

Exact Operator Solution of the Calogero–Sutherland Model

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Abstract: The wave functions of the Calogero–Sutherland model are known to be expressible in terms of Jack polynomials. A formula which allows to obtain the wave functions of the excited states by acting with a string of creation operators on the wave function of the ground state is presented and derived. The creation operators that enter in this formula of Rodrigues-type for the Jack polynomials involve Dunkl operators.

1. Introduction

Exactly solvable models are of great help in the understanding of quantum many-body physics. The Calogero–Sutherland (CS) [1, 2, 3] model, which describes a system of N particles on a circle interacting pairwise through long range potentials, is generating a lot of attention in this connection, in particular because it provides a fully solvable model in which the ideas of fractional statistics can be tested [4]. There is thus considerable interest in identifying the algebraic structure responsible for the solvability of this model.

The spectrum of the CS Hamiltonian can be interpreted as the energy of a collection of free quasi-particles obeying a generalized exclusion principle. Recent computations [5, 6, 7] of some correlation functions have confirmed this point of view and shown that the exclusion statistics of quasi-particles and quasi-holes is consistent with the anyon statistics of the real particles. The calculation of these quantities proved possible because of the following circumstance: the eigenfunctions of the CS Hamiltonian are given in terms of Jack polynomials [8, 9, 10]. These polynomials form a basis for the ring of symmetric functions and enjoy algebraic properties that allow to carry out analytically the computation of various dynamical functions of the CS model. These polynomials also appear in related areas like the characterization of the Virasoro and W_N algebras singular vectors [11, 12] and in the construction of Yangian modules [13, 14].

We will show in this paper that the wave functions of the CS Hamiltonian and hence the Jack polynomials can be obtained by applying a string of creation

operators on the ground state wave function. After reviewing in Sect. 2, the spectrum and the eigenstates of the CS Hamiltonian, we shall present and discuss this operator solution of the CS model in Sect. 3. Proofs will be deferred mainly to Sect. 4. Conclusion and outlook will form the content of Sect. 5.

2. Spectrum and Eigenstates of the Hamiltonian

The CS model describes a system of N particles on a circle. We shall denote by L the perimeter of that circle and by $x_i, i = 1, \dots, N; 0 \leq x_i \leq L$, the positions of the particles.

2.1. *Hamiltonian and Ground State.* The quantum Hamiltonian H_{CS} is

$$H_{CS} = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2\beta(\beta - 1) \sum_{j < k} \frac{1}{d^2(x_j - x_k)}, \tag{2.1}$$

where β is a constant and

$$d(x_j - x_k) = \frac{L}{\pi} \sin \frac{\pi}{L}(x_j - x_k), \tag{2.2}$$

is the chord length between the positions of the particles j and k . Note the symmetry under the exchange of β and $1 - \beta$. For β real, that is $\beta(\beta - 1) \geq -1/4$, H_{CS} is known to be stable and to have no bound states. The momentum operator P_{CS} is $P = -i \sum_{j=1}^N \partial/\partial x_j$. The operators P and H_{CS} are self-adjoint with respect to the inner product $(f, g) = \int_0^L dx_1 \cdots \int_0^L dx_N f(x) \overline{g(x)}$. Moreover, since H_{CS} can be written in the form [12]

$$H_{CS} = \sum_{j=1}^N A_j^+(\beta) A_j(\beta) + E_0, \tag{2.3}$$

where

$$A_j(\beta) = -i \frac{\partial}{\partial x_j} + i \frac{\pi}{L} \beta \sum_{k \neq j} \cot \left[\frac{\pi}{L}(x_j - x_k) \right], \tag{2.4}$$

we see that H_{CS} is bounded from below with

$$E_0 = \frac{1}{3} \left(\frac{\pi}{L} \right)^2 \beta^2 N(N^2 - 1), \tag{2.5}$$

the ground state energy. There are a priori two wavefunctions satisfying $H_{CS}\psi_0 = E_0\psi_0$, one being annihilated by the operators $A_j(\beta)$ and the other by the operators $A_j(1 - \beta)$. Only the first is normalizable, however, for all values of β . It is of Jastrow type and given explicitly by

$$\psi_0(x) = \prod_{j < k} \sin \left[\frac{\pi}{L}(x_j - x_k) \right]^\beta. \tag{2.6}$$

The coupling β controls the statistics of the real particles in the ground state.

2.2. *Excited States.* The wave functions of the excited states are written in the form $\psi(x) = \phi(x)\psi_0(x)$, where $\phi(x)$ is required to be symmetric in order for ψ to behave like ψ_0 under the exchange of particles. It proves convenient to use the variables

$$z_j = e^{2\pi i x_j/L}. \tag{2.7}$$

In these coordinates, the ground state wave function is (up to a constant) $\Delta^\beta(z) = \prod_{j < k} (z_j - z_k)^\beta \prod_k z_k^{-\beta(N-1)/2}$ and the Schrödinger equation $H_{CS}\psi = E\psi$ is transformed into the following equation for ϕ :

$$H\phi = \left(\frac{L}{2\pi}\right)^2 (E - E_0)\phi, \tag{2.8}$$

where

$$H = \left(\frac{L}{2\pi}\right)^2 \Delta^{-\beta} H_{CS} \Delta^\beta = H_1 + \beta H_2, \tag{2.9}$$

$$H_1 = \sum_{j=1}^N \left(z_j \frac{\partial}{\partial z_j}\right)^2, \tag{2.10a}$$

$$H_2 = \sum_{j < k} \left(\frac{z_j + z_k}{z_j - z_k}\right) \left(z_j \frac{\partial}{\partial z_j} - z_k \frac{\partial}{\partial z_k}\right). \tag{2.10b}$$

The momentum operator becomes $P = \Delta^{-\beta} P \Delta^\beta = 2\pi/L \sum_j z_j \partial/\partial z_j$ and commutes with H . We thus supplement (2.8) with

$$P\phi = \kappa\phi. \tag{2.11}$$

Let $\phi' = G^q\phi$ with

$$G = \prod_j z_j, \quad q \in \mathbb{R}. \tag{2.12}$$

If ϕ obeys (2.8) and (2.11), it is easy to see that ϕ' will also be an eigenfunction of H and P with eigenvalues $(L/2\pi)^2(E - E_0) + 2Nq(L/2\pi)\kappa + (Nq)^2$ and $\kappa + Nq$ respectively. Multiplication by G thus implements Galilei boosts. Finally, one readily notices that H is S_N -invariant. This implies that the space of symmetric functions of degree $n \in \mathbb{N}$ is stable under the action of H . In fact, it is shown [5, 6, 7] that apart from factors of the form (2.12), the eigenfunctions ϕ of H and P are given in terms of a specific basis for these symmetric functions which is known as that of the Jack polynomials $J_\lambda(z; 1/\beta)$.

2.3. *Jack Polynomials.* These polynomials are labelled by partitions λ of their degree n , that is sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \dots$ such that $n = \lambda_1 + \lambda_2 + \dots$. Let λ and μ be two partitions of n . In the dominance ordering, we have $\lambda \geq \mu$ if $\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i$, for all i . Two natural bases for the space of symmetric functions are conventionally used to define the Jack polynomials:

(i) the power sum symmetric functions p_λ which in terms of the power sums

$$p_\lambda = \sum_k z_k^{\lambda_k}, \tag{2.13}$$

are given by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots, \tag{2.14}$$

(ii) the monomial symmetric functions m_λ which are

$$m_\lambda = \sum_{\text{distinct permutations}} z_1^{\lambda_1} z_2^{\lambda_2} \dots \tag{2.15}$$

To the partition λ with m_i parts equal to i , we associate the number

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots \tag{2.16}$$

We then introduce the following scalar product on the space of symmetric functions:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} z_\lambda \beta^{-l(\lambda)}, \tag{2.17}$$

where $l(\lambda)$ is the number of parts of λ . The Jack polynomials $J_\lambda(z_1, \dots, z_N; 1/\beta)$ are then uniquely defined [8,9,10] as the symmetric polynomials satisfying the following two conditions:

$$\langle J_\lambda, J_\mu \rangle = 0 \quad \text{if } \lambda \neq \mu, \tag{2.18a}$$

$$J_\lambda(z; 1/\beta) = m_\lambda + \sum_{\mu < \lambda} v_{\lambda\mu}(\beta) m_\mu. \tag{2.18b}$$

While no explicit formula has yet been obtained for these polynomials, they have been shown to obey a number of interesting properties. For instance, they are also orthogonal under the norm [10]

$$(\phi_1, \phi_2) = \int \frac{d\theta_1}{2\pi} \dots \frac{d\theta_N}{2\pi} \prod_{j < k} |(z_j - z_k)|^{2\beta} \phi_1(z) \overline{\phi_2(z)}, \quad z_j = e^{i\theta_j}, \tag{2.19}$$

which is induced from the scalar product associated to the original quantum mechanical problem ($\theta_i = 2\pi x_i/L$). We can therefore replace $\langle J_\lambda, J_\mu \rangle$ by (J_λ, J_μ) in (2.18a) to define the Jack functions, these two scalar products being proportional. Moreover, when $l(\lambda) \leq N$, the Jack polynomials $J_\lambda(z_1, \dots, z_N; 1/\beta)$ are shown [8,9,10] to obey a differential equation which coincides with (2.8)–(2.10).

2.4. Diagonalization of H . This last result can be presented following Sutherland’s argument [3] on how to triangulate the CS Hamiltonian. Since the ground state incorporates correlations, it is reasonable to expect when writing the wave functions in the form $\psi = \phi\psi_0$, that ϕ will look much like the wave functions of free particles which are symmetric monomials $m_\lambda = \sum_{\text{perm}} \prod_j z_j^{\lambda_j}$. It is easy to see that $H = H_1 + \beta H_2$ is self-adjoint with respect to the scalar product (2.19). It thus has a set of orthogonal eigenfunctions which form a basis for the symmetric functions. Consider now the action of H on m_λ . One readily finds, using (2.10) that

$$H_1 m_\lambda = \left(\sum_j \lambda_j^2 \right) m_\lambda, \quad H_2 m_\lambda = \varepsilon_2 m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu, \tag{2.20}$$

where $c_{\lambda\mu}$ are some coefficients and

$$\varepsilon_2 = \sum_{j < k} (\lambda_j - \lambda_k) = \sum_j (N + 1 - 2j) \lambda_j. \tag{2.21}$$

We thus find that H is triangular in the symmetric monomial basis. Its eigenvalues are thus

$$\left(\frac{L}{2\pi}\right)^2 (E - E_0) = \sum_{j=1}^N (\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j). \tag{2.22}$$

The corresponding eigenfunctions ϕ_λ will have an expansion of the form $\phi_\lambda = m_\lambda + \sum_{\mu < \lambda} v_{\lambda\mu} m_\mu$ and satisfy $(\phi_\lambda, \phi_\mu) = 0$. Since these two conditions uniquely define the Jack polynomials, we must have $\phi_\lambda(z) = J_\lambda(z; 1/\beta)$.

To sum up, allowing for the Galilean boosts, the eigenfunctions of the Calogero–Sutherland Hamiltonian are

$$\psi_{\lambda,q}(z) = \left(\prod_{i=1}^N z_i\right)^{q-(N-1)\beta/2} \prod_{i < j} (z_i - z_j)^\beta J_\lambda(z; 1/\beta), \tag{2.23}$$

where q is an arbitrary real number and λ ranges over all partitions of all non-negative integers such that $l(\lambda) \leq N - 1$. The number of parts in λ is restricted to be strictly smaller than the number of particles in order to avoid double counting. Indeed, it is immediate to convince oneself that the solution $(\prod_{i=1}^N z_i) J_\lambda(z; 1/\beta)$ of (2.18) is actually the solution $J_{\lambda+1}(z; 1/\beta)$, where $\lambda + 1 = (\lambda_1 + 1, \lambda_2 + 1, \dots)$. We therefore set $\lambda_N = 0$ absorbing any value that this part might take in the arbitrary Galilei transformation parametrized by q .

