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Time-Dependent Quantum Scattering in 2+1 Dimensional Gravity

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Abstract: The propagation of a localized wave packet in the conical space-time created by a pointlike massive source in 2+1 dimensional gravity is analyzed. The scattering amplitude is determined and shown to be finite along the classical scattering directions due to interference between the scattered and the transmitted wave functions. The analogy with diffraction theory is emphasized.

1. Introduction

The time-dependent scattering problem was solved in the case of the Aharonov-Bohm interaction in [1]. This author considered the time evolution of an electrically charged well-localized wave packet in the presence of a magnetic vortex. The main result in that work is the analysis of the forward direction, where the wave packet undergoes a self-interference; the probability density current was shown to be finite.

The question arises if a similar analysis can be carried out in 2+1 dimensional gravity. By this we mean to consider the scattering of a wave packet by a static source in planar gravity, to find the scattering amplitude, and to determine the behaviour of the wave packet along the directions where self-interference effects are significant.

The classical theory of 2+1 dimensional gravity, as well as its interpretation as a conical space-time, was presented in [2]. The quantum-mechanical scattering problem for two scalar particles interacting only gravitationally in 2+1 dimensions was first solved in [3] by reducing the problem to the motion of a free particle on a cone. A closely related procedure was put forward in [4], this time derived from a partial wave decomposition. Needless to say, both methods yield the same scattering amplitude. These works showed that in the case of 2+1 dimensional gravity the forward direction is not exceptional; it is at the classical scattering angles where self-interferences take place.

A further step was taken in [5]. These authors not only generalized the previous results to the case of spinning sources, but also pointed out an interesting analogy

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between scattering in 2+1 dimensional gravity and classical diffraction theory. Even though their discussion of this point is qualitative, they were able to interpret the main features of the scattering amplitude as a diffractive effect.

Albeit these works provided a thorough understanding of the scattering process, none of them addresses the time-dependent scattering problem as posed before. In this work we present a solution based in the optical analogy noted in [5].

This paper is organized as follows. In Sect. 2 we recall the propagator found in [4] for the conical Schrödinger equation, and analyze its behaviour close to the classical scattering angles. This is accomplished by means of a method developed by W. Pauli in the context of classical diffraction theory [6]. In Sect. 3 we introduce an incoming Gaussian wave packet, with vanishing impact parameter, and study its propagation by using the results of the previous section. The result is free from the singularities in the scattering amplitude found in [3, 4]. We find a cancellation of finite discontinuities along the classical scattering angles due to interference between scattered and transmitted waves. This can be considered a quantitative version of the qualitative analysis presented in [5]. In Sect. 4 we perform a similar analysis for a wave packet with non-zero impact parameter. Finally, in Sect. 5 we present our conclusions. In the Appendix the same method is applied to time-independent scattering.

2. Calculation of the Propagator

In this section we shall discuss the quantal propagator for a test-particle of mass m moving in the conical space created by a static mass M at the origin of our coordinate system. We refer the reader to [2] and [4] for a full exposition of these points.

Let us summarize the geometrical structure of the space-time in question. An intrinsic characterization [2] uses a Euclidean metric with incomplete angular range to describe the two-dimensional geometry of space:

$$(dl)^2 = (dr)^2 + r^2(d\varphi)^2 \ , \qquad -\pi\alpha \leq \varphi \leq \pi\alpha \ , \tag{1}$$

where $0 \leq (1-\alpha) = 4MG < 1$ and G is "Newton's constant." We recall that in this situation (quantal scattering of a test-particle by a static mass) the time-component of the metric does not play any role. An alternative characterization of this conical space is based on embedded coordinates [4]

$$(dl)^2 = \alpha^{-2}(dr)^2 + r^2(d\theta)^2 \ , \qquad -\pi \leq \theta \leq \pi \ . \tag{2}$$

We shall use these coordinates in the following because the full angular range allows for conventional partial-wave analysis and identification of phase shifts in the wave functions. The Hamiltonian of a test particle of mass m in this conical space-time is

$$H = -\frac{\hbar^2}{2m}\left[\alpha^2 \frac{1}{r}\,\partial_r(r\partial_r) + \frac{1}{r^2}\,\partial_\theta^2\right] \ . \tag{3}$$

This operator is diagonalized by eigenfunctions proportional to Bessel functions; the dependence in the angle θ factorizes in a single-valued exponential:

$$\begin{aligned} \Psi_{n,k}(r,\theta) &= \sqrt{\frac{1}{2\pi}}\, e^{in\theta}\, u_n(kr) \ , \\ u_n(kr) &= (-1)^{\frac{n-|n|}{2}} J_{\frac{|n|}{\alpha}}(kr) \ , \end{aligned} \tag{4}$$

where $k^2 = 2mE/\hbar^2\alpha^2$, E is the energy eigenvalue, and n is an integer. The radial eigenfunctions $u_n(kr)$ are regular at the origin and have the following asymptotic behaviour:

$$u_n(kr) \stackrel{kr\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi kr}} \cos\left(kr - \frac{|n|\pi}{2\alpha} - \frac{\pi}{4} + \frac{(|n|-n)\pi}{2}\right) \ . \tag{5}$$

Thus the phase shifts are independent of the energy of the incoming particle, as a consequence of non-relativistic conformal invariance, and increase with $|n|$,

$$\delta_n = -\frac{|n|\pi}{2}(\alpha^{-1} - 1) \ . \tag{6}$$

Since we are interested in a time-evolution problem, we need the Feynman propagator

$$G(\mathbf{r}, \mathbf{r}'; t) = <\mathbf{r}'|\mathrm{e}^{-\mathrm{i}Ht}|\mathbf{r}> \ , \tag{7}$$

having a spatial delta function as boundary condition at $t = 0$. Using the complete set of energy eigenstates and taking as initial and final points $\mathbf{r} = (r, \theta)$ and $\mathbf{r}' = (r', \theta')$, we have the representation (the imaginary time $T = it$ makes well-defined the integration)

$$G(\mathbf{r}, \mathbf{r}'; -iT) = \frac{1}{2\pi} \int_0^\infty k\, dk\, \mathrm{e}^{-\frac{\hbar\alpha^2 T k^2}{2m}} \sum_n J_{\frac{|n|}{\alpha}}(kr)\, J_{\frac{|n|}{\alpha}}(kr')\, \mathrm{e}^{\mathrm{i}n(\theta'-\theta)}. \tag{8}$$

