z}\right) = \frac{(\frac{w}{z}|q, pq^{-1}, p^2; p^2)_\infty}{(\frac{w}{z}|1, pq, p^2q^{-1}; p^2)_\infty} .$$

Formula (7.1) is valid for $|z| \gg |w|$ (see Lemma 1) and it shows that the composition $\Lambda(z)\Lambda(w)$ can be analytically continued to a meromorphic operator-valued function on $\mathbb{C} \times \mathbb{C}$, given by the right-hand side of the formula.

Likewise, the composition $\Lambda(w)\Lambda(z)$ converges when $|w| \gg |z|$, and we have:

$$\Lambda(w)\Lambda(z) = f_{1,2}\left(\frac{z}{w}\right)^{-1} : \Lambda(w)\Lambda(z) : . \tag{7.2}$$

Since $: \Lambda(z)\Lambda(w) :=: \Lambda(w)\Lambda(z) :$, by definition of the normal ordering, we obtain from formulas (7.1) and (7.2) the following relation on the analytic continuations:

$$\Lambda(z)\Lambda(w) = \varphi\left(\frac{w}{z}\right)\Lambda(w)\Lambda(z)\,, \tag{7.3}$$

where

$$\varphi(x) = \frac{f_{1,2}(x^{-1})}{f_{1,2}(x)} = \frac{\theta_{p^2}(x)\theta_{p^2}(xpq)\theta_{p^2}(xp^2q^{-1})}{\theta_{p^2}(xq)\theta_{p_2}(xpq^{-1})\theta_{p^2}(xp^2)}\,. \tag{7.4}$$

and

$$\theta_a(x) = \prod_{n=0}^{\infty}(1 - xa^n)\prod_{n=1}^{\infty}(1 - x^{-1}a^n)\prod_{n=1}^{\infty}(1 - a^n)\,.$$

We can also write:

$$\varphi(x) = \frac{\theta_{p^2}(xp)\theta_{p^2}(xq^{-1})\theta_{p^2}(xp^{-1}q)}{\theta_{p^2}(xp^{-1})\theta_{p^2}(xq)\theta_{p^2}(xpq^{-1})}\,.$$

Formula (7.3) can be rewritten in a more symmetric form as

$$\gamma\left(\frac{w}{z}\right)\Lambda(z)\Lambda(w) = \gamma\left(\frac{z}{w}\right)\Lambda(w)\Lambda(z)\,, \tag{7.5}$$

where $\gamma(x)$ satisfies: $\varphi(x) = \gamma(x^{-1})/\gamma(x)$. Apart from $\gamma(x) = f_{1,2}(x)$, there exist other choices for the function $\gamma(x)$, in particular,

$$\gamma(x) = \frac{\theta_{p^2}(xp^{-1})\theta_{p^2}(xq)\theta_{p^2}(xpq^{-1})}{\theta_{p^2}(x)^3}\,,$$

and

$$\gamma(x) = \frac{\theta_{p^2}(xq)\theta_{p^2}(xpq^{-1})}{\theta_{p^2}(x)\theta_{p^2}(xp)}\,.$$

Any two such choices differ by a multiple, which is invariant under the change $x \to x^{-1}$.

Remark 5. Formula (7.5) should be compared with the property of locality in vertex operator algebras [16, 17] (see also [18]). Recall that two power series $A(z)$ and $B(w)$ are called local if $A(z)B(w)$ converges for $|z| > |w|, B(w)A(z)$ converges for $|w| > |z|$, and their analytic continuations satisfy: $A(z)B(w) = B(w)A(z)$. □

The function $\varphi(x)$ is an elliptic function, i.e.

$$\varphi(xp^2) = \varphi(x)\,, \tag{7.6}$$

and it satisfies the functional equation

$$\varphi(x)\varphi(xp) = 1 \,. \tag{7.7}$$

Equations (7.6), (7.7) provide a new understanding for the algebra $\mathcal{W}_{q,p}(\mathfrak{sl}_2)$. Let us explain that.

