Commun. Math. Phys. 178, 27–60 (1996)

Communications in Mathematical Physics

Residue Formulas for the Large k Asymptotics of Witten's Invariants of Seifert Manifolds. The Case of $SU(2)$

L. Rozansky[1]

Physics Department, University of Miami, P.O. Box 248046, Coral Gables, FL 33124, U.S.A.

Received: 3 April 1995/Accepted: 17 July 1995

Abstract: We derive the large k asymptotics of the surgery formula for $SU(2)$ Witten's invariants of general Seifert manifolds. The contributions of connected components of the moduli space of flat connections are identified. The contributions of irreducible connections are presented in the residue form. This allows us to express them in terms of intersection numbers on their moduli spaces.

1. Introduction

Let A_μ be a connection on an $SU(2)$ bundle E over a 3-dimensional manifold M. The Chern–Simons action is a functional of this connection:

$$S_{\mathrm{CS}} = \frac{1}{2}\,\mathrm{Tr}\,\varepsilon^{\mu\nu\rho}\int_M d^3x\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right), \tag{1.1}$$

here Tr denotes a trace in the fundamental representation of $SU(2)$.

Consider an n-component link $\mathscr{L}$ in M. Let us attach α-dimensonal irreducible representations V_{α_j} to the components $\mathscr{L}_j$ of $\mathscr{L}$. A partition function of the quantum Chern–Simons theory with the Planck constant

$$\hbar = \frac{2\pi}{k}, \quad k \in \mathbb{Z} \tag{1.2}$$

can be presented as a path integral taken with an appropriate measure over the gauge equivalence classes of A_μ:

$$Z_{\{\alpha\}}(M,\mathscr{L};k) = \int[\mathscr{D}A_\mu]\,e^{\frac{i}{\hbar}S_{\mathrm{CS}}[A_\mu]}\prod_{j=1}^{n}\mathrm{Tr}_{\alpha_j}\,\mathrm{Pexp}\left(\oint_{\mathscr{L}_j}A_\mu dx^\mu\right), \tag{1.3}$$

here $\mathrm{Pexp}(\oint_{\mathscr{L}_j}A_\mu dx^\mu)\in SU(2)$ is a holonomy of A_μ along the contour $\mathscr{L}_j$ and Tr_α is the trace in the α-dimensional representation V_α. We also use the following

[1] Address after September 25: L. Rozansky, School of Mathematics, Institute for Advanced Study, Princeton, N.J. 08540, USA.

general notation: $\{x\}$ denotes a set of n numbers $x_1,\ldots,x_n$. E. Witten noticed in [1] that the partition function (1.3) is a topological invariant of the (framed) manifold M and link $\mathscr{L}$. He also showed that the ratio

$$J_{\{\alpha\}}(\mathscr{L};k)=\frac{Z_{\{\alpha\}}(S^3,\mathscr{L};k)}{\sqrt{\frac{2}{K}}\sin\left(\frac{\pi}{K}\right)},\quad K=k+2 \tag{1.4}$$

is equal to the Jones polynomial for $q=e^{\frac{\pi i}{K}}$.

Another important result of [1] is that $Z_{\{\alpha\}}(M,\mathscr{L};k)$ can be exactly calculated through the surgery formula. Let us first define a rational (p,q) surgery on a knot $\mathscr{K}$ belonging to a manifold M. We choose a pair of cycles C_1,C_2 on the boundary of the tubular neighborhood Tub($\mathscr{K}$). C_1 is a meridian, it is contractible through Tub($\mathscr{K}$). C_2 is a parallel, it is defined by a condition that it has a unit intersection number with C_1. The parallel C_2 is defined only modulo C_1. The (p,q) surgery on $\mathscr{K}$ is produced by cutting Tub($\mathscr{K}$) out of M and then gluing in back in such a way that the cycles C_1 and C_2 on $\partial\,\mathrm{Tub}(\mathscr{K})$ are identified with $C_1'=pC_1+qC_2$ and $C_2'=rC_1+sC_2$ on $\partial(M\setminus\mathrm{Tub}(\mathscr{K}))$. The numbers $r,s\in\mathbb{Z}$ are defined modulo p,q by a condition

$$ps-qr=1\,, \tag{1.5}$$

which follows from the fact that C_1' and C_2' should also have a unit intersection number. The topological class of the new manifold M' constructed by the (p,q) surgery does not depend on a particular choice of r and s.

Let M' be a manifold produced by rational (p_j,q_j) surgeries on the first m components of the link $\mathscr{L}$ in M. M' still contains a link $\mathscr{L}'$ consisting of the remaining components $\mathscr{L}_{m+1},\ldots,\mathscr{L}_n$ of $\mathscr{L}$. According to [1], the invariant of the new pair $M',\mathscr{L}'$ can be expressed in terms of the old one through the surgery formula

$$Z_{\alpha_{m+1},\ldots,\alpha_n}(M',\mathscr{L}';k)=e^{if_{\mathrm{fr}}}\sum_{1\leqq\alpha_1,\ldots,\alpha_m\leqq K-1}Z_{\alpha_1,\ldots,\alpha_n}(M,\mathscr{L};k)\prod_{j=1}^{m}\tilde{U}_{\alpha_j1}^{(p,q)}\,, \tag{1.6}$$

here f_{fr} is a framing correction phase and the matrices $\tilde{U}_{\alpha\beta}^{(p,q)}$generate a K–1-dimensional representation of the surgery matrices

$$\tilde{U}^{(p,q)}=\begin{pmatrix}p & q\\ q & s\end{pmatrix}\in SL(2,\mathbb{Z})\,. \tag{1.7}$$

The formula for $\tilde{U}_{\alpha\beta}^{(p,q)}$ was derived by L. Jeffrey in [2]:

$$\begin{aligned}\tilde{U}_{\alpha\beta}^{(p,q)}&=i\frac{\mathrm{sign}\,(q)}{\sqrt{2K|q|}}e^{-\frac{i\pi}{4}\Phi(U^{(p,q)})}\\&\quad\times\sum_{\mu=\pm1}\sum_{n=0}^{q-1}\mu\exp\left[\frac{i\pi}{2Kq}(p\alpha^2-2\alpha(2Kn+\mu\beta)+s(2Kn+\mu\beta)^2)\right]\,,\end{aligned} \tag{1.8}$$

here $\Phi(U^{(p,q)})$ is the Rademacher function defined as follows:

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q)\,, \tag{1.9}$$

$s(p,q)$ is a Dedekind sum

$$s(p,q) = \frac{1}{4q}\sum_{j=1}^{|q|-1} \cot\left(\pi\frac{j}{q}\right)\cot\left(\pi\frac{pj}{q}\right)\,. \tag{1.10}$$

N. Reshetikhin and V. Turaev showed in [3] that the surgery formula (1.6) defines a topological invariant, without relying on the path integral representation (1.3). This made the whole theory mathematically rigorous. They also formulated a set of general conditions on the components of Eq. (1.6) for it to define an invariant. The problem with the surgery formula (1.6) is however that it does not make obvious the relation between the "quantum" invariant $Z_{\{\alpha\}}(M,\mathscr{L};k)$ and the well-known classical invariants of M and $\mathscr{L}$ such as Betti numbers, linking numbers, etc. A possible remedy is to study the large k asymptotic behavior of $Z_{\{\alpha\}}(M,\mathscr{L};k)$ by applying a stationary phase approximation to the path integral (1.3). The stationary points of the phase (1.1) are $SU(2)$ flat connections on M. Let $\mathscr{M}$ be their moduli space, $\mathscr{M}_c$ being its connected components numbered by the index c. Each component $\mathscr{M}_c$ gives its own contribution $Z^{(c)}_{\{\alpha\}}(M,\mathscr{L};k)$ to the total invariant:

$$Z_{\{\alpha\}}(M,\mathscr{L};k) = \sum_c Z^{(c)}_{\{\alpha\}}(M,\mathscr{L};k)\,. \tag{1.11}$$

The individual contributions are presented as asymptotic series in $\hbar$ (or the exponentials thereof):

$$Z^{(c)}_{\{\alpha\}}(M,\mathscr{L};k) = (2\pi\hbar)^{\frac{N_{\text{zero}}}{2}} \exp\left(\frac{i}{\hbar}S^{(c)}_{\text{CS}}\right)\left[\sum_{n=1}^{\infty}\hbar^{n-1}\Delta^{(c)}_n\right]\,, \tag{1.12}$$

or, equivalently,

$$Z^{(c)}_{\{\alpha\}}(M,\mathscr{L};k) = (2\pi\hbar)^{\frac{N_{\text{zero}}}{2}} \exp\left[\frac{i}{\hbar}\left(S^{(c)}_{\text{CS}} + \sum_{n=1}^{\infty}S^{(c)}_n\hbar^n\right)\right]\,. \tag{1.13}$$

Here $S^{(c)}_{\text{CS}}$ is a Chern–Simons action of connections of $\mathscr{M}_c$ and

$$N_{\text{zero}} = \dim H^0_c - \dim H^1_c\,, \tag{1.14}$$

$H^{0,1}_c$ being the cohomologies of the covariant (with respect to A_μ) derivative D. The coefficients $\Delta^{(c)}_n, S^{(c)}_n$ are called n-loop corrections. The expression for the 1-loop correction was derived in [1, 4 and 2] (some details were added in [5]):

$$\Delta^{(c)}_1 \equiv e^{iS^{(c)}_1} = \frac{1}{\text{Vol}(H_c)}\exp\left[\frac{i}{\pi}\left(S^{(c)}_{\text{CS}} - \frac{i\pi}{8}N_{\text{ph}}\right)\right]$$
$$\times\int_{\mathscr{M}_c}\sqrt{|\tau_R|}\prod_{j=1}^{n}\text{Tr}_{\alpha_j}\,\text{Pexp}\left(\oint_{\mathscr{L}_j}A_\mu dx^\mu\right)\,. \tag{1.15}$$

In this formula H_c is an isotropy group of $\mathcal{M}_c$ (i.e. a subgroup of $SU(2)$ which commutes with the holonomies of connections of $\mathcal{M}_c$), N_{ph} is expressed in [4] as

$$N_{\text{ph}} = 2I_c + \dim H_c^0 + \dim H_c^1 + 3(1 + b_M^1)\,. \tag{1.16}$$

Here I_c is a spectral flow of the operator $L_- = \star D + D\star$ acting on 1- and 3-forms on M, b_M^1 is the first Betti number of M. τ_R is the Reidemeister–Ray–Singer torsion. L. Jeffrey observed in [2] that $\sqrt{|\tau_R|}$ defines a ratio of volume forms on $\mathcal{M}_c$ and H_c.

The higher loop corrections $\Delta_n^{(c)}, S_n^{(c)}$ come from the n-loop Feynman diagrams. They are expressed as multiple integrals of the products of propagators taken over the manifold M and the link $\mathcal{L}$ (see, e.g. [6, 7] and references therein for details).

The asymptotic formulas (1.11)–(1.13) follow from the path integral of Eq. (1.4) and can not be derived directly (at least, at this point) from the surgery formula (1.6). In other words, the asymptotic properties of the r.h.s. of Eq. (1.6) are not immediately obvious. Therefore it is interesting to take the surgery formula for the invariant of a particular simple manifold and try to find its large k asymptotics in order to compare it with Eq. (1.15) and multiloop Feynman diagrams. This program was initiated by D. Freed and R. Gompf in [4]. They observed a numerical correspondence between the invariants of some lens spaces and Seifert manifolds calculated through surgery formula and the predictions of Eqs. (1.11), (1.15) for large values of k. L. Jeffrey worked out the full asymptotic expansion of the invariants of lens spaces as well as some mapping class tori in [2]. She checked analytically that the classical and 1-loop parts of the flat connection contributions were equal to the Chern–Simons action and the r.h.s. of Eq. (1.15).

In our previous paper [5] we studied the large k asymptotics of the invariant of Seifert manifolds constructed by rational surgeries on the fibers of $S^2 \times S^1$. We demonstrated the consistency between our results and Eqs. (1.11), (1.15) for the case of 3-fibered spaces. We also found that the contributions of irreducible flat connections were finite loop exact. This means that (up to minor details) the asymptotic series $\sum_{n=1}^{\infty} \Delta_n^{(c)} \hbar^{n-1}$ of Eq. (1.12) appeared to be finite polynomials for the case when $\dim H_c = 0$. Such behavior is similar to the one observed in [8] for the 2d Yang–Mills theories and explained there by a non-abelian localization.

In this paper we study the large k asymptotics of $SU(2)$ Witten's invariant of general Seifert manifolds $X_{g,\{\frac{p}{q}\}}$. We calculate all contributions $Z^{(c)}(X_{g,\{\frac{p}{q}\}};k)$ (Proposition 3.1) and relate them to connected components of the moduli space of flat connections (Proposition 4.3). Our formulas express the contributions of irreducible connections as residues, which makes them look similar to the non-abelian localization formulas of [9] and [10]. By comparing our expressions with the residue formulas for intersection numbers derived in [9] and conjectured in [10] we express the contributions of irreducible connections in terms of intersection numbers on their moduli spaces (Proposition 5.3). As a byproduct of our calculations we derive the full asymptotic expansion of the partition function of 2d $SU(2)$ Yang–Mills theory on a Riemann surface with punctures, including the contributions of constant curvature reducible connections (Proposition 5.2). In Appendix 6 we discuss the alternative way of deriving the asymptotics of Witten's invariants of Seifert manifolds which relates them to Kostant's partition function (this is analogous to the relation between the intersection numbers and Duistermaat–Heckman polynomial discussed in [9]). In Appendix 6 we use the moduli space of twisted flat $SU(2)$

connections in order to get rid of singularities of the moduli space of untwisted connections and to simplify some residue and intersection number formulas.

