Commun. Math. Phys. 177, 451 – 468 (1996)

Communications in
Mathematical
Physics

Character Expansion Methods for Matrix Models of Dually Weighted Graphs

Vladimir A. Kazakov, Matthias Staudacher★, Thomas Wynter★

Laboratoire de Physique Théorique de l'École Normale Supérieure★★, Université de Paris Sud, 24 rue, Lhomond, F-75231 Paris Cedex 05, France

Received: 27 March 1995

Abstract: We consider generalized one-matrix models in which external fields allow control over the coordination numbers on both the original and dual lattices. We rederive in a simple fashion a character expansion formula for these models originally due to Itzykson and Di Francesco, and then demonstrate how to take the large N limit of this expansion. The relationship to the usual matrix model resolvent is elucidated. Our methods give as a by-product an extremely simple derivation of the Migdal integral equation describing the large N limit of the Itzykson–Zuber formula. We illustrate and check our methods by analysing a number of models solvable by traditional means. We then proceed to solve a new model: a sum over planar graphs possessing even coordination numbers on both the original and the dual lattice. We conclude by formulating equations for the case of arbitrary sets of even, self-dual coupling constants. This opens the way for studying the deep problem of phase transitions from random to flat lattices.
January 1995

1. Introduction

After the considerable success of two dimensional quantum field theory and statistical mechanics – integrable models on 2D regular lattices, conformal field theories, Liouville theory and matrix models of 2D gravity and non-critical strings – progress in analytical results in this field has slowed down.

Among the principal questions remaining unsolved are, first, the so-called $c = 1$ barrier for non-critical strings (c is the central charge of the matter), and, second, the mysterious connection between the physical properties of various integrable 2D models coupled and non-coupled to gravity. The first problem is usually attributed to the absence of a stable vacuum for $c > 1$, though it has never been clearly

★ This work is supported by funds provided by the European Community, Human Capital and Mobility Programme

★★ Unité Propre du Centre National de la Recherche Scientifique, associée à l'École Normale Supérieure et à l'Université de Paris-Sud understood. Indeed, in terms of matrix models, the obstacles seem to be purely technical. The second problem concerns the observation of many intriguing relations between 2D physical systems with and without coupling to 2D quantum gravity.

This relation is clearly established on the level of critical exponents by the use of the continuous formulation of 2D gravity [1]. The conformal dimensions of matter fields undergo a simple quadratic transformation as a result of gravitational dressing. The same phenomenon is observed, of course, in the matrix model formalism. However, if one goes away from the critical point, the relation between the physical properties with and without gravitational coupling, although persisting, becomes much more tricky and fragmentary. For example there is the description of 2D gravitational systems in terms of KDV hierarchies of classical 2D integrable systems [2], as well as a strange coincidence between the amplitude for the open string in the SOS formalism and the S-matrix of the two dimensional Sine–Gordon theory [3].

It seems that two-dimensional physics is more united than one would think at first sight. An interesting question to ask would be the existence of some interpolating models connecting the gravitational and "flat" phases of the same matter fields.

Our paper is inspired by this physical idea, though it concerns mostly the elaboration of a technique for the solution of a new type of matrix model. The model describes, in the large N limit, planar graphs having arbitrary coordination number dependent weights for both the vertices and faces. In other words, we introduce a set of couplings $t_1, t_2, \dots t_q, \dots$, the weights of vertices with $1, 2, \dots, q, \dots$ neighbours, and $t_1^*, t_2^*, \dots t_q^*, \dots$, the weights of the dual vertices (or faces) with appropriate coordination numbers. In Fig. 1 below is a typical graph with, for illustration, some vertices on the original and dual lattice labeled with their associated weights. The matrix models under consideration allow us to generate the following partition function of closed planar graphs G:

$$Z(t, t^*) = \sum_G \prod_{v_q, v_q^* \in G} t_q^{\# v_q} t_q^{*\# v_q^*} , \tag{1.1}$$

where v_q, v_q^* are the vertices with q neighbours on the original and dual graph, respectively, and $\# v_q$, $\# v_q^*$ are the numbers of such vertices in the given graph G. We propose to call this the model of dually weighted graphs (DWG).

It is clear that this model opens the way to understanding the very interesting transition mentioned above. If we set $t_4 = t_4^* = 1$ and $t_q = t_q^* = 0$, for $q \neq 4$, only regular square lattices (graphs) will exist in Eq. (1.1). Hence, there are trajectories in the coupling space of this model, interpolating between pure gravity (for example, when all $t_q^* = 1$) and the regular "flat" lattice.

We will show in this paper that the underlying matrix model describing the DWG is solvable. Our solution is based on an elegant representation of this model in terms of the group character expansion found in [4]. It allows us to reduce the N^2 degrees of freedom of the original matrix model to the N degrees of freedom labeling a representation. We then apply the saddle point approximation to find the most probable group representation in the corresponding sum over characters, specified by the distribution of its highest weights. A similar approximation was first successfully used in [7] for the calculation of the QCD_2 partition function on the sphere. We conclude with a well defined (though complicated) integral equation for this distribution. Though we have not yet been able to extract the physical picture

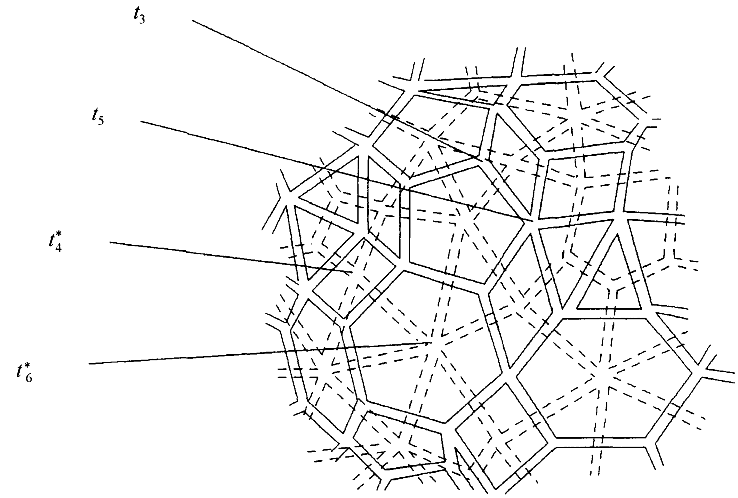


Fig. 1. A typical surface and some of the associated weights.

corresponding to the "flattening transition" we demonstrate on a number of simpler examples that our method is consistent and correct.

Furthermore, we solve a model apparently inaccessible by standard methods: we calculate the number of planar graphs having only an even number of neighbours for both original and dual vertices $t_{2q} = t_{2q}^* = 1$, and $t_{2q-1} = t_{2q-1}^* = 0$, for any q.

We hope that our methods will lead to new progress in solving many physically interesting 2D systems. A natural step forward would be the introduction of matter on DWG, a tempting opportunity, whose success is, of course, not automatically guaranteed. Since our matrix model is a generalized matrix external field problem, it could also be useful for new studies in random (mesoscopic) systems.

We present below explicit details of our technique as we feel it is a general and powerful method for matrix models.

