Commun. Math. Phys. 177, 561–582 (1996)

Communications in
Mathematical
Physics

Solitons and Vertex Operators in Twisted Affine Toda Field Theories

Marco A.C. Kneipp, David I. Olive

Department of Physics, University of Wales, Swansea, Swansea SA2 8PP, Wales, UK

Received: 8 April 1994 / Accepted: 12 September 1995

Abstract: Affine Toda field theories in two dimensions constitute families of integrable, relativistically invariant field theories in correspondence with the affine Kac–Moody algebras. The particles which are the quantum excitations of the fields display interesting patterns in their masses and coupling which have recently been shown to extend to the classical soliton solutions arising when the couplings are imaginary. Here these results are extended from the untwisted to the twisted algebras. The new soliton solutions and their masses are found by a folding procedure which can be applied to the affine Kac–Moody algebras themselves to provide new insights into their structures. The relevant foldings are related to inner automorphisms of the associated finite dimensional Lie group which are calculated explicitly and related to what is known as the twisted Coxeter element. The fact that the twisted affine Kac–Moody algebras possess vertex operator constructions emerges naturally and is relevant to the soliton solutions.

1. Introduction

Affine Toda theories in two dimensions are integrable and possess an infinite number of local conversion laws [1, 2] whose charges generate what can be considered as an infinite dimensional Poincaré algebra,

$$[P^{(M)}, P^{(N)}] = 0\,, \tag{1.1a}$$

$$[K, P^{(M)}] = iMP^{(M)}\,. \tag{1.1b}$$

The Lorentz boost, K, measures the Lorentz spin, M, of the "momentum," $P^{(M)}$. The values of the integers M for which the momenta $P^{(M)}$ are non-zero form the set of exponents of the associated affine Kac Moody algebra whose root system appears in the original equations of motion. The affine Toda field theory possesses critical points with W-symmetry and the symmetries (1.1) can be regarded as the relics of this which survive when the critical theory is deformed in the appropriate integrable manner.

A remarkable but well known mathematical fact is that the underlying affine Kac Moody algebra possesses a subalgebra called the "principal Heisenberg subalgebra" which, when augmented by the "principal derivation," is precisely isomorphic to (1.1) at level zero. Recent studies [3, 4, 5] of the classical soliton solutions of the affine Toda field theories for imaginary couplings (thereby extending the familiar theory of sine–Gordon solitons) have revealed a formulation in which the above mentioned subalgebra does indeed act on the solutions as the Poincaré algebra (1.1). It is this coupling of space-time and internal symmetries which essentially explains why so many of the space time properties of the particles and solitons of the theory (masses, coupling and scattering matrices) possess a Lie algebraic nature [3–10]. However this formalism has so far included only the untwisted affine Kac Moody algebras and not the twisted ones.

The untwisted algebras has been traditionally easier to understand and, up to now, found more physical applications. The twisted algebras can, in any case, be understood as subalgebras of the simple simply-laced untwisted ones. In fact there are two natural but distinct ways of achieving this. One way has been well studied in the mathematics literature [11] whereas the other has been found to be useful in the physics literature dealing with affine Toda field theories [12, 13].

Here we shall find it worthwhile to relate these two hitherto distinct procedures. From the mathematical point of view we shall illuminate the "twisted Coxeter element" [14] which plays a rôle in the grading of the twisted algebra, as well as the vertex operator construction which is valid for the twisted algebras despite the fact that the Dynkin diagrams conventionally associated with them are not simply laced.

More physically, the results will clarify both the spectrum and the coupling of the particles of the twisted affine Toda field theories. Moreover the general formalism for the soliton solutions will be found to extend in a straightforward manner. Recent observations of Dorey [15, 16] will be understood better and set in a more general framework. Part of the interest of the twisted theories is that they constitute a different sort of integrable deformation of the conformal Toda theories as compared to the untwisted theories, in the sense of Zamolodchikov [18–20].

Section 2 reviews the usual classification of the twisted affine Kac–Moody algebras and the Dynkin diagrams associated with them as explained by Kac in his book. In the second construction these are obtained by "folding" an untwisted simply laced extended Dynkin diagram $\Delta(\mathbf{g}^{(1)})$ making use of a special symmetry of $\Delta(\mathbf{g}^{(1)})$ which has the property that it can be lifted to an inner automorphism of the finite dimensional Lie algebra $\mathbf{g}$. Such symmetry forms the finite group [12, 4]

$$W_0(\mathbf{g}) \cong W(\mathbf{g}) \cap \operatorname{Aut}\Delta\left(\mathbf{g}^{(1)}\right) \cong Z(G)\,, \tag{1.2}$$

where $W(\mathbf{g})$ is the Weyl group of $\mathbf{g}$, and $Z(G)$ the centre of the simply connected Lie Group, G, whose Lie algebra is $\mathbf{g}$. All twisted Kac–Moody algebras, except $\mathbf{A}^{(2)}_{2n}$, can be so obtained in an essentially unique way. $\mathbf{A}^{(2)}_{2n}$ is exceptional in having a Dynkin diagram with three different root lengths and needs special treatment in both approaches and so will be ignored, as we seek general arguments.

These folding procedures are also applied to the corresponding affine Toda field theory equations and results concerning masses and solutions are immediately deduced.

Section 3 presents the main ideas in rather general form. The lift of any element $\tau \in W_0$, (1.2), to an inner automorphism of $\mathbf{g}$ is considered. Its fixed point set, $\mathbf{g}_\tau$, is defined, and shown to be acted upon a natural way under conjugation by S, the lift of the Coxeter element of $\mathbf{g}$. Thus S acts as an outer automorphism of $\mathbf{g}_\tau$ which will be related to the twisted Coxeter element of the semisimple part of $\mathbf{g}_\tau$ when this is simple. It is further shown that if τ has order k, then S^k lies in the Lie group G_τ itself and this observation provides a crucial link with the alternative construction described by Kac.

Section 4 describes a concrete expression for the group element, S, conjugation by which yields the inner automorphism of G corresponding to τ in the cases $\mathbf{g}$ is simply laced and τ is direct. From this is deduced the precise structure of G_τ. As this has to have the same rank as G it is unclear how it is related to the fold of $\Delta(\mathbf{g}^{(1)})$. It turns out that G_τ is never semisimple as it possesses $(k-1)$ invariant $U(1)$ factors. When the remaining semisimple factor is actually simple, we recognize a relationship to the twisted Dynkin diagrams appearing in the classification described by Kac [11]. The remaining possibilities are interesting but were not considered in [11].

The construction of Sect. 4 has several geometrical features which are described in Sect. 5 and used to view the root system of the simple part of $\mathbf{g}_\tau$ as being acted upon by a twisted Coxeter element following from the action of S mentioned above.

We then see how a basis for the twisted affine Kac Moody algebra can be formulated in terms of a twisted principal Heisenberg subalgebra and the associated quantities ad-diagonalising them. Viewed this way, the basis for the twisted algebra is simply a subset of the corresponding basis for $\mathbf{g}^{(1)}$, the untwisted simply-laced affine Kac–Moody algebra. As a consequence it is shown to inherit its vertex operator construction.

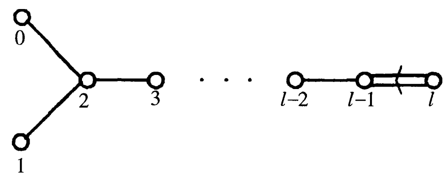
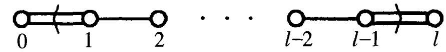
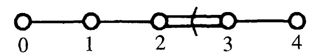
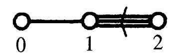
In Sect. 6 we discuss the application of these results to understanding the properties of the particles and solitons in the twisted affine Toda theories. In particular we find that there is a sense in which the energy momentum tensor is unchanged by folding. As a consequence, the twisted soliton mass spectrum is a subset of the spectrum of the unfolded $\mathbf{g}^{(1)}$ theory, in the same manner as for the spectrum of masses of the quantum field excitations.

For completeness, we also discuss how our general results apply to the other kind of folding, namely that which yields the untwisted non-simply laced affine Kac–Moody algebras. Here the results for the two kinds of mass spectrum differ and our method gives a simple explanation of this, thereby confirming previous results.

2. Folding and Twisted Theories

The conventional notation for the twisted affine Kac–Moody algebra is the designation $\mathbf{X}_n^{(k)}, k \geqq 2$. As explained in Kac's book [11], $\mathbf{X}_n$ denotes a simple, simply laced Lie algebra of rank n endowed with a diagram automorphism (of the Dynkin diagram $\Delta(\mathbf{X}_n)$) of order k. It turns out that the construction can be summarised by the statement that the Dynkin diagram $\Delta(\mathbf{X}_n^{(k)})$ has as Cartan matrix $K((\mathbf{Y}^\vee)^{(1)})^T$, where $\mathbf{Y}$ is the subalgebra of $\mathbf{X}_n$ fixed by the automorphism of $\mathbf{X}_n$ which is the lift of the diagram automorphism; $\mathbf{Y}^\vee$ is its dual, that is the algebra with the roots and co-roots interchanged. Thus $K((\mathbf{Y}^\vee)^{(1)})^T$ is the transpose of the extended Cartan matrix of $\mathbf{Y}^\vee$. Following Kac's book, Table 1 lists the possibilities with the exception of

Table 1. Kac's notation for twisted affine algebras

$X_n^{(k)}$	$\Delta(X_n^{(k)})$
$A_{2l-1}^{(2)}$ $(l \geqq 3)$	
$D_{l+1}^{(2)}$ $(l \geqq 2)$	
$E_6^{(2)}$	
$D_4^{(3)}$	

$\mathbf{A}_{2n}^{(2)}$ which has to be treated as a special case in both the conventional approach and in ours which follows.

