Commun. Math. Phys. 178, 391–398 (1996)

Communications in Mathematical Physics

On the Breakdown of Axisymmetric Smooth Solutions for the 3-D Euler Equations

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Received: 16 June 1995/Accepted: 28 September 1995

Abstract: We refine the Beale–Kato–Majda criterion for the breakdown of smooth solutions of the 3-D incompressible Euler equations in the case of axisymmetry. In this case the angular component of vorticity in the cylindrical coordinates alone controls blow-up of the higher Sobolev norms of the velocity.

1. Introduction

The Euler equations for homogeneous inviscid incompressible fluid flows in $\mathbf{R}^3$ are

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p \quad \text{in } \mathbf{R}^3 \times \mathbf{R}_+ , \tag{1}$$

$$\nabla \cdot v = 0 \quad \text{in } \mathbf{R}^3 \times \mathbf{R}_+ , \tag{2}$$

$$v(\,\cdot\,, 0) = v_0 \quad \text{in } \mathbf{R}^3 . \tag{3}$$

Here $v = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the velocity of the fluid flow, $p = p(x,t)$ is the scalar pressure, and v_0 is the initial velocity satisfying $\nabla \cdot v_0 = 0$. Taking the curl of (1), we obtain the equation for the vorticity $\omega = \nabla \times v$,

$$\frac{\partial \omega}{\partial t} + v \cdot \nabla \omega = \omega \cdot \nabla v . \tag{4}$$

For the existence of local in time smooth solutions we have the following result by Kato [3]: Suppose an initial velocity field $v_0 \in V^m$, $m \geqq 3$, is given. Then, there exists $T_0 = T_0(\|v_0\|_{H^3})$ such that the system of equations (1)–(3) has the unique solution

$$v \in C([0,T]; V^m) \cap C^1([0,T]; V^{m-1}) \tag{5}$$

for all $T \in (0, T_0)$, where we used the function space

$$V^m = \{v \in H^m(\mathbf{R}^3) \,|\, \nabla \cdot v = 0\} .$$

On the other hand in [1] Beale, Kato and Majda showed that if the local solution satisfies

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty \tag{6}$$

for some $T > 0$, then the solution v can be continued in the class (5) to the interval $[0, T]$. We are concerned here with the axisymmetric solution of the Euler equations. By an axisymmetric solution of the equations (1)–(3) we mean a solution of the form

$$v(x,t) = v_r(r,x_3,t)e_r + v_\theta(r,x_3,t)e_\theta + v_3(r,x_3,t)e_3$$

in the cylindrical coordinates system. Here we use

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \qquad e_\theta = \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right), \qquad e_3 = (0,0,1), \quad r = \sqrt{x_1^2 + x_2^2}\,.$$

If the initial data v_0 is axisymmetric, belonging to V^m, $m \geqq 3$, then due to the rotational covariance properties of the Euler equations, and by the uniqueness of the local classical solution as described above, the solution remains axisymmetric during the time of the local existence. In the particular axisymmetric case of $v_\theta(\,\cdot\,,0) = 0$, Majda [4] and Raymond [5] showed that the global smooth solution for smooth initial data exists. This is due to the fact that $\frac{\omega_\theta}{r}$ is conserved along the particle trajectories in this case. He proved, in particular, that the Beale–Kato–Majda criterion (6) is satisfied in this case.

In the case of $v_\theta(\,\cdot\,,0) \neq 0$ there is no longer conservation of $\frac{\omega_\theta}{r}$ along the particle trajectories, and there is numerical evidence suggesting for possible finite time breakdown of smooth solutions (see e.g. [2]). In this paper we show that if the finite time breakdown of axisymmetric smooth solution occurs for the 3-D Euler equations, then necessarily some norms of $\omega_\theta = \frac{\partial v_r}{\partial x_3} - \frac{\partial v_3}{\partial r}$ must blow up at that time. (See Theorem 1 and its corollary in Sect. 3 for a precise statement of our result.) Our result could be useful to test numerically the breakdown of the smooth solution of the 3-D Euler equation in the axisymmetric case by computing the growth in time of the norms of ω_θ in the integral appearing in the theorem.