According to (7.1) the series $\Lambda(z)$ satisfies the relations:

$$f_{1,2}\left(\frac{w}{z}\right)\Lambda(z)\Lambda(w) = f_{1,2}\left(\frac{z}{w}\right)\Lambda(w)\Lambda(z)\,. \tag{7.8}$$

There is a difference between relations (7.3) and (7.8). The first is a relation on analytic continuations of the compositions of two operators, while the second is a relation on formal power series. A relation of the second type implies a relation of the first type – it can be obtained by multiplying it by a suitable meromorphic function. But different relations of the second type may give rise to the same relation of the first type as we will see below.

Remark 6. Similar phenomenon occurs in vertex operator algebras. Consider for example the Heisenberg algebra with generators $\beta_n, \gamma_n, n \in \mathbb{Z}$, and relations

$$[\beta_n, \gamma_m] = \kappa\delta_{n,-m}\,,$$

where $\kappa \in \mathbb{C}$. These relations imply the following formal power series relations:

$$\beta(z)\gamma(w) - \gamma(w)\beta(z) = \kappa\delta\left(\frac{w}{z}\right)\,.$$

But we can also write

$$(z-w)\beta(z)\gamma(w) = (z-w)\gamma(w)\beta(z)\,,$$

regardless of the value of κ. $\square$

In order to get a relation of the second type from the relation (7.3) of the first type, we have to "factorize" the function $\varphi(x)$, i.e. to represent it as $\varphi(x) = g_1(x^{-1})g_2(x)$, where $g_1(x)$ and $g_2(x)$ are formal Taylor power series in x. Then we obtain a relation of the second type such as (7.8).

Factorization of $\varphi(x)$ is not unique in general. In our case, we can write $\varphi(x) = f_{1,2}(x^{-1})f_{1,2}(x)^{-1}$, but we can also write $\varphi(x) = f_{1,2}(x^{-1}p)^{-1}f_{1,2}(xp^{-1})$, by virtue of the functional relation (7.7). This non-uniqueness leads to interesting consequences, and in a sense it "explains" the structure of the quantum $\mathcal{W}$-algebra.

Let us discuss this structure in more detail Let $\Lambda_1(z)$ and $\Lambda_2(z)$ satisfy the relation

$$\Lambda_i(z)\Lambda_i(w) = f_{1,2}\left(\frac{w}{z}\right)^{-1} :\Lambda_i(z)\Lambda_i(w):\,, \tag{7.9}$$

$$\Lambda_1(z)\Lambda_2(w) = f_{1,2}\left(\frac{w}{zp}\right) :\Lambda_1(z)\Lambda_2(w):\,, \tag{7.10}$$

$$\Lambda_2(z)\Lambda_1(w) = f_{1,2}\left(\frac{wp}{z}\right) :\Lambda_2(z)\Lambda_1(w):\,. \tag{7.11}$$

Then we obtain the power series relations:

$$f_{1,2}\left(\frac{w}{z}\right) \Lambda_i(z)\Lambda_i(w) = f_{1,2}\left(\frac{z}{w}\right) \Lambda_i(w)\Lambda_i(z)\,, \tag{7.12}$$

$$f_{1,2}\left(\frac{w}{zp}\right)^{-1} \Lambda_1(z)\Lambda_2(w) = f_{1,2}\left(\frac{zp}{w}\right)^{-1} \Lambda_2(w)\Lambda_1(z)\,. \tag{7.13}$$

By analytically continuing these relations and dividing by the appropriate functions, we obtain:

$$\Lambda_i(z)\Lambda_j(w) = \varphi\left(\frac{w}{z}\right) \Lambda_j(w)\Lambda_i(z) \tag{7.14}$$

for all $i, j, \in \{1,2\}$. The elliptic relations (7.14) do not depend on i and j while the formal power series relations (7.12)–(7.13) do. However, formulas (7.14) have different meanings for $i = j$ and $i \neq j$.[3]

Consider first this formula in the case when $i = j$. In this case we write:

$$\varphi(x) = f_{1,2}(x^{-1}) f_{1,2}(x)^{-1}\,. \tag{7.15}$$

Let us assume that $|p| < |q| < 1$. Then the function $f_{1,2}(x^{-1})$ is analytic in the region $|x| > 1$, while the function $f_{1,2}(x)^{-1}$ is analytic in the region $|x| < 1$. Thus, formula (7.15) gives us a solution of the Riemann problem on the circle $|x| = 1$. In other words, we represent the function $\varphi(x)$ on the circle $|x| = 1$ as a product of two functions, one of which has analytic continuation to the exterior of the circle, and the other has analytic continuation to the interior of the circle.