The integration leads to the Deser-Jackiw propagator. Going back to real time t this propagator can be written as

$$G(r, \theta; r', \theta'; t) = \frac{m}{2\pi \mathrm{i}\hbar t\alpha^2} \exp\left\{\frac{im}{2\hbar t\alpha^2}(r^2 + r'^2)\right\} \sum_n \mathrm{e}^{\mathrm{i}n(\theta'-\theta)} I_{\frac{|n|}{\alpha}}\left(\frac{mrr'}{\mathrm{i}\hbar t\alpha^2}\right). \tag{9}$$

The partial wave sum can be evaluated with the help of the Schläfli contour integral representation for the Bessel function, whose contour of integration is shown in Fig. 1,

$$I_\nu(x) = \frac{1}{2\pi} \int_C dz\, \mathrm{e}^{x\cos z + \mathrm{i}\nu z} \ . \tag{10}$$

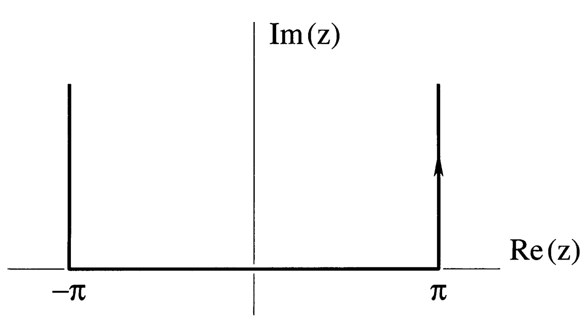

Fig. 1. The Schläfli contour

After the summation the propagator $G(r, \theta; r', \theta'; t)$ can be written as a sum of two different terms, namely G_1 and G_2, corresponding respectively to the transmitted and the scattered wave:

$$
\begin{aligned}
&G_1(r,\theta;r',\theta';t) = \\
&\quad \frac{m}{2\pi \mathrm{i}\hbar t\alpha} {\sum_n}' \exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}\left[r^2 + r'^2 - 2rr'\cos\alpha(\theta' - \theta - 2\pi n)\right]\right\}, \\
&G_2(r,\theta;r',\theta';t) = \\
&\quad \frac{m}{8\pi^2 \mathrm{i}\hbar t\alpha^2} \int_{-\infty}^{\infty} dy \Big\{ \cot\left[\frac{\mathrm{i}y}{2\alpha} - \frac{\pi}{2\alpha} + \frac{\theta' - \theta}{2}\right] - \\
&\quad - \cot\left[\frac{\mathrm{i}y}{2\alpha} + \frac{\pi}{2\alpha} + \frac{\theta' - \theta}{2}\right]\Big\} \exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r^2 + r'^2 + 2rr'\cosh y)\right\}, \qquad (11)
\end{aligned}
$$

where the primed sum includes only n such that $\alpha(\theta' - \theta - 2\pi n) \in (-\pi, \pi)$. The propagator G_1 is presented in a closed form, but G_2 is given as an integral representation. We are going to elaborate the latter in order to make it useful for calculations. The propagator G_2 can be written in an alternative way by means of a trigonometric identity:

$$
\begin{aligned}
G_2(r,\theta;r',\theta';t) &= \frac{m}{4\pi^2 \mathrm{i}\hbar t\alpha^2} \int_{-\infty}^{\infty} dy \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha} - \cos\left(\frac{\mathrm{i}y}{\alpha} + \theta' - \theta\right)} \\
&\quad \times \exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r^2 + r'^2 + 2rr'\cosh y)\right\} . \qquad (12)
\end{aligned}
$$

Therefore, if α^{-1} is an integer this contribution to the propagator vanishes. Otherwise the integral can be performed in the limit of large $mrr'/\hbar t$, where the leading contribution comes from the small-y region:

$$
\begin{aligned}
G_2(r,\theta;r',\theta';t) &\approx \frac{m\sin\frac{\pi}{\alpha}}{4\pi^2 \mathrm{i}\hbar t\alpha^2} \exp\left\{\frac{\mathrm{i}m}{2\hbar t\alpha^2}(r^2 + r'^2)\right\} \int_{-\infty}^{\infty} dy\ \exp\left\{\frac{\mathrm{i}mrr'}{2\hbar t\alpha^2}y^2\right\} \\
&\quad \times \frac{1}{\cos\frac{\pi}{\alpha} - \cos(\theta' - \theta) + \frac{\mathrm{i}y}{\alpha}\sin(\theta' - \theta) + \mathcal{O}(y^2)} . \qquad (13)
\end{aligned}
$$

To proceed with the integration we need an additional assumption. We shall consider the case $\theta' - \theta \neq \pm\pi/\alpha \bmod(2\pi)$, i.e., we keep away from the classical scattering angles [4]. This allows to approximate the integral by a Gaussian. The final result is

$$
G_2(r,\theta;r',\theta';t) \approx \left(\frac{m}{8\pi^2 \hbar t\alpha^2 \mathrm{i} rr'}\right)^{\frac{1}{2}} \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha} - \cos(\theta' - \theta)} \exp\left\{\frac{\mathrm{i}m}{2\hbar t\alpha^2}(r + r')^2\right\}, \qquad (14)
$$

which can be used immediately to find the scattering amplitude. The result is that of [3] and [4].