In Table 1 the vertices of $\Delta(\mathbf{X}_n^{(k)})$ are numbered for future reference. Note that the integer l is chosen so that the folded Dynkin diagrams $\Delta(\mathbf{D}_{l+1}^{(2)})$ and $\Delta(\mathbf{A}_{2l-1}^{(2)})$ each possess precisely $(l+1)$ vertices.

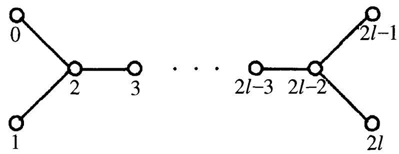
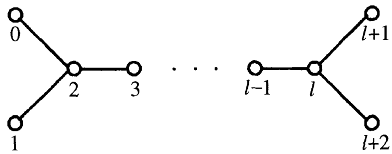
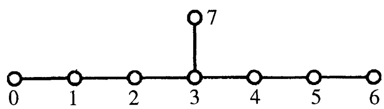
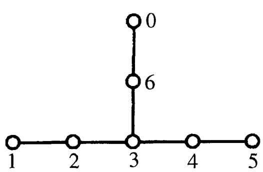
In considering affine Toda field theories, it has been found helpful to view the diagrams in Table 1 as arising in a different way, namely by "folding" [12] the extended Dynkin diagrams of simply laced simple Lie algebras. Thus, by virtue of a symmetry of the extended Dynkin diagram $\Delta(\mathbf{D}_{2l}^{(1)})$ it can be folded to $\Delta(\mathbf{A}_{2l-1}^{(2)})$ by identifying points related by the symmetry. It is sufficient to denote these symmetries as a permutation of the tip points of the extended Dynkin diagram, namely those symmetrically related to the vertex (0). Each diagram in Table 1 can be found as indicated in Table 2. The symmetry to be used for the folding is specified in the last column, using the numbering of vertices of $\Delta(\mathbf{g}^{(1)})$ and the conventional notation for permutations.

Notice that the automorphism of $\Delta(\mathbf{g}^{(1)})$ needed to obtain the diagrams of Table 2 possess two important properties: they are direct (i.e. never relate two linked vertices) and they are elements of $W_0(\mathbf{g})$, (1.2). This means that unlike the element of $\mathrm{Aut}\,\Delta(\mathbf{X_n})$ used in the construction of Table 1, as described by Kac, these automorphisms of $\Delta(\mathbf{g}^{(1)})$ can be lifted to inner automorphisms of the Lie algebra $\mathbf{g}$ (as well as the corresponding Lie group G). This inner automorphism will be important in what follows and will be constructed explicitly. It has been shown that [12, 4]

$$W_0(\mathbf{g}) \cong Z(G)\,, \tag{2.1}$$

where $Z(G)$ is the centre of the simply connected Lie group G whose Lie algebra is $\mathbf{g}$. The result (2.1) makes it easy to scan all the possibilities of elements of $W_0(\mathbf{g})$ when $\mathbf{g}$ is simple and simply laced. We find that two possibilities remain

Table 2. Twisted affine algebras obtained by folding $\Delta(g^{(1)})$ by τ

$X_n^{(k)}$	$\Delta(g^{(1)})$	$g^{(1)}$	$\tau \in W_0(g)$
$A_{2l-1}^{(2)}$ $(l \geqq 3)$		$D_{2l}^{(1)}$	$(0, 2l-1)(1, 2l)$
$D_{l+1}^{(2)}$ $(l \geqq 2)$		$D_{l+2}^{(1)}$	$(0,1)(l+1, l+2)$
$E_6^{(2)}$		$E_7^{(1)}$	$(0, 6)$
$D_4^{(3)}$		$E_6^{(1)}$	$(0, 1, 5)$

beyond those in Table 2. $W_0(\mathbf{D}_{2n+1})$ possesses an element of order 4 not listed above. It is, however, not "direct" and so excluded from most of our argument. The remaining case turns out to be of considerable interest,

$$W_0(\mathbf{SU(N)}) \cong W_0(\mathbf{A}_{N-1}) \cong Z_N \,. \tag{2.2}$$

If $N = mn$, we have $\tau \in W_0$ defined by

$$\tau(i) = i + n (\operatorname{mod} mn)\,, \tag{2.3}$$

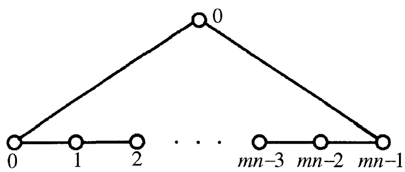
using the numbering of vertices in Table 3. As $\tau^m(i) = i + mn = i$, this has order m. In this case, "folding" gives $\Delta(\mathbf{A_{n-1}}^{(1)})$.

We shall now explain why the folding procedure of Table 2 is useful in affine Toda field theory and deduce the form of the general soliton solution for the twisted theories, generalising those of the untwisted theories.

Following the treatment of [4], let $\langle i \rangle$ denote the set of vertices of $\Delta(\mathbf{g}^{(1)})$ related to the vertex i by the action of $\tau \in \operatorname{Aut}\Delta(\mathbf{g}^{(1)})$, that is, its orbit. Then, if τ is direct, so that no two points of $\langle i \rangle$ are ever linked directly,

$$K_{\langle i \rangle \langle j \rangle} = \sum_{j \in \langle j \rangle} K_{ij} \tag{2.4}$$

Table 3.

$g^{(1)}$	$\Delta(g^{(1)})$	$\tau \in W_0(g)$
$A^{(1)}_{mn-1}$		$(0, n, 2n, \ldots, (m-1)n)$

defines a new, folded, Cartan matrix. This is the precise way in which the folded diagrams of Table 1 were found by folding the diagrams in Table 2, using the specific element τ listed in the last column. The set of coprime integers m_i, defined as $\sum_{j=0} K_{ij} m_j = 0$, obviously fold according to

$$m_{\langle i \rangle} = m_i \tag{2.5}$$

as $\sum_{\langle j \rangle} K_{\langle i \rangle \langle j \rangle} m_{\langle j \rangle} = 0$, and the $m_{\langle i \rangle}$ remain coprime integers.

If we define the variables

$$\phi_i = \alpha_i \cdot \phi, \quad i = 0, 1, \ldots, r\,, \tag{2.6}$$

where $(\alpha_0, \alpha_1, \ldots, \alpha_r)$ are an extended set of simple roots, and

$$K_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}\,, \tag{2.7}$$

we have the relations

$$\sum_{i=0}^{r} \frac{2m_i}{\alpha_i^2} \phi_i = 0, \qquad \sum_{i=0}^{r} \frac{2m_i}{\alpha_i^2} K_{ij} = 0\,. \tag{2.8}$$

The advantage of introducing a redundant variable is that the affine Toda field equation of motion associated with the Cartan matrix K_{ij} can be written

$$\partial^2 \phi_i + \frac{\mu^2}{\beta} \sum_{j=0}^{r} K_{ij} m_j e^{\beta \phi_j} = 0, \quad i = 0, 1, \ldots, r\,, \tag{2.9}$$

subject to (2.8), and that this version better displays the symmetry of the Dynkin diagram whose Cartan matrix is K. This can be done whether the algebra is twisted or not. Equation (2.9) can be linearised to

$$\partial^2 \phi_i + \mu^2 \sum_{j=0}^{r} K_{ij} m_j \phi_j = 0\,. \tag{2.10}$$

Thus the squared masses of the quantum particles excitations of the fields equal μ^2 times the eigenvalues of the matrix $C'_{ij} = K_{ij} m_j$ which is similar to $C_{ij} = m_i K_{ij}$, so sharing the same eigenvalues (cf. [4]). Because of the condition (2.8), the eigenvalue 0 is excluded leaving precisely r values.

The basic result concerning the folding of Eq. (2.9) via (2.4) is that if $\phi_{\langle i \rangle}$ is a solution of the folded equation, with $C'_{\langle i \rangle \langle j \rangle}$ replaced by C'_{ij} in (2.9), then

$$\phi_i = \phi_{\langle i \rangle} \tag{2.11}$$

is a solution to the unfolded equation that displays the symmetry $\phi_i = \phi_{\tau(i)}$. Conversely, all such symmetric solutions furnish solutions via (2.11) to the folded equations.

In particular, the mass spectrum of the quantum field excitation particles of the folded theory automatically forms a subset of the unfolded mass spectrum. These results hold for any diagram automorphism $\tau \in \mathrm{Aut}\,\Delta(\mathbf{g}^{(1)})$ as long as it is direct. The group $\mathrm{Aut}\Delta(\mathbf{g}^{(1)})$ is actually a semidirect product of two subgroups $\mathrm{Aut}\Delta(\mathbf{g})$ and $W_0(\mathbf{g})$ ([12, 4]). Both these subgroups are relevant. Folding with $\tau \in \mathrm{Aut}\Delta(\mathbf{g})$ yields a non-simply laced untwisted affine Kac Moody algebra. Such τ can be lifted to an outer automorphism of the Lie algebra $\mathbf{g}$, preserving the principal $\mathbf{SO(3)}$ subalgebra. All this was discussed as in detail in [4] and will not be pursued again here. Instead, as explained above, we consider the effect of elements of $W_0(\mathbf{g})$. These can be lifted to inner automorphisms of $\mathbf{g}$ which do not preserve the principal $\mathbf{SO(3)}$ subalgebra and fold to give twisted algebras.

