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Classification of Local Generalized Symmetries for the Vacuum Einstein Equations

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Dedicated to the memory of H. Rund

Abstract: A local generalized symmetry of a system of differential equations is an infinitesimal transformation depending locally upon the fields and their derivatives which carries solutions to solutions. We classify all local generalized symmetries of the vacuum Einstein equations in four spacetime dimensions. To begin, we analyze symmetries that can be built from the metric, curvature, and covariant derivatives of the curvature to any order; these are called natural symmetries and are globally defined on any spacetime manifold. We next classify first-order generalized symmetries, that is, symmetries that depend on the metric and its first derivatives. Finally, using results from the classification of natural symmetries, we reduce the classification of all higher-order generalized symmetries to the first-order case. In each case we find that the local generalized symmetries are infinitesimal generalized diffeomorphisms and constant metric scalings. There are no non-trivial conservation laws associated with these symmetries. A novel feature of our analysis is the use of a fundamental set of spinorial coordinates on the infinite jet space of Ricci-flat metrics, which are derived from Penrose's "exact set of fields" for the vacuum equations.

1. Introduction

Symmetry plays an important role throughout theoretical physics and one of central importance in field theory [1, 2]. Indeed, in the construction of a field theory physical considerations usually demand that the field equations (or the Lagrangian) possess certain symmetries. These symmetries include Poincaré symmetry, gauge symmetry, diffeomorphism symmetry, various discrete symmetries, and a host of specialized symmetries needed to ensure the conservation of appropriate quantum numbers. Symmetries also play an important role in the mathematical analysis of differential equations [3, 4]. Originating with the work of Lie, symmetry group methods and their recent generalizations have proved useful in understanding conservation laws, in constructing exact solutions, and in establishing complete integrability of certain systems of differential equations.

The symmetries encountered in field theory are usually of the type commonly referred to as point symmetries. A point symmetry of a system of differential equations is a 1-parameter group of transformations of the underlying space of independent and dependent variables that carries any solution of the equations to another solution. If a point symmetry preserves an underlying Lagrangian for the system of equations, then there is a corresponding conservation law. However, not all conservation laws stem from point symmetries. To account for all local conservation laws in Lagrangian field theory one must enlarge the notion of symmetry to include generalized symmetries [5]. In this paper we will define a *generalized symmetry* to be an infinitesimal transformation, constructed *locally* from the independent variables, the dependent variables, and the *derivatives* of the dependent variables, that carries any solution of the differential equations to a nearby solution. The importance of generalized symmetries is underscored by their role in completely integrable systems of non-linear differential equations. In particular, the integrability of a system of differential equations is often (but not always) reflected by the existence of "hidden" generalized symmetries [3, 6, 7].

In recent years considerable attention has been devoted to applications of symmetry group methods to a variety of non-linear partial differential equations, but relatively few complete results have been obtained for the Einstein equations. It is, of course, natural to inquire whether or not the Einstein equations admit any hidden generalized symmetries, but the apparent complexity of the ensuing analysis has, to date, precluded substantive progress. The existence of hidden symmetries of the Einstein equations would lead to solution generating–classification techniques, and perhaps even information about the general solution to the Einstein equations. There are hints that such symmetries may exist. The two Killing vector reduction of the Einstein equations leads to an integrable system of partial differential equations [8, 9]; the self-dual Einstein equations exhibit an infinite number of symmetries and can be integrated using twistor methods [10, 11, 12]. A complete generalized symmetry analysis provides a systematic and rigorous way to unravel some aspects of the integrable behavior of the gravitational field equations. In particular, such an analysis indicates whether the rich structure of special reductions of the Einstein equations extends to the full theory via local symmetry transformations.

An equally important consequence of a generalized symmetry analysis stems from the fact that the existence of generalized symmetries of the Einstein equations is a necessary condition for the existence of local differential conservation laws for the gravitational field [13]. If such conservation laws could be found, they would lead to observables for the gravitational field [14]. It has long been an open problem in relativity theory to exhibit such observables, and the lack thereof currently hampers progress in canonical quantization of general relativity [15].

In this paper we will give a *complete* classification of all arbitrary-order local generalized symmetries for the vacuum Einstein equations in four spacetime dimensions. We shall show that the only generalized symmetries admitted by the vacuum Einstein equations consist of the diffeomorphism symmetry that is inherent in the Einstein equations and a trivial scaling symmetry. More precisely, we will prove the following theorem.

Theorem. *Let*

$$h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,h_1}, \ldots, g_{ij,h_1\cdots h_k})$$

be the components of a k^{th}-order generalized symmetry of the vacuum Einstein equations $R_{ij} = 0$ in four spacetime dimensions. Then there is a constant c and a generalized vector field

$$X^i = X^i(x^i, g_{ij}, g_{ij,h_1}, \ldots, g_{ij,h_1\cdots h_{k-1}})$$

such that, modulo the Einstein equations,

$$h_{ab} = cg_{ab} + \nabla_a X_b + \nabla_b X_a \ .$$

This result was announced in [16].

The plan of this paper is as follows. In Sect. 2 we begin with a summary of the theory of generalized symmetries, and we present elementary applications of this theory to the Einstein equations. The technical machinery needed for our analysis is then summarized. A complete account of this machinery can be found in [17]. Section 3 is devoted to applying our techniques to a model problem, namely, classifying a relatively simple class of third-order generalized symmetries. All our subsequent analysis follows the pattern of this example. In Sect. 4 we classify natural symmetries, which are symmetries built from the metric, curvature and covariant derivatives of the curvature to any order. In Sect. 5 we classify first-order generalized symmetries, which require a considerably more intricate analysis than that needed for natural generalized symmetries. In Sect. 6 we extend the analysis of Sect. 4 to obtain a classification of all generalized symmetries. The analysis of Sect. 6 uses an induction argument to reduce the classification to that of first-order generalized symmetries. The Appendix contains various results from spinor and tensor algebra which we use repeatedly.

We believe the methods that are used to prove these results are of no less importance than the results themselves. In classifying the generalized symmetries of the Einstein equations we have developed an effective spinor–jet bundle formalism for analyzing mathematical properties of the Einstein equations and related equations [17]. By far, the most important ingredient in this formalism is the use of what Penrose calls an "exact set of fields" for the field equations [18, 19]. These are spinor fields which allow us to parametrize the jet space of vacuum Einstein metrics. In future work we will apply these spinor–jet techniques to related aspects of general relativity. Specifically, our methods can be used to classify systematically (i) all closed p-forms that are built locally from a Ricci-flat metric, (ii) all symplectic forms for the Einstein equations, and (iii) all divergence-free symmetric tensors built locally from Einstein metrics. Finally, it is worth pointing out that the existence of an exact set of fields is not limited to the Einstein equations. For example, the generalized symmetries of the Yang–Mills equations are amenable to analysis using these techniques [20].

2. Preliminaries

In Sect. 2A we briefly review the geometric theory of generalized symmetries for differential equations and their role in constructing local conservation laws. For more on generalized symmetries and their applications, see [3]. In Sect. 2B we derive the defining equations for the generalized symmetries of the vacuum Einstein equations and present some preliminary results concerning solutions to these defining equations. We then summarize in Sect. 2C the technical machinery needed to compute the generalized symmetries of the Einstein equations. A complete presentation of the results in this latter section can be found in [17].

2A. Generalized Symmetries for Classical Field Theories. In classical field theory, the fields are usually identified with sections $\varphi\colon M \to E$ of a fiber bundle $\pi\colon E \to M$. In general relativity, M is a 4-dimensional manifold and π is the bundle $\pi\colon \mathcal{G} \to M$ of quadratic forms on the tangent space TM with signature $(-+++)$. A section $g\colon M \to \mathcal{G}$ is a choice of Lorentz metric on M.