Now we analyze Eq. (12) in the vicinity of the classical scattering angle. A straightforward saddle-point calculation is not possible now because the integrand develops a singularity precisely at the saddle-point $y = 0$. Hence the problem arises to obtain an explicit formula for the scattered propagator G_2 in the limit of large $mrr'/\hbar t$, which will be valid also at the classical scattering directions. It is here that the method developed in [6] comes into play. Pauli considered the problem of the diffraction of light by a wedge limited by two perfectly reflecting planes. The diffracted wave can be calculated by means of an integral representation similar to

Eq. (12), whose singularity lies in the boundary between the "illuminated" region and the "shadow" of geometrical optics. He was able to show that the transition from shadow to light is completely smooth. Our problem is to show that the apparent singularity present in Eq. (14) when $\theta' - \theta = \pm\pi/\alpha \bmod(2\pi)$ does not actually exist, so that the wave function is regular everywhere. The formal similarity between these two problems makes it possible to apply Pauli's method in our case.

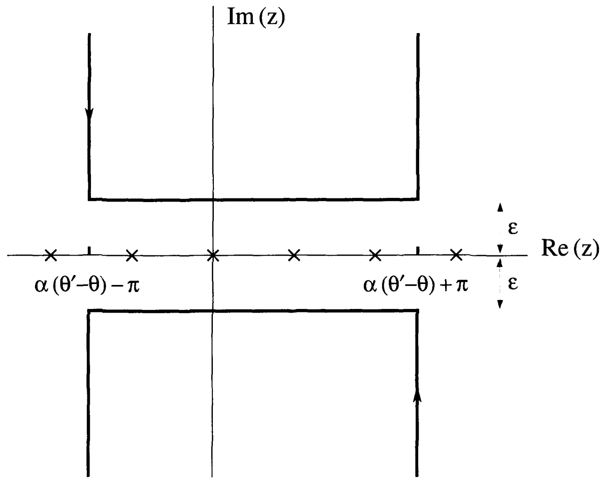

Fig. 2. The contour for the propagator

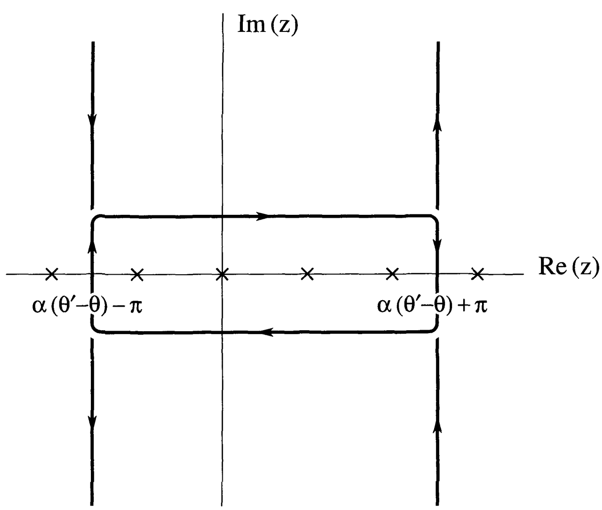

Fig. 3. An equivalent contour

Let us first examine the solution given in [4] to a similar difficulty in the time-independent scattering of plane waves in 2+1 dimensional gravity. These authors started with an integral representation for the wave function whose integration path is that of Fig. 2 or, equivalently, that of Fig. 3 (see [4] for details). The equivalence between these contours follows from the cancellation of the vertical sides of the closed contour in Fig. 3 with the adjacent segments of the straight lines. All the singularities of the integrand are poles which lie on the real axis at $z = 2\pi\alpha N$, with N an integer. The closed contour in Fig. 3 corresponds to a sum of Cauchy residues, which yields the transmitted wave; the two straight lines correspond to the scattered wave.

This construction is rigorous as long as the contours can be deformed to avoid the singularities. If $\theta' - \theta = \pm\pi/\alpha \bmod(2\pi)$ the contours cross over one of the poles, which therefore cannot be avoided. In other words, the decomposition of the wave function in "transmitted" (closed contour in Fig. 3) and "scattered" (straight lines, ibid.) components must be re-examined at the classical scattering angles. In [4] it is assumed that the pole that is now present at the boundary of the closed contour contributes only half its residue, and that the two straight lines in Fig. 3 exclude $\mathrm{Im}(y) \in [-\epsilon, \epsilon]$, thereby not cancelling the vertical sides of the closed contour. If we apply this idea, the integration in Eq. (11) would be interpreted as a principal value.

Under this assumption the angular dependence in Eq. (14) reduces to $-\cot(\pi/\alpha)$. Nevertheless, the contours in Figs. 2 and 3 cannot be identified if the vertical sides of the closed contour in Fig. 3 remain uncancelled. We conclude that this analysis of the dominant (at large t) portion of the propagator close to the classical scattering directions is not adequate for analyzing the physics: a subdominant term in the scattered wave, found below, is essential.

In order to apply the method proposed in [6] we go back to Eq. (12) and change to a new set of variables

$$\begin{aligned} y &= \mathrm{i}\eta \ , \\ \frac{mrr'}{\hbar t\alpha^2} &= \rho \ , \\ \theta' - \theta &= -\frac{\phi}{\alpha} \ , \end{aligned} \tag{15}$$

which gives an integral representation for G_2 more suitable for the following analysis:

$$\begin{aligned} G_2(r,\theta;r',\theta';t) &= -\frac{m\sin\frac{\pi}{\alpha}}{4\pi^2\hbar t\alpha^2}\exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r^2+r'^2)\right\} \\ &\times \int_{-\mathrm{i}\infty+\gamma}^{\mathrm{i}\infty-\gamma} d\eta \frac{\mathrm{e}^{\mathrm{i}\rho\cos\eta}}{\cos\frac{\pi}{\alpha}-\cos\left(\frac{\phi+\eta}{\alpha}\right)} \ , \end{aligned} \tag{16}$$

where γ is any angle between zero and π. The physically interesting case is that of large ρ, where the method of steepest descent can be applied. This requires the introduction of the variable

$$s = \mathrm{e}^{\mathrm{i}\pi/4}2^{\frac{1}{2}}\sin\frac{\eta}{2} \ . \tag{17}$$