In untwisted affine Toda theories, the quantum field particle excitations are in one to one correspondence with points of the ordinary Dynkin diagram $\Delta(\mathbf{g})$. Mass degeneracies reflect the symmetries of the diagram and are broken by foldings by $\tau \in \mathrm{Aut}\Delta(\mathbf{g})$. We have just seen that for $\tau \in W_0$ a subset of particles survive folding and one of our results will be to determine precisely which in terms of a simple general formula. The surviving particles correspond to the subset of vertices first found in the explicit calculations of Braden et al. [13], and rederived in Sect. 6 to follow.

The general soliton solution to (2.9) for imaginary coupling β can be written

$$e^{-\beta\phi_i} = \prod_{j=0}^{n} (M_j)^{K_{ij}}\ , \tag{2.12}$$

where

$$M_j = \langle \Lambda_j | g(t) | \Lambda_j \rangle \tag{2.13}$$

is a tau function, namely an expectation value with respect to the j^{th} fundamental highest weight state of the algebra. $g(t)$ is a group element constructed in a specific way from the soliton data. For the untwisted theories (2.12) is derived by a simple rearrangement of the general expression of [3]. We shall now show that, unlike the formula of [3], (2.12) has the virtue of applying to the twisted affine Toda theories as well.

First note that, if M_j symmetric in the sense $M_j = M_{\tau(j)}$, we can define $M_{\langle j \rangle}$ to be equal to it and use (2.4) and (2.11), to indeed find

$$e^{-\beta\phi_{\langle i \rangle}} = \prod_{\langle j \rangle = \langle 0 \rangle} (M_{\langle j \rangle})^{K_{\langle i \rangle \langle j \rangle}}\ . \tag{2.14}$$

It will be verified in Sect. 6, as a result of our construction, that indeed $M_j = M_{\tau(j)}$ in the twisted Kac–Moody algebra.

3. Action of $W_0(\mathbf{g})$ on the Principal Coxeter Element S

The first, and main, part of our argument is quite simple and general. It applies to any element of $\tau \in W_0(\mathbf{g})$, (1.2), whether or not $\mathbf{g}$ is simply laced, and whether or not τ is direct.

Recall that any element τ of $W_0(\mathbf{g})$ is uniquely characterised by its action, $\tau(0)$, on the vertex 0 of the extended Dynkin diagram, $\Delta(\mathbf{g}^{(1)})$, whose deletion yields the ordinary Dynkin diagram $\Delta(\mathbf{g})$ [4]. Moreover if $\tau \neq 1$ it moves each tip point of $\Delta(\mathbf{g}^{(1)})$. Suppose $\tau \in W_0(\mathbf{g})$ has order k:

$$\tau^k = 1 . \tag{3.1}$$

Because $W_0(\mathbf{g}) \subset W(\mathbf{g})$, the Weyl group of $\mathbf{g}$, τ can be lifted to an inner automorphism $\tilde{\tau}$ of $\mathbf{g}$:

$$\tilde{\tau}(p) = TpT^{-1}, \quad p \in \mathbf{g},\ T \in H' \subset G , \tag{3.2}$$

where, as stated T lies in H' the maximal torus of G whose Lie algebra is the Cartan subalgebra in apposition, $\mathbf{h}'$. This is because

$$\tilde{\tau}(E_1) = E_1, \quad E_1 = \sum_{i=0}^{r} \sqrt{m_i} E_{\alpha_i} \tag{3.3}$$

and H' is the centraliser of E_1. From (3.1) it follows that

$$T^k \in Z(G) , \tag{3.4}$$

where $Z(G)$ is the centre of the simply connected Lie group G whose Lie algebra is $\mathbf{g}$.

The definition (3.2) of $\tilde{\tau}$ leaves some ambiguity in T, to be discussed later, but, irrespective of this, it was shown that [4]

$$TST^{-1}S^{-1} = e^{-2\pi i \lambda^v_{\tau(0)} \cdot H} \equiv z(\tau) , \tag{3.5}$$

where S is the principal element of G

$$S = \exp \frac{2\pi i T^3}{h(\mathbf{g})} , \tag{3.6}$$

and $\lambda_i^v = 2\lambda_i/\alpha_i^2$ is a fundamental coweight. The angular momentum T^3 lies in the intersection of the principal $\mathbf{SO(3)}$ subalgebra of $\mathbf{g}$ with the original Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$. $h(\mathbf{g})$ denotes the Coxeter number of $\mathbf{g}$. As

$$S^{h(\mathbf{g})} \in Z(G) , \tag{3.7}$$

conjugation by S grades $\mathbf{g}$ into $h(\mathbf{g})$ eigenspaces

$$\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_1 \oplus \cdots \oplus \mathbf{g}_{h(\mathbf{g})-1}, \quad \text{where } S\mathbf{g}_m S^{-1} = e^{\frac{2\pi i m}{h(\mathbf{g})}} \mathbf{g}_m , \tag{3.8}$$

and therefore

$$\mathbf{g}_0 = \mathbf{h}; \qquad E_1 \in \mathbf{g}_1 . \tag{3.9}$$

The known form of the zero curvature potentials of the affine Toda field theories indicates that these eigenspaces are of crucial importance in understanding the integrability of these theories.

The group element $z(\tau)$ in (3.5) lies in $Z(G)$. Thus (3.5) is a key equation, stating that T and S almost commute, despite the fact that they lie in maximum tori in apposition. It follows from (3.5) that, for any pair of integers m and n,

$$T^m S^n = z(\tau)^{mn} S^n T^m . \tag{3.10}$$

Our first deduction is that $z(\tau)$ has precisely the same order, k, (3.1), as τ and that k necessarily divides the Coxeter number $h(\mathbf{g})$.

By (3.4) and (3.10), $z(\tau)^k = 1$, but if $z(\tau)$ had a lower order, k', then $T^{k'}$ would commute with S, and so, by (3.2) and (3.8), would lie in the intersection of the two maximal tori in apposition with each other, H and H'. As this intersection is simply $Z(G)$, this would imply τ had order k', contrary to the hypothesis. It now follows that k provides the smallest power of S that commutes with T. By (3.7) it then follows that k divides $h(\mathbf{g})$.

Now let us define G_τ, the fixed point subgroup of G with respect to $\tilde{\tau}$,

$$G_\tau = \{a \in G; TaT^{-1} = a\} . \tag{3.11}$$

Since $z(\tau)^k = 1$, from (3.10) we see that S^k commutes with T and hence lies in G_τ while S itself does not when $k \geqq 2$. Nevertheless

$$SaS^{-1} \in G_\tau \quad \text{for all } a \in G_\tau \tag{3.12}$$

as T commutes with SaS^{-1} by virtue of (3.5) and (3.11). Thus conjugation by S produces an outer automorphism of G_τ, with the property that its k^{th} power is inner.

The remaining argument is to relate this action of S to that of the so-called twisted Coxeter element. The problem to be addressed is that G_τ necessarily has the same rank as G as it contains H' by virtue of by (3.11) and the fact that T lies in H'. Thus G_τ would therefore not relate simply to the folded Dynkin diagram which has fewer points than this as also does the rank of $\mathbf{X}_n$ in Tables 1 and 2.

The resolution of the apparent paradox will be that G_τ defined by (3.11) is not semisimple. Rather it is composed of $(k-1)$ invariant $U(1)$ factors times a semisimple factor which relates straightforwardly to the folding when it is simple.

To understand this we need to establish the structure of T, (3.2) in more detail. As $T \in H'$,

$$T = e^{-2\pi i Y \cdot h} , \tag{3.13}$$

where Y is a vector to be determined and $(h_1, h_2, \ldots, h_r)$ denote an orthonormal basis of the Cartan subalgebra in apposition $\mathbf{h}'$ namely that containing E_1, (3.3) and generates H'. This basis is chosen to be conjugate to the basis of the original Cartan subalgebra $\mathbf{h}$, $(H_1, H_2, \ldots, H_r)$,

$$h_i = PH_iP^{-1}, \quad i = 1, 2, \ldots, r, \;\; P \in G . \tag{3.14}$$

Using this we can write $z(\tau)$ in (3.5) also as

$$z(\tau) = e^{-2\pi i \lambda^v_{\tau(0)} \cdot h} , \tag{3.15}$$

remembering that $z(\tau)$ lies in the centre of G and therefore commutes with P. Since

$$SY \cdot hS^{-1} = \sigma(Y) \cdot h , \tag{3.16}$$

where σ is the Coxeter element of $W(\mathbf{g})$ in the form $\sigma_-\sigma_+$, [6], we can evaluate $z(\tau)$, (3.15), in another way, using (3.5) and (3.13), and find

$$z(\tau) = e^{-2\pi i(1-\sigma)Y\cdot h}\,. \tag{3.17}$$

Comparing these two expressions, (3.15) and (3.17), we have

$$(1-\sigma)Y = \lambda^v_{\tau(0)} + \Lambda_R(\mathbf{g}^v)\,, \tag{3.18}$$

where the element of the coroot lattice $\Lambda_R(\mathbf{g}^v)$ is undetermined.