2. Preliminary Estimates

For an axisymmetric flow $v(r,z)$ we can write the vorticity

$$\omega = \omega_r e_r + \omega_3 e_3 + \omega_\theta e_\theta\,,$$

where

$$\omega_r = -\frac{\partial v_\theta}{\partial x_3}, \qquad \omega_3 = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta), \qquad \omega_\theta = \frac{\partial v_r}{\partial x_3} - \frac{\partial v_3}{\partial r}\,.$$

Below we use the notations

$$\tilde{v} = v_r e_r + v_3 e_3, \qquad \tilde{\omega} = \omega_r e_r + \omega_3 e_3$$

and

$$\tilde{\nabla} = e_r \frac{\partial}{\partial r} + e_3 \frac{\partial}{\partial x_3}\,.$$

We set

$$\tilde{\nabla}\tilde{v} = \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_r}{\partial x_3} \\ \frac{\partial v_3}{\partial r} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} .$$

On the other hand we use

$$\nabla\tilde{v} = \left(\frac{\partial \tilde{v}_i}{\partial x_j}\right)_{i,j=1}^{3} .$$

We begin with the following elementary lemma:

Lemma 1.

$$\|\tilde{\nabla}\tilde{v}\|_{L^\infty} \leqq \|\nabla\tilde{v}\|_{L^\infty} . \tag{7}$$

Proof. Let us denote

$$\cos\theta = \frac{x_1}{r}, \qquad \sin\theta = \frac{x_2}{r}, \quad \text{where } r = \sqrt{x_1^2 + x_2^2} .$$

Then, we have

$$\begin{aligned}
\left|\frac{\partial v_r}{\partial r}\right| &= \left|\frac{\partial}{\partial r}(\tilde{v}_1\cos\theta + \tilde{v}_2\sin\theta)\right| \leqq \left|\frac{\partial \tilde{v}_1}{\partial r}\right| + \left|\frac{\partial \tilde{v}_2}{\partial r}\right| \\
&\leqq \left|\frac{\partial \tilde{v}_1}{\partial x_1}\cos\theta + \frac{\partial \tilde{v}_1}{\partial x_2}\sin\theta\right| + \left|\frac{\partial \tilde{v}_2}{\partial x_1}\cos\theta + \frac{\partial \tilde{v}_2}{\partial x_2}\sin\theta\right| \\
&\leqq \left|\frac{\partial \tilde{v}_1}{\partial x_1}\right| + \left|\frac{\partial \tilde{v}_1}{\partial x_2}\right| + \left|\frac{\partial \tilde{v}_2}{\partial x_1}\right| + \left|\frac{\partial \tilde{v}_2}{\partial x_2}\right|
\end{aligned}$$

and

$$\left|\frac{\partial v_r}{\partial x_3}\right| = \left|\frac{\partial}{\partial x_3}(\tilde{v}_1\cos\theta + \tilde{v}_2\sin\theta)\right| \leqq \left|\frac{\partial \tilde{v}_1}{\partial x_3}\right| + \left|\frac{\partial \tilde{v}_2}{\partial x_3}\right|$$

$$\left|\frac{\partial v_3}{\partial r}\right| = \left|\frac{\partial \tilde{v}_3}{\partial x_1}\cos\theta + \frac{\partial \tilde{v}_3}{\partial x_2}\sin\theta\right| \leqq \left|\frac{\partial \tilde{v}_3}{\partial x_1}\right| + \left|\frac{\partial \tilde{v}_3}{\partial x_2}\right|$$

and obviously,

$$\left|\frac{\partial v_3}{\partial x_3}\right| = \left|\frac{\partial \tilde{v}_3}{\partial x_3}\right| .$$

Summing up these inequalities we have

$$\left|\frac{\partial v_r}{\partial r}\right| + \left|\frac{\partial v_r}{\partial x_3}\right| + \left|\frac{\partial v_3}{\partial r}\right| + \left|\frac{\partial v_3}{\partial x_3}\right| \leqq \sum_{i,j=1}^{3}\left|\frac{\partial \tilde{v}_i}{\partial x_j}\right|$$

thus obtaining (7). This completes the proof of the lemma.