As a path of integration, the real s axis can be taken, so that G_2 becomes

$$\begin{aligned} G_2(r,\theta;r',\theta';t) &= \frac{m\sin\frac{\pi}{\alpha}}{4\pi^2\hbar t\alpha^2}\exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r^2+r'^2)\right\}\mathrm{e}^{-\mathrm{i}\pi/4}2^{\frac{1}{2}} \\ &\times \int_{-\infty}^{\infty} ds \frac{\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho s^2}}{\cos\frac{\pi}{\alpha}-\cos\left(\frac{\phi+\eta}{\alpha}\right)}\left(1+\frac{\mathrm{i}}{2}s^2\right)^{-\frac{1}{2}} \ . \end{aligned} \tag{18}$$

The purpose of the preceding changes of variable was to extract the Gaussian factor $\exp(-\rho s^2)$ now present in Eq. (18). The obvious procedure would be to expand the integrand, except the Gaussian factor, in powers of s and evaluate the integrals. The result obtained in this way would become ill-defined if $\theta' - \theta = \pm\pi/\alpha \bmod(2\pi)$, which corresponds to the classical scattering directions.

The method presented in [6] avoids this difficulty by developing not the whole integrand, but only a factor regular at the saddle point. If we introduce the notation $-a = 1 + \cos\phi$, the propagator G_2 can be written as

$$\begin{aligned} G_2(r,\theta;r',\theta';t) &= -\frac{m\sin\frac{\pi}{\alpha}}{4\pi^2\hbar t\alpha^2}\exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r^2+r'^2)\right\} \\ &\quad \times \mathrm{e}^{\mathrm{i}(\rho+\pi/4)}2^{\frac{1}{2}}\int_{-\infty}^{\infty} ds\ \mathrm{e}^{-\rho s^2}\frac{f(s,\phi)}{\mathrm{i}a+s^2}\ , \end{aligned} \tag{19}$$

where the function $f(s,\phi)$ is defined as

$$f(s,\phi) = \frac{\cos\eta(s)+\cos\phi}{\cos\frac{\pi}{\alpha} - \cos\frac{\phi+\eta(s)}{\alpha}}\ \frac{1}{\cos\frac{\eta(s)}{2}}\ . \tag{20}$$

This function is regular at the saddle point $\eta(s) = 0$ even if $\phi = \pm\pi$. Its only singularities at $\eta(s) = 0$ occur if $\phi = \pi + 2\pi\alpha N$ with N integer but αN not integer. Nevertheless these cases will not be relevant in our problem, since we are mainly interested in $\phi \approx \pm\pi$.

Let us expand $f(s,\phi)$ in powers of s,

$$f(s,\phi) = \sum_{m=0}^{\infty}\mathrm{e}^{\mathrm{i}m\frac{\pi}{4}}A_m(\phi)s^m \tag{21}$$

and insert this series in Eq. (19). The values of $A_0(\phi)$ and $A_2(\pi)$, which will be used below, are:

$$\begin{aligned} A_0(\phi) &= \frac{1+\cos\phi}{\cos\frac{\pi}{\alpha} - \cos\frac{\phi}{\alpha}}\ , \\ (2a)^{-\frac{1}{2}}A_0(\phi)\Big|_{\phi=\pi\pm} &= \pm\frac{\mathrm{i}\alpha}{2\sin\frac{\pi}{\alpha}}\ , \\ A_2(\pi) &= -\frac{\cos\frac{\pi}{\alpha}}{2\sin^2\frac{\pi}{\alpha}}\ . \end{aligned} \tag{22}$$

Notice that the evaluation of $(2a)^{-1/2}A_0(\phi)$ when $\phi = \pm\pi$ is actually a limit ($-a = 1+\cos\phi \approx 0$). It is possible to show that all the $A_{2m}(\phi)$ are finite at $\phi = \pm\pi$. The terms with odd s in Eq. (21) cancel when integrating, while the terms with even s give after the substitution $s = \tau\rho^{-1/2}$ a confluent hypergeometric function. In terms of $S_m(x)$ functions, defined by Pauli [6], the propagator G_2 reads

$$\begin{aligned} G_2(r,\theta;r',\theta';t) &= -\frac{m\sin\frac{\pi}{\alpha}}{4\pi^2\hbar t\alpha^2}\exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r^2+r'^2)\right\}\mathrm{e}^{\mathrm{i}(\rho+\pi/4)} \\ &\quad \times\left(\frac{2}{a}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\mathrm{i}^m\Gamma\left(m+\frac{1}{2}\right)A_{2m}(\phi)S_m(a\rho)\rho^{-m}\ . \end{aligned} \tag{23}$$

The behaviour of these $S_m(x)$ functions for large and small x are

$$\begin{array}{lll} S_m(x) & \approx -\mathrm{i}x^{-\frac{1}{2}}\left[1-\left(m+\frac{1}{2}\right)(\mathrm{i}x)^{-1}+\cdots\right] , & |x| \gg 1 , \\ S_m(x) & \approx \left(m-\frac{1}{2}\right)^{-1} x^{\frac{1}{2}} , & |x| \approx 0 \text{ and } m > 0 , \\ S_0(x) & \approx \pi^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\pi/4} , & |x| \approx 0 . \end{array} \tag{24}$$

We can now proceed with the analysis of the propagator G_2. There are two interesting cases:

2.1. $\rho(1+\cos\phi) \to \infty$.

This represents the large $mrr'/\hbar t\alpha^2$ limit, away from the classical scattering angles,i.e., $\theta' - \theta \neq \pm\pi/\alpha$. Taking into account Eqs. (22), (23), (24) and going back to the original variables r, θ, the result is

$$\begin{aligned} G_2(r,\theta;r',\theta';t) &= \frac{1}{2\pi}\left(\frac{m}{2\pi\mathrm{i}\hbar t\alpha^2 rr'}\right)^{\frac{1}{2}} \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha}-\cos(\theta'-\theta)} \\ &\quad \times \exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r+r')^2\right\} + \mathscr{O}(r^{-3/2}) . \end{aligned} \tag{25}$$

2.2. $\phi = \pi^{\pm}$.

