Commun. Math. Phys. 178, 281 – 299 (1996)

Communications in Mathematical Physics
© Springer-Verlag 1996

The Gervais–Neveu–Felder Equation and the Quantum Calogero–Moser Systems

J. Avan, O. Babelon, E. Billey

L.P.T.H.E. Université Paris VI (CNRS UA 280), Box 126, Tour 16, 1[er] étage, 4 place Jussieu, F-75252 Paris Cedex 05, France

Received: 23 May 1995 / Accepted: 20 September 1995

Abstract: We quantize the spin Calogero–Moser model in the R-matrix formalism. The quantum R-matrix of the model is dynamical. This R-matrix has already appeared in Gervais–Neveu's quantization of Toda field theory and in Felder's quantization of the Knizhnik–Zamolodchikov–Bernard equation.

1. Introduction

Integrable systems of N particles on a line with pairwise interaction have recently attracted much attention. After the famous works of Calogero and Moser [1], many generalizations have been proposed. These include the relativistic generalization of Ruijsenaars [2], the spin generalization of the non-relativistic models [3,4] and finally the spin generalization of the relativistic models [5]. They have many relations to harmonic analysis [6], algebraic geometry [7], topological field theory [8], conformal field theory [9,10], string field theory [11].

In this paper we consider yet another aspect of these models, i.e. their embedding into the R-matrix formalism, both at the classical and quantum levels. In this respect the essentially new feature which emerges is that the R-matrix turns out to be a dynamical one. At the classical level, the r-matrix was computed for the usual Calogero–Moser models in [12]. It was computed in [13] for their spin generalization, while it was calculated first in the Sine–Gordon soliton case [14], then in the general case [15] for the Ruijsenaars systems. We address here the issue of the quantum formulation of these models within an R-matrix framework. We are going to show that the quantum Yang–Baxter equation has to be generalized. At present this new equation stands at the crossroads of three seemingly distinct topics: quantization of Toda field theory, quantization of KZB equations, and quantization of Calogero–Moser–Ruijsenaars models.

In Sect. 2 we explain the above connections at the classical level. The classical r-matrix of the Calogero–Moser model, the KZB connection for the WZW model on the torus and the r-matrix of the exchange algebra in Toda field theory all satisfy the same generalized Yang–Baxter equation. In Sect. 3 we take advantage of these identifications to define the commutation relations obeyed by the quantum

Lax operator of the Calogero–Moser model. In Sect. 4 we use this quantum algebra to construct a set of commuting operators which are the quantum analogs of $\operatorname{tr} L^n$, where L is the Lax matrix of the system. Finally in Sect. 5 we give examples of such operators built for specific representations of the quantum algebra.

One should stress again that our concern here was to embed the Calogero–Moser systems into the R-matrix formalism. Many other different approaches exist for these models. In particular the works [4, 10] are probably closely related to our results, but the precise connexions are yet to be clarified.

2. The Generalized Classical Yang–Baxter Equation

2.1. The Calogero–Moser Model and its Classical r-Matrix. The Calogero–Moser system is a system of N particles on a line with positions x_i and momenta p_i. The Hamiltonian is:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}g^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{N} V(x_{ij}), \quad x_{ij} = x_i - x_j\,, \tag{1}$$

where the two-body potential $V(x)$ is the Weierstrass function $\wp(x)$ or its trigonometric limit $1/\sinh^2(x)$, or its rational limit $1/x^2$. The Poisson bracket is the canonical one:

$$\{p_i, x_j\} = \delta_{ij}\,.$$

Rather than considering the Calogero–Moser model in this standard version, it will be important to consider instead its spin generalization

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\sum_{\substack{i,j=1 \\ i \neq j}}^{N} h_{ij}h_{ji}V(x_{ij})\,, \tag{2}$$

where the Poisson bracket on the new dynamical variables h_{ij} is given by

$$\{h_{ij}, h_{kl}\} = \delta_{li}h_{kj} - \delta_{jk}h_{il}\,.$$

The above Poisson bracket is degenerated. We have to choose particular symplectic leaves which we parametrize as

$$h_{ij} = \sum_{\alpha=1}^{l} b_i^\alpha a_j^\alpha$$

with

$$\{a_i^\alpha, b_j^\beta\} = -\delta_{\alpha\beta}\delta_{ij}\,.$$

Remark that $\{H, h_{ii}\} = 0$ for all i. The standard Calogero–Moser model is then obtained by a Hamiltonian reduction of the spin model for $l = 1$ under this symmetry. Indeed in this case we have $h_{ij} = a_i b_j$; the reduced manifold is characterized by the value of the momentum $h_{ii} = a_i b_i = g$ for all i. Then $h_{ij}h_{ji} = g^2$ and we recover Eq. (1). For general l, the spin model is integrable only on the reduced manifold.

The standard Calogero–Moser model is well known to be integrable. It has a Lax matrix depending on a spectral parameter λ [16],

$$L_{ij}(\lambda, x, p) = \delta_{ij}p_i + (1 - \delta_{ij})\Phi(x_{ij}, \lambda)\,,$$

with

$$\Phi(x,\lambda) = \frac{\sigma(\lambda - x)}{\sigma(x)\sigma(\lambda)},$$

where σ is the Weierstrass σ function.

This yields conserved quantities $I_n = \operatorname{tr} L^n$. However Liouville integrability requires that these quantities be in involution. This is equivalent [17] to the existence of an r-matrix (we use the standard notation $L_1 = L \otimes \mathrm{Id} \ldots$)

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{3}$$

This r-matrix was computed in [12] and is given by

$$\begin{aligned} r_{12}^{\mathrm{Cal}}(\lambda,\mu,x) = &- \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Phi(x_{ij}, \lambda - \mu)\, e_{ij} \otimes e_{ji} + \zeta(\lambda - \mu) \sum_{i=1}^{N} e_{ii} \otimes e_{ii} \\ &+ \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Phi(x_{ij}, \mu)\, e_{ii} \otimes e_{ij}. \end{aligned} \tag{4}$$

The important new feature of this model is that the r-matrix depends on the dynamical variables x_i.

Occurrence of the last term in Eq. (4) jeopardizes the eventual quantization of Eq. (3). It is in this context that the consideration of the "spin" model is advantageous. Defining

$$L_{ij}(\lambda, x, p) = \delta_{ij} p_i + (1 - \delta_{ij})\, h_{ij} \Phi(x_{ij}, \lambda), \tag{5}$$

we find

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2] + [\mathcal{D}, r_{12}] \tag{6}$$

with

$$r_{12}(\lambda,\mu,x) = - \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \Phi(x_{ij}, \lambda - \mu)\, e_{ij} \otimes e_{ji} + \zeta(\lambda - \mu) \sum_{i=1}^{N} e_{ii} \otimes e_{ii} \tag{7}$$

and

$$\mathcal{D} = \sum_{i=1}^{N} h_{ii} \frac{\partial}{\partial x_i}. \tag{8}$$

The last term $[\mathcal{D}, r_{12}]$ reflects the non-integrability of the non-reduced system. Since the matrix r only depends on the differences $x_{ij} = x_i - x_j$, the last term takes the explicit form

$$[\mathcal{D}, r_{12}(\lambda,\mu,x)] = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} (h_{ii} - h_{jj}) \frac{\partial}{\partial x_{ij}} r_{12}(\lambda,\mu,x).$$

Its contributions eventually vanish on the reduced phase space $h_{ii} =$ constant. One can recover the r-matrix (4) from (7) using the reduction procedure [13].

Proposition. *The r-matrix Eq.* (7) *is antisymmetric*: $r_{12}(\lambda,\mu,x) = -r_{21}(\mu,\lambda,x)$ *and satisfies the equation*

$$-\{L_1, r_{23}\} + \{L_2, r_{13}\} - \{L_3, r_{12}\} + [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{9}$$

In particular, this implies the Jacobi identity.