2. Reduction of the DWG Model to a Sum Over Characters

The partition function for the dually weighted graphs can be formulated as the following matrix model (see for example [8]):

$$Z(t,t^*) = \lambda^{-\frac{N^2}{2}} \int \mathscr{D}M \; e^{-\frac{N}{2\lambda}\operatorname{Tr} M^2 + \sum_{k=1}^{\infty}\frac{1}{k}\operatorname{Tr} B^k \operatorname{Tr}(MA)^k} . \tag{2.1}$$

The matrices A and B are external matrices. In the perturbative expansion of the above integral, the matrix B weights a vertex of coordination number q with the factor $\operatorname{Tr} B^q$, while each face bounded by q vertices is weighted by a factor $\operatorname{Tr} A^q$. We can therefore make the connection

$$t_q = \frac{1}{q}\frac{1}{N}\operatorname{Tr} B^q \quad \text{and} \quad t_q^* = \frac{1}{q}\frac{1}{N}\operatorname{Tr} A^q . \tag{2.2}$$

Note that it is impossible to solve the above matrix integral by standard methods since it is unclear, for $A \neq 1$, how to perform the angular integration, i.e. how to evaluate unitary matrix integrals of the form $\int (d\Omega)_H \exp(\sum_k \beta_k \mathrm{Tr}\,(M\Omega A\Omega^\dagger)^k)$ with M and A diagonal. We circumvent this difficulty by expanding the potential in terms of the characters on the group:

$$e^{\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{Tr}\,B^k\ \mathrm{Tr}(MA)^k} = \prod_{i,j=1}^{N}\frac{1}{(1-B_i(MA)_j)} = \frac{1}{N^N}\sum_R \chi_R(B)\,\chi_R(MA)\,. \tag{2.3}$$

Here B_i and $(MA)_j$ are the eigenvalues of the matrices B and MA. The first step involves rewriting the sum over k as a double sum over all the eigenvalues of the matrices B and MA of $-\ln(1-B_i(MA)_j)$. Exponentiating the log then gives the product in the numerator. The second step uses a group theoretic result to write the product in terms of a sum over characters. The character is defined by the Weyl formula:

$$\chi_{\{h\}}(A) = \frac{\det_{(k,l)}(a_k^{h_l})}{\Delta(a)}\,, \tag{2.4}$$

where the set of $\{h\}$ are a set of ordered, increasing, non-negative integers, $\Delta(a)$ is the Vandermonde determinant, and the sum over R is the sum over all such sets. The R's label representations of the group $U(N)$ and the sets of integers, $\{h\}$, have the correspondence with the Young tableaux shown in Fig. 2.

Note that the restriction on the allowed Young tableaux that any row must have at least as many boxes as the row below implies that the $\{h_i\}$ are a set of increasing integers

$$h_{i+1} > h_i\,. \tag{2.5}$$

Substituting Eq. (2.3) into the integral in Eq. (2.1) we can now do the angular integration using the identity $\int (\mathcal{D}\Omega)_H \chi_R(\Omega M\Omega^\dagger A) = \chi_R(M)\chi_R(A)/d_R$ (where d_R is the dimension of the representation given by $d_R = \Delta(h)/\prod_{i=1}^{N-1} i!$), and arrive at the expression

$$Z = \frac{\lambda^{-\frac{N^2}{2}}}{N^N}\sum_R \frac{1}{d_R}\chi_R(A)\,\chi_R(B)\int\prod_{i=1}^{N}\mathcal{D}M_i\Delta(M)\det_{(k,l)}(M_k^{h_l})e^{-\frac{N}{2\lambda}\mathrm{Tr}\,M^2}\,. \tag{2.6}$$

The gaussian integral can be done explicitly and we arrive at the final formula

$$Z = c\sum_{\{h^e,h^o\}}\frac{\prod_i (h_i^e-1)!!h_i^o!!}{\prod_{i,j}(h_i^e-h_j^o)}\chi_{\{h\}}(A)\chi_{\{h\}}(B)\left(\frac{\lambda}{N}\right)^{-\frac{1}{4}N(N-1)+\frac{1}{2}\sum_i(h_i^e+h_i^o)}\,, \tag{2.7}$$

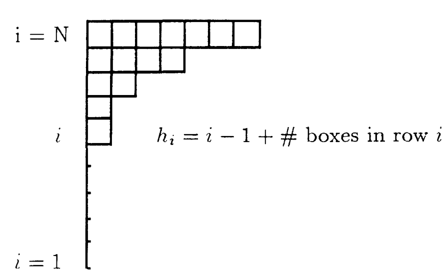


Fig. 2. Connection between Young tableaux and the integers h_i.

where c is some numerical constant that we can drop, the $\{h^e\}$ are a set of $N/2$ even, increasing, non-negative integers and the $\{h^o\}$ are $N/2$ odd, increasing, positive integers, and the sum is over all such sets. In other words the original sum is now restricted to the subsets of $\{h\}$ with equal numbers of even and odd integers. This is an exact result. The sum is in general divergent, as is the original matrix integral, and should be thought of as a generating function for graphs of arbitrary genus. This formula was originally derived by Itzykson and Di Francesco [4], using abstract combinatorial reasoning. For the rest of this paper we restrict our attention solely to the genus 0 contribution. In other words we will be studying the large N limit of Eq. (2.7).

There is a second useful formula for the character given in terms of Schur polynomials, $P_n(\theta)$, defined by

$$e^{\sum_{i=1}^{\infty} z^i \theta_i} = \sum_{n=0}^{\infty} z^n P_n(\theta) . \tag{2.8}$$

In terms of Schur polynomials the character is

$$\chi_{\{h\}}(A) = \det_{(k,l)}(P_{h_k+1-l}(\theta)) , \tag{2.9}$$

where $\theta_i = \frac{1}{i}\mathrm{Tr}\, A^i$. In general, the explicit expressions for the characters are very complicated. Certain specific cases however yield simple results which we will state as we need them.

3. Relations Between Highest Weight and Matrix Model Quantities

Before we look for the large N limit of Eq. (2.7), we derive some explicit expressions relating useful quantities from matrix models to quantities encountered in the language of highest weights. In the large N limit of Eq. (2.7), we assume that the sum over all representations will be dominated by a single contribution, or a single Young tableau, $\{h_i\}$, and introduce a density $\rho(h)$ defined in the standard way by $\rho(h) = \frac{1}{N}\frac{\partial i}{\partial h}$. In order to define a sensible density we have to rescale the integers $\{h_i\}$ by dividing them by N. For the rest of this paper an h_i with an index refers to one of the original integers and h without a subscript refers to the rescaled continuous parameter.

All the formulae in this section have their root in the simple observation that

$$\mathrm{Tr}\, A^q = \sum_k \frac{\chi_{\{\tilde{h}\}}(A)}{\chi_{\{h\}}(A)} , \quad \text{where } \tilde{h}_i = h_i + q\delta_{i,k} . \tag{3.1}$$

