

The Momentum Constraints of General Relativity and Spatial Conformal Isometries

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Abstract: Transverse-tracefree (TT-) tensors on (\mathbf{R}^3, g_{ab}) , with g_{ab} an asymptotically flat metric of fast decay at infinity, are studied. When the source tensor from which these TT tensors are constructed has fast fall-off at infinity, TT tensors allow a multipole-type expansion. When g_{ab} has no conformal Killing vectors (CKV's) it is proven that any finite but otherwise arbitrary set of moments can be realized by a suitable TT tensor. When CKV's exist there are obstructions—certain (combinations of) moments have to vanish—which we study.

1. Introduction

In this paper we consider transverse-tracefree (TT-) tensors on \mathbf{R}^3 with an asymptotically flat metric g_{ab} , i.e. tensors P_{ab} satisfying

$$D^a P_{ab} = 0, \quad \text{trace } P = 0 \text{ on } (\mathbf{R}^3, g_{ab}), \quad (1.1)$$

where D is the covariant derivative associated with g . The interest in this problem comes first of all from (vacuum) general relativity, where Eq. (1.1) is the momentum constraint for an initial data set $(\mathbf{R}^3, g_{ab}, P_{ab})$

$$D^a(P_{ab} - g_{ab} \text{ trace } P) = 0 \quad (1.2)$$

in the maximal (i.e. trace $P = 0$) case. As is well-known, Eq. (1.2) is just the expression of the invariance of the theory under diffeomorphisms of three space. Thus our study of Eq. (1.1) is relevant to a much larger class of theories than Einstein's.

In the standard conformal approach to solving the constraints [13], Eq. (1.1) is not solved on the physical metric g_{ab} , but a conformally related metric g'_{ab} having faster decay at infinity than g_{ab} . One is here using the fact that P_{ab} being TT is

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invariant under $g'_{ab} = \omega^2 g_{ab}$, $P'_{ab} = \omega^{-1} P_{ab}$, $\omega > 0$. We call g'_{ab} , P'_{ab} again g_{ab} , P_{ab} . Our assumptions on g_{ab} are that g_{ab} is smooth and, in standard coordinates x^a on \mathbf{R}^3 , satisfies

$$g_{ab} - \delta_{ab} = O^\infty \left(\frac{1}{r^{K-1+\varepsilon}} \right), \quad 0 < \varepsilon < 1 \tag{1.3}$$

for some $K = 1, 2, \dots$, where $r = (x^a x^b \delta_{ab})^{1/2}$ and $F = O^\infty(f(r))$ means that $F = O(|f(r)|)$, $\partial F = O(|f'(r)|)$, $\partial \partial F = O(|f''(r)|)$, a.s.o. In addition we require a condition of conformal smoothness for g_{ab} , as follows: there are functions

$$f^a(x) = O^\infty \left(\frac{1}{r^{K-2+\varepsilon}} \right),$$

such that, with $\tilde{x}^a = x^a + f^a(x)$, $\Omega^{-1} = \delta_{ab} \tilde{x}^a \tilde{x}^b$, the tensor field $\tilde{g}_{ab} = \Omega^2 g_{ab}$ admits a smooth extension in coordinates $\tilde{x}^a = x^a/\Omega$ to $\tilde{x}^a = 0$. For example, these assumptions will be valid for all K when g_{ab} equals the flat metric outside a compact subset of \mathbf{R}^3 . For P_{ab} we require that

$$P_{ab} = O^\infty \left(\frac{1}{r^2} \right). \tag{1.4}$$

We shall impose one more condition on P_{ab} which arises as follows. Any smooth, trace-free tensor Q_{ab} satisfying (1.4) can be written as (York [13] and Chaljub–Simon [3])

$$Q_{ab} = P_{ab} + (LW)_{ab}, \tag{1.5}$$

where

$$(LW)_{ab} := D_a W_b + D_b W_a - \frac{2}{3} g_{ab} D_c W^c, \tag{1.6}$$

i.e. the conformal Killing operator associated with the vector field W^a satisfying

$$W_a = O^\infty \left(\frac{1}{r} \right) \tag{1.7}$$

and P_{ab} being TT . Thus

$$D^b(LW)_{ab} = \Delta W_a + \frac{1}{3} D_a(D^b W_b) + \mathcal{R}_a^b W_b = D^b Q_{ab}, \tag{1.8}$$

where \mathcal{R}_a^b is the Ricci tensor of g_{ab} . Given Q_{ab} , W_a and whence P_{ab} is unique. Thus the decomposition (1.5) can be used to find TT -tensors and, clearly, all TT -tensors arise this way (just take $Q_{ab} = P_{ab}$, $W_a = 0!$). We call Q_{ab} a “source tensor” for P_{ab} . It now seems natural to restrict P_{ab} further by imposing asymptotic conditions on the source tensor from which it arises. We assume

$$Q_{ab} = O^\infty \left(\frac{1}{r^{1+K+\varepsilon}} \right), \tag{1.9}$$

where ε and K are the same numbers as the ones appearing in (1.3). As K increases we shall obtain more detailed information on the multipole behaviour of W_a , and whence P_{ab} , near infinity. We will then ask and answer the question whether, given arbitrary values for the relevant multipole moments, whose number depends on K , a source tensor Q_{ab} satisfying (1.9) can be found, yielding a P_{ab} having precisely these moments. Since the map sending Q_{ab} to P_{ab} is many-to-one: Q_{ab} and $Q_{ab} + (Ls)_{ab}$ for any s_a satisfying $s_a = O^\infty(1/r^{1+K+\varepsilon})$ give the same TT -tensor,

one wonders what, if anything, condition (1.9) means in terms of P_{ab} . The answer is that (1.3,5,7,9) imply

$$\Delta D_{[a}P_{b]c} = O^\infty \left(\frac{1}{r^{4+K+\varepsilon}} \right), \tag{1.10}$$

and we state without proof that the converse also holds. (The appearance of third derivatives in Eq. (1.10) is no accident. If we viewed P_{ab} as a linearization of the metric g_{ab} , the left-hand side in (1.10) is essentially the linearization of the Cotton tensor, whose vanishing is equivalent to conformal flatness of a metric. The connection between TT-tensors and the linearized Cotton tensor will be further studied in forthcoming work by one of us (R.B.) Conditions such as (1.9), while natural from the viewpoint of the York decomposition (1.5), do not have an obvious physical interpretation. As a model one could look at the Gauß constraint of electrodynamics

$$D^a E_a = 0 \quad \text{on } (\mathbf{R}^3, \delta_{ab}) \tag{1.11}$$

with the ansatz

$$E_a = q_a + D_a \phi, \tag{1.12}$$

where q_a has fast fall-off, say compact support. Thus $D_{[a}E_{b]}$ has compact support too. If the same assumption is made for the magnetic fields B_a one would find that (E_a, B_a) are data for an electromagnetic field which is stationary in the domain of dependence of a neighbourhood of infinity. Presumably, in some approximate sense, a similar interpretation could be given for (1.9), when supplemented by some additional conditions on the metric g_{ab} (see e.g. Reula [10]).