2. A Surgery Formula for Seifert Manifolds

The simplest way to construct a Seifert manifold $X_{g,\left\{\frac{p}{q}\right\}}$ is to perform n rational surgeries on the manifold $\Sigma_g \times S^1$, Σ_g being a g-handled Riemann surface. Choose n points P_j, $1 \leqq j \leqq n$ on Σ_g and consider an n-component link $\mathscr{L}$ in $\Sigma_g \times S^1$ formed by the loops $P_j \times S^1$. The Seifert manifold $X_{g,\left\{\frac{p}{q}\right\}}$ is constructed by n rational (p_j, q_j) surgeries on the link components $\mathscr{L}_j$. The surgery formula (1.6) tells us that

$$Z(X_{g,\left\{\frac{p}{q}\right\}}; k) = e^{if_{\mathrm{fr}}} \sum_{1 \leqq \{\alpha\} \leqq K} Z_{\{\alpha\}}(\Sigma_g \times S^1, \mathscr{L}; k) \prod_{j=1}^{n} \tilde{U}_{\alpha_j 1}^{(p_j, q_j)} . \tag{2.1}$$

The framing correction f_{fr} for this surgery was calculated in [4]:

$$f_{\mathrm{fr}} = \frac{\pi}{4}\left(1 - \frac{2}{K}\right)\left[\sum_{j=1}^{n} \Phi(U^{(-q_j, p_j)}) + 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right] , \tag{2.2}$$

here we used a notation

$$P = \prod_{j=1}^{n} p_j , \qquad H = P \sum_{j=1}^{n} \frac{q_j}{p_j} . \tag{2.3}$$

The invariant $Z_{\{\alpha\}}(\Sigma_g \times S^1, \mathscr{L}; k)$ is equal to the Verlinde number, i.e. to the number of conformal blocks of the $SU(2)$ WZW theory at level k for the surface Σ_g with n insertions of the primary fields $\mathcal{O}_{\alpha_j}$ which correspond to the representations V_{α_j}. The number $N^g_{\{\alpha\}}$ is given by the Verlinde formula

$$Z_{\{\alpha\}}(\Sigma_g \times S^1, \mathscr{L}; k) = N^g_{\{\alpha\}} = \sum_{\beta=1}^{K-1} \frac{\prod_{j=1}^{n} \tilde{S}_{\alpha_j \beta}}{\tilde{S}_{\beta 1}^{n+2g-2}} , \tag{2.4}$$

here S is an $SL(2, \mathbb{Z})$ matrix which interchanges a parallel and a meridian:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{2.5}$$

and $\tilde{S}_{\alpha\beta}$ is its $K-1$-dimensional representation:

$$\tilde{S}_{\alpha\beta} = \sqrt{\frac{2}{K}} \sin\left(\frac{\pi}{K}\alpha\beta\right) . \tag{2.6}$$

By substituting Eqs. (2.2) and (2.4) into Eq. (2.1) and using an obvious relation $SU^{(p,q)} = U^{(-q,p)}$ we arrived at the following equation:

$$Z(X_{g,\{\frac{p}{q}\}};k) = e^{if_{\mathrm{fr}}}\sum_{\beta=1}^{K-1}\frac{\prod_{j=1}^{n}\tilde{U}_{\beta 1}^{(-q_j,p_j)}}{\tilde{S}_{\beta 1}^{n+2g-2}} = \frac{i^n K^{g-1}}{2^{n+g-1}}\frac{\mathrm{sign}\,(P)}{\sqrt{|P|}}e^{\frac{3}{4}i\pi\,\mathrm{sign}(\frac{H}{P})}$$

$$\times\exp\left[\frac{i\pi}{2K}\left(\frac{H}{P}-12\sum_{j=1}^{n}s(q_j,p_j)-3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right]Z_s(X_{g,\{\frac{p}{q}\}};k), \quad (2.7)$$

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = \sum_{\beta=1}^{K-1}\frac{\exp\left(-\frac{i\pi}{2K}\frac{H}{P}\beta^2\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)}\sum_{m_j=0}^{p_j-1}\sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^{n}\mu_j\right)$$

$$\times\exp\left(2\pi i\sum_{j=1}^{n}\frac{r_j}{p_j}\left(Km_j^2+\mu_j m_j\right)\right)\exp\left(-\frac{i\pi}{K}\beta\sum_{j=1}^{n}\frac{2Km_j+\mu_j}{p_j}\right). \quad (2.8)$$

Here we split the invariant $Z(X_{g,\{\frac{p}{q}\}};k)$ into a product of lengthy numerical factors and a sum $Z_s(X_{g,\{\frac{p}{q}\}};k)$ whose large k asymptotics has to be determined. Note that this sum takes a slightly different form if we substitute $\hat{m}_j = \mu_j m_j$ for m_j:

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = (-2i)^n\sum_{\hat{m}_j=0}^{p_j-1}\exp\left[2\pi i\sum_{j=1}^{n}\frac{r_j}{p_j}\left(K\hat{m}_j^2+\hat{m}_j\right)\right]$$

$$\times\sum_{\beta=1}^{K-1}\frac{\exp\left(-\frac{i\pi}{2K}\frac{H}{P}\beta^2\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)}\prod_{j=1}^{n}\sin\left(2\pi\beta\frac{\hat{m}_j+\frac{1}{2K}}{p_j}\right). \quad (2.9)$$

This expression bears a close resemblance to the following two objects: Verlinde numbers and a partition function of the 2d Yang–Mills theory. With the substitution of Eq. (2.6), Verlinde formula (2.4) turns into

$$N_{\{\alpha\}}^{g} = \left(\frac{K}{2}\right)^{g-1}\sum_{\beta=1}^{K-1}\frac{\prod_{j=1}^{n}\sin\left(\frac{\pi}{K}\beta\alpha_j\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)}. \quad (2.10)$$

According to [11], a partition function of a 2d Yang–Mills theory with the coupling constant a defined on a unit area surface Σ_g, which has n punctures with the holonomies

$$\mathrm{Pexp}(\oint A_\mu dx^\mu) = \exp(2\pi i\sigma_3\theta_j), \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad (2.11)$$

around them, is equal to

$$Z_{\{\theta\}}(\Sigma_g;a) = \frac{1}{2^{g-1}\pi^{n+2g-2}}\sum_{\beta\geqq 1}\frac{e^{-a\beta^2}}{\beta^{n+2g-2}}\prod_{j=1}^{n}\sin(2\pi\beta\theta_j). \quad (2.12)$$

The similarity between the sums in Eqs. (2.9), (2.10) and (2.12) becomes apparent if we put

$$\alpha_j = 2K\frac{\hat{m}_j+\frac{1}{2K}}{p_j}, \qquad \theta_j = \frac{\hat{m}_j+\frac{1}{2K}}{p_j}, \qquad a = \frac{i\pi}{K}\frac{H}{P}. \quad (2.13)$$

The sum (2.9) is a generalization of the other two sums: it has a quadratic exponent of Eq. (2.12) and a sine in denominator of Eq. (2.10). The difference between the ranges of summation in the sums (2.9) and (2.11) does not affect the similarity of calculation of their asymptotics as we will see in the next section. Note however that we cannot multiply the summand of Eq. (2.10) by an arbitrary quadratic exponential. The exponent of Eq. (2.9) is special: the exponential is periodic in β after the sum over m_j is taken.

3. A Residue Calculation of Asymptotics

Now we turn directly to the asymptotic calculation of the sum (2.8). We convert it into a sum over $\beta \in \mathbb{Z}$ in two steps. By slightly shifting the argument of the denominator along the imaginary axis we can double the range of summation:

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = \sum_{m_j=0}^{p_j-1} \exp\left(2\pi i K \sum_{j=1}^{n} \frac{r_j}{p_j} m_j^2\right) \lim_{\xi\to 0^+} \frac{1}{2} \sum_{\beta=-K+1}^{K} \frac{\exp\left(-\frac{i\pi}{2K}\frac{H}{P}\beta^2\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}(\beta - i\xi)\right)}$$
$$\times \sum_{\{\mu\}=\pm 1} \left\{ \prod_{j=1}^{n} \mu_j \exp\left[\frac{2\pi i}{p_j}\left(r_j \mu_j m_j - \beta(m_j + \frac{\mu_j}{2K})\right)\right]\right\} . \quad (3.1)$$

Indeed, the product of sines kills the summand at $\beta = 0$ (to see that the same happens at $\beta = K$ combine the terms at m_j and $q_j - m_j$). If the product of sines is absent (as it happens for the sums (2.10) and (2.12) if $n = 0$) we may add an extra factor

$$\frac{\sin\left(\frac{\pi}{K}\beta\right)}{\sin\left[\frac{\pi}{K}(\beta - i\xi)\right]} \quad (3.2)$$

that will take care of $\beta = 0, K$.

As the next step, we extend the sum over β to all integer numbers by using the following simple lemma:

Lemma 3.1 *If the function $f(\beta)$ defined on $\mathbb{Z}$ has a period T then*

$$\sum_{\beta=0}^{T-1} f(\beta) = \frac{1}{T} \lim_{\varepsilon\to 0} \sqrt{\varepsilon} \sum_{\beta\in\mathbb{Z}} f(\beta) e^{-\pi\varepsilon\beta^2} . \quad (3.3)$$

As a result,

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = \frac{1}{2K} \lim_{\varepsilon\to 0} \sqrt{\varepsilon} \sum_{m_j=0}^{p_j-1} \exp\left(2\pi i K \sum_{j=1}^{n} \frac{r_j}{p_j} m_j^2\right) \lim_{\xi\to 0^+} \frac{1}{2} \sum_{\beta\in\mathbb{Z}} e^{-\pi\varepsilon\beta^2}$$
$$\times \frac{\exp\left(-\frac{i\pi}{2K}\frac{H}{P}\beta^2\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}(\beta - i\xi)\right)}$$
$$\times \sum_{\{\mu\}=\pm 1} \left\{ \prod_{j=1}^{n} \mu_j \exp\left[\frac{2\pi i}{p_j}\left(r_j \mu_j m_j - \beta(m_j + \frac{\mu_j}{2K})\right)\right]\right\} . \quad (3.4)$$

Thus we eliminated the difference in the summation range between Eqs. (2.9), (2.10) and Eq. (2.12).

At this point we can use the Poisson resummation formula

$$\sum_{\beta\in\mathbb{Z}} f(\beta) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{+\infty} d\beta \exp(2\pi i m\beta) f(\beta)\,, \tag{3.5}$$

which tells us that

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = \frac{1}{2K} \lim_{\varepsilon\to 0} \sqrt{\varepsilon} \sum_{m_j=0}^{p_j-1} \sum_{m_0\in\mathbb{Z}} \sum_{\{\mu\}=\pm 1} \left(\prod_{j=1}^{n} \mu_j\right)$$

$$\times \exp\left(2\pi i \sum_{j=1}^{n} \frac{r_j}{p_j}\left(Km_j^2 + \mu_j m_j\right)\right) \lim_{\varepsilon\to 0} \frac{1}{2} \int_{-\infty}^{+\infty} d\beta e^{-\pi\varepsilon\beta^2} F(\beta; m_0, \{m\}, \{\mu\})\,, \tag{3.6}$$

$$F(\beta; m_0, \{m\}, \{\mu\}) = \frac{\exp\left[-\frac{i\pi}{2K}\frac{H}{P}\beta^2 + 2\pi i\beta\left(m_0 - \sum_{j=1}^{n} \frac{m_j + \frac{\mu_j}{2K}}{p_j}\right)\right]}{\sin^{n+2g-2}\left[\frac{\pi}{K}(\beta - i\xi)\right]}\,. \tag{3.7}$$

A substituition $\beta = K\tilde{\beta}$ in the integral (3.6) would demonstrate explicitly the applicability of the stationary phase approximation in the limit $K \to \infty$. The stationary phase point for the phase of the integrand (3.7) is

$$\beta_{\mathrm{st}} = 2K\frac{P}{H}\left(m_0 - \sum_{j=1}^{n} \frac{m_j}{p_j}\right)\,. \tag{3.8}$$

The steepest descent contour $C_{\mathrm{sd}}(\beta_{\mathrm{st}})$ in the complex β plane is the line

$$\operatorname{Im}\beta = -\operatorname{sign}\left(\frac{H}{P}\right)(\operatorname{Re}\beta - \beta_{\mathrm{st}})\,. \tag{3.9}$$

In the process of being deformed from its original form $\operatorname{Im}\beta = 0$ to (3.9) the integration contour crosses those poles

$$\beta_l = K(l + i\xi) \tag{3.10}$$

of the integrand (3.7) for which

$$\operatorname{sign}\left(\frac{H}{P}\right)(\beta_{\mathrm{st}} - Kl) > 0\,. \tag{3.11}$$

Therefore to the leading order in ε

$$\int_{-\infty}^{+\infty} d\beta e^{-\pi\varepsilon\beta^2} F(\beta; m_0, \{m\}, \{\mu\}) = e^{-\pi\varepsilon\beta_{\mathrm{st}}^2} \int_{C_{\mathrm{sd}}(\beta_{\mathrm{st}})} d\beta F(\beta; m_0, \{m\}, \{\mu\})$$

$$+2\pi i \sum_{\substack{l\in\mathbb{Z} \\ \operatorname{sign}\left(\frac{H}{P}\right)(\beta_{\mathrm{st}} - Kl) > 0}} \operatorname{Res}_{\beta=\beta_l} F(\beta; m_0, \{m\}, \{\mu\})\,. \tag{3.12}$$

Let us substitute this expression into Eq. (3.6) and take the sum over m_j, μ_j and m_0. The function $Z_s(X_{g,\{\frac{p}{q}\}};k)$ will be presented as a sum of the contributions of all stationary phase points (3.8) with m_0 and m_j belonging to the summation range of Eq. (3.6) as well as the contributions of the poles (3.10). Both stationary points and poles form 1-dimensional lattices Λ_{st} and Λ_{p}, which are invariant under the shift

$$\beta \to \beta + 2K \tag{3.13}$$

and (if we put $\xi = 0$ in Λ_{p}) a reflection

$$\beta \to -\beta \,. \tag{3.14}$$

The function

$$\exp\left(2\pi i \sum_{j=1}^{n} \frac{r_j}{p_j}\left(Km_j^2 + \mu_j m_j\right)\right) F(\beta; m_0, \{m\}, \{\mu\}) \tag{3.15}$$

is invariant under the same transformations in the limit $\xi \to 0$ if we combine the shift (3.13) with the shift of m_j

$$m_j \to m_j - q_j, \quad 1 \leqq j \leqq n\,, \tag{3.16}$$

and the reflection (3.14) with the reflections

$$m_0 \to -m_0, \qquad m_j \to -m_j, \qquad \mu_j \to -\mu_j, \quad 1 \leqq j \leqq n\,. \tag{3.17}$$

An extra symmetry

$$m_0 \to m_0 - 1, \qquad m_j \to m_j + p_j \tag{3.18}$$

helps us to keep m_j within their summation range. Thus we conclude that the contributions of the stationary points Λ_{st} and poles Λ_{p} have the symmetries (3.13) and (3.14). Now we can apply Lemma 3.1 "backwards" to the contributions of Λ_{st} and Λ_{p}. We remove $\frac{1}{2K}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}$ from Eq. (3.6) while taking only the contributions of the poles β_0 and β_1 and of the stationary points $0 \leqq \beta_{\mathrm{st}} \leqq K$ (if $\beta_{\mathrm{st}} \neq 0, K$, then their contributions should be doubled in view of the symmetry (3.14)):

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = Z_{\mathrm{s,polar}} + Z_{\mathrm{s,st.ph.}}\,, \tag{3.19}$$

$$Z_{\mathrm{s,polar}} = \pi i \sum_{l=0,1}\sum_{m_j=0}^{p_j-1}\sum_{\{\mu\}=\pm 1}\sum_{\substack{m_0\in\mathbb{Z}\\ \operatorname{sign}\left(\frac{H}{P}\right)(\beta_{\mathrm{st}}-Kl)>0}}\left(\prod_{j=1}^{n}\mu_j\right)\exp\left(2\pi i\sum_{j=1}^{n}\frac{r_j}{p_j}\left(Km_j^2+\mu_j m_j\right)\right)$$
$$\times \lim_{\xi\to 0^+}\operatorname{Res}_{\beta=\beta_l} F(\beta; m_0, \{m\}, \{\mu\}), \tag{3.20}$$

$$Z_{\mathrm{s,st.ph.}} = \sum_{m_j=0}^{p_j-1}\sum_{\{\mu\}=\pm 1}\sum_{\substack{m_0\in\mathbb{Z}\\ 0\leqq\beta_{\mathrm{st}}\leqq K}}\frac{1}{\operatorname{Sym}_{\mathbb{Z}_\pm}\left(\frac{\beta_{\mathrm{st}}}{2K}\right)}\left(\prod_{j=1}^{n}\mu_j\right)\exp\left(2\pi i\sum_{j=1}^{n}\frac{r_j}{p_j}\left(Km_j^2+\mu_j m_j\right)\right)$$
$$\times \lim_{\xi\to 0^+}\int_{C_{\mathrm{sd}}(\beta_{\mathrm{st}})} d\beta F(\beta; m_0, \{m\}, \{\mu\})\,. \tag{3.21}$$

Here we used the following notation: let G be a group acting on a set X. For $x \in X$, we denote by $\mathrm{Sym}_G(x)$ the number of elements of G which leave x invariant. In the future we will need two groups: the group of reflections $\pm$ (its only nontrivial element multiplies real numbers by -1) and the group of "affine" reflections $\mathbb{Z}_\pm$ which combines reflections of $\pm$ with the shifts by integer numbers.