Consider now the case $i = 1, j = 2$. In that case we write:

$$\varphi(x) = f_{1,2}(x^{-1}p)^{-1} f_{1,2}(xp^{-1})\,. \tag{7.16}$$

The function $f_{1,2}(x^{-1})^{-1}$ is analytic in the region $|x| > |p|$, while the function $f_{1,2}(xp^{-1})$ is analytic in the region $|x| < |p|$. Hence formula (7.16) gives a solution of the Riemann problem on the circle $|x| = |p|$. The fact that formulas (7.14) are defined on different circles for $i = j$ and $i \neq j$ leads to the appearance of δ-functions in elliptic relations.

Indeed, let us set $T_1(z) = \Lambda_1(z) + \Lambda_2(z)$. Naive application of formula (7.14) tells us that $T_1(z)$ satisfies the same elliptic relations as $\Lambda_i(z)$'s:

$$T_1(z)T_1(w) = \varphi\left(\frac{w}{z}\right) T_1(w)T_1(z)\,.$$

These relations are correct if z and w are generic. However, there are additional δ-function terms if $|z| = |w|p^{\pm 1}$.

[3] We thank N. Reshetikhin for a discussion of this issue, which led us to a better understanding of elliptic relations and helped to correct some of the formulas below.

To see that, let us write the relations between $T_1(z)$ and $T_1(w)$ as formal power series. Using formulas (7.9)–(7.11), we obtain, in the same way as in the proof of formula (6.2):

$$f_{1,2}\left(\frac{w}{z}\right) T_1(z)T_1(w) - f_{1,2}\left(\frac{z}{w}\right) T_1(w)T_1(z)$$
$$= \frac{(1-q)(1-p/q)}{1-p}\left(\delta\left(\frac{w}{zp}\right) : \Lambda_1(z)\Lambda_2(zp) : -\delta\left(\frac{wp}{z}\right) : \Lambda_1(w)\Lambda_2(wp) :\right). \tag{7.17}$$

The appearance of the δ-function terms in the right-hand side is due to the following functional relations:

$$f_{1,2}(x) = f_{1,2}(xp)^{-1}\frac{(1-xq)(1-xp/q)}{(1-x)(1-xp)},$$

$$f_{1,2}(x) = f_{1,2}(xp^{-1})^{-1}\frac{(1-xq^{-1})(1-xq/p)}{(1-x)(1-xp^{-1})}.$$

If we divide both sides of formula (7.17) by $f_{1,2}(w/z)$, we obtain:

$$T_1(z)T_1(w) = \varphi\left(\frac{w}{z}\right) T_1(w)T_1(z) + \frac{(q;p^2)_\infty(pq^{-1};p^2)_\infty}{(pq;p^2)_\infty(p^2q^{-1};p^2)_\infty}$$
$$\times\left(\delta\left(\frac{w}{zp}\right) : \Lambda_1(wp^{-1})\Lambda_2(w) : +\delta\left(\frac{wp}{z}\right) : \Lambda_1(w)\Lambda_2(wp) :\right). \tag{7.18}$$

The right-hand side contains extra δ-function terms as expected.

We see from formulas (7.9)–(7.11) that $:\Lambda_1(z)\Lambda_2(zp):$ is a central element in the algebra generated by $\Lambda_1(z)$ and $\Lambda_2(z)$; note that this fact is equivalent to the functional relation (7.7). Hence we may set it equal to any number. If we set it equal to 1, then we obtain relations (6.3), but actually we can put an arbitrary overall factor in the right-hand side of formula (6.3). In particular, if this factor is 0, we obtain the original defining relations (7.8) of the Heisenberg algebra. Hence the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$ is a central extension of the Heisenberg algebra $\mathscr{H}_{q,p}(\mathfrak{sl}_2)$.