Proof. Denoting $Z_{12} = [\mathcal{D}, r_{12}]$, the Jacobi identity reads

$$\begin{aligned} 0 &= \{L_1,\{L_2,L_3\}\} + \{L_2,\{L_3,L_1\}\} + \{L_3,\{L_1,L_2\}\} \\ &= [L_1,[r_{12},r_{23}] + [r_{12},r_{13}] + [r_{32},r_{13}] + \{L_2,r_{13}\} - \{L_3,r_{12}\}] + \text{cycl. perm.} \\ &\quad + [r_{23},Z_{12}] + [r_{31},Z_{23}] + [r_{12},Z_{31}] - [r_{32},Z_{13}] - [r_{13},Z_{21}] - [r_{21},Z_{32}] \\ &\quad + \{L_1,Z_{23}\} + \{L_2,Z_{31}\} + \{L_3,Z_{12}\}\,. \end{aligned} \tag{10}$$

Using the antisymmetry of r, we find

$$\begin{aligned} &[r_{23},Z_{12}] + [r_{31},Z_{23}] + [r_{12},Z_{31}] - [r_{32},Z_{13}] - [r_{13},Z_{21}] - [r_{21},Z_{32}] \\ &\quad = -[\mathcal{D},[r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}]]\,. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \{L_1,Z_{23}\} &= \{L_1,[\mathcal{D},r_{23}]\} = [\mathcal{D},\{L_1,r_{23}\}] + [\{L_1,\mathcal{D}\},r_{23}] \\ &= [\mathcal{D},\{L_1,r_{23}\}] - \sum_{i=1}^{N} [[L_1,e_{ii}^{(1)}]\partial_{x_i},r_{23}] \\ &= [\mathcal{D},\{L_1,r_{23}\}] - [L_1,\{L_1,r_{23}\}]\,, \end{aligned}$$

so that Eq. (10) becomes

$$\begin{aligned} 0 &= [L_1,[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]-\{L_1,r_{23}\}+\{L_2,r_{13}\}-\{L_3,r_{12}\}]+\text{cycl. perm.} \\ &\quad - [\mathcal{D},[r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] - \{L_1,r_{23}\} + \{L_2,r_{13}\} - \{L_3,r_{12}\}]\,. \end{aligned}$$

Hence the Jacobi identity is satisfied if Eq. (9) holds, which is easily checked by a direct calculation. □

From the facts that $\{L_1,r_{23}\} = \sum_{i=1}^N e_{ii}^{(1)} \otimes \partial_{x_i} r_{23}$ and r depends only on the differences $x_i - x_j$, we can rewrite Eq. (9) as

$$\begin{aligned} &[r_{12},r_{13}] + [r_{12},r_{23}] + [r_{13},r_{23}] - \sum_\nu h_\nu^{(1)} \frac{\partial}{\partial x_\nu} r_{23} \\ &\quad + \sum_\nu h_\nu^{(2)} \frac{\partial}{\partial x_\nu} r_{13} - \sum_\nu h_\nu^{(3)} \frac{\partial}{\partial x_\nu} r_{12} = 0\,, \end{aligned} \tag{11}$$

where $\{h_\nu\}$ is an orthonormal basis of the Cartan subalgebra of diagonal matrices of sl_N and $x = \sum_\nu x_\nu h_\nu$.

Let us comment on the trigonometric limit of the classical r-matrix $r_{12}(\lambda,\mu,x)$ defined in Eq. (7). We remark that if $r_{12}(\lambda,\mu,x)$ is a solution of Eq. (11) such that $\forall \nu,\ [h_\nu \otimes 1 + 1 \otimes h_\nu, r_{12}(\lambda,\mu,x)] = 0$, then

$$\widetilde{r}_{12}(\lambda,\mu,x) = \mathrm{e}^{\alpha(\lambda)x} \otimes \mathrm{e}^{\alpha(\mu)x}\, r_{12}(\lambda,\mu,x)\ \mathrm{e}^{-\alpha(\lambda)x} \otimes \mathrm{e}^{-\alpha(\mu)x} - (\alpha(\lambda)-\alpha(\mu))\sum_\nu h_\nu \otimes h_\nu$$

is also a solution of Eq. (11) for any function $\alpha(\lambda)$. Using this freedom we see that the trigonometric limit of Eq. (7) may be recast into the form

$$r_{12}(\lambda,\mu,x) = \coth(\lambda-\mu) \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} - \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \coth(x_{ij}) e_{ij} \otimes e_{ji}\,. \tag{12}$$

Equation (11) will be the cornerstone of our quantization procedure of the Calogero–Moser model. It also appeared in two other contexts which we now briefly recall.

2.2. Relation to the Knizhnik–Zamolodchikov–Bernard Equation. It is well known that there is a relation between conformal field theories and the classical Yang–Baxter equation through the Knizhnik–Zamolodchikov equation [18]. Let $r_{ij}(z) = -r_{ji}(-z)$ be a skew-symmetric solution of the classical Yang–Baxter equation taking values in the tensor product $\mathscr{G}^{(i)} \otimes \mathscr{G}^{(j)}$, where $\mathscr{G}$ is a simple Lie algebra. Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{G}$. Then the KZ connexion

$$\nabla_i = \partial_{z_i} - \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ij}(z_i - z_j)$$

has zero curvature. Hence the system of equations

$$\partial_{z_i} u = \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ij}(z_i - z_j) u \tag{13}$$

for a function $u(z_1, \dots, z_N)$ on $\mathbf{C}^{\mathbf{N}} - \bigcup_{\mathbf{i}<\mathbf{j}} \{z; z_i = z_j\}$ with values in $V \otimes \cdots \otimes V$, where V is a representation space for $\mathscr{G}$, has a solution. Equation (13) characterizes conformal blocks of the Wess–Zumino–Witten model on the sphere. On a higher genus Riemann surface the corresponding equations are the Knizhnik–Zamolodchikov–Bernard equations (in our case of interest $g = 1$); they are equations for functions $u(z_1, \dots, z_N, x)$ taking values in the weight zero subspace of a tensor product of irreducible finite dimensional representations of a simple Lie algebra $\mathscr{G}$ i.e.

$$\forall v, \left(\sum_{i=1}^{N} h_v^{(i)} \right) u = 0 .$$

(Here and in the following the superscript in $h_v^{(i)}$ denotes the space on which h_v acts and the subscript v denotes an element in a basis of $\mathscr{H}$.) In the case of a torus, they take the form

$$\kappa \partial_{z_i} u = -\sum_{v} h_v^{(i)} \partial_{x_v} u + \sum_{\substack{j=1 \\ j \neq i}}^{N} r_{ij}(z_i - z_j, x) u \tag{14}$$

with additional equations involving derivatives w.r.t. the modular parameters. The compatibility condition of Eq. (14) is exactly Eq. (11) [9].

2.3. Relation to Toda Field Theory. The Toda field equations associated to a simple Lie algebra $\mathscr{G}$ read

$$\Box \phi_v = \sum_{\alpha \text{ simple}} \alpha_v \, \mathbf{e}^{2\alpha(\phi)} ,$$

where $\phi = \sum_v \phi_v h_v$ is a field taking values in a Cartan subalgebra $\mathscr{H}$ of a Lie algebra $\mathscr{G}$. As above, $\{h_v\}$ is an orthonormal basis of this Cartan subalgebra. In the case $\mathscr{G} = \mathrm{sl}_2$, this becomes the Liouville equation.

Leznov and Saveliev [19] found a generalization of Liouville's solution to the Liouville equation. It takes the form

$$\mathbf{e}^{-\Lambda(\phi)} = \Psi(z) \cdot \overline{\Psi}(\bar{z}) ,$$

where $|\Lambda\rangle$ is a highest weight vector, $\Psi(z)$ and $\overline{\Psi}(\bar{z})$ are chiral fields ($z = \sigma + \tau$ and $\bar{z} = \sigma - \tau$ are the light-cone coordinates)

$$\Psi = \langle\Lambda|Q_+, \quad \overline{\Psi} = Q_-|\Lambda\rangle$$

with $Q_\pm$ solutions of the linear systems

$$\partial_z Q_+ = (P + \mathscr{E}_+)Q_+, \qquad \partial_{\bar{z}} Q_- = Q_-(\overline{P} + \mathscr{E}_-)\,.$$

P and $\overline{P}$ are chiral fields with values in a Cartan subalgebra of $\mathscr{G}$ and $\mathscr{E}_\pm = \sum_{\alpha\,\text{simple}} E_{\pm\alpha}$ with E_α the root vectors in the corresponding Cartan decomposition of $\mathscr{G}$.