2. The Asymptotic Expansion

Let, first, g_{ab} be the flat metric δ_{ab} on \mathbf{R}^3 and consider the elliptic equation

$$\partial^2 W_a + \frac{1}{3} \partial_a (\partial^b W_b) = j_a, \tag{2.1}$$

where j_a is smooth and $j_a = O^\infty(1/(r^{2+K+\varepsilon}))$, $K = 1, 2, \dots$. The unique solution W_a to (2.1) going to zero at infinity is given by

$$W_a(x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} F_{ab}(x-x') j^b(x') d^3x', \tag{2.2}$$

where

$$F_{ab}(x) = \frac{1}{8} \left(7 \frac{\delta_{ab}}{r} + \frac{x_a x_b}{r^3} \right). \tag{2.3}$$

It is straightforward to see from (2.1, 2, 3) that $W_a(x)$ admits an expansion

$$W_a^K(x) = \sum_{k=1}^K \frac{\omega_a^k(n)}{r^k} + O^\infty \left(\frac{1}{r^{K+\varepsilon}} \right), \tag{2.4}$$

where ω_a^k are smooth on S^2 ($n^a =: x^a/r$). Instead of computing the vectors ω_a^k in terms of j_a directly from (2.2,3) we prefer an apparently more roundabout but actually more efficient way, as follows. The vectors ω_a^k can be decomposed into parts orthogonal and tangential to S^2 , i.e.

$$\omega_a^k = n_a \sigma^k + \mu_a^k. \tag{2.5}$$

The tangential parts $\overset{k}{\mu}_a$, in turn, can be expanded as

$$\overset{k}{\mu}_a = \nabla_a \overset{k}{\varphi} + \varepsilon_a^b \nabla_b \overset{k}{\psi}, \tag{2.6}$$

where ∇ is the derivative on S^2 and ε_{ab} the volume element on S^2 . The scalars $\overset{k}{\sigma}$, $\overset{k}{\varphi}$ and $\overset{k}{\psi}$ can now be expanded in terms of spherical harmonics, e.g.

$$\overset{k}{\varphi} = \sum_{l=0}^{\infty} \overset{k}{m}_{a_1 \dots a_l} n^{a_1} \dots n^{a_l}, \tag{2.7}$$

where $\overset{k}{m}_{a_1 \dots a_l}$ are symmetric, trace-free tensors. Putting all this back into $\overset{k}{\omega}_a$ we see that

$$\begin{aligned} \overset{k}{\omega}_a = & \sum_{l=0}^{\infty} (n_a \overset{k}{M}_{b_1 \dots b_l} n^{b_1} \dots n^{b_l} + \overset{k}{O}_{ab_1 \dots b_l} n^{b_1} \dots n^{b_l} \\ & + \varepsilon_a^{bc} n_b \overset{k}{L}_{cb_1 \dots b_l} n^{b_1} \dots n^{b_l}), \quad 1 \leq k \leq K, \end{aligned} \tag{2.8}$$

where all of $\overset{k}{M}$, $\overset{k}{O}$, $\overset{k}{L}$ are symmetric and trace-free. We now insert (2.4, 8) into (2.1). It follows that the first term in Eq. (2.4) has to satisfy (2.1) with $j_a = 0$. This results in a coupling between the number k in (2.8) and the l -values which can give a contribution. More precisely, we find after a straightforward computation that $\overset{k}{L}_{ab_1 \dots b_l} = 0$, except for $k = l + 2$. We also find that $\overset{k}{M}_{b_1 \dots b_l}$ and $\overset{k}{O}_{ab_1 \dots b_l}$ both = 0 except for $k = l$ in which case

$$(8 - k)\overset{k}{M}_{b_1 \dots b_k} - (2k - 1)\overset{k}{O}_{b_1 \dots b_k} = 0, \quad k \geq 1 \tag{2.9}$$

or $k = l + 2$ in which case

$$(k - 2)\overset{k}{M}_{b_1 \dots b_{k-2}} + (2k - 3)\overset{k}{O}_{b_1 \dots b_{k-2}} = 0. \tag{2.10}$$

Thus $\overset{K}{W}_a$ can be written as a sum of three terms plus a remainder, i.e.

$$\overset{K}{W}_a = \overset{(1)}{W}_a + \overset{(2)}{W}_a + \overset{(3)}{W}_a + O^\infty \left(\frac{1}{r^{K+\varepsilon}} \right), \tag{2.11}$$

where

$$\overset{(1,K)}{W}_a = \sum_{k=2}^K \frac{\varepsilon_a^{bc} n_b \overset{k}{L}_{cb_1 \dots b_{k-2}} n^{b_1} \dots n^{b_{k-2}}}{r^k}, \tag{2.12}$$

$$\overset{(2,K)}{W}_a = \sum_{k=1}^K \frac{(2k - 1)n_a \overset{k}{M}_{b_1 \dots b_k} n^{b_1} \dots n^{b_k} + (8 - k)\overset{k}{M}_{ab_1 \dots b_{k-1}} n^{b_1} \dots n^{b_{k-1}}}{r^k}, \tag{2.13}$$

$$\overset{(3,K)}{W}_a = \sum_{k=2}^K \frac{(2k - 3)n_a \overset{k}{N}_{b_1 \dots b_{k-2}} n^{b_1} \dots n^{b_{k-2}} - (k - 2)\overset{k}{N}_{ab_1 \dots b_{k-3}} n^{b_1} \dots n^{b_{k-3}}}{r^k}. \tag{2.14}$$

We now show, using a simple bootstrap argument, that the estimates (2.11–14) remain valid, when the l.h. side of (2.1) is replaced by (1.8), where the flat metric is replaced by one satisfying (1.3) and all of (2.12,13,14) is understood with respect to the flat background metric at infinity.