We can also set $:\Lambda_1(z)\Lambda_2(zp):=1$ in formula (7.18). Then we obtain the following relation:[4]

$$T_1(z)T_1(w) = \varphi\left(\frac{w}{z}\right) T_1(w)T_1(z)$$
$$+\frac{(q;p^2)_\infty(pq^{-1};p^2)_\infty}{(pq;p^2)_\infty(p^2q^{-1};p^2)_\infty}\left(\delta\left(\frac{w}{zp}\right)+\delta\left(\frac{wp}{z}\right)\right).$$

The function given by (7.4) is just one of solutions of Eq. (7.7). It is interesting whether other solutions give rise to algebras similar to $\mathscr{W}_{q,p}(\mathfrak{sl}_2)$.

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[4] Analogous relation was also obtained by S. Lukyanov [21].

7.2. The case of $\mathfrak{sl}_N$. From formulas (4.7)–(4.9) we find in the same way as in the case of $\mathfrak{sl}_2$:

$$\Lambda_i(z)\Lambda_j(w) = \varphi_N\left(\frac{w}{z}\right)\Lambda_j(w)\Lambda_i(z)\,, \tag{7.19}$$

for all i, j, where

$$\varphi_N(x)=\frac{\theta_{p^N}(x)\theta_{p^N}(xp)\theta_{p^N}(xp^{N-1}q)\theta_{p^N}(xp^Nq^{-1})}{\theta_{p^N}(xq)\theta_{p^N}(xpq^{-1})\theta_{p^N}(xp^{N-1})\theta_{p^N}(xp^N)}=\frac{\theta_{p^N}(xp)\theta_{p^N}(xq^{-1})\theta_{p^N}(xp^{-1}q)}{\theta_{p^N}(xp^{-1})\theta_{p^N}(xq)\theta_{p^N}(xpq^{-1})}\,.$$

It is the last equality that ensures that the elliptic relations between $\Lambda_i(z)$ and $\Lambda_j(w)$ are the same for $i=j$ and $i\neq j$, although the "factorized" relations between them are different, see formulas (4.7)–(4.9).

Relations (7.19) can be rewritten in a more symmetric form:

$$\gamma_N\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_j(w) = \gamma_N\left(\frac{z}{w}\right)\Lambda_j(w)\Lambda_i(z)\,, \tag{7.20}$$

where

$$\gamma_N(x) = \frac{\theta_{p^N}(xp^{-1})\theta_{p^N}(xq)\theta_{p^N}(xpq^{-1})}{\theta_{p^N}(x)^3}\,,$$

or

$$\gamma_N(x) = \frac{\theta_{p^N}(xq)\theta_{p^N}(xpq^{-1})}{\theta_{p^N}(x)\theta_{p^N}(xp)}\,.$$

Although relations (7.19) do not depend on i and j, they have different meanings for $i=j$ and $i\neq j$, as in the case of $\mathfrak{sl}_2$ (see above). In particular, this leads to the appearance of non-trivial δ-function terms in elliptic relations between $T_i(z)$, $i=1,\ldots,N-1$. For generic z and w we have:

$$T_i(z)T_j(w) = \prod_{k=1}^{i}\prod_{l=1}^{j}\varphi_N\left(\frac{w}{z}p^{l-k}\right)T_j(w)T_i(z)\,.$$

But when $|z| = |w|p^k$ there may appear additional δ-function terms. For example, we obtain from formula (6.2):

$$\begin{aligned}T_1(z)T_m(w) &= \prod_{k=0}^{m-1}\varphi_N\left(\frac{wp^k}{z}\right)T_m(w)T_1(z)\\ &+\frac{(p^m;p^N)_\infty(p^{m+1};p^N)_\infty(q;p^N)_\infty(pq^{-1};p^N)_\infty}{(p^mq;p^N)_\infty(p^{m+1}q^{-1};p^N)_\infty(p^N;p^N)_\infty(p;p^N)_\infty}\delta\left(\frac{w}{zp}\right)T_{m+1}(z)\,,\end{aligned}$$