These values of ϕ correspond to the classical scattering angles. The parameter $-a = 1 + \cos\phi$ is now vanishing. That notwithstanding, the singularity $1/\sqrt{a}$ in Eq. (23) is compensated by $A_0(\phi)$ if $m = 0$ and by $S_m(a\rho)$ if $m > 0$. This implies that the asymptotic limit can be performed without finding any singularities at the classical scattering angles, in contrast with the result of applying the asymptotic limit directly to Eq. (12).

In this kinematic region we find a finite discontinuity:

$$\begin{aligned} &G_2(r,\theta'(\pi^{\pm}/\alpha);r',\theta';t) = \\ &\quad \pm\frac{m}{4\pi\mathrm{i}\hbar t\alpha}\exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r+r')^2\right\} \\ &\quad +\frac{\mathrm{i}}{2\pi}\left(\frac{\mathrm{i}m}{8\pi\hbar t\alpha^2 rr'}\right)^{\frac{1}{2}}\cot\frac{\pi}{\alpha}\exp\left\{\mathrm{i}\frac{m}{2\hbar t\alpha^2}(r+r')^2\right\} \\ &\quad +\mathscr{O}(r^{-3/2}) . \end{aligned} \tag{26}$$

The two first terms in this expansion will be denoted by G_{21} and G_{22} respectively. It should be noticed that the second term has the radial structure of a scattered wave, and coincides with the result of taking the principal value in the integral representation of G_2 given in Eq. (11). The other terms, however, would be lost in so doing. In particular, the first term represents a discontinuous wave transmitted along the classical scattering angle, which will be called "subdominant" because of its dependence on time. The wave propagated by G_{22} will be called "leading." The terms not included in Eq. (26) can be calculated by taking more elements in the expansion (21). These terms can be shown to be continuous, and therefore do not contribute to the discontinuity of the scattered wave at the classical scattering directions.

If $\phi = -\pi^{\pm}$ a similar analysis shows that the result is identical. Therefore we shall not consider this case explicitly.

3. Scattering of a Wave Packet: Zero Impact Parameter

In this section we consider the scattering of a Gaussian wave packet by means of the propagator calculated in the previous section. For the moment we assume that the impact parameter is zero, and that the wave packet is centered at $r \approx r_0$ and $\theta \approx \pi$. Its initial momentum is $(k_0, 0)$ in Cartesian coordinates; we will consider that $r_0 \gg k_0^{-1}$:

$$\Psi_0(r',\theta',0) = \frac{1}{\sqrt{2\pi}\xi} \exp\left\{ik_0 r' \cos\theta' - \frac{1}{4\xi^2}\left(r'^2 + r_0^2 + 2r' r_0 \cos\theta'\right)\right\} . \tag{27}$$

It is convenient to distinguish whether θ is different from or equal to the classical scattering angle $\theta' \pm (\pi/\alpha)$, since in the first situation the relevant propagator is Eq. (25), whereas in the second one we need G_1 and Eq. (26).

3.1. $\theta \neq \theta' \pm (\pi/\alpha)$.

As stated before, the propagator is Eq. (25), so that the integration to be done is

$$\begin{aligned}\Psi(r,\theta,t) &= \frac{1}{2\pi}\left(\frac{m}{2\pi i\hbar t\alpha^2 r}\right)^{\frac{1}{2}} \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha} + \cos\theta} \int_0^{2\pi} d\theta' \int_0^{\infty} \sqrt{r'}dr' \frac{1}{\sqrt{2\pi}\xi} \\ &\quad \times \exp\left\{ik_0 r'\cos\theta' - \frac{1}{4\xi^2}\left(r'^2 + r_0^2 + 2r' r_0\cos\theta'\right)\right\} \\ &\quad \times \exp\left\{i\frac{m}{2\hbar t\alpha^2}(r + r')^2\right\}. \end{aligned} \tag{28}$$

We have approximated $\theta' = \pi$ in the propagator but not in the initial wave function. Following standard procedures we find that in the limit $k_0 \gg r_0^{-1}$ the final wave function can be written as

$$\Psi(r,\theta,t) = \sqrt{\frac{i}{r}}\, \frac{1}{\sqrt{2\pi k_0}}\, \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha} + \cos\theta}\, \Psi_{\text{free}}(r, \alpha^2 t), \tag{29}$$

where Ψ_{free} denotes a freely propagating radial wave packet,

$$\begin{aligned}\Psi_{\text{free}}(r,\alpha^2 t) &= \frac{1}{\sqrt{2\pi}\xi} \int_0^{\infty} dr' \exp\left\{ - ik_0 r' - \frac{1}{4\xi^2}\left(r' - r_0\right)^2\right\} \\ &\quad \times \left(\frac{m}{2\pi i\hbar t\alpha^2}\right)^{\frac{1}{2}} \exp\left\{i\frac{m}{2\hbar t\alpha^2}(r + r')^2\right\}. \end{aligned} \tag{30}$$

Note that the dependence on t is through $\alpha^2 t$. This can be interpreted as a time delay in the propagation of the scattered wave packet. The delay $\Delta(t)$ of a scattered wave is usually due to the dependence of the phase shifts on the energy, as explained by Wigner's formula [7] (see also [5]):

$$\Delta(t) = 2\frac{\partial}{\partial E}\delta_n(E) \ . \tag{31}$$

This cannot account for the time delay of Ψ_{free} because the partial wave analysis of this problem shows that the phase shifts, Eq. (6), do not depend on the energy [4]. We leave this question open for future clarification.