Theorem 1. *Let W_a be a solution of*

$$\Delta W_a + \frac{1}{3}D_a(D^b W_b) + \mathcal{R}_a^b W_b = j_a, \tag{2.15}$$

$$g_{ab} - \delta_{ab} = O^\infty\left(\frac{1}{r^{K-1+\varepsilon}}\right) \quad \text{and} \quad j_a = O^\infty\left(\frac{1}{r^{2+K+\varepsilon}}\right), \quad K = 1, \dots,$$

with W_a going to zero at infinity. Then there is a string of “multipole moments” L, M and N , such that Eqs. (2.11–14) are valid.

Proof. From the work [3] it follows that the (unique) solution W_a to (2.15) has $W_a = O^\infty(1/r^{\varepsilon'})$ for all $0 < \varepsilon' < 1$. Putting the terms coming from $g_{ab} - \delta_{ab}$ in Eq. (2.15) to the r.h. side, this implies that (2.15) can be written as the flat-space equation (2.1) with source $j'_a = O^\infty(1/r^{K+1+\varepsilon+\varepsilon'})$. Taking $\varepsilon' > 1 - \varepsilon$ and using again the representation (2.2, 3) this yields that $W_a = O^\infty(1/r)$, and inserting this back into j'_a that j'_a is in fact $O^\infty(1/r^{2+K+\varepsilon})$. Now the above computation gives Theorem 1.

At this stage it is important to remark that the source j_a in Eq. (2.1) does not necessarily come from a Q_{ab} , s. th. $j_a = D^b Q_{ab}$ satisfying (1.9) for the respective K . As an example consider Q_{ab} of the form (F_{abc} are symmetric, trace-free constants)

$$\bar{Q}_{ab} = \frac{2F_{abc}n^c - 6n_{(a}F_{b)cd}n^c n^d + 3(\delta_{ab} - n_a n_b)F_{cde}n^c n^d n^e}{r^2} \tag{2.16}$$

for $r > R > 0$, and extended smoothly as a tracefree tensor to all of \mathbf{R}^3 . \bar{Q}_{ab} so chosen satisfies the flat-space equation $\partial^a \bar{Q}_{ab} = 0$ for $r > R$. Thus $j_a = D^b \bar{Q}_{ab}$ is $O^\infty(1/(r^{2+K+\varepsilon}))$ and $P_{ab} = \bar{Q}_{ab} + (LW)_{ab}$ satisfies the momentum constraints together with (1.4). But it is not of the form $(L\bar{W})_{ab} + O^\infty(1/(r^{1+K+\varepsilon}))$ for $K = 1, 2, \dots$, since \bar{Q}_{ab} is only $O^\infty(1/r^2)$. Thus neither P_{ab} , nor \bar{Q}_{ab} , satisfy (1.10) for any K , and this can of course also be checked by direct computation.

There is a second and more fundamental way in which (2.1) can fail to solve our original problem. This can occur when j_a is such that we have difficulty in finding a trace-free Q_{ab} for which $j_a = D^b Q_{ab}$. This can occur when (M, g_{ab}) has conformal isometries, i.e. conformal Killing vectors (CKV's) ξ^a :

$$(L\xi)_{ab} = D_a \xi_b + D_b \xi_a - \frac{2}{3}g_{ab} D_c \xi^c = 0. \tag{2.17}$$

Let ξ^a be any such vector field. Then

$$-\int_{\mathbf{R}^3} \xi^a D^b Q_{ab} dV + \oint_{r=\infty} \xi^a Q_{ab} dS^b = \int_{\mathbf{R}^3} (D^a \xi^b) Q_{ab} dV. \tag{2.18}$$

Thus, from Eq. (2.17), the left-hand side of (2.18) is zero. If we were to do this analysis on a compact manifold without boundary, the surface term in Eq. (2.18) would not appear and we get the immediate restriction that j_a must be L^2 -orthogonal to ζ^a . In the asymptotically flat case this restriction gets softened to the requirement that if j_a is not orthogonal to ζ^a then the falloff of Q_{ab} must be slow enough that the surface integral in (2.18) does not vanish.

Equation (2.18) has a second use. If Q_{ab} is source-free, i.e., is a TT-tensor, we see that the surface integral in Eq. (2.18) must vanish, irrespective of the decay rate of Q_{ab} . This will have further consequences.

The possible existence of CKV's will be important in our next goal, which is trying to find Q_{ab} 's, for which the moments appearing in (2.11,12,13) assume arbitrary values. To see this we write out the lowest two orders in this expansion, i.e. for $K = 2$,

$$W_a^{(1,2)} = \frac{\varepsilon_c{}^{bc} n_b L_c^2}{r^2} + O^\infty \left(\frac{1}{r^{2+\varepsilon}} \right), \tag{2.19}$$

$$W_a^{(2,2)} = \frac{n_a M_b n^b + 7M_a^1}{r} + \frac{3n_a M_{bc}^2 n^b n^c + 6M_{ab} n^b}{r^2} + O^\infty \left(\frac{1}{r^{2+\varepsilon}} \right), \tag{2.20}$$

$$W_a^{(3,2)} = \frac{n_a N^2}{r^2} + O^\infty \left(\frac{1}{r^{2+\varepsilon}} \right). \tag{2.21}$$

Let ζ_T^a be an asymptotic translation, i.e. a vector field of the form

$$\zeta_T^a = \mu^a + O^\infty \left(\frac{1}{r^\varepsilon} \right), \tag{2.22}$$

where the μ^a 's are constants. Then, using the decay of Q_{ab} ,

$$\oint_{r=\infty} P_{ab} \zeta_T^a dS^b = \oint_{r=\infty} (L W)^{(2,2)}_{ab} \zeta_T^a dS^b = -32\pi M_a \mu^a. \tag{2.23}$$

Thus M_a^1 is essentially the (conserved) ADM 3-momentum. (In order to compare with the standard definition one has to check that the same value is obtained, when one takes P_{ab} in (2.23) to be the physical extrinsic curvature \bar{P}_{ab} related to P_{ab} by $\bar{P}_{ab} = \phi^{-2} P_{ab}$, where ϕ is the solution to the Lichnerowicz equation.) If ζ_T^a happens to be a CKV and if we have no source-current, the l.h. side of (2.22) is zero, and we obtain the obstruction $M_a \mu^a = 0$.