If we substitute Eqs. (3.19)–(3.21) into Eq. (2.7) we will see that the whole invariant $Z(X_{g,\{\frac{p}{q}\}};k)$ turns into a sum of polar and stationary phase contributions. Let us first calculate the contribution of the stationary phase points $\beta_{\mathrm{st}} \neq 0, K$. We introduce a new integration variable

$$x = \frac{\beta - \beta_{\mathrm{st}}}{K} e^{-\frac{i\pi}{4}\,\mathrm{sign}\left(\frac{H}{P}\right)}, \tag{3.22}$$

so that

$$\begin{aligned}
&\lim_{\xi \to 0^+} \int_{C_{\mathrm{sd}}(\beta_{\mathrm{st}})} d\beta F(\beta; m_0, \{m\}, \{\mu\}) \\
&= K e^{\frac{i\pi}{4}\,\mathrm{sign}\left(\frac{H}{P}\right)} \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P}\beta_{\mathrm{st}}^2 - 2\beta_{\mathrm{st}}\sum_{j=1}^{n}\frac{\mu_j}{p_j}\right)\right] \\
&\quad \times \int_{-\infty}^{+\infty} dx \frac{\exp\left(-\frac{\pi K}{2}\left|\frac{H}{P}\right| x^2 - i\pi x e^{\frac{i\pi}{4}\,\mathrm{sign}\left(\frac{H}{P}\right)}\sum_{j=1}^{n}\frac{\mu_j}{p_j}\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta_{\mathrm{st}} + \pi x e^{\frac{i\pi}{4}\,\mathrm{sign}\left(\frac{H}{P}\right)}\right)} \\
&= 2K \exp\left(\frac{i\pi}{2K}\frac{H}{P}\beta_{\mathrm{st}}^2\right) \sum_{l'=0}^{\infty} \frac{\Gamma\left(l' + \frac{1}{2}\right)}{(2l')!} \left(\frac{1}{2\pi i K}\frac{P}{H}\right)^{l'+\frac{1}{2}} \partial_\phi^{(2l')} \left.\frac{e^{-2\pi i \phi \sum_{j=1}^{n}\frac{\mu_j}{p_j}}}{\sin^{n+2g-2}(2\pi\phi)}\right|_{\phi = \frac{\beta_{\mathrm{st}}}{2K}} .
\end{aligned} \tag{3.23}$$

Here we expanded the preexponential factor of the intergrand in powers of x and integrated the series term by term.

The case of $\beta_{\mathrm{st}} = 0, K$ requires a more careful consideration because the stationary phase point coincides with one of the poles (3.10). First of all, we introduce new variables

$$\beta' = \beta - Kl\,, \qquad m'_j = m_j + \frac{1}{2}q_j l\,, \tag{3.24}$$

in which the contribution of $\beta_{\mathrm{st}} = Kl$ is equal to

$$\begin{aligned}
Z^{(l)}_{\mathrm{spec.}} = \frac{(-1)^{nl}}{2} &\sum_{\substack{0 \le m'_j < p_j \\ m'_j \in \mathbb{Z} + \frac{1}{2}q_j l}} \sum_{\{\mu\} = \pm 1} \left(\prod_{j=1}^{n}\mu_j\right) \exp\left[2\pi i K \sum_{j=1}^{n}\left(\frac{r_j}{p_j}m'^2_j - \frac{1}{4}s_j q_j l^2\right)\right] \\
&\times \exp\left[2\pi i \sum_{j=1}^{n}\mu_j\left(\frac{r_j}{p_j}m'_j - \frac{1}{2}s_j l\right)\right] \lim_{\xi \to 0^+} \int_{C(\xi)} d\beta' G\left(\beta'; \mu_j\right) .
\end{aligned} \tag{3.25}$$

Here

$$G\left(\beta';\mu_j\right)=\frac{\left[-\frac{i\pi}{2K}\left(\frac{H}{P}\beta'^2+2\beta'\sum_{j=1}^{n}\frac{\mu_j}{p_j}\right)\right]}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta'\right)} \tag{3.26}$$

and the contour $C(\xi)$ is described by equation

$$\operatorname{Im}\beta'=-\operatorname{sign}\left(\frac{H}{P}\right)\operatorname{Re}\beta'-\xi\,. \tag{3.27}$$

Let us split the function $G\left(\beta';\mu_j\right)$ into odd and even parts:

$$G^{\pm}\left(\beta';\mu_j\right)=\frac{1}{2}\left(G\left(\beta';\mu_j\right)\pm G(-\beta';\mu_j)\right)\,. \tag{3.28}$$

To calculate the integral of $G^-\left(\beta';\mu_j\right)$ we double the integration contour and then close it:

$$\begin{aligned}\lim_{\xi\to 0^+}\int_{C(\xi)} d\beta' G^-\left(\beta';\mu_j\right)&=\frac{1}{2}\lim_{\xi\to 0^+}\left[\int_{C(\xi)}-\int_{C(-\xi)}\right]d\beta' G^-\left(\beta';\mu_j\right)\\ &=\pi i\operatorname{Res}_{\beta'=0}G^-\left(\beta';\mu_j\right)=\pi i\operatorname{Res}_{\beta'=0}G\left(\beta';\mu_j\right)\,. \end{aligned} \tag{3.29}$$

We substituted $G\left(\beta';\mu_j\right)$ for $G^-\left(\beta';\mu_j\right)$ because $G^+\left(\beta';\mu_j\right)$ has zero residue.

To integrate $G^+\left(\beta';\mu_j\right)$ we introduce a new integration variable

$$x=\left(\frac{\beta'}{K}e^{-\frac{i\pi}{2}\operatorname{sign}\left(\frac{H}{P}\right)}\right)^2\,, \tag{3.30}$$

so that the integration contour $C(\xi)$ folds into two branches: one over and one under the positive semi-axis in the complex x plane. The expansion of the preexponential factor $\sin^{2-2g-n}\left(\frac{\pi}{K}\beta'\right)$ in powers of x leads to Γ-function type integrals:

$$\begin{aligned}\lim_{\xi\to 0^+}\int_{C(\xi)} d\beta G^+\left(\beta';\mu_j\right)&=\frac{2K}{(2\pi)^{n+2g-2}}\sum_{\substack{l'\geqq 0\\ l'-n\in 2\mathbb{Z}}}\left(\frac{1}{2\pi iK}\frac{P}{H}\right)^{\frac{l'-n-2g+3}{2}}\frac{\Gamma\left(\frac{l'-n-2g+3}{2}\right)}{l'!}\\ &\times\partial_\phi^{l'}\left[e^{-2\pi i\phi\sum_{j=1}^{n}\frac{\mu_j}{p_j}}\left(\frac{2\pi\phi}{\sin(2\pi\phi)}\right)^{n+2g-2}\right]\Bigg|_{\phi=0}\,. \end{aligned} \tag{3.31}$$

The Γ-function in this equation is well-defined even if its argument is negative, because it is always half-integer.

It remains now to substitute Eq. (3.23) into Eq. (3.21) and (3.29), (3.31) into (3.25). Recall that $Z^{(0,1)}_{\text{spec.}}$ represents the contributions of $\beta_{\text{st}}=0,K$ to $Z_{\text{s,st.ph.}}$. The sum over m_0 in Eq. (3.21) is finite due to the condition $0\leqq\beta_{\text{st}}\leqq K$.

Now we turn to the polar contributions. The calculation of the residue in Eq. (3.20) is straightforward. The problems come from the condition (3.11). Consider a contribution of a general pole β_l. We introduce the new variables (3.24), so that the pole at $\beta = \beta_l$ corresponds to the pole at $\beta' = \beta'_0 = i\xi$. In the new variables the contribution of β_l to $Z_{\mathrm{s,polar}}$ is equal to

$$Z^{(l)}_{\mathrm{s,polar}} = \pi i(-1)^{nl} \sum_{\substack{0 \leqq m'_j < p_j \\ m'_j \in \mathbb{Z}+\frac{1}{2}q_j l}} \sum_{\{\mu\}=\pm 1} \sum_{\substack{m_0 \in \mathbb{Z} \\ \beta'_{\mathrm{st}}\,\mathrm{sign}\left(\frac{H}{P}\right)>0}} \left(\prod_{j=1}^{n} \mu_j\right)$$

$$\times \exp\left[2\pi i K \sum_{j=1}^{n}\left(\frac{r_j}{p_j} m_j'^2 - \frac{1}{4} s_j q_j l^2\right)\right] \exp\left[2\pi i \sum_{j=1}^{n} \mu_j \left(\frac{r_j}{p_j} m'_j - \frac{1}{2} s_j l\right)\right]$$

$$\times \lim_{\xi \to 0^+} \mathrm{Res}_{\beta'=i\xi} F(\beta'; m_0, m'_j, \mu_j)\,, \tag{3.32}$$

here

$$\beta'_{\mathrm{st}} = 2K \frac{P}{H}\left(m_0 - \sum_{j=1}^{n} \frac{m'_j}{p_j}\right). \tag{3.33}$$

We used the symmetry (3.18) in order to reduce the range of summation over m'_j to $0 \leqq m'_j < p_j$. The numbers m'_j are integer or half-integer depending on the parity of q_j and l. The same symmetry (3.18) allows us to further transform the sum $\sum_{\substack{0 \leqq m'_j < p_j \\ m'_j \in \mathbb{Z}+\frac{1}{2}q_j l}}$ into

$$\sum_{\{\mu'\}=\pm 1} \sum_{\substack{0 \leqq m'_j \leqq \frac{p_j}{2} \\ m'_j \in \mathbb{Z}+\frac{1}{2}q_j l}} \frac{1}{\prod_{j=1}^{n} \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m'_j}{p_j}\right)} \tag{3.34}$$

if we substitute $\mu'_j m'_j$ for m'_j everywhere in Eq. (3.20). After taking a sum over $\{\mu\} = \pm 1$, we arrive at the following expression:

$$Z^{(l)}_{\mathrm{s,polar}} = \pi i(-1)^{nl}(2i)^n \sum_{\substack{0 \leqq m'_j \leqq \frac{p_j}{2} \\ m'_j \in \mathbb{Z}+\frac{1}{2}q_j l}} \frac{1}{\prod_{j=1}^{n} \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m'_j}{p_j}\right)} \exp\left[2\pi i K \sum_{j=1}^{n}\left(\frac{r_j}{p_j} m_j'^2 - \frac{1}{4} s_j q_j l^2\right)\right]$$

$$\times \sum_{\{\mu'\}=\pm 1} \sum_{m_0 > \sum_{j=1}^{n} \frac{\mu'_j m'_j}{p_j}} \lim_{\xi \to 0^+} \mathrm{Res}_{\beta'=i\xi} \frac{\exp\left[-\frac{i\pi}{2K}\frac{H}{P}\beta'^2 + 2\pi i \beta'\left(m_0 - \sum_{j=1}^{n} \frac{\mu'_j m'_j}{p_j}\right)\right]}{\sin^{n+2g-2}\left[\frac{\pi}{K}(\beta' - i\xi)\right]}$$

$$\times \prod_{j=1}^{n} \mu'_j \sin\left[2\pi\left(\frac{r_j}{p_j} m'_j - \frac{\beta'}{2K}\frac{\mu'_j}{p_j} - \frac{1}{2} s_j l\right)\right]. \tag{3.35}$$

We can extend the sum over m_0 to

$$\sum_{m_0 \geqq \sum_{j=1}^{n} \frac{\mu'_j m'_j}{p_j}} \frac{1}{\mathrm{Sym}_\pm\left(m_0 - \sum_{j=1}^{n} \frac{\mu'_j m'_j}{p_j}\right)}, \tag{3.36}$$

if we transfer the polar contributions (3.29) from $Z_{\mathrm{s,st.ph.}}$ to $Z_{\mathrm{s,polar}}$. Then this sum can be split in two parts with the help of the formula

$$\sum_{m_0 \geqq a} \frac{f(m_0)}{\mathrm{Sym}_\pm(m_0 - a)} = \sum_{m_0 \geqq 0} \frac{f(m_0)}{\mathrm{Sym}_\pm(m_0)} - \sum_{0 \leqq m_0 \leqq |a|} \frac{\mathrm{sign}(a)}{\mathrm{Sym}_\pm(m_0)\,\mathrm{Sym}_\pm(m_0 - |a|)} \times f(\mathrm{sign}(a) m_0)\,. \tag{3.37}$$

The residue in Eq. (3.35) is calculated at $\beta' = i\xi$, so we can assume that $\mathrm{Im}\,\beta' > 0$. Then the sum $\sum_{m_0 \geqq 0} \frac{1}{\mathrm{Sym}_\pm(m_0)}$ can be easily calculated:

$$\sum_{m_0 \geqq 0} \frac{e^{2\pi i \beta' m_0}}{\mathrm{Sym}_\pm(m_0)} = \frac{i}{2}\cot(\pi\beta')\,. \tag{3.38}$$

Let us introduce the variable $\beta'' = \beta' - i\xi$. The residue in β'' is calculated at 0. A dependence on ξ in the vicinity of this point is nonsingular except for the factor $\cot[\pi(\beta'' + i\xi)]$ coming from the sum (3.38). Since the range of summation in $\sum_{m_0 \geqq 0}$ does not depend on μ'_j, the sum $\sum_{\{\mu'\}=\pm}$ in Eq. (3.35) can be calculated explicitly:

$$\begin{aligned}
&\sum_{\{\mu'\}=\pm 1} \mu'_j \exp\left(-2\pi i \beta' \frac{\mu'_j m'_j}{p_j}\right) \sin\left[2\pi\left(\frac{r_j}{p_j} m'_j - \frac{\beta'}{2K}\frac{\mu'_j}{p_j} - \frac{1}{2} s_j l\right)\right] \\
&= -2i \sin\left[2\pi\left(\frac{r_j}{p_j} m'_j - \frac{1}{2} s_j l\right)\right] \cos\left(\frac{\pi}{K}\frac{\beta'}{p_j}\right) \sin\left(2\pi\beta' \frac{m'_j}{p_j}\right) \\
&\quad -2\cos\left[2\pi\left(\frac{r_j}{p_j} m'_j - \frac{1}{2} s_j l\right)\right] \sin\left(\frac{\pi}{K}\frac{\beta'}{p_j}\right) \cos\left(2\pi\beta' \frac{m'_j}{p_j}\right).
\end{aligned} \tag{3.39}$$

