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Crystallizing the Spinon Basis

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Abstract: The quasi-particle structure of the higher spin XXZ model is studied. We obtained a new description of crystals associated with the level k integrable highest weight $U_q(\widehat{sl_2})$ modules in terms of the creation operators at $q = 0$ (the crystalline spinon basis). The fermionic character formulas and the Yangian structure of those integrable modules naturally follow from this description. We have also derived the conjectural formulas for the multi quasi-particle states at $q = 0$.

0. Introduction

In this paper, we consider the integrable XXZ spin chain with spin $k/2$ of sl_2. The space of states is the infinite tensor product (the space of local fields)

$$\mathscr{W} = \cdots \otimes \mathbf{C}^{k+1} \otimes \mathbf{C}^{k+1} \otimes \mathbf{C}^{k+1} \otimes \mathbf{C}^{k+1} \otimes \cdots .$$

In remarkable papers [7, 25], using the Bethe Ansatz, Faddeev and Takhatajan discovered that the one-particle excitation in the anti-ferromagnetic regime of XXX chain is always (a kink of) spin 1/2. According to this picture, one can expect another description of space of states [7, 23] such as (the space of asymptotic particles)

$$\mathscr{F} = \sum_{n=0}^{\infty} \left[\sum_{p \in \text{path}} \mathbf{C}((z_1, \ldots, z_n)) \otimes [\otimes^n \mathbf{C}^2]_p \right]^{\text{Symm}} ,$$

where Symm is the symmetrization with respect to the S-matrix. On the other hand, stimulated by the deep results by Smirnov [24], the third description of the space of states in terms of the representations of $U_q(\widehat{sl_2})$ (the space of non-local symmetries) was proposed [5, 10]

$$\mathscr{H} = \sum_{i,j=0}^{k} V(\lambda_i) \hat{\otimes} V(\lambda_j)^* .$$

Here $V(\lambda_i)$ is the integrable highest weight representation of $U_q(\widehat{sl_2})$ and $\hat{\otimes}$ means some extended tensor product. The particle picture is recovered using the q-vertex operators acting among them.

In [3, 4], a new description of the Hilbert space of the (chiral) WZW conformal field theory was obtained. The spin 1/2 vertex operators were identified with the particles (spinons) and the basis of the integrable $\widehat{sl_2}$ modules are determined in terms of the spinons (spinon basis). Their picture is clear, but technically it is quite hard because of the very complicated algebra of spinons.

The aim of this paper is to establish the remarkable equivalence

$$\mathscr{H} = \mathscr{F} ,$$

rigorously, focusing on its combinatorial aspects, that is at the level of crystals. In a sense this can be considered as the crystallization of the q-deformation of [3, 4].

One of the (and the most remarkable) advantages to consider the q-analogue is that one can go into $q \to 0$ limit where everything becomes clear and transparent (the "crystal" theory). It is expected that in the limit $q \to 0$ the resulting algebraic structures for the "crystalline spinon" will be much simpler than that of $q = 1$.

Here we must recall that there are two kinds of q-deformation of the vertex operators, called type I and type II in the terminology of [5]. They almost have the same properties as far as we consider them separately. However, since it is crucial, in the study of spin chains, to consider them simultaneously, they inevitably reveal different faces.

While the type I q-vertex operators have a well defined $q = 0$ limit and play the central role for the equivalence $\mathscr{H} = \mathscr{W}$, the existence of $q \to 0$ limits of the creation operators (those represented as type II q-VO) is not clear. Indeed it is known that type II q-VO produces poles at $q = 0$, hence the $q \to 0$ limit does *not* exist in a naive sense.

Nevertheless, it is conjectured [5] that those poles summed up to a meromorphic function in spectral parameters and the $q \to 0$ limits of the creation operators are well-defined, if they act on the *true vacuum*, the ground state of the model. In fact what we need here is this type II q-vertex operators.

One way to avoid this subtle problem is to construct the creation operators and their action on the path basis directly at $q = 0$, rather than taking the limit $q \to 0$. For the level $k = 1$ case, this kind of description of $q = 0$ creation operators was considered previously in [5], and we will generalize it to higher levels $k \geqq 2$. Though there still remains a deep gap between $q = 0$ and $q \neq 0$ descriptions, the thus obtained statements of combinatorial nature (such as character formulas) are independent of q and available to generic q including 1. This is reminiscent of the "Combinatorial Bethe Ansatz" due to [19].

The paper is organized as follows. Section 1 is a quick review of the path realization that shows the equivalence $\mathscr{H} = \mathscr{W}$. Then in Sect. 2, we define and study the algebra of the creation operators at $q = 0$, and give a precise combinatorial description of the space $\mathscr{F}$ including subtleties on the statistics. In Sect. 3, we formulate and prove our main theorem (Theorem 3) that gives bijection (isomorphism of crystals) between the space of fields $\mathscr{W}$ and the space of particles $\mathscr{F}$. In Sects. 4 and 5, we discuss the Yangian structure and spinon (or fermionic) character formulas. These kinds of structures, that are conjectured by recent TBA analysis (see for example [17, 20, 18]), will be obtained as corollaries of the results in Sect. 3. In Sect. 6 the conjectural formulas for the $q = 0$ limit of the quasi-particles of the spin $k/2$ XXZ model are given in terms of path basis. This is also an easy consequence of the commutation relations of creation operators at $q = 0$. Section 7 is devoted to comments and discussions.

1. Review of the Path Realization

Here we review a path realization of crystals. Unless otherwise stated we follow the notations of Sect. 2 of [10]. For other notations and fundamental properties of crystals which we use in this paper we refer to [16, 22]. We denote by $B^{(k)} = \{b_l^{(k)} | 0 \leqq l \leqq k\}$ the associated crystal to the crystal base of the $k+1$ dimensional $U_q(\widehat{sl_2})$ module $V^{(k)}$. The actions of $\tilde{e}_i$, $\tilde{f}_i$ $(i=0,1)$ on $B^{(k)}$ are specified by

$$\tilde{f}_1 b_l^{(k)} = \tilde{e}_0 b_l^{(k)} = b_{l+1}^{(k)}, \quad 0 \leqq l \leqq k-1 ,$$

$$\tilde{e}_1 b_l^{(k)} = \tilde{f}_0 b_l^{(k)} = b_{l-1}^{(k)}, \quad 1 \leqq l \leqq k ,$$

and $\tilde{x}_i b_l^{(k)} = 0$ (otherwise) for $x = e, f$, $i = 0, 1$. We call a crystal defined from $U_q(\widehat{sl_2})$ a $\widehat{sl_2}$ crystal. An $\widehat{sl_2}$ crystal is called the $(\widehat{sl_2})_i$ crystal $(i = 0, 1)$ if we forget the actions of $\tilde{e}_{1-i}$, $\tilde{f}_{1-i}$. Let P and P^* denote the weight and the dual weight lattice of $\widehat{sl_2}$, $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta$, $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d$, $\langle h_i, \Lambda_j \rangle = \delta_{ij}$, $\langle d, \delta \rangle = 1$, $\langle h_i, \delta \rangle = \langle d, \Lambda_j \rangle = 0$, $\{h_i\}$ being the simple coroots.

Let us set $\mathscr{P}^{(m,m')} = B(\lambda_m) \otimes B(\lambda_{m'})^*$, which is the crystal associated to $V(\lambda_m) \otimes V(\lambda_{m'})^{*a}$. The ground state path $\bar{p}_m$ is defined by

$$\bar{p}_m(l) = m + (k-2m)\varepsilon(l) \quad l \in \mathbf{Z} ,$$

where $\varepsilon(l) = 0(l : even) = 1(l : odd)$. Define the (m, m') ground state path by

$$\bar{p}_{m,m'}(l) = \begin{cases} \bar{p}_m(l) & l \geqq 1 , \\ \bar{p}_{m'}(l) & l < 0 . \end{cases}$$

Then

Proposition 1. *There is a bijection*

$$\mathscr{P}^{(m,m')} = \{p = (p(l))_{l \in \mathbf{Z}} \mid p(l) \in \{0, \dots, k\}, p(l) = \bar{p}_{m,m'}(l) \ (|l| \gg 0)\} .$$

We call an element in the right-hand side of this an (m, m') path. From now on we identify $\mathscr{P}^{(m,m')}$ with the set of (m, m') paths.

A (m, m') path can be described in terms of domain walls. For $p \in \mathscr{P}^{(m,m')}$, we can associate the uniquely determined data

$$n \in \mathbf{Z}_{\geqq 0}, \quad (l_n, \dots, l_1) \in \mathbf{Z}^n, \quad \text{and} \quad (m_n, \dots, m_0) \in \{0, \dots, k\}^{n+1}$$

such that

$$l_n \geqq \cdots \geqq l_1, \quad m_n = m, \quad m_0 = m', \quad |m_j - m_{j-1}| = 1 \quad \text{for any } j,$$

$$\text{if } l_{s+s'} > l_{s+s'-1} = \cdots = l_s > l_{s-1}, \text{ then } |m_{s+s'-1} - m_s| = s',$$

$$p(l) = \bar{p}_{m_r}(l) \quad l_{r+1} \geqq l \geqq l_r + 1. \tag{1}$$

If $l_r = l_{r+1}$, we understand that there are no corresponding conditions of the form (1). Conversely these data uniquely determines the (m, m') path. For a fixed sequence $(m_n, \dots, m_0)$ we denote this path by $[[l_n, \dots, l_1]]$.