if $m=1,\ldots,N-2$, and

$$\begin{aligned}T_1(z)T_{N-1}(w) &= \prod_{k=0}^{m-1}\varphi_N\left(\frac{wp^k}{z}\right)T_{N-1}(w)T_1(z)\\ &+\frac{(p^{N-1};p^N)_\infty(q;p^N)_\infty(pq^{-1};p^N)_\infty}{(p^{N-1}q;p^N)_\infty(p^Nq^{-1};p^N)_\infty(p;p^N)_\infty}\left(\delta\left(\frac{w}{zp}\right)+\delta\left(\frac{wp^{N-1}}{z}\right)\right)\,.\end{aligned}$$

The function $\varphi_N(x)$ is elliptic, i.e. $\varphi_N(xp^N) = \varphi_N(x)$, and it satisfies the functional equation

$$\varphi_N(x)\varphi_N(xp)\cdots\varphi_N(xp^{N-1}) = 1 . \tag{7.21}$$

Equation (7.21) implies that $:\Lambda_1(z)\Lambda_2(zp)\cdots\Lambda_N(zp^{N-1}):$ is a central element of $\mathscr{H}_{q,p}(\mathfrak{sl}_N)$. Formula (4.3) shows that we have set it equal to 1, but we could set it equal to any number. If we do not set it equal to a number, we obtain another algebra, which is natural to call the quantum $\mathscr{W}$-algebra associated to $\mathfrak{gl}_N$.

We see that in a certain sense the structure of the algebra $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$ is "encoded" in Eq. (7.21), as it is in the case of $\mathfrak{sl}_2$.

8. Elliptic Relations for the Screening Currents

8.1. The Screening Currents of $\mathscr{W}_{q,p}(\mathfrak{sl}_N)$. Let us we assume that p and q are generic and $|q| < 1$. Then we have the following relations for the screening currents when $|z| \gg |w|$:

$$S_i^+(z)S_i^+(w) = z^{2\beta}\frac{(\frac{w}{z}|1,p;q)_\infty}{(\frac{w}{z}|q,p^{-1}q;q)_\infty} : S_i^+(z)S_i^+(w):,$$

$$S_i^+(z)S_j^+(w) = z^{-\beta}\frac{(\frac{w}{z}|p^{-1/2}q;q)_\infty}{(\frac{w}{z}|p^{1/2};q)_\infty} : S_i^+(z)S_j^+(w):, \qquad A_{ij} = -1,$$

$$S_i^+(z)S_j^+(w) =: S_i^+(z)S_j^+(w):, \qquad A_{ij} = 0 .$$

Remark 7. If we set $p = q^{1-\beta}$, then in the limit $q \to 1$ these relations give us the well-known relations between the ordinary screening currents:

$$S_i^+(z)S_j^+(w) = (z-w)^{A_{ij}\beta} : S_i^+(z)S_j^+(w):$$

when $|z| > |w|$. $\square$

From the formulas above we obtain, in the analytic continuation sense,

$$S_i^+(z)S_i^+(w) = -\left(\frac{w}{z}\right)^{1-2\beta}\frac{\theta_q(\frac{w}{z}p)}{\theta_q(\frac{z}{w}p)}S_i^+(w)S_i^+(z),$$

$$S_i^+(z)S_j^+(w) = \left(\frac{w}{z}\right)^{-2+\beta}\frac{\theta_q(\frac{w}{z}p^{-1/2})}{\theta_q(\frac{z}{w}p^{-1/2})}S_j^+(w)S_i^+(z), \qquad A_{ij} = -1,$$

$$S_i^+(z)S_j^+(w) = S_j^+(w)S_i^+(z), \qquad A_{ij} = 0 .$$

We can rewrite these formulas as follows:

$$S_i^+(z)S_j^+(w) = (-1)^{A_{ij}-1}\left(\frac{w}{z}\right)^{A_{ij}-A_{ij}\beta-1}\frac{\theta_q(\frac{w}{z}p^{A_{ij}/2})}{\theta_q(\frac{z}{w}p^{A_{ij}/2})}S_j^+(w)S_i^+(z) . \tag{8.1}$$