The factors $\sin\left[2\pi(\beta'' + i\xi)\frac{m'_j}{p_j}\right]$ and $\sin\left(\frac{\pi}{K}\frac{\beta'' + i\xi}{p_j}\right)$ cancel the singularity of $\cot[\pi(\beta'' + i\xi)]$ if $n \geqq 1$. Otherwise the singularity will be canceled by the extra factor (3.2). As a result, we may simply put $\xi = 0$ in Eq. (3.35):

$$\begin{aligned}
Z^{(l)}_{\mathrm{s,polar}} &= \pi i (-1)^{nl} (2i)^n \sum_{\substack{0 \leqq m'_j \leqq \frac{p_j}{2} \\ m'_j \in \mathbb{Z} + \frac{1}{2} q_j l}} \frac{1}{\prod_{j=1}^n \mathrm{Sym}_{\mathbb{Z}\pm}\left(\frac{m'_j}{p_j}\right)} \\
&\quad \times \exp\left[2\pi i K \sum_{j=1}^n \left(\frac{r_j}{p_j} m'^2_j - \frac{1}{4} s_j q_j l^2\right)\right] \left(Z^{(l)}_{\mathrm{s,polar,reg.}} + Z^{(l)}_{\mathrm{s,polar,sing.}}\right),
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
Z^{(l)}_{\mathrm{s,polar,reg.}} &= \frac{i}{2}(-2)^n \operatorname{Res}_{\beta=0} \left\{ \frac{\exp\left(-\frac{i\pi}{2K}\frac{H}{P}\beta^2\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)} \cot(\pi\beta) \right. \\
&\quad \times \prod_{j=1}^n \left[i \sin\left[2\pi\left(\frac{r_j}{p_j} m'_j - \frac{1}{2} s_j l\right)\right] \cos\left(\frac{\pi}{K}\frac{\beta}{p_j}\right) \sin\left(2\pi\beta \frac{m'_j}{p_j}\right) \right. \\
&\quad \left.\left. + \cos\left[2\pi\left(\frac{r_j}{p_j} m'_j - \frac{1}{2} s_j l\right)\right] \sin\left(\frac{\pi}{K}\frac{\beta}{p_j}\right) \cos\left(2\pi\beta \frac{m'_j}{p_j}\right)\right]\right\},
\end{aligned} \tag{3.41}$$

$$Z^{(l)}_{\mathrm{s,polar,sing.}} = - \sum_{\{\mu'\}=\pm 1} \sum_{0 \leqq m_0 \leqq \left|\sum_{j=1}^n \frac{\mu'_j m'_j}{p_j}\right|} \frac{\mathrm{sign}\left(\sum_{j=1}^n \frac{\mu'_j m'_j}{p_j}\right)}{\mathrm{Sym}_\pm(m_0)\,\mathrm{Sym}_\pm\left(m_0 - \left|\sum_{j=1}^n \frac{\mu'_j m'_j}{p_j}\right|\right)}$$

$$\times \mathrm{Res}_{\beta=0} \frac{\exp\left[-\frac{i\pi}{2K}\frac{H}{P}\beta^2 + 2\pi i\beta\,\mathrm{sign}\left(\sum_{j=1}^n \frac{\mu'_j m'_j}{p_j}\right)\left(m_0 - \left|\sum_{j=1}^n \frac{\mu'_j m'_j}{p_j}\right|\right)\right]}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)}$$

$$\times \prod_{j=1}^n \mu'_j \sin\left[2\pi\left(\frac{r_j}{p_j}m'_j - \frac{\beta}{2K}\frac{\mu'_j}{p_j} - \frac{1}{2}s_j l\right)\right]. \tag{3.42}$$

The reason why we call the sum (3.42) singular (apart from the apparent ugliness of the sum over m_0) is that it seems to be related to a singularity in the "underlying" moduli space. Note that $Z^{(l)}_{\mathrm{s,polar,reg.}} = 0$ if $g = 0$ because the function whose residue is calculated in Eq. (3.41) is nonsingular at $\beta = 0$.

Now it just remains to combine together Eqs. (2.7), (3.19), (3.21), (3.23), (3.31), (3.40)–(3.42) into one proposition:

Proposition 3.1 *The large k asymptotics of Witten's invariant of a Seifert manifold is a sum of a finite number of contributions:*

$$Z(X_{g,\{\frac{p}{q}\}};k) = \sum_{m_j=0}^{p_j-1} \sum_{\substack{m_0\in\mathbb{Z} \\ 0<\beta_{\mathrm{st}}<K}} Z^{(\mathrm{red.})}_{\{m\};m_0} + \sum_{l=0,1} \sum_{\substack{0\leqq m'_j \leqq \frac{p_j}{2} \\ m'_j\in\mathbb{Z}+\frac{1}{2}q_j l,\ \sum_{j=1}^n \frac{\pm m'_j}{p_j}\notin\mathbb{Z}}} Z^{(\mathrm{irr.})}_{\{m'\};l}$$

$$+ \sum_{l=0,1} \sum_{\substack{0\leqq m'_j \leqq \frac{p_j}{2} \\ m'_j\in\mathbb{Z}+\frac{1}{2}q_j l,\ \exists\mu'_j=\pm 1:\ \sum_{j=1}^n \frac{\mu'_j m'_j}{p_j}\in\mathbb{Z}}} Z^{(\mathrm{irr.sp.})}_{\{m'\};l}, \tag{3.43}$$

$$Z^{(\mathrm{red.})}_{\{m\};m_0} = (-1)^n \frac{2}{\sqrt{|H|}}\left(\frac{K}{2}\right)^{g-\frac{1}{2}} \mathrm{sign}(P)\, e^{i\frac{\pi}{2}\mathrm{sign}\left(\frac{H}{P}\right)}$$

$$\times \exp\left[2\pi i K\left(\sum_{j=1}^n \frac{r_j}{p_j}n_j^2 + \frac{1}{4}\frac{H}{P}\left(\frac{\beta_{\mathrm{st}}}{K}\right)^2\right)\right]$$

$$\times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^n s(q_j,p_j) - 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right]$$

$$\times \sum_{l'=0}^{\infty} \frac{1}{l'!}\left(\frac{1}{8\pi i K}\frac{P}{H}\right)^{l'} \partial_\phi^{(2l')} \left.\frac{\prod_{j=1}^n \sin\left(\frac{2\pi}{p_j}(r_j m_j - \phi)\right)}{\sin^{n+2g-2}(2\pi\phi)}\right|_{\phi=\frac{\beta_{\mathrm{st}}}{2K}}, \tag{3.44}$$

$$Z^{(\text{irr.})}_{\{m'\};l} = \frac{(-1)^{n(l+1)} i\pi}{\prod_{j=1}^n \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m_j'}{p_j}\right)} \left(\frac{K}{2}\right)^{g-1} \frac{\operatorname{sign}(P)}{\sqrt{|P|}} e^{i\frac{3}{4}\pi \operatorname{sign}\left(\frac{H}{P}\right)}$$

$$\times \exp\left[2\pi i K \sum_{j=1}^n \left(\frac{r_j}{p_j} m_j'^2 - \frac{1}{4} s_j q_j l^2\right)\right]$$

$$\times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^n s(q_j, p_j) - 3\operatorname{sign}\left(\frac{H}{P}\right)\right)\right]$$

$$\times \left\{ \frac{i}{2}(-2)^n \operatorname{Res}_{\beta=0} \left\{ \frac{\exp\left(-\frac{i\pi}{2K}\frac{H}{P}\beta^2\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)} \cot(\pi\beta) \right.\right.$$

$$\times \prod_{j=1}^n \left[i \sin\left[2\pi\left(\frac{r_j}{p_j} m_j' - \frac{1}{2} s_j l\right)\right] \cos\left(\frac{\pi}{K}\frac{\beta}{p_j}\right) \sin\left(2\pi\beta\frac{m_j'}{p_j}\right)\right.$$

$$\left.\left. + \cos\left[2\pi\left(\frac{r_j}{p_j} m_j' - \frac{1}{2} s_j l\right)\right] \sin\left(\frac{\pi}{K}\frac{\beta}{p_j}\right) \cos\left(2\pi\beta\frac{m_j'}{p_j}\right)\right]\right\}$$

$$- \sum_{\{\mu'\}=\pm 1} \sum_{0 \leqq m_0 \leqq \left|\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right|} \frac{\operatorname{sign}\left(\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right)}{\mathrm{Sym}_\pm(m_0)\, \mathrm{Sym}_\pm\left(m_0 - \left|\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right|\right)}$$

$$\times \operatorname{Res}_{\beta=0} \frac{\exp\left[-\frac{i\pi}{2K}\frac{H}{P}\beta^2 + 2\pi i\beta \operatorname{sign}\left(\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right)\left(m_0 - \left|\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right|\right)\right]}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)}$$

$$\left.\times \prod_{j=1}^n \mu_j' \sin\left[2\pi\left(\frac{r_j}{p_j} m_j' - \frac{\beta}{2K}\frac{\mu_j'}{p_j} - \frac{1}{2} s_j l\right)\right]\right\}, \tag{3.45}$$

$$Z^{(\text{irr.sp.})}_{\{m'\};\,l} = Z^{(\text{irr.})}_{\{m'\};l} + \frac{(-1)^{n(l+1)} K^g}{2^{n+3g-3}\pi^{n+2g-2}} \frac{\operatorname{sign}(P)}{\sqrt{|P|}} \frac{e^{i\frac{3}{4}\pi \operatorname{sign}\left(\frac{H}{P}\right)}}{\prod_{j=1}^n \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m_j'}{p_j}\right)}$$

$$\times \exp\left[2\pi i K \sum_{j=1}^n \left(\frac{r_j}{p_j} m_j'^2 - \frac{1}{4} s_j q_j l^2\right)\right]$$

$$\times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^n s(q_j, p_j) - 3\operatorname{sign}\left(\frac{H}{P}\right)\right)\right]$$

$$\times \sum_{\substack{l'=0 \\ l'-n\in 2\mathbb{Z}}}^{\infty} \left(\frac{1}{2\pi i K}\frac{P}{H}\right)^{\frac{l'-n-2g+3}{2}} \frac{\Gamma\left(\frac{l'-n-2g+3}{2}\right)}{l'!}$$

$$\times \partial_\phi^{l'} \left\{ \left(\frac{2\pi\phi}{\sin(2\pi\phi)}\right)^{n+2g-2} \sum_{\substack{\mu_j'=\pm 1 \\ \sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\in\mathbb{Z}}} \prod_{j=1}^n \sin\left[2\pi\left(\frac{r_j\mu_j' m_j' - \phi}{p_j} - \frac{1}{2}s_j l\right)\right]\right\}\Bigg|_{\phi=0} . \tag{3.46}$$

A condition $\sum_{j=1}^n \frac{\pm m_j'}{p_j} \notin \mathbb{Z}$ in the second sum of Eq. (3.43) means that for any choice of signs $\pm$ in front of the numbers m_j' the sum is never integer. The condition $\exists \mu_j' = \pm 1 : \sum_{j=1}^n \frac{\mu_j' m_j'}{p_j} \in \mathbb{Z}$ means on the contrary that there exists a choice of signs such that the sum is integer.

4. Flat Connections and Asymptotic Contributions

4.1. Connected Components of Moduli Space

Our goal is to relate the terms $Z^{(\text{red.})}_{\{m\};m_0}$, $Z^{(\text{irr.})}_{\{m'\};l}$ and $Z^{(\text{irr.sp.})}_{\{m'\};\, l}$ of the asymptotic formula (3.43) to connected components of the moduli space $\mathcal{M}(X_{g,\{\frac{p}{q}\}})$ of flat connections of the Seifert manifold $X_{g,\{\frac{p}{q}\}}$ in accordance with the quantum field theory prediction (1.11). In this subsection we describe the connected components of $\mathcal{M}(X_{g,\{\frac{p}{q}\}})$.

A flat connection A_μ on a manifold induces a homomorphism Hol_A of the fundamental group π_1 into the gauge group which in our case is $SU(2)$. This homomorphism maps an element $x \in \pi_1$ into a parallel transport along x:

$$\text{Hol}_A(x) = \text{Pexp}\left(\oint_x A_\mu dx^\mu\right) \in SU(2) . \tag{4.1}$$

Two flat connections A_μ and A_μ' are gauge equivalent iff there exists an element h of the gauge group which conjugates one homomorphism into another:

$$\text{Hol}_{A'} = h^{-1}\, \text{Hol}_A\, h . \tag{4.2}$$

Therefore $\mathcal{M}$ is also a moduli space of homomorphisms $\pi_1 \to SU(2)$ up to a global conjugation.

The Seifert manifold $X_{g,\{\frac{p}{q}\}}$ is constructed by the surgeries $U^{(p_i,q_i)}$ on the loops $P_j \times S^1$ of the manifold $\Sigma_g \times S^1$ as it was described in Sect. 2. The fundamental group of $X_{g,\{\frac{p}{q}\}}$ is generated by the following elements: the loop b along S^1, the loops $a_1,\dots,a_n$ around n punctures P_j on Σ_g and the standard generators $c_1,d_1,\dots,c_g,d_g$ of $\pi_1(\Sigma_g)$. These elements satisfy relations

$$a_j^{p_j} b^{q_j} = 1, \quad 1 \leqq j \leqq n , \tag{4.3}$$

$$a_1 \cdot \cdots \cdot a_n = c_1 d_1 c_1^{-1} d_1^{-1} \cdot \cdots \cdot c_g d_g c_g^{-1} d_g^{-1} , \tag{4.4}$$

and the requirement that b commutes with all other elements of π_1.

There is another set of important elements in π_1. These elements represent the middle cycles of the solid tori (i.e. their parallels) which we glued in during the surgeries:

$$f_j = a_j^{r_j} b^{s_j}, \quad 1 \leqq j \leqq n . \tag{4.5}$$

Consider a homomorphism $\mathrm{Hol}_A : \pi_1 \to SU(2)$. We introduce a function $\phi : \pi_1 \to [0, \frac{1}{2}]$ such that for $x \in \pi_1$ both $\mathrm{Hol}_A(x)$ and $\exp[2\pi i \sigma_3 \phi(x)]$ belong to the same conjugation class of $SU(2)$. Since b commutes with a_j, Eqs. (4.3) and (4.5) imply that for some numbers $\tilde{m}, \tilde{m}' \in \mathbb{Z}$,

$$\phi(a_j) = \left| \frac{\tilde{m}_j + q_j \phi(b)}{p_j} \right| , \tag{4.6}$$

$$\phi(f_j) = \left| \frac{\phi(b) - r_j \tilde{m}_j}{p_j} + \tilde{m}_j' \right| . \tag{4.7}$$

The remaining analysis depends on the value of $\phi(b)$. If $\phi(b) \neq 0, \frac{1}{2}$, then $\mathrm{Hol}_A(b)$ does not belong to the center of $SU(2)$. Therefore since b belongs to the center of $\pi_1(X_{g,\{\frac{p}{q}\}})$, all the holonomies should belong to the same $U(1)$ subgroup of $SU(2)$, in particular,

$$\mathrm{Hol}_A(b) = \exp[2\pi i \sigma_3 \phi(b)] , \qquad \mathrm{Hol}_A(a_j) = \exp\left(2\pi i \sigma_3 \frac{\tilde{m}_j + q_j \phi(b)}{p_j} \right) . \tag{4.8}$$

This means that the connection is reducible: the isotropy group H_c, which commutes with the holonomies, is equal to $U(1)$. Also since all the holonomies now commute, the r.h.s. of Eq. (4.4) is trivial. Therefore for some $\tilde{m}_0 \in \mathbb{Z}$,

$$\tilde{m}_0 + \sum_{j=1}^{n} \frac{\tilde{m}_j + q_j \phi(b)}{p_j} = 0 . \tag{4.9}$$

Substituting here Eq. (4.6) we find that

$$\phi(b) = \frac{P}{H} \left(\tilde{m}_0 - \sum_{j=1}^{n} \frac{\tilde{m}_j}{p_j} \right) . \tag{4.10}$$

As for the phases $\phi(c_j), \phi(d_j)$, $1 \leqq j \leqq g$, they are totally unrestricted. The only condition on $\mathrm{Hol}_A(c_j)$ and $\mathrm{Hol}_A(d_j)$ is that they belong to the same subgroup $U(1) \subset SU(2)$ as all other holonomies.