Below we use the notation

$$\|u\|_{C^\gamma} = \sup_{x \neq y,\ x,y\in\mathbf{R}^3} \frac{|u(x) - u(y)|}{|x - y|^\gamma} ,$$

where $\gamma \in (0, 1)$. The space $C^{0,\gamma}(\mathbf{R}^3)$ is the usual Hölder space equipped with the norm

$$\|u\|_{C^{0,\gamma}} = \|u\|_{L^\infty} + \|u\|_{C^\gamma} .$$

The following is a refinement of the similar type of inequality in [1] for the case of the axisymmetric vector fields.

We note that we will use the same notation C for the constants appearing in the inequalities afterwards.

Lemma 2. *Let* $(\gamma, p) \in (0,1) \times [1, \infty)$ *be given. Suppose* $\omega_\theta \in C^{0,\gamma}(\mathbf{R}^3) \cap L^p(\mathbf{R}^3)$, *and let* $\tilde{v}$ *be an axisymmetric vector field in* $\mathbf{R}^3$ *with the axis of symmetry* x_3-*axis such that*

$$\nabla \cdot \tilde{v} = 0, \qquad \nabla \times \tilde{v} = \omega_\theta e_\theta \,.$$

Then, we have the following inequality:

$$\|\nabla \tilde{v}\|_{L^\infty} \leqq C\{1 + \|\omega_\theta \ln^- r\|_{L^\infty} + \|\omega_\theta\|_{L^\infty}(1 + \ln^+(\|\omega_\theta\|_{C^\gamma}\|\omega_\theta\|_{L^p}))\} \,,$$

where C *is a constant depending only on* γ *and* p; $r(\cdot)$ *denotes the distance function from the* x_3-*axis, i.e.*

$$r(x) = \sqrt{x_1^2 + x_2^2}$$

and $\ln^- r = -\ln r$ *if* $r < 1$, *and* $\ln^- r = 0$ *otherwise.*

Proof. It is well known (see e.g. [1], or [4])

$$\nabla \tilde{v}(x) = C\omega_\theta(x) e_\theta(x) + [K * (\omega_\theta e_\theta)](x) \,,$$

where C is a constant matrix and $K(\cdot) \in C^\infty(\mathbf{R}^3 \backslash \{0\})$ is a matrix kernel defining a singular integral operator in $\mathbf{R}^3$, satisfying the estimate

$$|K(y)| \leqq \frac{C}{|y|^3} \quad \forall y \in \mathbf{R}^3$$

and, having the cancellation property [6];

$$\int_{a \leqq |x-y| \leqq b} K(x-y)dy = 0 \quad \forall a, b \quad \text{with } 0 < a \leqq b \leqq \infty \,.$$

We have

$$\|\nabla \tilde{v}\|_{L^\infty} \leqq C\|\omega_\theta\|_{L^\infty} + \|K * (\omega_\theta e_\theta)\|_{L^\infty}$$

for an absolute constant C. Using the cancellation property of K, we can decompose

$$|[K * (\omega_\theta e_\theta)](x)| \leqq \left| \int_{\mathbf{R}^3} K(x-y)\omega_\theta(y) e_\theta(y) dy \right|$$

$$\leqq \left| \int_{|x-y|<\varepsilon} K(x-y)(\omega(x) - \omega(y)) e_\theta(y) dy \right| + |\omega_\theta(x)| \left| \int_{|x-y|<\varepsilon} K(x-y) e_\theta(y) dy \right|$$

$$+ \left| \int_{\varepsilon<|x-y|<R} K(x-y)\omega_\theta(y) e_\theta(y) dy \right| + \left| \int_{|x-y|>R} K(x-y)\omega_\theta(y) e_\theta(y) dy \right|$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \,.$$