Suppose, next, that we have an asymptotic rotation vector ζ_R^a , i.e.

$$\zeta_R^a = \varepsilon^{abc} x_b \kappa_c + O^\infty \left(\frac{1}{r^\varepsilon} \right). \tag{2.24}$$

Then

$$\oint_{r=\infty} P_{ab} \zeta_R^a dS^b = \int_{r=\infty} (L W)^{(1,2)}_{ab} \zeta_R^a dS^b = -8\pi L_a \kappa^a. \tag{2.25}$$

Thus L_a^2 is essentially the conserved ADM 3-angular momentum. When ζ_R^a is a CKV and the matter is at rest, we have the obstruction $L_a \kappa^a = 0$.

We will show in the next section that the quantities M_a^1, L_a^2, N^2 and N_a^3 appearing in $W_a^{(3,3)}$ are the only ones which can possibly not be specified arbitrarily. The essential step will be a description of all moments in terms of surface integrals like (2.23), which will however not be expressible just in terms of P_{ab} , but will involve both P_{ab} and W_a .

3. The λ -Fields

Define the following collections of vector fields:

$${}_{o\lambda}_a^{(1,k)} = \varepsilon_a^{bc} x_b \kappa_{cb_1 \dots b_{k-2}} x^{b_1} \dots x^{b_{k-2}}, \quad k \geq 2, \tag{3.1}$$

$${}_{o\lambda}_a^{(2,k)} = \mu_{ab_1 \dots b_{k-1}} x^{b_1} \dots x^{b_{k-1}}, \quad k \geq 1, \tag{3.2}$$

$$\begin{aligned} {}_{o\lambda}_a^{(3,k)} &= 2(7k - 11) x_a \nu_{b_1 \dots b_{k-2}} x^{b_1} \dots x^{b_{k-2}} \\ &\quad - (k - 2)(k + 7) r^2 \nu_{ab_1 \dots b_{k-3}} x^{b_1} \dots x^{b_{k-3}}, \quad k \geq 2, \end{aligned} \tag{3.3}$$

where, again, the flat background metric is used and all of κ, μ, ν are constants which are symmetric and trace-free with respect to that metric. These fields have the following properties: they are globally regular (although they blow up at infinity) and they are annihilated by the flat space operator (2.1). Thus

$$D^b (L {}_{o\lambda}_{ab}^{(\alpha,k)}) = O^\infty \left(\frac{1}{r^{2+\varepsilon}} \right) \tag{3.4}$$

for $\alpha = 1, 2, 3$, provided that $K \geq k$. Thus, using [3], we can uniquely solve the equations

$$\Delta \delta \lambda_a^{(\alpha,k)} + \frac{1}{3} D_a D^b \delta \lambda_b^{(\alpha,k)} + \mathcal{R}_a^b \delta \lambda_b^{(\alpha,k)} = -D^b (L {}_{\lambda}_{ab}^{(\alpha,k)}) \tag{3.5}$$

with $\delta \lambda_a^{(\alpha,k)} = O^\infty(1/r^\varepsilon)$. Calling $\lambda_a^{(\alpha,k)} = {}_{o\lambda}_a^{(\alpha,k)} + (\delta \lambda)_a^{(\alpha,k)}$, we obtain

Theorem 2. *For any non-zero choice of symmetric, trace-free constants in (3.1, 2, 3) there exist unique non-zero vector fields $\lambda_a^{(\alpha,k)}$ with*

$$\lambda_a^{(\alpha,k)} = {}_{o\lambda}_a^{(\alpha,k)} + O^\infty \left(\frac{1}{r^\varepsilon} \right), \tag{3.6}$$

satisfying

$$D^b (L {}_{\lambda}_{ab}^{(\alpha,k)}) = 0. \tag{3.7}$$

Particularly interesting in this list are the “special” fields $\lambda^{(2,1)}, \lambda^{(1,2)}, \lambda^{(3,2)}, \lambda^{(3,3)}$ which we call—in this order—*asymptotic translations, rotations, dilations and conformal boosts*.

Lemma 1. *If the manifold has a CKV and if $K \geq 3$, it must be a linear combination of $\overset{(\alpha,k)}{\lambda}_a$.*

Proof. We know that CKV’s cannot go to zero at infinity (Christodoulou and Ó Murchadha [4]). The CKV, call it ζ , since it satisfies $L\zeta = 0$, must satisfy $D(L\zeta) = 0$. A decomposition such as used in the proof of Theorem 1 shows that the leading part of ζ must be a $\overset{(\alpha,k)}{\lambda}_a$. If we now demand that the first order condition ($L\zeta = 0$) be satisfied, we find that only the “special” fields listed above can survive. The lemma follows.

An alternative way of saying this is:

Lemma 2. *Let us have a linear combination $\overset{K}{\lambda} = \sum_{k=1}^K c(\alpha, k) \overset{(\alpha,k)}{\lambda}$, $k \leq K$, $c(\alpha, k) = \text{const}$, which, in addition to (3.7), satisfies the stronger condition $(L\overset{K}{\lambda})_{ab} = 0$, i.e. is a CKV. Then, trivially, for $K = 1$ it is an asymptotic translation. For $K = 2$, $c(2, 2) = 0$ and, for $K \geq 3$, all further $c(\alpha, k)$, $k \leq K$, vanish except $c(3, 3)$. In other words, $\overset{K}{\lambda}$ can only be a linear combination of the special vector fields.*

The proof of this lemma is a straightforward computation. Clearly, when g_{ab} is conformally flat, the special λ -fields are all CKV’s. When g_{ab} is not conformally flat, almost the opposite is true. We cannot have either the translation or the dilation CKV’s. We can have (at most) one rotation CKV and up to three conformal boosts. Namely, we have

Theorem 3. *Let g_{ab} not be conformally flat. Then none of the $\overset{(2,1)}{\lambda}$ ’s are CKV’s. For $K \geq 2$, there is at most one linear combination of $\overset{(2,1)}{\lambda}$, $\overset{(1,2)}{\lambda}$, $\overset{(3,2)}{\lambda}$ which can be a CKV, and this has to satisfy $\overset{2}{v} = 0$ (which means $\overset{(3,2)}{\lambda} \equiv 0$) and there exists a vector d^a , such that $\varepsilon^a_{bc} d^b \overset{2}{\kappa}^c = \overset{1}{\mu}^a$. In other words this CKV has to be an asymptotic rotation, possibly after a shift of origin.*