Proposition 4.1 *The connected components of reducible flat connections with* $\phi(b) \neq 0, \frac{1}{2}$ *are* $\mathcal{M}^{(\mathrm{red.})}_{\{\tilde{m}\};\tilde{m}_0}$. *Their holonomies are described by Eqs.* (4.8), (4.10) *and* (4.6). *The choice of numbers* $\tilde{m}_1, \ldots, \tilde{m}_n, \tilde{m}_0$ *is limited by a condition*

$$0 \leqq \phi(a_1), \ldots, \phi(a_n), \phi(b) \leqq \frac{1}{2} . \tag{4.11}$$

If $\phi(b) = 0, \frac{1}{2}$ then $\mathrm{Hol}_A(b)$ belongs to the center of $SU(2)$ and the connection can be irreducible. Equations (4.3) restrict the possible conjugation classes of the holonomies $\mathrm{Hol}_A(a_j)$. Since this time $\mathrm{Hol}_A(b)$ is invariant under the reflection $e^{2\pi i\phi\sigma_3} \to e^{-2\pi i\phi\sigma_3}$, we find that

$$\phi(a_j) = \frac{\tilde{m}_j + q_j\phi(b)}{p_j} \,. \tag{4.12}$$

If $g = 0$, then Eq. (4.4) degenerates into

$$a_1 \cdot \cdots \cdot a_n = 1 \,. \tag{4.13}$$

This condition imposes a quantum group version of the polygon (e.g., triangle for $n = 3$) inequalities on the phases $\phi(a_j)$. If however $g \geqq 1$, then since the commutants $h_1 h_2 h_1^{-1} h_2^{-1}$, $h_{1,2} \in SU(2)$ cover the whole group $SU(2)$, Eq. (4.4) does not restrict the phases $\phi(a_j)$.

Proposition 4.2 *The connected components of irreducible flat connections are* $\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}$. *The conjugation classes of some of their holonomies are determined by Eq.* (4.12) *with* $\phi(b) = \frac{\tilde{l}}{2}$, $\tilde{l} = 0, 1$. *The choice of the numbers* $\tilde{m}_j$ *is limited by the condition* (4.11).

If there exist the numbers $\tilde{\mu}_j = \pm 1$ *such that* $\sum_{j=1}^n \tilde{\mu}_j \phi(a_j) \in \mathbb{Z}$, (*cf. Eq.* (4.9)) *then some of the connections of the connected component* $\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}$ *are reducible and we denote it as* $\mathcal{M}^{(\mathrm{irr.sp.})}_{\{\tilde{m}\};\tilde{l}}$.

4.2. Identification of Asymptotic Contributions

We are going to identify the contributions that the connected components of the moduli space $\mathcal{M}(X_{g,\{\frac{p}{q}\}})$ make to Witten's invariant $Z(X_{g,\{\frac{p}{q}\}}; k)$.

Proposition 4.3 *The contribution to Witten's invariant* $Z(X_{g,\{\frac{p}{q}\}}; k)$ *of a reducible component* $\mathcal{M}^{(\mathrm{red.})}_{\{\tilde{m}\};\tilde{m}_0}$ *is* $Z^{(\mathrm{red.})}_{\{m\};m_0}$ *of Eq.* (3.44) *such that*

$$m_j = \tilde{m}_j (\mathrm{mod}\, p_j), \qquad m_0 = \tilde{m}_0 + \sum_{j=1}^{n} \frac{m_j - \tilde{m}_j}{p_j} \,. \tag{4.14}$$

The contribution of an irreducible component $\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}$ *is* $Z^{(\mathrm{irr.})}_{\{m'\};l}$ *such that*

$$m'_j = \tilde{m}_j + \frac{1}{2} q_j \tilde{l}, \qquad l = \tilde{l} \,. \tag{4.15}$$

The contribution of a special irreducible component $\mathcal{M}^{(\mathrm{irr.sp.})}_{\{\tilde{m}\};\tilde{l}}$ (*which also contains some reducible connections*) *is* $Z^{(\mathrm{irr.sp.})}_{\{m'\};\,l}$ *whose indices are given by Eq.* (4.15).

One possible way of verifying these claims is to use Eqs. (1.12) and (1.15). One has to compare the already known Chern–Simons actions of flat connections to the leading exponentials of Eqs. (3.44)–(3.46). One-loop corrections can also be compared if at least some of the parameters in the r.h.s. of Eq. (1.15) can be independently calculated. We carried out this program for 3-fibered Seifert manifolds $X_0\left(\frac{p_1}{q_1},\frac{p_2}{q_2},\frac{p_3}{q_3}\right)$ in [5] by using the 1-loop calculations of [4].

A more direct way of identifying the asymptotic contributions is to "measure" (or, in the language of quantum theory, "observe") directly the holonomies of flat connections along some elements of the fundamental group of the manifold. Suppose that we know that for an $x\in\pi_1$ the conjugation class of $\mathrm{Hol}_A(x)$ is the same for all connections of a connected component $\mathcal{M}_c$. Let us introduce a knot (that is, a Wilson line) along x carrying a γ-dimensional representation of $SU(2)$. In other words, we multiply the integrand of Eq. (1.3) by an extra factor $\mathrm{Tr}_\gamma\,\mathrm{Pexp}(\oint_x A_\mu dx^\mu)$. According to Eq. (1.15), at the 1-loop level in $1/K$ expansion the contribution of $\mathcal{M}_c$ will be multiplied by

$$\mathrm{Tr}_\gamma\,\mathrm{Pexp}\left(\oint_x A_\mu dx^\mu\right)=\mathrm{Tr}_\gamma\exp[2\pi i\sigma_3\phi(x)]\equiv\frac{\sin[2\pi\gamma\phi(x)]}{\sin[2\pi\phi(x)]}\,. \qquad (4.16)$$

Therefore the knot is an observable which measures the conjugation class of the holonomy.

We introduce the following link into the Seifert manifold $X_{g,\left\{\frac{p}{q}\right\}}$: a line along b with γ-dimensional representation and n lines along a_j with γ_j dimensional representations. The new Witten's invariant $Z_{\{\gamma\};\gamma}(X_{g,\left\{\frac{p}{q}\right\}},\mathcal{L};k)$ can be easily calculated with the help of the lemma whose simple proof can be traced back to [1]:

Lemma 4.1 *Let $\mathcal{K}$ be a knot in a manifold M and let $\mathcal{K}_m$ be the meridian of $\mathcal{K}$. If $\mathcal{K}$ carries an α-dimensional representation and $\mathcal{K}_m$ carries a γ-dimensional representation, then*

$$Z_{\alpha,\gamma}(M,\mathcal{K},\mathcal{K}_m;k)=\frac{\tilde{S}^{-1}_{\gamma\alpha}}{\tilde{S}^{-1}_{1\alpha}}Z_\alpha(M,\mathcal{K};k)\equiv\frac{\sin\left(\frac{\pi}{K}\alpha\gamma\right)}{\sin\left(\frac{\pi}{K}\alpha\right)}Z_\alpha(M,\mathcal{K};k)\,. \qquad (4.17)$$

As a result,

$$\begin{aligned}Z_{\{\gamma\};\gamma}(X_{g,\left\{\frac{p}{q}\right\}},\mathcal{L};k)&=e^{if_{\mathrm{fr}}}\sum_{\alpha_j=1}^{K-1}N^g_{\alpha_1,\dots,\alpha_n,\gamma}\prod_{j=1}^{n}\frac{\sin\left(\frac{\pi}{K}\alpha_j\gamma_j\right)}{\sin\left(\frac{\pi}{K}\alpha_j\right)}\tilde{U}^{(p_j,q_j)}_{\alpha_j1}\\&=e^{if_{\mathrm{fr}}}\sum_{\beta=1}^{K-1}\frac{\sin\left(\frac{\pi}{K}\beta\gamma\right)}{\sin\left(\frac{\pi}{K}\beta\right)}\frac{\prod_{j=1}^{n}\sum_{\alpha_j=1}^{K-1}\frac{\sin\left(\frac{\pi}{K}\alpha_j\gamma_j\right)}{\sin\left(\frac{\pi}{K}\alpha_j\right)}\tilde{S}_{\beta\alpha_j}\tilde{U}^{(p_j,q_j)}_{\alpha_j1}}{\tilde{S}^{n+2g-2}_{\beta1}}\,. \qquad (4.18)\end{aligned}$$

Instead of going through the detailed asymptotic calculation of the sums of this equation along the lines of the previous section (which is possible but tedious) we will present a simple argument which will show how the extra factors

$$\frac{\sin\left(\frac{\pi}{K}\beta\gamma\right)}{\sin\left(\frac{\pi}{K}\beta\right)} \qquad (4.19)$$

and

$$\frac{\sin\left(\frac{\pi}{K}\alpha_j\gamma_j\right)}{\sin\left(\frac{\pi}{K}\alpha_j\right)} \tag{4.20}$$

affect the asymptotic formulas (3.44) and (3.46). Note that all the terms in Eq. (3.43) came as local contributions of some special points β^*: $Z^{(\mathrm{red.})}_{\{m\};m_0}$ came from the stationary phase points $\beta^* = \beta_{\mathrm{st}}$, $Z^{(\mathrm{irr.})}_{\{m'\};l}$ came from residues at $\beta^* = Kl$ and $Z^{(\mathrm{irr.sp.})}_{\{m'\};\,l}$ came from both stationary phase and residue at $\beta^* = Kl$. Therefore to the leading order in K the effect of the factor (4.19) is to multiply these contributions by $\frac{\sin(\frac{\pi}{K}\beta^*\gamma)}{\sin(\frac{\pi}{K}\beta^*)}$. Comparing this factor with the r.h.s. of Eq. (4.16) we conclude that

$$\phi(b) = \frac{\beta^*}{2K}\,. \tag{4.21}$$

This means that for $Z^{(\mathrm{red.})}_{\{m\};m_0}$

$$\phi(b) = \frac{P}{H}\left(m_0 - \sum_{j=1}^{n}\frac{m_j}{p_j}\right)\,, \tag{4.22}$$

while for $Z^{(\mathrm{irr.})}_{\{m'\};l}$ and $Z^{(\mathrm{irr.sp.})}_{\{m'\};\,l}$

$$\phi(b) = \frac{l}{2}\,, \tag{4.23}$$

in full agreement with the Proposition 4.3.

To find the effect of the factor (4.20) consider the calculation of the sum

$$\sum_{\alpha_j=1}^{K-1}\tilde{S}_{\beta\alpha_j}\tilde{U}^{(p_j,q_j)}_{\alpha_j 1}\,, \tag{4.24}$$

which produces the factor $\tilde{U}^{(-q_j,p_j)}_{\beta 1}$ of Eq. (2.7). The relevant part of this sum is

$$\sum_{\alpha_j=1}^{K-1}\exp\left[\frac{i\pi}{2K}\left(\frac{p_j}{q_j}\alpha_j^2 - 2\alpha_j\left(\frac{2Km_U+\mu_U}{q_j}+\beta\mu_S\right)\right)\right]\,, \tag{4.25}$$

here m_U and μ_U are m and μ coming from Eq. (1.8) while μ_S comes from the formula

$$\sin\left(\frac{\pi}{K}\beta\alpha_j\right) = \frac{i}{2}\sum_{\mu_S=\pm 1}\mu_S\exp\left(-i\frac{\pi}{K}\mu_S\beta\alpha_j\right)\,. \tag{4.26}$$

The sum (4.25) can be calculated along the lines of the previous section. It will turn into a purely gaussian integral over α_j. The stationary phase point which dominates this integral is

$$\alpha_j^{(\mathrm{st})} = \frac{2Km_U + \mu_S q_j\beta}{p_j}\,. \tag{4.27}$$

On the other hand, comparing an integral over α_j of the summand in Eq. (4.25) with the exponentials of Eq. (2.8) we conclude that

$$m_j = \mu_S m_U, \qquad \mu_j = \mu_S \mu_U , \tag{4.28}$$

so that

$$\alpha_j^{(\mathrm{st})} = \mu_S \frac{2Km_j + q_j\beta}{p_j} . \tag{4.29}$$

Therefore to the leading order in K, the effect of the factor (4.20) is to multiply the contributions by

$$\frac{\sin\left(\frac{\pi}{K}\alpha_j^{(\mathrm{st})}\gamma_j\right)}{\sin\left(\frac{\pi}{K}\alpha_j^{(\mathrm{st})}\right)} , \tag{4.30}$$

with $\alpha_j^{(\mathrm{st})}$ coming from Eq. (4.29) in which we should substitute $\beta = \beta^*$. Then Eq. (4.16) tells us that

$$\phi(a_j) = \left| \frac{m_j + q_j \frac{\beta^*}{2K}}{p_j} \right| , \tag{4.31}$$

which is again in full agreement with the Proposition 4.3.

Finally as a result of our identifications we can recognize the presence of the factors $\prod_{j=1}^n \sin[2\pi\phi(f_j)]$ in all the formulas (3.44)–(3.46), e.g. the factors $\prod_{j=1}^n \sin\left(\frac{2\pi}{p_j}(r_j m_j - \phi)\right)$ in Eq. (3.44) and $\prod_{j=1}^n \sin\left(\frac{r_j}{p_j}m_j' - \frac{1}{2}s_j l\right)$ in Eq. (3.45).

5. Intersection Numbers on Moduli Space

Consider again the asymptotic formulas (3.43)–(3.46). Whereas the contributions of reducible connections $Z^{(\mathrm{red.})}_{\{m\};m_0}$ are presented as infinite asymptotic series in $1/K$, it turns out that the contributions of irreducible connections are in fact finite polynomials in $1/K$. This follows easily from the residue formula (3.45). The situation seems similar to that of the Yang–Mills partition fuction calculation of [8] and the calculation of Verlinde numbers in [12, 13]. In all these cases the moduli spaces contributing the polynomials to the partition functions are isomorphic. In particular, it is easy to see that

$$\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}(X_{g,\{\frac{p}{q}\}}) = \mathcal{M}_{\{\theta\}}(\Sigma_g) , \qquad \theta_j = \frac{\tilde{m}_j + \frac{1}{2}q_j\tilde{l}}{p_j} \equiv \frac{m_j'}{p_j} , \tag{5.1}$$

here $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ is a moduli space of $SU(2)$ flat connections of a g-handle surface with n punctures P_j and holonomies around them fixed by Eq. (2.11). Both the Yang–Mills partition function (2.12) and Verlinde number (2.10) were expressed in terms of the intersection numbers on $\mathcal{M}_{\{\theta\}}(\Sigma_g)$. We will derive a similar expression for $Z^{(\mathrm{irr.})}_{\{m'\};l}$ by comparing the asymptotic formulas for these three objects and using the localization formulas of [8, 9 and 10].

We start by presenting the residue formulas for the partition functions (2.10) and (2.12).

Proposition 5.1 *A number of conformal blocks for the $SU(2)$ WZW model on Σ_g with n insertions of the primary fields $\mathcal{O}_{\alpha_j}$ is equal to*

$$N^g_{\{\alpha\}} = -4\pi\left(\frac{K}{2}\right)^g\left[\mathrm{Res}_{\phi=0}\frac{\prod_{j=1}^n\sin(2\pi\alpha_j\phi)}{\sin^{n+2g-2}(2\pi\phi)}\cot(2\pi K\phi)\right.$$

$$-\left(\frac{i}{2}\right)^{n-1}\sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^n\mu_j\right)\mathrm{sign}\left(\sum_{j=1}^n\mu_j\alpha_j\right)\sum_{0\leqq m<\frac{1}{2K}\left|\sum_{j=1}^n\mu_j\alpha_j\right|}\frac{1}{\mathrm{Sym}_\pm(m)}$$

$$\left.\times\,\mathrm{Res}_{\phi=0}\frac{\exp\left[2\pi i\phi\,\mathrm{sign}\left(\sum_{j=1}^n\mu_j\alpha_j\right)\left(2Km-\left|\sum_{j=1}^n\mu_j\alpha_j\right|\right)\right]}{\sin^{n+2g-2}(2\pi\phi)}\right], \quad (5.2)$$

if $n+2g-2>0$ and $n+\sum_{j=1}^n\alpha_j$ is even. If $n+\sum_{j=1}^n\alpha_j$ is odd then $N^g_{\{\alpha\}}=0$.