Let $\pi^k_M\colon J^k(E) \to M$ be the bundle of k^{th} order jets of local sections of E. A point $\sigma \in J^k(E)$ is, by definition, an equivalence class of local sections defined in a neighborhood U of the point $x = \pi^k_M(\sigma)$; two local sections $\varphi_1, \varphi_2\colon U \to E$ are equivalent if φ_1 and φ_2 and all their partial derivatives to order k agree at x. If $\varphi\colon U \to E$ is a local section of E, then the canonical lift

$$j^k(\varphi)\colon U \to J^k(E)$$

is the map that assigns to each point $x \in U$ the k-jet $j^k(\varphi)(x)$ represented by φ at x. There are also canonical projections

$$\pi^k_l\colon J^k(E) \to J^l(E)\,,$$

defined for all $k \geqq l$. When $l = 0$, we write $\pi^k_E\colon J^k(E) \to E$. The infinite jet bundle $\pi^\infty_M\colon J^\infty(E) \to M$ is similarly defined. For a more detailed presentation of jet bundles, see [3, 21].

A differential form ω on $J^\infty(E)$ is called a *contact form* if, for every local section $\varphi\colon U \to E$,

$$[j^\infty(\varphi)]^*(\omega) = 0\,.$$

The set of all contact forms on $J^\infty(E)$ is a differential ideal in the ring $\Omega^*(J^\infty(E))$ of all differential forms on $J^\infty(E)$, and we denote this ideal by $\mathcal{C}(J^\infty(E))$.

A *generalized vector field* Z on E is a vector field along the map π^∞_E, that is, for each point $\sigma \in J^\infty(E)$, Z_σ is a tangent vector in $T_p(E)$, where $p = \pi^\infty_E(\sigma)$. If Z is a generalized vector field on E, then there is a unique vector field $\operatorname{pr} Z$ on $J^\infty(E)$, called the *infinite prolongation* of Z such that

(i) for each $\sigma \in J^\infty(E)$, $(\pi^\infty_E)_*[(\operatorname{pr} Z)_\sigma] = Z_{\pi^\infty_E(\sigma)}$, and
(ii) $\operatorname{pr} Z$ preserves the contact ideal, that is, under Lie differentiation

$$\mathcal{L}_{\operatorname{pr} Z}\,\mathcal{C}(J^\infty(E)) \subset \mathcal{C}(J^\infty(E)).$$

We shall give local expressions for Z and $\operatorname{pr} Z$ shortly. A generalized vector field Y on E that is π-vertical, i.e.,

$$\pi_*(Y_\sigma) = 0\,,$$

for all $\sigma \in J^\infty(E)$, is called an *evolutionary vector field.* Evolutionary vector fields determine "infinitesimal field variations," and their prolongations determine the induced variations in the derivatives of the fields. Finally, a *generalized vector field X on M* is a vector field along the map π^∞_M, and a *generalized tensor field A of type (p, q) on M* is a smooth map

$$A\colon J^\infty(E) \to T^p_q(M)$$

along π^∞_M, where $T^p_q(M)$ is the bundle of tensors of type (p,q) over M. Note that if Z is a generalized vector field on E, then $Z_M = \pi_*(Z)$ is a generalized vector field on M.

Every generalized vector field X on M defines a unique vector field $\operatorname{tot} X$ on $J^\infty(E)$, called the *total vector field* of X, with the properties

(i) $(\pi^\infty_M)_*[(\operatorname{tot} X)_\sigma] = X_{\pi^\infty_M(\sigma)}$, and

(ii) $\operatorname{tot} X$ annihilates all contact 1-forms, that is, if ω is a contact 1-form, then $\operatorname{tot} X \lrcorner\, \omega = 0$.

We remark that if X is a generalized vector field on M and $X_E = (\pi^\infty_E)_*(\operatorname{tot} X)$, then

$$\operatorname{pr} X_E = \operatorname{tot} X \ .$$

In other words, $\operatorname{tot} X$ is also a prolongation of a vector field and therefore $\operatorname{tot} X$ preserves the contact ideal.

If Z_1 and Z_2 are generalized vector fields on E, then there exists a generalized vector field Z_3 such that $[\operatorname{pr} Z_1, \operatorname{pr} Z_2] = \operatorname{pr} Z_3$. We call Z_3 the generalized Lie bracket of Z_1 and Z_2 and write

$$[Z_1, Z_2] = Z_3 \ .$$

It is also straightforward to verify that if $\operatorname{tot} X_1$ and $\operatorname{tot} X_2$ are two total vector fields, then $[\operatorname{tot} X_1, \operatorname{tot} X_2]$ is also a total vector field, $[\operatorname{tot} X_1, \operatorname{tot} X_2] = \operatorname{tot} X_3$. (Hence the set of all total vector fields on $J^\infty(E)$ is a connection of general type on $J^\infty(E) \longrightarrow M$.)

Now suppose a system of differential equations for the sections of E is given. These are the field equations for the classical field theory. If these equations are of order k (typically $k = 2$), then they determine a smooth subbundle

$$\mathscr{R}^k \hookrightarrow J^k(E)$$

with projection $\pi^k_M : \mathscr{R}^k \to M$. We call $\mathscr{R}^k$ the *equation manifold* for the classical field theory. The total derivatives of the field equations to order l then define the l^{th} *prolonged equation manifold*

$$\mathscr{R}^{k+l} \hookrightarrow J^{k+l}(E) \ .$$

The field equations, together with all their total derivatives, determine the *infinitely prolonged equation manifold*

$$\mathscr{R}^\infty \hookrightarrow J^\infty(E) \ .$$

It is customary to assume [22, 23] that the maps

$$\pi^{l+1}_l : \mathscr{R}^{l+1} \to \mathscr{R}^l$$

are surjective for all $l \geqq k$ and have constant rank. The fiber dimension of π^{l+1}_l represents the number of "degrees of freedom" available in constructing a formal power series solution for the field equations to order $l+1$ from a given solution to order l. Roughly speaking, equations that are not "over-determined" will satisfy the surjectivity assumption. As we shall see, the vacuum Einstein equations satisfy these surjectivity and constant rank assumptions [17].

Definition 2.1. *A generalized vector field Z on E is called a generalized symmetry of the given field equations if* $\operatorname{pr} Z$ *is tangent to the infinitely prolonged equation manifold $\mathcal{R}^\infty$, that is, for all $\sigma \in \mathcal{R}^\infty$,*

$$(\operatorname{pr} Z)_\sigma \in T_\sigma(\mathcal{R}^\infty) .$$

Generalized symmetries are sometimes called *Lie-Bäcklund symmetries*. If Z_1 and Z_2 are two generalized symmetries for $\mathcal{R}^\infty$, then the generalized Lie bracket $[Z_1, Z_2]$ is also a generalized symmetry.

We now give local coordinate descriptions of these various notions. If $(x^i, \varphi^\alpha) \to (x^i)$, $i = 1, 2, \ldots, n$ and $\alpha = 1, 2, \ldots, m$, are local coordinates on $\pi \colon E \to M$, then the standard local coordinates for $J^\infty(E)$ are

$$(x^i, \varphi^\alpha, \varphi^\alpha_{i_1}, \varphi^\alpha_{i_1 i_2}, \ldots, \varphi^\alpha_{i_1 i_2 \cdots i_k}, \ldots) ,$$

where, for a given local section $\varphi^\alpha = \varphi^\alpha(x^i)$,

$$\varphi^\alpha_{i_1 \cdots i_k}(j^\infty(\varphi)(x)) = \frac{\partial^k \varphi^\alpha(x)}{\partial x^{i_1} \cdots \partial x^{i_k}} .$$

The contact ideal $\mathcal{C}(J^\infty(E))$ is spanned locally by the contact 1-forms

$$\theta^\alpha_{i_1 \cdots i_k} = d\varphi^\alpha_{i_1 \cdots i_k} - \varphi^\alpha_{i_1 \cdots i_k j} dx^j$$

for $k = 0, 1, 2, \ldots$. These forms satisfy the structure equations

$$d\,\theta^\alpha_{i_1 \cdots i_k} = dx^j \wedge \theta^\alpha_{i_1 \cdots i_k j} .$$

A generalized vector field Z on E assumes the form

$$Z = A^i \frac{\partial}{\partial x^i} + B^\alpha \frac{\partial}{\partial \varphi^\alpha} ,$$

where

$$A^i = A^i(x^j, \varphi^\beta, \varphi^\beta_{i_1}, \ldots, \varphi^\beta_{i_1 \cdots i_k}), \quad \text{and} \quad B^\alpha = B^\alpha(x^j, \varphi^\beta, \varphi^\beta_{i_1}, \ldots, \varphi^\beta_{i_1 \cdots i_k}) .$$