Let us describe the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $[[l_n,\dots,l_1]]$. Set

$$s_j = \mathrm{sign}(p(l_j) + p(l_j+1) - k) \quad 1 \leqq j \leqq n \tag{2}$$

and $r_j = \frac{1}{2}(1+s_j)$, where $\mathrm{sign}(j) = 1(j>0) = -1(j<0)$. Note that $p(l_j) + p(l_j+1) - k \neq 0$. Let us associate the element b of $B^{(1)\otimes n}$ with the path p by

$$b = b^{(1)}_{r_n} \otimes \cdots \otimes b^{(1)}_{r_1} . \tag{3}$$

Suppose that

$$\tilde{x}_i b = b^{(1)}_{r_n} \otimes \cdots \otimes \tilde{x}_i b^{(1)}_{r_j} \otimes \cdots \otimes b^{(1)}_{r_1} . \tag{4}$$

Then

$$\tilde{x}_i[[l_n,\dots,l_1]] = \begin{cases} 0, & \text{if } \tilde{x}_i b = 0 , \\ [[l'_n,\dots,l'_1]], & \text{if } \tilde{x}_i b \neq 0 , \end{cases} \tag{5}$$

where $l'_s = l_s \ (s \neq j)$ and

$$l'_j = \begin{cases} l_j + 1, & \text{for } x = f , \\ l_j - 1, & \text{for } x = e . \end{cases} \tag{6}$$

The weight of a (m,m') path p is given by

$$wt(p) = (m - m' + 2s(p))(\Lambda_1 - \Lambda_0) - \omega(p)\delta ,$$

$$s(p) = \sum_{l \in \mathbf{Z}} (\bar{p}_{m,m'}(l) - p(l)) ,$$

$$\omega(p) = \sum_{l \in \mathbf{Z}} l(\tilde{H}(p(l+1), p(l)) - \tilde{H}(\bar{p}_{m,m'}(l+1), \bar{p}_{m,m'}(l))) ,$$

where the function $\tilde{H}(j,j')$ is defined by

$$\tilde{H}(j,j') = \begin{cases} -j' & \text{if } j + j' \leqq k , \\ j - k & \text{if } j + j' > k . \end{cases}$$

2. Creation Algebra at $q = 0$

Definition 1. *The algebra $\mathscr{A}$ is generated by $\{\varphi^{*p}_j \mid j \in \mathbf{Z},\ p \in \{0,1\}\} \cup \{1\}$ over $\mathbf{Z}$ subject to the following defining relations*:

$$\varphi^{*p_1}_{j_1}\varphi^{*p_2}_{j_2} + \varphi^{*p_1}_{j_2}\varphi^{*p_2}_{j_1} = 0, \quad j_1 = j_2 \,\mathrm{mod}.2 \quad \text{and} \quad (p_1,p_2) = (1,0) ,$$

$$\varphi^{*p_1}_{j_1}\varphi^{*p_2}_{j_2} + \varphi^{*p_1}_{j_2+1}\varphi^{*p_2}_{j_1-1} = 0, \quad j_1 \neq j_2 \,\mathrm{mod}.2 \quad \text{and} \quad (p_1,p_2) = (1,0) ,$$

$$\varphi^{*p_1}_{j_1}\varphi^{*p_2}_{j_2} + \varphi^{*p_1}_{j_2-2}\varphi^{*p_2}_{j_1+2} = 0, \quad j_1 = j_2 \,\mathrm{mod}.2 \quad \text{and} \quad (p_1,p_2) \neq (1,0) ,$$

$$\varphi^{*p_1}_{j_1}\varphi^{*p_2}_{j_2} + \varphi^{*p_1}_{j_2-1}\varphi^{*p_2}_{j_1+1} = 0, \quad j_1 \neq j_2 \,\mathrm{mod}.2 \quad \text{and} \quad (p_1,p_2) \neq (1,0) ,$$

$$\varphi^{*p}_j 1 = 1\varphi^{*p}_j = \varphi^{*p}_j .$$

As a special case of the above defining equations we have

$$\varphi_j^{*p_1}\varphi_j^{*p_2} = \varphi_j^{*p_1}\varphi_{j-1}^{*p_2} = 0, \quad \text{if } (p_1,p_2) = (1,0)\,,$$

$$\varphi_j^{*p_1}\varphi_{j+1}^{*p_2} = \varphi_j^{*p_1}\varphi_{j+2}^{*p_2} = 0, \quad \text{if } (p_1,p_2) \neq (1,0)\,.$$

We call $\mathscr{A}$ the crystalline creation algebra. The derivation of the above commutation relations is explained in Appendix A. The algebra $\mathscr{A}$ is naturally graded by

$$\mathscr{A} = \bigoplus_{n=0}^{\infty}\mathscr{A}_n\,,$$

$$\mathscr{A}_n = \sum_{(p_n,\dots,p_1),(j_n,\dots,j_1)} \mathbf{Z}\varphi_{j_n}^{*p_n}\cdots\varphi_{j_1}^{*p_1}, \quad \mathscr{A}_0 = \mathbf{Z}\,. \tag{7}$$

Let us introduce the functions H, K, ε by

$$H(0,0) = H(0,1) = H(1,1) = 0, \quad H(1,0) = 1\,,$$

$$K(p_1,p_2) = 1 - H(p_1,p_2), \quad \varepsilon(j) = \frac{1-(-1)^j}{2}\,.$$

Then the commutation relations can be written compactly as

$$\varphi_{j_2}^{*p_2}\varphi_{j_1}^{*p_1} + \varphi_{j_1-2K(p_2,p_1)+\varepsilon(j_1+j_2)}^{*p_2}\varphi_{j_2+2K(p_2,p_1)-\varepsilon(j_1+j_2)}^{*p_1} = 0\,.$$

Let us set

$$B = \bigcup_{n=1}^{\infty}\{\varphi_{j_n}^{*p_n}\cdots\varphi_{j_1}^{*p_1} \mid j_l \in \mathbf{Z},\ p_l \in \{0,1\}\}\,,$$

$$B(p_n,\dots,p_1) = \{\varphi_{j_n}^{*p_n}\cdots\varphi_{j_1}^{*p_1} \mid (j_n,\dots,j_1) \text{ satisfies the condition (8)}\}\,,$$

$$B_{\geqq 0}(p_n,\dots,p_1) = \{\varphi_{j_n}^{*p_n}\cdots\varphi_{j_1}^{*p_1} \in B(p_n,\dots,p_1) \mid j_1 \geqq 0\}\,.$$

If $n = 0$, we define $(p_n,\dots,p_1) = \phi$ and $B(\phi) = \{1\}$. The condition is

$$j_n - 2I_n(p_n,\dots,p_1) \geqq \cdots \geqq j_2 - 2I_2(p_2,p_1) \geqq j_1, \tag{8}$$

$$I_l(p_l,\dots,p_1) = \sum_{s=1}^{l-1} H(p_{s+1},p_s)\,.$$

Theorem 1. $\sqcup_{(p_n,\dots,p_1)\in\{0,1\}^n} B(p_n,\dots,p_1)$ *is a* $\mathbf{Z}$ *linear base of* $\mathscr{A}_n$.

We prove the theorem in a more general setting. Let J be a set. Let us consider the associative algebra over $\mathbf{Z}$ with unit generated by $\{\psi_A(n) \mid A \in J, n \in \mathbf{Z}\}$ whose defining relations are

$$\psi_A(n)\psi_B(m) + \psi_A(m+s_{AB})\psi_B(n-s_{AB}) = 0\,,$$

where s_{AB} is some fixed integer. In what follows, we fix a sequence $(A_1,\dots,A_n)$ and put $s_{i,i+1} = s_{A_i,A_{i+1}}$. In general let us define

$$s_{i,j} = \begin{cases} s_{i,i+1} + s_{i+1,i+2} + \cdots + s_{j-1,j}, & (i < j)\,, \\ 0, & (i = j)\,, \\ -s_{j,i}, & (i > j)\,. \end{cases}$$

Lemma 1. *For any permutation σ of $(1,2,\ldots,n)$,*

$$\psi_{A_1}(m_1)\cdots\psi_{A_n}(m_n) = \mathrm{sgn}(\sigma)\psi_{A_1}(m_{1\sigma(1)})\cdots\psi_{A_n}(m_{n\sigma(n)}) ,$$

where

$$m_{i,j} = m_j + s_{i,j} .$$

Proof. By the induction on the length of σ, the lemma is easily proved. □

Definition 2. *In a product*

$$\Psi = \psi_{A_1}(m_1)\cdots\psi_{A_n}(m_n) ,$$

the index pair (m_i, m_j) $(i < j)$ is said to be normal (zero, abnormal) iff

$$m_i > m_j + s_{i,j} \quad (m_i = m_j + s_{i,j}, \quad m_i < m_j + s_{i,j}) .$$

Furthermore, Ψ is normal iff all pairs are normal.