As $(1-\sigma)$ never vanishes, it has a unique inverse. Further it maps the coweight lattice of $\mathbf{g}$ into its coroot lattice [7] by virtue of the identity:

$$\gamma_i = (\sigma - 1)\sigma^{-(1+c(i))/2}\lambda_i\,. \tag{3.19}$$

Therefore,

$$T = e^{-2\pi i(1-\sigma)^{-1}\lambda^v_{\tau(0)}\cdot h} \tag{3.20}$$

up to an element of $Z(G)$ dependent on the undetermined element of the coweight lattice of $\mathbf{g}$. This undetermined factor is innocuous as it has no effect on the adjoint action of T, (3.2).

Nevertheless we shall take advantage of this ambiguity in the following way: replace $\lambda^v_{\tau(0)}$ in (3.18) by $w(\lambda^v_{\tau(0)})$, where $w \in W(\mathbf{g})$ is a Weyl group element. Then, instead of (3.20),

$$T = e^{-2\pi i(1-\sigma)^{-1}w\lambda^v_{\tau(0)}\cdot h}\,, \tag{3.21}$$

where, by our previous comment, the dependence on w is carried by an innocuous central factor. The point is that we shall find a choice of w, dependent on τ, which simplifies (3.21) considerably.

4. Concrete Expression for T and the Structure of G_τ

We now add the assumptions that $\mathbf{g}$ is simply laced and that $\tau \in W_0(\mathbf{g})$ is direct. The latter condition excludes the element of order 4 in $W_0(\mathbf{D}_{2n+1}) \cong Z_4$. The former condition means that all roots can be taken to have length $\sqrt{2}$. Then the fundamental weight λ_i and the fundamental coweight $2\lambda_i/\alpha_i^2 = \lambda_i^v$ are identical. With these assumptions we shall verify that there exists a choice of $w \in W(\mathbf{g})$ such that T is given by (3.13) with

$$Y = (1-\sigma)^{-1}w\lambda_{\tau(0)} \equiv wY' = \frac{1}{k}w(\lambda_{\tau(0)} + \lambda_{\tau^2(0)} + \cdots + \lambda_{\tau^{(k-1)}(0)})\,. \tag{4.1}$$

With this we can determine the structure of G_τ, (3.11), and relate it to the Dynkin diagram obtained by folding $\Delta(\mathbf{g}^{(1)})$ with τ.

Let us define

$$\sigma' = w^{-1}\sigma w \tag{4.2}$$

which, being conjugate to σ, is equally a Coxeter element of $W(\mathbf{g})$, but not the one whose action is induced by S. Then (4.1) is equivalent to

$$Y' = (1-\sigma')^{-1}\lambda_{\tau(0)} = \frac{1}{k}(\lambda_{\tau(0)} + \lambda_{\tau^2(0)} + \cdots + \lambda_{\tau^{(k-1)}(0)})\,. \tag{4.3}$$

This will be proven by constructing σ' such that

$$\lambda_{\tau^p(0)} = (1 + \sigma' + \sigma'^2 + \cdots + \sigma'^{p-1})\lambda_{\tau(0)}, \quad p = 1, \ldots, k . \tag{4.4}$$

Formally $\lambda_{\tau^k(0)} = \lambda_0 = 0$ and this is guaranteed by (4.4) if

$$(1 - \sigma'^k)\lambda_{\tau(0)} = 0 . \tag{4.5}$$

Unfortunately the verification of (4.4) and (4.5) has to be done on a case by case basis and is therefore relegated to Appendix A. The case of $\mathbf{g} = \mathbf{SU(mn)}$ and τ given by (2.3) is particularly instructive and easy to verify. It follows from (4.4) that

$$\sigma' \lambda_{\tau^p(0)} = \lambda_{\tau^{p+1}(0)} - \lambda_{\tau(0)} . \tag{4.6}$$

Summing from values of p running from 1 to $(k-1)$ yields

$$\sigma' Y' = Y' - \lambda_{\tau(0)} ,$$

which confirms (4.3). Moreover, if we define

$$q'\left(\frac{ph(\mathbf{g})}{k}\right) = \frac{1}{k} \sum_{m=1}^{k-1} e^{\frac{-2\pi i p m}{k}} \lambda_{\tau^m(0)} , \tag{4.7}$$

we likewise find

$$\sigma' q'\left(\frac{ph(\mathbf{g})}{k}\right) = e^{\frac{2\pi i p}{k}} q'\left(\frac{ph(\mathbf{g})}{k}\right) . \tag{4.8}$$

As (4.7) manifestly does not vanish, being composed of k linearly independent quantities, and as $\frac{p}{k} = (\frac{ph(\mathbf{g})}{k})\frac{1}{h(\mathbf{g})}$ we deduce that the $(k-1)$ integers $\frac{h}{k}, \frac{2h}{k}, \ldots, \frac{(k-1)h}{k}$ are all exponents of $\mathbf{g}$. Combined with the statement that all the integers between 1 and $(h(\mathbf{g}) - 1)$ coprime to $h(\mathbf{g})$ are also exponents, we have a economical means of calculating the exponents of $\mathbf{E}_6, \mathbf{E}_7$ and $\mathbf{E}_8$. Note that

$$Y' + \sum_{p=1}^{k-1} q'\left(\frac{ph(\mathbf{g})}{k}\right) = 0 . \tag{4.9}$$

The structure of G_τ, (3.11), can be determined by considering the isomorphic group conjugate to G_τ within G

$$G'_\tau = W^{-1} G_\tau W$$

whose elements commute with

$$T' = W^{-1} T W = e^{-2\pi i Y' \cdot h} ,$$

where Y' was given in (4.3). The generators of $\mathbf{g}'_\tau$ comprise the complete Cartan subalgebra in apposition of $\mathbf{g}$, and step operators $F^{\alpha'}$ for roots α' with respect to this Cartan subalgebra:

$$F^{\alpha'} = W^{-1} F^\alpha W \sim F^{w^{-1}(\alpha)}$$

which satisfy

$$e^{-2\pi i Y' \cdot w^{-1}\alpha} = 1 . \tag{4.10}$$

But by (4.3) and the fact that the weights $\lambda_{\tau^p(0)}$ are all minimal so that for any root β, $\lambda_{\tau^p(0)}\cdot\beta = 0$ or ± 1, we have

$$|Y'\cdot\beta| \leqq 1 - \frac{1}{k} < 1\,. \tag{4.11}$$

Hence the only solutions to condition (4.10) satisfy

$$Y'\cdot w^{-1}\alpha = Y\cdot\alpha = 0\,. \tag{4.12}$$

If $w^{-1}\alpha$ is positive this implies that it is orthogonal to each of the minimal fundamental weights $\lambda_{\tau(0)}, \lambda_{\tau^2(0)}, \ldots, \lambda_{\tau^{k-1}(0)}$. This, in turn, means that

$$G_\tau = G_\tau^0 \otimes U(1)^{k-1}\,, \tag{4.13}$$

where G_τ^0 is semisimple. As G_τ has the same rank as G

$$\mathrm{rank}\,(G_\tau^0) = \mathrm{rank}\,(G) + 1 - k\,. \tag{4.14}$$

Furthermore, the semisimple factor G_τ^0 has a Dynkin diagram, $\Delta(\mathbf{g}_\tau^0)$, obtained by deleting from $\Delta(\mathbf{g}^{(1)})$ the k vertices $0, \tau(0), \ldots, \tau^{(k-1)}(0)$. Thus, referring to Table 2, we see that deleting vertices $0, 2l-1$ from $\Delta(\mathbf{D}_{2l}^{(1)})$ yields $\Delta(\mathbf{A}_{2l-1})$, deleting vertices $0, 1$ from $\Delta(\mathbf{D}_{l+2}^{(1)})$ yields $\Delta(\mathbf{D}_{l+1})$, deleting vertices $0, 6$ from $\Delta(\mathbf{E}_7^{(1)})$ yields $\Delta(\mathbf{E}_6)$, while deleting vertices $0, 1, 5$ from $\Delta(\mathbf{E}_6^{(1)})$ yields $\Delta(\mathbf{D}_4)$. Thus, in these cases, the Lie algebra $\mathbf{g}_\tau^0$ is indeed isomorphic to $\mathbf{X}_n$ defined by the first column in Table 2. In particular, we see that $\mathbf{g}_\tau^0$ is actually simple in all cases except the final one cited in Table 3 when deleting vertices $0, n, 2n, \ldots, (m-1)n$ from $\Delta(\mathbf{A}_{mn-1}^{(1)})$ yields m copies of $\Delta(\mathbf{A}_{n-1})$.

Notice also that it follows from (4.12) that if F^α is a generator of $\mathbf{g}_\tau^0$,

$$w^{-1}\alpha\cdot q'\left(\frac{ph}{k}\right) = \alpha\cdot q\left(\frac{ph}{k}\right) = 0, \quad p = 1, 2, \ldots, k-1\,. \tag{4.15}$$

Thus the roots of $\mathbf{g}_\tau^0$ are all perpendicular to the $(k-1)$ dimensional subspace spanned by the $(k-1)$ eigenvectors of σ, (the Coxeter element induced by S) corresponding to the $(k-1)$ exponents $h(\mathbf{g})/k, 2h(\mathbf{g})/k \ldots$. This was first observed by Dorey [15] for the particular case of $\mathbf{g} = \mathbf{E}_6$.