Proposition 5.2 *A partition function of the 2d Yang–Mills theory on a g-handled surface Σ_g with n punctures P_j, the holonomies around which are fixed by Eqs.* (2.11), *has the following asymptotic representation in the limit of small gauge coupling constant a: if $\sum_{j=1}^n\pm\theta_j\notin\mathbb{Z}$, then*

$$Z_{\{\theta\}}(\Sigma_g;a)=Z^{(\mathrm{irr.})}_{\{\theta\}}(\Sigma_g;a)+\sum_{\{\mu\}=\pm 1}\sum_{\substack{m\in\mathbb{Z}\\ m-\sum_{j=1}^n\mu_j\theta_j>0}}Z^{(\mathrm{red.})}_{\{\theta\}}(\Sigma_g;a)\,, \quad (5.3)$$

$$Z^{(\mathrm{irr.})}_{\{\theta\}}(\Sigma_g;a)=-\frac{1}{2^g\pi^{n+2g-3}}\left[\mathrm{Res}_{\phi=0}\frac{e^{-a\phi^2}}{\phi^{n+2g-2}}\cot(\pi\phi)\prod_{j=1}^n\sin(2\pi\theta_j\phi)\right.$$

$$-\left(\frac{i}{2}\right)^{n-1}\sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^n\mu_j\right)\sum_{0\leqq m<\left|\sum_{j=1}^n\mu_j\theta_j\right|}\frac{\mathrm{sign}\left(\sum_{j=1}^n\mu_j\theta_j\right)}{\mathrm{Sym}_\pm(m)}$$

$$\left.\times\,\mathrm{Res}_{\phi=0}\frac{\exp\left[-a\phi^2+2\pi i\phi\,\mathrm{sign}\left(\sum_{j=1}^n\mu_j\theta_j\right)\left(m-\left|\sum_{j=1}^n\mu_j\theta_j\right|\right)\right]}{\phi^{n+2g-2}}\right], \quad (5.4)$$

$$Z^{(\mathrm{red.})}_{\{\theta\}}(\Sigma_g;a)=\left(\prod_{j=1}^n\mu_j\right)\exp\left[-\frac{\pi^2}{a}\left(m-\sum_{j=1}^n\mu_j\theta_j\right)^2\right]\sum_{l'=0}^{\infty}\frac{(-1)^{g-1+l'}a^{n+2g+l'-\frac{5}{2}}}{2^{n+2g+2l'}\pi^{2n+4g+2l'-\frac{9}{2}}l'!}$$

$$\times\frac{(n+2g+2l'-3)!}{(n+2g-3)!}\frac{1}{\left(m-\sum_{j=1}^n\mu_j\theta_j\right)^{n+2g+l'-2}}\,. \quad (5.5)$$

If n is even and $\exists \mu_j = \pm 1$ such that $\sum_{j=1}^{n} \mu_j \theta_j \in \mathbb{Z}$, then $Z_{\{\theta\}}^{(\mathrm{irr.})}(\Sigma_g; a)$ in Eq. (5.3) *should be substituted by $Z_{\{\theta\}}^{(\mathrm{irr.sp.})}(\Sigma_g; a)$:*

$$Z_{\{\theta\}}^{(\mathrm{irr.sp.})}(\Sigma_g; a) = Z_{\{\theta\}}^{(\mathrm{irr.})}(\Sigma_g; a) + \frac{i^n a^{\frac{2n+2g-3}{2}}}{2^{n+g}\pi^{n+2g-2}} \Gamma\left(\frac{3-2g-n}{2}\right) \sum_{\substack{\mu_j=\pm 1 \\ \sum_{j=1}^{n}\mu_j\theta_j \in \mathbb{Z}}} \left(\prod_{j=1}^{n} \mu_j\right)$$

$$\equiv Z_{\{\theta\}}^{(\mathrm{irr.})}(\Sigma_g; a) - (-1)^{g+\frac{n}{2}} \frac{i^n 2^{g-2}}{\pi^{n+2g-\frac{5}{2}}} \frac{\left(\frac{n+2g-2}{2}\right)!}{(n+2g-2)!} a^{n+g-\frac{3}{2}} \sum_{\substack{\mu_j=\pm 1 \\ \sum_{j=1}^{n}\mu_j\theta_j \in \mathbb{Z}}} \left(\prod_{j=1}^{n} \mu_j\right). \tag{5.6}$$

The contributions $Z_{\{\theta\}}^{(\mathrm{red.})}(\Sigma_g; a)$ come from constant curvature $U(1)$ connections, the contribution $Z_{\{\theta\}}^{(\mathrm{irr.})}(\Sigma_g; a)$ comes from irreducible flat connections.

Equation (5.2) was derived (for the case of $n = 0$) in the papers [12, 13]. E. Witten derived Eqs. (5.5) and Eq. (5.6) in [8].

According to [8],

$$2Z_{\{\theta\}}^{(\mathrm{irr.})}(\Sigma_g; a) = \int_{\mathcal{M}_{\{\theta\}}(\Sigma_g)} \exp(\omega + 4a\Theta) ; \tag{5.7}$$

here Θ is a 4-form defined in [8] and ω is a symplectic form on $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ normalized in the following way: if a_μ and b_μ are two $su(2)$ valued 1-forms representing the tangent vectors at a point on $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ then

$$\omega(a_\mu, b_\mu) = \frac{1}{4\pi^2} \operatorname{Tr} \int_{\Sigma_g} a_\mu \wedge b_\mu . \tag{5.8}$$

The moduli space $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ is a bundle over a moduli space $\mathcal{M}(\Sigma_g)$ of flat connections on Σ_g without punctures[2] (let us forget for a moment that $\mathcal{M}(\Sigma_g)$ has a singularity, we also assume that θ_j are small and $\sum_{j=1}^{n} \pm\theta_j \notin \mathbb{Z}$). The symplectic form ω is a sum of forms

$$\omega = \omega_0 + 2\sum_{j=1}^{n} \theta_j \omega_j ; \tag{5.9}$$

here ω_0 is a pull-back of the symplectic form on $\mathcal{M}(\Sigma_g)$ while ω_j are closed 2-forms normalized so that

$$\int_{S_i^2} \omega_j = \delta_{ij} , \tag{5.10}$$

S_i^2 $(q \leqq i \leqq n)$ are the 2-dimensional spheres which make up the fibers of the bundle $\mathcal{M}_{\{\theta\}}(\Sigma_g) \to \mathcal{M}(\Sigma_g)$.

The Verlinde number (5.2) is a dimension of the Chern–Simons Hilbert space for Σ_g with n insertions of primary fields $\mathcal{O}_{\alpha_j}$. In other words, it is a number of holomorphic sections of a certain line bundle over $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ with

$$\theta_j = \frac{\alpha_j - 1}{2k} . \tag{5.11}$$

---

[2] I am thankful to L. Jeffrey and A. Szenes for explaining to me the properties of this bundle and its symplectic structure.

Therefore it is given by the Riemann–Roch formula

$$N^g_{\{\alpha\}} = \int_{\mathcal{M}_{\{\theta\}}(\Sigma_g)} e^{k\omega}\,\mathrm{Td}(\mathcal{M}_{\{\theta\}}(\Sigma_g)) \tag{5.12}$$

(see, e.g. [11, 10] and references therein). Note that a natural symplectic form coming from Eqs. (1.1) and (1.3) is

$$\omega' = 4\pi^2\omega\,. \tag{5.13}$$

Therefore the semiclassical formula for the dimension of the Hilbert space should contain the exponent $\exp\left(\frac{\omega'}{2\pi\hbar}\right) = \exp\left(\frac{k\omega'}{4\pi^2}\right)$ in full agreement with Eq. (5.12).

The Todd class $\mathrm{Td}(\mathcal{M}_{\{\theta\}}(\Sigma_g))$ can be expressed as

$$\mathrm{Td}(\mathcal{M}_{\{\theta\}}(\Sigma_g)) = \exp\left(2\omega_0 + \sum_{j=1}^{n}\omega_j\right)\hat{A}(\mathcal{M}_{\{\theta\}}(\Sigma_g)) \tag{5.14}$$

(see, e.g.[10] and references therein). Upon substituting this expression in Eq. (5.12) we get

$$N^g_{\{\alpha\}} = \int_{\mathcal{M}_{\{\theta\}}(\Sigma_g)} \exp\left(K\omega_0 + \sum_{j=1}^{n}\alpha_j\omega_j\right)\hat{A}(\mathcal{M}_{\{\theta\}}(\Sigma_g))\,. \tag{5.15}$$

The pairs of Eqs. (5.2), (5.15) and (5.4), (5.7) are particular cases of the following conjecture which can be deduced from the calculations of [8], the main theorem of [9] and the calculations and conjecture of [10]:

Conjecture 5.1 *For the numbers $\theta_j, 1 \leqq j \leqq n$ such that $\sum_{j=1}^n \pm\theta_j \notin \mathbb{Z}$, let $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ be the moduli space of flat $SU(2)$ connections on Σ_g with n punctures and holonomies* (2.11) *around them. Then for the two (not necessarily integer) numbers K, a*

$$\begin{aligned}
&\int_{\mathcal{M}_{\{\theta\}}(\Sigma_g)} \exp\left[K\left(\omega_0 + 2\sum_{j=1}^{n}\theta_j\omega_j\right) + 4a\Theta\right]\hat{A}(\mathcal{M}_{\{\theta\}}(\Sigma_g))\\
&= -2\pi\left(\frac{K}{2}\right)^g\left[\mathrm{Res}_{\phi=0}\frac{\exp(-a\phi^2)}{\sin^{n+2g-2}(\pi\phi)}\cot(\pi K\phi)\prod_{j=1}^{n}\sin(2\pi K\theta_j\phi)\right.\\
&\quad -\left(\frac{i}{2}\right)^{n-1}\sum_{\{\mu\}=\pm1}\left(\prod_{j=1}^{n}\mu_j\right)\mathrm{sign}\left(\sum_{j=1}^{n}\mu_j\theta_j\right)\sum_{0\leqq m<\left|\sum_{j=1}^n\mu_j\theta_j\right|}\frac{1}{\mathrm{Sym}_\pm(m)}\\
&\quad\left.\times\,\mathrm{Res}_{\phi=0}\frac{\exp\left[-a\phi^2 + 2\pi iK\phi\,\mathrm{sign}\left(\sum_{j=1}^n\mu_j\theta_j\right)\left(m - \left|\sum_{j=1}^n\mu_j\theta_j\right|\right)\right]}{\sin^{n+2g-2}(\pi\phi)}\right]\,.
\end{aligned} \tag{5.16}$$

Suppose that we change the phases θ_j by small amounts $\Delta\theta_j$ such that for any $t \in [0,1]$, $\sum_{j=1}^n \pm(\theta_j + t\Delta\theta_j) \notin \mathbb{Z}$. The topological class of the manifold $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ does not change. As a result,

$$\begin{aligned}
&\int_{\mathcal{M}_{\{\theta\}}(\Sigma_g)} \exp\left[K\left(\omega_0 + 2\sum_{j=1}^n(\theta_j + \Delta\theta_j)\omega_j\right) + 4a\Theta\right]\hat{A}(\mathcal{M}_{\{\theta\}}(\Sigma_g)) \\
&= -2\pi\left(\frac{K}{2}\right)^g \left[\mathrm{Res}_{\phi=0}\frac{\exp(-a\phi^2)}{\sin^{n+2g-2}(\pi\phi)}\cot(\pi K\phi)\prod_{j=1}^n \sin(2\pi K(\theta_j+\Delta\theta_j)\phi)\right. \\
&\quad - \left(\frac{i}{2}\right)^{n-1}\sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^n \mu_j\right)\mathrm{sign}\left(\sum_{j=1}^n \mu_j\theta_j\right)\sum_{0\le m<\left|\sum_{j=1}^n \mu_j\theta_j\right|}\frac{1}{\mathrm{Sym}_\pm(m)} \\
&\quad \left.\times\ \mathrm{Res}_{\phi=0}\frac{\exp\left[-a\phi^2 + 2\pi i K\phi\,\mathrm{sign}\left(\sum_{j=1}^n \mu_j\theta_j\right)\left(m - \left|\sum_{j=1}^n \mu_j(\theta_j+\Delta\theta_j)\right|\right)\right]}{\sin^{n+2g-2}(\pi\phi)}\right].
\end{aligned} \tag{5.17}$$

It is easy to put the r.h.s. of Eq. (3.45) in a form similar to the r.h.s. of Eq. (5.17) for the case when $\sum_{j=1}^n \pm\frac{m_j'}{p_j} \notin \mathbb{Z}$:

$$\begin{aligned}
Z^{(\mathrm{irr.})}_{\{m'\};l} &= -\frac{(-1)^{nl}\pi}{\prod_{j=1}^n \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m_j'}{p_j}\right)}\left(\frac{K}{2}\right)^g \frac{\mathrm{sign}\,(P)}{\sqrt{|P|}} e^{i\frac{3}{4}\pi\,\mathrm{sign}\left(\frac{H}{P}\right)} \\
&\quad \times \exp\left[2\pi i K\sum_{j=1}^n\left(\frac{r_j}{p_j}m_j'^2 - \frac{1}{4}s_j q_j l^2\right)\right] \\
&\quad \times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^n s(q_j,p_j) - 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right] \\
&\quad \times \sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^n \mu_j\right)\exp\left[2\pi i\sum_{j=1}^n \mu_j\left(\frac{r_j}{p_j}m_j' - \frac{1}{2}s_j l\right)\right] \\
&\quad \times\left\{\mathrm{Res}_{\phi=0}\frac{\exp\left(-\frac{i\pi K}{2}\frac{H}{P}\phi^2\right)}{\sin^{n+2g-2}(\pi\phi)}\cot(\pi K\phi)\prod_{j=1}^n \sin\left[2\pi K\phi\left(\frac{m_j' + \frac{\mu_j}{2K}}{p_j}\right)\right]\right. \\
&\quad - \left(\frac{i}{2}\right)^{n-1}\sum_{\{\mu'\}=\pm 1}\left(\prod_{j=1}^n \mu_j'\right)\sum_{0\le m_0<\left|\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right|}\frac{\mathrm{sign}\left(\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right)}{\mathrm{Sym}_\pm(m_0)} \\
&\quad \left.\times\mathrm{Res}_{\phi=0}\frac{\exp\left[-\frac{i\pi K}{2}\frac{H}{P}\phi^2 + 2\pi i K\phi\,\mathrm{sign}\left(\sum_{j=1}^n \frac{\mu_j' m_j'}{p_j}\right)\left(m_0 - \left|\sum_{j=1}^n \frac{\mu_j'}{p_j}\left(m_j' + \frac{\mu_j}{2K}\right)\right|\right)\right]}{\sin^{n+2g-2}(\pi\phi)}\right\}.
\end{aligned} \tag{5.18}$$

Comparing this expression with the intersection number formula (5.17) we come to the following conclusion:

Proposition 5.3 *The contribution of a connected component $\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}$ of the moduli space of irreducible flat connections to Witten's invariant $Z(X_{g,\{\frac{p}{q}\}};k)$ can be expressed in terms of the intersection numbers of the forms on this component:*

$$
\begin{aligned}
Z^{(\mathrm{irr.})}_{\{m'\};l} = & \frac{(-1)^{nl}}{2} \frac{e^{i\frac{3}{4}\pi\,\mathrm{sign}(\frac{H}{P})}}{\prod_{j=1}^{n}\mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m'_j}{p_j}\right)} \frac{\mathrm{sign}\,(P)}{\sqrt{|P|}} \\
& \times \exp\left[2\pi i K \sum_{j=1}^{n}\left(\frac{r_j}{p_j}m'^2_j - \frac{1}{4}s_j q_j l^2\right)\right] \\
& \times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^{n}s(q_j,p_j) - 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right] \\
& \times \sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^{n}\mu_j\right)\exp\left[2\pi i\sum_{j=1}^{n}\mu_j\left(\frac{r_j}{p_j}m'_j - \frac{1}{2}s_j l\right)\right] \\
& \times \int_{\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\},\tilde{l}}} \exp\left[K\left(\omega_0 + 2\sum_{j=1}^{n}\frac{m'_j + \frac{\mu_j}{2K}}{p_j}\omega_j + 2\pi i\frac{H}{P}\Theta\right)\right]\hat{A}\,(\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}})\,, \qquad (5.19)
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
Z^{(\mathrm{irr.})}_{\{m'\};l} = & \frac{(-1)^{nl}}{2} \frac{e^{i\frac{3}{4}\pi\,\mathrm{sign}(\frac{H}{P})}}{\prod_{j=1}^{n}\mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m'_j}{p_j}\right)} \frac{\mathrm{sign}\,(P)}{\sqrt{|P|}} \\
& \times \exp\left[2\pi i K \sum_{j=1}^{n}\left(\frac{r_j}{p_j}m'^2_j - \frac{1}{4}s_j q_j l^2\right)\right] \\
& \times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^{n}s(q_j,p_j) - 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right] \\
& \times \int_{\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\},\tilde{l}}} \exp\left[K\left(\omega_0 + 2\sum_{j=1}^{n}\frac{m'_j}{p_j}\omega_j + 2\pi i\frac{H}{P}\Theta\right)\right]\hat{A}\,(\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}) \\
& \times \prod_{j=1}^{n} 2i\,\sin 2\pi\left(\frac{r_j}{p_j}m'_j - \frac{1}{2}s_j l + \frac{1}{2\pi i}\frac{\omega_j}{p_j}\right)\,. \qquad (5.20)
\end{aligned}
$$

The numbers m'_j and $\tilde{m}_j$ are related by Eq. (4.15), *also $\mathcal{M}^{(\mathrm{irr.})}_{\{\tilde{m}\};\tilde{l}}$ is isomorphic to $\mathcal{M}_{\{\frac{m'}{p}\}}(\Sigma_g)$.*

Note that the last product in Eq. (5.20) looks like the "equivariantized" Reidemeister torsion of the Seifert manifold.