A generalized vector field X on M and an evolutionary vector field Y on E take the form

$$X = A^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha} ,$$

where, again, the coefficients A^i and B^α are functions of x^i, φ^α and the derivatives $\varphi^\alpha_{i_1 \cdots i_k}$ to some arbitrary but finite order. The vector field $\operatorname{tot} X$ is given by

$$\operatorname{tot} X = A^i D_i ,$$

where D_i is the total derivative operator

$$D_i = \frac{\partial}{\partial x^i} + \varphi^\alpha_i \frac{\partial}{\partial \varphi^\alpha} + \varphi^\alpha_{i i_1} \frac{\partial}{\partial \varphi^\alpha_{i_1}} + \varphi^\alpha_{i i_1 i_2} \frac{\partial}{\partial \varphi^\alpha_{i_1 i_2}} + \cdots .$$

We write

$$D_{i_1 i_2 \cdots i_k} = D_{i_1} D_{i_2} \cdots D_{i_k} .$$

The prolongation of Z is given by the prolongation formula [3]

$$\operatorname{pr} Z = A^i D_i + \sum_{k=0}^{\infty} D_{i_1 i_2 \cdots i_k}(B^\alpha - \varphi^\alpha_i A^i) \frac{\partial}{\partial \varphi^\alpha_{i_1 i_2 \cdots i_k}} . \tag{2.1}$$

Note that, in particular, the prolongation of the evolutionary vector field $Y = B^\alpha \frac{\partial}{\partial\varphi^\alpha}$ is

$$\operatorname{pr} Y = \sum_{k=0}^{\infty} (D_{i_1 i_2 \cdots i_k} B^\alpha) \frac{\partial}{\partial \varphi^\alpha_{i_1 i_2 \cdots i_k}} . \tag{2.2}$$

We now remark that (2.1) and (2.2) together prove the following theorem.

Theorem 2.2. *Let Z be a generalized vector field on E. Then there exists a unique evolutionary vector field Z_{ev} such that*

$$\operatorname{pr} Z = \operatorname{tot} Z_M + \operatorname{pr} Z_{\text{ev}} , \tag{2.3}$$

where $Z_M = \pi_*(Z)$.

The evolutionary vector field in (2.3) is

$$Z_{\text{ev}} = (B^\alpha - \varphi^\alpha_i A^i) \frac{\partial}{\partial \varphi^\alpha} . \tag{2.4}$$

If $X_1 = A^i_1 \frac{\partial}{\partial x^i}$ and $X_2 = A^i_2 \frac{\partial}{\partial x^i}$ are generalized vector fields on M, then

$$[X_1, X_2] = [A^i_1(D_i A^j_2) - A^i_2(D_i A^j_1)] \frac{\partial}{\partial x^j} .$$

If $Y_1 = B^\gamma_1 \frac{\partial}{\partial \varphi^\alpha}$ and $Y_2 = B^\alpha_2 \frac{\partial}{\partial \varphi^\alpha}$ are evolutionary vector fields on E, then

$$[Y_1, Y_2] = [\operatorname{pr} Y_1(B^\alpha_2) - \operatorname{pr} Y_2(B^\alpha_1)] \frac{\partial}{\partial \varphi^\alpha} .$$

An evolutionary vector field $Y = B^\alpha \frac{\partial}{\partial \varphi^\gamma}$ defines "infinitesimal field variations" $\delta\varphi^\alpha_{i_1 \cdots i_l}$, $l = 0, 1, \ldots$, which depend locally on the fields and their derivatives. Explicitly, $\delta\varphi^\alpha_{i_1 \cdots i_l}$ is defined by letting the prolonged vector field $\operatorname{pr} Y$ act on the coordinates $\varphi^\gamma_{i_1 \cdots i_l}$, which are viewed as functions on $J^\infty(E)$:

$$\delta\varphi^\alpha_{i_1 \cdots i_l} = \operatorname{pr} Y(\varphi^\alpha_{i_1 \cdots i_l}) = (D_{i_1 \cdots i_l} B^\gamma)(x^i, \varphi^\alpha, \varphi^\alpha_i, \ldots, \varphi^\alpha_{i_1 \cdots i_{l+k}}) .$$

If

$$\Delta_\beta(x^i, \varphi^\alpha, \varphi^\alpha_{i_1}, \ldots, \varphi^\alpha_{i_1 \cdots i_k}) = 0, \qquad \beta = 1, \ldots, m \tag{2.5}$$

is a system of field equations for the fields φ^α, then $\mathscr{R}^k \subset J^k(E)$ is the manifold defined by these equations. The infinitely prolonged equation manifold $\mathscr{R}^\infty$ is defined by the Eqs. (2.5) together with the equations

$$D_{i_1 i_2 \cdots i_l} \Delta_\beta = 0$$

for $l = 1, 2, \ldots$.

It now follows that if X is a generalized vector field on M, then $\operatorname{tot} X$ (or more precisely $X_E = \pi^\infty_E(\operatorname{tot} X)$) is always a generalized symmetry for any system of equations. Total vector fields are therefore viewed as trivial symmetries. A generalized symmetry Z is also considered trivial if Z vanishes on the prolonged equation manifold $\mathscr{R}^\infty$. Two generalized symmetries are said to be equivalent if their difference is a trivial symmetry. Theorem 2.2 implies that *every generalized symmetry*

Z of a given system of equations is equivalent to a generalized symmetry Y which is π-vertical, that is, to an evolutionary generalized symmetry.

The evolutionary vector field $Y = B^\gamma \frac{\partial}{\partial \varphi^\alpha}$ is, according to the tangency condition in Definition 2.1, a generalized symmetry of (2.5) if and only if the coefficient functions B^γ satisfy the linear total differential equation

$$\sum_{l=0}^{k} \frac{\partial \Delta_\beta}{\partial \varphi^\alpha_{i_1 \cdots i_l}} [D_{i_1 \cdots i_l} B^\gamma] = 0 \quad \text{on } \mathscr{R}^\infty . \tag{2.6}$$

This equation is called the *formal linearization* of (2.5), or the *defining equation* for the generalized symmetry Y.

Let us remark that when Z is an ordinary vector field on E, that is,

$$Z = A^i(x^j, \varphi^\beta) \frac{\partial}{\partial x^i} + B^\alpha(x^j, \varphi^\beta) \frac{\partial}{\partial \varphi^\alpha} ,$$

and $(\operatorname{pr} Z)(\Delta_\beta) = 0$ on the equation manifold $\Delta_\beta = 0$, then Z is called a *point symmetry* of the equations. Point symmetries are in one-to-one correspondence with first-order evolutionary symmetries

$$Y = B^\beta(x^i, \varphi^\alpha, \varphi^\alpha_i) \frac{\partial}{\partial \varphi^\beta} ,$$

with B^α a collection of affine linear functions of the first derivatives φ^α_i.