For compactness of notation we have omitted labeling the $\tilde{h}$ with an index k. This formula follows directly from the Weyl formula for the character, (2.4). The character can be written in terms of the Itzykson–Zuber integral, $I(h,\alpha) = \det_{(k,l)}(e^{h_k\alpha_l})/(\Delta(h)\Delta(\alpha))$, as

$$\chi_h(A) = I(h,\alpha)\Delta(h)\frac{\Delta(\alpha)}{\Delta(a)} , \tag{3.2}$$

where α_l is defined through the eigenvalues of A by $a_l = e^{\alpha_l}$. This allows us to write

$$\mathrm{Tr}\, A^q = \sum_k \frac{\Delta(\tilde{h})}{\Delta(h)} e^{q\frac{\ln I(\tilde{h},\alpha) - \ln I(h,\alpha)}{q}} \sim \lim_{N\to\infty} \sum_k \prod_{j(\neq k)} \left(1 + \frac{q}{h_k - h_j}\right) e^{qF(h_k)} , \tag{3.3}$$

where

$$F(h_k) = \frac{\partial}{\partial h_k} \ln I(h, \alpha) \tag{3.4}$$

is the derivative of the Itzykson–Zuber integral. In the last step we assume that the Itzykson–Zuber integral has a well defined large N limit: $I(h,\alpha) = e^{N^2 F_0[\rho(h),\rho(\alpha)] + O(N^0)}$. We now notice that we can replace the sum by a contour integral in h (encircling all the h_k's) if we also unrestrict the product allowing it to be a product over all j. This contour integration trick was originally invented in [5] for a much simpler model ($c = -2$ gravity), and more recently was used in [6] for the large N limit of the heat kernel. Applying it in this case we obtain:

$$\frac{1}{N} \mathrm{Tr}\, A^q = \frac{1}{q} \oint \frac{dh}{2\pi i} e^{q(H(h)+F(h))} \quad \text{with } H(h) = \int dh' \frac{\rho(h')}{h - h'} \,. \tag{3.5}$$

$F(h)$ is initially defined only on the support of h; we have then analytically continued $F(h)$ into the whole complex plane, so that the contour integral, which circles the cut of the resolvent, $H(h)$, is well-defined. We have thus rederived in a very compact way the large N limit of the Itzykson–Zuber integral [9, 10]. To make the connection with the result in [9] more explicit it is simple to expand (3.5) as a power series in q and then resum the series to express the result in terms of the resolvent for the eigenvalues α (see also [6]):

$$\Theta(\alpha) = -\oint \frac{dh}{2\pi i} \ln(\alpha - H(h) - F(h)) \quad \text{with } \Theta(\alpha) = \int \frac{d\alpha' \sigma(\alpha')}{\alpha - \alpha'} \,, \tag{3.6}$$

where $\sigma(\alpha)$ is the density for the eigenvalues α.

Next we look at the expectation value of $\frac{1}{N}\mathrm{Tr}\, M^{2q}$. Placing this term into the M integrand in Eq. (2.6) and using similar steps as for $\frac{1}{N}\mathrm{Tr}\, A^q$ we arrive at

$$\frac{1}{N} \mathrm{Tr}\, M^{2q} = \frac{\lambda^q}{q} \oint \frac{dh}{2\pi i} h^q e^{qH(h)} \,. \tag{3.7}$$

This formula is derived only for traces of even powers of the matrix M. The expectation value of $\frac{1}{N}\mathrm{Tr}(MA)^{2q}$ is derived in a like manner. This time we substitute $\frac{1}{N}\mathrm{Tr}(MA)^{2q}$ into an earlier step in the derivation of the Itzykson–Di Francesco formula, namely into Eq. (2.3). Using (3.1) we see that this time we shift both $\chi_R(A)$ and $\chi_R(M)$, which leads to the final result

$$\frac{1}{N} \mathrm{Tr}((MA)^{2q}) = \frac{\lambda^q}{q} \oint \frac{dh}{2\pi i} h^q e^{q(H(h)+2F(h))} \,. \tag{3.8}$$

Again this is derived only for traces of the even powers of MA. We assume from here on that we are working with an even potential so that it is only the even traces that remain. Summing up Eqs. (3.7) and (3.8) over all q, assuming also that the potential is even, we arrive at two formulae for the two types of resolvents one can define for the original matrix model:

$$\begin{aligned} W(P) &= \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{P - M} \right\rangle = \frac{1}{P} - \frac{1}{P} \oint \frac{dh}{2\pi i} \ln(P^2 - \lambda h e^{H(h)}) \,, \\ W_A(P) &= \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{P - MA} \right\rangle = \frac{1}{P} - \frac{1}{P} \oint \frac{dh}{2\pi i} \ln(P^2 - \lambda h e^{H(h)+2F(h)}) \,. \end{aligned} \tag{3.9}$$

The first equation can be solved by Lagrange inversion since we know that the only singularity of $H(h)$ is the cut circled by the contour. Performing the inversion gives the very simple pair of equations

$$\begin{aligned} PW(P) &= \frac{P^2}{\lambda} - h\,, \\ \lambda h e^{H(h)} &= P^2\,. \end{aligned} \tag{3.10}$$

Knowing $H(h)$ we perform a functional inversion to obtain the resolvent of the original matrix model. We cannot in general do the same inversion for the resolvent $W_A(P)$ since we do not know in advance the singularities of $F(h)$.

4. The Gaussian Model: Straightening Random Loops

We will now check the power of our method on the simplest non-trivial case of our external field problem: We simply set $B = 0$, i.e. all $t_q = 0$ in Eq. (2.1). Now there is no potential at all and thus no weights are excited: The only contribution to Eq. (2.7) is the empty Young tableau:

$$H(h) = \ln \frac{h}{h-1}\,. \tag{4.1}$$

One immediately checks that the inversion formulae (3.10) correctly give the Wigner semi-circle law (we may set $\lambda = 1$ here)

$$W(P) = \frac{1}{2}(P - \sqrt{P^2 - 4})\,. \tag{4.2}$$

Clearly $W(P)$ and $H(h)$ are unchanged even in the presence of non-trivial coupling constants t_q^*, but now the interesting quantity is the resolvent $W_A(P)$:

$$W_A(P) = \frac{1}{Z}\int \mathscr{D}M\, e^{-\frac{N}{2}\mathrm{Tr}\, M^2}\frac{1}{N}\mathrm{Tr}\frac{1}{P - MA}\,. \tag{4.3}$$

We can find it by substituting $H(h)$ into (3.5),

$$\sum_{q=1}^{\infty} q t_q \omega^q = -\oint \frac{dh}{2\pi i} \ln[h - 1 - \omega\, h\, e^{F(h)}] - 1\,, \tag{4.4}$$

where we also summed up the moments constructing their generating function with an auxiliary variable ω. One now sees that in this case the Lagrange inversion can be performed by picking up a pole term *inside* the contour, giving immediately

$$h - 1 = \sum_{q=1}^{\infty} q t_q^* \omega^q \quad \text{and} \quad h - 1 = \omega\, h\, e^{F(h)}\,. \tag{4.5}$$

In addition, from the inversion formula (3.9) for $W_A(P)$ we find, using the same method,

$$h = PW_A(P) \quad \text{and} \quad h - 1 = \frac{1}{P^2} h^2 e^{2F(h)}\,. \tag{4.6}$$

Eliminating $F(h)$ we find the exact solution of our problem:

$$P^2\omega^2 = \sum_{q=1}^{\infty} q t_q^* \omega^q \quad \text{and} \quad PW_A(P) = 1 + P^2\omega^2\,. \tag{4.7}$$

Indeed, this set of functional equations determines, for any set of couplings t_q^*, after elimination of ω, the desired resolvent. Let us remark that these equations can alternatively be derived using Schwinger–Dyson techniques, yielding a non-trivial check of our functional methods. It is interesting to observe that we may obtain arbitrarily complicated resolvents by freely choosing the t_q^*'s. On the other hand, it is seen that a finite number of non-zero coupling constants always leads to an algebraic resolvent. An amusing toy system consists in only activating the first three coupling constants. Now the resolvent $W_A(P)$ is interpreted as the generating function of rainbow graphs with face-valency not larger than three, see Fig. 3 below. It is, from (4.7), given explicitly by (we have absorbed the factors q in the couplings)

$$W_A(P) = \frac{1}{P} + \frac{P}{2t_3^{*2}}\left[(P^2 - t_2^*)^2 - 2t_1^* t_3^* - (P^2 - t_2^*)\sqrt{(P^2 - t_2^*)^2 - 4t_1^* t_3^*}\right] . \quad (4.8)$$

It is interesting to investigate what happens in this toy system if we tune away the faces with negative boundary curvature, i.e. $t_3^* \to 0$:

$$W_A(P) = \frac{1}{P} + \frac{Pt_1^{*2}}{(P^2 - t_2^*)^2} . \quad (4.9)$$

Now we obtain merely the "cigar-like" diagrams below.