We shall now consider the action of S on G_τ^0.

5. The Twisted Coxeter Element from the Action of S on G_τ^0

We shall now assemble the preceding results. In Sect. 3 we saw that the conjugation by S acted on G_τ as an outer automorphism. In Sect. 4 we saw how G_τ factored into a semisimple group G_τ^0 times an Abelian group of dimension $(k-1)$ with generators

$$q\left(\frac{h(\mathbf{g})}{k}\right)\cdot h, q\left(\frac{2h(\mathbf{g})}{k}\right)\cdot h, \ldots, q\left(\frac{(k-1)h(\mathbf{g})}{k}\right)\cdot h\,. \tag{5.1}$$

Because $q(\frac{ph(\mathbf{g})}{k})$ are eigenvectors of σ, S acts on the Abelian factor diagonally:

$$Sq\left(\frac{ph(\mathbf{g})}{k}\right)\cdot hS^{-1} = \sigma q\left(\frac{ph(\mathbf{g})}{k}\right)\cdot h = e^{\frac{2\pi ip}{k}}\, q\left(\frac{ph(\mathbf{g})}{k}\right)\cdot h\,.$$

As these eigenvectors are orthogonal to all the roots of $\mathbf{g}_\tau^\mathbf{0}$, (4.15), S acts directly on $\mathbf{g}_\tau^\mathbf{0}$. So

$$SF^\alpha S^{-1} = F^{\sigma(\alpha)} . \tag{5.2}$$

In fact, the roots of $\mathbf{g}_\tau^\mathbf{0}$ fall into complete orbits of $h(\mathbf{g})$ elements under σ. This is because if one element of an orbit is orthogonal to the eigenvectors of σ mentioned above then so are all the other roots in the orbit. If there are l such orbits then $\mathbf{g}_\tau^\mathbf{0}$ has $lh(\mathbf{g})$ roots. Thus

$$lh(\mathbf{g}) = r(\mathbf{g}_\tau^\mathbf{0})h(\mathbf{g}_\tau^\mathbf{0}) = (r(\mathbf{g}) + 1 - k)h(\mathbf{g}_\tau^\mathbf{0}) \tag{5.3}$$

equals the number of roots of $\mathbf{g}_\tau^\mathbf{0}$ counted two different ways. This makes it clear that the action of σ on the roots of $\mathbf{g}_\tau^\mathbf{0}$ differs from the action of its own Coxeter element. This different action is that of the "twisted Coxeter" element. By the result of Sect. 3, σ^k is inner as far as G_τ^0 is concerned even though σ is outer. Thus σ, being an automorphism of the $\mathbf{g}_\tau^\mathbf{0}$ root system, is composed of a $\Delta(\mathbf{g}_\tau^\mathbf{0})$ diagram automorphism of order k times an element of $W(\mathbf{g}_\tau^\mathbf{0})$. Furthermore, as it has the same order as the Coxeter element of $W(\mathbf{g})$, namely $h(\mathbf{g})$, it follows by the discussion in Sect. 7 of the paper by Springer, [14], that σ is $W(\mathbf{g}_\tau^\mathbf{0})$ conjugate to his definition of a twisted Coxeter element. Thus we can think of our construction as providing an alternative construction of the twisted Coxeter element to that presented by Springer.

It can also be checked from (5.3) that, when $\mathbf{g}_\tau^\mathbf{0}$ is simple, the Dynkin diagram $\Delta(\mathbf{X}_n^{(k)}) = \Delta((\mathbf{g}_\tau^\mathbf{0})_n^{(k)})$ has precisely $(l+1)$ vertices so that our two usages of the symbol l indeed agree.

By extension of the usual definition of an exponent, one can define the twisted exponents of $\mathbf{g}_\tau^\mathbf{0}$ as being those powers of $\exp 2\pi i/h(\mathbf{g})$ which occur as eigenvalues of σ applied to the Cartan subalgebra in apposition of $\mathbf{g}_\tau^\mathbf{0}$. By our construction, the twisted exponents are given by the set of exponents of $\mathbf{g}$ less the $(k-1)$ exponents $h(\mathbf{g})/k, 2h(\mathbf{g})/k, \dots, (k-1)h(\mathbf{g})/k$ associated with the abelian subalgebra of $\mathbf{g}_\tau$, (5.1). Again this agrees with the results described in [11].

The action of S applied to the Lie algebra $\mathbf{g}_\tau^\mathbf{0}$ furnishes a grading of order $h(\mathbf{g})$. It is precisely the grading (3.8) defined for $\mathbf{g}$ applied to the subalgebra $\mathbf{g}_\tau^\mathbf{0}$. When $\mathbf{g}_\tau^\mathbf{0}$ is simple it coincides with the grading defined in book of Kac by a different method and when $\mathbf{g}_\tau^\mathbf{0}$ is not simple (see Table 3) it provides an interesting new possibility.

It is this grading, the twisted principal grading, which can be used to define the twisted affine Kac–Moody algebra $\mathbf{X}_n^{(k)}$. In the present manner of construction, the natural basis for the algebra is a subset of the basis of $\mathbf{g}^{(1)}$ written in terms of its principal Heisenberg subalgebra $\hat{E}_M$ (M equals an exponent of $\mathbf{g}^{(1)}$) and the quantities $\hat{F}(\alpha,z)$ ad-diagonalising the principal Heisenberg subalgebra. This subset is

$$\begin{aligned}
&\hat{E}_M : \quad M = \text{a twisted exponent of } \mathbf{g}_\tau^\mathbf{0} (\operatorname{mod} h(\mathbf{g})) , \\
&\hat{F}(\alpha,z) : \quad \text{for roots } \alpha \text{ of } \mathbf{g} \text{ satisfying } \alpha \cdot q\left(\frac{ph}{k}\right) = 0; \quad p = 1,2,\dots,(k-1) .
\end{aligned} \tag{5.4}$$

Since $\mathbf{g}^{(1)}$ was simply laced, the roots, α all have the same length ($\sqrt{2}$ say) and this applies equally to $\mathbf{X}_n^{(k)}$. Thus, despite the Dynkin diagram (Table 1) not being simply laced, the algebra is nevertheless simply laced in the sense just described. The most striking property of $\mathbf{X}_n^{(k)}$ that follows as a consequence is that the vertex operator construction for $\mathbf{g}^{(1)}$ still applies to $\mathbf{X}_n^{(k)}$. This was originally demonstrated by appealing to character formulae for the irreducible representations [11] but we shall see how it follows naturally in our approach.

First we must consider how the fundamental highest weight representations of $\mathbf{X}_n^{(k)}$ follows from those of $\mathbf{g}^{(1)}$ via folding. What happens is that the inequivalent fundamental representations of $\mathbf{g}^{(1)}$ with highest weights $\Lambda_j, \Lambda_{\tau(j)}, \Lambda_{\tau^2(j)}, \dots$, become identified as a single fundamental representation of $\mathbf{X}_n^{(k)}$ whose highest weight is denoted $\Lambda_{\langle j\rangle}$. This is because $X_{\langle i\rangle} = \sum_{i\in\langle i\rangle} X_i$, $X_i = e_i, f_i, h_i$ have the same action on each of these states and they generate $\mathbf{X}_n^{(k)}$. Actually we expect the rest of the principal Heisenberg subalgebra of $\mathbf{g}^{(1)}$, namely $\hat{E}_M$ for $M = (n + \frac{p}{k})h(\mathbf{g})$, $n \in Z, p = 1, 2, \dots, (k-1)$ to be represented. As $m_j = m_{\tau(j)}$ the level of $\Lambda_{\langle j\rangle}$ is the same as the levels of $\Lambda_j, \Lambda_{\tau(j)}, \Lambda_{\tau^2(j)}, \dots$. Recall that as $\mathbf{g}^{(1)}$ is simply laced

$$\hat{F}(\alpha, z)^{m_j+1} = 0 \tag{5.5}$$

in the $\mathbf{g}^{(1)}$ irrep with highest weight Λ_j. This therefore remains true in the $\Lambda_{\langle j\rangle}$ irrep of $\mathbf{X}_n^{(k)}$ for the surviving $\hat{F}(\alpha, z)$, namely those satisfying (5.4). Furthermore, $\hat{F}(\alpha, z)^{m_j}/m_j!$ is a vertex operator [5], that is a normal ordered exponential of the principal Heisenberg subalgebra $\hat{E}_M$.

Without loss of generality we can replace α by γ_i the standard representative on its orbit under the action of σ as explained in [3]. Then if $\hat{F}^i(z) \equiv \hat{F}(\gamma_i, z)$ in the representation with highest weight Λ_j, we have the generalised vertex operator construction [5],

$$\frac{\hat{F}^i(z)^{m_j}}{m_j!} = e^{-2\pi i \lambda_i \cdot \lambda_j} Y^i Z^i \,, \tag{5.6}$$

where

$$Y^i = \exp \sum_{M>0} \frac{\gamma_i \cdot q([M]) z^M \hat{E}_{-M}}{M}, \qquad Z^i = \exp \sum_{M>0} -\frac{\gamma_i \cdot q([M])^* z^{-M} \hat{E}_M}{M}\,. \tag{5.7}$$

In order to establish that (5.6) and (5.7) makes sense in $\mathbf{X}_n^{(k)}$ as well as $\mathbf{g}^{(1)}$ we need to check two things.