Lemma 2. *The notion of normal (zero, abnormal) of two indices (m_i, m_j) is independent of the position of them as far as their order is preserved. If the order is exchanged, then the normal pair is transformed into abnormal one and zero pair to zero pair.*

Proof. Under any permutation σ such that $\sigma^{-1}(i) = i'$ and $\sigma^{-1}(j) = j'$, one has

$$m_{i',i} = m_i + s_{i',i}, \quad m_{j',j} = m_j + s_{j',j} .$$

Then, for the new pair $(m_{i',i}, m_{j',j})$ at position (i', j'), one has

$$m_{i',i} - m_{j',j} - s_{i',j'} = m_i - m_j + s_{i',i} - s_{i'j'} - s_{j',j} = m_i - m_j - s_{i,j} ,$$

and

$$m_{j',j} - m_{i',i} - s_{j',i'} = -(m_i - m_j - s_{i,j}) .$$

Hence the normal (zero, abnormal) pair is transformed into normal (zero, abnormal) for $i' < j'$, or abnormal (zero, normal) for $i' > j'$. □

Corollary 1. *A product of the form*

$$\Psi = \psi_{A_1}(m_1)\cdots\psi_{A_n}(m_n) ,$$

is 0 iff there exist (at least) a pair (m_i, m_j) such that

$$m_i = m_j + s_{i,j} .$$

Otherwise, it can be transformed into normal form.

Corollary 2. *The set of normal forms $\{\psi_{A_1}(m_1)\cdots\psi_{A_n}(m_n)\}$ forms a* **Z** *linear base of the algebra.*

Proof. The corollary is proved by the standard argument of constructing a representation of the algebra using Corollary 1. □

Proof of Theorem 1. Take $J = \{0,1\}^2$ and

$$s_{(p_1,i_1)(p_2,i_2)} = -K(p_1,p_2) + H(i_2,i_1) .$$

Then the above algebra is isomorphic to $\mathscr{A}$ by $\varphi^{*p}_{2n+i} = \psi_{(p,i)}(n)$, where $i = 0,1$. In this case the normality condition is exactly the condition (8). Hence Theorem 1 follows from Corollary 2. □

Definition 3. *Let us define the weight of a nonzero element of B by*

$$wt(\varphi_{j_n}^{*p_n}\cdots\varphi_{j_1}^{*p_1}) = \sum_{l=1}^{n} wt(\varphi_{j_l}^{*p_l}),$$

$$wt(\varphi_j^{*p}) = -\left[\frac{j}{2}\right]\delta + (1-2\varepsilon(j))(\Lambda_1 - \Lambda_0),$$

where $[j]$ is the Gauss symbol. We set $wt1 = 0$.

Let us introduce the actions of $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1)$ on the set $B(p) = \{\varphi_j^{*p} | j \in \mathbf{Z}\}$ $(p = 0, 1)$ by

$$\tilde{f}_1\varphi_{2j}^{*p} = \varphi_{2j+1}^{*p}, \qquad \tilde{f}_0\varphi_{2j-1}^{*p} = \varphi_{2j}^{*p},$$

$$\tilde{e}_1\varphi_{2j+1}^{*p} = \varphi_{2j}^{*p}, \qquad \tilde{e}_0\varphi_{2j}^{*p} = \varphi_{2j-1}^{*p},$$

$$\tilde{x}_i\varphi_j^{*p} = 0 \quad \text{otherwise},$$

where $x = e, f$. By these actions and the weight of φ_j^{*p}, $B(p)$ is an affine crystal [16, 22] isomorphic to $\mathrm{Aff}(B^{(1)})$. In general

Theorem 2. *Let $n \geqq 1$ and $(p_n, \ldots, p_1) \in \{0,1\}^n$. There is a unique $\widehat{sl_2}$ crystal structure in $B(p_n, \ldots, p_1)$ such that the natural map*

$$\mathrm{Aff}(B^{(1)})^{\otimes n} \longrightarrow (B(p_n,\ldots,p_1) \cup -B(p_n,\ldots,p_1) \cup \{0\})\big/\pm$$

$$\varphi_{j_n}^{*p_n} \otimes \cdots \otimes \varphi_{j_1}^{*p_1} \longmapsto \varphi_{j_n}^{*p_n}\cdots\varphi_{j_1}^{*p_1}$$

commutes with the actions of $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1)$.

We consider $B(\phi) = \{1\}$ to be the trivial crystal, $\tilde{x}_i 1 = 0$ for $x = e, f$ and $i = 0, 1$.

Proof of Theorem 2. Take any $(p_2, p_1) \in \{0,1\}^2$ and fix it. Let us set, as a subset of the tensor algebra generated by $B(p)$ $(p = 0, 1)$,

$$\mathscr{I} = \{\varphi_{j_2}^{*p_2} \otimes \varphi_{j_1}^{*p_1} + \varphi_{j_1-2K(p_2,p_1)+\varepsilon(j_1+j_2)}^{*p_2} \otimes \varphi_{j_2+2K(p_2,p_1)-\varepsilon(j_1+j_2)}^{*p_1} | j_1, j_2 \in \mathbf{Z}\}.$$

Then it is sufficient to prove that

$$\tilde{x}_i\mathscr{I} \subset \mathscr{I} \sqcup \{0\}$$

for $x = e, f$, $i = 0, 1$. Here in general for the elements b_1, b_2 of a crystal we understand $\tilde{x}_i(b_1 + b_2) = \tilde{x}_i b_1 + \tilde{x}_i b_2$. By direct calculations this property is easily proved. □

3. Crystalline Spinon Base

Definition 4. *Let us call $(p_n, \ldots, p_1) \in \{0,1\}^n$ a level k restricted path from 0 to l of length n if*

$$(-1)^{p_1} + \cdots + (-1)^{p_s} \in \{0, \ldots, k\} \quad \textit{for } 1 \leqq s \leqq n,$$

$$(-1)^{p_1} + \cdots + (-1)^{p_n} = l.$$

We denote by $\mathscr{P}^k_{\mathrm{res},\,n}(0,l)$ *the set of restricted paths from* 0 *to* l *of length* n *and set*

$$\mathscr{P}^k_{\mathrm{res},\,n} = \bigsqcup_{l=0}^{k} \mathscr{P}^k_{\mathrm{res},\,n}(0,l)\,.$$

We understand that $\mathscr{P}^k_{\mathrm{res},\,0} = \{\phi\}$.

The following theorem provides a new parametrization of the crystal base of the integrable highest or lowest weight $U_q(\widehat{sl_2})$ modules.

Theorem 3. *There is an isomorphism of affine crystals*

$$\bigsqcup_{n=0}^{\infty} \bigsqcup_{(p_n,\ldots,p_1)\in\mathscr{P}^k_{\mathrm{res},\,n}(0,l)} B(p_n,\ldots,p_1) \simeq B(\lambda_l)\otimes B(\lambda_0)^*$$

given by

$$1 \longmapsto [[\ ]] = b_{\lambda_0}\otimes b_{-\lambda_0}\,,$$

$$\varphi^{*p_n}_{j_n}\cdots\varphi^{*p_1}_{j_1} \longmapsto [[j_n - p_n,\ldots,j_1 - p_1]]\,.$$

Corollary 3. *The map in Theorem 3 induces the isomorphism of* $(\widehat{sl_2})_1$ *crystals with affine weights*:

$$\bigsqcup_{n=0}^{\infty} \bigsqcup_{(p_n,\ldots,p_1)\in\mathscr{P}^k_{\mathrm{res},\,n}(0,l)} B_{\geqq 0}(p_n,\ldots,p_1) \simeq B(\lambda_l)\,,$$

where we make the identification:

$$B(\lambda_l) \simeq B(\lambda_l)\otimes b_{-\lambda_0}\,,$$

$$b \longrightarrow b\otimes b_{-\lambda_0}\,.$$

Here $b_{-\lambda_0}$ *is the lowest weight element in* $B(\lambda_0)^*$.

Note that there are no naturally defined $\widehat{sl_2}$ crystal structure on $B_{\geqq 0}(p_n,\ldots,p_1)$.

The map in Theorem 3 determines a representation of $\mathscr{A}$ as in the proof of Corollary 2. The action of φ^{*p}_j on $\oplus_b \mathbf{Z}b$ is described in the following manner, where b runs over all the elements in $\sqcup_{l=0}^k B(\lambda_l)\otimes B(\lambda_0)^*$. Let $b = [[j_{n-1} - p_{n-1},\ldots, j_1 - p_1]]$ be as in Theorem 3 and $\varphi = \varphi^{*p}_j \varphi^{*p_{n-1}}_{j_{n-1}}\cdots\varphi^{*p_1}_{j_1}$. If $\varphi \neq 0$ then, by Lemma 1 and Lemma 2, there exist unique $\mu\in\{\pm 1\}$ and normal sequence $(j'_n,\ldots,j'_1)$ such that

$$\varphi = \mu\varphi^{*p}_{j'_n}\varphi^{*p_{n-1}}_{j'_{n-1}}\cdots\varphi^{*p_1}_{j'_1}\,.$$

We define $\varphi^{*p}_j b = 0$ if $(p, p_{n-1},\ldots,p_1)$ is not a restricted path. Suppose that $(p, p_{n-1},\ldots,p_1)\in\mathscr{P}^k_{\mathrm{res},\,n}$. Then

$$\varphi^{*p}_j b = \begin{cases} 0, & \text{if } \varphi = 0\,,\\ \mu[[j'_n - p, j'_{n-1} - p_{n-1},\ldots,j'_1 - p_1]], & \text{if } \varphi \neq 0\,. \end{cases}$$

In particular for the element in $B(p_n,\ldots,p_1)$ we have

$$\varphi^{*p_n}_{j_n}\cdots\varphi^{*p_1}_{j_1}[[\ ]] = [[j_n - p_n,\ldots,j_1 - p_1]]\,.$$

This representation is considered as a Fock type representation of the crystalline creation algebra, the vacuum being [[ ]].