The scattering amplitude can be read from Eq. (29), which is the well-known result [3, 4].

$$f(k,\theta) = \frac{1}{\sqrt{2\pi k}} \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha} + \cos\theta} \ , \tag{32}$$

3.2. $\theta \approx \theta' \pm (\pi/\alpha)$.

This angular range involves three main contributions: G_1 and the two terms of G_2 shown in Eq. (26). The contribution of G_{22} to the final wave function, denoted by Ψ_{22} can be easily calculated:

$$\Psi_{22}(r,\theta' \pm \frac{\pi}{\alpha}, t) = -\frac{1}{2}\sqrt{\frac{\mathrm{i}}{r}}\,\frac{1}{\sqrt{2\pi k_0}}\,\cot\frac{\pi}{\alpha}\,\Psi_{\text{free}}(r,\alpha^2 t) \ . \tag{33}$$

Let us denote by Ψ_{21} and Ψ_1 the contribution of G_{21} and G_1 to the final wave function. G_{21} presents a discontinuous behaviour in $\phi = \pi$ which exactly compensates the discontinuity in G_1 due to the "absorption" of a new pole into the closed contour in Fig. 3. We shall show this explicitly. Let δ be a small positive angle; the discontinuity in Ψ_{21} is

$$\Psi_{21}(r,\pi + \frac{\pi}{\alpha} + \delta, t) - \Psi_{21}(r,\pi + \frac{\pi}{\alpha} - \delta, t) =$$

$$\frac{m}{2\pi\mathrm{i}\hbar t\alpha}\int_0^{2\pi} d\theta' \int_0^{\infty} r'dr'\ \Psi_0(r',\theta',0)\exp\left\{\frac{\mathrm{i}m}{2\pi\hbar t\alpha^2}(r+r')^2\right\} + \mathcal{O}(\delta), \tag{34}$$

while the discontinuity in Ψ_1 is

$$\Psi_1(r,\pi + \frac{\pi}{\alpha} + \delta, t) - \Psi_1(r,\pi + \frac{\pi}{\alpha} - \delta, t) =$$

$$\frac{m}{2\pi\mathrm{i}\hbar t\alpha}\int_0^{2\pi} d\theta' \int_0^{\infty} r'dr'\ \Psi_0(r',\theta',0)\exp\left\{\frac{\mathrm{i}m}{2\pi\hbar t\alpha^2}(r^2+r'^2)\right\}$$

$$\times\Big[\sum_n{}' \exp\left\{\frac{-\mathrm{i}mrr'}{\hbar t\alpha^2}\cos\Big(\pi + \alpha\delta - 2\pi\alpha n\Big)\right\}$$

$$-\sum_n{}' \exp\left\{\frac{-\mathrm{i}mrr'}{\hbar t\alpha^2}\cos\Big(\pi - \alpha\delta - 2\pi\alpha n\Big)\right\}\Big]. \tag{35}$$

Each sum includes all n such that the argument of the cosine is in $(-\pi,\pi)$. The range is different in each sum due to the presence of δ. More precisely, the maximum and minimum values of n are

$$\begin{aligned} n_{max} &= \left[\frac{1}{\alpha} \pm \frac{\delta}{2\pi}\right] \approx \left[\frac{1}{\alpha}\right] \ , \\ n_{min} &= \left[\pm\frac{\delta}{2\pi}\right] + 1 = \begin{cases} 1 & \text{if } + \\ 0 & \text{if } - \end{cases} \ . \end{aligned} \tag{36}$$

Therefore, if we expand Eq. (35) in powers of δ all leading terms cancel, except the one that comes from $n = 0$ in the second sum. The discontinuity in Ψ_1 is

$$\Psi_1(r, \pi + \frac{\pi}{\alpha} + \delta, t) - \Psi_1(r, \pi + \frac{\pi}{\alpha} - \delta, t) =$$

$$-\frac{m}{2\pi i \hbar t \alpha} \int_0^{2\pi} d\theta' \int_0^{\infty} r' dr' \; \Psi_0(r', \theta', 0) \exp\left\{\frac{im}{2\hbar t \alpha^2}(r^2 + r'^2 + 2rr' \cos(\alpha\delta)\right\}$$

$$\approx -\frac{m}{2\pi i \hbar t \alpha} \int_0^{2\pi} d\theta' \int_0^{\infty} r' dr' \; \Psi_0(r', \theta', 0) \exp\left\{\frac{im}{2\hbar t \alpha^2}(r + r')^2\right\} + \mathcal{O}(\delta). \quad (37)$$

It is clear that the discontinuities in Ψ_{21} and Ψ_1 cancelled out. Therefore we have shown that the wave function is continuous along the classical scattering direction due to the interference between the subdominant part of the scattered wave and the transmitted wave. It can be shown that not only the discontinuities in the scattered wave function, but also in its derivatives, are compensated by those in the transmitted wave function. The leading part of the scattered wave does not play any significant role in this interference. This situation is reminiscent of Young's theory of optical diffraction [8].

There is another relation which can be proven within this framework: if we approach the classical scattering angle $\pi + \pi/\alpha$ from below we can write, in the limit of large $mrr'/\hbar t \alpha^2$,

$$\Psi_{21}(r, \pi + \pi/\alpha - \delta, t) =$$

$$-\frac{m}{4\pi i \hbar t \alpha} \int_0^{2\pi} d\theta' \int_0^{\infty} r' dr' \; \Psi_0(r', \theta', 0) \exp\left\{\frac{im}{2\hbar t \alpha^2}(r + r')^2\right\} + \mathcal{O}(\delta),$$

$$\Psi_1(r, \pi + \pi/\alpha - \delta, t) =$$

$$\frac{m}{2\pi i \hbar t \alpha} \int_0^{2\pi} d\theta' \int_0^{\infty} r' dr' \; \Psi_0(r', \theta', 0) \exp\left\{\frac{im}{2\hbar t \alpha^2}(r + r')^2\right\} + \mathcal{O}(\delta), \quad (38)$$

where in Ψ_1 only the $n = 0$ term has been retained. The remaining terms are negligible in the asymptotic limit. Also, Ψ_{22} is much smaller than Ψ_{21} or Ψ_1 in that limit. Of course, δ is a correspondingly small angle. Therefore we can conclude that

$$\Psi_{21}(r, \pi + \pi/\alpha - \delta, t) = -\frac{1}{2}\Psi_1(r, \pi + \pi/\alpha - \delta, t) \; . \quad (39)$$