The function

$$\psi_{ij}(x) = (-1)^{A_{ij}-1}x^{A_{ij}-A_{ij}\beta-1}\frac{\theta_q(xp^{A_{ij}/2})}{\theta_q(x^{-1}p^{A_{ij}/2})} \tag{8.2}$$

appearing in the right-hand side of formula (8.1) can be written as

$$\psi_{ij}(x) = \frac{\phi_{ij}(x)}{\phi_{ij}(x^{-1})}, \tag{8.3}$$

where

$$\phi_{ij}(x) = x^{-\beta A_{ij}/2} \frac{\theta_q(xp^{A_{ij}/2})}{\theta_q(xq^{A_{ij}/2})}. \tag{8.4}$$

The functions $\psi_{ij}(x)$ and $\phi_{ij}(x)$ are quasi-periodic:

$$\psi_{ij}(xq) = \psi_{ij}(x), \qquad \psi_{ij}(xe^{2\pi i}) = e^{-2\pi i A_{ij}\beta}\psi_{ij}(x),$$

$$\phi_{ij}(xq) = \phi_{ij}(x), \qquad \phi_{ij}(xe^{2\pi i}) = e^{-\pi i A_{ij}\beta}\phi_{ij}(x).$$

Note also that $\psi_{ij}(x) = \phi_{ij}(x) = 1$, if $p = q$.

To obtain relations on the screening currents $S_i^-(z)$, $i = 1, \dots, N-1$, let us assume that p and q are generic and $|p/q| < 1$. Then we obtain in the same way as above:

$$S_i^-(z)S_j^-(w) = (-1)^{A_{ij}-1} \left(\frac{w}{z}\right)^{A_{ij}-A_{ij}/\beta-1} \frac{\theta_{p/q}(\frac{w}{z}p^{A_{ij}/2})}{\theta_{p/q}(\frac{z}{w}p^{A_{ij}/2})} S_j^-(w)S_i^-(z). \tag{8.5}$$

We also have:

$$S_i^+(z)S_i^-(w) = \frac{1}{(z-wq)(z-wp^{-1}q)} : S_i^+(z)S_i^-(w) : ,$$

$$S_i^+(z)S_j^-(w) = (z - wp^{-1/2}q) : S_i^+(z)S_j^-(w) : , \qquad A_{ij} = -1,$$

$$S_i^+(z)S_j^+(w) =: S_i^+(z)S_j^-(w) :, \qquad A_{ij} = 0.$$

8.2. The Screening Currents of $U_q(\widehat{\mathfrak{sl}}_N)$. The screening currents involved in the Wakimoto realization of $U_q(\widehat{\mathfrak{sl}}_N)$ [2] also satisfy elliptic relations.

These screening currents $S_i(z)$, $i = 1, \dots, N-1$, are given in [2] by the formula

$$S_i(z) =: \exp\left(\sum_{n \neq 0} \frac{a_n^i}{[vn]_q} q^{-v|n|/2} z^{-n} - \frac{1}{v}\hat{q}^i - \frac{1}{v}\hat{p}^i \log z\right) : \widetilde{S}_i(z).$$

The following relations hold [2]:

$$[a_n^i, a_m^j] = \frac{1}{n}[vn]_q[A_{ij}]_q\delta_{n,-m}, \qquad [\hat{p}^i, \hat{q}^j] = vA_{ij},$$

$$\widetilde{S}_i(z)\widetilde{S}_j(w) = \frac{q^{-A_{ij}}z - w}{z - wq^{-A_{ij}}}\widetilde{S}_i(w)\widetilde{S}_i(z),$$

in the sense of analytic continuation, and a_n^i commute with $\widetilde{S}_j(z)$. Here v is $k + g$ in the notation of [2].

Let us assume that q is generic and $|q^{2v}| < 1$. Then the relations above give us the following relations on $S_i(z)$'s:

$$S_i(z)S_j(w) = -\left(\frac{w}{z}\right)^{-A_{ij}/v-1} \frac{\theta_t(\frac{w}{z}q^{-A_{ij}})}{\theta_t(\frac{z}{w}q^{A_{ij}})} S_j(w)S_i(z)\,, \tag{8.6}$$

where $t = q^{2v}$. These relations should be understood in the analytic continuation sense.