We estimate the first term

$$\mathrm{I} \leqq \|\omega_\theta\|_{C^\gamma} \int_{|x-y|<\varepsilon} |K(x-y)||x-y|^\gamma dy$$

$$\leqq \|\omega_\theta\|_{C^\gamma} \int_{|x-y|<\varepsilon} |x-y|^{-3+\gamma} dy = C\varepsilon^\gamma \|\omega_\theta\|_{C^\gamma} ,$$

where C is a constant depending on γ. The third term is easily estimated as

$$\mathrm{III} \leqq C\|\omega_\theta\|_{L^\infty} \int_{\varepsilon \leqq |x-y| \leqq R} |x-y|^{-3} dy \leqq C\|\omega_\theta\|_{L^\infty} \ln^+\left(\frac{R}{\varepsilon}\right) .$$

By Hölder's inequality we have

$$\mathrm{IV} \leqq C\left(\int_{|x-y|>R} \frac{1}{|x-y|^{\frac{3p}{p-1}}} dy\right)^{\frac{p-1}{p}} \|\omega_\theta\|_{L^p}$$

$$\leqq C\left(\int_R^\infty r^{-\frac{p+2}{p-1}} dr\right)^{\frac{p-1}{p}} \|\omega_\theta\|_{L^p} \leqq C\|\omega_\theta\|_{L^p} R^{-\frac{3}{p}} .$$

For the second term, using the cancellation property and the mean value theorem, we have

$$\left|\int_{|x-y|<\varepsilon} K(x-y)e_\theta(y)dy\right| = \left|\int_{|y|<\varepsilon} K(y)e_\theta(x-y)dy\right|$$

$$\leqq \left|\int_{|y|<\frac{r(x)}{2}} K(y)(e_\theta(x-y)-e_\theta(x))dy\right| + \int_{\frac{r(x)}{2}<|y|<\varepsilon} |e_\theta(x-y)||K(y)|dy$$

$$\leqq \int_{|y|<\frac{r(x)}{2}} |y|^{-2}|\nabla e_\theta(x-sy)|dy + C\ln^+\left(\frac{2\varepsilon}{r(x)}\right) ,$$

where $s \in (0,1)$. By direct computation from

$$e_\theta(z) = \left(\frac{z_2}{r(z)}, -\frac{z_1}{r(z)}, 0\right) ,$$

we have

$$|\nabla e_\theta(z)| \leqq \frac{C}{r(z)} .$$

For $|y| < \frac{r(x)}{2}$ we observe

$$r(x-sy) \geqq r(x) - r(y) \geqq r(x) - |y| \geqq r(x) - \frac{r(x)}{2} = \frac{r(x)}{2} .$$

Thus

$$|\nabla e_\theta(x-sy)| \leqq \frac{C}{r(x-sy)} \leqq \frac{2C}{r(x)}\,.$$

These observations lead to an estimate for the second term

$$\mathrm{II} \leqq C|\omega_\theta(x)|\left(1+\ln^+\left(\frac{\varepsilon}{r(x)}\right)\right) \leqq C\|\omega_\theta\|_{L^\infty} + C\|\omega_\theta \ln^- r\|_{L^\infty}$$

if we assume $\varepsilon \leqq 1$. Summing up the estimates for I, II, III, IV, and taking

$$\varepsilon = \min\{1, \|\omega_\theta\|_{C^\gamma}^{-\frac{1}{\gamma}}\} \quad \text{and} \quad R = \|\omega_\theta\|_{L^p}^{\frac{p}{3}}\,,$$

we obtain the desired result. This completes the proof of the lemma.