The eigenvalues κ_λ and E_λ that P and H_{CS} have when acting on the wavefunctions (2.23) can be nicely presented if one introduces the quantities

$$\kappa_i = \frac{2\pi}{L} [\lambda_i + \beta(N + 1 - 2i) + q]. \tag{2.24}$$

In terms of these

$$\kappa_\lambda = \frac{2\pi}{L} (n + Nq) = \sum_{i=1}^N \kappa_i, \tag{2.25}$$

$$E_\lambda = \sum_{i=1}^N \kappa_i^2. \tag{2.26}$$

The spectrum of the CS model is that of free quasi-particles with quasi-momenta κ_i . The neighboring quasi-momenta satisfy $\kappa_i - \kappa_{i+1} \geq 2\pi\beta/L$ which indicates that the quantum excitations of the CS system obey a generalized exclusion statistics.

3. Operator Solution

We shall now present a formula that gives the eigenfunctions of the CS Hamiltonian and hence the Jack polynomials, through the action of a string of creation operators on the ground state wave functions. The proof of this formula will be given in the next section.

3.1. Dunkl Operators. The creation operators that will provide this operator solution of the CS model will be constructed in terms of the so-called Dunkl operators ∇_i [15]. These operators are defined as follows:

$$\nabla_i = \frac{\partial}{\partial z_i} + \beta \sum_{\substack{j=1 \\ j \neq i}}^N \frac{1}{(z_i - z_j)} (1 - K_{ij}), \tag{3.1}$$

where $K_{ij} = K_{ji}, K_{ij}^2 = 1$, is the operator that permutes the variables z_i and z_j :

$$K_{ij}z_j = z_iK_{ij} . \tag{3.2}$$

The Dunkl operators are easily found to have the following properties:

$$[\nabla_i, \nabla_j] = 0 , \tag{3.3a}$$

$$K_{ij}\nabla_j = \nabla_iK_{ij} , \tag{3.3b}$$

$$[\nabla_i, z_j] = \delta_{ij} \left(1 + \beta \sum_{l=1}^N K_{il} \right) - \beta K_{ij} . \tag{3.3c}$$

We now define

$$D_i \equiv z_i \nabla_i . \tag{3.4}$$

In terms of these operators, the operator H given in (2.9) and (2.10) takes a remarkably simple form:

$$H = \text{Res} \sum_{i=1}^N D_i^2 , \tag{3.5}$$

where $\text{Res}X$ means that the action of X is restricted to symmetric functions of the variables z_1, \dots, z_N . Actually, H belongs to a set of mutually commuting operators. Being completely integrable, the CS system admits N functionally independent constants of motion that are in involution [16]. Modulo conjugation by Δ^β (see (2.9)), these are

$$L_j = \text{Res} \sum_{i=1}^N (D_i)^j, \quad j = 1, \dots, N ,$$

$$[L_k, L_j] = 0, \quad L_2 = H . \tag{3.6}$$

3.2. Main Result. We need additional notation to present our main result. Let $J = \{j_1, j_2, \dots, j_l\}$ be a set of cardinality $|J| = l$ made of integers such that

$$j_\kappa \in \{1, 2, \dots, N\}, \quad j_1 < j_2 < \dots < j_l . \tag{3.7}$$

Introduce the operators $D_{k,J}$ labelled by these sets J and by non-negative integers k

$$D_{k,J} = (D_{j_1} + k\beta)(D_{j_2} + (k+1)\beta) \cdots (D_{j_l} + (k+l-1)\beta) . \tag{3.8}$$

We now set

$$B_{i,J}^+ = \sum_{J' \subset J; |J'|=i} z_{J'} D_{1,J'} , \quad i < N , \tag{3.9}$$

with

$$z_{J'} = \prod_{i \in J'} z_i . \tag{3.10}$$

The sum in (3.9) is over all subsets J' of J that are of cardinality i . When $J = \{1, \dots, N\}$, we shall write in short form:

$$B_i^+ \equiv B_{i, \{1, \dots, N\}}^+ , \quad i < N . \tag{3.11}$$

We shall also take

$$B_N^+ = z_1 z_2 \cdots z_N . \tag{3.12}$$

This last operator is identical to the Galilei boost operator of (2.12).

We can now state the following result.

Theorem 3.1. *The Jack polynomials $J_\lambda(z; 1/\beta)$ associated to partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{N-1})$ are given by*

$$J_\lambda(z; 1/\beta) = c_\lambda^{-1} (B_{N-1}^+)^{\lambda_{N-1}} \dots (B_2^+)^{\lambda_2 - \lambda_3} (B_1^+)^{\lambda_1 - \lambda_2} \cdot 1, \tag{3.13}$$

with the constant c_λ equal to

$$c_\lambda = \prod_{k=1}^{N-1} c_k(\lambda_1, \dots, \lambda_{k+1}; \beta), \tag{3.14}$$

where $\lambda_N \equiv 0$ and

$$c_k(\lambda_1, \dots, \lambda_{k+1}; \beta) = (\beta)_{\lambda_k - \lambda_{k+1}} (2\beta + \lambda_{k-1} - \lambda_k)_{\lambda_k - \lambda_{k+1}} \dots (k\beta + \lambda_1 - \lambda_k)_{\lambda_k - \lambda_{k+1}}. \tag{3.15}$$

In (3.15), $(\beta)_n$ stands for the Pochhammer symbol that is $(\beta)_n = \beta(\beta + 1) \dots (\beta + n - 1)$, $(\beta)_0 \equiv 1$.

We can of course conjugate the creation operators with the ground state wave function $\psi_0(z) = (\prod_{i=1}^N z_i)^{-(N-1)\beta/2} \prod_{i < j} (z_i - z_j)^\beta$ and write

$$\tilde{B}_i^+ = \psi_0 B_i^+ \psi_0^{-1}. \tag{3.16}$$

In view of (2.23) and (3.13), we then have the following formula.

Corollary 3.2. *The eigenfunctions of the CS Hamiltonian are given by*

$$\psi_{\lambda,q}(z) = c_\lambda^{-1} (\tilde{B}_N^+)^q (\tilde{B}_{N-1}^+)^{\lambda_{N-1}} \dots (\tilde{B}_2^+)^{\lambda_2 - \lambda_3} (\tilde{B}_1^+)^{\lambda_1 - \lambda_2} \psi_0(z). \tag{3.17}$$

This shows how the wave functions $\psi_{\lambda,q}$ of the excited states can be obtained by applying iteratively creation operators on the ground state wave function of the system.

3.3. Remarks. Formulas (3.13) and (3.14) will be proved in the next section. Let us here comment on some of their features.

3.3.1. It is not immediately obvious that the right-hand side of (3.13) yields symmetric polynomials of degree $n = \lambda_1 + \dots + \lambda_{N-1}$ in the variables z_1, \dots, z_N . This however is easily seen to follow from the properties of the Dunkl operators. We shall often use the notation $\text{Res}^{\{i,j,k,\dots\}} X$ or $\text{Res}^J X$ to indicate that X is taken to act on functions that are symmetric in the variables z_i, z_j, z_k, \dots or z_{j_1}, z_{j_2}, \dots with $j_\kappa \in J$, respectively. When the restriction will be taken over all the N variables of the system, we shall simply use $\text{Res} X$. From the identities (3.3), it is checked that the operators D_i satisfy the commutation relations

$$[D_i, D_j] = \beta(D_j - D_i)K_{ij}. \tag{3.18}$$

With m some integer, it is then straightforward to verify that

$$\text{Res}^{\{i,j\}} (D_i + m\beta)(D_j + (m + 1)\beta) = \text{Res}^{\{i,j\}} (D_j + m\beta)(D_i + (m + 1)\beta). \tag{3.19}$$

It follows that $\text{Res}^J D_{J,k} = \text{Res}^J (D_{j_1} + k\beta) \dots (D_{j_l} + (k + l - 1)\beta)$ is invariant under the permutations of the variables $z_{j_\kappa}, j_\kappa \in J$ and that this operator therefore

leaves invariant the space of symmetric functions in these variables. Recalling how the B_i^+ are constructed in (3.9) in terms of the operators $D_{1,J}$, it is clear that $\phi_\lambda = (B_{N-1}^+)^{\lambda_{N-1}} \dots (B_2^+)^{\lambda_2 - \lambda_3} (B_1^+)^{\lambda_1 - \lambda_2} \cdot 1$ is a symmetric function of the variables z_1, \dots, z_N . That it is a homogeneous polynomial of degree n is readily seen by observing that the operators $D_{J,k}$ have scaling dimension zero and hence that $B_i^+ \rightarrow \rho^i B_i^+$ when $z_i \rightarrow \rho z_i, i = 1, \dots, N$. The degree of ϕ_λ is thus $\lambda_1 - \lambda_2 + 2(\lambda_2 - \lambda_3) + \dots + (N - 1)\lambda_{N-1} = n$.

3.3.2. It might be useful to give an example. To this end, let us take the number of particles $N = 3$ and consider a solution of degree $n = 4$. According to formulas (3.13)–(3.15), the Jack polynomial associated to the partition $\lambda = (3, 1)$ is given by

$$J_{(3,1)}(z_1, z_2, z_3; 1/\beta) = c_{(3,1)}^{-1} B_2^+ (B_1^+)^2 \cdot 1, \tag{3.20}$$

with

$$\begin{aligned} B_1^+ &= z_1(D_1 + \beta) + z_2(D_2 + \beta) + z_3(D_3 + \beta), \\ B_2^+ &= z_1 z_2 (D_1 + \beta)(D_2 + 2\beta) + z_1 z_3 (D_1 + \beta)(D_3 + 2\beta) \\ &\quad + z_2 z_3 (D_2 + \beta)(D_3 + 2\beta), \end{aligned} \tag{3.21}$$

and

$$c_{(3,1)} = 2\beta^2(\beta + 1)^2. \tag{3.22}$$

3.3.3. We shall conclude this section by describing yet another way to obtain the spectrum and to characterize the wave functions of the CS model with the help of operators closely related to the operators D_i introduced in (3.4). Let

$$\hat{D}_i = D_i + \beta(i - 1) - \beta \sum_{j < i} (1 - K_{ij}), \quad i = 1, \dots, N. \tag{3.23}$$

Remarkably, this defines a set of N commuting operators

$$[\hat{D}_i, \hat{D}_j] = 0. \tag{3.24}$$

We also have

$$\begin{aligned} [K_{i+1}, \hat{D}_k] &= 0 \quad \text{if } k \neq i, i + 1, \\ K_{i+1} \hat{D}_i - \hat{D}_{i+1} K_{i+1} &= \beta. \end{aligned} \tag{3.25}$$

In terms of these the Hamiltonian H of (2.9) reads

$$H = \text{Res } \hat{H}, \tag{3.26}$$

with

$$\hat{H} = \sum_{i=1}^N \{ \hat{D}_i^2 - (N - 1)\beta \hat{D}_i \} + \frac{1}{6} N(N - 1)(N - 2)\beta^2. \tag{3.27}$$

Clearly, $[\hat{H}, \hat{D}_i] = 0$ for all i . We may thus obtain the eigenfunctions of \hat{H} by diagonalizing simultaneously all the \hat{D}_i . The eigenvalues and eigenfunctions of these operators are simply constructed by observing that they are triangular on the set of monomials $\hat{m}_\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \dots z_N^{\lambda_N}$ associated to the partitions $\lambda = (\lambda_1, \dots, \lambda_N)$ of n . (Note that the monomials \hat{m}_λ are not symmetrized.) The functions χ_λ such that

$$\hat{D}_i \chi_\lambda = \delta_i^\lambda \chi_\lambda, \tag{3.28}$$

have the form

$$\chi_\lambda = \hat{m}_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} \hat{m}_\mu, \quad |\lambda| = |\mu| = n, \tag{3.29}$$

and from the action of \hat{D}_i on \hat{m}_λ , the eigenvalues δ_i^λ are found to be

$$\delta_i^\lambda = \lambda_i + \beta(N - i). \tag{3.30}$$

The eigenvalues of \hat{H} are then readily evaluated from (3.27) and one gets

$$\hat{H}\chi_\lambda = \sum_{j=1}^N (\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j)\chi_\lambda. \tag{3.31}$$

Comparing with (2.22), we see that H and \hat{H} have the same spectrum.