Finally, we cite a version of Noether's theorem as it applies to generalized symmetries [3]. Recall that a *local differential conservation law* V for the field equations $\Delta_\beta = 0$ is a generalized vector density

$$V = V^i(x^k, \varphi^\alpha, \varphi^\alpha_{i_1}, \ldots, \varphi^\alpha_{i_1 \cdots i_k}) \frac{\partial}{\partial x^i}$$

on M such that the total divergence

$$\operatorname{Div} V = D_i V^i = 0 \quad \text{on } \mathscr{R}^\infty .$$

A conservation law V is said to be trivial if there is a generalized skew-symmetric tensor density

$$S^{ij} = S^{ij}(x^k, \varphi^\alpha, \varphi^\gamma_{i_1}, \varphi^\alpha_{i_1 i_2}, \ldots, \varphi^\alpha_{i_1 \cdots i_l})$$

such that

$$V^i = D_j S^{ij} \quad \text{on } \mathscr{R}^\infty .$$

Two conservation laws are said to be equivalent if their difference is a trivial conservation law. Following Olver [3], an evolutionary vector field $Y = B^\alpha \frac{\partial}{\partial \varphi^\alpha}$ is called a *characteristic vector field* for the conservation law V if

$$\operatorname{Div} V = B^\gamma \Delta_\alpha \tag{2.7}$$

identically. Under mild regularity conditions on the equations $\Delta_\beta = 0$, it can be shown that every conservation law V' is equivalent to a conservation law V whose divergence satisfies (2.7). It is a simple result from the variational calculus that if Δ_α are the components of the Euler–Lagrange operator $E_\alpha(L)$ for some Lagrangian

$L = L(x^i, \varphi^\alpha, \varphi^\alpha_{i_1}, \ldots, \varphi^\gamma_{i_1 \cdots i_k})$,

$$E_\alpha(L) = \frac{\partial L}{\partial \varphi^\alpha} - D_{i_1} \frac{\partial L}{\partial \varphi^\alpha_{i_1}} + \cdots \pm D_{i_1 \cdots i_k} \frac{\partial L}{\partial \varphi^\gamma_{i_1 \cdots i_k}} ,$$

then *every characteristic vector field Y for a local differential conservation law for the equations $\Delta_\alpha = 0$ defines a generalized symmetry*. The converse need not be true. For example, scaling symmetries of Euler-Lagrange equations typically will not lead to conservation laws.

2B. The Formal Linearization of the Einstein Equations. To study the generalized symmetries of the Einstein field equations, we let $\pi: \mathcal{G} \to M$ be the bundle of Lorentz metrics over the spacetime manifold M. Standard local coordinates for $J^k(\mathcal{G})$ are

$$(x^i, g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1 i_2 \cdots i_k}) .$$

The Christoffel symbols Γ^k_{ij}, the curvature tensor $R_i{}^h{}_{jk}$, and their derivatives are now considered functions on $J^k(\mathcal{G})$. The covariant derivatives of a generalized tensor field on M are defined in terms of total derivatives. For example, if

$$A_a = A_a(x^i, g_{ij}, g_{ij,i_1}, g_{ij,i_1 i_2}, \ldots, g_{ij,i_1 i_2 \cdots i_k})$$

are the components of a generalized 1-form on M, then

$$\begin{aligned} \nabla_b A_a &= D_b A_a - \Gamma^c_{ab} A_c \\ &= \frac{\partial A_a}{\partial x^b} + \frac{\partial A_a}{\partial g_{ij}} g_{ij,b} + \frac{\partial A_a}{\partial g_{ij,i_1}} g_{ij,i_1 b} + \cdots + \frac{\partial A_a}{\partial g_{ij,i_1 \cdots i_k}} g_{ij,i_1 \cdots i_{+kb}} - \Gamma^c_{ab} A_c . \end{aligned}$$

We now compute the formal linearization (2.6) of the vacuum Einstein equations.

Proposition 2.3. *Let*

$$Y = h_{ab}(x^i, g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1 i_2 \cdots i_k}) \frac{\partial}{\partial g_{ab}}$$

be an evolutionary vector field on the bundle $\mathcal{G}$ of Lorentz metrics. Then Y is a generalized symmetry of the Einstein equations $R_{ij} = 0$ if and only if

$$\left[-g^{cd} \delta^a_i \delta^b_j - g^{ab} \delta^c_i \delta^d_j + g^{ac} \left(\delta^b_i \delta^d_j + \delta^b_j \delta^d_i\right)\right] \nabla_c \nabla_d h_{ab} = 0 \tag{2.8}$$

whenever R_{ij} and its covariant derivatives to order k vanish.

Proof. This is an easy computation based upon the identities

$$(\mathrm{pr}\, Y)(\Gamma^l_{ij}) = \tfrac{1}{2} g^{lm} [\nabla_i h_{mj} + \nabla_j h_{mi} - \nabla_m h_{ij}] , \tag{2.9}$$

and

$$(\mathrm{pr}\, Y)(R_i{}^l{}_{jk}) = \nabla_k (\mathrm{pr}\, Y(\Gamma^l_{ij})) - \nabla_j (\mathrm{pr}\, Y(\Gamma^l_{ik})) . \tag{2.10}$$

These formulas are, of course, familiar from the variational calculus. We emphasize that now (2.9) and (2.10) are to be viewed as identities on $J^k(\mathcal{G})$, where they are direct consequences of the prolongation formula

$$\mathrm{pr}\, Y = h_{ab}\frac{\partial}{\partial g_{ab}} + (D_i h_{ab})\frac{\partial}{\partial g_{ab,i}} + (D_{ij} h_{ab})\frac{\partial}{\partial g_{ab,ij}} + \cdots .$$

We remark that Proposition 2.3 could also be formulated in terms of the Einstein tensor G_{ij} and its derivatives. The symmetry conditions so-obtained are equivalent to (2.8). □

Let $X = X^a(x)\frac{\partial}{\partial x^a}$ be a vector field on M with local flow $\phi_t: M \to M$. Then ϕ_t induces a local flow on $\mathscr{G}$ with corresponding vector field $\widetilde{X}$ on $\mathscr{G}$ given by

$$\widetilde{X} = X^a\frac{\partial}{\partial x^a} - \left(\frac{\partial X^a}{\partial x^i} g_{aj} + \frac{\partial X^a}{\partial x^j} g_{ai}\right)\frac{\partial}{\partial g_{ij}} .$$

The associated evolutionary vector field is, by (2.4),

$$\widetilde{X}_{\mathrm{ev}} = -(\nabla_i X_j + \nabla_j X_i)\frac{\partial}{\partial g_{ij}} .$$

It is well-known [24] that $\widetilde{X}$, or equivalently $\widetilde{X}_{\mathrm{ev}}$, represents a point symmetry of the Einstein equations corresponding to the diffeomorphism invariance of the Einstein equations. This observation motivates the following definition.

Definition 2.4. *Let*

$$X = X^a(x^i, g_{ij}, g_{ij,i_1}, \ldots, g_{ij,i_1 i_2 \cdots i_l})\frac{\partial}{\partial x^a}$$

be a generalized vector field on M. We call the evolutionary vector field

$$\mathscr{K}_X = (\nabla_i X_j + \nabla_j X_i)\frac{\partial}{\partial g_{ij}} ,$$

where $X_i = g_{ij}X^j$, the associated generalized diffeomorphism vector field on $\mathscr{G}$.

We remark that if X_1 and X_2 are generalized vector fields on M, then

$$[\mathscr{K}_{X_1}, \mathscr{K}_{X_2}] = \mathscr{K}_{[X_1, X_2]} .$$

Proposition 2.5. *For any generalized vector field X on M, the associated generalized diffeomorphism vector field $\mathscr{K}_X$ is a generalized symmetry of the vacuum Einstein equations.*

Proof. By virtue of (2.9), we find that

$$(\mathrm{pr}\,\mathscr{K}_X)(\Gamma^l_{ij}) = \nabla_j\nabla_i X^l + R_i{}^l{}_{jp}X^p ,$$

and hence, by (2.10),

$$(\mathrm{pr}\,\mathscr{K}_X)R_{ij} = (\nabla_p R_{ij})X^p + R_{pj}\nabla_i X^p + R_{ip}\nabla_j X^p ,$$

which vanishes when $R_{ij} = 0$ and $\nabla_k R_{ij} = 0$. □

We call the symmetry $\mathscr{K}_X$ a *generalized diffeomorphism symmetry* of the Einstein equations. Note that the generalized diffeomorphism vector fields $\mathscr{K}_X$ will be symmetries for any generally covariant set of field equations on $\mathscr{G}$. In particular, Proposition 2.5 generalizes to the Einstein equations with cosmological constant.

There is one more obvious symmetry of the vacuum Einstein equations $R_{ij} = 0$.

Proposition 2.6. *For any constant c, the vector field*

$$\mathscr{S}_c = c\, g_{ij} \frac{\partial}{\partial g_{ij}} \tag{2.11}$$

is a point symmetry of the vacuum Einstein equations $R_{ij} = 0$.