In fact, as it was argued in Appendix A of [12], the universal continuum limit (with string susceptibility $\gamma_{\rm str} = \frac{1}{2}$ due to the square-root singularity in Eq. (4.8)) of the model (4.8) can be interpreted as two-dimensional *topological* quantum gravity: The expectation value of the metric tensor is zero in the bulk, leading to a theory of quantum loops. "Flattening" in such a theory is thus just "straightening," and indeed the continuum limit of (4.9) (see the right half of Fig. 2) is simply a straightened loop with two curvature defects. Note that the cross-over from the "quantum" to the "straight" phase is simply a catastrophe in the algebraic sense: As soon as we turn on the negative curvature coupling t_3^*, the defects proliferate and the straight line disorders. It is not excluded that a similar rather trivial mechanism will govern the crossover from random to flat graphs. However, this is not the most likely scenario; indeed one is reminded of the fundamental difference between one and two-dimensional systems with regard to the absence, respectively presence, of phase transitions. At any rate, let us turn to real planar graphs.

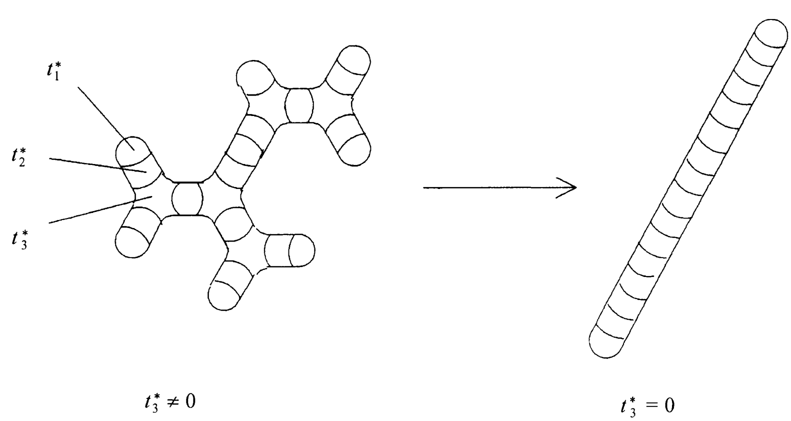


Fig. 3. "Rainbow"⟶ "cigar-like" diagrams in the gaussian model.

5. Saddlepoint Equations and Planar Graphs

Success was guaranteed in the case of the Gaussian model since we knew from the start the trivial, linear distribution of weights. We now have to establish that non-trivial distributions $H(h)$ can be found and that we are able to reproduce planar graphs from the character expansion. One finds that saddlepoint techniques may be successfully applied if certain precautions, to be elaborated below, are taken. Let us illustrate the method and its subtleties on a number of examples:

In our new language, the simplest model generating planar graphs turns out to be the case $A = 1$ and $B = \mathcal{J}$, where $\operatorname{Tr} \mathcal{J}^q$ equals one if q is even, and zero otherwise. Thus we obtain a traditional one matrix model with the "even-log" potential $-\frac{1}{2} \ln (1 - M^2)$, generating planar graphs with arbitrary even vertices. It is easy to explicitly work out – with the help of the Schur character formula (2.9) (see Appendix) – the characters for this case:

$$\chi_{\{h\}}(1) \sim \Delta(h) \quad \text{and} \quad \chi_{\{h\}}(\mathcal{J}) \sim \Delta(h^o)\Delta(h^e) \operatorname{sgn} \prod_{i,j} (h_i^e - h_j^o) . \tag{5.1}$$

From here on we will omit irrelevant numerical constants. One now sees that the character expansion (2.7) for the partition function becomes

$$Z \sim \lambda^{-\frac{1}{4}N(N-1)} \sum_{\{h^e, h^o\}} \prod_i (h_i^e - 1)!! h_i^o !! \Delta(h^o)^2 \Delta(h^e)^2 \left(\frac{\lambda}{N}\right)^{\frac{1}{2}\sum_i (h_i^e + h_i^o)} . \tag{5.2}$$

We thus observe that even and odd weights completely factorize! By symmetry, they should have the same statistical distribution. This partition sum is ideally suited for a saddlepoint analysis: the Vandermondes repel the weights from each other while the potential attracts – for small coupling λ – to the origin. It is therefore natural to write down, in the large N limit, the saddlepoint equation,

$$⨍ dh' \frac{\rho(h')}{h - h'} = -\frac{1}{2} \ln (h \lambda) , \tag{5.3}$$

obtained in the standard fashion from Eq. (5.2). The density $\rho(h)$ and the continuous variables h were defined in Sect. 3 and one also uses Sterling's formula: $\ln h!! \sim \frac{h}{2}(\ln(h) - 1)$. This equation is easily solved but leads to the *wrong* result. The phenomenon is identical to the one previously encountered in [7]: The naive saddle point equation fails to take into account the constraint $\rho(h) \leqq 1$ which follows from Eq. (2.5). Imposing the condition that the density is saturated at its maximum value $\rho(h) = 1$ on the interval $[0, b]$, we write down the modified saddlepoint equation,

$$⨍_b^a dh' \frac{\rho(h')}{h - h'} = -\frac{1}{2} \ln (h \lambda) - \ln \left(\frac{h}{h - b}\right) , \tag{5.4}$$

determining the non-trivial piece of the density on the interval $[b, a]$. We generate the full analytic function $H(h) = \int dh' \rho(h')/(h - h')$ from $⨍ dh' \rho(h')/(h - h')$ by performing the contour integral,

$$H(h) = \ln \left(\frac{h}{h - b}\right) + \sqrt{(h - a)(h - b)} \oint_C \frac{ds}{2\pi i} \frac{\frac{1}{2} \ln (s\lambda) + \ln(\frac{s}{s-b})}{(s - h)\sqrt{(s - a)(s - b)}} , \tag{5.5}$$

where the contour encircles the cut $[a,b]$. Inflating the contour and catching instead the cuts $[\infty,0]$ and $[0,b]$ we arrive at

$$H(h) = \ln\left[\frac{\sqrt{a}-\sqrt{b}}{\sqrt{\lambda}}\frac{h+\sqrt{ab}+\sqrt{(h-a)(h-b)}}{(a+b)h-2ab+2\sqrt{ab}\sqrt{(h-a)(h-b)}}\right], \tag{5.6}$$

with

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = \frac{1}{4\lambda}(1-\sqrt{1-8\lambda}),$$
$$\left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2 = \frac{1}{8\lambda}(1-4\lambda-\sqrt{1-8\lambda}), \tag{5.7}$$

the constants a and b being fixed by the condition that $H(h) = 1/h + O(1/h^2)$. This is the solution to the "even-log" matrix model in the language of highest weights, reproducing the correct critical coupling $\lambda_c = \frac{1}{8}$. Below we show a plot of $\rho(h)$ for various values of λ.