To reconstruct periodic solutions of the Toda field equation, it is natural to consider the quasi-periodic basis for Ψ and $\overline{\Psi}$,

$$\Psi(\sigma + 2\pi) = \Psi(\sigma)\exp(x), \qquad \overline{\Psi}(\bar{\sigma} + 2\pi) = \exp(-x)\overline{\Psi}(\bar{\sigma})\,,$$

where $x = \sum_\nu x_\nu h_\nu$ is the quasi-momentum (zero mode), belonging to the Cartan subalgebra.

The Poisson bracket (at equal time $\tau = 0$)

$$\{P(\sigma), P(\sigma')\} = \delta'(\sigma - \sigma')\sum_\nu h_\nu \otimes h_\nu$$

induces a Poisson bracket on Ψ [21]

$$\{\Psi_1(\sigma), \Psi_2(\sigma')\} = \Psi_1(\sigma)\Psi_2(\sigma')r_{12}^{\pm}(x), \quad \pm = \text{sign}(\sigma - \sigma')\,, \tag{15}$$

where [20] (in the sl_N case)

$$r_{12}^{\pm}(x) = \pm\sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} - \sum_{\substack{i,j=1\\ i\neq j}}^{N} \coth(x_{ij})\, e_{ij} \otimes e_{ji}\,. \tag{16}$$

Taking into account that

$$\{\Psi(\sigma), x_\nu\} = \Psi(\sigma)h_\nu\,,$$

the Jacobi identity $\{\Psi_1, \{\Psi_2, \Psi_3\}\} + \text{cycl. perm.} = 0$ implies exactly Eq. (11) on $r^{\pm}(x)$.

The solutions (16) of Eq. (11) are related to the solution (12) by the formula:

$$r_{12}(\lambda, x) = \frac{e^{\lambda}\, r_{12}^{+}(x) - e^{-\lambda}\, r_{12}^{-}(x)}{e^{\lambda} - e^{-\lambda}}\,.$$

3. The Gervais–Neveu–Felder Equation

In this section, we give the quantum version of Eq. (11). This results into a deformed version of the Quantum Yang–Baxter equation, which first appeared in [21] and later in [9].

We need to introduce some notations. If $\mathscr{G}$ is a simple Lie algebra and $\mathscr{H}$ a Cartan subalgebra of $\mathscr{G}$, let $x = \sum_\nu x_\nu h_\nu$ be an element of $\mathscr{H}$. For any function

$f(x) = f(\{x_\nu\})$ with values in $\mathbf{C}$, we denote

$$f(x + \gamma h^{(i)}) = \mathrm{e}^{\gamma \mathscr{D}^{(i)}} f(x)\, \mathrm{e}^{-\gamma \mathscr{D}^{(i)}} ,$$

where

$$\mathscr{D}^{(i)} = \sum_\nu h_\nu^{(i)} \partial_{x_\nu} .$$

Suppose $V^{(1)}, \dots, V^{(n)}$ are finite dimensional diagonalizable $\mathscr{H}$-modules; the Gervais–Neveu–Felder equation is an equation for a function $R_{ij}(\lambda, x)$ meromorphic in the spectral parameter λ, depending on x, and taking values in $\mathrm{End}(V^{(i)} \otimes V^{(j)})$. It reads

$$\begin{aligned} &R_{12}(\lambda_{12}, x + \gamma h^{(3)}) R_{13}(\lambda_{13}, x - \gamma h^{(2)}) R_{23}(\lambda_{23}, x + \gamma h^{(1)}) \\ &\quad = R_{23}(\lambda_{23}, x - \gamma h^{(1)}) R_{13}(\lambda_{13}, x + \gamma h^{(2)}) R_{12}(\lambda_{12}, x - \gamma h^{(3)}) . \end{aligned} \tag{17}$$

We have used the notation $\lambda_{ij} = \lambda_i - \lambda_j$.

The classical limit of the Gervais–Neveu–Felder equation is obtained as usual by expanding R in powers of $\hbar = -2\gamma$,

$$R_{12}(\lambda, x) = \mathrm{Id} - 2\gamma\, r_{12}(\lambda, x) + \mathrm{O}(\gamma^2) .$$

The first non-trivial term of Eq. (17) is of order γ^2 and stems from:

$$\begin{aligned} &\gamma(r_{12}(x - \gamma h^{(3)}) - r_{12}(x + \gamma h^{(3)}) + r_{13}(x + \gamma h^{(2)}) - r_{13}(x - \gamma h^{(2)}) + r_{23}(x - \gamma h^{(1)}) \\ &\quad - r_{23}(x + \gamma h^{(1)})) + \gamma^2([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) + \mathrm{O}(\gamma^3) = 0 . \end{aligned}$$

The term of order γ^2 yields exactly Eq. (11).

Gervais and Neveu first obtained Eq. (17) in [21] as a result of the quantization of Liouville field theory (this result was later extended to sl_N Toda field theory in [22]). In this quantization procedure, the quantum version of Eq. (15) was shown to take the form of an exchange algebra:

$$\Psi_1(\sigma)\Psi_2(\sigma') = \Psi_2(\sigma')\Psi_1(\sigma) R^{\pm}_{GN}(x, q), \quad \pm = \mathrm{sign}(\sigma - \sigma'), \quad q = \mathrm{e}^{-2\gamma} , \tag{18}$$

where for sl_N [22, 23]

$$\begin{aligned} R^{\pm}_{GN}(x, q) = q^{\mp 1/N} \Bigg[& q^{\pm 1} \sum_{i=1}^N e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{q \mathrm{e}^{x_{ij}} - q^{-1} \mathrm{e}^{-x_{ij}}}{\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}}} e_{ii} \otimes e_{jj} \\ & - (q - q^{-1}) \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\mathrm{e}^{\mp x_{ij}}}{\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}}} e_{ij} \otimes e_{ji} \Bigg] . \end{aligned} \tag{19}$$

Taking into account the shift property of the fields Ψ, that is, for any scalar function $f(x)$:

$$f(x)\ \Psi_1(\sigma) = \Psi_1(\sigma)\ f(x - 2\gamma h^{(1)}), \quad q = \mathrm{e}^{-2\gamma} , \tag{20}$$

the associativity of the Ψ fields algebra yields

$$R_{12}(x) R_{13}(x - 2\gamma h^{(2)}) R_{23}(x) = R_{23}(x - 2\gamma h^{(1)}) R_{13}(x) R_{12}(x - 2\gamma h^{(3)}) .$$

This is equivalent to Eq. (17) if $R_{ij}(x)$ satisfies the relation

$$[\mathscr{D}^{(i)} + \mathscr{D}^{(j)}, R_{ij}(x)] = 0\,, \tag{21}$$

which is true for the R-matrix given by Eq. (19).

Felder [9] interpreted Eq. (17) as a compatibility condition for the algebra of L-operators (following the well known Leningrad school approach [24]):

$$\begin{aligned} &R_{12}(\lambda_{12}, x + \gamma h^{(q)})\widetilde{L}_{1q}(\lambda_1, x - \gamma h^{(2)})\widetilde{L}_{2q}(\lambda_2, x + \gamma h^{(1)}) \\ &\quad = \widetilde{L}_{2q}(\lambda_2, x - \gamma h^{(1)})\widetilde{L}_{1q}(\lambda_1, x + \gamma h^{(2)})R_{12}(\lambda_{12}, x - \gamma h^{(q)})\,. \end{aligned} \tag{22}$$

Here we assume that the matrix elements of $\widetilde{L}_{1q}$ act on a quantum space $V^{(q)}$ which is a $\mathscr{H}$-module so that the action of $h^{(q)}$ is defined. In the following we will be interested in R-matrices and representations $\widetilde{L}_{iq}$ satisfying the properties

$$[h^{(i)} + h^{(j)}, R_{ij}(\lambda_{ij}, x)] = 0\,, \tag{23}$$

$$[h^{(i)} + h^{(q)}, \widetilde{L}_{iq}] = 0\,. \tag{24}$$

From Eq. (20) we see that $\Psi(z)$ in the exchange algebra (18) naturally contains the shift operator $\mathrm{e}^{2\gamma\mathscr{D}}$. By analogy we define a Lax operator:

$$L_{iq}(\lambda, x) = \mathrm{e}^{\gamma\mathscr{D}^{(i)}}\widetilde{L}_{iq}(\lambda, x)\,\mathrm{e}^{\gamma\mathscr{D}^{(i)}}\,. \tag{25}$$

In the limit when $\gamma \to 0$, and assuming that $\widetilde{L}(\lambda, x) = \mathrm{Id} + 2\gamma\widetilde{l}(\lambda, x) + \mathrm{O}(\gamma^2)$, the behaviour of L is

$$L(\lambda, x) = \mathrm{Id} + 2\gamma\left(\sum_\nu h_\nu \frac{\partial}{\partial x_\nu} + \widetilde{l}(\lambda, x)\right) + \mathrm{O}(\gamma^2)\,,$$

which is the typical form (see Eq. (5)) of the Lax matrix of the Calogero–Moser system. The shift operator $\mathrm{e}^{\gamma\mathscr{D}}$ thus contributes to reintroducing the momentum $p_\nu = \partial_{x_\nu}$ on the diagonal.