Proof. This proposition follows from the fact that $(\mathrm{pr}\, \mathscr{S}_c)(\Gamma^k_{ij}) = 0$, and hence $(\mathrm{pr}\, \mathscr{S}_c)(R_{ij}) = 0$. Alternatively, $h_{ij} = c\, g_{ij}$ clearly satisfies (2.8). □

On a 4-dimensional manifold M we have

$$(\mathrm{pr}\, \mathscr{S}_c)(\sqrt{g}R) = c\sqrt{g}R\ .$$

Thus the scaling symmetry $\mathscr{S}_c$ of the Einstein equations does not preserve the Hilbert Lagrangian (even up to a divergence) and therefore does not generate a conservation law. The generalized diffeomorphism symmetry $\mathscr{K}_X$ is a characteristic for a conservation law for the Einstein equations, namely,

$$\nabla_j(2\sqrt{g}X_iG^{ij}) = (\nabla_iX_j + \nabla_jX_i)\sqrt{g}G^{ij}\ .$$

But the conserved vector density $V^j = 2\sqrt{g}X_iG^{ij}$ is trivial.

We remark that the scaling symmetry $\mathscr{S}_c$ and the point diffeomorphism symmetry $\widetilde{X}$ are the only *point* symmetries of the vacuum Einstein equations [24].

2C. Spinor Coordinates for Prolonged Einstein Equation Manifolds. Let $\mathscr{E}^k \subset J^k(\mathscr{G})$ be the set of k-jets that satisfy the Einstein equations and the covariant derivatives of the Einstein equations to order $k-2$,

$$\mathscr{E}^k = \{\, j^k(g)(x_0) \in J^k(\mathscr{G}) \mid G_{ij} = 0, G_{ij|i_1} = 0, \ldots, G_{ij|i_1\cdots i_{k-2}} = 0 \text{ at } j^k(g)(x_0) \,\}\ .$$

Here and in what follows, we will either use the vertical bar or ∇ to indicate covariant differentiation. If $h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,j_1}, \ldots, g_{ij,j_1\cdots j_k})$ is a generalized symmetry of the vacuum Einstein equations, then the linearized equations (2.8) must hold identically at each point of $\mathscr{E}^{k+2}$. To solve these equations we shall need explicit coordinates for these prolonged equation manifolds [17].

To this end, we let Γ^i_{jk} be the Christoffel symbols of the metric g_{ij} and inductively define *higher-order Christoffel symbols* by

$$\Gamma^i_{j_0j_1\cdots j_k} = \Gamma^i_{(j_0j_1\cdots j_{k-1},j_k)} - (k-1)\Gamma^i_{m(j_1\cdots j_{k-2}}\Gamma^m_{j_{k-1}j_k)}\ , \tag{2.12}$$

for $k \geqq 1$. These higher-order symbols arise naturally from the prolongations of the geodesic equations and play a prominent role in T.Y. Thomas' theory of normal extensions [25]. We will, on occasion, denote the generalized Christoffel symbols (2.12) simply by Γ^k. Note that $\Gamma^i_{j_0j_1\cdots j_k}$ is completely symmetric in the indices $j_0j_1\cdots j_k$ and depends on the metric and its first k derivatives.

Next, let [18]

$$Q_{ij,j_1\cdots j_k} = g_{ir}g_{js}R^r{}_{(j_1}{}^s{}_{j_2|j_3\cdots j_k)}\ , \tag{2.13}$$

for $k \geqq 2$. This tensor is a generalized tensor on M of order k, which we denote by Q^k. Note that $Q_{ij,j_1\cdots j_k}$ is symmetric in ij and $j_1 \cdots j_k$, and satisfies the cyclic identity

$$Q_{i(j,j_1\cdots j_k)} = 0 . \tag{2.14}$$

It is then possible to prove [17] that the variables

$$(x^i, g_{ij}, \Gamma^i_{j_0 j_1}, \ldots, \Gamma^i_{j_0 j_1\cdots j_k}, Q_{ij,j_1 j_2}, \ldots, Q_{ij,j_1\cdots j_k}) \tag{2.15}$$

can be used as coordinates for the bundle $J^k(\mathscr{G})$. Furthermore, if $[Q_{ab,j_1\cdots j_k}]_{\text{tracefree}}$ is the completely trace-free part of $Q_{ab,j_1\cdots j_k}$ (trace-free with respect to g_{ij}), then local coordinates for $\mathscr{E}^k$ are given by

$$(x^i, g_{ij}, \Gamma^i_{j_0 j_1\cdots j_l}, [Q_{ij,\, j_1\cdots j_l}]_{\text{tracefree}}) \quad \text{for } l \leqq k . \tag{2.16}$$

Now we consider the spinor representation of the curvature tensor [19],

$$R_{abcd} \longleftrightarrow R^{A'B'C'D'}_{ABCD} ,$$

where

$$\begin{aligned} R^{A'B'C'D'}_{ABCD} &= \Psi_{ABCD}\varepsilon^{A'B'}\varepsilon^{C'D'} + \overline{\Psi}^{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &\quad + \Phi^{C'D'}_{AB}\varepsilon_{CD}\varepsilon^{A'B'} + \Phi^{A'B'}_{CD}\varepsilon_{AB}\varepsilon^{C'D'} \\ &\quad + 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon^{A'C'}\varepsilon^{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon^{A'D'}\varepsilon^{B'C'}) . \end{aligned} \tag{2.17}$$

The totally symmetric spinors Ψ_{ABCD} and $\overline{\Psi}^{A'B'C'D'}$ correspond to the spinor representation of the Weyl tensor. The symmetric spinor $\Phi^{C'D'}_{AB}$ corresponds to the trace-free Ricci tensor, and the scalar Λ corresponds to the scalar curvature. If we set

$$[Q_{ab,J_1\cdots J_k}]_{\text{tracefree}} \longleftrightarrow Q^{A'B'J'_1\cdots J'_k}_{ABJ_1\cdots J_k} ,$$

then it is not too difficult to show that

$$Q^{A'B'J'_1\cdots J_k}_{ABJ_1\cdots J_k} = \varepsilon^{A'(J'_1}\varepsilon^{|B'|J'_2}\Psi^{J'_3\cdots J'_k)}_{ABJ_1\cdots J_k} + \varepsilon_{A(J_1}\varepsilon_{|B|J_2}\overline{\Psi}^{A'B'J'_1\ \ J'_k}_{\ \ J_3\cdots J_k)} , \tag{2.18}$$

where

$$\Psi^{J'_1\ \ J'_{k-2}}_{J_1\cdots J_{k+2}} = \nabla^{(J'_1}_{(J_1}\cdots\nabla^{J'_{k-2})}_{J_{k-2}}\Psi_{J_{k-1}J_kJ_{k+1}J_{k+2})} ,$$

and

$$\overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\ \ J_{k-2}} = \nabla^{(J'_1}_{(J_1}\cdots\nabla^{J'_{k-2}}_{J_{k-2})}\overline{\Psi}^{J'_{k-1}J'_kJ'_{k+1}J'_{k+2})} .$$

In summary, *the spinor coordinates for the prolonged Einstein equation manifold* $\mathscr{E}^k$ *are*

$$(x^i, g_{ij}, \Gamma^i_{j_0 j_1}, \ldots, \Gamma^i_{j_0\cdots j_k}, \Psi_{J_1J_2J_3J_4}, \overline{\Psi}^{J'_1J'_2J'_3J'_4}, \ldots, \Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}, \overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\cdots J_{k-2}}) . \tag{2.19}$$

For instance, the spinor coordinates for $\mathscr{E}^2$ and $\mathscr{E}^3$ are

$$(x^i, g_{ij}, \Gamma^i_{j_0 j_1}, \Gamma^i_{j_0 j_1 j_2}, \Psi_{J_1J_2J_3J_4}, \overline{\Psi}^{J'_1J'_2J'_3J'_4}) ,$$

and

$$(x^i, g_{ij}, \Gamma^i_{j_0 j_1}, \Gamma^i_{j_0 j_1 j_2}, \Gamma^i_{j_0 j_1 j_2 j_3}, \Psi_{J_1J_2J_3J_4}, \overline{\Psi}^{J'_1J'_2J'_3J'_4}, \Psi^{J'_1}_{J_1J_2J_3J_4J_5}, \overline{\Psi}^{J'_1J'_2J'_3J'_4J'_5}_{J_1}) .$$

The symmetrized covariant derivatives $\Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}$ and $\overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\cdots J_{k-2}}$ corresponding to Penrose's notion of an exact set of fields for the vacuum Einstein equations [18]. Henceforth we refer to these spinors as the *Penrose fields* for the vacuum Einstein equations, and we denote them by Ψ^k and $\bar{\Psi}^k$. We remark that to pass between the coordinates (2.19) and (2.16) we use any soldering form $\sigma_a^{AA'}$ such that

$$g_{ij} = \sigma_i^{AA'}\sigma_{jAA'}\,.$$

We have the following important structure equation for the Penrose fields [18].