We call the basis given in Theorem 3 and Corollary 3 the crystalline spinon basis.

Proof of Theorem 3. The bijectivity of the map is obvious by the description of the paths in terms of domain walls and the normality condition (8). So what we should prove is the following two statements:

(i) The map commutes with the actions of $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1)$.
(ii) The map preserves weights.

Let us prove (i) first.

Lemma 3. *Let* $p = (p(n))_{n \in \mathbf{Z}} = [[j_n - p_n, \dots, j_1]]$ *be an element of* $\sqcup_{l=0}^{k} B(\lambda_l) \otimes B(\lambda_0)^*$. *If*

$$j_{r+l} - p_{r+l} > j_{r+l-1} - p_{r+l-1} = \cdots = j_r - p_r > j_{r-1} - p_{r-1} ,$$

then

$$p(j_r - p_r) + p(j_r - p_r + 1) = k - l(-1)^{j_r} .$$

Note that the assumption of the lemma implies $p_{r+l-1} = \cdots = p_r$ and $j_{r+l-1} = \cdots = j_r$. By the rule (2)–(4), the above domain wall corresponds to $b^{(1)\otimes l}_{\varepsilon(j_r)} \in B^{(1)\otimes l}$.

Proof. The statement is easily proved by direct calculations. □

Let us consider the classical crystal morphism

$$cl^n : \mathrm{Aff}(B^{(1)\otimes n}) \longrightarrow B^{(1)\otimes n} ,$$

$$\varphi^{*p_n}_{j_n} \otimes \cdots \otimes \varphi^{*p_1}_{j_1} \longmapsto b^{(1)}_{\varepsilon(j_n)} \otimes \cdots \otimes b^{(1)}_{\varepsilon(j_1)} .$$

Since cl^n commutes with the actions of $\tilde{e}_i$ and $\tilde{f}_i$

$$\tilde{x}_i(\varphi^{*p_n}_{j_n} \otimes \cdots \otimes \varphi^{*p_1}_{j_1}) = \varphi^{*p_n}_{j_n} \otimes \cdots \otimes \tilde{x}_i \varphi^{*p_l}_{j_l} \otimes \cdots \otimes \varphi^{*p_1}_{j_1}$$

is equivalent to

$$\tilde{x}_i(b^{(1)}_{\varepsilon(j_n)} \otimes \cdots \otimes b^{(1)}_{\varepsilon(j_1)}) = b^{(1)}_{\varepsilon(j_n)} \otimes \cdots \otimes \tilde{x}_i b^{(1)}_{\varepsilon(j_l)} \otimes \cdots \otimes b^{(1)}_{\varepsilon(j_1)} .$$

Hence (i) follows from the definition of the actions of $\tilde{e}_i$, $\tilde{f}_i$ on $B(p)$ and (2)–(6). □

Next let us prove (ii).

Lemma 4. *Let* $p \in \sqcup_{m,m'} \mathscr{P}^{(m,m')}$ *be a path with n domain walls counting multiple ones. Then there exists P which is a composition of* $\tilde{e}_i$ *and* $\tilde{f}_i$ $(i = 0, 1)$ *such that*

$$Pp = [[l_n, \dots, l_1]], \quad l_n > \cdots > l_1 \geqq 0 .$$

Proof. Since $B^{(1)}$ is a perfect crystal of level 1, $B^{(1)\otimes n}$ is connected. Hence there exists a composition P_1 of $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1)$ such that

$$cl^n(P_1 p) = b_0^{(1)\otimes n} .$$

If we denote $P_1 p = [[l_n, \ldots, l_1]]$ then

$$\tilde{f}_0^n \tilde{f}_1^n P_1 p = [[l_n + 2, \ldots, l_1 + 2]] .$$

Hence p is connected to $[[r_n, \ldots, r_1]]$ such that

$$r_1 \geqq 0, \quad cl^n[[r_n, \ldots, r_1]] = b_0^{(1)\otimes n} .$$

So let us assume that p is of this form from the beginning. Then

$$(\tilde{f}_0 \tilde{f}_1^2)(\tilde{f}_0^3 \tilde{f}_1^4) \cdots (\tilde{f}_0^{n-2} \tilde{f}_1^{n-1}) p = [[r_n + n - 1, \ldots, r_2 + 1, r_1]] \quad n\text{: odd} ,$$

$$\tilde{f}_1 (\tilde{f}_0^2 \tilde{f}_1^3) \cdots (\tilde{f}_0^{n-2} \tilde{f}_1^{n-1}) p = [[r_n + n - 1, \ldots, r_2 + 1, r_1]] \quad n\text{: even} ,$$

which proves the lemma. □

We prove the weight preservation by induction on the number of domain walls. By Lemma 4 and the statement (i) above it is sufficient to prove the weight preservation for such a path $[[l_n, \ldots, l_1]]$ that

$$l_n > \cdots > l_1 \geqq 0 .$$

0) For the vacuum (=zero domain wall) state, the statement is obvious since $wt([[\ ]]) = wt(1) = 0$. Now, we will prove that

$$wt(p_2) - wt(p_1) = wt(\mathcal{O}) ,$$

for any (b,c)- and (a,c)-path, p_1 and p_2 such as

$$p_1(l) = \begin{cases} \bar{p}_b(l), & n < l, \\ q(l), & l \leqq n, \end{cases} \quad \text{and} \quad p_2(l) = \begin{cases} \bar{p}_a(l), & m < l , \\ \bar{p}_b(l), & n < l \leqq m , \\ q(l), & l \leqq n , \end{cases}$$

and the corresponding creation operator $\mathcal{O}$ (see 3) below). Here we can assume $m > n \geqq 0$.

1) First, calculate the ω-function. For $\omega(p_1)$, one decomposes the summation into four parts $\omega(p_1) = h_1^{(4)} + h_1^{(3)} + h_1^{(2)} + h_1^{(1)}$ as $(n < l)$, $(n = l)$, $(n > l > 0)$ and $(0 > l)$, then

$$h_1^{(4)} = \sum_{n<l} l[\tilde{H}(p_1(l+1), p_1(l)) - \tilde{H}(\bar{p}_b(l+1), \bar{p}_b(l))] = 0 ,$$

$$h_1^{(3)} = n[\tilde{H}(\bar{p}_b(n+1), q(n)) - \tilde{H}(\bar{p}_b(n+1), \bar{p}_b(n))] ,$$

$$h_1^{(2)} = \sum_{n>l>0} l[\tilde{H}(q(l+1), q(l)) - \tilde{H}(\bar{p}_b(l+1), \bar{p}_b(l))] ,$$

$$h_1^{(1)} = \sum_{0>l} l[\tilde{H}(q(l+1), q(l)) - \tilde{H}(\bar{p}_c(l+1), \bar{p}_c(l))] .$$

Similarly, by decomposing to six parts, one has $\omega(p_2) = h_2^{(6)} + \cdots + h_2^{(1)}$, where

$$h_2^{(6)} = \sum_{m<l} l[\tilde{H}(p_2(l+1), p_2(l)) - \tilde{H}(\bar{p}_a(l+1), \bar{p}_a(l))] = 0\,,$$

$$h_2^{(5)} = m[\tilde{H}(\bar{p}_a(m+1), \bar{p}_b(m)) - \tilde{H}(\bar{p}_a(m+1), \bar{p}_a(m))]\,,$$

$$h_2^{(4)} = \sum_{m>l>n} l[\tilde{H}(\bar{p}_b(l+1), \bar{p}_b(l)) - \tilde{H}(\bar{p}_a(l+1), \bar{p}_a(l))]\,,$$

$$h_2^{(3)} = n[\tilde{H}(\bar{p}_b(n+1), q(n)) - \tilde{H}(\bar{p}_a(n+1), \bar{p}_a(n))]\,,$$

$$h_2^{(2)} = \sum_{n>l>0} l[\tilde{H}(q(l+1), q(l)) - \tilde{H}(\bar{p}_a(l+1), \bar{p}_a(l))]\,,$$

$$h_2^{(1)} = \sum_{0>l} l[\tilde{H}(q(l+1), q(l)) - \tilde{H}(\bar{p}_c(l+1), \bar{p}_c(l))]\,.$$

Taking the difference of these two, one obtains

$$\omega(p_2) - \omega(p_1) = m[\tilde{H}(\bar{p}_a(m+1), \bar{p}_b(m)) - \tilde{H}(\bar{p}_a(m+1), \bar{p}_a(m))] + \sum_{m>l>0} l[\tilde{H}(\bar{p}_b(l+1), \bar{p}_b(l)) - \tilde{H}(\bar{p}_a(l+1), \bar{p}_a(l))]\,.$$

The first line of the R.H.S. is evaluated as

$$m[\tilde{H}(\bar{p}_a(m+1), \bar{p}_b(m)) - \tilde{H}(\bar{p}_a(m+1), \bar{p}_a(m))] = \begin{cases} m[\bar{p}_a(m) - \bar{p}_b(m)], & \bar{p}_a(m) \geqq \bar{p}_b(m)\,, \\ 0, & \bar{p}_a(m) < \bar{p}_b(m)\,. \end{cases}$$