This operator (25) now obeys the following equation:

$$\begin{aligned} &R_{12}(\lambda_{12}, x + \gamma h^{(q)})L_{1q}(\lambda_1, x)L_{2q}(\lambda_2, x) \\ &\quad = L_{2q}(\lambda_2, x)L_{1q}(\lambda_1, x)R_{12}(\lambda_{12}, x - \gamma h^{(q)})\,, \end{aligned} \tag{26}$$

provided one has

$$[\mathscr{D}^{(1)} + \mathscr{D}^{(2)}, R_{12}(\lambda_{12}, x)] = 0\,. \tag{27}$$

Equation (24) translates into the following shift properties for a scalar function $f(x)$

$$f(x - \gamma h^{(q)})\,L_{iq} = L_{iq}\,f(x - \gamma h^{(q)} - 2\gamma h^{(i)})\,, \tag{28}$$

$$f(x + \gamma h^{(q)} + 2\gamma h^{(i)})\,L_{iq} = L_{iq}\,f(x + \gamma h^{(q)})\,. \tag{29}$$

As in the classical case, if $R_{12}(\lambda_{12}, x)$ is a solution of Eq. (17) having the property $[h^{(1)} + h^{(2)}, R_{12}(\lambda_{12}, x)] = 0$, then

$$\begin{aligned} \widetilde{R}_{12}(\lambda_{12}, x) = {} & \mathrm{e}^{[\alpha(\lambda_1)+\beta]x} \otimes \mathrm{e}^{[\alpha(\lambda_2)-\beta]x}\,\mathrm{e}^{\gamma[\alpha(\lambda_1)-\alpha(\lambda_2)-\beta]h\otimes h}\,R_{12}(\lambda_{12}, x) \\ & \times \mathrm{e}^{\gamma[\alpha(\lambda_1)-\alpha(\lambda_2)+\beta]h\otimes h}\,\mathrm{e}^{-[\alpha(\lambda_1)-\beta]x} \otimes \mathrm{e}^{-[\alpha(\lambda_2)+\beta]x}\,, \end{aligned} \tag{30}$$

defines another solution of Eq. (17) with $\alpha(\lambda)$ an arbitrary function of λ and β an arbitrary parameter. A solution of Eq. (17), the classical limit of which – up to a redefinition of type (30) – is the r-matrix (7), was given in [9]. It reads

$$R_F(\lambda,x) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\sigma(\lambda)\sigma(2\gamma - x_{ij})}{\sigma(2\gamma - \lambda)\sigma(x_{ij})} e_{ii} \otimes e_{jj}$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\sigma(2\gamma)\sigma(x_{ij} - \lambda)}{\sigma(2\gamma - \lambda)\sigma(x_{ij})} e_{ij} \otimes e_{ji} . \qquad (31)$$

Just as in the classical case, the relation between (31) and (19) is obtained by using transformation (30) and by taking the trigonometric limit. One gets

$$R_F(\lambda,x) = \frac{e^{\lambda} q^{1/N} R^{+}_{GN}(x,q) - e^{-\lambda} q^{-1/N} R^{-}_{GN}(x,q)}{q e^{\lambda} - q^{-1} e^{-\lambda}}, \quad q = e^{-2\gamma} . \qquad (32)$$

Let us recall at this point some known facts about the matrices $R^{\pm}_{GN}(x,q)$. These matrices are related to Drinfeld's matrices $R^{\pm}_{D}$ by:

$$R^{\pm}_{GN}(x,q) = F^{-1}_{21}(x)\, R^{\pm}_{D}\, F_{12}(x) ,$$

where $R^{-}_{D,12} = (R^{+}_{D,21})^{-1}$,

$$R^{+}_{D} = \sum_{\substack{i,j=1\\ i\neq j}}^{N} e_{ii} \otimes e_{jj} + q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + (q - q^{-1}) \sum_{\substack{i,j=1\\ i<j}}^{N} e_{ij} \otimes e_{ji} \qquad (33)$$

and

$$F_{12}(x) = \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{1}{e^{x_{ij}} - e^{-x_{ij}}} e_{ii} \otimes e_{jj} + \sum_{\substack{i,j=1\\ i>j}}^{N} \frac{1}{q e^{-x_{ij}} - q^{-1} e^{x_{ij}}} e_{ii} \otimes e_{jj}$$

$$- (q - q^{-1}) \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{e^{x_{ij}}}{(e^{x_{ij}} - e^{-x_{ij}})(q e^{x_{ij}} - q^{-1} e^{-x_{ij}})} e_{ij} \otimes e_{ji} .$$

In the sl_2 case, a universal formula for $F_{12}(x)$ is available [25].

In the framework of Toda field theory, it is known that one can eliminate the x dependence from the exchange algebra (18) by a suitable change of basis [26, 27]: defining

$$\xi(\sigma) = \Psi(\sigma) M(x)$$

with

$$M(x) = \sum_{i,j=1}^{N} e^{2j(x_i - \frac{1}{N}\Sigma x_k)} e_{ij}$$

we get

$$\xi_1(\sigma)\xi_2(\sigma') = \xi_2(\sigma')\xi_1(\sigma) R^{\pm}_{CG}(q), \quad \pm = \mathrm{sign}(\sigma - \sigma')$$

with an R-matrix $R^{\pm}_{CG}(q)$ independent of x [27]:

$$R^{+}_{CG}(q) = q^{-1/N}\Bigg(q\sum_{i=1}^{N} e_{ii}\otimes e_{ii} + q\sum_{\substack{i,j=1\\ i>j}}^{N} q^{-2(i-j)/N} e_{ii}\otimes e_{jj} + q^{-1}\sum_{\substack{i,j=1\\ i<j}}^{N} q^{-2(i-j)/N} e_{ii}\otimes e_{jj}$$
$$- (q-q^{-1})\sum_{\substack{i,j=1\\ i<j}}^{N}\sum_{r=1}^{j-i-1} q^{2r/N} e_{j-r,i}\otimes e_{i+r,j}$$
$$+ (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N}\sum_{r=0}^{i-j-1} q^{-2r/N} e_{j+r,i}\otimes e_{i-r,j}\Bigg)$$

and of course $R^{-}_{CG,12}(q) = [R^{+}_{CG,21}(q)]^{-1}$.

We may wonder whether the x dependence in (26) may not be eliminated by a similar change of variables. Starting from Eq. (26) with the R-matrix given by (32), we set

$$\mathscr{L}(\lambda,x) = M^{-1}(x+\gamma h^{(q)})L(\lambda,x)M(x-\gamma h^{(q)})\,.$$

Then Eq. (26) becomes

$$R^{\star}_{CG}(\lambda-\mu,q)\mathscr{L}_1(\lambda,x)\mathscr{L}_2(\mu,x) = \mathscr{L}_2(\mu,x)\mathscr{L}_1(\lambda,x)R_{CG}(\lambda-\mu,q)$$

with

$$R_{CG,12}(\lambda,q) = \frac{e^{\lambda}R^{+}_{CG}(q) - e^{-\lambda}R^{-}_{CG}(q)}{q\,e^{\lambda} - q^{-1}\,e^{-\lambda}}, \qquad R^{\star}_{CG,12}(\lambda,q) = R_{CG,21}(-\lambda,q^{-1})\,.$$

This equation is reminiscent of the equation studied in [28].

4. Construction of Commuting Operators

We now present a set of commuting operators quantizing the classical quantities $\operatorname{tr} L^n$. We consider in this section an abstract algebraic setting. Examples will be provided in the next section when there is no spectral parameter ($\lambda = \infty$). In that case one can restrict oneself to finite dimensional quantum groups. In the case with spectral parameter, one should consider full affine quantum groups, and this will be left for further investigations.