The formula (5.19) looks very similar to Eq. (5.7) and also to Eq. (5.12) if we recall that

$$N^g_{\{\alpha\}} = Z_{\{\alpha\}}(\Sigma_g \times S^1, \mathcal{L}; k), \tag{5.21}$$

the n-component link $\mathcal{L}$ consists of n loops which go along S^1 of $\Sigma_g \times S^1$. E. Witten proved Eq. (5.7) in [8] by applying the equivariant localization arguments to the path integral representation of the 2d Yang–Mills theory. It seems likely that there should be a path integral localization proof for Eq. (5.19) as well. We came to Eq. (5.19) through the back door: by working out the large k asymptotics of the surgery formula and then cooking up an intersection number that would match the contribution of an irreducible connection. A localization argument would derive the r.h.s. of Eq. (5.19) directly from the path integral (1.3). Note, however, that even for the seemingly simpler case of Eq. (5.12) there is no localization proof yet. M. Blau and R. Thompson [14] could only use abelian localization in order to establish Verlinde formula (2.4). At the present time in order to prove the formula (5.12) one has to show that the path integral for $Z_{\{\alpha\}}(\Sigma_g \times S^1, \mathcal{L}; k)$ is equal to the number of sections of a certain holomorphic line bundle and then use the Riemann–Roch theorem to calculate that number.

6. Conclusion

An extensive use of path integral arguments puts the theory of Witten's invariants somewhere between mathematics and physics. The path integral calculations are tested in physics against the data coming from experiments with elementary particles. In a similar way we can say that the asymptotic expansion of the surgery formula (1.6) provides us with experimental data about Seifert manifolds. This data has to be compared with the asymptotic expansion (1.13) of the path integral.

Being viewed in this way, the annoying complexity of the formulas (3.43)–(3.46) should be encouraging. It means that there is plenty of experimental data (i.e. topological invariants of 3d manifolds) hidden in them. As we already know, this data includes Chern–Simons invariants, Reidemeister–Ray–Singer torsion and spectral flows at the 1-loop level. The Casson–Walker invariant appears as a 2-loop correction to the contribution of the trivial connection to Witten's invariant of rational homology spheres (and Seifert manifolds $X_{0,\{\frac{p}{q}\}}$ among them, see e.g. [5]). The full trivial connection contribution in the general case of $X_{g,\{\frac{p}{q}\}}$ was studied in [15] with the help of Eq. (3.46). We do not repeat this analysis here.

In this paper we were mostly interested in the contributions of irreducible flat connections. These contributions appear to be finite loop exact. Our main result is Eq. (5.20) which expresses these contributions in terms of the intersection numbers on the moduli space of flat connections. Similar expressions were obtained by J. Andersen [16] for the case of Seifert manifolds with $H = 0$. He expressed Witten's invariant as a trace of an operator acting in the space of sections of a certain line bundle and then used the Lefschetz–Riemann–Roch theorem in order to calculate that trace. We have not completely reconciled our formulas yet.

The form of Eq. (5.20) is very suggestive. It combines symplectic form on the moduli space, 4-form Θ (which appeared in Witten's study of 2d Yang–Mills theory [8]), $\hat{A}$-genus and an "equivariantized" Reidemeister torsion of the Seifert manifold. However Eq. (5.19) was derived "through the back door," that is, by comparing the residue expression (3.45) coming from the surgery formula (5.17) with the residue formula (5.17) for the intersection numbers. It would be much better to derive Eq. (5.19) directly by applying some sort of localization arguments in the spirit of [8] to the Chern–Simons path integral (1.3). This still remains an unsolved problem.

Acknowledgements. I am thankful to J. Andersen, L. Jeffrey, A. Szenes, A. Vaintrob and E. Witten for valuable discussions and advice. This work was supported by the National Science Foundation under Grant No. PHY-92 09978.

Appendix 1

There is an alternative way of calculating the sum (3.1) which is similar to the one used in [5]. This method is a Fourier transform of the method used in Sect. 3. It involves gaussian integrals instead of residues and boundary contributions instead of stationary phase contributions.

We start by expanding the denominator of Eq. (3.1) in an analog of geometric series:

$$\frac{1}{\sin^{n+2g-2}\left(\frac{\pi}{K}(\beta - i\xi)\right)} = (2i)^{n+2g-2} e^{-\frac{i\pi}{K}(n+2g-2)(\beta-i\xi)} \sum_{\substack{\gamma\in\mathbb{Z}\\ \gamma\geq 0}} K_{n+2g-2}(\gamma) e^{-\frac{2\pi i}{K}\gamma(\beta-i\xi)} \,. \tag{A1.1}$$

Here $K_n(m)$ is the $SU(2)$ Kostant's partition function:

$$K_n(m) = \binom{m+n-1}{n-1} \equiv \frac{(m+n-1)!}{(n-1)!m!} = 2\pi i \operatorname{Res}_{x=0} \frac{e^{2\pi i m x}}{(1-e^{-2\pi i x})^n} \,. \tag{A1.2}$$

In other words, the polynomial $K_n(m)$ is equal to the number of ways in which an integer number m can be split into a sum of n ordered nonnegative integers.

The expression (A1.1) can be put in a different form if we use a "shifted" Kostant's polynomial

$$\tilde{K}_n(m) = K_n(m - \frac{n}{2}) = \frac{1}{(n-1)!} \prod_{\substack{0\leq j\leq \frac{n}{2}-1\\ j\in\mathbb{Z}+\frac{n}{2}}} (m^2 - j^2)^{\frac{1}{\mathrm{Sym}\pm(j)}}$$

$$= \frac{\pi}{(2i)^{n-2}} \operatorname{Res}_{x=0} \frac{e^{2\pi i m x}}{\sin^n(\pi x)} \quad \text{for } m \geq 0\,,$$

$$\tilde{K}_n(m) = 0 \quad \text{for } m < 0\,. \tag{A1.3}$$

Since $K_n(m) = 0$ (as defined by $K_n(m) = \frac{1}{(n-1)!}\prod_{j=1}^{n-1}(m+j)$) if $m \in \mathbb{Z}$, $1-n \leq m \leq -1$, we can shift the range of summation in Eq. (A1.1) so that

$$\frac{1}{\sin^{n+2g-2}\left(\frac{\pi}{K}(\beta - i\xi)\right)} = (2i)^{n+2g-2} \sum_{\substack{\gamma>0\\ \gamma\in\mathbb{Z}+\frac{n}{2}}} \tilde{K}_{n+2g-2}(\gamma) \exp\left(-\frac{2\pi i}{K}\gamma(\beta - i\xi)\right) \,. \tag{A1.4}$$

The Poisson resummation formula

$$\sum_{m\in\mathbb{Z}+\frac{n}{2}} \delta(\gamma - m) = \sum_{l=0,1} e^{\pi i l n} \sum_{\beta'\in\mathbb{Z}} e^{2\pi i(l-2\beta')\gamma} \tag{A1.5}$$

allows us to convert the sum in Eq. (A1.4) into an integral over γ:

$$\frac{1}{\sin^{n+2g-2}\left(\frac{\pi}{K}(\beta - i\xi)\right)} = (2i)^{n+2g-2} \sum_{l=0,1} e^{\pi i l n} \int_0^\infty d\gamma\, \tilde{K}_{n+2g-2}(\gamma)$$

$$\times \exp\left(-\frac{2\pi i}{K}(\beta + 2K\beta' - Kl - i\xi)\right). \tag{A1.6}$$

We substitute this expression into Eq. (3.1). Since the summand of Eq. (3.1) is invariant under the shift $\beta \to \beta + 2K$, we can combine the sums $\sum_{\beta=-K+1}^{K}\sum_{\beta'\in\mathbb{Z}}$ into one sum $\sum_{\beta\in\mathbb{Z}}$, which we transform into an integral with the help of the Poisson formula (3.5):

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = \frac{(2i)^{n+2g-2}}{2} \sum_{l=0,1} e^{i\pi l n} \sum_{m_j=0}^{p_j-1} \sum_{\{\mu\}=\pm 1} \left(\prod_{j=1}^{n} \mu_j\right)$$

$$\times \exp\left(2\pi i \sum_{j=1}^{n} \frac{r_j}{p_j}\left(Km_j^2 + \mu_j m_j\right)\right)$$

$$\times \lim_{\xi\to 0^+} \sum_{m_0\in\mathbb{Z}} \int_0^\infty d\gamma\, \tilde{K}_{n+2g-2}(\gamma) \exp\left[2\pi i\gamma\left(l + \frac{i\xi}{K}\right)\right]$$

$$\times \int_{-\infty}^{+\infty} d\beta \exp\left[-\frac{2\pi i}{K}\left(\frac{1}{4}\frac{H}{P}\beta^2 + \beta\left(\gamma - Km_0 + \sum_{j=1}^{n}\frac{Km_j + \frac{\mu_j}{2}}{p_j}\right)\right)\right]. \tag{A1.7}$$

The integral over β is purely gaussian and straightforward to calculate. We go from m_j to m_j' according to Eq. (3.24) and transform a sum $\sum_{m_j=0}^{p_j-1}$ into

$$\sum_{\substack{0\le m_j'\le \frac{p_j}{2} \\ m_j'\in\mathbb{Z}+\frac{1}{2}q_j l}} \frac{1}{\prod_{j=1}^{n} \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m_j'}{p_j}\right)} \sum_{\{\mu'\}=\pm 1} \tag{A1.8}$$

by substituting $\mu_j' m_j'$ for m_j'. Since $\tilde{K}_{n+2g-2}(\gamma) = 0$ for $\gamma < 0$ we can extend the integration range to all γ. We also substitute

$$\gamma + Km_0 - \sum_{j=1}^{n} \frac{\mu_j'}{p_j}\left(Km_j' + \frac{\mu_j}{2}\right) \tag{A1.9}$$

for γ. After all these transformations we end up with the following expression:

$$Z_s(X_{g,\{\frac{p}{q}\}};k) = (2i)^{n+2g-2}\left(\frac{K}{2}\left|\frac{P}{H}\right|\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4}\,\mathrm{sign}\left(\frac{H}{P}\right)} \sum_{l=0,1} e^{i\pi l n} \sum_{\substack{0\leqq m'_j\leqq \frac{p_j}{2}\\ m'_j\in\mathbb{Z}+\frac{1}{2}q_j l}} \frac{1}{\prod_{j=1}^{n}\mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m'_j}{p_j}\right)}$$
$$\times \exp\left[2\pi i K\sum_{j=1}^{n}\left(\frac{r_j}{p_j}m_j'^2 - \frac{1}{4}s_j q_j l^2\right)\right]$$
$$\times \sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^{n}\mu_j\right)\exp\left[2\pi i\sum_{j=1}^{n}\mu_j\left(\frac{r_j}{p_j}m'_j - \frac{1}{2}s_j l\right)\right]$$
$$\times\int_{-\infty}^{+\infty}\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma;m'_j,\mu_j)\exp\left(\frac{2\pi i}{K}\frac{P}{H}\gamma^2\right), \tag{A1.10}$$

here

$$\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma;m'_j,\mu_j) = \lim_{\xi\to 0^+}\sum_{m_0\in\mathbb{Z}}\sum_{\{\mu'\}=\pm 1}\left(\prod_{j=1}^{n}\mu'_j\right)$$
$$\times\tilde{K}_{n+2g-2}\left(\gamma + Km_0 - \sum_{j=1}^{n}\frac{\mu'_j}{p_j}\left(Km'_j + \frac{\mu_j}{2}\right)\right)$$
$$\times\exp\left[-\frac{2\pi}{K}\xi\left(\gamma + Km_0 - \sum_{j=1}^{n}\frac{\mu'_j}{p_j}\left(Km'_j+\frac{\mu_j}{2}\right)\right)\right]. \tag{A1.11}$$

The function $\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma;m'_j,\mu_j)$ is locally polynomial in γ but it (or its derivatives) has a break at the points

$$\gamma_{\mathrm{br}} = -Km_0 + \sum_{j=1}^{n}\frac{\mu'_j}{p_j}\left(Km'_j + \frac{\mu_j}{2}\right),\quad m_0\in\mathbb{Z}, \tag{A1.12}$$

because the shifted Kostant's partition function $\tilde{K}_{n+2g-2}(\gamma)$ (or its derivatives) has a break at $\gamma = 0$.

The sum $\sum_{m_0\in\mathbb{Z}}$ in Eq. (A1.11) can be limited to

$$m_0 \geqq -\frac{\gamma}{K} + \sum_{j=1}^{n}\frac{\mu'_j}{p_j}\left(m'_j + \frac{\mu_j}{2K}\right), \tag{A1.13}$$

because $\tilde{K}_{n+2g-2}(\gamma) = 0$ if $\gamma < 0$. The remaining semi-infinite sum over m_0 is regularized by the factor $e^{-2\pi m_0\xi}$ which is present in Eq. (A1.11). Actually, if $g = 0$, then the alternating sum over μ'_j is similar to the ones which express the weight multiplicities of tensor products through Kostant's partition functions. Therefore the sums

$$\sum_{\{\mu'\}=\pm 1}\left(\prod_{j=1}^{n}\mu'_j\right)\tilde{K}_{n-2}\left(\gamma + Km_0 - \sum_{j=1}^{n}\frac{\mu'_j}{p_j}\left(Km'_j + \frac{\mu_j}{2}\right)\right) \tag{A1.14}$$

are equal to zero if $\gamma + Km_0$ is big enough. As a result, only a finite number of terms contribute to the sum over m_0. If $g \geqq 1$, the number of terms is infinite but the limit at $\xi \to 0^+$ is still finite.