Proposition 2.7. *The spinorial covariant derivative of* $\Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}$, *when evaluated on* $\mathscr{E}^{k+1}$, *is given by*

$$\nabla^{A'}_{A}\Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}} = \Psi^{A'J'_1\cdots J'_{k-2}}_{A\,J_1\cdots J_{k+2}} + \{\bigstar\}\,, \tag{2.20}$$

where $\{\bigstar\}$ *denotes a spinor-valued function of the Penrose fields* $\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}$.

The fact that the lower-order terms $\{\bigstar\}$ are of order less than or equal to $k-1$ is essential to much of our symmetry analysis.

It now follows that the restriction to $\mathscr{E}^k$ of any tensor on $J^k(\mathscr{G})$ built locally from the metric, curvature, and covariant derivatives of curvature, say

$$T_{a_1\cdots a_p}(g_{ij}, g_{ij,j_1},\ldots, g_{ij,j_1\cdots j_k})\,,$$

may be uniquely expressed as a function of the Penrose fields, that is,

$$T_{a_1\cdots a_p} \longleftrightarrow T^{A'_1\cdots A'_p}_{A_1\cdots A_p}(\Psi_{J_1J_2J_3J_4}, \overline{\Psi}^{J'_1J'_2J'_3J'_4}, \ldots, \Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}, \overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\cdots J_{k-2}})\,. \tag{2.21}$$

Under an arbitrary $SL(2,\mathbf{C})$ transformation Λ^A_B, the spinor T satisfies the identity

$$T^{A'_1\cdots A'_p}_{A_1\cdots A_p}[\Lambda\cdot\Psi] = \Lambda^{B_1}_{A_1}\cdots\Lambda^{B_p}_{A_p}\overline{\Lambda}^{A'_1}_{B'_1}\cdots\overline{\Lambda}^{A'_p}_{B'_p}T^{B'_1\cdots B'_p}_{B_1\cdots B_p}[\Psi]\,, \tag{2.22}$$

where $\Lambda\cdot\Psi$ denotes the action of $SL(2,\mathbf{C})$ on the Penrose fields, for example,

$$(\Lambda\cdot\Psi)_{ABCD} = \Lambda^J_A\Lambda^K_B\Lambda^L_C\Lambda^M_D\Psi_{JKLM}\,.$$

We call spinors (2.21) that satisfy (2.22) *natural spinors of the Penrose fields* $\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k$.

We let $\partial_{\Psi^{J_1\cdots J_{k+2}}_{J'_1\cdots J'_{k-2}}}$, $\partial_{\overline{\Psi}^{J_1\cdots J_{k-2}}_{J'_1\cdots J'_{k+2}}}$, and $\partial_{\Gamma^{j_0\cdots j_k}_i}$ denote the (symmetrized) partial differential operators with respect to the coordinates $\Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}$, $\overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\cdots J_{k-2}}$, and $\Gamma^i_{j_0\cdots j_k}$. For example,

$$\partial_{\Psi}{}^{J_1J_2J_3J_4}(\Psi_{ABCD}) = \delta^{(J_1}_A\delta^{J_2}_B\delta^{J_3}_C\delta^{J_4)}_D\,.$$

As a consequence of (2.22) we have the following result [17].

Proposition 2.8. *Let* $T^{A'_1\cdots A'_p}_{A_1\cdots A_q}$ *be a natural spinor of the fields* $\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k$. *The spinorial covariant derivative of* $T^{A'_1\cdots A'_p}_{A_1\cdots A_q}$ *is a natural spinor of the Penrose* *fields* $\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k+1}, \overline{\Psi}^{k+1}$, *and is given by*

$$\nabla^{B'}_{B} T^{A'_1\cdots A'_p}_{A_1\cdots A_q} = \sum_{l=2}^{k} \big[\partial_{\Psi^{J_1\cdots J_{l+2}}_{J'_1\cdots J'_{l-2}}} T^{A'_1\cdots A'_p}_{A_1\cdots A_q}\big] \nabla^{B'}_{B} \Psi^{J'_1\cdots J'_{l-2}}_{J_1\cdots J_{l+2}} + \sum_{l=2}^{k} \big[\partial_{\overline{\Psi}^{J_1\cdots J_{l-2}}_{J'_1\cdots J'_{l+2}}} T^{A'_1\cdots A'_p}_{A_1\cdots A_q}\big] \nabla^{B'}_{B} \overline{\Psi}^{J'_1\cdots J'_{l+2}}_{J_1\cdots J_{l-2}} .$$

We close this section by deriving a spinor expression for the linearized Einstein equations (2.8) that we shall use to compute generalized symmetries. Starting from (2.8), and using the spinor correspondence

$$\nabla_c\nabla_d \longleftrightarrow \nabla_{CC'}\nabla_{DD'} \qquad h_{ab} \longleftrightarrow h_{ABA'B'} \qquad g^{cd} \longleftrightarrow \varepsilon^{CD}\varepsilon^{C'D'} ,$$

the defining equation (2.8) takes the form

$$[-\varepsilon^{CD}\varepsilon^{C'D'}\delta^A_M\delta^{A'}_{M'}\delta^B_N\delta^{B'}_{N'} - \varepsilon^{AB}\varepsilon^{A'B'}\delta^C_M\delta^{C'}_{M'}\delta^D_N\delta^{D'}_{N'} + \varepsilon^{AC}\varepsilon^{A'C'}(\delta^B_M\delta^{B'}_{M'}\delta^D_N\delta^{D'}_{N'} + \delta^D_M\delta^{D'}_{M'}\delta^B_N\delta^{B'}_{N'})]\nabla_{CC'}\nabla_{DD'}h_{ABA'B'} = 0 . \tag{2.23}$$

Since $h_{ABA'B'} = h_{BAB'A'}$, we have that

$$h_{ABA'B'} = h_{BAA'B'} + \tfrac{1}{2}\varepsilon_{AB}\varepsilon_{A'B'}h , \tag{2.24}$$

where the trace h of $h_{ABA'B'}$ is given by

$$h = \varepsilon^{AB}\varepsilon^{A'B'}h_{ABA'B'} .$$

Substituting (2.24) into the last two terms of (2.23), we find that all the trace terms cancel leaving us with

$$[-\varepsilon^{CD}\varepsilon^{C'D'}\delta^A_M\delta^{A'}_{M'}\delta^B_N\delta^{B'}_{N'} + \varepsilon^{BC}\varepsilon^{A'C'}\delta^A_M\delta^{B'}_{M'}\delta^D_N\delta^{D'}_{N'} + \varepsilon^{BC}\varepsilon^{A'C'}\delta^D_M\delta^{D'}_{M'}\delta^A_N\delta^{B'}_{N'}] \times \nabla_{CC'}\nabla_{DD'}h_{ABA'B'} = 0 .$$

We now multiply this expression with arbitrary spinors α^M, $\overline{\alpha}^{M'}$, β^N, $\overline{\beta}^{N'}$ to get our final spinor form of the linearized equations.