In the context of the non-shifted Yang–Baxter equation $R_{12}L_1L_2 = L_2L_1R_{12}$, the quantum analogs of the conserved quantities $\operatorname{tr} L^n$ are to be defined [29] as

$$I_n = \operatorname{Tr}_{1\cdots n}\,[L_1\cdots L_n\hat{R}_{12}\hat{R}_{23}\cdots\hat{R}_{n-1,n}]\,,$$

where

$$\hat{R}_{ij} = P_{ij}R_{ij}$$

and P_{ij} are the permutation operators of the spaces i and j.

In the Gervais–Neveu–Felder case, we have the following

Theorem 4.1. *Let $R(x)$ and L be as in Eqs.* (17,26) *with the shift properties as in Eqs.* (28,29), *and condition* (21) *be satisfied.*

We define the operators

$$I_n = \mathrm{Tr}_{1\cdots n}\ [L_1(x)\cdots L_n(x)\hat{R}_{12}(x-2\gamma h^{(3,n)})\cdots \times \hat{R}_{k,k+1}(x-2\gamma h^{(k+2,n)})\cdots \hat{R}_{n-1,n}(x)]\,, \tag{34}$$

where

$$h^{(k,l)} = \sum_{i=k}^{l} h^{(i)}\,.$$

Then:

1) *The operators* I_n *leave the subspace of zero weight vectors invariant* (*vectors* $|V\rangle$ *such that* $h^{(q)}|V\rangle = 0$).

2) *The restrictions of the operators* I_n *to the zero weight subspaces form a set of commuting quantities.*

Proof. To prove (1) we have to prove that $[I_n, h_v^{(q)}] = 0$. This follows immediately from the relations

$$[h^{(q)}, L_i] = -[h^{(i)}, L_i]\,, \qquad [h^{(q)}, \hat{R}_{ij}] = 0\,,$$
$$[h^{(i)} + h^{(j)}, \hat{R}_{ij}] = 0\,.$$

We will decompose the proof of (2) into several lemmas. We will need the important shift properties of L given by Eq. (28, 29).

Lemma 4.1. *On the zero weight subspace, one can write*

$$I_n I_m = \mathrm{Tr}[L_1(x)\cdots L_n(x)L_{n+1}(x)\cdots L_{n+m}(x)\mathscr{I}^{(1,n)}(x-2\gamma h^{(n+1,n+m)})\mathscr{I}^{(n+1,n+m)}(x)]\,,$$

where

$$\mathscr{I}^{(i,j)}(x) = \hat{R}_{i,i+1}(x-2\gamma h^{(i+2,n)})\cdots \hat{R}_{k,k+1}(x-2\gamma h^{(k+2,n)})\cdots \hat{R}_{j-1,j}(x)\,.$$

Proof. Since I_m leaves the zero weight subspace invariant, it is possible to rewrite

$$I_n I_m = \mathrm{Tr}[L_1(x)\cdots L_n(x)\mathscr{I}^{(1,n)}(x)]\ \mathrm{Tr}[L_{n+1}(x)\cdots L_{n+m}(x)\mathscr{I}^{(n+1,n+m)}(x)]$$

as

$$I_n I_m = \mathrm{Tr}[L_1(x)\cdots L_n(x)\mathscr{I}^{(1,n)}(x-\gamma h^{(q)})]\,\mathrm{Tr}[L_{n+1}(x)\cdots L_{n+m}(x)\mathscr{I}^{(n+1,n+m)}(x)]\,.$$

We now push $\mathscr{I}^{(1,n)}(x-\gamma h^{(q)})$ through $L_{n+1}\cdots L_{n+m}$ using Eq. (28). Applying the expression found to the zero weight subspace gives the result. □

Lemma 4.2. *We can rewrite*

$$I_m I_n = \mathrm{Tr}[L_1(x)\cdots L_{n+m}(x)Q_n^{-1}(x)\cdots Q_1^{-1}(x)\mathscr{I}^{(n+1,n+m)}(x-2\gamma h^{(1,n)}) \cdot \mathscr{I}^{(1,n)}(x)Q_1(x)\cdots Q_n(x)]\,,$$

where

$$Q_i(x) = \overleftarrow{\prod_{j=1}^{m}} R_{i,n+j}(x-2\gamma h^{(i+1,n)} - 2\gamma h^{(n+j+1,n+m)})\,.$$

Proof. According to Lemma 4.1,

$$I_m I_n = \mathrm{Tr}[L_{n+1}(x)\cdots L_{n+m}(x)L_1(x)\cdots L_n(x)\mathscr{I}^{(n+1,n+m)} \times (x-\gamma h^{(q)} - 2\gamma h^{(1,n)})\mathscr{I}^{(1,n)}(x)]\,,$$

and using

$$L_{n+m}(x)L_1(x) = R_{1,n+m}(x+\gamma h^{(q)})L_1(x)L_{n+m}(x)R^{-1}_{1,n+m}(x-\gamma h^{(q)})$$

we have

$$I_mI_n = \mathrm{Tr}[L_{n+1}(x)\cdots L_{n+m-1}(x)R_{1,n+m}(x+\gamma h^{(q)})L_1(x)L_{n+m}(x)R^{-1}_{1,n+m}(x-\gamma h^{(q)})$$
$$\times L_2(x)\cdots L_n(x)\mathscr{I}^{(n+1,n+m)}(x-\gamma h^{(q)}-2\gamma h^{(1,n)})\mathscr{I}^{(1,n)}(x)]\,.$$

Using once more Eq. (28,29), Eq. (27) and the cyclicity property of the trace, we get

$$I_mI_n = \mathrm{Tr}\;[L_{n+1}(x)\cdots L_{n+m-1}(x)L_1(x)L_{n+m}(x)L_2(x)\cdots L_n(x)$$
$$\times R^{-1}_{1,n+m}(x-\gamma h^{(q)}-2\gamma h^{(2,n)})\mathscr{I}^{(n+1,n+m)}(x-\gamma h^{(q)}-2\gamma h^{(1,n)})$$
$$\times \mathscr{I}^{(1,n)}(x)R_{1,n+m}(x+\gamma h^{(q)}-2\gamma h^{(2,n)})]\,.$$

Then pushing L_1 through $L_{n+m-1}, L_{n+m-2}, \ldots, L_{n+1}$ gives

$$I_mI_n = \mathrm{Tr}[L_1(x)L_{n+1}(x)\cdots L_{n+m}(x)L_2(x)\cdots L_n(x)Q_1^{-1}(x)$$
$$\times \mathscr{I}^{(n+1,n+m)}(x-\gamma h^{(q)}-2\gamma h^{(1,n)})\mathscr{I}^{(1,n)}(x)Q_1(x)]\,.$$

The result is obtained by repeating the procedure with $L_2, L_3, \ldots, L_n$. □

Comparing Lemmas 4.1 and 4.2, commutation of I_m and I_n will be proved if

$$Q_1(x)\cdots Q_n(x)\mathscr{I}^{(1,n)}(x-2\gamma h^{(n+1,n+m)})\mathscr{I}^{(n+1,n+m)}(x)$$
$$= \mathscr{I}^{(n+1,n+m)}(x-2\gamma h^{(1,n)})\mathscr{I}^{(1,n)}(x)Q_1(x)\cdots Q_n(x)\,.$$

Since $\mathscr{I}^{(1,n)}$ and $\mathscr{I}^{(n+1,n+m)}$ act on different spaces and since $[h^{(n+1,n+m)}, \mathscr{I}^{(n+1,n+m)}(x)] = 0$, this last relation is equivalent to

$$Q_1(x)\cdots Q_n(x)\mathscr{I}^{(n+1,n+m)}(x)\mathscr{I}^{(1,n)}(x-2\gamma h^{(n+1,n+m)})$$
$$= \mathscr{I}^{(n+1,n+m)}(x-2\gamma h^{(1,n)})\mathscr{I}^{(1,n)}(x)Q_1(x)\cdots Q_n(x)\,.$$

We shall prove this relation in two steps:

$$(*)\;\; Q_1(x)\cdots Q_n(x)\mathscr{I}^{(n+1,n+m)}(x) = \mathscr{I}^{(n+1,n+m)}(x-2\gamma h^{(1,n)})Q_1(x)\cdots Q_n(x),$$
$$(**)\;\; Q_n^{-1}(x)\cdots Q_1^{-1}(x)\mathscr{I}^{(1,n)}(x) = \mathscr{I}^{(1,n)}(x-2\gamma h^{(n+1,n+m)})Q_n^{-1}(x)\cdots Q_1^{-1}(x).$$

Relation $(*)$ is a straightforward consequence of the following lemma.