It is easy to check the correctness of this solution by perturbation theory. Even better, we can independently calculate the eigenvalue resolvent $W(P)$ of this model by traditional methods and demonstrate its exact coincidence with the resolvent obtained from the inversion formulae (3.10).

To demonstrate the power of our method we will next consider a case that is not a traditional one matrix model and thus has not yet been solved with other methods: Consider $A = \mathscr{J}$ and $B = \mathscr{J}$, i.e. planar graphs with even coordination numbers for vertices *and* faces. Here Eqs. (2.7) and (5.1) lead to

$$Z \sim \sum_{\{h^e,h^o\}} \prod_i (h_i^e-1)!!h_i^o!!\frac{\Delta(h^o)^2\Delta(h^e)^2}{\prod_{ij}(h_i^o-h_j^e)}\left(\frac{\lambda}{N}\right)^{-\frac{1}{4}N(N-1)+\frac{1}{2}\sum_i(h_i^e+h_i^o)}. \tag{5.8}$$

Here even and odd weights no longer factorize. However, it is natural to assume that they are equally distributed. Thus the crossproduct should precisely cancel one power of a Vandermonde, leading to the saddlepoint equation,

$$\fint_b^a dh'\frac{\rho(h')}{h-h'} = -\ln(h\lambda) - \ln\left(\frac{h}{h-b}\right). \tag{5.9}$$

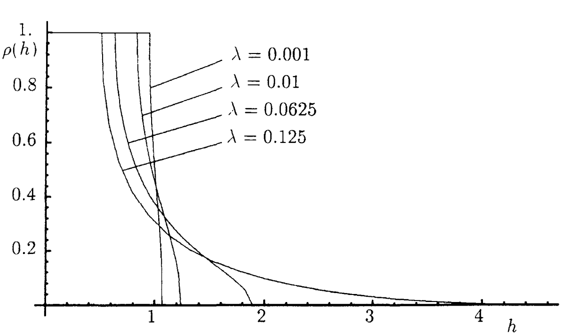


Fig. 4. Highest weight density $\rho(h)$ for potential $-\ln(1-M^2)$.

This equation is solved exactly as the previous case and one finds the weight resolvent,

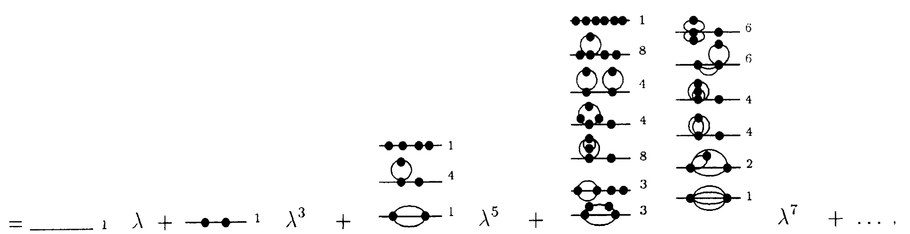
$$H(h) = \ln\left[\frac{(a-b)}{h\lambda(\sqrt{a}+\sqrt{b})^2}\,\frac{2h^2 - h(\sqrt{a}-\sqrt{b})^2 + 2ab + 2(h+\sqrt{ab})\sqrt{(h-a)(h-b)}}{(a+b)h - 2ab + 2\sqrt{ab}\sqrt{(h-a)(h-b)}}\right], \tag{5.10}$$

with the interval boundaries being determined through the quantities $\xi = (\frac{\sqrt{a}+\sqrt{b}}{4})^2$ and $\eta = (\frac{\sqrt{a}-\sqrt{b}}{4})^2$ by

$$3\lambda^2\xi^3 - \xi + 1 = 0 \quad \text{and} \quad \eta = \frac{1}{3}(\xi - 1)\,. \tag{5.11}$$

One easily finds the critical coupling to be $\lambda_c = \frac{2}{9}$. It is satisfying to observe that this is very slightly less than two times the value of the previous case; this is as expected since the asymptotic growth of the number of graphs with n edges is $\sim \lambda_c^{-n}$. It is straightforward to verify that this solution indeed correctly counts the graphs under consideration; e.g. from Eq. (5.10) with the help of (3.9):

$$\frac{1}{N}\mathrm{Tr}\,M^2 = \lambda + \lambda^3 + 6\lambda^5 + 54\lambda^7 + \cdots$$



(5.12)

where the dots correspond to insertions of the matrix $\mathcal{J}$.

These examples correspond to the ensembles of planar graphs simplest in the weight language. It is natural to ask for the description of the simplest original even model [11] of pure gravity, i.e. the one matrix model with the action $-\frac{1}{2}\mathrm{Tr}\,M^2 + \frac{\lambda}{4}\mathrm{Tr}\,M^4$. It is again straightforward to explicitly calculate the characters here (see Appendix); the weights are now grouped into four blocks $h_i^{(\varepsilon)}$, where $\varepsilon \in \{0,1,2,3\}$ denotes their congruence, modulo four. This leads to the expansion

$$Z \sim \lambda^{-\frac{N^2}{8}} \sum_{\{h^0,h^2\}} \Delta(h^{(0)})^2 \Delta(h^{(2)})^2 \prod_{i,j}(h_i^{(2)} - h_j^{(0)})\, e^{\sum_k^{\varepsilon=0,2} \frac{h_k^{(\varepsilon)}}{4}(\ln(\lambda h_k^{(\varepsilon)}/N)-1)}$$

$$\times \sum_{\{h^1,h^3\}} \Delta(h^{(1)})^2 \Delta(h^{(3)})^2 \prod_{i,j}(h_i^{(3)} - h_j^{(1)})\, e^{\sum_k^{\varepsilon=1,3} \frac{h_k^{(\varepsilon)}}{4}(\ln(\lambda h_k^{(\varepsilon)}/N)-1)}, \tag{5.13}$$

where for convenience we have substituted in Sterling's formula for the factorials. One observes that even and odd weights factorize, but not the congruence classes $(0,2)$ and $(1,3)$. In fact, each of the non-factorizing sectors has a structure identical to the case $A = B = 1$ (i.e. the one matrix model with action $-\frac{1}{2\lambda}\mathrm{Tr}\,M^2 -$

$\ln(1-M)$), since here the character expansion (2.7) gives, together with (5.1),

$$Z \sim \lambda^{-\frac{N^2}{4}} \sum_{\{h^e,h^o\}} \Delta(h^o)^2 \Delta(h^e)^2 \prod_{ij} (h_i^o - h_j^e)\, e^{\sum_k \frac{h_k}{2}(\ln(\lambda h_k/N)-1)} \,. \tag{5.14}$$