First consider the sum in the exponential. In $\mathbf{g}^{(1)}$ the sum includes all the exponents of $\mathbf{g}^{(1)}$ but in $\mathbf{X}_n^{(k)}$ the sum should be restricted to the twisted exponents of $\mathbf{X}_n^{(k)}$. This restriction is automatically guaranteed as, when $\hat{F}^i(z)$ lies in $\mathbf{X}_n^{(k)}$, γ_i satisfies condition (5.4). Thus the surviving $\hat{F}^i(z) \in \mathbf{X}_n^{(k)}$ are obtained by exponentiating the twisted principal Heisenberg subalgebra ones.

The second point concerns the phase factor in (5.6) [5] which we know plays an important rôle in determining the asymptotic behaviour of the soliton solution. We have to show that this phase is unaltered if λ_j is replaced by $\lambda_{\tau(j)}$ so that it is independent of the representative Λ_j or $\Lambda_{\tau(j)}$ chosen for $\Lambda_{\langle j\rangle}$. Consider the expectation value

$$\left\langle \Lambda_{\tau(j)} \left| \frac{(\hat{F}^i(z))^{m_j}}{m_j!} \right| \Lambda_{\tau(j)} \right\rangle = e^{-2\pi i \lambda_i \cdot \lambda_{\tau(j)}}\,. \tag{5.8}$$

Lifting τ to the outer automorphism $\hat{\tau}$ of $\mathbf{g}^{(1)}$ expression (5.8) equals

$$\left\langle \Lambda_j \left| \frac{(\hat{\tau}^{-1}(\hat{F}^i(z)))^{m_j}}{m_j!} \right| \Lambda_j \right\rangle . \tag{5.9}$$

But $\hat{\tau}^{-1}(\hat{F}^i(z)) = \hat{F}^i(z)$ precisely when $\hat{F}^i(z) \in \mathbf{X}_n^{(k)}$. Thus this expression equals expression (5.8) with $\tau(j)$ replaced by j and the desired conclusion follows.

6. Particles and Solitons in Twisted Affine Toda Field Theories

We can now use the results of the preceding work to determine the mass spectrum (and other properties) of both the quantum particles and the classical soliton solutions in the twisted affine Toda field theories. The results are quite simple and further indicate that the twisted theories more resemble the simply-laced untwisted theories than the non-simply laced untwisted theories despite having non-simply laced Dynkin diagrams.

In the simply-laced untwisted theories based on $\mathbf{g}^{(1)}$ with which we start there is a one to one correspondence between the $r \equiv \operatorname{rank} \mathbf{g}$ species of quantum particles and the same number of species of classical soliton solution. Furthermore the ratio between the corresponding masses is given by

$$\frac{\text{Mass(Soliton } i)}{\text{Mass(Particle } i)} = \frac{4h(\mathbf{g})}{|\beta^2|\hbar\gamma_i^2} . \tag{6.1}$$

Therefore, for the simply laced theories, this number is universal in that it is independent of the species i in question. This was discovered for $\mathbf{A}_n$ by Hollowood, [21], and later generalised to other untwisted theories [4, 24, 25]. When such a theory is folded to give a twisted theory, subsets of the quantum particles and classical solitons survive preserving this correspondence. The same mass values survive unchanged, thereby maintaining the universal ratio (6.1), independent of i, but with $h(\mathbf{g})$ equalling the twisted Coxeter number of $\mathbf{g}_\tau^0 \equiv \mathbf{X}_n$.

On the other hand, when such a theory is folded to give an untwisted non-simply-laced theory the result is somewhat different. Subsets of quantum particles and classical solitons again survive, and can still be put into correspondence. However some of the soliton masses will change so that the mass ratio is still given by (6.1) but no longer universal as the squared lengths of the roots γ_i differ. This is the result of [4] which will be confirmed below by a simpler argument.

To see all this in greater detail, we recall that the quantum particle of species i in the $\mathbf{g}^{(1)}$ theory is associated with the orbit under the action of σ of the $h(\mathbf{g})$ roots $\gamma^i, \sigma\gamma^i, \ldots$ through the corresponding step operators of the finite dimensional Lie algebra $\mathbf{g}$, F^{γ^i}, etc. A similar association applies to the classical soliton species except that, according to the explicit construction, the correspondence is with the generators of the affine Kac–Moody algebra, $\hat{F}^{\gamma^i}(z), \hat{F}^{\sigma(\gamma^i)}(z) = \hat{F}^{\gamma^i}(ze^{\frac{-2\pi i}{h}}) \ldots$. Under the folding, the generators surviving to the subalgebra $\mathbf{g}_\tau^0 \in \mathbf{g}$ or to its affinisation are, by definition, those invariant under the lift of $\tau \in W_0(\mathbf{g})$. These are determined by the result of [4]:

$$\tilde{\tau}(F^{\gamma^i}) = e^{-2\pi i \lambda_i \cdot \lambda_{\tau(0)}} F^{\gamma^i} ,$$

$$\hat{\tau}(\hat{F}^{\gamma^i}(z)) = e^{-2\pi i \lambda_i \cdot \lambda_{\tau(0)}} \hat{F}^{\gamma^i}(z) ,$$

and the fact that, if F^{γ^i} survives, so do all the step operators for the roots on the σ orbit of γ^i. Thus the condition for the survival under folding of either a quantum particle of species i or a soliton of species i is exactly the same:

$$e^{-2\pi\imath\lambda_i\cdot\lambda_{\tau(0)}} = 1 \,. \tag{6.2}$$

It is easy to evaluate this condition. Using the numbering of Table 2, the results are

$$\begin{aligned}
&\mathbf{E}_6^{(1)}(\tau(0)=1) : i = 3,6\,,\\
&\mathbf{E}_7^{(1)}(\tau(0)=6) : i = 1,2,3,5\,,\\
&\mathbf{D}_{l+2}^{(1)}(\tau(0)=1) : i = 1,2,\ldots,l\,,\\
&\mathbf{D}_{2l}^{(1)}(\tau(0)=2l) : i = 2,4,\ldots,2l-2,2l \quad \text{for } l \text{ even}\\
&\qquad\qquad\qquad\quad\;\; i = 2,4,\ldots,2l-2,2l-1 \quad \text{for } l \text{ odd}
\end{aligned}$$

and agree with the calculations of Braden et al. [13] for the quantum particles for the twisted theories.

We saw in Sect. 4 that the condition (6.2) for survival could be written in another way, Eq. (4.15) or (5.4). The significance of this presentation, namely that the root to the surviving generators be orthogonal to the space spanned by $q(h(\mathbf{g})/k), q(2h(\mathbf{g})/k), \ldots$, is that the subset of surviving particles (be they quantum particles or solitons) is closed, and hence self consistent, under the operations of

i. antiparticle conjugation,
ii. fusing.

(i) follows because the orbit associated with the antiparticle of species i consists of the negatives of the roots in the orbit containing γ_i and so if condition (4.15) is satisfied by the particle it is satisfied by the antiparticle. (In fact, the surviving species equal their anti-species.) If species i and j fuse to give k, then, by Dorey's rule, γ_k can be expressed as the sum of two roots in the orbits containing γ_i and γ_j respectively. Evidently if orbits i and j satisfy (4.15) so does species k. Thus the spectrum (of either quantum particles or solitons) of the twisted affine Toda field theories will contain its own antiparticles and will *still* couple by means of Dorey's fusing rule. This means that, in the quantum theory, the S-matrix in the untwisted theory, restricted to the quantum particles states of the twisted theory, will satisfy unitarity, crossing and bootstrap property, as argued in the case of $\mathbf{D}_4^{(3)}$ by Fring and Koberle [17]. However, as Dorey [9, 10] has noted, there may be doubt about the positivity properties. Certainly the quantum corrections to the same quantum particle mass will differ in the two theories due to the different spectrum of intermediate states allowed.

There has been much discussion of the proposal that the quantum twisted theory be related to the untwisted non-simply laced theories whose Dynkin diagram is obtained by reversing all the arrows [22, 15, 23]. It is tantalising that these two theories do share isomorphic Poincaré algebras (1.1) as their generalised exponents coincide [11] (Corollary 14.3).

Whatever the kind of folding involved, as long as it is direct, it was shown in Sect. 2, that the masses of the surviving quantum particles are unchanged in the folding but we still have to check what happens to the classical soliton masses, or indeed the multisoliton solutions more generally. First we need to consider the expectation values M_i and check how the condition $M_i = M_{\tau(i)}$ of Sect. 2, needed to define the corresponding quantities $M_{\langle i\rangle} = M_i$ for the twisted theory, is satisfied whenever τ is a direct symmetry of $\Delta(\mathbf{g}^{(1)})$. Recall that for a multisoliton solution of the $\mathbf{g}^{(1)}$ theory

$$M_j = \langle \Lambda_j | g(t) | \Lambda_j \rangle \,, \tag{6.3}$$

where

$$g(t) = V(t)\, g(0)\, V(t)^{-1} \,,$$

$$g(0) = \prod_j \exp\left[Q_{n(j)} \hat{F}^{n(j)}(z_{(n(j))})\right] ,$$

$$V(t) = \exp(\mu \hat{E}_{-1} t_{-1}) \exp(-\mu \hat{E}_1 t_1) \,.$$

From this we see that the symmetry condition $M_i = M_{\tau(i)}$ is satisfied whenever the group element satisfies $\hat{\tau}(g(0)) = g(0)$, i.e. it is generated by the folded algebra. If $\tau \in W_0(\mathbf{g})$, this means that the species of $\hat{F}$ satisfy (6.2), or, equivalently, (5.4).