Lemma 4.3. *Defining* $T_i(x) = \mathscr{I}^{(n+1,n+m)}(x-2\gamma h^{(i,n)})$, *we have*

$$T_i(x)Q_i(x) = Q_i(x)T_{i+1}(x)\,.$$

Proof. Lemma 4.3 will be proved if we show that

$$\hat{R}_{n+k,n+k+1}(x-2\gamma h^{(i,n)}-2\gamma h^{(n+k+2,n+m)})Q_i(x)$$
$$= Q_i(x)\hat{R}_{n+k,n+k+1}(x-2\gamma h^{(i+1,n)}-2\gamma h^{(n+k+2,n+m)})\,.$$

Let us write $Q_i(x) = A_{i,k}(x)B_{i,k}(x)C_{i,k}(x)$ with

$$A_{i,k}(x) = \overleftarrow{\prod_{j=k+2}^{m}} R_{i,n+j}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+j+1,n+m)}),$$

$$B_{i,k}(x) = R_{i,n+k+1}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+k+2,n+m)}) \times R_{i,n+k}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+k+1,n+m)}),$$

$$C_{i,k}(x) = \overleftarrow{\prod_{j=1}^{k-1}} R_{i,n+j}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+j+1,n+m)}).$$

Since $R_{i,n+k+2} \cdots R_{i,n+m}(x)$ and $\hat{R}_{n+k,n+k+1}(x)$ act on different spaces and since $[h^{(i)} + h^{(j)}, R_{ij}] = 0$, we have

$$\hat{R}_{n+k,n+k+1}(x - 2\gamma h^{(i,n)} - 2\gamma h^{(n+k+2,n+m)})A_{i,k}(x) = A_{i,k}(x)\hat{R}_{n+k,n+k+1}(x - 2\gamma h^{(i,n)} - 2\gamma h^{(n+k+2,n+m)}).$$

Using then the Yang–Baxter equation

$$\hat{R}_{23}(x - 2\gamma h^{(1)})R_{13}(x)R_{12}(x - 2\gamma h^{(3)}) = R_{13}(x)R_{12}(x - 2\gamma h^{(3)})\hat{R}_{23}(x),$$

we see that

$$\hat{R}_{n+k,n+k+1}(x - 2\gamma h^{(i,n)} - 2\gamma h^{(n+k+2,n+m)})B_{i,k}(x) = B_{i,k}(x)\hat{R}_{n+k,n+k+1}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+k+2,n+m)}).$$

For the same reasons as for $A_{i,k}$,

$$\hat{R}_{n+k,n+k+1}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+k+2,n+m)})C_{i,k}(x) = C_{i,k}(x)\hat{R}_{n+k,n+k+1}(x - 2\gamma h^{(i+1,n)} - 2\gamma h^{(n+k+2,n+m)}),$$

which ends the proof. □

Relation $(**)$ is proved in a similar way: writing $Q_n^{-1} \cdots Q_1^{-1} = S_{n+1}^{-1} \cdots S_{n+m}^{-1}$, where

$$S_j^{-1}(x) = \overleftarrow{\prod_{k=1}^{n}} R_{k,j}^{-1}(x - 2\gamma h^{(k+1,n)} - 2\gamma h^{(j+1,n+m)})$$

and introducing

$$T_j'(x) = \mathscr{I}^{(n)}(x - 2\gamma h^{(j+1,n+m)})$$

we have, similarly to Lemma 4.3,

$$S_j^{-1}(x)T_j'(x) = T_{j-1}'(x)S_j^{-1}(x).$$

This ends the proof of Theorem (4.1). □

5. Examples of Commuting Hamiltonians

We give two applications of the above theorem. In the first one we construct a Ruijsenaars type Hamiltonian with scalar coefficients. The limit $q \to 1$ yields the usual trigonometric Calogero–Moser Hamiltonian. In the second example, we construct a set of commuting finite difference operators with matrix coefficients. Their limit $q \to 1$ is related to the spin generalization of the Calogero–Moser model.

To avoid problems of handling infinite dimensional representations of affine Lie algebras, we restrict ourselves to the trigonometric case where we need to consider only a finite dimensional matrix algebra. We shall construct here the quantum analogs of the classical quantities $\operatorname{tr} L^n(\lambda = +\infty)$. We thus apply Theorem 4.1 with the R-matrix $R^+_{GN}(x, q)$ which is the limit of Eq. (32) when $\lambda \to +\infty$. In the spin Calogero–Moser case, we recall that these commuting Hamiltonians are precisely those which are Yangian-invariant [4, 13].

5.1. The Scalar Case. As required by Theorem 4.1, we need representations of the algebra (22) admitting a non-trivial subspace of zero weights. We shall first consider the representation of algebra (22) analogous to the representation by a completely symmetrized tensor product $N^{\otimes N}$ of the Lie algebra sl_N.

By comparison of Eq. (22) and Eq. (17) we see that $\widetilde{L}^{(N)}_{1q} = R_{1q}$ is a solution of Eq. (22). $\widetilde{L}^{(N)}_{1q}$ is a matrix in an auxiliary space (1)

$$\widetilde{L}^{(N)}_{1q} = \sum_{i,j=1}^{N} e^{(1)}_{ij} \widetilde{L}^{(N)}_{ij} ,$$

the elements of which are quantum operators represented as the following matrices $\widetilde{L}^{(N)}_{ij}$:

$$\widetilde{L}^{(N)}_{ii} = e_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{e^{x_{ij}} - q^{-2} e^{-x_{ij}}}{e^{x_{ij}} - e^{-x_{ij}}} e_{jj} , \tag{35}$$

$$\widetilde{L}^{(N)}_{ij} = -(1 - q^{-2}) \frac{e^{-x_{ij}}}{e^{x_{ij}} - e^{-x_{ij}}} e_{ji} \quad \text{for } i \neq j . \tag{36}$$

Choosing on the auxiliary space

$$h^{(1)}_i = \left(e_{ii} - \frac{1}{N} \mathrm{Id} \right) , \quad i = 1, \ldots, N ,$$

the solution (35, 36) satisfies Eq. (24) with

$$h^{(q)}_i = h^{(N)}_i = \left(e_{ii} - \frac{1}{N} \mathrm{Id} \right) , \quad i = 1, \ldots, N . \tag{37}$$

This is the analog of the vector representation N of sl_N. Next, following [9] one can construct the tensor product of N such representations

$$\widetilde{L}^{(N^{\otimes N})} = \prod_{j=2}^{N+1} \widetilde{L}_{1j} \left(x - \gamma \sum_{1<i<j} h^{(j)} + \gamma \sum_{j<i \leqq N+1} h^{(j)} \right) .$$

As in the Lie algebra case it turns out that there is a unique symmetric zero weight vector

$$|V\rangle = e_1 \otimes e_2 \otimes \cdots \otimes e_N + \text{permutations} .$$

Hence

$$(\operatorname{tr} L^{(N^{\otimes N})}) |V\rangle = \widetilde{\mathscr{H}} |V\rangle .$$

We find after some calculation

$$\widetilde{\mathscr{H}} = \sum_{i=1}^{N} e^{2\gamma p_i} \prod_{j \neq i} \left(\frac{q^2 e^{x_{ij}} - q^{-2} e^{-x_{ij}}}{q e^{x_{ij}} - q^{-1} e^{-x_{ij}}} \right) .$$

One can perform a similarity transformation

$$\mathscr{H} = f(x) \widetilde{\mathscr{H}} \frac{1}{f(x)}; \qquad f(x) = \prod_{k<l} \frac{e^{x_{kl}} - e^{-x_{kl}}}{(q e^{x_{kl}} - q^{-1} e^{-x_{kl}})(q^{-1} e^{x_{kl}} - q e^{-x_{kl}})}$$

to get

$$\mathscr{H} = \sum_{i=1}^{N} e^{2\gamma p_i} \prod_{j \neq i} \left(\frac{(q e^{x_{ij}} - q^{-1} e^{-x_{ij}})(q^{-1} e^{x_{ij}} - q e^{-x_{ij}})}{(e^{x_{ij}} - e^{-x_{ij}})^2} \right) .$$

In this form the limit $q = e^{-2\gamma} \to 1$ becomes simple. We find

$$\mathscr{H} = N + 2\gamma \sum_{i=1}^{N} p_i + (2\gamma)^2 \left(\frac{1}{2} \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{\sinh^2 x_{ij}} \right) + O(\gamma^3) .$$

Thus we recover the usual trigonometric Calogero–Moser Hamiltonian.