Proof. Let us assume that there exists a CKV which blows up like r at infinity. From our conformal smoothness assumption on g_{ab} it follows (Geroch [6]) that this CKV λ^a extends to a smooth CKV $\tilde{\lambda}^a$ for some smooth metric \tilde{g}_{ab} on $\mathbf{R}^3 \cup \{r = \infty\} \cong S^3$. From the asymptotic condition we have that $\tilde{\lambda}^a$ vanishes at the point-at-infinity A , i.e. $\tilde{\lambda}^a|_A = 0$. Furthermore

$$\tilde{\lambda}^a = \tilde{r}^2 \overset{1}{\mu}^a - 2\tilde{x}^a(\tilde{x}_b \overset{1}{\mu}^b) + \varepsilon^a_{bc} \tilde{x}^b \overset{2}{\kappa}^c + 6 \overset{2}{v} \tilde{x}^a + O^\infty(\tilde{r}^{2+\varepsilon}). \tag{3.8}$$

Invariantly, we have that

$$\begin{aligned} \tilde{D}_a \tilde{\lambda}^a|_A &= 18 \overset{2}{v}, \\ (\tilde{D}_a \tilde{\lambda}_b)|_A &= \varepsilon_{ab}{}^c \overset{2}{\kappa}_c =: F_{ab}, \\ (\tilde{D}_a \tilde{D}_b \tilde{\lambda}^b)|_A &= -2 \overset{1}{\mu}_a. \end{aligned} \tag{3.9}$$

Now we recall the notion of an inessential (resp. essential) CKV. A CKV is called inessential, if there exists a metric $g'_{ab} = \omega^2 \tilde{g}_{ab}$, $\omega > 0$, so that it is a Killing vector w.r. to g'_{ab} . Otherwise the CKV is called essential. Suppose the CKV $\tilde{\lambda}^a$ was

inessential. This would imply that

$$D'_a \tilde{\lambda}^a|_A = 0, \quad (D'_a D'_b \tilde{\lambda}^b)|_A = 0 \tag{3.10}$$

for some suitable conformal metric g' . But under conformal rescalings, using $\tilde{\lambda}^a|_A = 0$,

$$D'_a \tilde{\lambda}^a|_A = \tilde{D}_a \tilde{\lambda}^a|_A \tag{3.11}$$

and

$$(D'_a D'_b \tilde{\lambda}^b)|_A = (\tilde{D}_a \tilde{D}_b \tilde{\lambda}^b)|_A + 3F_a^b (\omega^{-1} \tilde{D}_b \omega)|_A. \tag{3.12}$$

Thus, if $\tilde{\lambda}^a$ is inessential, we would have that $\tilde{\nu}$ is zero and $\tilde{\mu}$ is of the form $\tilde{\mu}^a = \epsilon^a_{bc} d^b \tilde{\kappa}^c$ for some vector d^a . The only alternative is that $\tilde{\lambda}^a$ is essential. But it is shown in Appendix A that this is impossible except if $(\tilde{M}, \tilde{g}_{ab})$ is conformally diffeomorphic to S^3 with the standard metric. This also follows from a famous result of Obata [9], and Appendix A goes some way towards giving an independent proof of the full Obata theorem in 3 dimensions.

In order to show the “at most one”-statement in Theorem 3, suppose there was a second CKV $\tilde{\lambda}^a$ vanishing at A . By taking the commutator between the two, we obtain a third such CKV. Now, using (the full force of) the Obata theorem, their action, when \tilde{g}_{ab} is not conformal to the standard metric on S^3 , would again have to be inessential, i.e. isometric after a conformal rescaling. Since A is fixed, this would have to be an action under $SO(3)$ with S^2 principal orbits and thus (Fischer [5]) a standard spherical action on S^3 with all orbits S^2 except for two fixed points. Consequently, \tilde{g}_{ab} would have the standard rotational symmetry and thus be conformal to the standard metric. This contradiction ends the proof of Theorem 3.

We add the following remark: When $\tilde{\lambda}^{(1,2)}$ (a rotation) is a CKV and, in addition, satisfies

$$\mathcal{L}_{\tilde{\lambda}^{(1,2)}} P_{ab} = -(D^c \tilde{\lambda}^c) P_{ab}, \tag{3.13}$$

it follows that for the physical initial-data set $\tilde{P}_{ab} = \phi^{-2} P_{ab}$, $\tilde{g}_{ab} = \phi^4 g_{ab}$, with ϕ being the Lichnerowicz conformal factor, $\tilde{\lambda}^{(1,2)}$ is an isometry, i.e. $\mathcal{L}_{\tilde{\lambda}^{(1,2)}} \tilde{g}_{ab} =$

$\mathcal{L}_{\tilde{\lambda}^{(1,2)}} \tilde{P}_{ab} = 0$. But (3.13) implies that $\tilde{\kappa}_a$ is parallel to \tilde{L}_a in the center of energy,

whereas, from (2.25), we have that $\tilde{L}_a \tilde{\kappa}^a$ is zero. Thus \tilde{L}_a vanishes. It is in fact a known result, although we are not aware of a place in the literature where this is explicitly stated, that an asymptotically flat, topologically trivial vacuum spacetime with a $U(1)$ -isometry has zero angular momentum in the centre of energy.

4. The Product $\langle \tilde{\lambda} | W \rangle$

We now use the $\tilde{\lambda}$ -vector fields to obtain a useful description of the moments of W_a in terms of surface integrals. Consider the following antisymmetric scalar product:

$$\langle \tilde{\lambda}^{(x,l)} | W^K \rangle := \oint_{r=\infty} [\tilde{\lambda}^{(x,l)a} (LW^K)_{ab} - W^a (L\tilde{\lambda}^{(x,l)})_{ab}] dS^b \tag{4.1}$$

for $l \leqq K$. Using (2.14) and (3.7) we see that

$$\langle \overset{(x,l)}{\lambda} \mid \overset{K}{W} \rangle = \int_{\mathbf{R}^3} \overset{(x,l)}{\lambda}^a j_a dV. \tag{4.2}$$