Theorem 2.9. *If* $h^{A\,B}_{A'B'}$ *are the spinor components of a generalized symmetry of the vacuum Einstein equations, then for all spinors* α^M, $\overline{\alpha}^{M'}$, β^N, $\overline{\beta}^{N'}$,

$$[-\varepsilon_{CD}\varepsilon^{C'D'}\alpha_A\beta_B\overline{\alpha}^{A'}\overline{\beta}^{B'} + \varepsilon_{BC}\varepsilon^{A'C'}\alpha_A\beta_D\overline{\alpha}^{B'}\overline{\beta}^{D'} + \varepsilon_{BC}\varepsilon^{A'C'}\alpha_D\beta_A\overline{\alpha}^{D'}\overline{\beta}^{B'}]\nabla^C_{C'}\nabla^D_{D'}h^{A\,B}_{A'B'} = 0 \qquad \text{on } \mathscr{E}^{k+2} . \tag{2.25}$$

In general $h^{A\,B}_{A'B'}$ *is a function of the coordinates* (2.19), *that is,*

$$h^{A\,B}_{A'B'} = h^{A\,B}_{A'B'}(x^i, \sigma^{aB'}_{\ \ A}, \Gamma^i_{j_0j_1}, \ldots, \Gamma^i_{j_0\cdots j_k}, \Psi_{J_1J_2J_3J_4}, \overline{\Psi}^{J'_1J'_2J'_3J'_4}, \ldots, \Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}, \overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\cdots J_{k-2}}) .$$

When $h^{A\,B}_{A'B'}$ *is a natural generalized symmetry,*

$$h^{A\,B}_{A'B'} = h^{A\,B}_{A'B'}(\Psi_{J_1J_2J_3J_4}, \overline{\Psi}^{J'_1J'_2J'_3J'_4}, \ldots, \Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}}, \overline{\Psi}^{J'_1\cdots J'_{k+2}}_{J_1\cdots J_{k-2}}) .$$

In both cases, $h^{AB}_{A'B'}$ *satisfies the* $SL(2,\mathbf{C})$ *invariance properties*

$$\Lambda^C_A \Lambda^D_B \Lambda^{A'}_{C'} \Lambda^{B'}_{D'} h^{AB}_{A'B'}(x,\sigma,\Gamma,\Psi) = h^{CD}_{C'D'}(x,\Lambda \cdot \sigma,\Gamma,\Lambda \cdot \Psi)\,,$$

where $\Lambda \cdot \sigma$, *and* $\Lambda \cdot \Psi$ *denote the action of* $SL(2,\mathbf{C})$ *on the soldering form and Penrose fields.*

3. Third-Order Symmetries of the Einstein Equations: A Model Problem

The complete classification of higher-order generalized symmetries of even the simplest partial differential equations, let alone the Einstein field equations, is almost always a daunting computational task. In this section we shall characterize a particularly simple class of third-order generalized symmetries of the vacuum Einstein equations. Subsequent sections of this paper will extend this analysis in full generality. Our goals here are principally to elucidate our basic computational scheme and to introduce notation and techniques which will be used repeatedly in what follows.

Recall that a third-order generalized symmetry of the Einstein equations is a symmetric rank-2 spacetime tensor

$$h_{ab} = h_{ab}(x^i,\, g_{ij},\, g_{ij,h},\, g_{ij,hk},\, g_{ij,hkl}) \tag{3.1}$$

that satisfies the linearized equations (2.8). The standard jet coordinates on $J^3(\mathscr{G})$,

$$(x^i,\, g_{ij},\, g_{ij,k},\, g_{ij,hk},\, g_{ij,hkl})\,,$$

are ill-suited to the problem of solving Eq. (2.8) because they are not well-adapted to the structure of the Einstein field equations. In Sect. 2 we showed that any function (3.1) can also be expressed as

$$h_{ab} = h_{ab}(x^i,\, g_{ij},\, \Gamma^i_{jk},\, \Gamma^i_{jkh},\, \Gamma^i_{jkhl},\, Q_{ij,kh},\, Q_{ij,khl})\,,$$

where the generalized Christoffel symbols $\Gamma^i_{jkh}, \Gamma^i_{jkhl}$ are defined by (2.12), and the curvature tensors $Q_{ij,kh}, Q_{ij,khl}$ are defined by (2.13). However, a generalized symmetry of the Einstein equations is only defined up to terms which vanish when $R_{ij} = 0$, $R_{ij|k} = 0$, …, and so, with no loss of generality, we may replace the dependencies of h_{ab} on $Q_{ij,kh}$ and $Q_{ij,khl}$ by their trace-free parts. These trace-free tensors are best represented by the Penrose fields and so we can assume that the general form of the third-order symmetry is given by

$$h_{ab} = h_{ab}(x^i,\, \sigma^{A'}_{iA},\, \Gamma^i_{jk},\, \Gamma^i_{jkh},\, \Gamma^i_{jkhl},\, \Psi_{A_1A_2A_3A_4},\, \Psi^{A'_1}_{A_1A_2A_3A_4A_5},\, \overline{\Psi}^{A'_1A'_2A'_3A'_4},\, \overline{\Psi}^{A'_1A'_2A'_3A'_4A'_5}_{A_1})\,. \tag{3.2}$$

In the next section we classify arbitrary-order symmetries depending upon the Penrose fields alone (natural symmetries). In Sects. 5 and 6 we complete the generalized symmetry analysis of the Einstein equations by considering higher-order symmetries with dependencies typified by (3.2). In this section we shall simply analyze natural symmetries whose spinor components are of the general form

$$h^{AB}_{A'B'} = h^{AB}_{A'B'}(\Psi_{A_1A_2A_3A_4},\, \Psi^{A'_1}_{A_1A_2A_3A_4A_5})\,. \tag{3.3}$$

Admittedly, this is a somewhat artificial problem, but it serves us well for the purposes of this section. We shall prove that if (3.3) satisfies the linearized equations

$$[-\varepsilon^{CD}\varepsilon^{C'D'}\delta^A_M\delta^{A'}_{M'}\delta^B_N\delta^{B'}_{N'} + \varepsilon^{BC}\varepsilon^{A'C'}\delta^A_M\delta^{B'}_{M'}\delta^D_N\delta^{D'}_{N'} + \varepsilon^{BC}\varepsilon^{A'C'}\delta^D_M\delta^{D'}_{M'}\delta^A_N\delta^{B'}_{N'}] \times \nabla_{CC'}\nabla_{DD'}h_{ABA'B'} = 0\,, \tag{3.4}$$

when $R_{ij} = 0$, $R_{ij|k} = 0$, $R_{ij|kl} = 0$, and $R_{ij|klm} = 0$, then there is a vector field

$$X^B_{B'} = X^B_{B'}(\Psi_{A_1A_2A_3A_4})\,,$$

and a constant c such that ,

$$h^{A\,B}_{A'B'} = c\,\varepsilon^{AB}\varepsilon_{A'B'} + \nabla^A_{A'}X^B_{B'} + \nabla^B_{B'}X^A_{A'}\,.$$

To begin the analysis of (3.4), we first expand the covariant derivatives of $h^{A\,B}_{A'B'}$ by the chain rule (Proposition 2.8) to find that, because of (3.3),

$$\begin{aligned}\nabla^D_{D'}h^{A\,B}_{A'B'} &= \frac{\partial h^{A\,B}_{A'B'}}{\partial\Psi_{A_1A_2A_3A_4}}(\nabla^D_{D'}\Psi_{A_1A_2A_3A_4}) + \frac{\partial h^{A\,B}_{A'B'}}{\partial\Psi^{A_1'}_{A_1A_2A_3A_4A_5}}(\nabla^D_{D'}\Psi^{A_1'}_{A_1A_2A_3A_4A_5})\\ &= (\partial_\Psi{}^{A_1A_2A_3A_4}h^{A\,B}_{A'B'})\Psi^D{}_{A_1A_2A_3A_4D'}\\ &\quad + (\partial_{\Psi}{}^{A_1A_2A_3A_4A_5}_{A_1'}h^{A\,B}_{A'B'})(\Psi^{A_1'D}{}_{A_1A_2A_3A_4A_5D'} + \{\bigstar\})\,,\end{aligned} \tag{3.5}$$

where we have used the structure equations (2.20), and $\{\bigstar\}$ reflects terms quadratic in Ψ_{ABCD}. We take another covariant derivative and, retaining for the moment only the terms involving $\Psi^{A_1'CD}_{A_1\cdots A_5C'D'}$ and $\Psi^{A_1'}_{A_1\cdots A_5}\Psi^{B_1'}_{B_1\cdots B_5}$, we find that

$$\begin{aligned}\nabla^C_{C'}\nabla^D_{D'}h^{A\,B}_{A'B'} &= (\partial_{\Psi}{}^{A_1\cdots A_5}_{A_1'}h^{A\,B}_{A'B'})\Psi^{A_1'CD}{}_{A_1\cdots A_5C'D'}\\ &\quad + (\partial_{\Psi}{}^{A_1\cdots A_5}_{A_1'}\partial_{\Psi}{}^{B_1\cdots B_5}_{B_1'}h^{A\,B}_{A'B'})\Psi^{A_1'C}{}_{A_1\cdots A_5C'}\Psi^{B_1'D}{}_{B_1\cdots B_5D'} + \{\bigstar\}\,,\end{aligned}$$

where $\{\bigstar\}$ denotes terms depending upon the Penrose fields $\Psi_{A_1\cdots A_4}$, $\Psi^{A_1'}_{A_1\cdots A_5}$, and terms linear in $\Psi^{A_1'A_2'}_{A_1\cdots A_6}$.