If, instead, $\tau \in \operatorname{Aut} \Delta(\mathbf{g})$, so that the folding leads to a non-simply laced untwisted theory, the condition is again satisfied but in a somewhat different way. In this case it was shown in [4] that the surviving $\hat{F}^{\langle i\rangle}(z)$ were linear combinations of the original ones:

$$\hat{F}^{\langle i\rangle}(z) = \sum_{i \in \langle i\rangle} \hat{F}^i(z) \,. \tag{6.4}$$

We shall see that this implies that the single soliton of species $\langle i\rangle$ of the folded theory will arise from folding a special $|\langle i\rangle|$-soliton solution of the unfolded theory.

We have been using the result of Sect. 2 whereby, given a "direct" symmetry of the Dynkin diagram $\Delta(\mathbf{g}^{(1)})$, the symmetric solution of the unfolded theory yields solutions of the folded theory and vice-versa. In order to understand the resultant soliton masses we shall present a similar theorem for the Lagrangian and for the energy momentum tensor: that a symmetric configuration for the unfolded theory (not necessarily a solution) yields a configuration of the folded theory such that the respective Lagrangian (and energy momentum tensor) assume identical values.

In order to see this we first note that the folding of the Cartan matrix (2.4) is satisfied by folding the roots as follows:

$$\alpha_{\langle i\rangle} = \sum_{i \in \langle i\rangle} \frac{\alpha_i}{|\langle i\rangle|} \,, \tag{6.5}$$

where $|\langle i\rangle|$ is the number of roots in the orbit of i. It follows that

$$\frac{2m_i}{\alpha_i^2} = \frac{2m_{\langle i\rangle}}{|\langle i\rangle| \alpha_{\langle i\rangle}^2} \,, \tag{6.6}$$

so that the constraint condition (2.8) is preserved. We now apply this to the interaction term in the Lagrangian (and the energy momentum tensor)

$$V(\phi) = \frac{2\mu^2}{\beta^2} \sum_{i=0}^{r} \frac{m_i}{\alpha_i^2} (e^{\beta\phi_i} - 1) \tag{6.7}$$

and see immediately that for symmetric configurations it equals

$$\frac{2\mu^2}{\beta^2}\sum_{\langle i\rangle}^{r}\frac{m_{\langle i\rangle}}{\alpha^2_{\langle i\rangle}}(e^{\beta\phi_{\langle i\rangle}}-1)\,, \tag{6.8}$$

which is the corresponding interaction term for the folded theory. The same result applies to the kinetic terms $\partial_\mu\phi\cdot\partial^\mu\phi$ and $\partial_\mu\phi\cdot\partial_\nu\phi$ in the Lagrangian and energy-momentum tensor, but the proof is more complicated and relegated to Appendix 2. The main difficulty is to express the kinetic term in term of the variables ϕ_i, (2.6), in a way which manifestly respects the symmetry under Aut $\Delta(\mathbf{g}^{(1)})$.

In particular, if we have a symmetric multisoliton solution of the unfolded theory, it will yield a multisoliton solution of the folded theory with the same energy and momentum. Nevertheless we cannot conclude that the solitons have the same masses in the folded and unfolded theory as soliton number is not conserved and necessarily preserved in the folding as we now see.

In the case that $\tau\in W_0(\mathbf{g})$, so that the folded theory is twisted, a multisoliton solution of species $n(1),n(2),\dots,n(k)$ satisfying (6.2) is symmetric and survives as a soliton with the same interpretation. According to the result of [3], its energy and momentum is given by

$$\sqrt{2}P^{\pm}=\sum_{i=1}^{k}M_{n(i)}e^{\pm\eta_{n(i)}}\,.$$

As this result applies to the folded theory also, by our theorem, the folded soliton solution therefore possesses the same mass as the unfolded one. This is the result mentioned at the beginning.

On the other hand, if $\tau\in$ Aut $\Delta(\mathbf{g})$ so that the folded theory is untwisted but non-simply laced, the symmetric solution could involve an exponential of (6.4),

$$e^{Q\hat{F}^{\langle i\rangle}(z)}=\prod_{i\in\langle i\rangle}e^{Q\hat{F}^{i}(z)}\,. \tag{6.9}$$

The left-hand side would create a single soliton of the folded theory but the right-hand side would create a superposition of $|\langle i\rangle|$ solitons of the unfolded theory all with the same coordinate and rapidity. We have used the fact that the pieces in (4.3) mutually commute since they have the same rapidity.

When we equate the consequent contributions to the energy and momentum tensor of the soliton we have

$$\sqrt{2}P^{\pm}=M_{\langle i\rangle}e^{\pm\eta}=\sum_{i\in\langle i\rangle}M_ie^{\pm\eta}\,.$$

Hence we conclude that the mass of the soliton of species $|\langle i\rangle|$,

$$M_{\langle i\rangle}=|\langle i\rangle|M_i$$

(as $M_i=M_{\tau(i)}$). This confirms the result (6.1) of [4] with the bonus of a more physical understanding. Note that this explains why MacKay and McGhee [24] overlooked some of the soliton species in the untwisted nonsimply-laced theories. They considered folding only single solutions and so ignored configurations such as (6.9).

Acknowledgements. We are grateful for discussions with H. Braden, E. Corrigan, P. Dorey and J.W.R. Underwood. Marco Kneipp wishes to thank $\mathrm{CNP_q}$ (Brazil) for financial support.

Appendix A: Element T of G Corresponding to $\tau \in W_0(\mathbf{g})$

The lift of the diagram automorphism $\tau \in W_0(g)$ is provided by the inner automorphism (3.2) in which τ has the form $\exp(-2\pi i Y \cdot h)$, (3.13). We shall now verify, on a case by case basis, the existence of an element w, of the Weyl group, $W(g)$, such that Y has the form (4.1). It is instructive to start with the case $\mathbf{g} = \mathbf{A}_{mn-1}$ in Table 3. As this is the only case in which τ may have an order greater than 3, namely m, it is the most complicated.

We recall that the $mn-1$ fundamental weights λ_l can be expressed in terms of mn unit vectors ε_i:

$$\lambda_l = \sum_{i=1}^{l} \varepsilon_i - \frac{l}{mn} \sum_{i=1}^{mn} \varepsilon_i, \quad l = 1, 2, \ldots, mn-1 \; .$$

The Weyl group, $W(\mathbf{A}_{mn-1})$, is isomorphic to the group of permutation of the unit vectors, any element can be denoted by the standard permutation notation. The following element cyclically permutes the mn unit vectors and hence is a Coxeter element

$$\sigma' = (1, n+1, 2n+1, \ldots, (m-1)n+1, 2, n+2, 2n+2, \ldots, (m-1)n+2, \ldots, \\ n, n+n, \ldots, (m-1)n+n) \; .$$

It is easy to check that σ' satisfies (4.4) and (4.5) so that the desired result follows.

Now we turn to the four cases in Table 2. When τ has order 2, as it does unless $\mathbf{g} = \mathbf{E}_6$, all that has to be shown is the existence of a Coxeter element σ' conjugate to a standard one σ satisfying

$$\sigma' \lambda_{\tau(0)} = -\lambda_{\tau(0)} \; .$$

Let us now consider $\mathbf{D}_\tau$ with its simple roots constructed out of the unit vectors in the usual way

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \ldots, r-1 \; ,$$
$$\alpha_r = \varepsilon_{r-1} + \varepsilon_r \; ,$$

so that the fundamental weights are

$$\lambda_i = \sum_{i=1}^{r} \varepsilon_i \quad i = 1, 2, \ldots, r-2 \; ,$$

$$\lambda_{r-1} = \frac{1}{2}\left(\sum_{i=1}^{r-1} \varepsilon_i - \varepsilon_r\right), \qquad \lambda_r = \frac{1}{2}\sum_{i=1}^{r} \varepsilon_i \; .$$

Then it is easy to see that the Coxeter element $\sigma'' = \sigma_1 \sigma_2 \cdots \sigma_r$ (with σ_i the reflection in α_i) has the following action on the unit vectors:

$$\varepsilon_1 \longrightarrow \varepsilon_2 \longrightarrow \cdots \longrightarrow \varepsilon_{r-1} \longrightarrow -\varepsilon_1 \longrightarrow -\varepsilon_2 \longrightarrow \cdots \longrightarrow -\varepsilon_{r-1} \longrightarrow \varepsilon_1 \; ,$$

$$\varepsilon_r \longrightarrow -\varepsilon_r \; .$$

Hence as $\sigma_{\varepsilon_1+\varepsilon_r}(\varepsilon_r) = -\varepsilon_1$,

$$\sigma' = \sigma_{\varepsilon_1+\varepsilon_r}\sigma''\sigma_{\varepsilon_1+\varepsilon_r}$$

has the action $\sigma'\lambda_1 = -\lambda_1$. This is the desired result if $\tau(0) = 1$ as in the second line of Table 2.