The second line is

$$\sum_{n>l>0} l[\tilde{H}(\bar{p}_b(l+1), \bar{p}_b(l)) - \tilde{H}(\bar{p}_a(l+1), \bar{p}_a(l))] = \begin{cases} -(a-b)m/2, & \text{even } m\,, \\ (a-b)(m-1)/2, & \text{odd } m\,. \end{cases}$$

Hence

$$\omega(p_2) - \omega(p_1) = \begin{cases} (a-b)m/2, & \text{even } m, \quad a > b\,, \\ -(a-b)m/2, & \text{even } m, \quad a < b\,, \\ (a-b)(m-1)/2, & \text{odd } m, \quad a > b\,, \\ -(a-b)(m+1)/2, & \text{odd } m, \quad a < b\,. \end{cases}$$

2) By similar and easier calculation, one can show

$$s(p_2) - s(p_1) = \sum_{m \geqq l > 0} [\bar{p}_a(l) - \bar{p}_b(l)] = \begin{cases} 0, & \text{even } m\,, \\ b-a, & \text{odd } m\,. \end{cases}$$

Together with the results of 1), one obtains

$$wt(p_2) - wt(p_1) = \begin{cases} (a-b)(\Lambda_1 - \Lambda_0) - (a-b)m/2\,\delta, & \text{even } m, \quad a > b, \\ (a-b)(\Lambda_1 - \Lambda_0) + (a-b)m/2\,\delta, & \text{even } m, \quad a < b, \\ -(a-b)(\Lambda_1 - \Lambda_0) - (a-b)(m-1)/2\,\delta, & \text{odd } m, \quad a > b, \\ -(a-b)(\Lambda_1 - \Lambda_0) + (a-b)(m+1)/2\,\delta, & \text{odd } m, \quad a < b. \end{cases}$$

3) On the other hand, in the spinon basis, the path p_2 is obtained from p_1 by applying the following (products of) operators:

$$\mathcal{O} = \begin{cases} (\varphi_m^{*0})^{a-b}, & a > b\,, \\ (\varphi_{m+1}^{*1})^{b-a}, & a < b\,. \end{cases}$$

It is easy to see that the difference of wt, $wt(p_2) - wt(p_1)$ is given exactly by the weight of these operators, hence the theorem is proved. □

4. Yangian-like Structure

In this section we give a decomposition of the crystal of integrable irreducible highest weight $U_q(\widehat{sl_2})$ modules as $(\widehat{sl_2})_1$ crystals which has a natural description in terms of crystalline spinon basis. Those $(\widehat{sl_2})_1$ crystals can be considered as describing the Yangian module structure in the $q \to 1$ limit. In fact one half of the character formula (11) corresponds to this structure which is known to describe the Yangian contribution [4].

Definition 5. *For* $(p_n, \ldots, p_1) \in \{0,1\}^n$ *and* $(j_n, \ldots, j_1)$ *which satisfy the condition*

$$j_{l+1} - j_l \geqq H(p_{l+1}, p_l) \quad 1 \leqq l \leqq n-1 \tag{9}$$

define

$$\mathscr{B}_{j_n,\ldots,j_1}^{p_n,\ldots,p_1} = \{\varphi_{2j_n+i_n}^{*p_n} \cdots \varphi_{2j_1+i_1}^{*p_1} \in B(p_n,\ldots,p_1) | i_1,\ldots,i_n = 0,1\}\,.$$

Here we set $(j_n,\ldots,j_1) = \phi$ *if* $n = 0$, *and* $\mathscr{B}_\phi^\phi = \{1\}$.

Note that the set $\mathscr{B}_{j_n,\ldots,j_1}^{p_n,\ldots,p_1} \sqcup \{0\}$ is preserved by the actions of $\tilde{e}_1$ and $\tilde{f}_1$ and all nonzero elements of it has the same weight with respect to d. As a corollary of Theorem 3 we have

Theorem 4. *There is an isomorphism of the* $(\widehat{sl_2})_1$ *crystal*

$$B(\lambda_l) \simeq \bigsqcup_{n=0}^{\infty} \bigsqcup_{(p_n,\ldots,p_1)\in\mathscr{P}^k_{\text{res},\,n}(0,l)} \bigsqcup_{(j_n,\ldots,j_1)} \mathscr{B}_{j_n,\ldots,j_1}^{p_n,\ldots,p_1}\,,$$

where the disjoint union in $(j_n,\ldots,j_1)$ *is over all sets which satisfy the condition* (9) *and* $j_1 \geqq 0$.

Let us determine the structure of $\mathscr{B}_{j_n,\ldots,j_1}^{p_n,\ldots,p_1}$ as a $(\widehat{sl_2})_1$ crystal. The crucial point for it is that an element of the form $\varphi_{2j_n+i_n}^{*p_n} \cdots \varphi_{2j_1+i_1}^{*p_1}$ can be zero. We first study the most degenerate cases.

Proposition 2. *If $j_l - j_{l-1} = H(p_l, p_{l-1})$ for any $l \geqq 2$, then*

$$\mathscr{B}_{j_n,\dots,j_1}^{p_n,\dots,p_1} \simeq B^{(n)}$$

as a $(\widehat{sl_2})_1$ crystal.

Proof. Using

$$\varphi_{2j_l}^{*p_l} \varphi_{2j_{l-1}+1}^{*p_{l-1}} = 0$$

and the definition of $\tilde{f}_1$, $\tilde{e}_1$, the statement is easily verified. □

In order to state the general case we need to introduce some terminology.

Definition 6. *For $(p_n, \dots, p_1) \in \{0,1\}^n$ the $(j_n, \dots, j_1) \in \mathbf{Z}^n$ is said to be a string if*

$$j_l - j_{l-1} = H(p_l, p_{l-1}) \quad 2 \leqq l \leqq n . \tag{10}$$

We call n the length of the string.

For a fixed $(p_n, \dots, p_1)$ we can uniquely decompose $(j_n, \dots, j_1)$ into strings in such a way that

(i) there exist $M \in \mathbf{Z}_{\geqq 1}$ and $0 = r_0 < \cdots < r_M = n$ such that $(j_{r_s}, \dots, j_{r_{s-1}+1})$ is a string for any $s = 1, \dots, M$,
(ii) $j_{r_s+1} - j_{r_s} > H(p_{r_s+1}, p_{r_s})$ for any $s = 1, \dots, M-1$.

Setting $m_s = r_s - r_{s-1}$, we call this decomposition the string decomposition of type $(m_M, \dots, m_1)$.

Theorem 5. *Let us fix $(p_n, \dots, p_1)$. If the string decomposition of $(j_n, \dots, j_1)$ is of type $(m_M, \dots, m_1)$, then there is an isomorphism*

$$\mathscr{B}_{j_n,\dots,j_1}^{p_n,\dots,p_1} \simeq B^{(m_M)} \otimes \cdots \otimes B^{(m_1)}$$

of $(\widehat{sl_2})_1$ crystals.

Proof. Let us use the notations of the above definition for the string decomposition of $(j_n, \dots, j_1)$ such as r_s. Set

$$\mathscr{B} = \{\varphi_{2j_n+i_n}^{*p_n} \otimes \cdots \otimes \varphi_{2j_1+i_1}^{*p_1} | i_n, \dots, i_1 = 0, 1\} .$$

By definition the product map $\mathscr{B} \longrightarrow \mathscr{B}_{j_n,\dots,j_1}^{p_n,\dots,p_1} \sqcup \{0\}$ is factorized as

$$\begin{array}{ccc} \mathscr{B} \sqcup \{0\} & \longrightarrow & \mathscr{B}_{j_n,\dots,j_1}^{p_n,\dots,p_1} \sqcup \{0\} \\ \downarrow & & \| \\ \mathscr{B}_{j_n,\dots,j_{r_M+1}}^{p_n,\dots,p_{r_M+1}} \otimes \cdots \otimes \mathscr{B}_{j_{r_1},\dots,j_1}^{p_{r_1},\dots,p_1} \sqcup \{0\} & \xrightarrow{F} & \mathscr{B}_{j_n,\dots,j_1}^{p_n,\dots,p_1} \sqcup \{0\} . \end{array}$$

By the definition of the crystal structure on $\mathscr{B}_{j_n,\dots,j_1}^{p_n,\dots,p_1}$, each map of the above diagram commutes with $\tilde{e}_1$ and $\tilde{f}_1$. It is easily verified that F is a bijection. Then the theorem follows from Proposition 2. □

To each $\mathscr{B}^{p_n,\dots,p_1}_{j_n,\dots,j_1} \simeq B^{(m_M)} \otimes \cdots \otimes B^{(m_1)}$, let us define the $U_q(\widehat{sl_2})_1$ module $V^{p_n,\dots,p_1}_{j_n,\dots,j_1}$ by

$$V^{p_n,\dots,p_1}_{j_n,\dots,j_1} = V^{(m_M)} \otimes \cdots \otimes V^{(m_1)} ,$$

where $U_q(\widehat{sl_2})_1$ is the subalgebra generated by e_1, f_1, h_1. Then, by the uniqueness of the crystal base (Theorem 3 of [13]), as a corollary of Theorem 4 and Theorem 5 we have

Corollary 4. *Suppose that* $q \in \{q \in \mathbf{C} | q^n \neq 1,\, n = 1,2,\dots\} \sqcup \{1\}$. *Then there is an isomorphism of* $U_q(\widehat{sl_2})_1$ *modules,*

$$V(\lambda_l) \simeq \bigoplus_{n=0}^{\infty} \bigoplus_{(p_n,\dots,p_1)\in\mathscr{P}^k_{\mathrm{res},\,n}(0,l)} \bigoplus_{(j_n,\dots,j_1)} V^{p_n,\dots,p_1}_{j_n,\dots,j_1} .$$

So far we do not have a natural full Yangian type operation on each set $\mathscr{B}^{p_n,\dots,p_1}_{j_n,\dots,j_1}$ or on the $U_q(\widehat{sl_2})_1$ module $V^{p_n,\dots,p_1}_{j_n,\dots,j_1}$. One reason for this will be explained in the following way. We consider not a $\widehat{sl_2}$ module but a $U_q(\widehat{sl_2})$ module. The full Yangian action may not survive in the deformation but it is possible for the sl_2 structure to outlive. If we see appropriately we can extract the full Yangian structure from it.