5.2. The Spin Case. We shall now construct the representation of algebra (22) analogous to the representations $\bar{N}$ of the Lie algebra sl_N and take its tensor product with the representation N. As in the Lie algebra case, the tensor product will have a structure similar to the standard decomposition, $N \otimes \bar{N} = 1 + \mathrm{ad}$, and admit a subspace of zero weight vectors of dimension N. The Hamiltonians we will construct act in this zero weight subspace.

One can find another solution of Eq. (22), given by (see also [23])

$$\widetilde{L}_{ii}^{(\bar{N})} = e_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mathrm{e}^{x_{ij}} - q^2 \mathrm{e}^{-x_{ij}}}{\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}}} e_{jj} , \tag{38}$$

$$\widetilde{L}_{ij}^{(\bar{N})} = (q - q^{-1}) \frac{\mathrm{e}^{-x_{ij}}}{q \mathrm{e}^{x_{ij}} - q^{-1} \mathrm{e}^{-x_{ij}}} e_{ij} \quad \text{for } i \neq j . \tag{39}$$

Remark that $\widetilde{L}^{(\bar{N})}$ is essentially the transposed of $\widetilde{L}^{(N)}$. In this case we have

$$h_i^{(q)} = h_i^{(\bar{N})} = -\left(e_{ii} - \frac{1}{N} \mathrm{Id} \right) , \quad i = 1, \ldots, N . \tag{40}$$

Notice the sign difference between Eq. (37) and Eq. (40). Following [9] one now constructs the tensor product of the two representations:

$$\widetilde{L}_{ij}^{(N \otimes \bar{N})}(x) = \sum_{k=1}^{N} \widetilde{L}_{ik}^{(N)}(x + \gamma h^{(\bar{N})}) \widetilde{L}_{kj}^{(\bar{N})}(x - \gamma h^{(N)}) \tag{41}$$

and

$$h_i^{(q)} = h_i^{(N \otimes \bar{N})} = \left(e_{ii} - \frac{1}{N} \mathrm{Id} \right) \otimes \mathrm{Id} - \mathrm{Id} \otimes \left(e_{ii} - \frac{1}{N} \mathrm{Id} \right) \quad \text{for } i = 1, \ldots, N . \tag{42}$$

From this last formula, we see that the subspace of zero weight vectors admits $\{E_i = e_i \otimes e_i\}_{i=1 \cdots N}$ as a basis. We introduce the canonical basis $\{E_{ij}\}_{i,j=1 \cdots N}$ of matrices acting on this subspace by $E_{ij} E_j = E_i$.

Applying Theorem 4.1 to the L operator

$$L_{1q} = \sum_{i,j=1}^{N} e_{ij}^{(1)} \; [\mathrm{e}^{\gamma p_i} \widetilde{L}_{ij}^{(N\otimes\bar{N})}(x)\, \mathrm{e}^{\gamma p_j}]$$

given by formulas (35, 36, 38, 39, 41), we find

$$I_1 = \mathscr{H}_1 ,$$

$$I_2 = I_1^2 - (1+q^{-2})\mathscr{H}_2 ,$$

$$I_3 = -\frac{1}{1+q^{-2}} I_1^3 + \left(1 + \frac{1}{1+q^{-2}}\right) I_2 I_1 + (1+q^{-2}+q^{-4})\mathscr{H}_3 ,$$

where the operators $\mathscr{H}_{1,2,3}$ are

$$\mathscr{H}_1 = \sum_{i=1}^{N} \mathrm{e}^{2\gamma p_i} \left[\mathrm{Id} + q^2(1-q^{-2})^2 \sum_{\substack{j=1\\ j\neq i}}^{N} V_{ji}(x)(E_{ji} - E_{jj}) \right] ,$$

$$\mathscr{H}_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathrm{e}^{2\gamma(p_i+p_j)} \left[\frac{1}{2}\mathrm{Id} + (1-q^{-2})^2 \sum_{\substack{k=1\\ k\neq i,j}}^{N} V_{kji}(x)(E_{ki} - E_{kk}) \right] ,$$

$$\mathscr{H}_3 = \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{N} \mathrm{e}^{2\gamma(p_i+p_j+p_k)} \left[\frac{1}{6}\mathrm{Id} + \frac{1}{2} q^{-2}(1-q^{-2})^2 \sum_{\substack{l=1\\ l\neq i,j,k}}^{N} V_{lkji}(x)(E_{li} - E_{ll}) \right] ,$$

and

$$V_{ji}(x) = \frac{1}{(\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}})^2} ,$$

$$V_{kji}(x) = \frac{1}{(\mathrm{e}^{x_{ik}} - \mathrm{e}^{-x_{ik}})^2} \frac{(q^2\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}})(q^2\mathrm{e}^{x_{jk}} - \mathrm{e}^{-x_{jk}})}{(\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}})(\mathrm{e}^{x_{jk}} - \mathrm{e}^{-x_{jk}})} ,$$

$$V_{lkji}(x) = \frac{1}{(\mathrm{e}^{x_{il}} - \mathrm{e}^{-x_{il}})^2} \frac{(q^2\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}})(q^2\mathrm{e}^{x_{jl}} - \mathrm{e}^{-x_{jl}})(q^2\mathrm{e}^{x_{ik}} - \mathrm{e}^{-x_{ik}})(q^2\mathrm{e}^{x_{kl}} - \mathrm{e}^{-x_{kl}})}{(\mathrm{e}^{x_{ij}} - \mathrm{e}^{-x_{ij}})(\mathrm{e}^{x_{jl}} - \mathrm{e}^{-x_{jl}})(\mathrm{e}^{x_{ik}} - \mathrm{e}^{-x_{ik}})(\mathrm{e}^{x_{kl}} - \mathrm{e}^{-x_{kl}})} .$$

Generalizing the preceding formulas, we introduce quantities $\{\mathscr{H}_n\}_{n=1,\ldots,N}$ defined as

$$\mathscr{H}_n = \sum_{\substack{i_1,\ldots,i_n=1\\ i_1\neq i_2\neq\cdots\neq i_n}}^{N} e^{2\gamma\Sigma_{k=1}^{n} p_{i_k}} \left(\frac{1}{n!}\mathrm{Id} + \frac{q^{-2(n-2)}(1-q^{-2})^2}{(n-1)!} \times \sum_{\substack{i_0=1\\ i_0\neq i_1,\ldots,i_n}}^{N} V_{i_0 i_n \cdots i_1}(x)(E_{i_0 i_1} - E_{i_0 i_0}) \right) \tag{43}$$

with

$$V_{i_0 i_n \cdots i_1}(x) = V_{i_0 i_{n-1} \cdots i_1}(x) \frac{(q^2 e^{x_{i_1 i_n}} - e^{-x_{i_1 i_n}})(q^2 e^{x_{i_n i_0}} - e^{-x_{i_n i_0}})}{(e^{x_{i_1 i_n}} - e^{-x_{i_1 i_n}})(e^{x_{i_n i_0}} - e^{-x_{i_n i_0}})}, \tag{44}$$

$$V_{i_0 i_1}(x) = \frac{1}{(e^{x_{i_0 i_1}} - e^{-x_{i_0 i_1}})^2}. \tag{45}$$

We have checked directly, up to 5 particles, that the $\{\mathscr{H}_n\}_{n=1,\dots,N}$ form a set of commuting operators.

The occurrence of the matrices $E_{ji} - E_{jj}$ immediately shows that the vector

$$|s\rangle = \sum_{i=1}^{N} e_i \otimes e_i$$

is left invariant by all the Hamiltonians $\mathscr{H}_n$. Their restriction to this one dimensional subspace is the abelian algebra of the symmetric polynomials in $e^{2\gamma p_i}$.