In particular, since $j_a = O^\infty(1/(r^{2+K+\varepsilon}))$ and $\overset{(x,l)}{\lambda} = O^\infty(r^{l-1})$, the surface integrals in (4.1) converge. The remainder terms in (2.10) and (3.6) do not contribute to these integrals so that they can be evaluated explicitly in terms of the constants entering $\overset{(x,K)}{W}$ and $\overset{(x,k)}{\lambda}$. This is a somewhat tedious exercise. We need the following crucial facts. Any integral of the form

$$I(A, B) = \int_{S^2} A_{a_1 \dots a_k} n^{a_1} \dots n^{a_k} B_{b_1 \dots b_l} n^{b_1} \dots n^{b_l} d^2S \tag{4.3}$$

is zero for $k \neq l$, by virtue of orthogonality of spherical harmonics (A and B are symmetric and trace-free). For $k = l$, (4.3) can be computed (Appendix B), to give

$$I(A, B) = 4\pi \frac{2^l (l!)^2}{(2l + 1)!} A \cdot B \delta_{kl}, \tag{4.4}$$

where $A \cdot B := A_{a_1 \dots a_l} B^{a_1 \dots a_l}$. It is furthermore easy to see that an integral of the form

$$J(A, B) = \int_{S^2} \varepsilon^{abc} A_{aa_1 \dots a_k} n^{a_1} \dots n^{a_k} B_{bb_1 \dots b_l} n^{b_1} \dots n^{b_l} n_c d^2S \tag{4.5}$$

is zero for all (k, l) . It follows from the last remark that $\langle \overset{(x,l)}{\lambda} \mid \overset{(\beta,K)}{W} \rangle$ is zero, when one of (α, β) is equal to one and the other is not. Using (4.4) we find for $\alpha = 2, \beta = 3$ and for $\alpha = 3, \beta = 2$ that all terms which remain after using orthogonality of spherical harmonics in fact cancel. Thus

$$\langle \overset{(x,l)}{\lambda} \mid \overset{(\beta,K)}{W} \rangle = \delta_{x\beta} F(\alpha, l), \quad l \leqq K, \tag{4.6}$$

where for $F(\alpha, l)$ we finally obtain

$$F(1, l) = -4\pi \frac{2^{l-1} [(l-2)!]^2}{(2l-3)!} \binom{l}{L} \cdot \binom{l}{K}, \quad l \geqq 2, \tag{4.7}$$

$$F(2, l) = -4\pi \frac{2^{l+2} [(l-1)!]^2}{(2l-2)!} \binom{l}{M} \cdot \binom{l}{\mu}, \quad l \geqq 1, \tag{4.8}$$

$$F(3, l) = -4\pi \frac{(l^3 + 2l^2 - 29l + 54) 2^{l-2} [(l-3)!]^2}{(2l-5)!} \binom{l}{N} \cdot \binom{l}{\nu}, \quad l \geqq 3, \tag{4.9}$$

$$F(3, 2) = -4\pi \cdot 36 \binom{2}{N} \binom{2}{\nu}.$$

In particular, $F(\alpha, l)$ are all non-zero. Thus the product $\langle \lambda \mid W \rangle$ gives rise to a pairing between the moments contained in $\overset{(x,l)}{\lambda}$ and those contained in $\overset{(\beta,K)}{W}$. So we can take two sets of basis vectors for the two sets of symmetric, trace-free tensors involved, which are dual with respect to this pairing. The dimension of each set

can be computed e.g. from (3.1, 2, 3): any symmetric, trace-free tensor with k indices contributes $2k + 1$ dimensions. This gives

$$\sum_{k=2}^K [2(k - 1) + 1] = K^2 - 1 \quad \text{for } \overset{(1,k)}{\lambda},$$

$$\sum_{k=1}^K (2k + 1) = K^2 + 2K \quad \text{for } \overset{(2,k)}{\lambda}$$

and

$$\sum_{k=2}^K [2(k - 2) + 1] = K^2 - 2K + 1 \quad \text{for } \overset{(3,k)}{\lambda},$$

adding up to $3K^2$ dimensions. This is of course consistent with the three linear momentum components at order $1/r$ and the nine independent moments at order $1/r^2$ in Eqs. (2.19, 20, 21). We thus have λ -fields $\overset{A}{\lambda}_a$, with $1 \leq A \leq 3K^2$. The $3K^2$ moments $\overset{A}{M}$ encoded by the terms of order $1/r$ up to order $1/r^K$ in W , in the above basis, are

$$\overset{A}{M} = \langle \overset{A}{\lambda} | W \rangle, \tag{4.10}$$

provided j in (2.14) is $O^\infty(1/(r^{2+K+\epsilon}))$.

We now ask the question. Given a set of moments $\overset{A}{M}$: does there exist a source j_a having the required fall-off, so that the unique W solving Eq. (2.14) has exactly these moments? The answer is affirmative, as the following consideration shows. Take, for j_a , the linear combination

$$j_a = \frac{1}{(1 + r^2)^{(2+2K'+\epsilon)/2}} \sum_{A=1}^{3K^2} c^A \overset{A}{\lambda}_a, \quad K' \geq K, \quad c^A = \text{const}. \tag{4.11}$$

This is clearly $O^\infty(1/(r^{2+K+\epsilon}))$. Inserting (4.11) into (4.10), using (4.2), we are now left with the finite-dimensional linear equation

$$\overset{A}{M} = \sum_{B=1}^{3K^2} D_{AB} c^B, \tag{4.12}$$

with $D_{AB} = D_{(AB)}$ given by

$$D_{AB} = \int_{\mathbf{R}^3} \frac{1}{(1 + r^2)^{(2+2K'+\epsilon)/2}} \overset{A}{\lambda}_a \overset{B}{\lambda}^a dV. \tag{4.13}$$

The matrix D is clearly positive definite: $\sum_{A,B} D_{AB} c^A c^B = 0$ would imply $\sum c^A \overset{A}{\lambda}_a \equiv 0$, and this, by using (3.1,2,3) at increasing orders in $1/r$ and using the orthogonality of spherical harmonics, can only happen when $c^A = 0$. Thus Eq. (4.12) can be uniquely solved for c^A , given arbitrary $\overset{A}{M}$.