5. Character Formulas

Here, we derive the character formulas for the level k integrable $\widehat{sl_2}$-modules $V(\lambda_j)$ ($\lambda_j = (k-j)\Lambda_0 + j\Lambda_1,\ 0 \leqq j \leqq k$) using the description of the crystalline spinon basis in Sects. 3 and 4.

Let us denote $(p_n,\dots,p_1) \in \{0,1\}^n$ the level k restricted path $(m_n,\dots,m_0)$ beginning $m_0 = 0$ ending $m_n = j$ by making identification as $m_l = m_{l-1} + 1 - 2p_l$. In Sect. 3 (Corollary 3), we obtained a spinon parametrization of the basis of $B(\lambda_j)$

$$\varphi^{*p_n}_{j_n} \cdots \varphi^{*p_1}_{j_1}, \quad n \geqq 0 ,$$

where the conditions for the indices are

$$\begin{aligned} j_l &\geqq j_{l-1} + 2 \quad && \text{if } (p_l, p_{l-1}) = (1,0) , \\ j_l &\geqq j_{l-1} && \text{otherwise} , \\ j_1 &\geqq 0 . \end{aligned}$$

The weight wt of which is given by[1]

$$\begin{aligned} -d \equiv \langle -d, wt \rangle &= \sum_{l=1}^{n} n_l, \\ h_1 \equiv \langle h_1, wt \rangle &= \sum_{l=1}^{n} (1 - 2r_l) = n - 2m(r) , \end{aligned}$$

where $j_l = 2n_l + r_l$, ($r_l \in \{0,1\}$), and $m(r) = \sharp\{l \mid r_l = 1\}$.

[1] In conformal field theory language, $L_0 = -d + \Delta(j)$, ($\Delta(j) = j(j+2)/4(k+2)$) is the conformal weight and $J_0^0 = h_1/2$ is the $U(1)$ charge.

For fixed $p = (p_n, \dots, p_1)$ and $r = (r_n, \dots, r_1)$, the above conditions on $(j_n, \dots, j_1)$ can be rewritten in terms of mode sequences $(n_n, \dots, n_1)$ as

$$n_l + r_l \geqq n_{l-1} + r_{l-1} + H(r_l, r_{l-1}) + H(p_l, p_{l-1}), \quad (1 \leqq l \leqq n),$$

where $r_0 = p_0 = 0$ and the H-function is

$$H(0,0) = H(0,1) = H(1,1) = 0, \quad H(1,0) = 1 .$$

Especially, the minimal allowed mode sequence is

$$n_{\min,l} = \sum_{i=1}^{l} (H(r_i, r_{i-1}) + H(p_i, p_{i-1})) - r_l .$$

The general mode sequence can be parametrized as

$$n_l = n_{\min,l} + i_1 + \cdots + i_l, \quad (i_l \geqq 0),$$

and weight $-d$ of which is given by

$$\sum_{l=1}^{n} n_l = \sum_{l=1}^{n} l i_l + \sum_{l=1}^{n} [(n-l+1)H(r_l, r_{l-1}) + (n-l+1)H(p_l, p_{l-1}) - r_l] .$$

To sum up, the crystalline spinon basis can be parametrized by three sequences $i = (i_n, \dots, i_1) \in (\mathbf{Z}_{\geqq 0})^n$, $p = (p_n, \dots, p_1) \in \{0,1\}^n$ (restricted) and $r = (r_n, \dots, r_1) \in \{0,1\}^n$, with weights

$$\begin{aligned} -d &= g(i) + h(p) + h(r) - m(r) , \\ h_1 &= n - 2m(r) , \\ g(i) &= \sum_{l=1}^{n} l i_l , \\ h(p) &= \sum_{l=1}^{n} (n-l+1)H(p_l, p_{l-1}) , \\ h(r) &= \sum_{l=1}^{n} (n-l+1)H(r_l, r_{l-1}) . \end{aligned}$$

Using this description of basis, one obtains the following character formula:

Theorem 6. *The character* $\mathrm{ch}_j(q,z) \equiv \mathrm{tr}_{V(\lambda_j)}(q^{-d} z^{h_1})$ *is given by*

$$\mathrm{ch}_j(q,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(q)_{n-m}(q)_m} \sum_p q^{h(p)} z^{n-2m} , \tag{11}$$

where $p = (p_n, \dots, p_1)$ *is level* k *restricted fusion path from* $m_0 = 0$ *to* $m_n = j$.

Proof. Using the parametrization of the crystalline spinon basis as above, one obtains

$$\begin{aligned} \mathrm{ch}_j(q,z) &= \sum_{n=0}^{\infty} \frac{1}{(q)_n} \sum_p \sum_r q^{h(p)+h(r)} q^{-m(r)} z^{n-2m(r)} , \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(q)_{n-m}(q)_m} \sum_p q^{h(p)} z^{n-2m} , \end{aligned}$$

where we have used the formula

$$\sum_{i_l \geqq 0} q^{i_1+2i_2+\cdots+ni_n} = \frac{1}{(q)_n},$$

and

$$\sum_{r=(r_n,\ldots,r_1)} q^{h(r)} q^{-m(r)} z^{-2m(r)} = \sum_{m=0}^{n} \frac{(q)_n}{(q)_{n-m}(q)_m} z^{-2m}.$$

Thus the formula is proved. □

This result is exactly the spinon character formula proposed in [4].

In the paper [4], the remaining sum over the restricted fusion path p has also been done explicitly. We quote their final result

$$\mathrm{ch}_j(q,z) = q^{-j/4} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(q)_{n-m}(q)_m} z^{n-2m} q^{-n^2/4} \Phi^n_{A_k}(u_j, q),$$

and

$$\Phi^{m_1}_K(u,q) = \sum_{m_2,\ldots,m_k} q^{\frac{1}{4} m \cdot K \cdot m} \prod_{i \geqq 2} \begin{bmatrix} \frac{1}{2}((2-K)\cdot m + u)_i \\ m_i \end{bmatrix}.$$

Here K is the Cartan matrix of A_k-type, $(u_j)_i = \delta_{i,j+1}$ and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q)_n}{(q)_{n-m}(q)_m}.$$

The m sum is taken over all non-negative odd [resp. even] integers for $(m_j, m_{j-2}, \ldots)$ [resp. otherwise].

6. Quasi-Particles at $q = 0$

In this section, as another application of the crystalline creation algebra, we derive the conjectural formula for the multi quasi-particle states of the higher spin XXZ models at $q = 0$. In the case of spin 1/2 XXZ chain, the formulas are consistent with the results of the Bethe Ansatz calculations [5].

Proposition 3. *For* $(p_n, \ldots, p_1) \in \mathscr{P}^k_{\mathrm{res},n}$ *and* $(\varepsilon_n, \ldots, \varepsilon_1) \in \{0,1\}^n$ *we have*

$$\varphi^{*p_n}_{\varepsilon_n}(z_n) \cdots \varphi^{*p_1}_{\varepsilon_1}(z_1)[[\ ]] = \sum_{(m_1,\ldots,m_n)\in M} C(m_n \cdots m_1)[[2m_n + \varepsilon_n - p_n, \ldots, 2m_1 + \varepsilon_1]],$$

where

$$C(m_n \cdots m_1) = \prod_{l=1}^{n} z_l^{v_l} \det(z_i^{m_j - v_j})_{1 \leqq i,j \leqq n},$$

and

$$v_j = \sum_{l=j}^{n-1} (-K(p_{l+1}, p_l) + H(\varepsilon_l, \varepsilon_{l+1})).$$

The sum in $(m_1, \ldots, m_n)$ *is over the modes* M *such as*

$$M = \{(m_1, \ldots, m_n) \mid 2m_n + \varepsilon_n - 2I_n \geqq \cdots \geqq 2m_1 + \varepsilon_1\}.$$