Here it is the even and odd weights that remain coupled. We see directly that the partition function in Eq. (5.13) is the square of the partition function in Eq. (5.14). We thus rediscover in the highest weight language the well known connection between these two models. At the diagrammatic level this can be seen by placing a vertex of the M^4 model at the midpoint of every edge of the $-\ln(1-M^2)$ diagrams so that the face centres of the M^4 model are the vertices and face centres of the $-\ln(1-M^2)$ model. It is tempting to make a saddlepoint ansatz like in (5.4), multiplying in this equation the principal part integral by an extra factor of $\frac{3}{2}$ due to the variation of the extra factor $\prod_{ij}(h_i^o - h_j^e)$. However, here the solution of this equation does *not* lead to the correct result. We can gain some insight into this failure by computing, by the usual means, the eigenvalue resolvent $W(P)$ and then deducing $H(h)$ from (3.10). The result of this calculation leads to a third order algebraic equation for $e^{H(h)}$:

$$H(h) = \ln\left(\frac{X(h)}{h}\right) , \tag{5.15}$$

with $X(h)$ defined through the solution of

$$\lambda X^3 - \lambda(1+h)X^2 + \left(\frac{8}{9} - h + \gamma(\lambda)\right) X + h^2 = 0 \,, \tag{5.16}$$

with $\gamma(\lambda) = \frac{1}{54}\frac{(1-\sqrt{1-12\lambda})(1-12\lambda)}{12\lambda}$. One then finds on inspecting $H(h)$ that the saddlepoint configuration of weights is *complex*: The rapid sign-changes of the product $\prod_{ij}(h_i^o - h_j^e)$ destabilize the reality of the saddlepoint. It is worth pointing out that the saddle point nevertheless *exists*, even though it is much harder to find. As we have seen in the previous example, the presence of this factor in the denominator of the expansion is however without danger. A rough intuitive "explanation" is that in the numerator the product acts to repulse the different distributions, destabilizing the saddle point, whereas in the denominator it attracts and stabilizes. We will see in the next section that the stability of the saddlepoint can be preserved in the case of greatest physical interest: the gradual flattening of the random surface.

6. Flattening Random Lattices

Before we flatten our surface it is worth understanding how the flat lattice is represented in the language of highest weights. In this case $\frac{1}{N}\mathrm{Tr}\,A^q = \frac{1}{N}\mathrm{Tr}\,B^q = \delta_{q,4}$ and it is simple to derive the characters from Eq. (2.9) (see Appendix), to obtain the partition function

$$Z \sim \lambda^{-\frac{N^2}{4}} \sum_{\{h^0,h^1,h^2,h^3\}} \frac{\Delta(h^{(0)})^2 \Delta(h^{(1)})^2 \Delta(h^{(2)})^2 \Delta(h^{(3)})^2\, e^{\sum_{k,\varepsilon} \frac{1}{2} h_k^{(\varepsilon)} \ln(\lambda)}}{\prod_{i,j}(h_i^{(1)} - h_j^{(0)})(h_i^{(3)} - h_j^{(0)})(h_i^{(1)} - h_j^{(2)})(h_i^{(3)} - h_j^{(2)})} \,, \tag{6.1}$$

where again we have substituted in Sterling's formula for the factorials. The potential term, $e^{\sum_{k,\varepsilon} \frac{1}{2} h_k^{(\varepsilon)} \ln(\lambda)}$, attracts to the origin for $\lambda < 1$, and repulses and is unstable for $\lambda > 1$. The critical point is therefore now $\lambda_c = 1$. The repulsion of the Vandermondes in the numerator is now precisely balanced by the attractive effect of the products in the denominator. Indeed, using our rule of thumb that the variation of a product $\prod_{i,j}(h_i^{(\varepsilon_2)} - h_j^{(\varepsilon_1)})$ in the denominator precisely cancels the variation of one power of a Vandermonde, we arrive at the trivial saddle point equation

$$-\frac{1}{2}\ln(\lambda) = 0\,. \tag{6.2}$$

We observe that the saddle point equation is not satisfied anywhere (except possibly for λ_c) i.e. none of the weights are excited and, as in the gaussian case, only the empty Young tableau contributes:

$$H(h) = \ln\frac{h}{h-1}\,. \tag{6.3}$$

To order N^2, only the original gaussian term can contribute to the partition function and expectation values of $\mathrm{Tr}M^q$. The potential term cannot contribute, since it is impossible for all loops to have 4 A matrices running around them. This is simply the statement of the fact that it is impossible to put a completely flat lattice on the sphere. Positive curvature defects have to be introduced to close the surface.

Since the order N^2 contribution is trivially zero, it is interesting to investigate the first non-trivial order: the order N^0 contribution. We will give some qualitative arguments that the flat case is described by N fermions with an equidistant spectrum and, from this starting point, calculate the number of flat graphs in the first 2 orders in $1/N$.

One may notice that, in the large N limit, all the factorials and products of differences of various highest weights cancel (assuming all 4 groups of h's to be distributed in the same way). One can hypothesize that this will be true even for the next order in $1/N^2$ which describes the graphs with the topology of a torus (we will check this assumption below). We are then left with the partition function of fermionic type:

$$Z(\lambda) = \lambda^{N(N-1)/2} \sum_{h_1 > h_2 > \cdots > h_N} \lambda^{\Sigma h}\,. \tag{6.4}$$

A standard calculation for the free energy $f(\lambda) = 1/N^2 \ln Z(\lambda)$ gives:

$$\begin{aligned} f(\lambda) = -\sum_{n=1}^{N} \ln(1-\lambda^n) &= -N^{-2}\sum_{n=1}^{\infty}\ln(1-\lambda^n) + O(N^{-4}) \\ &= N^{-2}\lambda^{-1/24}\eta(\lambda) + O(N^{-4})\,, \end{aligned} \tag{6.5}$$

where $\eta(\lambda)$ is the Dedekind function. The order N^2 contribution, which counts the number of flat graphs with spherical topology, is zero since, as already stated above, no such graphs exist, at least not without defects. The next order counts the number of flat graphs that can be fitted onto a torus. It is easy to verify this calculation by directly counting the number of possible graphs. A general graph on the torus consists of $m \times n$ squares glued into a rectangle (m columns and n rows). Opposite sides are then glued together: first we glue together the two sides of the length n and then, with m possible twists, the two sides of length m. The symmetry factor is $1/(mn)$. We obtain:

$$f(\lambda) = \sum_{n,m=1}^{\infty}\frac{m}{mn}\lambda^{mn} = -\sum_{n=1}^{\infty}\ln(1-\lambda^n)\,, \tag{6.6}$$

which coincides exactly with our result obtained from the sum over highest weights treated as free fermions. Since completely flat graphs exist only for the torus topology, we can be sure that there exist no higher order contributions. Let us note that this calculation on the torus is similar to what we would have for QCD2 on a toroidal target space [13].

We conclude that, in the limit of flat graphs, the highest weights play the role of the energy levels of N free fermions. This fact may be useful for the understanding of the mechanism of the flattening phase transition.

We are now in a position to discuss the real problem of interest. We have observed that, for models (5.8) and (6.1), where both the vertex and face coordination numbers are even, the simple saddle point equation is valid. In fact, all the simple models with even face and vertex coordination numbers which we have investigated (for brevity we do not discuss them here) can be solved correctly with the simple saddle point formulation. We thus hypothesize that if we restrict our attention to the "even-even" models (models where the coordination numbers for both faces and vertices are even), the destabilization of the saddle point discussed at the end of Sect. 5 will be avoided and we can correctly interpolate all the way from the $A = B = \mathscr{I}$ case (5.8) to the flat lattice (6.1) using the saddle point approximation. Any results we obtain can be checked against perturbation theory to verify this assumption. For the interpolating model we thus choose

$$A = B = \begin{bmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{bmatrix} , \tag{6.7}$$

so that all odd traces equal zero, calculate the character from (2.4) (see Appendix)

$$\chi_{\{h\}}(A) = \chi_{\{h\}}(B) = \chi_{\{\frac{h^e}{2}\}}(b)\chi_{\{\frac{h^o-1}{2}\}}(b)\,\mathrm{sgn}\left[\prod_{i,j}(h_i^e - h_j^o)\right] , \tag{6.8}$$