Using the relations (3.25), we show that

$$K_{ii+1}\hat{H} = \hat{H}K_{ii+1}. \tag{3.32}$$

Define now

$$\phi_\lambda = \sum_{\text{permutations}} \chi_\lambda. \tag{3.33}$$

Since all permutations can be expressed as products of transpositions, owing to (3.32), we see that

$$\hat{H}\phi_\lambda = \sum_{\text{permutations}} \hat{H}\chi_\lambda. \tag{3.34}$$

We thus find that the symmetric function ϕ_λ which has triangular expansion on the monomial basis $\{m_\lambda; |\lambda| = n\}$ is actually an eigenfunction of \hat{H} , and also, of $H = \text{Res } \hat{H}$ obviously, with eigenvalue equal to $\sum_{j=1}^N (\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j)$. We thus conclude that it must be proportional to the Jack polynomial

$$J_\lambda(z_1, \dots, z_N; 1/\beta) \sim \sum_{\text{permutations}} \chi_\lambda(z_1, \dots, z_N; 1/\beta). \tag{3.35}$$

This provides an alternative to the method described in Sect. 2 for obtaining the spectrum and eigenfunctions of the CS Hamiltonian.

4. Proofs

4.1. Outline. After preliminary remarks, we shall proceed to give the proof of Theorem 3.1 which we stated in the last section. To this end, we shall require additional definitions. Keeping with the notation of Sect. 3, we introduce

$$N_{i,J} = \sum_{\substack{J' \subset J \\ |J'|=i}} D_{0,J'}, \quad i = 1, \dots, N, \tag{4.1}$$

and the shorthand notations:

$$N_i \equiv N_{i,\{1, \dots, N\}}, \quad \tilde{N}_{i,J} \equiv N_{i,\{J_1, \dots, J_i\}} = D_{0,\{J_1, \dots, J_i\}},$$

$$\tilde{N}_i \equiv N_{i,\{1, \dots, i\}} = D_{0,\{1, \dots, i\}}. \tag{4.2}$$

We remind the reader that the notation $\text{Res}^J X$ indicates that the operator is taken to act on functions that are symmetric under the exchanges of the variables $z_{j\kappa}$, $j\kappa \in J$. At times, we shall need to vary the set of variables entering in the Dunkl operators ∇_i from which the operators $B_{i,J}^+$ and $N_{i,J}$ are built. To specify this, if S is the set of integers labelling the variables, we shall introduce a superscript S writing for instance

$$\nabla_i^{(S)} = \frac{\partial}{\partial z_i} + \beta \sum_{\substack{j \in S \\ j \neq i}} \frac{1}{z_i - z_j} (1 - K_{ij}), \quad i \in S. \tag{4.3}$$

The same superscript will be added, when necessary, to the symbols of the operators constructed from these $\nabla_i^{(S)}$. When S consists of the first M integers, $S = \{1, 2, \dots, M\}$ we shall write $N_{i,J}^{(S)} = N_{i,J}^{\{M\}}$. Generally, when this superscript is omitted, it is understood that the operators depend on the original N variables z_1, z_2, \dots, z_N . The only exception will occur in Subsect. 4.2 where we shall need to use $M \geq N$ variables and shall also drop the superscript at some point.

By the reasoning of 3.3.1, it is clear that the operators $\text{Res}^J B_{i,J}^+$ and $\text{Res}^J N_{i,J}$ are invariant under the permutations of the variables $z_{j\kappa}$, $j\kappa \in J$. We may also remark that

$$[\text{Res}^J N_{i,J}, \text{Res}^J N_{k,J}] = 0. \tag{4.4}$$

This is seen as follows. The operators $\text{Res}^J N_{i,J}^{(J)}$ are completely symmetric under the permutations of the indices of the operators $D_i^{(J)}$. They must therefore be combinations of the invariants $L_i^{(J)} = \sum_{k \in J} (D_k^{(J)})^i$, $[L_i^{(J)}, L_j^{(J)}] = 0$ and as a result, must also commute among themselves, $[\text{Res}^J N_{i,J}^{(J)}, \text{Res}^J N_{k,J}^{(J)}] = 0$. Since $[\text{Res}^J N_{i,J}, \text{Res}^J N_{k,J}] = \text{Res}^J [N_{i,J}, N_{k,J}]$ and $[\text{Res}^J N_{i,J}^{(J)}, \text{Res}^J N_{k,J}^{(J)}] = \text{Res}^J [N_{i,J}^{(J)}, N_{k,J}^{(J)}]$ have the same form in terms of Dunkl operators and since the commutation relations between these operators are not affected by the number of variables, (4.4) must then be true.

Let

$$\varphi_{(\lambda_1, \dots, \lambda_t, 0, \dots)} \equiv (B_i^+)^{\lambda_i} \dots (B_1^+)^{\lambda_1 - \lambda_2} \varphi_0, \tag{4.5}$$

with $\varphi_0 = 1$. We shall show that $J_{(\lambda_1, \dots, \lambda_{N-1})}(z_i; 1/\beta) = c_\lambda^{-1} \varphi(\lambda_1, \dots, \lambda_{N-1})$ by proving that functions $\varphi_{(\lambda_1, \dots, \lambda_{N-1})}$ thus constructed are simultaneous eigenfunctions of the Hamiltonian H given in (2.8)–(2.10) and of the momentum operator $P = \frac{2\pi}{L} \sum_j z_j \partial / \partial z_j$. This is in essence the content of Theorem 3.1. There will remain to prove that the constant C_λ is indeed given by (3.14) and (3.15). This will be done in Subsect. 4.4. We have already observed in Subsect. 3.3.1 that $\varphi_\lambda \equiv \varphi_{(\lambda_1, \dots, \lambda_{N-1})}$ is S_N -invariant and that it is a homogeneous polynomial of degree $|\lambda| = n$ in the variables z_1, \dots, z_N : $P\varphi_\lambda = \frac{2\pi n}{L} \varphi_\lambda$. Since φ_λ is symmetric, this last property can be expressed as follows:

$$\left(\sum_{j=1}^N D_j \right) \varphi_{(\lambda_1, \dots, \lambda_{N-1})} = \left(\sum_{j=1}^{N-1} \lambda_j \right) \varphi_{(\lambda_1, \dots, \lambda_{N-1})}. \tag{4.6}$$

Theorem 3.1 will be proved by establishing the following proposition.

Proposition 4.1.

$$[H, B_i^+] \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = B_i^+ \left\{ 2 \sum_{j=1}^{N-1} \lambda_j + i + \beta i(N - i) \right\} \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)},$$

$$\forall i \in \{1, \dots, N - 1\}.$$
(4.7)

If we assume that this proposition is true, it is then readily seen that the successive applications of the operators $B_k^+, k = 1, \dots, N - 1$, on φ_0 build the eigenfunctions $\varphi_{(\lambda_1, \lambda_2, \dots)}$ of H . Let $\varepsilon_{(\lambda_1, \lambda_2, \dots)}$ be their eigenvalues. Since

$$B_i^+ \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = \varphi_{(\lambda_1+1, \dots, \lambda_i+1, 0, \dots)},$$
(4.8)

by definition, (4.7) is equivalent to

$$H \varphi_{(\lambda_1+1, \dots, \lambda_i+1, 0, \dots)}$$

$$= \left\{ \varepsilon_{(\lambda_1, \dots, \lambda_i, 0, \dots)} + 2 \sum_{j=1}^{N-1} \lambda_j + i + \beta i(N - i) \right\} \varphi_{(\lambda_1+1, \dots, \lambda_i+1, 0, \dots)}.$$
(4.9)

Iterating with this formula, starting from the function $\varphi_0 = 1$ for which $H\varphi_0 = 0$, one recursively finds that

$$H \varphi_{(\lambda_1, \dots, \lambda_{N-1})} = \left\{ \sum_{j=1}^{N-1} (\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j) \right\} \varphi_{(\lambda_1, \dots, \lambda_{N-1})},$$
(4.10)

which is the desired result.

Instead of proving Proposition 4.1 directly, we observe that it follows from Propositions 4.2 and 4.3 given below.

Proposition 4.2.

$$\tilde{N}_{i+1, J} \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = 0, \quad 1 \leq i \leq N - 1.$$
(4.11)

Proposition 4.3.

$$[H, B_i^+] \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = \left\{ B_i^+ \left[2 \sum_{j=1}^N D_j + i + \beta i(N - i) \right] + \sum_J G_J \tilde{N}_{i+1, J} \right\} \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)},$$

$$1 \leq i \leq N - 1,$$
(4.12)

where G_J are certain unspecified expressions.

It is indeed seen with the help of (4.6), that Proposition 4.3 is equivalent to Proposition 4.1 if Proposition 4.2 is true. The next two subsections are devoted to the proofs of these Propositions (4.2 and 4.3) which imply Proposition 4.1. We proceed by induction mostly; the symmetry properties of our constructs are extensively used and formula (3.19) is for instance called upon repeatedly. The constants c_λ which relate the functions φ_λ to the monic Jack polynomials J_λ are determined in the last subsection to complete the proof of Theorem 3.1.

4.2. *Proof of Proposition 4.2.* Owing to the fact that $\varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)}$ is a symmetric function of the variables z_1, \dots, z_N , it suffices to show that

$$\tilde{N}_{i+1} \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} = 0, \tag{4.13}$$

in order to prove (4.11). Equation (4.13) in turn, is seen to follow from the relation

$$\text{Res}[\tilde{N}_{i+1}, B_k^+] \sim \text{Res} \tilde{N}_{k+1}, \quad \forall k < i + 1, \tag{4.14}$$

where by \sim we mean that the term on the right-hand side can be multiplied on the left by some non-singular operator. In fact, using (4.14) iteratively, one finds that $\tilde{N}_{i+1} \varphi_{(\lambda_1, \dots, \lambda_i, 0, \dots)} \sim \tilde{N}_{j+1} \varphi_0$ with j the smallest integer such that $\lambda_j - \lambda_{j+1} \neq 0$. Since $\tilde{N}_{j+1} \varphi_0 = 0, \forall j$, (4.13) is thus implied by (4.14).