Here $I_n = I_n(p_n, \ldots, p_1)$, *etc.*

Proof. Let us use the notations in the proof of Theorem 1. Namely, for $\mu = 0, 1$,

$$\varphi^{*p}_{2n+\mu} = \psi_{(p,\mu)}(n).$$

Let us set $A_j = (p_{n-j+1}, \varepsilon_{n-j+1})$. Then, using Lemma 1, we have

$$\begin{aligned}
&\varphi^{*p_n}_{\varepsilon_n}(z_n)\cdots\varphi^{*p_1}_{\varepsilon_1}(z_1)[[\ ]] \\
&= \sum_{m_1,\dots,m_n\in\mathbf{Z}} \psi_{A_1}(m_1)\cdots\psi_{A_n}(m_n)[[\ ]]\, z_n^{m_1}\cdots z_1^{m_n} \\
&= \sum_{(m_1,\dots,m_n)\in M}\ \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) z_n^{m_{1,\sigma(1)}}\cdots z_1^{m_{n,\sigma(n)}}\psi_{A_1}(m_1)\cdots\psi_{A_n}(m_n)[[\ ]] \\
&= \sum_{(m_1,\dots,m_n)\in M} \det\left(z_{n-i+1}^{s_{i,j}+m_{n-j+1}}\right)[[2m_n+\varepsilon_n-p_n,\dots,2m_1+\varepsilon_1]]\,.
\end{aligned}$$

Noting that

$$\det\left(z_{n-i+1}^{s_{i,j}+m_{n-j+1}}\right) = \det\left(z_i^{m_j+s_{n-i+1,\,n-j+1}}\right) = \prod_{l=1}^{n} z_l^{s_{n-l+1,1}} \det\left(z_i^{m_j+s_{1,n-j+1}}\right)$$

and $v_j = s_{n-j+1,1} = -s_{1,n-j+1}$ we obtain the desired formula. □

Let $\{G(b)|b \in B(\lambda_l)\otimes B(\lambda_0)^*\}$ be the global crystal base of $V(\lambda_l)\otimes V(\lambda_0)^{*a}$ [15] and $\{\langle G(b)|\}$ the dual base, $\langle G(b)|G(b')\rangle = \delta_{bb'}$. Let $\varphi^{*\mu}_{\lambda,\varepsilon}(z)$ be the creation operator defined in [10] acting on the actual representation. We denote by $|\mathrm{vac}\rangle_0$ the ground state in $V(\lambda_0)\hat{\otimes}V(\lambda_0)^{*a}$ [10]. Then

Conjecture 1. *Assume the same data as in Proposition* 3. *Let* $b = [[2m_n+\varepsilon_n - p_n,\dots,2m_1+\varepsilon_1]]$ *be an element of* $\sqcup_{l=0}^{k} B(\lambda_l)\otimes B(\lambda_0)^*$. *Then the n-quasi-particles have the well defined* $q\to 0$ *limit and*

$$\langle G(b)|\varphi^{*\lambda_{i_n}}_{\lambda_{i_{n-1}}}(z_n)\cdots\varphi^{*\lambda_{i_1}}_{\lambda_{i_0}}(z_1)|\mathrm{vac}\rangle_0|_{q\to 0} = \prod_{j=1}^{n} z_j^{r_j}\, C(m_n,\dots,m_1)\,,$$

where $i_0 = 0$, $i_l - i_{l-1} = 1-2p_l$ *and* $r_j = \Delta(i_j)-\Delta(i_{j-1})$.

7. Discussion

In this paper we have given a spinon type description of the crystal of level k integrable highest or lowest weight $U_q(\widehat{sl_2})$ modules. The algebra of creation operators for the higher spin XXZ model at $q = 0$ has been introduced and it plays a central role. As consequences of this description we derived the fermionic type character formulas and found the Yangian like structure in the integrable highest weight modules or in the level zero modules. Moreover this description clarifies the structure of the crystal associated with the tensor products of the integrable highest and lowest weight modules of the same level in a different manner from that in [10]. In the following part we shall discuss the possible generalizations and remaining problems.

To generalize our formalism to the case of arbitrary affine quantum algebras will be possible. The problems there are the identification of creation operators with type II VOs and to calculate the commutation relations of them. As to the first problem Reshetikhin [23] gave a conjecture that the elementary excitations of the model defined by the simply laced affine Lie algebra $\hat{\mathscr{G}}$ will be parametrized by the fundamental weights of $\mathscr{G}$. We remark that the second process will be skipped with the aid of the general theory of crystals [16, 6]. The treatment of the RSOS models will also be possible [12].

So far we could not prove the existence of the $q \to 0$ limit of the creation operators. However it is reasonable and certain that this limit actually exists (see for example [21]). If the limit exists, the corresponding components in the products of q-vertex operators to our crystalline spinon basis form a base of the $U_q(\widehat{sl_2})$ module. To understand the "Yangian" representation $V^{p_n,\dots,p_1}_{j_n,\dots,j_1}$ in the integrable highest weight modules the study of these problems will be important.

Note added in proof. While preparing this manuscript we came to know that Nakanishi et al. [27] also proved the character formula which is equivalent to one in Sect. 5 rigorously based on the path description of the character. They also discuss the Yangian structure from the combinatorial point of view which is closely related to our results in Sect. 4. We would like to thank T. Nakanishi for sending us his private note.

A. Appendix

Here we consider the naive limit of the q-vertex operators at $q = 0$. The consideration here motivates the definition of Sect. 2, though it tells nothing about the existence.

In the vertex operator formalism, the creation operators are represented as the type II vertex operator $\varphi^{*\nu}_{\mu\ \varepsilon}(z)$. The commutation relation of them are given by

$$\varphi^{*\nu}_{\mu\ \varepsilon_1}(z_1)\varphi^{*\mu}_{\lambda\ \varepsilon_2}(z_2) = R_{V^*V^*}{}^{\varepsilon_1'\ \varepsilon_2'}_{\varepsilon_1\ \varepsilon_2}\left(\frac{z_1}{z_2}\right)\varphi^{*\nu}_{\mu'\ \varepsilon_2'}(z_2)\varphi^{*\mu'}_{\lambda\ \varepsilon_1'}(z_1) W\begin{pmatrix}\lambda & \mu \\ \mu' & \nu\end{pmatrix}\left|\frac{z_1}{z_2}\right) .$$

For the definitions and notations see [5, 10, 12]. $R_{V^*V^*}$ is the trigonometric R-matrix of $U_q(\widehat{sl(2)})$ for 2-dimensional representation V^*. W is essentially the elliptic Boltzmann weight due to [1]. Infinitely many poles in W corresponds to the multivaluedness of usual ($q = 1$) CFT chiral vertex operators [26].[2]

Being regarded as a commutation relation, it looks too terrible to treat. For instance when $q = 1$, the algebra brings some generalized commutation relation in the sense of the vertex operator algebra which takes a very complicated form in general. However, if one considers the opposite limit $q \to 0$, a remarkable simple structure comes out. In fact one can show that

$$R_{V^*V^*}(z) \longrightarrow -z^{-1/2}\begin{bmatrix}1 & & & \\ & & z & \\ & 1 & & \\ & & & 1\end{bmatrix} ,$$

[2] One can take another $q \to 1$ limit in which R and W reduce to rational and trigonometric R respectively by rescaling the spectral parameter $z = q^{-2\beta/i\pi}$. This limit is exactly what we expected for the "particles," that is (nilpotent half of) the Faddeev–Zamolodchikov algebra.

$$W\begin{pmatrix} \lambda & \lambda_{\pm} \\ \lambda_{\pm} & \lambda \end{pmatrix}\Big| z\Big) \longrightarrow z^{\pm 1/2} ,$$

$$W\begin{pmatrix} \lambda & \lambda_{\pm} \\ \lambda_{\pm} & \lambda_{\pm\pm} \end{pmatrix}\Big| z\Big) \longrightarrow z^{-1/2} ,$$

and other W's vanish. Note that the W are independent of the initial weight λ in this limit, hence, one can regard it as a vertex type weight. Accordingly, we change the notation as

$$\varphi^{*\lambda_+}_{\lambda\ (-1)^\varepsilon}(z) \longrightarrow \varphi^{*0}_{\varepsilon}(z), \quad \varphi^{*\lambda_-}_{\lambda\ (-1)^\varepsilon}(z) \longrightarrow \varphi^{*1}_{\varepsilon}(z) ,$$

where $\varepsilon \in \{0,1\}$ and we identify $\pm$ with ± 1. Then the commutation relation takes the form

$$\varphi^{*p_1}_{\varepsilon_1}(z_1)\varphi^{*p_2}_{\varepsilon_2}(z_2) = -\left(\frac{z_1}{z_2}\right)^e \varphi^{*p_1}_{\varepsilon_1}(z_2)\varphi^{*p_2}_{\varepsilon_2}(z_1) ,$$

where the exponent e is given by

$$e = H(p_1, p_2) + H(\varepsilon_2, \varepsilon_1) - 1 .$$

This algebra can be regarded as a twisted version of the anti-commutation relation. The algebra in the main text (Definition 1) can be derived by mode expansion as follows:

$$\varphi^{*p}_0(z) = \sum_{n\in\mathbf{Z}} \varphi^{*p}_{2n} z^n ,$$

$$\varphi^{*p}_1(z) = \sum_{n\in\mathbf{Z}} \varphi^{*p}_{2n+1} z^n .$$

B. Appendix

Table 1. Crystalline spinon basis of $V(\Lambda_0)$

$(-d, h_1)$	crystal base	[[domain wall]]	(fusion path)	[j-path]
(0,0)	$\cdots$ 010101	[[ ]]	( )	[ ]
(1,2)	$\cdots$ 01010/0/	[[1,0]]	(1,0)	[2;0]
(1,0)	$\cdots$ 0101/10/	[[2,0]]	(1,0)	[3;0]
(1,−2)	$\cdots$ 0101/1/1	[[2,1]]	(1,0)	[3;1]
(2,2)	$\cdots$ 010/010/	[[3,0]]	(1,0)	[4;0]
(2,0)	$\cdots$ 010/01/1	[[3,1]]	(1,0)	[4;1]
(2,0)	$\cdots$ 01/1010/	[[4,0]]	(1,0)	[5;0]
(2,−2)	$\cdots$ 01/101/1	[[4,1]]	(1,0)	[5;1]
...				
(4,4)	$\cdots$ 010/0/0/0/	[[3,2,1,0]]	(1,0,1,0)	[4;2,2;0]