The best way to find an expression for $\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma; m'_j, \mu_j)$ is to use the residue part of Eq. (A1.3). The sum over (A1.13) can be calculated with the help of Eqs. (3.12) and (3.38):

$$\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma; m'_j, \mu_j) = \frac{(-1)^{n+g}\pi}{2^{2g-2}} \left\{ \mathrm{Res}_{\phi=0} \frac{\exp(2\pi i\phi\gamma)}{\sin^{n+2g-2}(\pi\phi)} \cot(\pi K\phi) \right.$$

$$\times \prod_{j=1}^{n} \sin\left[\frac{2\pi\phi}{p_j}\left(Km'_j + \frac{\mu_j}{2}\right)\right] - \left(\frac{i}{2}\right)^{n-1} \sum_{\{\mu'\}=\pm 1} \left(\prod_{j=1}^{n} \mu'_j\right) \left[\sum_{0 \leqq m_0 < a} \frac{\mathrm{sign}\,(a)}{\mathrm{Sym}_{\pm}(m_0)}\right.$$

$$\left.\left.\times \mathrm{Res}_{\phi=0} \frac{e^{2\pi i\gamma\phi} \exp[2\pi i\phi\, \mathrm{sign}\,(a)(m - |a|)]}{\sin^{n+2g-2}(\pi\phi)}\right]\right|_{a=-\frac{\gamma}{K}+\sum_{j=1}^{n}\frac{\mu'_j}{p_j}\left(m'_j + \frac{\mu_j}{2K}\right)} \Bigg\}. \quad \text{(A1.15)}$$

It is clear from this formula that $\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma; m'_j, \mu_j)$ is indeed a local polynomial in γ, the breaks at the points (A1.12) come from the "singular" sum $\sum_{0 \leqq m < a}$.

The calculation of the integral over γ in Eq. (A1.10) is now straightforward (but tedious). The integral is a sum of the contributions of the stationary phase point $\gamma = 0$ associated with irreducible connections and break points (A1.12) associated with reducible connections. To calculate the former one has to take the polynomial which is equal to $\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma; m'_j, \mu_j)$ in the vicinity of $\gamma = 0$ and substitute it in Eq. (A1.10) instead of $\tilde{K}^{(\mathrm{tot})}_{n+2g-2}(\gamma; m'_j, \mu_j)$. To calculate a contribution of a point (A1.12) one may substitute the term of the sum $\sum_{m_0 \in \mathbb{Z}}$ of Eq. (A1.11) which has the break at that point, in the similar way. These calculations lead ultimately to Eqs. (3.43)–(3.46). We do not discuss them here but the examples for the case of a 3-fibered rational homology sphere $X_0\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$ can be found in [5].

As we see, the residue calculations of Sect. 3 are simpler and more straightforward. However the calculations involving the Kostant partition function present a clear group theoretical picture by relating the surgery formula to multiplicities of irreducible representations in tensor products of representations of quantum groups (for more details see [5]). This simplifies the analysis of reducibility of connections providing the contributions to Witten's invariant based on general simple Lie groups.

Appendix 2

In Sect. 4 we used the fact that the moduli space $\mathscr{M}_{\{\theta\}}(\Sigma_g)$ of flat connections on a punctured surface is a bundle over the moduli space $\mathscr{M}(\Sigma_g)$. However the space $\mathscr{M}(\Sigma_g)$ is singular. Its singularity results in the "ugly" sums like $\sum_{0 \leqq m < \left|\sum_{j=1}^{n} \mu_j\theta_j\right|}$ in Eq. (5.4) and in the requirement that the sums $\sum_{j=1}^{n} \pm\theta_j$ should not be integer.

In order to avoid the singularity of $\mathscr{M}(\Sigma_g)$ E. Witten suggested in [8] to consider the twisted $SO(3)$ bundle over Σ_g for which the moduli space of flat connections is nonsingular. Since we are dealing with punctured surfaces, we may even avoid using $SO(3)$ directly although our formulas will be very similar to those of [8].

The base for our bundles is the moduli space $\tilde{\mathcal{M}}(\Sigma_g)$ of flat connections on Σ_g with one puncture, the holonomy around which is equal to $e^{i\pi\sigma_3}$. Note that $\dim \tilde{\mathcal{M}}(\Sigma_g) = \dim \mathcal{M}(\Sigma_g) = 3g-3$. In fact, $\tilde{\mathcal{M}}(\Sigma_g)$ is a 2^g times folded covering of the moduli space of flat connections on the twisted $SO(3)$ bundle over Σ_g. For the set of phases

$$\theta_1 = \frac{1}{2} - \tilde{\theta}_1, \qquad \theta_2 = \tilde{\theta}_2, \ldots, \theta_n = \tilde{\theta}_n; \qquad \tilde{\theta}_j \ll 1 , \tag{A2.1}$$

the moduli space $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ which we will also denote as simply as $\tilde{\mathcal{M}}_n(\Sigma_g)$, is a bundle over $\tilde{\mathcal{M}}(\Sigma_g)$ in much the same way as it was a bundle over $\mathcal{M}(\Sigma_g)$ when θ_1 rather than $\tilde{\theta}_1$ was very small. The reason why we can use notation $\tilde{\mathcal{M}}_n(\Sigma_g)$ for $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ is that in contrast to the case of $\theta_j \ll 1$, the topological class of $\mathcal{M}_{\{\theta\}}(\Sigma_g)$ does not depend on the phases θ_j as long as $\tilde{\theta}_j \ll 1$.

Rewriting the r.h.s. of Eq. (2.12) in terms of $\tilde{\theta}_j$ we find that

$$Z_{\{\theta\}}(\Sigma_g; a) = -\frac{1}{2^{g-1}\pi^{n+2g-2}} \sum_{\beta \geqq 1} (-1)^\beta \frac{e^{-a\beta^2}}{\beta^{n+2g-2}} \prod_{j=1}^{n} \sin(2\pi\beta\tilde{\theta}_j) . \tag{A2.2}$$

The extra factor $(-1)^\beta$ translates into shifting the summation from integer to half-integer m in the Poisson resummation formula:

$$\sum_{\beta\in\mathbb{Z}} (-1)^\beta \delta(\beta - x) = \sum_{m\in\mathbb{Z}} +\tfrac{1}{2} e^{2\pi i m x} . \tag{A2.3}$$

As a result, instead of Eq. (3.38) we should use

$$\sum_{\substack{m>0 \\ m\in\mathbb{Z}+\frac{1}{2}}} e^{2\pi i\beta m} = \frac{i}{2} \frac{1}{\sin(\pi\beta)} . \tag{A2.4}$$

We can also drop the second sum in Eq. (3.37) if $|a| < \frac{1}{2}$, which is indeed the case if $\tilde{\theta}_j \ll 1$. Thus we get

$$Z^{(\mathrm{irr.})}_{\{\theta\}}(\Sigma_g; a) = \frac{1}{2^g\pi^{n+2g-3}} \operatorname{Res}_{\phi=0} \frac{e^{-a\phi^2}}{\phi^{n+2g-2}} \frac{\prod_{j=1}^{n} \sin(2\pi\phi\tilde{\theta}_j)}{\sin(\pi\phi)} , \tag{A2.5}$$

instead of Eq. (5.4).

If we introduce a set of integer numbers $\tilde{\alpha}_j$ related to α_j

$$\alpha_1 = K - \tilde{\alpha}_1, \qquad \alpha_2 = \tilde{\alpha}_2, \ldots, \alpha_n = \tilde{\alpha}_n , \tag{A2.6}$$

then apparently

$$N^g_{\{\alpha\}} = -\left(\frac{K}{2}\right)^{g-1} \sum_{\beta=1}^{K-1} (-1)^\beta \frac{\prod_{j=1}^{n} \sin\left(\frac{\pi}{K}\beta\tilde{\alpha}_j\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}\beta\right)} . \tag{A2.7}$$

As a result, if $\tilde{\alpha}_j \ll K$, then instead of Eq. (5.2)

$$N^g_{\{\alpha\}} = 4\pi \left(\frac{K}{2}\right)^g \operatorname{Res}_{\phi=0} \frac{\prod_{j=1}^{n} \sin(2\pi\phi\tilde{\alpha}_j)}{\sin^{n+2g-2}(2\pi\phi)} \frac{1}{\sin(2\pi K\phi)} . \tag{A2.8}$$

Finally, if we introduce the new numbers

$$m_1' = \frac{p_1}{2} - \tilde{m}_1', \qquad m_2' = \tilde{m}_2', \ldots, m_n' = \tilde{m}_n', \tag{A2.9}$$

and assume that $\tilde{m}_j' \ll p_j$, then Eq. (5.18) can be rewritten as

$$\begin{aligned} Z^{(\mathrm{irr.})}_{\{m'\};l} = &-\frac{(-1)^{nl+r_1} i^{Kr_1p_1}\pi}{\prod_{j=1}^n \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m_j'}{p_j}\right)} \left(\frac{K}{2}\right)^g \frac{\mathrm{sign}\,(P)}{\sqrt{|P|}} e^{i\frac{3}{4}\pi\,\mathrm{sign}(\frac{H}{P})} \\ &\times \exp\left[2\pi i K \sum_{j=1}^n \left(\frac{r_j}{p_j}\tilde{m}_j'^2 - \frac{1}{4}s_j q_j l^2\right)\right] \\ &\times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^n s(q_j,p_j) - 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right] \\ &\times \sum_{\{\mu\}=\pm 1}\left(\prod_{j=1}^n \mu_j\right) \exp\left[2\pi i \sum_{j=1}^n \mu_j \left(\frac{r_j}{p_j}\tilde{m}_j' - \frac{1}{2}s_j l\right)\right] \\ &\times \mathrm{Res}_{\phi=0} \frac{\exp\left(-\frac{i\pi K}{2}\frac{H}{P}\phi^2\right)}{\sin^{n+2g-2}(\pi\phi)} \frac{\prod_{j=1}^n \sin\left(2\pi K\phi \frac{\tilde{m}_j' + \frac{\mu_j}{2K}}{p_j}\right)}{\sin(\pi K\phi)}\,. \end{aligned} \tag{A2.10}$$

Since Eqs. (5.17) and (5.15) still hold:

$$2Z^{(\mathrm{irr.})}_{\{\theta\}}(\Sigma_g;a) = \int_{\tilde{\mathscr{M}}_n(\Sigma_g)} \exp(\omega + 4a\Theta), \qquad \omega = \omega_0 + 2\sum_{j=1}^n \tilde{\theta}_j\omega_j\,, \tag{A2.11}$$

$$N^g_{\{\alpha\}} = \int_{\tilde{\mathscr{M}}_n(\Sigma_g)} \exp\left(K\omega_0 + \sum_{j=1}^n \tilde{\alpha}_j\omega_j\right)\hat{A}(\tilde{\mathscr{M}}_n(\Sigma_g))\,, \tag{A2.12}$$

we conjecture that

$$\begin{aligned} &\int_{\tilde{\mathscr{M}}_n(\Sigma_g)} \exp\left[K\left(\omega_0 + 2\sum_{j=1}^n \tilde{\theta}_j\omega_j\right) + 4a\Theta\right]\hat{A}(\tilde{\mathscr{M}}_n(\Sigma_g)) \\ &= 2\pi\left(\frac{K}{2}\right)^g \mathrm{Res}_{\phi=0}\frac{e^{-a\phi^2}}{\sin^{n+2g-2}(\pi\phi)}\frac{\prod_{j=1}^n \sin(2\pi K\phi\tilde{\theta}_j)}{\sin(\pi K\phi)}\,. \end{aligned} \tag{A2.13}$$

We do not need to generalize this equation further to the analog of Eq. (5.17) because the manifold $\tilde{\mathscr{M}}_n(\Sigma_g)$ is manifestly independent of $\tilde{\theta}_j$.

Combining Eqs. (A2.10) and (A2.13) we obtain the nonsingular version of Eq. (5.19) which holds for $\tilde{m}'_j \ll p_j$:

$$Z^{(\mathrm{irr.})}_{\{m'\};l} = -\frac{(-1)^{nl+r_1} i^{Kr_1 p_1}}{2\prod_{j=1}^{n} \mathrm{Sym}_{\mathbb{Z}_\pm}\left(\frac{m'_j}{p_j}\right)} \frac{\mathrm{sign}\,(P)}{\sqrt{|P|}} e^{i\frac{3}{4}\pi\,\mathrm{sign}\left(\frac{H}{P}\right)} \tag{A2.14}$$

$$\times \exp\left[2\pi i K \sum_{j=1}^{n}\left(\frac{r_j}{p_j}\tilde{m}'^2_j - \frac{1}{4}s_j q_j l^2\right)\right]$$

$$\times \exp\left[\frac{i\pi}{2K}\left(\frac{H}{P} - 12\sum_{j=1}^{n} s(q_j, p_j) - 3\,\mathrm{sign}\left(\frac{H}{P}\right)\right)\right]$$

$$\times \int_{\tilde{\mathscr{M}}_n(\Sigma_g)} \exp\left[K\left(\omega_0 + 2\sum_{j=1}^{n}\frac{\tilde{m}'_j}{p_j}\omega_j + 2\pi i\frac{H}{P}\Theta\right)\right]\hat{A}\,(\tilde{\mathscr{M}}_n(\Sigma_g))$$

$$\times \prod_{j=1}^{n} 2i \sin 2\pi\left(\frac{r_j}{p_j}\tilde{m}'_j - \frac{1}{2}s_j l + \frac{1}{2\pi i}\frac{\omega_j}{p_j}\right). \tag{A2.15}$$

References

1. Witten, E.: Commun. Math. Phys. **121**, 351 (1989)
2. Jeffrey, L.: Commun. Math. Phys. **147**, 563 (1992)
3. Reshetikhin, N., Turaev, V.: Invent. Math. **103**, 547 (1991)
4. Freed, D., Gompf, R.: Commun. Math. Phys. **141**, 79 (1991)
5. Rozansky, L.: A Large *k* Asymptotics of Witten's Invariant of Seifert Manifolds. Commun. Math. Phys. **171**, 279–322 (1995)
6. Axelrod, S., Singer, I.: Chern–Simons Perturbation Theory. Proceedings of XXth Conference on Differential Geometric Methods in Physics, Baruch College, C.U.N.Y., NY, NY
7. Bar-Natan, D.: Perturbative Chern–Simons Theory. Princeton Preprint, August 23, 1990
8. Witten, E.: J. Geom. Phys. **9**, 303 (1992)
9. Jeffrey, L., Kirwan, R.: Localization for Nonabelian Group Actions. Preprint, June 1993, (to be published in Topology)
10. Szenes, A.: The Combinatorics of the Verlinde Formulas. Preprint, alg-geom/9402003
11. Witten, E.: Commun. Math. Phys. **141**, 153 (1991)
12. Szenes, A.: Topology **32**, 587 (1993)
13. Bertram, A., Szenes, A.: Topology **32**, 599 (1993)
14. Blau, M., Thompson, R.: Equivariant Kähler Geometry and Localization in the *G*/*G* Model. Preprint IC/94/108, ENSLAPP-L-469/94, June 1994
15. Rozansky, L.: A Contribution of the Trivial Connection to the Jones Polynomial and Witten's Invariant of 3d Manifolds I. Preprint UMTG-172-93, UTTG-30-93, hep-th/9401061
16. Andersen, J.: The Witten Invariant of Finite Order Mapping Tori I. Preprint Aarhus Universitet, Matematisk Institut, May 1995, Preprint series 1995, No. 13

Communicated by G. Felder