If now r is even, equal to $2l$, say, σ'' also reverses the signs of $\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots + \varepsilon_{2l-1}$. Hence it reverses the sign of one half this plus or minus ε_{2l}. But this is a spinor weight Weyl conjugate to λ_{2l} or λ_{2l-1} according to the sign chosen. This establishes the existence of σ' reversing the sign of λ_{2l-1} as needed when $\tau(0) = 2l - 1$ as in the first line of Table 2.

For $\mathbf{E}_6$ let us consider the element of the Weyl group

$$w = \sigma_\beta\sigma_{\alpha_1}\sigma_{\alpha_5}, \qquad \beta \equiv \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 \,.$$

One can check that

$$w\lambda_1 = -\lambda_1 + \lambda_5 \,, \qquad w\lambda_5 = -\lambda_2 + \lambda_4 \,.$$

Now using the bicolouration such that $c_1 = c_5 = -1$ and the identities [7]

$$\sigma\lambda_{i\_} = \lambda_{i\_} - \alpha_{i\_} \,, \qquad \sigma_\pm\alpha_{i\mp} = \alpha_{i\mp} - K_{i\mp j}\alpha_j \,, \qquad \sigma_\pm\alpha_{i\pm} = -\alpha_{i\pm} \,,$$

it can be proven that

$$(1+\sigma)w\lambda_1 = w\lambda_5 \,, \qquad (1+\sigma+\sigma^2)w\lambda_1 = 0 \,,$$

which correspond to (4.4) for $\tau(0) = 1$.

Finally, for $\mathbf{E}_7$, let $w = \sigma_\beta$, where $\beta \equiv \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_5 + \alpha_6 + \alpha_7$. Then,

$$w\lambda_6 = -\lambda_4 + \lambda_6 + \lambda_7 \,.$$

Now, like before, using the bicolouration such that $c_4 = c_6 = c_7 = -1$ it is easy to prove that

$$(1+\sigma)w\lambda_6 = 0 \,.$$

This completes the claimed result.

Appendix B: Folding of the Energy Momentum Tensor

Here we complete the argument of Sect. 6 and show that, for a field configuration symmetric under $\tau \in \mathrm{Aut}\,\Delta(\mathbf{g}^{(1)})$, the energy momentum tensors of the folded and unfolded affine Toda field theories are equal, providing τ is direct.

Given the Lie algebra $\mathbf{g}$, we define the quantities

$$\mu_i = \lambda_i^v - \frac{2m_i\rho^v}{\alpha_i^2 H} \quad i = 0,1,2\ldots,r \,, \tag{B1}$$

where $\lambda_1^v, \lambda_2^v, \ldots, \lambda_r^v$ are the fundamental coweights, ρ^v their sum, while λ_0^v vanishes. The number

$$H \equiv \sum_{i=0}^{r} \frac{2m_i}{\alpha_i^2} \,. \tag{B2}$$

If the long roots have length $\sqrt{2}, H$ denotes either the Coxeter number or twisted Coxeter number, whichever is relevant. Then, because of the constraint (2.8), the field ϕ can be written

$$\phi = \sum_{i=0}^{r} \phi_i \mu_i \, . \tag{B3}$$

The significance of this is that it provides the way of introducing all $r+1$ variables $\phi_0, \phi_1, \ldots, \phi_r$, (2.6), which will respect the full symmetry of the extended Dynkin diagram, Aut $\Delta(\mathbf{g}^{(1)})$. To see this, first note that

$$\alpha_i \cdot \mu_j = \delta_{ij} - \frac{2m_i}{H\alpha_i^2} \quad i,j = 0,1,\ldots,r \, . \tag{B4}$$

Since

$$\tau(\alpha_i) \equiv \alpha_{\tau(i)}$$

defines the linear map τ, given $\tau \in \text{Aut}\, \Delta(\mathbf{g}^{(1)})$, we can easily check that

$$\alpha_i \cdot \tau(\mu_j) = \alpha_i \cdot \mu_{\tau(j)} \, , \tag{B5}$$

and hence

$$\mu_i \cdot \mu_j = \mu_{\tau(i)} \cdot \mu_{\tau(j)} \, . \tag{B6}$$

So the term $\partial_\mu \phi \cdot \partial_\nu \phi$ in the energy momentum tensor can be written

$$\sum_{i \cdot j=0}^{r} \mu_i \cdot \mu_j \partial_\mu \phi_i \partial_\nu \phi_j \, , \tag{B7}$$

a form which explicitly respects all the symmetries of the extended Dynkin diagram.

Using (6.5) one sees that H is unchanged by the folding if it is direct. Hence, by (B4),

$$\mu_{\langle j \rangle} = \sum_{j \in \langle j \rangle} \mu_j \, . \tag{B8}$$

Hence, for symmetric field configurations, expression (B7) equals

$$\sum_{\langle i \rangle, \langle j \rangle} \mu_{\langle i \rangle} \cdot \mu_{\langle j \rangle} \partial_\mu \phi_{\langle i \rangle} \partial_\nu \phi_{\langle j \rangle} \, .$$

The same argument evidently applies to the kinetic term in the Lagrangian, $\frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi$.

References

1. Olive, D.I., Turok, N.: Local conserved densities and zero curvature conditions for Toda lattice field theories. Nucl. Phys. **B257** [FS14], 277–301 (1985)
2. Wilson, G.: The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras. Ergod. Th. & Dynam. Sys. **1**, 361–380 (1981)
3. Olive, D.I., Turok, N., Underwood, J.W.R.: Solitons and the energy-momentum tensor for affine Toda theory. Nucl. Phys. **B401**, 663–697 (1993)
4. Olive, D.I., Turok, N., Underwood, J.W.R.: Affine Toda solitons and vertex operators. Nucl. Phys. **B409**, 509–546 (1993), hep-th/9305160

5. Kneipp, M.A.C., Olive, D.I.: Crossing and antisolitons in affine Toda theories. Nucl. Phys. **B408**, 565–578 (1993), hep-th/9305154
6. Fring, A., Liao, H.C., Olive, D.I.: The mass spectrum and coupling in affine Toda theories. Phys. Lett. **B266**, 82–86 (1991)
7. Fring, A., Olive, D.I.: The fusing rule and the scattering matrix of affine Toda theory. Nucl. Phys. **B379**, 429–447 (1992)
8. Freeman, M.D.: On the mass spectrum of affine Toda field theory. Phys. Lett. **B217**, 57 (1991)
9. Dorey, P.E.: Root systems and purely elastic S-matrices. Nucl. Phys. **B358**, 654 (1991)
10. Dorey, P.E.: Root systems and purely elastic S-matrices 2. Nucl. Phys. **B374**, 741 (1992), hep-th/9110058
11. Kac, V.G.: Infinite Dimensional Lie Algebras. Cambridge: Cambridge University Press, 3rd ed., 1990
12. Olive, D.I., Turok, N.: The symmetries of Dynkin diagrams and the reduction of Toda field equations. Nucl. Phys. **B215** [FS7], 470–494 (1983)
13. Braden, H.W., Corrigan, E., Dorey, P.E., Sasaki, R.: Affine Toda field theory and exact S-matrices. Nucl. Phys. **B338**, 689 (1990)
14. Springer, T.A.: Regular elements of finite reflection groups. Inv. Math. **25**, 159 (1974)
15. Dorey, P.E.: Hidden geometrical structures in integrable models. In: Integrable Quantum Field Theories. Bonora, L., Mussardo, G., Schwimmer, A., Girardello, L., Martellini, M., (eds.) Plenum 1993, 83-97 hep-th/9212143
16. Dorey, P.E.: A Remark on the coupling dependence in affine Toda field theories. Phys. Lett. **B312**, 291–298 (1993), hep-th/9304149
17. Fring, A., Koberle, R.: On exact S-matrices for non-simply laced affine Toda field theories. USP-IFQSC/TH/93-13
18. Zamolodchikov, A.B.: Integrals of motion in scaling 3-state Potts model quantum theory. Int. J. Mod. Phys. **A3**, 743–750 (1988)
19. Zamolodchikov, A.B.: Integrals of motion and S-matrix of (Scaled) $T = T_c$ Ising model with magnetic field. Int. J. Mod. Phys. **A4**, 4235–4248 (1989)
20. Zamolodchikov, A.B.: Integrable field theory from conformal field theory. In: Integrable Systems in Quantum Field Theory and Statistical Mechanics. Advanced Studies in Pure Mathematics **19**, New York: Academic Press, 1989, pp. 641–674
21. Hollowood, T.J.: Solitons in affine Toda field theories. Nucl. Phys. **B384**, 523–540 (1992)
22. Delius, G.W., Grisaru, M.T., Zanon, D.: Exact S-matrices for non-simply laced affine Toda theories. Nucl. Phys. **B382**, 365 (1992), hep-th/9112007
23. Corrigan, E., Dorey, P.E., Sasaki, R.: On a generalised bootstrap principle. Nucl. Phys. **B408**, 579–599 (1993)
24. MacKay, N.J., McGhee, W.A.: Affine Toda solitons and automorphisms of Dynkin diagrams. Int. J. Mod. Phys. **A8**, 2791–2808 (1993), hep-th/9208057
25. Aratyn, H., Constantinidis, C.P., Ferreira, L.A., Gomes, J.F., Zimerman, A.H.: Hirota's solitons in the affine and the conformal affine Toda model. Nucl. Phys. **B406**, 727–770 (1993), hep-th/9212086

Communicated by M. Jimbo