We now come to the question of existence of Q_{ab} , so that W_a , solving (2.14) with $j_a = D^b Q_{ab}$ has arbitrary moments up to some finite order. The answer is

afforded by

Theorem 4.

a) All moments other than the “exceptional moments” M_a^1, L_a^2, N^2 for $K \geq 2$, plus N_a^3 appearing in $W^{(3,3)}$ for $K \geq 3$ can be prescribed by a suitable choice of Q_{ab} satisfying (1.9).

b) When g_{ab} is conformally flat, the exceptional moments are constrained to vanish.

c) Suppose g_{ab} is not conformally flat and $K \geq 2$: when there are no conformal isometries, all of M_a^1, L_a^2, N^2 and N_a^3 can be prescribed. Otherwise, by Theorem 3, there is at most one CKV which has $v_a^3 = 0$. This has $v^2 = 0$ and, possibly after a shift of coordinates, $\mu_a^1 = 0$. In these coordinates M_a^1, L_a^2 and N^2 can still be prescribed with the only condition that $L_a^2 \kappa^a = 0$.

Proof. Again we start from

$$M^A = \langle \hat{\lambda}^A | W \rangle = \int_{\mathbf{R}^3} \hat{\lambda}^A j_a dV . \tag{4.14}$$

But, since $j_a = D^b Q_{ab}$, one more integration by parts yields

$$M^A = - \int_{\mathbf{R}^3} (L \hat{\lambda}^A)^{ab} Q_{ab} dV . \tag{4.15}$$

When g_{ab} is conformally flat, using the A -values corresponding to the special λ -fields, we find that all exceptional moments are zero, which proves b). To prove a) and c), we make the ansatz

$$Q_{ab} = \frac{1}{(1+r^2)^{(2K'+\epsilon)/2}} \sum_{A=1}^{3K^2} c^A (L \hat{\lambda}^A)_{ab}, \quad K' \geq K, \quad c^A = \text{const} , \tag{4.16}$$

and try to solve

$$M^A = \sum_{B=1}^{3K^2} E_{AB} c^B , \tag{4.17}$$

where $E_{AB} = E_{(AB)}$ is defined by

$$E_{AB} = \int_{\mathbf{R}^3} \frac{1}{(1+r^2)^{(2K'+\epsilon)/2}} (L \hat{\lambda}^A)^{ab} (L \hat{\lambda}^B)_{ab} dV . \tag{4.18}$$

Equation (4.17) can be solved provided $\sum M^A f^A = 0$, where $\sum E_{AB} f^B = 0$. But the latter condition, by (4.18), implies that $\sum f^A \hat{\lambda}_a^A$ is a CKV. Choosing, successively, for $\hat{\lambda}^A$ all possibilities except for the special λ -fields and using the pairing (4.6–9) and Lemma 2, we see that a) is true. Using Theorem 3 statement c) follows similarly.

Remark. If we insist on prescribing a non-zero value for N_a^3 , we have to take into account the possibility of CKV's with $v_a^3 \neq 0$. Such cases do in fact exist (see Beig, Husa [2]), and give rise to more conditions on the exceptional moments.

We end this paper with a “compact-support version” of Theorem 4, namely

Theorem 4'. *Let g_{ab} be a metric on \mathbf{R}^3 which is flat outside a compact set. Then all statements on arbitrariness of moments in Theorem 4 remain valid, when Q_{ab} is constrained to have compact support, rather than the fall-off of Eq. (1.9).*

Proof. The ansatz (4.16) is now replaced by

$$Q_{ab} = \rho \sum_{A=1}^{3K^2} c(L^A \lambda)_{ab}, \tag{4.19}$$

where $\rho \in C_0^\infty(\mathbf{R}^3)$, $\rho \geq 0$ and $\rho > 0$ in a region \mathcal{B} strictly containing the support of $g_{ab} - \delta_{ab}$. Then the null space of E_{AB} consists of vectors f^A such that $\sum f^A \lambda_a^A = \lambda_a$ satisfies $(L\lambda)_{ab} = 0$ in \mathcal{B} . But, outside \mathcal{B} , λ_a satisfies the flat-space version of $D^a(L\lambda)_{ab} = 0$. Taking one more divergence of this equation we see that $D_a \lambda^a$ is harmonic and, inserting back, that each component λ_a is harmonic. Thus λ_a , whence $(L\lambda)_{ab}$ is analytic outside \mathcal{B} . It follows that $(L\lambda)_{ab} = 0$ everywhere, and λ^a is a CKV. Now the statements of Theorem 4 follow, again, from Lemma 2 and Theorem 3.

Appendix A

Let \tilde{M} be a connected 3 dimensional compact manifold without boundary, with a smooth metric \tilde{g}_{ab} . Let ζ^a be a CKV on $(\tilde{M}, \tilde{g}_{ab})$. For discussing whether ζ^a can be essential, we distinguish between two cases, based on the sign of $\lambda_1(\tilde{g})$, the lowest eigenvalue of the conformal Laplacian $L_{\tilde{g}} = -\Delta_{\tilde{g}} + \frac{1}{8}\mathcal{R}[\tilde{g}]$, where \mathcal{R} is the scalar curvature of \tilde{g} . The first case is, from our present viewpoint, the unphysical case, since the Hamiltonian constraint cannot be solved for maximal data, when the background metric is conformally extendable to a metric \tilde{g} on the compactified manifold \tilde{M} with $\lambda_1(\tilde{g}) \leq 0$.

Theorem A.1. *Let $\lambda_1(\tilde{g}) \leq 0$. Then ζ^a is inessential.*

Proof. Let \bar{g} be a metric conformal to \tilde{g} with $\mathcal{R}[\bar{g}] = \text{const}$. This exists by the easier part of the solution to the Yamabe problem (Trudinger [12]). The rest is an argument due to Lichnerowicz [8]. By straightforward computation we find from

$$\tilde{D}_a \tilde{\zeta}^a + \tilde{D}_b \tilde{\zeta}^b = \frac{2}{3} \tilde{g}_{ab} \tilde{D}_c \tilde{\zeta}^c \tag{A.1}$$

and $\bar{\mathcal{R}} \equiv \mathcal{R}_0 = \text{const}$, that

$$\left(\Delta_{\bar{g}} + \frac{\mathcal{R}_0}{2} \right) \bar{D}_a \zeta^a = 0. \tag{A.2}$$

Thus $\mathcal{R}_0 \leq 0$ and the maximum principle implies that $\bar{D}_a \zeta^a = \text{const}$ and $\mathcal{R}_0 = 0$. Integrating $\bar{D}_a \zeta^a$ over (\tilde{M}, \bar{g}) gives zero, by the Gauß theorem. Thus $\bar{D}_a \zeta^a = 0$, and ζ^a is a Killing vector of \bar{g}_{ab} .