Table 2. Crystalline spinon basis of $V(\Lambda_1)$

$(-d, h_1)$	crystal base	[[domain wall]]	(fusion path)	[j-path]
(0,1)	··· 101010/	[[0]]	(0)	[0]
(0,−1)	··· 10101/1	[[1]]	(0)	[1]
(1,1)	··· 1010/01	[[2]]	(0)	[2]
(1,−1)	··· 101/101	[[3]]	(0)	[3]
(2,3)	··· 1010/0/0/	[[2,1,0]]	(0,1,0)	[2,2;0]
(2,1)	··· 10/0101	[[4]]	(0)	[4]
(2,1)	··· 101/10/0/	[[3,1,0]]	(0,1,0)	[3,2;0]
(2,−1)	··· 1/10101	[[5]]	(0)	[5]
(2,−1)	··· 101/1/10/	[[3,2,0]]	(0,1,0)	[3,3;0]
(2,−3)	··· 101/1/1/1	[[3,2,1]]	(0,1,0)	[3,3;1]

Table 3. Crystalline spinon basis of $V(2\Lambda_0)$

$(-d, h_1)$	crystal base	[[domain wall]]	(fusion path)	[j-path]
(0,0)	··· 020202	[[ ]]	( )	[ ]
(1,2)	··· 02020/1/	[[1,0]]	(1,0)	[2;0]
(1,0)	··· 0202/11/	[[2,0]]	(1,0)	[3;0]
(1,−2)	··· 0202/1/2	[[2,1]]	(1,0)	[3;1]
(2,4)	··· 02020//0//	[[1,1,0,0]]	(1,1,0,0)	[2,2;0,0]
(2,2)	··· 020/111/	[[3,0]]	(1,0)	[4;0]
(2,2)	··· 0202/1/0//	[[2,1,0,0]]	(1,1,0,0)	[3,2;0,0]
(2,0)	··· 02/1111/	[[4,0]]	(1,0)	[5;0]
(2,0)	··· 0202//20//	[[2,2,0,0]]	(1,1,0,0)	[3,3;0,0]
(2,0)	··· 020/11/2	[[3,1]]	(1,0)	[4;1]

Table 4. Crystalline spinon basis of $V(\Lambda_0 + \Lambda_1)$

$(-d, h_1)$	crystal base	[[domain wall]]	(fusion path)	[j-path]
(0,1)	··· 111111/	[[0]]	(0)	[0]
(0,−1)	··· 11111/2	[[1]]	(0)	[1]
(1,3)	··· 1111/0//	[[1,0,0]]	(1,0,0)	[2;0,0]
(1,1)	··· 1111/02	[[2]]	(0)	[2]
(1,1)	··· 1111/20//	[[2,0,0]]	(1,0,0)	[3;0,0]
(1,−1)	··· 111/202	[[3]]	(0)	[3]
(1,−1)	··· 1111/0/1/	[[2,1,0]]	(1,0,0)	[3;1,0]
(1,−3)	··· 1111/2//2	[[2,1,1]]	(1,0,0)	[3;1,1]
(2,3)	··· 111/020//	[[3,0,0]]	(1,0,0)	[4;0,0]
(2,3)	··· 1111/0/1/	[[2,1,0]]	(0,1,0)	[2,2;0]

Table 5. Crystalline spinon basis of $V(2\Lambda_1)$

$(-d, h_1)$	crystal base	[[domain wall]]	(fusion path)	[j-path]
(0,2)	$\cdots$ 202020//	[[0,0]]	(0,0)	[0,0]
(0,0)	$\cdots$ 20202/1/	[[1,0]]	(0,0)	[1,0]
(0,−2)	$\cdots$ 20202//2	[[1,1]]	(0,0)	[1,1]
(1,2)	$\cdots$ 2020/11/	[[2,0]]	(0,0)	[2,0]
(1,0)	$\cdots$ 202/111/	[[3,0]]	(0,0)	[3,0]
(1,0)	$\cdots$ 2020/1/2	[[2,1]]	(0,0)	[2,1]
(1,−2)	$\cdots$ 202/11/2	[[3,1]]	(0,0)	[3,1]
...				
(3,4)	$\cdots$ 2020//0/1/	[[2,2,1,0]]	(0,0,1,0)	[2,2,2;0]

References

1. Andrews, G., Baxter, R., Forrester, P.: Eight-vertex SOS model and generalized Rogers–Ramanujan type identities. J. Stat. Phys. **35**, 193–266 (1984)
2. Bernard, D., Pasquier, V., Serban, D.: Spinons in Conformal Field Theory. Nucl. Phys. **B428**, 612–628 (1994)
3. Bouwknegt, P., Ludwig, A., Schoutens, K.: Spinon bases, Yangian symmetry and fermionic representations of Virasoro characters in conformal field theory. Phys. Lett. **B338**, 448–456 (1994)
4. Bouwknegt, P., Ludwig, A., Schoutens, K.: Spinon bases for higher level SU(2) WZW models. Preprint, hep-th/9412108 (1994)
5. Davies, B., Foda, O., Jimbo, M., Miwa, T., Nakayashiki, A.: Diagonalization of the XXZ Hamiltonian by Vertex Operators. CMP **151**, 89–153 (1993)
6. Date, E., Jimbo, M., Okado, M.: Crystal base and q vertex operators. CMP **155**, 47–69 (1993)
7. Faddeev, L., Takhatajan, L.: Spectrum and scattering of excitations in the one dimensional Heisenberg model. J. Soviet Math. **24**, 241–267 (1984)
8. Frenkel, I.B., Reshetikhin, N.Yu.: Quantum affine algebras and holonomic difference equations. CMP **146**, 1–60 (1992)
9. Haldane, F., Ha, Z., Tastra, J., Bernard, D., Pasquier, V.: Yangian symmetry of integrable quantum chains with long-range interactions and a new description of states in conformal field theory. Phys. Rev. Lett. **69**, 2021–2025 (1992)
10. Idzumi, M., Iohara, K., Jimbo, M., Miwa, T., Nakashima, T., Tokihiro, T.: Quantum affine symmetry in vertex models. IJMPA **8**, 1479–1511 (1993)
11. Jimbo, M., Misra, K., Miwa, T., Okado, M.: Combinatorics of representation of $U_q(\widehat{sl(n)})$ at $q = 0$. CMP **136**, 543–566 (1991)
12. Jimbo, M., Miwa, T., Ohta, Y.: Structure of the space of states in RSOS models. IJMPA **8**, 1457–1477 (1993)
13. Kashiwara, M.: On crystal bases of the q-analogue of universal enveloping algebras. Duke Math. J. **63**, No.2, 465–516 (1991)
14. Kashiwara, M.: The crystal base and Littelmann's refined Demazure character formula. Duke Math. J. **71**, No.3, 839–858 (1993)
15. Kashiwara, M.: Crystal bases of modified quantized enveloping algebras. Duke Math. J. **73**, No.2, 383–413 (1994)
16. Kang, S-J., Kashiwara, M., Misra, K., Miwa, T., Nakashima, T., Nakayashiki, A.: Affine crystals and vertex models. Int. J. Mod. Phys. **A7**, Suppl. 1A, 449–484 (1992)
17. Kedem, R., McCoy, B.M., Melzer, E.: The sums of Rogers, Schur and Ramanujan and the Bose–Fermi correspondence in $1 + 1$-dimensional quantum field theory. hep-th/9304056 (1993)
18. Kirillov, A.N.: Dilogarithm identities. Preprint, hep-th/9408113 (1994)
19. Kirillov, A.N., Reshetikhin, N.Yu.: The Bethe Ansatz and the combinatorics of Young tableaux. J. Soviet Math. **41**, 925–955 (1988)

20. Kuniba, A., Nakanishi, T., Suzuki, J.: Characters in conformal field theories from thermodynamic Bethe Ansatz. Mod. Phys. Lett. **A8** (1993)
21. Miki, K.: Creation/Annihilation operators and form factors of the XXZ model. Phys. Lett. **A186**, 217–224 (1994)
22. Nakayashiki, A.: Quasi-particle structure in solvable vertex models. Kyushu Univ. preprint (1995)
23. Reshetikhin, N.: S-matrices in integrable models of isotropic magnetic chains I. J. Phys. A: Math. Gen. **24**, 3299–3309 (1991)
24. Smirnov, F.: Dynamical symmetries of massive integrable models. Int. J. Mod. Phys. **A7**, Suppl. 1B, 813–837, 839–858 (1992)
25. Takhatajan, L.: The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins. Phys. Lett. **A87**, 479–482 (1982)
26. Tsuchiya, A., Kanie, Y.: Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid group. Adv. Studies in Pure Math. **16**, 297–372 (1988)
27. Arakawa, T., Nakanishi, T., Ooshima, K., Tsuchiya, A.: Talk in the meeting of Japan Mathematical Society at Ritsumeikan University, March (1995)

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