With the help of (3.19), it is easy to show that

$$\text{Res}^{\{i+1\}} \tilde{N}_{i+1} = \text{Res}^{\{i+1\}} \left(D_1 K_{1i+1} + \beta \sum_{j=2}^i K_{ji+1} + \beta \right) \tilde{N}_i, \tag{4.15}$$

and hence that

$$\text{Res}^{\{i+1\}} \tilde{N}_{i+1} \sim \text{Res}^{\{i+1\}} \tilde{N}_i. \tag{4.16}$$

Equation (4.14) will therefore be true if

$$\text{Res}[\tilde{N}_{k+1}, B_k^+] \sim \text{Res} \tilde{N}_{k+1}, \tag{4.17}$$

holds. We shall actually prove a stronger result, namely:

Proposition 4.4.

$$\text{Res}^{\{N\}} [\tilde{N}_{k+1}^{\{M\}}, B_k^{\{M\}}] \sim \text{Res}^{\{N\}} \tilde{N}_{k+1}^{\{M\}}, \quad \forall M \geq N. \tag{4.18}$$

As already mentioned, the superscript $\{M\}$ indicates that we are using Dunkl operators that not only depend on the variables z_1, \dots, z_N but also on the variables z_{N+1}, \dots, z_M . (It should be noted that $B_k^{\{M\}}$ is symmetric under S_N but not under S_M .) Clearly, proving Proposition 4.4 is tantamount to proving Proposition 4.2.

For the remainder of Subsect. 4.2, the Dunkl operators entering our various expressions will always be taken to depend on the variables z_1, \dots, z_M . With this understood, we shall omit the superscript $\{M\}$ in the following.

It is possible to isolate the parts of B_k^+ involving the indices $\{1, \dots, k + 1\}$. From the definition of B_k^+ , one has the result:

Lemma 4.5.

$$B_k^+ = \sum_{l=0}^k \sum_{\substack{J \subset \{k+2, \dots, N\} \\ |J|=l}} z_J B_{k-l, \{1, \dots, k+1\}}^+ D_{k-l+1, J}, \tag{4.19}$$

with $B_{0,J}^+ \equiv 1$ and when $|J| = 0, z_J \equiv 1$ and $D_{k+1, J} \equiv 1$.

We shall also make use of the following formulas.

Lemma 4.6.

$$(i) [D_i, z_J] = -\beta z_i \sum_{j \in J} z_{J \setminus \{j\}} K_{ij}, \quad i \notin J, \tag{4.20a}$$

$$(ii) [D_i, z_J] = z_J \left(1 + \beta \sum_{\substack{j \in J \\ j \in \{1, \dots, M\}}} K_{ij} \right), \quad i \in J, \tag{4.20b}$$

$$(iii) \text{Res}^{\{k+1\}}[\tilde{N}_{k+1}, z_l] \\ = \text{Res}^{\{k+1\}}(-\beta)(z_1 K_{12} K_{23} \cdots K_{kk+1} K_{k+1l} + \cdots + z_{k+1} K_{k+1l}) \tilde{N}_k \\ \sim \text{Res}^{\{k+1\}} \tilde{N}_k, \quad l \notin \{1, \dots, k+1\}. \tag{4.20c}$$

Equations (4.20a) and (4.20b) are immediately obtained from the definition of the operators D_i . Result (i) expresses the fact that the commutator of D_i with products of the form $z_{j_1} z_{j_2} \cdots z_{j_l}$ that exclude z_i , yields expressions which have as factors on the left, similar products with z_i replacing one of the initial variables. Formula (ii) shows that the commutator of D_i with products of the form $z_{j_1} z_{j_2} \cdots z_{j_l}$ involving z_i , gives expressions having as a factor on the left the same products $z_{j_1} z_{j_2} \cdots z_{j_l}$ of the variables.

Property (iii) is easily derived by induction. It shows that on symmetric functions of z_1, \dots, z_{k+1} , commuting \tilde{N}_{k+1} with a variable z_l not belonging to this set, has the effect of giving an operator having \tilde{N}_k as a factor on the right.

In the following, we shall need to identify in various expressions, the terms that do not have z_1 appearing as an explicit factor on the left. If X represents one such quantity of interest, the terms in question will be denoted by $X|_{z_1 \sim 0}$. A result that will soon be useful is:

Lemma 4.7.

$$([\tilde{N}_{l+1}, z_2 \cdots z_n] + z_2 \cdots z_n [D_1, D_{1, \{2, \dots, l+1\}}]) |_{z_1 \sim 0} \sim D_1, \\ \forall M \geq n \geq l+1. \tag{4.21}$$

The proof is done by induction. That (4.21) is true when $l = 1$ is easily seen from

$$([\tilde{N}_2, z_2 \cdots z_n] + z_2 \cdots z_n [D_1, D_2]) |_{z_1 \sim 0} \\ = z_2 \cdots z_n \left(1 + \beta K_{12} + \beta \sum_{i=3}^M K_{2i} \right) D_1 \sim D_1. \tag{4.22}$$

Now in general one has,

$$([\tilde{N}_{l+1}, z_2 \cdots z_n] + z_2 \cdots z_n [D_1, D_{1, \{2, \dots, l+1\}}]) |_{z_1 \sim 0} \\ = ([\tilde{N}_l, z_2 \cdots z_n](D_{l+1} + l\beta) + \tilde{N}_l [D_{l+1}, z_2 \cdots z_n] \\ + z_2 \cdots z_n [D_1, D_{1, \{2, \dots, l\}}](D_{l+1} + l\beta) \\ + z_2 \cdots z_n D_{1, \{2, \dots, l\}} [D_1, D_{l+1}]) |_{z_1 \sim 0}. \tag{4.23}$$

It is easy to check that

$$\begin{aligned}
 \tilde{N}_l[D_{l+1}, z_2 \cdots z_n] &= z_2 \cdots z_n \tilde{N}_l \left(1 + \beta K_{1l+1} + \beta \sum_{i=n+1}^M K_{il+1} \right) \\
 &\quad + [\tilde{N}_l, z_2 \cdots z_n] \left(1 + \beta K_{1l+1} + \beta \sum_{i=n+1}^M K_{il+1} \right) \\
 &= z_2 \cdots z_n \tilde{N}_l(\beta K_{1l+1}) + [\tilde{N}_l, z_2 \cdots z_n] \left(1 + \beta K_{1l+1} + \beta \sum_{i=n+1}^M K_{il+1} \right) \\
 &\quad + z_2 \cdots z_n (D_{1, \{2, \dots, l\}} D_1 + [D_1, D_{1, \{2, \dots, l\}}]) \left(1 + \beta \sum_{i=n+1}^M K_{il+1} \right), \tag{4.24}
 \end{aligned}$$

and that

$$\begin{aligned}
 D_{1, \{2, \dots, l\}} [D_1, D_{l+1}] &= D_{1, \{2, \dots, l\}} (\beta K_{1l+1} D_1) \\
 &\quad - \tilde{N}_l(\beta K_{1l+1}) + [D_1, D_{1, \{2, \dots, l\}}] (\beta K_{1l+1}). \tag{4.25}
 \end{aligned}$$

Substituting (4.24) and (4.25) in (4.23) and dropping terms which have D_1 already on the right, we get

$$\begin{aligned}
 &([\tilde{N}_{l+1}, z_2 \cdots z_n] + z_2 \cdots z_n [D_1, D_{1, \{2, \dots, l+1\}}]) \Big|_{z_1 \sim 0} \\
 &= \left(\{[\tilde{N}_l, z_2 \cdots z_n] + z_2 \cdots z_n [D_1, D_{1, \{2, \dots, l\}}]\} \right. \\
 &\quad \left. \times \left(D_{l+1} + l\beta + 1 + \beta K_{1l+1} + \beta \sum_{i=n+1}^M K_{il+1} \right) \right) \Big|_{z_1 \sim 0}. \tag{4.26}
 \end{aligned}$$

By hypothesis, the term in curly brackets has D_1 occurring on the right and since

$$D_1(D_{l+1} + \beta K_{1l+1}) = (D_{l+1} + \beta K_{1l+1})D_1, \tag{4.27}$$

Lemma 4.7 is thus shown to hold.

As a step toward establishing Proposition 4.4, we shall prove the following result.

Proposition 4.8.

$$\text{Res}^{\{k+1\}} [\tilde{N}_{k+1}, B_{k, \{1, \dots, k+1\}}^+] \sim \text{Res}^{\{k+1\}} \tilde{N}_{k+1}. \tag{4.28}$$

Note that (4.28) is a special case of (4.18) with $N = k + 1$.

Let,

$$z_1 \cdots \hat{z}_i \cdots z_{k+1} = \prod_{\substack{l=1 \\ l \neq i}}^{k+1} z_l. \tag{4.29}$$

From the identities (i) and (ii) of Lemma 4.6, it is immediate to see that

$$[\tilde{N}_{k+1}, z_1 \cdots \hat{z}_i \cdots z_{k+1}] = \sum_{j=1}^{k+1} z_1 \cdots \hat{z}_j \cdots z_{k+1} \mathcal{E}_j, \tag{4.30}$$

with \mathcal{E}_j quantities involving the operators D_l and $K_{lm}, 1 \leq l, m \leq k + 1$. It thus follows, given the definition of $B_{k,\{1,\dots,k+1\}}^+$, that all the terms in the expression of the commutator $[\tilde{N}_{k+1}, B_{k,\{1,\dots,k+1\}}^+]$ will have on the left a factor consisting in the product of k distinct variables taken among the set $\{z_1, z_2, \dots, z_{k+1}\}$. Since \tilde{N}_{k+1} and $B_{k,\{1,\dots,k+1\}}^+$ are invariant under permutations of these variables, in order to prove Proposition 4.8, it will suffice to show that the term in the expression of this commutator which is multiplied on the left by $z_2 \cdots z_{k+1}$ has on the right, the operator $\text{Res}^{\{k+1\}} \tilde{N}_{k+1}$. In other words, proving that

$$\text{Res}^{\{k+1\}}[\tilde{N}_{k+1}, B_{k,\{1,\dots,k+1\}}^+] |_{z_1 \sim 0} \sim \text{Res}^{\{k+1\}} \tilde{N}_{k+1}, \tag{4.31}$$

will establish Proposition 4.8. We proceed by induction. When $k = 1$ we have,

$$\text{Res}^{\{2\}}[\tilde{N}_2, B_{1,\{1,2\}}^+] |_{z_1 \sim 0} = \text{Res}^{\{2\}} z_2 \left(1 + \beta \sum_{j=3}^M K_{2j} \right) \tilde{N}_2 \sim \text{Res}^{\{2\}} \tilde{N}_2. \tag{4.32}$$

We now suppose that (4.31) is true for all k smaller than l hoping that it is satisfied for $k = l$ as a consequence. Let us cast $B_{l,\{1,\dots,l+1\}}^+$ in a form where the part involving z_1 explicitly is isolated:

$$B_{l,\{1,\dots,l+1\}}^+ = z_1 B_{l-1,\{2,\dots,l+1\}}^+ (D_1 + l\beta) + z_2 \cdots z_{l+1} D_{1,\{2,\dots,l+1\}}. \tag{4.33}$$

With the help of (4.33) and since $D_1 \cdot D_{1,\{2,\dots,l+1\}} = \tilde{N}_{l+1}$, we can write:

$$\begin{aligned} & \text{Res}^{\{l+1\}}[\tilde{N}_{l+1}, B_{1,\{1,\dots,l+1\}}^+] |_{z_1 \sim 0} \\ &= \text{Res}^{\{l+1\}}(D_1 [D_{1,\{2,\dots,l+1\}}, z_1] B_{l-1,\{2,\dots,l+1\}}^+ (D_1 + l\beta) \\ &+ \{[\tilde{N}_{l+1}, z_2 \cdots z_{l+1}] + z_2 \cdots z_{l+1} [D_1, D_{1,\{2,\dots,l+1\}}]\} D_{1,\{2,\dots,l+1\}}) |_{z_1 \sim 0}. \end{aligned} \tag{4.34}$$