It is well known that the sign of $M(\tilde{g})$ is the same as the sign of the constant scalar curvature one can conformally transform to.

Theorem A.2. *Let $\lambda_1(g) > 0$ and ξ^a be a CKV vanishing at $\Lambda \in \tilde{M}$. Then, either a) (\tilde{M}, \tilde{g}) is conformally diffeomorphic to S^3 with the standard metric $\overset{0}{g}_{ab}$. Or b) $-3\alpha := \tilde{D}_a \xi^a|_\Lambda = 0$ and $-6c_a := \tilde{D}_a \tilde{D}_b \xi^b|_\Lambda$ lies in the image of the linear map F_a^b with $F_{ab} := \tilde{D}_{[a} \tilde{\xi}_{b]}$.*

Proof. Since $\lambda_1(\tilde{g}) > 0$, the operator $L[\tilde{g}]$ has a positive Green function (see [7]), which we take to be centered at Λ , i.e.

$$L_{\tilde{g}}G = 4\pi\delta_\Lambda. \tag{A.3}$$

G has the asymptotic expansion [7]

$$G = \frac{1}{\|x\|} + \frac{m}{2} + O^\infty(\|x\|), \tag{A.4}$$

where $\|x\|$ is geodesic distance from Λ and m is the ADM mass of the asymptotically flat metric

$$g'_{ab} = G^b \tilde{g}_{ab} \quad \text{on } M = \tilde{M} \setminus \Lambda. \tag{A.5}$$

By virtue of (A.3) the metric g'_{ab} has vanishing scalar curvature, i.e. $\mathcal{R}[g'] = 0$. The expansion (A.4) can be improved: under conformal rescalings $\bar{g}_{ab} = \omega^2 g_{ab}$, $\omega > 0$, G changes according to $\bar{G} = \omega^{-1/2}|_\Lambda \omega^{-1/2}G$. Now $\omega > 0$ can be found so that $\mathcal{R}_{ab}[\bar{g}]$ is zero at Λ . Thus, in this conformal gauge, $\bar{g}_{ab} = \delta_{ab} + O(\|x\|^3)$ in Riemannian normal coordinates x^a centered at Λ . Expanding $L_{\bar{g}}$ accordingly and using standard estimates for the flat-space Green function $\overset{0}{G}(x, x') = (x - x', x - x')^{-1/2}$, where $(x, y) = \delta_{ab}x^a y^b$, it follows that $(|x| = (x, x)^{1/2})$

$$G = \frac{1}{|x|} + \frac{m}{2} + (d, x) + O^\infty(|x|^2), \quad d^a = \text{const}. \tag{A.6}$$

For ξ^a we have the expansion

$$\xi^a = -\alpha x^a + F_b^a x^b + c^a x^2 - 2x^a(c, x) + O^\infty(|x|^4). \tag{A.7}$$

(Note that, in the notation of (3.9), $\alpha = -6\frac{2}{y}$, $c^a = \frac{1}{3}\mu^a$.) The uniqueness of the Green function implies (see Beig [1]) that ξ^a is a homothetic vector field for the metric g'_{ab} . Equivalently,

$$\mathcal{L}_\xi G + \frac{1}{6}(\tilde{D}_a \xi^a)G = \gamma G, \quad \gamma = \text{const}. \tag{A.8}$$

Evaluating the l.h. side of Eq. (A.8) using (A.6,7), we find that

$$\mathcal{L}_\xi G = \frac{\alpha}{|x|} + \frac{(c, x)}{|x|} - \alpha(d, x) + F_b^a x^b d_a + O^\infty(|x|^2), \tag{A.9}$$

$$\frac{1}{6}(\tilde{D}_a \xi^a)G = -\frac{\alpha}{2|x|} - \frac{\alpha m}{4} - \frac{(c, x)}{|x|} - \frac{(\alpha d + mc, x)}{2} + O^\infty(|x|^2). \tag{A.10}$$

Comparing coefficients in (A.8), there results $\alpha = 2\gamma$ at order -1 in $|x|$ and $\alpha m = -2\gamma m$ at order 0. Thus, either

a) $m = 0$: Then, since $\mathcal{R}[g'] = 0$, the positive mass theorem [11] applies and yields that (M, g') is diffeomorphic to \mathbf{R}^3 with the standard metric, or (\tilde{M}, \tilde{g}) conformally diffeomorphic to the standard S^3 . Or,

b) $m > 0$: Then $\alpha = \gamma = 0$. The order 1 in (A.8) now gives

$$F_a^b d_b = \frac{m}{2} c_a . \tag{A.11}$$

This ends the proof of Theorem (A.2): Clearly, transforming to asymptotically flat coordinates $x'^a = x^a/|x|^2$, we see that α corresponds to the dilation part, F_{ab} the rotation part and c^a the translation part of the CKV on \mathbf{R}^3 associated with ζ .

Appendix B

To prove (4.4) it suffices to consider $A = B$,

$$I(B, B) = B^{a_1 \cdots a_k} B^{a_{k+1} \cdots a_{2k}} \int_{S^2} n_{a_1} \cdots n_{a_{2k}} d^2 S . \tag{B.1}$$

The integral in (B.1) is proportional to

$$\delta_{(a_1 a_2} \cdots \delta_{a_{2k-1} a_{2k})} .$$

Using the formula ($a \in \mathbf{R}^3$)

$$\int_{S^2} (a, n)^k d^2 S = \frac{4\pi |a|^2}{2k + 1} , \tag{B.2}$$

the proportionality constant is found to $4\pi/(2k + 1)$. It remains to evaluate

$$\frac{4\pi}{2k + 1} B^{a_1 \cdots a_k} B^{a_{k+1} \cdots a_{2k}} \delta_{(a_1 a_2} \cdots \delta_{a_{2k-1} a_{2k})} .$$

Of the $(2k)!$ terms in this expression, due to the vanishing trace of B , only those terms contribute for which, in each Kronecker delta, there is one i_l with $1 \leq l \leq k$ and one i_m with $k + 1 \leq m \leq 2k$, of which there are $2^k (k!)^2$. Thus

$$I(B, B) = \frac{4\pi 2^k (k!)^2}{(2k + 1)!} B_{a_1 \cdots a_k} B^{a_1 \cdots a_k} . \tag{B.3}$$

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