3. The Main Theorem

Using the lemmas in the previous section, we will prove the following main Theorem.

Theorem 1. *Let v be the local in time solution of Eqs.* (1)–(3) *corresponding to axisymmetric initial data $v_0 \in V^m$, $m \geqq 3$.*

Suppose there exists a number $(\gamma, p) \in (0,1)\times[1,\infty)$, and $M < \infty$ such that the following inequality holds:

$$\int_0^T \|\omega_\theta(t)\|_{L^\infty}dt + \int_0^T \exp\left[\int_0^t \{\|\omega_\theta(s)\|_{L^\infty}(1+\ln^+(\|\omega_\theta(s)\|_{C^\gamma}\|\omega_\theta(s)\|_{L^p})) + \|\omega_\theta(s)\ln^- r\|_{L^\infty}\}ds\right]dt \leqq M\,, \tag{8}$$

then the solution v can be continued in the class (5) *to the interval* $[0,T]$.

Remark. The above inequality (8) can be reduced to the simpler, but less sharper form as follows:

$$\int_0^T \{\|\omega_\theta(t)\|_{L^\infty}(1+\ln^+(\|\omega_\theta(t)\|_{C^\gamma}\|\omega_\theta(t)\|_{L^p})) + \|\omega_\theta(t)\ln^- r\|_{L^\infty}\}dt \leqq M\,.$$

Before proving this theorem we state an immediate corollary of it.

Corollary 1. *Let v be the local axisymmetric solution of the Eqs.* (1)–(3) *which belongs to the class in* (5) *for all $T < T^*$, and suppose*

$$\limsup_{t\nearrow T^*} \|v(t)\|_{H^m} = \infty\,,$$

then necessarily

$$\int_0^{T^*} \|\omega_\theta(t)\|_{L^\infty}dt = \infty$$

or

$$\int_0^{T^*} \exp\left[\int_0^t \{\|\omega_\theta(s)\|_{L^\infty}(1+\ln^+(\|\omega_\theta(s)\|_{C^\gamma}\|\omega_\theta(s)\|_{L^p})) + \|\omega_\theta(s)\ln^- r\|_{L^\infty}\}ds\right]dt = \infty$$

for any $(\gamma, p) \in (0,1)\times[1,\infty)$.

Proof of Theorem 1. We write the vorticity equation (4) in the axisymmetric case as

$$\nabla \cdot \tilde{v} = 0\,, \tag{9}$$

$$\frac{\tilde{D}\omega_r}{Dt} = \tilde{\omega}\cdot\tilde{\nabla}v_r\,, \tag{10}$$

$$\frac{\tilde{D}\omega_3}{Dt} = \tilde{\omega}\cdot\tilde{\nabla}v_3\,, \tag{11}$$

$$\frac{\tilde{D}}{Dt}\left(\frac{\omega_\theta}{r}\right) = \tilde{\omega}\cdot\tilde{\nabla}\left(\frac{v_\theta}{r}\right)\,, \tag{12}$$

where we denoted

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + \tilde{v}\cdot\tilde{\nabla} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + v_\theta\frac{\partial}{\partial\theta}\,.$$

We first claim

$$\|\tilde{\omega}(t)\|_{L^\infty} \leqq C\|v_0\|_{H^m}\exp\left(\int_0^t \|\nabla\tilde{v}(s)\|_{L^\infty}ds\right)\,, \tag{13}$$

where C is an absolute constant. Indeed, let $p \geqq 2$. Multiplying $|\tilde{\omega}|^{p-2}\omega_r$ and $|\tilde{\omega}|^{p-2}\omega_3$ on both sides of Eqs. (10), (11) respectively, adding them up, and integrating over $\mathbf{R}^3$, we have

$$\int_{\mathbf{R}^3} |\tilde{\omega}|^{p-2}\frac{d}{dt}|\tilde{\omega}|^2 + \frac{1}{p}\int_{\mathbf{R}^3}\tilde{v}\cdot\nabla|\tilde{\omega}|^p \leqq \int_{\mathbf{R}^3}|\tilde{\nabla}\tilde{v}||\tilde{\omega}|^p dx\,.$$