From Lemma 4.7, we see that the term in curly brackets has the operator D_1 as the last factor on the right. It thus remains to show that

$$\text{Res}^{\{l+1\}}(D_1 [D_{1,\{2,\dots,l+1\}}, z_1] B_{l-1,\{2,\dots,l+1\}}^+ (D_1 + l\beta)) |_{z_1 \sim 0} \sim \text{Res}^{\{l+1\}} \tilde{N}_{l+1}, \tag{4.35}$$

to complete the proof of Proposition 4.8. A result analogous to (4.20c) which is again straightforwardly proved by induction is now helpful:

$$\begin{aligned} & \text{Res}^{\{2,\dots,l+1\}} [D_{1,\{2,\dots,l\}}, z_1] \\ &= \text{Res}^{\{2,\dots,l+1\}} (-\beta)(z_2 K_{23} \cdots K_{ll+1} K_{l+1} + \cdots + z_{l+1} K_{l+1}) D_{1,\{2,\dots,l\}}. \end{aligned} \tag{4.36}$$

Since $B_{l-1,\{2,\dots,l+1\}}^+ (D_1 + l\beta)$ is invariant under the permutations of the indices $\{2, \dots, l + 1\}$, we find with the help of (4.36) that

$$\begin{aligned} & \text{Res}^{\{l+1\}}(D_1 [D_{1,\{2,\dots,l+1\}}, z_1] B_{l-1,\{2,\dots,l+1\}}^+ (D_1 + l\beta)) |_{z_1 \sim 0} \\ & \sim \text{Res}^{\{l+1\}}(\tilde{N}_{l,\{2,\dots,l+1\}} B_{l-1,\{2,\dots,l+1\}}^+ (D_1 + l\beta)) |_{z_1 \sim 0}, \end{aligned} \tag{4.37}$$

where in obtaining (4.37), we have used the identity

$$\begin{aligned} D_1 \operatorname{Res}^{\{2, \dots, l+1\}}(z_2 K_{23} \cdots K_{ll+1} K_{l+1} + \cdots + z_{l+1} K_{l+1}) D_{1, \{2, \dots, l\}}|_{z_1 \sim 0} \\ = \operatorname{Res}^{\{2, \dots, l+1\}}(z_2 K_{23} \cdots K_{ll+1} K_{l+1} + \cdots + z_{l+1} K_{l+1}) D_{l+1} D_{1, \{2, \dots, l\}}|_{z_1 \sim 0}, \end{aligned} \tag{4.38}$$

which follows from the fact that $\operatorname{Res}^{\{l+1\}} \operatorname{Res}^{\{2, \dots, l+1\}} = \operatorname{Res}^{\{l+1\}}$ and that every term with which D_1 is commuted contains the operator K_{l+1} .

From the induction hypothesis we have

$$\begin{aligned} \operatorname{Res}^{\{2, \dots, l+1\}}[\tilde{N}_{l, \{2, \dots, l+1\}}, B_{l-1, \{2, \dots, l+1\}}^+]|_{z_1 \sim 0} \\ \sim \operatorname{Res}^{\{2, \dots, l+1\}} \tilde{N}_{l, \{2, \dots, l+1\}}. \end{aligned} \tag{4.39}$$

Since $(D_1 + l\beta)$ is invariant under the permutations of the indices $\{2, \dots, l+1\}$ we finally obtain

$$\begin{aligned} \operatorname{Res}^{\{2, \dots, l+1\}}(\tilde{N}_{l, \{2, \dots, l+1\}} B_{l-1, \{2, \dots, l+1\}}^+ (D_1 + l\beta))|_{z_1 \sim 0} \\ \sim \operatorname{Res}^{\{2, \dots, l+1\}} \tilde{N}_{l, \{2, \dots, l+1\}} (D_1 + l\beta) \sim \operatorname{Res}^{\{2, \dots, l+1\}} \tilde{N}_{l+1}, \end{aligned} \tag{4.40}$$

which through (4.37) proves (4.35) and, as a consequence, Proposition 4.8.

A few more results will be required to prove Proposition 4.4. They can be stated as follows.

Lemma 4.9. *For sets $J = \{j_1, \dots, j_l\}$ of cardinality l such that $J \cap \{1, \dots, k+1\} = \emptyset$, we have*

$$(i) \operatorname{Res}^{\{k+1\}}[\tilde{N}_{k+1}, z_J] \sim \operatorname{Res}^{\{k+1\}} \tilde{N}_{k+1-l}, \tag{4.41a}$$

$$(ii) \operatorname{Res}^{\{N\}}(\operatorname{Res}^{\{k+1\}} \tilde{N}_{k+1-l} D_{k+1-l, J}) \sim \operatorname{Res}^{\{N\}} \tilde{N}_{k+1}, \tag{4.41b}$$

$$(iii) \operatorname{Res}^{\{N\}}[\tilde{N}_{k+1}, D_{k+1-l, J}] \sim \operatorname{Res}^{\{N\}} \tilde{N}_{k+1}. \tag{4.41c}$$

Property (i) is readily obtained by applying (4.20c) l times. Property (ii) is also easily derived. Since $D_{k+1-l, J}$ is invariant under the permutations of $\{1, \dots, k+1\}$, the restriction $\operatorname{Res}^{\{k+1\}}$ is redundant and can be dropped. By definition, $\tilde{N}_{k+1-l} D_{k+1-l, J} = \tilde{N}_{k+1, \{1, \dots, k+1-l\} \cup J}$. When restricting this operator to symmetric functions of z_1, \dots, z_N , we can relabel appropriately the variables to find $\operatorname{Res}^{\{N\}} \tilde{N}_{k+1}$. The last property is obtained as follows. Using the identity

$$\operatorname{Res}^{\{1, \dots, k+1, i\}}[\tilde{N}_{k+1}, D_i] \sim \operatorname{Res}^{\{1, \dots, k+1, i\}} \tilde{N}_{k+1}, \quad i \notin \{1, \dots, k+1\}, \tag{4.42}$$

which is easy to prove, one sees that

$$\begin{aligned} \operatorname{Res}^{\{N\}} \tilde{N}_{k+1} D_{k+1-l, J} \\ = \operatorname{Res}^{\{N\}} \operatorname{Res}^{\{1, \dots, k+1, J_1\}}(\tilde{N}_{k+1} (D_{J_1} + (k+1-l)\beta)) D_{k+2-l, J \setminus \{J_1\}} \\ \sim \operatorname{Res}^{\{N\}} \tilde{N}_{k+1} D_{k+2-l, J \setminus \{J_1\}}. \end{aligned} \tag{4.43}$$

Iterating (4.43), one thus arrives at (4.41c).

We are now ready to give the proof of Proposition 4.4. We proceed again by induction. The proposition is easily seen to hold when $k = 1$. Indeed,

$$\begin{aligned} \text{Res}^{\{N\}}[\tilde{N}_2, B_1^+] &= \text{Res}^{\{N\}}[\tilde{N}_2, B_{1,\{1,2\}}^+] \\ &+ \sum_{i=3}^N \text{Res}^{\{N\}}[\tilde{N}_2, z_i(D_i + \beta)]. \end{aligned} \tag{4.44}$$

From Proposition 4.8 we know that $\text{Res}^{\{N\}}[\tilde{N}_2, B_{1,\{1,2\}}^+] \sim \text{Res}^{\{N\}}\tilde{N}_2$. Using (4.20c), we find

$$\text{Res}^{\{N\}}[\tilde{N}_2, z_i](D_i + \beta) \sim \text{Res}^{\{N\}}D_1(D_i + \beta) \sim \text{Res}^{\{N\}}\tilde{N}_2, \tag{4.45}$$

and, with the help of (4.41c), we observe that

$$\text{Res}^{\{N\}}[\tilde{N}_2, D_i] \sim \text{Res}^{\{M\}}\tilde{N}_2. \tag{4.46}$$

All terms are therefore seen to have the required factor \tilde{N}_2 on the right.

Now, upon supposing that $\text{Res}^{\{N\}}[\tilde{N}_{k+1}, B_k^+] \sim \text{Res}^{\{N\}}\tilde{N}_{k+1}$ is true for all $k < m$, we wish to prove that this relation holds also for $k = m$. In view of formula (4.19) for B_m^+ , it is clear that Proposition 4.4 would be established if assuming (4.18), one could show for all $l \in \{0, \dots, m\}$ that

$$\text{Res}^{\{N\}}[\tilde{N}_{m+1}, z_J B_{m-l, \{1, \dots, m+1\}}^+ D_{m-l+1, J}] \sim \text{Res}^{\{N\}}\tilde{N}_{m+1}, \tag{4.47}$$

where J are subsets of $\{m + 2, \dots, N\}$ with cardinality l . In fact, property (4.41c) shows that it is sufficient to prove

$$\text{Res}^{\{N\}}[\tilde{N}_{m+1}, z_J B_{m-l, \{1, \dots, m+1\}}^+] D_{m-l+1, J} \sim \text{Res}^{\{N\}}\tilde{N}_{m+1}, \tag{4.48}$$

in order to establish (4.47).

The case $l = 0$ is the content of Proposition 4.8 and has thus already been proved. Remarkably, the cases of lower degree follow from the induction hypothesis which allows one to assume that

$$\text{Res}^{\{m+1\}}[\tilde{N}_{m-l+1}, B_{m-l, \{1, \dots, m+1\}}^+] \sim \text{Res}^{\{m+1\}}\tilde{N}_{m-l+1}, \tag{4.49}$$

for $l = 1, 2, \dots, m$.

Owing to the invariance of $B_{m-l, \{1, \dots, m+1\}}^+ D_{m-l+1, J}$ and of $D_{m-l+1, J}$ under the permutations of the indices $\{1, \dots, m + 1\}$, we may write

$$\begin{aligned} &\text{Res}^{\{N\}}[\tilde{N}_{m+1}, z_J B_{m-l, \{1, \dots, m+1\}}^+] D_{m-l+1, J} \\ &= \text{Res}^{\{N\}}(\text{Res}^{\{m+1\}}[\tilde{N}_{m+1}, z_J]) B_{m-l, \{1, \dots, m+1\}}^+ D_{m-l+1, J} \\ &+ \text{Res}^{\{N\}} z_J (\text{Res}^{\{m+1\}}[\tilde{N}_{m+1}, B_{m-l, \{1, \dots, m+1\}}^+]) D_{m-l+1, J}. \end{aligned} \tag{4.50}$$

We then use (4.41a), (4.49) and (4.41b) to conclude that (4.48) is true and thus to finally complete the proof of Proposition 4.4.

The reason why we needed to prove the stronger result (4.18) instead of the weaker one (4.17) should now have become clear. Indeed, the relations (4.49) with

precisely the restriction $\text{Res}^{\{m+1\}}$, are required for the induction proof of Proposition 4.4 to work. These relations obviously follow from assuming that $\text{Res}^{\{N\}}[\tilde{N}_{k+1}^{\{M\}}, B_{k,\{1,\dots,N\}}^{\{M\}}] \sim \text{Res}^{\{N\}}\tilde{N}_{k+1}^{\{M\}}, \forall k < m$ and $\forall M \geq N$. One simply takes $N = m + 1$ and $k = m - l, l = 1, 2, \dots, m$. They would not have been legitimate assumptions however, had the induction been performed on the relation $\text{Res}^{\{N\}}[\tilde{N}_{k+1}^{\{N\}}, B_{k,\{1,\dots,N\}}^{\{N\}}] \sim \text{Res}^{\{N\}}\tilde{N}_{k+1}^{\{N\}}$, since in this case, the restriction is tied to the number of variables N .

4.3. Proof of Proposition 4.3. The proof of Proposition 4.3 will be done in two steps. We shall first show that (4.11) holds when $i = N - 1$ and the number of variables z_1, z_2, \dots is equal to N . We shall then establish Proposition 4.3 in full generality by demonstrating that it is also true when the number of variables is taken to be arbitrarily larger than $N - 1$. We shall need the following results to proceed.