To recover the usual Calogero–Moser Hamiltonian we have to consider the expansion around $\gamma = 0$ of the above Hamiltonians. To order γ^2 we find

$$\mathscr{H}_n = C_N^n \,\mathrm{Id} + 2\gamma\, C_{N-1}^{n-1}\, \mathscr{H}_1^{CM} + 2\gamma^2 [C_{N-2}^{n-2} (\mathscr{H}_1^{CM})^2 + C_{N-2}^{n-1}\, \mathscr{H}_2^{CM}] + O(\gamma^3), \tag{46}$$

where C_N^n are the usual binomial coefficients and

$$\mathscr{H}_1^{CM} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}, \tag{47}$$

$$\mathscr{H}_2^{CM} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{1}{\sinh^2(x_{ij})} (E_{ij} + E_{ji} - E_{ii} - E_{jj}). \tag{48}$$

The matrices $E_{ij} + E_{ji} - E_{ii} - E_{jj}$ admit a simple interpretation in terms of the "spin operator" h_{ij} in the tensor product representation $N \otimes \bar{N}$,

$$h_{ij} = e_{ij} \otimes \mathrm{Id} - \mathrm{Id} \otimes e_{ji}.$$

Indeed we have

$$h_{ij} h_{ji}|_{\text{zero weight}} = E_{ii} + E_{jj} - E_{ij} - E_{ji},$$

hence in this representation we do recover the spin Calogero–Moser Hamiltonian

$$\mathscr{H}_2^{CM} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} h_{ij} h_{ji} \frac{1}{\sinh^2(x_{ij})} \tag{49}$$

as the first non-trivial order of $\mathscr{H}_n$.

We would like to stress that the above examples are built out of the simplest representations of (22) admitting a non-trivial zero-weight subspace. More general representations will affect, among other things, the value of the coupling constant. As indicated by the sl_2 case [25], the representation theory of Eq. (22) is intimately tied to the representation theory of quantum groups, but the link remains to be fully elucidated.

References

1. Calogero, F.: Exactly solvable one-dimensional many-body problems. Lett. Nuovo Cimento **13**, 411 (1975). On a functional equation connected with integrable many-body problems. Lett. Nuovo Cimento **16**, 77 (1976); Moser, J.: Three integrable Hamiltonian systems connected to isospectral deformations. Adv. Math. **16**, 1 (1976)
2. Ruijsenaars, S.N.M.: Complete integrability of relativistic Calogero–Moser systems and elliptic function identities. Commun. Math. Phys. **110**, 191 (1987); van Deijen, J.F.: Families of commuting difference operators. Ph.D. Thesis, University of Amsterdam (1994)
3. Gibbons, J., Hermsen, T.: A generalisation of the Calogero–Moser system. Physica **11D**, 337 (1984); Krichever, I.M., Babelon, O., Billey, E., Talon, M.: Spin generalization of Calogero–Moser system and the matrix KP equation. hep-th/9411160
4. Bernard, D., Gaudin, M., Haldane, F.D.M., Pasquier, V.: Yang–Baxter equation in spin chains with long range interactions. J. Phys. A: Math. Gen. **26**, 5219 (1993), hep-th/9301084
5. Krichever, I., Zabrodin, A.: Spin generalization of the Ruijsenaars–Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra. hep-th/9505039
6. Olshanetsky, M.A., Perelomov, A.M.: Classical integrable finite-dimensional systems related to Lie algebras. Phys. Rep. **71**, 314 (1981); Quantum integrable systems related to Lie algebras. Phys. Rep. **94**, 313 (1983)
7. Hitchin, N.: Stable bundles and integrable systems. Duke Math. J. **54**, 91 (1987); Nekrasov, N.: Holomorphic bundles and many-body systems. hep-th/9503157
8. Gorsky, A., Nekrasov, N.: Elliptic Calogero–Moser system from two-dimensional current algebra. Nucl. Phys. **B436**, 532 (1995), hep-th/9401021; Relativistic Calogero–Moser model as gauged WZW theory. Theor. Math. Phys. **100**, 874 (1994), hep-th/9401017; Gorsky, A.: Integrable many body systems in the field theories. hep-th/9410228
9. Felder, G.: Conformal field theory and integrable systems associated to elliptic curves. hepth/9407154
10. Etingof, P.I., Kirillov, A.A.: On the affine analogs of Jack's and Macdonald's polynomials. hepth/9403168. MacDonald's polynomials and representations of quantum groups. Math. Res. Lett. **1**, 279 (1994)
11. Avan, J., Jevicki, A.: Classical integrability and higher symmetries of collective string field theory. Phys. Lett. **B 266**, 35 (1991)
12. Avan, J., Talon, M.: Classical R-matrix structure for the Calogero model. Phys. Lett. **B 303**, 33 (1993); Sklyanin, E.K.: Dynamical r-matrices for the elliptic Calogero–Moser model. Alg. and Anal. **6**, 227 (1994), hep-th/9308060; Braden, H.W., Suzuki, T.: R-matrices for elliptic Calogero–Moser models. Lett. Math. Phys. **30**, 147 (1994), hep-th/9309033
13. Billey, E., Avan, J., Babelon, O.: The r-matrix structure of the Euler–Calogero–Moser model. Phys. Lett. **A 186**, 114 (1994), hep-th/9312042; Exact Yangian symmetry in the classical Euler–Calogero–Moser model. Phys. Lett. **A 188**, 263 (1994), hep-th/9401117
14. Babelon, O., Bernard, D.: The sine-Gordon solitons as an N-body problem. Phys. Lett. **B 317**, 363 (1993)
15. Avan, J., Rollet, G.: The classical r-matrix for the relativistic Ruijsenaars–Schneider system. Preprint Brown HET 1014 (1995)
16. Krichever, I.: Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles. Funct. Anal. Appl. **14**, 282 (1980)
17. Babelon, O., Viallet, C.M.: Hamiltonian structures and Lax equations. Phys. Lett. **B 237**, 411 (1990)
18. Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess–Zumino model in two dimensions. Nucl. Phys. **B 247**, 83 (1984); Bernard, D.: On the Wess–Zumino–Witten models on the torus. Nucl. Phys. **B 303**, 77 (1988). On the Wess–Zumino–Witten models on Riemann surfaces. Nucl. Phys. **B 309**, 145 (1988); Felder, G., Wieczerkowski, C.: Conformal field theory on elliptic curves and Knizhnik–Zamolodchikov–Bernard equation. hep-th/9411004
19. Leznov, A.N., Saveliev, M.V.: Representation of zero-curvature for the system of non-linear partial differential equations $x_{\alpha, z\bar{z}} = \exp(hx)_\alpha$ and its integrability. Lett. Math. Phys. **3**, 489 (1979)
20. Babelon, O., Bonora, L., Toppan, F.: Exchange algebra and the Drinfeld–Sokolov theorem. Commun. Math. Phys. **140**, 93 (1991); Balog, J., Dabrowski, L., Fehér, L.: Classical r-matrix and exchange algebra in WZNW and Toda theories. Phys. Lett. **B 244**, 227 (1990)

21. Gervais, J.L., Neveu, A.: Novel triangle relation and absence of tachyons in Liouville string field theory. Nucl. Phys. **B 238**, 125 (1984)
22. Bilal, A., Gervais, J.L.: Systematic construction of conformal theories with higher spin Virasoro symmetries. Nucl. Phys. **B 318**, 579 (1989)
23. Bonora, L., Bonservizi, V.: Quantum sl_n Toda field theories. Nucl. Phys. **B 390**, 205 (1993)
24. Faddeev, L.D.: Integrable models in $1+1$ dimensional quantum field theory. Recent Advances in Field Theory and Statistical Mechanics (Les Houches Summer School 1982), ed. by Zuber, J.B. and Stora, R., Amsterdam: North Holland, 1984
25. Babelon, O.: Universal exchange algebra for Bloch waves and Liouville theory. Commun. Math. Phys. **139**, 619 (1991)
26. Babelon, O.: Extended conformal algebra and the Yang–Baxter equation. Phys. Lett. **B 215**, 523 (1988)
27. Cremmer, E., Gervais, J.L.: The quantum group structure associated with non-linearly extended Virasoro algebras. Commun. Math. Phys. **134**, 619 (1990)
28. Foda, O., Iohara, K., Jimbo, M., Kedem, R., Miwa, T., Yan, H.: An elliptic quantum algebra for $\hat{sl}_2$. Lett. Math. Phys. **32**, 259 (1994), hep-th/9403094
29. Maillet, J.M.: Lax equations and quantum groups. Phys. Lett. **B 245**, 480 (1990)

Communicated by M. Jimbo