After integration by parts, using (9), we obtain

$$\frac{d}{dt}\|\tilde{\omega}\|_{L^p}^p \leqq p\int_{\mathbf{R}^3}|\tilde{\nabla}\tilde{v}||\tilde{\omega}|^p dx \leqq p\|\tilde{\nabla}\tilde{v}\|_{L^\infty}\|\tilde{\omega}\|_{L^p}^p \leqq p\|\nabla\tilde{v}\|_{L^\infty}\|\tilde{\omega}\|_{L^p}^p\,,$$

where we used (7) in the last inequality. Dividing by $\|\tilde{\omega}\|_{L^p}^{p-1}$, and using Grönwall's inequality we obtain

$$\|\tilde{\omega}(t)\|_{L^p} \leqq \|\tilde{\omega}_0\|_{L^p}\exp\left(\int_0^t\|\nabla\tilde{v}(s)\|_{L^\infty}ds\right) \tag{14}$$

for all $p \geqq 2$. Now let

$$A_\varepsilon = \{\, x\in\mathbf{R}^3 \mid |\tilde{\omega}| \geqq \|\tilde{\omega}\|_{L^\infty} - \varepsilon\,\} \cap \{\, x\in\mathbf{R}^3 \mid |x| < R\}\,,$$

where R is chosen large enough so that the Lebesgue measure $|A_\varepsilon|$ of A_ε is not zero. Then,

$$|A_\varepsilon|^{\frac{1}{p}}(\|\tilde{\omega}(t)\|_{L^\infty} - \varepsilon) \leqq \left(\int_{A_\varepsilon} |\tilde{\omega}(x,t)|^p dx \right)^{\frac{1}{p}} \leqq \|\tilde{\omega}(t)\|_{L^p} . \tag{15}$$

On the other hand, by an elementary interpolation, and Sobolev's inequality we have

$$\|\tilde{\omega}_0\|_{L^p} \leqq \|\tilde{\omega}_0\|_{L^\infty}^{\frac{p-2}{p}} \|\tilde{\omega}_0\|_{L^2}^{\frac{2}{p}} \leqq C\|v_0\|_{H^m}^{\frac{p-2}{p}} \|\tilde{\omega}_0\|_{L^2}^{\frac{2}{p}} . \tag{16}$$

Combining (15) and (16) with (14), we obtain

$$|A_\varepsilon|^{\frac{1}{p}}(\|\tilde{\omega}\|_{L^\infty} - \varepsilon) \leqq C^{\frac{p-2}{p}} \|v_0\|_{H^m}^{\frac{p-2}{p}} \|\tilde{\omega}_0\|_{L^2}^{\frac{2}{p}} \exp\left(\int_0^t \|\nabla \tilde{v}(s)\|_{L^\infty} ds \right) .$$

Passing first $p \to \infty$, and then $\varepsilon \to 0$, we obtain (13) as claimed.

Now, assume (8) holds. By Beale–Kato–Majda's criterion (6) it suffices to show that

$$\int_0^T (\|\omega_\theta\|_{L^\infty} + \|\tilde{\omega}\|_{L^\infty}) dt < \infty . \tag{17}$$

Since

$$\nabla \cdot \tilde{v}(t) = 0, \qquad \nabla \times \tilde{v}(t) = \omega_\theta(t) e_\theta ,$$

(17) follows immediately from the hypothesis by combining (13) and Lemma 2. This completes the proof of the theorem.

Acknowledgements. This research was supported partially by KOSEF(K94073), BSRI(N94121), E94287, GARC-KOSEF and SNU95-03-1038.

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Communicated by A. Jaffe