Lemma 4.10. *Let $z_1 \cdots \hat{z}_i \cdots z_N = \prod_{l=1}^N z_l$. The following relations are satisfied:*

$$\begin{aligned} \text{(i)} \quad & [D_j^2, z_1 \cdots \hat{z}_i \cdots z_N] \\ & = z_1 \cdots \hat{z}_i \cdots z_N \left\{ (1 + \beta K_{ij})^2 + (1 + \beta K_{ij})D_j + D_j(1 + \beta K_{ij}) \right\} \quad i \neq j, \end{aligned} \tag{4.51a}$$

$$\begin{aligned} \text{(ii)} \quad & [D_i^2, z_1 \cdots \hat{z}_i \cdots z_N] |_{z_1 \sim 0} \\ & = -\beta z_2 \cdots z_N (D_i K_{1i} + K_{1i} D_i + \beta K_{1i} + 1). \end{aligned} \tag{4.51b}$$

Both identities are obtained straightforwardly from (4.20a) and (4.20b). Let us introduce the notation

$$H_M = \sum_{i=1}^M D_i^2. \tag{4.52}$$

The Hamiltonian in N dimension $H^{\{N\}}$ (or H) is $H^{\{N\}} \equiv \text{Res} H_N^{\{N\}}$. The following result is an immediate consequence of Lemma 4.10.

Corollary 4.11. *For all i strictly smaller than N , the operators $[H_{N-1}^{\{N\}}, z_1 \cdots \hat{z}_i \cdots z_N] |_{z_1 \sim 0}$ and $z_N [H_{N-1}^{\{N-1\}}, z_1 \cdots \hat{z}_i \cdots z_{N-1}] |_{z_1 \sim 0}$ have the same symbol structure, that is, they have the same form in terms of the coordinates and the operators D_i .*

Lemma 4.12.

$$\text{(i)} \quad \text{Res}^{\{N\}} \left[\sum_{i=1}^{N-1} D_i^l, D_N \right] = \text{Res}^{\{N\}} \beta \left((N - 1)D_N^l - \sum_{i=1}^{N-1} D_i^l \right), \tag{4.53a}$$

$$\begin{aligned} \text{(ii)} \quad & \text{Res}^{\{N\}} [D_N^l, D_{1,\{1,\dots,N\}}] \\ & = \text{Res}^{\{N\}} \beta D_{1,\{1,\dots,N-1\}} \left(\sum_{i=1}^{N-1} D_i^l - (N - 1)D_N^l \right) \quad \forall N \geq 2. \end{aligned} \tag{4.53b}$$

Formula (4.53a) is readily obtained from observing that

$$[D_i^l, D_j] = \beta(D_j^l - D_i^l)K_{ij} . \tag{4.54}$$

The proof of (4.53b) is more involved and proceeds by induction. With the help of (4.54), (4.53b) is easily seen to hold in the first non-trivial case:

$$\begin{aligned} \text{Res}^{\{2\}}[D_2^l, (D_1 + \beta)(D_2 + 2\beta)] &= \text{Res}^{\{2\}}\beta(D_1^l - D_2^l)(D_1 + 2\beta) \\ &= \text{Res}^{\{2\}}\beta(D_1 + \beta)(D_1^l - D_2^l) . \end{aligned} \tag{4.55}$$

We show that (4.53b) follows from assuming that

$$\begin{aligned} &\text{Res}^{\{1, \dots, N-2, N\}}[D_N^l, D_{1, \{1, \dots, N-2, N\}}] \\ &= \text{Res}^{\{1, \dots, N-2, N\}}\beta D_{1, \{1, \dots, N-2\}} \left(\sum_{i=1}^{N-2} D_i^l - (N-2)D_N^l \right) \end{aligned} \tag{4.56}$$

is valid $\forall N > 3$.

Simple manipulations give

$$\begin{aligned} \text{Res}^{\{N\}}[D_N^l, D_{1, \{1, \dots, N\}}] &= \text{Res}^{\{N\}}[D_N^l, D_{1, \{1, \dots, N-2, N\}}(D_{N-1} + N\beta)] \\ &\quad - \beta \text{Res}^{\{N\}}D_{1, \{1, \dots, N-2\}}(D_{N-1} - D_N)(D_N^l - D_{N-1}^l) . \end{aligned} \tag{4.57}$$

One then easily finds that

$$\begin{aligned} &\text{Res}^{\{N\}}[D_N^l, D_{1, \{1, \dots, N-2, N\}}(D_{N-1} + N\beta)] \\ &= \text{Res}^{\{N\}}(\text{Res}^{\{1, \dots, N-2, N\}}[D_N^l, D_{1, \{1, \dots, N-2, N\}}])(D_{N-1} + N\beta) \\ &\quad + \text{Res}^{\{N\}}D_{1, \{1, \dots, N-2, N\}}[D_N^l, D_{N-1}] \\ &= \beta \text{Res}^{\{N\}}D_{1, \{1, \dots, N-2\}} \left\{ \left(\sum_{i=1}^{N-2} D_i^l - (N-2)D_N^l \right) (D_{N-1} + N\beta) \right. \\ &\quad \left. + (D_N + (N-1)\beta)(D_{N-1}^l - D_N^l) \right\} . \end{aligned} \tag{4.58}$$

The induction hypothesis (4.56) has been used in (4.58) to obtain the last equality. From (4.53a) we also get

$$\begin{aligned} &\text{Res} \left(\sum_{i=1}^{N-2} D_i^l - (N-2)D_N^l \right) (D_{N-1} + N\beta) \\ &= \text{Res} (D_{N-1} + (N-1)\beta) \left(\sum_{i=1}^{N-2} D_i^l - (N-2)D_N^l \right) . \end{aligned} \tag{4.59}$$

Inserting this last relation in (4.58) and reorganizing the terms yields

$$\begin{aligned} & \text{Res}^{\{N\}}[D_N^l, D_{1,\{1,\dots,N-2,N\}}(D_{N-1} + N\beta)] \\ &= \beta \text{Res}^{\{N\}} D_{1,\{1,\dots,N-1\}} \left(\sum_{i=1}^{N-1} D_i^l - (N-1)D_N^l \right) \\ & \quad - \beta \text{Res}^{\{N\}} D_{1,\{1,\dots,N-2\}}(D_{N-1} + (N-1)\beta)(D_{N-1}^l - D_N^l) \\ & \quad + \beta \text{Res}^{\{N\}} D_{1,\{1,\dots,N-2\}}(D_N + (N-1)\beta)(D_{N-1}^l - D_N^l). \end{aligned} \tag{4.60}$$

Using this result (4.57), one establishes that (4.53b) is an identity.

We are now ready to prove Proposition 4.3 in the case where $i = N - 1$.

Proposition 4.13. *In N dimensions,*

$$\text{Res}[H_N, B_{N-1}^+] = \text{Res} \left\{ B_{N-1}^+ \left(2 \sum_{i=1}^N D_i + (N-1)(1 + \beta) \right) + G_N \tilde{N}_N \right\}, \tag{4.61}$$

where G_N are some unspecified expressions.

Proof. Recall that $H_N = \sum_{i=1}^N D_i^2$. Since each term of B_{N-1}^+ has a product of $N - 1$ distinct variables as a factor on the left, it follows from Lemma 4.10 that the commutator of H_N and B_{N-1}^+ is of the form

$$[H_N, B_{N-1}^+] = \sum_{j=1}^N z_1 \cdots \hat{z}_j \cdots z_N \mathcal{F}_j \tag{4.62}$$

with \mathcal{F}_j expressions involving the D_l and K_{mn} . Since H_N and B_{N-1}^+ are invariant under the action of the symmetric group S_N , an argument similar to the one given in the proof of Proposition 4.8 shows that to compute the commutator $[H_N, B_{N-1}^+]$, one only needs to determine the terms with $z_2 \cdots z_N$ as factor on the left and to symmetrize the result. This is the approach that we shall take.

We initiate the induction by proving (4.61) when $N = 2$. In this special case we can write

$$H = \text{Res}(D_1^2 + D_2^2) = \text{Res}\{(D_1 + D_2)(D_1 + D_2 + \beta) - 2\tilde{N}_2\}. \tag{4.63}$$

It is then easy to see with the help say, of Proposition 4.8 and using $\text{Res} \sum_i D_i = \sum_i z_i \frac{\partial}{\partial z_i}$ that

$$\text{Res}[H_2, B_1^+] = \text{Res}\{B_1^+(2(D_1 + D_2) + 1 + \beta) + G_2 \tilde{N}_2\}. \tag{4.64}$$

We shall now show that (4.61) is satisfied in N dimensions if it is obeyed in dimensions lower than N . To this end, we shall need the following expression for B_{N-1}^+ :

$$B_{N-1}^+ = z_N B_{N-2,\{1,\dots,N-1\}}^+(D_N + (N-1)\beta) + z_1 \cdots z_{N-1} D_{1,\{1,\dots,N-1\}} \tag{4.65}$$

in which the dependence on z_N is isolated. In the notation (4.52), we shall also write

$$H_N = H_{N-1} + D_N^2, \quad H = \text{Res} H_N. \tag{4.66}$$

With the help of Lemma 4.10, this leads to

$$\begin{aligned} \text{Res}[H_N, B_{N-1}^+] |_{z_1 \sim 0} &= \text{Res}\{[H_{N-1}, z_N B_{N-2, \{1, \dots, N-1\}}^+](D_N + (N-1)\beta) \\ &\quad + z_2 \cdots z_N D_{1, \{2, \dots, N-1\}}[H_{N-1}, D_N] + [D_N^2, z_2 \cdots z_N D_{1, \{2, \dots, N\}}] \\ &\quad + [D_N^2, z_1 \cdots z_{N-1}] D_{1, \{1, \dots, N-1\}}\} |_{z_1 \sim 0}. \end{aligned} \tag{4.67}$$

We now invoke Corollary 4.11 and the definition of B_{N-2}^+ to assert that $[H_{N-1}, z_N B_{N-2, \{1, \dots, N-1\}}^+] |_{z_1 \sim 0}$ and $z_N [H_{N-1}^{\{N-1\}}, B_{N-2}^{+\{N-1\}}] |_{z_1 \sim 0}$ have the same symbol structure. This observation and the fact that $(D_N + (N-1)\beta)$ is invariant under S_{N-1} allow one to compute the first terms in the right-hand side (4.67) from the induction hypothesis. Indeed we have

$$\begin{aligned} &\text{Res}^{\{N-1\}}[H_{N-1}, z_N B_{N-2, \{1, \dots, N-1\}}^+] |_{z_1 \sim 0} \\ &= \text{Res}^{\{N-1\}} \left[z_N \left\{ z_2 \cdots z_{N-1} D_{1, \{2, \dots, N-1\}} \right. \right. \\ &\quad \left. \left. \times \left(2 \sum_{i=1}^{N-1} D_i + (N-2) + (N-2)\beta \right) + G_{N-1} \tilde{N}_{N-1} \right\} \right] \Bigg|_{z_1 \sim 0}, \end{aligned} \tag{4.68}$$

where it is understood that the Dunkl operators depend on the variables z_1, \dots, z_N .