and arrive at the partition function

$$Z \sim \sum_{\{h^e,h^o\}} \prod_i (h_i^e - 1)!!h_i^o!! \frac{\Delta(h^o)^2\Delta(h^e)^2}{\prod_{ij}(h_i^o - h_j^e)} \left(\frac{\lambda}{N}\right)^{\frac{1}{4}N(N+1)+\frac{1}{2}\sum_i(h_i^e+h_i^o)}$$
$$\times I\left(\frac{h^e}{2},b\right)^2 I\left(\frac{h^o-1}{2},b\right)^2 , \tag{6.9}$$

where $I(\frac{h^e}{2},b) = \chi_{\{\frac{h^e}{2}\}}(b)\Delta(b)/(\Delta(\beta)\Delta(\frac{h^e}{2}))$ is an Itzykson–Zuber integral, with $b = e^\beta$ and $I(\frac{h^o-1}{2},b)$ is defined similarly. The partition function is a generalisation of (5.8), the difference being the two Itzykson–Zuber integrals. Indeed, setting $b = 1$ we recover (5.8). For convenience we introduce the notation $\tilde{t}_q = \frac{2}{qN}\mathrm{Tr}\, b^q$ so that $\tilde{t}_q = t_{2q} = t_{2q}^*$. In this notation, flattening of the lattice corresponds to setting $\tilde{t}_2 \neq 0$ and $\tilde{t}_q = 0$ for $q \neq 2$. In fact, in complete analogy with the "rainbow" $\to$ "cigar-like" transition in the gaussian model, it is only necessary to set $\tilde{t}_q = 0$ for $q > 2$ to flatten the lattice. For convenience, we define $F(h_k^e) = 2\frac{\partial}{\partial h_k^e}\ln I(\frac{h^e}{2},b)$. It is then easy to derive the following pair of equations which in principle completely describe the model:

$$q\tilde{t}_q = \frac{2}{N}\mathrm{Tr}\, b^q = \frac{1}{q}\oint \frac{dh}{2\pi i} e^{q(H(h)+F(h))}\,, \tag{6.10}$$

$$\fint_b^a dh' \frac{\rho(h')}{h-h'} = -\ln(\lambda h) - \ln\frac{h}{h-b} - 2F(h)\,. \tag{6.11}$$

The first is derived just as (3.5) and the second is the saddle point equation. The whole complexity of the random to flat transition is succinctly encapsulated in these two equations. As in the toy gaussian model, we can capture the transition by turning on the first three coupling constants $\tilde{t}_q$. As it is a kind of Riemann–Hilbert problem, the highly non-trivial task is as always to deduce the analytical structure of the solution.

7. Conclusions and Discussion

In this paper, we developed a character expansion technique for a new kind of matrix model describing dually weighted planar graphs. We first reduced the N^2 degrees of freedom of the original matrix to the N highest weights specifying the irreducible representations of $U(N)$. This allowed us, with some precautions, to apply the saddle point method in the large N limit, reducing the problem to a set of integral equations.

We showed how to solve these integral equations in a number of known 1-matrix models and also found a new result: we are able to calculate the number of graphs having only even numbers of neighbours for both original and dual vertices.

We also understood the limiting case of the completely flat lattice. It is described by a system of free fermions (the highest weights) with an equidistant spectrum. This allowed us to reproduce correctly the partition function of regular graphs with toric topology (the only genus that can be realized from a completely flat graph).

The most important physical question still to be addressed is the description of the transition from completely random planar graphs (describing the 2D gravity) to the regular (flat, in our terminology) lattice.

We have not yet solved the corresponding integral equations. It is not an easy task as they are equivalent to a complicated Riemann–Hilbert problem, and the solution involves some non-trivial deductions about the analytical structure of the underlying functions.

Let us speculate possible physical pictures for the flattening phase transition. There are three different scenarios to consider:

1. The flattening could take place for a finite effective coupling constant in front of the R^2 (curvature squared) term in the 2D gravity action. This means that the characteristic flat size of an almost flat piece of a graph diverges at some finite critical coupling. This would be the most interesting scenario as it would mean the discovery of a completely new universal critical phenomenon.

An argument against this picture is the fact that the R^2 coupling is dimensionful, thus containing inverse powers of the cut-off. This is however a completely perturbative argument since the same reasoning could be applied to the 3D Ising model described as ϕ^4 scalar field theory. Here, the interaction term is also dimensionful, but it nevertheless exhibits a non-perturbative phase transition.

A more serious objection to a transition at finite effective coupling is the presence of macroscopic excitations on the background of a regular lattice generated by only a very small number of curvature defects. For example, on the square lattice, the introduction of four vertices of coordination number 6 and four of coordination number 2 allows a baby universe of arbitrary size to grow out from the flat lattice. This is in complete analogy with the tree-like structures studied in the toy gaussian model of Sect. 4, where the introduction of just one t_3^* defect and one t_1^* defect is enough to create a whole new branch.

2. The flattening could appear only in the limit of the infinite R^2 coupling. In the light of the previous arguments this is the most likely scenario. Another argument was given in [14] using the methods of Liouville theory, though again the argument was completely perturbative and thus unsatisfactory. This hypothesis also seems to be in agreement with numerical simulations [15]. Even if this scenario turns out to be correct, the asymptotic approach to the flat lattice could contain some interesting scaling behaviour and is worthy of study.

3. The third scenario is an extended version of the second one: we could have a flattening phenomenon for the R^2 coupling of the order of $\Lambda^2_{\mathrm{cutoff}}$. It will of course depend upon the type of regular lattice, and the critical exponents, if any, will be dependent on the symmetry of the lattice; triangular or square. In this case the phase transition would be better identified as a crystalization transition. The universal critical properties would then depend on the symmetry of the "crystal."

One could think of an analogy with the Berezinski–Kosterlitz–Thouless phase transition: the discrete curvatures on the lattice could be identified with quantized coulomb charges in two dimensions. This is indeed the picture in the conformal gauge of the continuous 2D gravity action.

Let us also make some technical comments. The character expansion method can be successfully applied to a more general type of matrix model, having the following action:

$$S(M) = \mathrm{tr}(W(M) + V(AM)), \tag{7.1}$$

instead of just $W(M) = M^2$ as in the present paper. The character expansion coefficients are more complicated as above, but there are no real obstacles: we still deal with only N highest weights as the principal degrees of freedom.

One can imagine the solution of a generalized 2-matrix model, such as

$$Z = \int d^{N^2}L\, d^{N^2}M \exp(\mathrm{Tr}(V(L) + V(M) + W(LM))), \tag{7.2}$$

with an arbitrary function $W(x)$, not just a merely linear one, or even a similar multi-matrix chain.

The most ambitious step would be to put "matter" on a random, but gradually flattening, lattice. For example, one can consider the matrix action:

$$S(M) = \mathrm{tr}(W(AL) + V(AM) + L^2 + M^2 + cLM). \tag{7.3}$$

It describes the Ising model on DWG. This model provides the interpolation between two solvable cases: the Onsager solution for a regular lattice and the Ising model on dynamical random graphs [16]. If $W \neq V$ it includes the non-zero magnetic field, which is still an unsolved case for the regular lattice. Unfortunately, the character expansion would be a much more complicated object in this case as it would contain some non-trivial Clebsch–Gordan coefficients.

In any case, we believe that the proposed approach could be fruitful for attacking many new combinatorial problems in 2D statistical mechanics and field theory.

8. Appendix

Below we present explicit formulae for some characters derived from the definitions in Sect. 2.