If we denote the total asymptotic wave function in this angular region, $\Psi_{21} + \Psi_1$, by Ψ_{total}, we find

$$\Psi_{\text{total}}(r, \pi + \pi/\alpha - \delta, t) = \frac{1}{2}\Psi_1(r, \pi + \pi/\alpha - \delta, t) \; . \quad (40)$$

This corresponds to the verification done in [6] of a general result in the theory of diffraction, due to Sommerfeld [8]: in the boundary between shadow and light the total light amplitude is half the transmitted amplitude.

There is a similar result for $\theta = \pi - \pi/\alpha$. The interpretation of these equations is clear: the wave packet hits the scattering centre and splits in two halves which propagate along the classical scattering angles. This is analogous to the classical motion of a bunch of particles approaching the scattering centre with zero average impact parameter.

4. Scattering of a Wave Packet: Non-Zero Impact Parameter

In this last section we generalize the previous results to non-vanishing impact parameters. The initial Gaussian wave packet is now centered at (ρ, θ_0) (polar coordinates); the impact parameter is $b = \rho \sin\theta_0$. The momentum is the same as in Eq. (27):

$$\Psi(r', \theta', 0) = \frac{1}{\sqrt{2\pi}\xi} \exp\left\{ ik_0 r' \cos\theta' - \frac{1}{4\xi^2}\left(r'^2 + \rho^2 - 2r'\rho\cos(\theta' - \theta_0)\right)\right\}. \quad (41)$$

The calculation follows the same steps as in the previous section: if we consider a scattering angle different from the classical one we must take G_2 as the relevant propagator; otherwise we take G_1, G_{21} and G_{22}.

Let us consider the first case. If the wave packet started its motion from a long distance, the scattered wave can be written as

$$\Psi(r, \theta, t) = \sqrt{\frac{i}{r}} \frac{1}{\sqrt{2\pi k_0}} \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha} + \cos\theta}\; e^{-\frac{b^2}{4\xi^2}}\, \Psi_{\text{free}}(r, \alpha^2 t) \quad . \quad (42)$$

Therefore, if $b \gg \xi$ there is no significant quantum scattering away from the classical scattering angles. If θ is equal to these angles, the relevant propagators are G_1, G_{21} and G_{22}. The contribution of G_{22} is similar to Eq. (42) and hence can be discarded, so that we are left with G_1 and G_{21}.

Let us consider that $\theta_0 = \pi{+}\delta$, where δ is a small but finite angle. When considering the wave packet in the remote past we will take $\delta \to 0$ but it will never be exactly zero. This prevents the contours in Fig. 3 from hitting the poles, and at the same time implies that G_{21} will not contribute. We recall here that this contribution to the propagator arises as a discontinuity in the integral representation of G_2 which occurs only if the contour cannot be deformed to avoid the poles in the real axis.

The contribution from G_1 depends on the sign of δ. If $\delta > 0$ the only contribution relevant in the asymptotic limit comes from $n = 0$ and $\theta = \pi{+}\pi/\alpha$ (other possibilities, like $n = 1$ and $\theta = -\pi{+}\pi/\alpha$ are physically equivalent). If $\delta < 0$ we need to take $n = 0$ and $\theta = \pi - \pi/\alpha$ instead, or any equivalent choice. This can be written compactly in the notation of Eq. (40):

$$\begin{aligned} \delta > 0 \quad &\Rightarrow \quad \Psi_{\text{total}}(r, \theta, t) = \Psi_1(r, -\pi + \pi/\alpha, t) \quad , \\ \delta < 0 \quad &\Rightarrow \quad \Psi_{\text{total}}(r, \theta, t) = \Psi_1(r, \pi - \pi/\alpha, t) \quad . \end{aligned} \quad (43)$$

These equations can be interpreted in the following way: the wave packet follows the classical trajectory of a particle with same initial position and velocity.

5. Conclusions

We can summarize our conclusions in four points:

1. The scattering amplitude coincides with the one found in [3] and [4].
2. The scattered wave packet is continuous everywhere. If the impact parameter b is not zero, it propagates like a classical particle. If $b = 0$, it hits the scattering centre and splits in two halves which propagate along the classical scattering angles, plus a scattered "spherical" wave, thus confirming the qualitative analysis in [5].
3. If the impact parameter is zero, the continuity is due to interference between the transmitted and the scattered parts of the wave function along the classical scattering directions. This is similar to the Aharonov-Bohm effect in the forward direction [1]; in both cases the wave function undergoes a self-interference at the classical scattering angles.
4. The time dependence of the scattered wave is modified due to the presence of the massive scattering centre, see for example Eq. (30). This calls for an explanation.

6. Appendix: Time-Independent Scattering

In this Appendix we show that the same method can be applied to the simpler case of time-independent scattering of plane waves in 2+1 dimensional gravity. We find a similar cancellation of discontinuities along the classical scattering angles but, this being a static problem, the cancelled terms are not subdominant in time.

Let us recall the Deser-Jackiw solution for the time-independent scattering problem [4]:

$$\begin{aligned} \Psi_{\mathrm{sc}}(r,\theta) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dy\, \mathrm{e}^{ikr\cosh y}\left[\tan\left(\frac{iy+\pi}{2\alpha}+\frac{\theta}{2}\right)-\tan\left(\frac{iy-\pi}{2\alpha}-\frac{\theta}{2}\right)\right], \\ \Psi_{\mathrm{in}}(r,\theta) &= \alpha\sum_n{}' \mathrm{e}^{-ikr\cos\alpha(\theta-(2n+1)\pi)} , \end{aligned} \tag{44}$$

where the primed sum includes only n such that $\alpha(\theta-(2n+1)\pi)\in(-\pi,\pi)$. The notation Ψ_{in} stands for the incoming wave, and Ψ_{sc} for the scattered wave. We are going to calculate the scattered wave following the procedure described in [6].