The other terms on the right-hand side of (4.67) are computed with the help of Lemma 4.12 (with $l = 2$) and Lemma 4.10. Putting everything together and using the fact that $\tilde{N}_{N-1}(D_N + (N-1)\beta) = \tilde{N}_N$, one sees that (4.67) reduces to

$$\begin{aligned} &\text{Res}[H_N, B_{N-1}^+] |_{z_1 \sim 0} \\ &= z_2 \cdots z_N \text{Res} \left\{ D_{1, \{2, \dots, N-1\}} \left(2 \sum_{i=1}^{N-1} D_i + (N-2)(1 + \beta) \right) (D_N + (N-1)\beta) \right. \\ &\quad + (2D_N + 1 + \beta - \beta(1 - K_{1N})(D_N - \beta) - 2\beta(1 - K_{1N})) D_{1, \{2, \dots, N\}} \\ &\quad \left. + \beta D_{1, \{2, \dots, N-1\}} (D_N^2 - D_1^2) + G'_N \tilde{N}_N \right\} \Bigg|_{z_1 \sim 0}, \end{aligned} \tag{4.69}$$

where G'_N is another unspecified quantity.

The first two terms between the curly brackets on the right-hand side of (4.69) can be rewritten in the form

$$\begin{aligned} &\left\{ D_{1, \{2, \dots, N\}} \left(2 \sum_{i=1}^{N-1} D_i + (N-2)(1 + \beta) \right) + D_{1, \{2, \dots, N-1\}} \left[2 \sum_{i=1}^{N-1} D_i, D_N \right] \right. \\ &\quad + D_{1, \{2, \dots, N-1\}} (2D_N + 1 + \beta) + 2[D_N, D_{1, \{2, \dots, N\}}] \\ &\quad \left. + (-\beta(1 - K_{1N})(D_N - \beta) - 2\beta(1 - K_{1N})) D_{1, \{2, \dots, N\}} \right\}. \end{aligned} \tag{4.70}$$

Using formulas (4.53a) and (4.53b) with $l = 1$ in (4.70), one can recast (4.69) as follows:

$$\begin{aligned} \text{Res}[H_N, B_{N-1}^+] |_{z_1 \sim 0} &= \text{Res } z_2 \cdots z_N \left\{ D_{1, \{2, \dots, N\}} \left(2 \sum_{i=1}^N D_i + (N-1)(1 + \beta) \right) \right. \\ &\quad \left. - \beta(1 - K_{1N})(D_N - \beta)D_{1, \{2, \dots, N\}} + \beta D_{1, \{2, \dots, N-1\}}(D_N^2 - D_1^2) + G'_N \tilde{N}_N \right\} |_{z_1 \sim 0}. \end{aligned} \tag{4.71}$$

With the help of (4.53b) with $l = 1$ again, we find

$$\begin{aligned} &\text{Res} \left(-\beta(1 - K_{1N})D_N D_{1, \{2, \dots, N\}} \right) \\ &= \text{Res} \left(-\beta D_{1, \{2, \dots, N-1\}}(D_N^2 - D_1^2) - \beta^2(1 - K_{1N})D_{1, \{2, \dots, N\}} \right), \end{aligned} \tag{4.72}$$

so that (4.71) becomes

$$\begin{aligned} &\text{Res}[H_N, B_{N-1}^+] |_{z_1 \sim 0} \\ &= \text{Res} \left\{ z_2 \cdots z_N D_{1, \{2, \dots, N\}} \left(2 \sum_{i=1}^N D_i^2 + (N-1)(1 + \beta) \right) + G''_N \tilde{N}_N \right\} \Big|_{z_1 \sim 0}. \end{aligned} \tag{4.73}$$

Upon symmetrizing the right-hand side of (4.73), we finally obtain

$$\text{Res}[H_N, B_{N-1}^+] = \text{Res} \left\{ B_{N-1}^+ \left(2 \sum_{i=1}^N D_i^2 + (N-1)(1 + \beta) \right) + G_N \tilde{N}_N \right\}, \tag{4.74}$$

thereby proving Proposition 4.13.

Given the Proposition 4.3 is true when $i = N - 1$, we shall fix i to be equal to $N - 1$ and extend the number of variables from N to an arbitrary larger number N' . With the symmetry of $H_{N'}$ and B_{N-1}^+ under the exchange of the variables $z_1, \dots, z_{N'}$ allowing to insert Res (meaning $\text{Res}^{\{N'\}}$) in front of every operator, Proposition 4.3 will follow from the next proposition.

Proposition 4.14. *In N' dimensions,*

$$\begin{aligned} [H, \text{Res } B_{N-1}^+] &= \text{Res } B_{N-1}^+ \left\{ 2 \text{Res} \sum_{i=1}^{N'} D_i + (N-1) + \beta(N-1)(N' - N + 1) \right\} \\ &\quad + \text{Res} \sum_{\substack{J \subset \{1, \dots, N'\} \\ |J|=N-1}} G_{N', J} \tilde{N}_{N, J}, \end{aligned} \tag{4.75}$$

with $G_{N', J}$ an unspecified expression.

In order to prove (4.75), we shall need the next three lemmas.

Lemma 4.15. *In N' dimensions, one has*

$$\begin{aligned} \text{Res } D_{k,J} &= \left(z_{j_1} \frac{\partial}{\partial z_{j_1}} + \beta \sum_{i \in \{2, \dots, l\}} \frac{z_{j_1}}{z_{j_1} - z_{j_i}} (1 - K_{j_1 j_i}) + k\beta \right) \\ &\quad \cdots \left(z_{j_l} \frac{\partial}{\partial z_{j_l}} + (k + l - 1)\beta \right) = \text{Res}^J D_{k,J}^{(J)} \\ k &= 0, 1, 2, \dots \end{aligned} \tag{4.76}$$

for all subsets $J \subset \{1, \dots, N'\}$ of cardinality $|J| = l$ such that, $j_\kappa < j_{\kappa+1}$ if $j_\kappa \in J$ and $1 \leq \kappa < l$.

Proof. This lemma follows from the definition of $D_{k,J}$ and the fact that $K_{ij} D_l = D_l K_{ij}$ if $l \neq i, j$. It expresses the fact that $\text{Res } D_{k,J}$ only depends upon the variables $z_i, i \in J$.

Corollary 4.16. *For all sets J defined in Lemma 4.15, $\text{Res } B_{N-1,J}^+$ only depends upon the variables $z_i, i \in J$.*

Let $p_i = z_i \partial / \partial z_i$ and $A_{ij} = \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$, and recall that the Hamiltonian reads

$$H^{(J)} = \sum_{i \in J} p_i^2 + \beta \sum_{\substack{i < j \\ i, j \in J}} A_{ij}, \tag{4.77}$$

in terms of the variables $z_i, i \in J$. This expression appears in the following decomposition of the commutator $[H, \text{Res } B_{N-1}^+]$.

Lemma 4.17. *For any $N' > N$,*

$$[H, \text{Res } B_{N-1}^+] = \sum_{m=1}^{N'+1-N} (-)^{m+1} \sum_{\substack{J \subset \{1, \dots, N'\} \\ |J|=N'-m}} [H^{(J)}, \text{Res } B_{N-1,J}^+], \tag{4.78}$$

with $B_{N-1,J}^+ \equiv z_J D_{1,J}$ if $|J| = N - 1$.

Proof. The proof is combinatorial. We need to show that the right-hand side of (4.78) contains each summand of $\text{Res } B_{N-1}^+$ commuted once and only once with all the parts of $H = H^{\{1, \dots, N'\}}$ given in (4.77). In view of the $S_{N'}$ -symmetry, it is sufficient to show that this is true for one summand say, $\text{Res } z_1 \cdots z_{N-1} D_{1, \{1, \dots, N-1\}}$. From Lemma 4.15, this operator depends only on the variable z_1, \dots, z_{N-1} and it thus suffices to look for the number of times $\text{Res } z_1 \cdots z_{N-1} D_{1, \{1, \dots, N-1\}}$ is commuted with p_i^2, A_{ij} and A_{ik} ($i, j \in \{1, \dots, N-1\}, k \notin \{1, \dots, N-1\}$). Since the sets J of (4.78) must in this case contain $\{1, \dots, N-1\}$, we see that the terms of the right-hand side of this equation that involve $\text{Res } z_1 \cdots z_{N-1} D_{1, \{1, \dots, N-1\}}$ are

$$\sum_{m=1}^{N'+1-N} (-)^{m+1} \sum_{\substack{J' \subset \{N, \dots, N'\} \\ |J'|=N'+1-N-m}} [H^{\{1, \dots, N-1\} \cup J'}, z_1 \cdots z_{N-1} \text{Res } D_{1, \{1, \dots, N-1\}}]. \tag{4.79}$$

Each $H^{\{1, \dots, N-1\} \cup J'}$ contains exactly one p_i^2 and one A_{ij} . Thus, for $|J'| = N' + 1 - N - m$, both p_i^2 and A_{ij} are seen to appear $\binom{N'+1-N}{N'+1-N-m} = \binom{N'+1-N}{m}$ times in the commutators of (4.79), for a total of $\sum_{m=1}^{N'+1-N} (-)^{m+1} \binom{N'+1-N}{m} = 1$ occurrence, as required. In the case of the A_{ik} , since k must be present in J' , it shows up $\binom{N'-N}{N'-N-m} = \binom{N'-N}{m}$ times in (4.79) if $m < N' + 1 - N$, and does not appear otherwise. Performing the sum over m , we also find $\sum_{m=1}^{N'-N} (-)^{m+1} \binom{N'-N}{m} = 1$, which proves the lemma. We see from the last summation that (4.78) does not hold for $N' = N$, since, in this case, A_{iN} never appears.

Let us finally prove.

Lemma 4.18. *For all sets J defined in Lemma 4.15,*

$$[H^{(J)}, \text{Res}_{z_J} D_{1,J}] = z_J D_{1,J} \left(2 \text{Res} \sum_{i \in J} D_i + l \right), \tag{4.80}$$

with $l = |J|$.

Proof. We have from (4.4) and Lemma 4.15 that

$$[H^{(J)}, \text{Res} D_{1,J}] = \text{Res}^J [H_i^{(J)}, D_{1,J}^{(J)}] = 0, \tag{4.81}$$

and from Lemma 4.10 that

$$[H^{(J)}, z_J] = \text{Res}^J [H_i^{(J)}, z_J] = z_J (2 \text{Res}^J \sum_{i \in J} D_i^{(J)} + l) = z_J (2 \text{Res} \sum_{i \in J} D_i + l). \tag{4.82}$$

Using (4.4) and Lemma 4.15 again to commute $\text{Res} \sum_{i \in J} D_i$ and $\text{Res} D_{1,J}$, (4.80) is seen to hold. We are now ready to prove Proposition 4.14. From Lemma 4.18 and Proposition 4.3, Proposition 4.14 is seen to be true for the cases $N' = N - 1$ and N .

Remarkably, when $N' > N$, Lemma 4.17 shows that $[H, \text{Res} B_{N-1}^+]$ can be decomposed in commutators involving less than N' variables. The induction process is thus greatly simplified. We use Corollary 4.16 to set $\text{Res}^J B_{N-1,J}^{+(J)} = \text{Res} B_{N-1,J}^+$ and let

$$[H, \text{Res} B_{N-1,J}^+] = \text{Res} B_{N-1,J}^+ \left\{ 2 \text{Res} \sum_{i \in J} D_i + (N - 1) + \beta(N - 1)(N' + 1 - m - N) \right\} + \text{Res} G_{N'-m,J} \tilde{N}_{N-1,J}, \tag{4.83}$$

for $m = 1, \dots, N' + 1 - N$, with $|J| = N' - m$, in (4.78). There thus remains to be shown that (4.75) is obtained upon performing the sum in (4.78).