8.1. $A = 1$, the unit matrix. In this case the character is just the dimension of the representation. The easiest way to derive this is to take the limit as $\varepsilon \to 0$ of the character formula, Eq. (2.4) with $a_k = e^{k\varepsilon}$. In this case

$$\chi_{\{h\}}(1) = \lim_{\varepsilon \to 0} \frac{\Delta((e^{\varepsilon h_l})^k)}{\Delta(e^{\varepsilon k})} = c\Delta(h_i)\,, \tag{8.1}$$

where c is the numerical constant $c = \prod_{i=1}^{N-1} i!$.

8.2. A_m, defined by $\frac{1}{N}\operatorname{Tr}(A_m^k) = \delta_{k,m}$. The traces of all positive powers of A_m are zero except $(A_m)^m$. We will sketch the derivation for the case $m = 2$. It is easy to generalize the derivation for arbitrary m. Using the second definition for the character, Eqs. (2.8), (2.9), we have

$$P_k = \begin{cases} \frac{1}{(k/2)!}\left(\frac{N}{2}\right)^{k/2} & k \text{ even and non-negative;} \\ 0 & \text{otherwise} \end{cases}\,. \tag{8.2}$$

If we substitute this into the determinant, we obtain a matrix structure in which every other entry in a row is zero. By interchanging rows and columns, the determinant can be put into block diagonal form, with one block for the h^e's and the other for the h^o's. The powers of $N/2$ factor out, and if we then factor out the product $(\prod_i(\frac{h_i^e}{2})!(\frac{h_i^o-1}{2})!)^{-1}$, the entries in the diagonal blocks become polynomials of ascending order in h^e or h^o, and the block determinants reduce to Vandermonde determinants. Taking into account all of the sign changes from reordering the rows and columns we obtain:

$$\chi_{\{h\}}(A_2) = c\left(\frac{N}{2}\right)^{\frac{1}{2}\sum_i h_i} \frac{\Delta(h^e)\Delta(h^o)}{\prod_i(\frac{h_i^e}{2})!(\frac{h_i^o-1}{2})!} \operatorname{sgn}\left[\prod_{i,j}(h_i^e - h_j^o)\right]\,, \tag{8.3}$$

where c is a numerical constant. For general m we have:

$$\chi_{\{h\}}(A_m) = c\left(\frac{N}{m}\right)^{\frac{1}{m}\sum_i h_i} \prod_{\varepsilon=0}^{m-1} \frac{\Delta(h^{(\varepsilon)})}{\prod_i(\frac{h_i^{(\varepsilon)}-\varepsilon}{m})!} \operatorname{sgn}\left[\prod_{0 \leqq \varepsilon_1 < \varepsilon_2 \leqq (m-1)} \prod_{i,j}(h_i^{(\varepsilon_2)} - h_j^{(\varepsilon_1)})\right]\,. \tag{8.4}$$

In this case, the integers h factor into m groups of $\frac{N}{m}$ integers $h^{(\varepsilon)}$ with $\varepsilon = 0, 1, \ldots, (m-1)$ denoting their congruence modulo m.

8.3. A_{bb} defined by $\operatorname{Tr}(A_{bb})^q = 0$ for odd q. Only the even powers of A_{bb} are non-zero. The matrix A_{bb} can be defined by an $\frac{N}{2}$ by $\frac{N}{2}$ matrix, b, (eigenvalues b_k) as follows:

$$A_{bb} = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}\,. \tag{8.5}$$

Again we just sketch the derivation. Calculating the determinant in Eq. (2.4) and rearranging the columns we notice that we can write it (up to some sign factors from the interchanges) as

$$\det_{(k,l)}(a_k^{h_l}) = \begin{vmatrix} b^{h^e} & b^{h^o} \\ b^{h^e} & -b^{h^o} \end{vmatrix} = (-2)^{\frac{N}{2}} \begin{vmatrix} b^{h^e} & 0 \\ 0 & b^{h^o} \end{vmatrix}, \tag{8.6}$$

where, for notational convenience, we denote by b^{h^e} the $\frac{N}{2}$ by $\frac{N}{2}$ matrix whose elements are $b_i^{h_j^e}$. The Vandermonde in Eq. (2.4) is just the special case of the above result, (8.6), with $h_j^e = 2j - 2$ and $h_l^o = 2l - 1$. Including the sign factors neglected earlier, we arrive at the formula

$$\chi_{\{h\}}(A_{bb}) = \chi_{\{\frac{h^e}{2}\}}(b^2)\chi_{\{\frac{h^o-1}{2}\}}(b^2)\,\mathrm{sgn}\left[\prod_{i,j}(h_i^e - h_j^o)\right]. \tag{8.7}$$

To obtain the character for the $\mathscr{J}$ matrix introduced in Sect. 5, we set $b = 1$ and obtain the expression given in (5.1).

It is easy to generalize to higher order cases. For example, to study the case where only every third power of the matrix A_{bbb} has a non-zero trace, we start with an $\frac{N}{3}$ by $\frac{N}{3}$ matrix, b, and define

$$A_{bbb} = \begin{bmatrix} b & 0 & 0 \\ 0 & \omega b & 0 \\ 0 & 0 & \omega^2 b \end{bmatrix}, \tag{8.8}$$

where ω is the third root of unity. This time the character factors into three characters, one for each of the congruence classes, modulo three, of h.

Acknowledgements. We would like to thank E. Brézin, J.-M. Daul, M. Douglas, A. Matytsin, A. Migdal, I. Kostov, D. Kutasov, and A. Zamolodchikov for many useful discussions. We would also especially like to thank Ph. Di Francesco and C. Itzykson for their enthusiastic interest during the early stages of this work.

References

1. Knizhnik, V.G., Polyakov, A.M., Zamolodchikov, A.B.: Mod. Phys. Lett. **A3**, 819 (1988); David, F.: Mod. Phys. Lett. **A3**, 1651 (1988); Distler, J., Kawai, H.: Nucl. Phys. **B321**, 509 (1989)
2. Douglas, M.R.: Phys. Lett. **B348**, 507 (1991)
3. Kazakov, V.A., Kostov, I.K.: Nucl. Phys. **B386**, 520 (1992)
4. Di Francesco, P., Itzykson, C.: Ann. Inst. Henri. Poincaré **59**, no. 2, 117 (1993)
5. Kazakov, V.A.: Nucl. Phys. **B354**, 614 (1991)
6. Kazakov, V.A., Wynter, T.: Preprint LPTENS-94/28 (1994)
7. Douglas, M.R., Kazakov, V.A.: Phys. Lett. **319B**, 219 (1993)
8. Das, S.R., Dhar, A., Sengupta, A.M., Wadia, S.R.: Mod. Phys. Lett. **A5**, 1041 (1990)
9. Migdal, A.A.: Mod. Phys. Lett. **A8**, No. 4, 359–371 (1993)
10. Matytsin, A.: Nucl. Phys. **B411**, 805 (1994)
11. Brézin, E., Itzykson, C., Parisi, G., Zuber, J.B.: Commun. Math. Phys. **59**, 35 (1978)
12. Moore, G., Seiberg, N., Staudacher, M.: Nucl. Phys. **B362**, 665 (1991)
13. Gross, D., Taylor, W.: Nucl. Phys. **B400**, 181 (1993)
14. Kawai, H., Nakayama, R.: Phys. Lett. **B306**, 224 (1993)
15. Boulatov, D.V., Kazakov, V.A.: Phys. Lett. **B184**, 247 (1986)
16. Kazakov, V.A.: Phys. Lett. **119A**, 140 (1986)

Communicated by R.H. Dijkgraaf