To apply this method, we write Ψ_{sc} in an alternative way by means of a trigonometric identity:

$$\Psi_{\mathrm{sc}}(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha}+\cos\left(\frac{iy}{\alpha}+\theta\right)}\,\mathrm{e}^{ikr\cosh y} . \tag{45}$$

As in the time-dependent case, if α^{-1} is an integer there is no scattered wave. If that is not the case the integral can be performed in the limit of large kr, where the leading contribution comes from the small y region. The result is

$$\Psi_{\mathrm{sc}}(r,\theta) \approx \frac{\sin\frac{\pi}{\alpha}}{2\pi}\mathrm{e}^{ikr}\int_{-\infty}^{\infty} dy\ \mathrm{e}^{\frac{1}{2}ikry^2}\frac{1}{\cos\frac{\pi}{\alpha}+\cos\theta-\frac{iy}{\alpha}\sin\theta+\mathcal{O}(y^2)} . \tag{46}$$

To proceed with the integration we will assume that $\theta \neq \pi \pm \pi/\alpha$ mod(2π), i.e., we keep away from the classical scattering angles [4]. In the large kr limit, a Gaussian integration yields

$$\Psi_{\rm sc}(r,\theta) \approx \sqrt{\frac{\rm i}{r}}\ {\rm e}^{{\rm i}kr}\frac{1}{\sqrt{2\pi k}}\ \frac{\sin\frac{\pi}{\alpha}}{\cos\frac{\pi}{\alpha}+\cos\theta}\ , \tag{47}$$

which gives the scattering amplitude found in [3] and [4]. The behaviour of the scattered wave close to the classical scattering angles can be determined as in the time-dependent analysis; the analog of the change of variables (15) in Eq. (45) is

$$\begin{aligned} y &= {\rm i}\eta\ , \\ kr &= \rho\ , \\ \theta &= -\frac{\phi}{\alpha}+\pi\ . \end{aligned} \tag{48}$$

In terms of the variables ρ and ϕ, the two physically interesting situations are:

6.1. $\rho(1+\cos\phi) \to \infty$.

This represents the large kr limit, away from the classical scattering angles,i.e., $\theta \neq \pi \pm \pi/\alpha$. The analysis in terms of S_m functions of this kinematic region coincides with the Gaussian integration of Eq. (46); the result is of course Eq. (47).

6.2. $\phi = \pi^{\pm}$.

These values of ϕ correspond to the classical scattering angles. In this case we find a discontinuous result. In the original variables r, θ, it reads

$$\begin{aligned} \Psi_{\rm sc}(r,\pi-(\pi^{\pm}/\alpha)) &= \pm\frac{1}{2}\alpha\ {\rm e}^{{\rm i}kr} - \sqrt{\frac{\rm i}{8\pi kr}}\ {\rm e}^{{\rm i}kr}\cot\frac{\pi}{\alpha} \\ &\quad +\mathcal{O}(r^{-3/2})\ . \end{aligned} \tag{49}$$

The second term has the radial structure of a scattered wave, and coincides with the result of taking the principal value in the integral representation of $\Psi_{\rm sc}$ shown in Eq. (44). The first term represents a discontinuous plane wave transmitted along the classical scattering angle, whose discontinuity will be cancelled by another contribution coming from $\Psi_{\rm in}$. The case $\phi = -\pi^{\pm}$ is no different.

The discontinuities cancel as in the time-dependent case. Let δ be a small positive angle. The discontinuity in Ψ_{sc} is

$$\Psi_{\rm sc}(r,\pi+\frac{\pi}{\alpha}+\delta) - \Psi_{\rm sc}(r,\pi+\frac{\pi}{\alpha}-\delta) = \alpha\ {\rm e}^{{\rm i}kr} + \mathcal{O}(\delta)\ , \tag{50}$$

while the discontinuity in Ψ_{in} is

$$\Psi_{\rm in}(r,\pi+\frac{\pi}{\alpha}+\delta)-\Psi_{\rm in}(r,\pi+\frac{\pi}{\alpha}-\delta)$$
$$=\alpha\Big[\sum_n{}' \exp\Big\{-ikr\cos\Big(\pi+\alpha\delta-2\pi\alpha n\Big)\Big\}$$
$$-\sum_n{}' \exp\Big\{-ikr\cos\Big(\pi-\alpha\delta-2\pi\alpha n\Big)\Big\}\Big] \quad . \tag{51}$$

Each sum includes all n such that the argument of the cosine is in $(-\pi,\pi)$. The range is different in each sum due to the presence of δ. More precisely, the maximum and minimum values of n are

$$\begin{aligned} n_{max} &= \left[\frac{1}{\alpha}\pm\frac{\delta}{2\pi}\right]\approx\left[\frac{1}{\alpha}\right] \ , \\ n_{min} &= \left[\pm\frac{\delta}{2\pi}\right]+1=\begin{cases} 1 & \text{if } + \\ 0 & \text{if } - \end{cases} \quad . \end{aligned} \tag{52}$$

Therefore the only uncancelled leading term corresponds to $n=0$ in the second sum. The discontinuity in $\Psi_{\rm in}$ is

$$\Psi_{\rm in}(r,\pi+\frac{\pi}{\alpha}+\delta)-\Psi_{\rm in}(r,\pi+\frac{\pi}{\alpha}-\delta)=-\alpha\, e^{ikr}+\mathcal{O}(\delta) \quad . \tag{53}$$

As expected, both discontinuities cancel. The wave function is continuous at the classical scattering directions. It can be shown that Sommerfeld's theorem holds also in this case, exactly as in the time-dependent case.

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