The relations (8.1) can be viewed as elliptic deformations of the quantum Serre relations of Drinfeld [11]:

$$E_i(z)E_j(w) = \frac{zq^{A_{ij}} - w}{z - wq^{A_{ij}}} E_j(w)E_i(z)\,,$$

which can be easily obtained from (8.6) in the limit $t \to 0$ with fixed q.

The function

$$\Psi_{ij}(x) = -x^{-A_{ij}/v-1} \frac{\theta_t(xq^{-A_{ij}})}{\theta_t(x^{-1}q^{-A_{ij}})}$$

in the right-hand side of formula (8.6) can be rewritten as

$$\Psi_{ij}(x) = \frac{\Phi_{ij}(x)}{\Phi_{ij}(x^{-1})}\,,$$

where

$$\Phi_{ij}(x) = x^{-A_{ij}/2v} \frac{\theta_t(xq^{-A_{ij}})}{\theta_t(x)}\,.$$

The functions $\Psi_{ij}(x)$ and $\Phi_{ij}(x)$ are quasi-periodic, and $\Psi_{ij}(x) = \Phi_{ij}(x) = 1$ in the limit when $q \to 1$ and t is fixed.

8.3. General Case. Let $\mathfrak{g}$ be a simply-laced simple Lie algebra. Let $\mathscr{H}'_{q,p}(\mathfrak{g})$ be the Heisenberg algebra with generators $a_i[n]$, $i = 1,\dots,l = \operatorname{rank}\mathfrak{g}; n \in \mathbb{Z}$, and relations (4.1), where (A_{ij}) is the Cartan matrix of $\mathfrak{g}$. We define the Fock representations π_μ and the completion $\mathscr{H}_{q,p}(\mathfrak{g})$ of $\mathscr{H}'_{q,p}(\mathfrak{g})$ in the same way as in Sect. 4.1.

We define the screening currents $S_i^\pm(z)$, $i = 1,\dots,l$, by formulas (5.1)–(5.4). We then define the algebra $\mathscr{W}_{q,p}(\mathfrak{g})$ as the subalgebra of $\mathscr{H}_{q,p}(\mathfrak{g})$ of elements which commute with the screening currents $S_i^+(w)$ up to a total $\mathscr{D}_q$-difference. It follows from the definition that $\mathscr{W}_{q,p}(\mathfrak{g})$ commutes with $S_i^-(w)$ up to a total $\mathscr{D}_{p/q}$-difference.

The relations between $S_i^+(z)$, $i = 1,\dots,l$, in the analytic continuation sense, are given by formula (8.1). The relations between $S_i^-(z)$, $i = 1,\dots,l$, in the analytic continuation sense, are given by formula (8.5). Note that formulas (8.1)–(8.5) make sense for an arbitrary integral symmetric matrix (A_{ij}).

If we set $p = q^{1-\beta}$, then in the limit $q \to 1$ the algebra $\mathscr{W}_{q,p}(\mathfrak{g})$ becomes the ordinary $\mathscr{W}$-algebra, $\mathscr{W}_{\sqrt{\beta}}(\mathfrak{g})$ in the notation of [18]. On the other hand, we expect that in the limit $\beta \to 0$, i.e. $p \to q$, the algebra $\mathscr{W}_{q,p}(\mathfrak{g})$ becomes isomorphic to the Poisson algebra $\mathscr{W}_q(\mathfrak{g})$ considered in [1].

Acknowledgements. We would like to thank T. Miwa for the invitation to visit R.I.M.S., and H. Awata and A. Odesskii for useful discussions.

Note added in proof. Shortly after this paper appeared on the q-alg electronic archive, the paper by H. Awata, H. Kubo, S. Odake and J. Shiraishi, "Quantum $\mathscr{W}_N$ algebras and Macdonald polynomials" was submitted to the same archive; it partially overlaps with our paper.

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Communicated by G. Felder