Inserting (4.83) in (4.78) gives

$$[H, \text{Res} B_{N-1}^+] = \sum_{m=1}^{N'+1-N} (-)^{m+1} \sum_{\substack{J \subset \{1, \dots, N'\} \\ |J|=N'-m}} \left\{ \text{Res} B_{N-1,J}^+ \left(2 \text{Res} \sum_{i \in J} D_i + (N - 1) + \beta(N - 1)(N' + 1 - m - N) \right) + \text{Res} G_{N'-m,J} \tilde{N}_{N-1,J} \right\}. \tag{4.84}$$

In order to compute the right-hand side, as in the proof of Lemma 4.17, we look at the part of $\text{Res } B_{N-1}^+$ involving $z_1 \cdots z_{N-1} \text{Res } D_{1,\{1,\dots,N-1\}}$ in (4.84). Considering the terms that appear on the left of this particular term in (4.84), we get

$$\begin{aligned} & \sum_{m=1}^{N'+1-N} (-1)^{m+1} \sum_{\substack{J' \subset \{N,\dots,N'\} \\ |J'|=N'+1-N-m}} \left(2 \text{Res} \sum_{i \in \{1,\dots,N-1\} \cup J'} D_i \right. \\ & \quad \left. + (N-1) + \beta(N-1)(N'+1-m-N) \right) \\ &= \sum_{m=1}^{N'+1-N} (-1)^{m+1} \binom{N'+1-N}{m} \left(2 \text{Res} \sum_{i=1}^{N-1} D_i + (N-1) \right. \\ & \quad \left. + \beta(N-1)(N'+1-m-N) \right) + \sum_{m=1}^{N'-N} (-1)^{m+1} \binom{N'-N}{m} 2 \text{Res} \sum_{i=N}^{N'} D_i \\ &= 2 \text{Res} \sum_{i=1}^{N'} D_i + (N-1) + \beta(N-1)(N'+1-N), \end{aligned} \tag{4.85}$$

where we used

$$\sum_{m=1}^{N'+1-N} (-1)^{m+1} \binom{N'+1-N}{m} m \propto \sum_{m=0}^{N'-N} (-1)^m \binom{N'-N}{m} = 0. \tag{4.86}$$

Since the result is the same for every part of B_{N-1}^+ , setting $\sum_{m=1}^{N'+1-N} (-1)^{m+1} \times \text{Res } G_{N'-m,J} = \text{Res } G_{N',J}$, we get the desired result, namely that

$$\begin{aligned} [H, \text{Res } B_{N-1}^+] &= \text{Res } B_{N-1}^+ \left\{ 2 \text{Res} \sum_{i=1}^{N'} D_i + (N-1) + \beta(N-1)(N'-N+1) \right\} \\ & \quad + \text{Res} \sum_{\substack{J \subset \{1,\dots,N'\} \\ |J|=N-1}} G_{N',J} \tilde{N}_{N,J}, \end{aligned} \tag{4.87}$$

thereby proving Proposition 4.14 and, as a consequence, Proposition 4.3.

4.4. The Coefficient c_λ . This last subsection is devoted to the proof of the expression (3.14)–(3.15) of the coefficient c_λ of Theorem 3.1.

Proposition 4.19. *For any partition $\lambda = (\lambda_1, \dots, \lambda_l)$,*

$$B_\lambda^+ m_\lambda = a_\lambda m_{\lambda+1} + \sum_{\lambda' < \lambda+1} d_{\lambda,\lambda'} m_{\lambda'} \tag{4.88}$$

with $a_\lambda = (\lambda_l + \beta) \cdots (\lambda_1 + l\beta)$, $\lambda + 1 = (\lambda_1 + 1, \dots, \lambda_l + 1)$ and the $d_{\lambda,\lambda'}$'s certain coefficients.

By the triangularity of the Jack polynomials on the monomial basis $\{m_\rho\}$, this proposition implies that $B_\lambda^+ J_\lambda = a_\lambda J_{\lambda+1} = (\lambda_l + \beta) \cdots (\lambda_1 + l\beta) J_{\lambda+1}$, which, by successive applications of the creation operators, gives the form (3.14) of c_λ .

The proof of Proposition 4.19 is done by looking for the coefficient appearing in (4.88) in front of the term $z_1^{\lambda_1+1} \dots z_l^{\lambda_l+1}$ of $m_{\lambda+1}$. We shall see that this term can be generated through simple operations only.

Since $z_1^{\lambda_1+1} \dots z_l^{\lambda_l+1}$ involves only the variables z_1, \dots, z_l , it can only be generated in (4.88) from the part $z_1 \dots z_l D_{1,\{1, \dots, l\}}$ of B_l^+ . Moreover,

$$m_\lambda = \sum_{\text{distinct}} z_1^{\lambda_{p(1)}} \dots z_l^{\lambda_{p(l)}} + \text{terms involving at least one other variable} \quad (4.89)$$

and $\text{Res} D_{1,\{1, \dots, l\}}$, from Lemma 4.15, depends only on the variables z_1, \dots, z_l . Thus, only the action of $z_1 \dots z_l \text{Res} D_{1,\{1, \dots, l\}}$ on $\sum_{\text{distinct}} z_1^{\lambda_{p(1)}} \dots z_l^{\lambda_{p(l)}}$ can generate $z_1^{\lambda_1+1} \dots z_l^{\lambda_l+1}$.

We now establish the following lemma.

Lemma 4.20. *For any $i < j$ and $P \in S_l$, the expansion of*

$$\frac{z_i}{z_i - z_j} (1 - K_{ij}) z_1^{\lambda_{p(1)}} \dots z_l^{\lambda_{p(l)}} \quad (4.90)$$

does not contain any term in $z_1^{\lambda_1} \dots z_l^{\lambda_l}$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$.

Proof. $\forall i < j$,

$$\begin{aligned} \frac{z_i}{z_i - z_j} (1 - K_{ij}) z_1^{\lambda_{p(1)}} \dots z_l^{\lambda_{p(l)}} &= z_1^{\lambda_{p(1)}} \dots \hat{z}_i \dots \hat{z}_j \dots z_l^{\lambda_{p(l)}} \\ &\times \begin{cases} -z_i^{\lambda_{p(j)}} z_j^{\lambda_{p(i)}} - \sum_{k=0}^{\lambda_{p(j)} - \lambda_{p(i)} - 2} z_i^{\lambda_{p(j)} - 1 - k} z_j^{\lambda_{p(i)} + 1 + k}, & \text{if } \lambda_{p(j)} > \lambda_{p(i)} \\ z_i^{\lambda_{p(i)}} z_j^{\lambda_{p(j)}} + \sum_{k=0}^{\lambda_{p(i)} - \lambda_{p(j)} - 2} z_i^{\lambda_{p(i)} - 1 - k} z_j^{\lambda_{p(j)} + 1 + k}, & \text{if } \lambda_{p(j)} < \lambda_{p(i)} \\ 0, & \text{if } \lambda_{p(j)} = \lambda_{p(i)}. \end{cases} \end{aligned} \quad (4.91)$$

Hence, the term associated to the partition $(\lambda_1, \dots, \lambda_l)$ in the right-hand side of (4.91) is

$$\begin{cases} -z_i^{\lambda_{p(1)}} \dots z_i^{\lambda_{p(j)}} \dots z_j^{\lambda_{p(i)}} \dots z_l^{\lambda_{p(l)}} & \text{if } \lambda_{p(j)} > \lambda_{p(i)} \\ z_1^{\lambda_{p(1)}} \dots z_i^{\lambda_{p(i)}} \dots z_j^{\lambda_{p(j)}} \dots z_l^{\lambda_{p(l)}}, & \text{if } \lambda_{p(j)} < \lambda_{p(i)} \\ 0, & \text{if } \lambda_{p(j)} = \lambda_{p(i)}, \end{cases} \quad (4.92)$$

from which we see that the exponent of z_j is always less than the one of z_i and that, consequently, the term $z_1^{\lambda_1} \dots z_l^{\lambda_l}$ cannot appear.

From Lemma (4.20), Lemma (4.15) in the case $J = \{1, \dots, l\}$, and the triangularity of the action of $\frac{z_i}{z_i - z_j} (1 - K_{ij})$ on $z_1^{\lambda_{p(1)}} \dots z_l^{\lambda_{p(l)}}$, we finally find that $z_1^{\lambda_1+1} \dots z_l^{\lambda_l+1}$ can only appear in the following way through the action of

$z_1 \cdots z_l \text{Res}^{\{N\}} D_{1,\{1,\dots,l\}}$ on m_λ :

$$\begin{aligned}
 & z_1 \cdots z_l \left(z_1 \frac{\partial}{\partial z_1} + \beta \right) \cdots \left(z_l \frac{\partial}{\partial z_l} + l\beta \right) z_1^{\lambda_1} \cdots z_l^{\lambda_l} \\
 &= (\lambda_l + \beta) \cdots (\lambda_1 + l\beta) z_1^{\lambda_l+1} \cdots z_l^{\lambda_1+1} = a_\lambda z_1^{\lambda_l+1} \cdots z_l^{\lambda_1+1}. \tag{4.93}
 \end{aligned}$$

This proves Proposition 4.19.

We may remark that the coefficient c_λ appears in (3.13) only because we use Jack polynomials that are monic. Stanley uses instead in [8] the normalization which is defined by taking $v_{\lambda,(1^n)} = n!$ if $|\lambda| = n$ in the expansion (2.18) of the Jack polynomials J_λ in terms of the symmetric monomials. We note that in this normalization $c_\lambda = \beta^n v_{\lambda,\lambda}$ therefore, had we used the normalization of Stanley and redefined the creation operators according to $B_i^+ \rightarrow 1/\beta^i B_i^+$, we would have found [18] that

$$J_\lambda(z; 1/\beta) = (B_{N-1}^+)^{\lambda_{N-1}} \cdots (B_2^+)^{\lambda_2 - \lambda_3} (B_1^+)^{\lambda_1 - \lambda_2} \cdot 1$$

without any proportionality constant.

5. Conclusion

Our general objective is to develop a completely algebraic treatment of the Calogero–Sutherland model. We would thus hope to identify the abstract structure of its full dynamical algebra and to work out its relevant representations. This should in principle allow one to obtain algebraically all physically interesting quantities.

We believe that the results presented in this paper provide important clues toward the resolution of these questions. In fact, they have already allowed us to make progress and to formulate in this connection remarkable conjectures. Clearly, the creation operators B_i^+ , $i = 1, \dots, N$, should be among the generators of the dynamical algebra. One would thus want to know their action on the wave functions of the CS model, that is on the Jack polynomials. However, since these operators do not commute among themselves and enter in a definite order in formula (3.13) (or (3.17)) for the excited wave functions, it is not straightforward to obtain these actions. We have nevertheless a conjecture for this. We also have expressions for annihilation operators B_i^- , $i = 1, \dots, N$ and a similar conjecture giving their action on the wave functions. These developments allow us to evaluate in particular the norm of the Jack polynomials in an algebraic fashion. We shall report on these results in a forthcoming publication [17]. A major problem that remains however is to determine the structure relations that the creation and annihilation operators obey.

The results of this paper also have mathematical applications. Various conjectures involving the Jack polynomials have been made. It turns out [18] that formula (3.13) readily implies a weak form of a famous conjecture due to Macdonald and Stanley, namely that in the normalization of Stanley the $v_{\lambda,\mu}$ in (2.18b) are polynomials in β^{-1} with integer coefficients (see the remark at the end of Sect. 4.3). We believe that this Rodrigues formula that we have obtained will provide a useful tool to further advance the proofs of the outstanding conjectures on the Jack polynomials.

Finally, it is expected that the results presented here extend to the relativistic generalization of the Calogero–Sutherland model [19]. This requires a q -deformation of our constructs and should yield a Rodrigues-type formula for the Macdonald

polynomials. It is also of interest to consider in the same vein models and special functions associated to root lattices other than A_{N-1} . We are currently studying these questions and hope to report on them in the near future.

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