The critical point to make now is that, in using the structure equations (2.20) to evaluate $\nabla^D_{D'}h^{A\,B}_{A'B'}$ and $\nabla^C_{C'}\nabla^D_{D'}h^{A\,B}_{A'B'}$, we have made full use of the Einstein equations. That is to say, the fields $\Psi^{A_1'A_2'A_3'}_{A_1A_2\cdots A_7}$ and $\Psi^{A_1'A_2'}_{A_1A_2\cdots A_6}$ are completely arbitrary and (3.4) is an identity in these higher-order Penrose fields. All the analysis that follows depends upon this fact.

Thus, in (3.4) the terms depending upon $\Psi^{A_1'A_2'A_3'}_{A_1A_2\cdots A_7}$ must vanish identically. Taking into account the symmetries of this spinor field, we conclude that the derivative $\partial_{\Psi}{}^{A_1\cdots A_5}_{A_1'}h^{A\,B}_{A'B'}$ satisfies the complicated algebraic equation

$$\varepsilon^{B(A_6}\varepsilon_{B'(A_2'}\partial_{\Psi}{}^{A_1\cdots A_5}_{A_1'}h^{A_7)A}_{|A'|A_3')} + \varepsilon^{A(A_6}\varepsilon_{A'(A_2'}\partial_{\Psi}{}^{A_1\cdots A_5}_{A_1'}h^{A_7)B}_{|B'|A_3')} = 0\,. \tag{3.6}$$

Our next task is to analyze this equation completely. This is almost impossible to do without first introducing some appropriate notation. Then we can bring to bear some decidedly non-trivial – but completely algebraic – results from the two-component spinor formalism. To begin, we set

$$[\partial^3_\Psi h](\psi^5,\overline{\psi};\alpha,\beta,\overline{\alpha},\overline{\beta}) = [\partial_\Psi{}^{A_1A_2A_3A_4A_5}_{A'_1} h^{A\,B}_{A'B'}]\psi_{A_1}\psi_{A_2}\psi_{A_3}\psi_{A_4}\psi_{A_5}\alpha_A\beta_B\overline{\psi}^{A'_1}\overline{\alpha}^{A'}\overline{\beta}^{B'} \; . \quad (3.7)$$

As we explain in the appendix, this notation indicates that $[\partial^3_\Psi h]$ is a spinor which is completely symmetric in its first 5 indices. We use a semi-colon here to separate the arguments of $[\partial^3_\Psi h]$ that correspond to the derivative indices from those attached to the spinor $h^{A\,B}_{A'B'}$. We emphasize that the values of $\partial_\Psi{}^{A_1\cdots A_5}_{A'_1} h^{A\,B}_{A'B'}$ are completely determined by the values of (3.7).

We now multiply (3.6) by $\psi_{A_1}\psi_{A_2}\cdots\psi_{A_7}\alpha_A\beta_B\overline{\psi}^{A'_1}\overline{\psi}^{A'_2}\overline{\psi}^{A'_3}\overline{\alpha}^{A'}\overline{\beta}^{B'}$ to arrive at the more palpable, but completely equivalent equation,

$$\begin{aligned}\langle\beta,\,\psi\rangle\langle\overline{\beta},\,\overline{\psi}\rangle[\partial^3_\Psi h](\psi^5,\,\overline{\psi};\psi,\,\alpha,\,\overline{\alpha},\,\overline{\psi})&\\ +\,\langle\alpha,\,\psi\rangle\langle\overline{\alpha},\,\overline{\psi}\rangle[\partial^3_\Psi h](\psi^5,\,\overline{\psi};\psi,\,\beta,\,\overline{\beta},\,\overline{\psi}) = 0\,.&\end{aligned} \quad (3.8)$$

If we set $\alpha=\beta$ and $\overline{\alpha}=\overline{\beta}$, this equation reduces to

$$[\partial^3_\Psi h](\psi^5,\,\overline{\psi};\psi,\,\alpha,\,\overline{\alpha},\,\overline{\psi}) = 0\,,$$

or, because of the symmetry of $h^{A\,B}_{A'B'}$,

$$[\partial^3_\Psi h](\psi^5,\,\overline{\psi};\alpha,\,\psi,\,\overline{\psi},\,\overline{\alpha}) = 0\,. \quad (3.9)$$

In components this first equation is equivalent to

$$\partial_\Psi{}^{(A_1A_2\cdots A_5}_{(A'_1} h^{A)B}_{|A'|B')} = 0\,. \quad (3.10)$$

We now recall (Proposition 7.2) that if $T^{A_1A_2\cdots A_5,A}$ is a spinor that is symmetric in $A_1A_2\cdots A_5$ and satisfies

$$T^{(A_1A_2\cdots A_5,A)} = 0\,,$$

then there is a symmetric spinor $S^{A_1A_2A_3A_4}$ such that

$$T^{A_1A_2\cdots A_5,A} = S^{(A_1A_2A_3A_4}\varepsilon^{A_5)A}\,.$$

In terms of our index-free notation, we write this equation as (see Proposition 7.2)

$$T(\psi^5,\alpha) = \langle\psi,\alpha\rangle S(\psi^4)\,.$$

We apply this result to (3.9) to conclude that

$$[\partial^3_\Psi h](\psi^5,\,\overline{\psi};\alpha,\,\beta,\,\overline{\alpha},\,\overline{\psi}) = \langle\psi,\,\alpha\rangle S(\psi^4,\,\overline{\psi}^2,\,\beta,\,\overline{\alpha})\,. \quad (3.11)$$

We can decompose $[\partial^3_\Psi h](\psi^5,\overline{\psi};\alpha,\beta,\overline{\alpha},\overline{\beta})$ into its symmetric and antisymmetric parts in the arguments corresponding to $\overline{\psi}$ and $\overline{\beta}$, and then use (3.11) to conclude

$$[\partial^3_\Psi h](\psi^5,\,\overline{\psi};\alpha,\,\beta,\,\overline{\alpha},\,\overline{\beta}) = \langle\psi,\,\alpha\rangle S(\psi^4,\,\overline{\psi}\,\overline{\beta},\,\beta,\,\overline{\alpha}) + \langle\overline{\psi},\,\overline{\beta}\rangle T(\psi^5,\,\alpha,\,\beta,\,\overline{\alpha})\,.$$

Unfortunately, this representation of $[\partial^3_\Psi h]$ does not incorporate the symmetry

$$[\partial^3_\Psi h](\psi^5,\,\overline{\psi}^1;\alpha,\,\beta,\,\overline{\alpha},\,\overline{\beta}) = [\partial^3_\Psi h](\psi^5,\,\overline{\psi}^1;\beta,\,\alpha,\,\overline{\beta},\,\overline{\alpha})\,. \quad (3.12)$$

It is a rather difficult algebraic problem to simultaneously impose both (3.9) and (3.12). Theorem 7.7, in the appendix to this paper, solves this problem and we may write

$$[\partial^3_{\Psi} h](\psi^5, \overline{\psi}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = \langle \psi, \alpha \rangle \langle \psi, \beta \rangle A(\psi^3, \overline{\psi}\overline{\alpha}\overline{\beta}) + \langle \psi, \alpha \rangle \langle \overline{\alpha}, \overline{\psi} \rangle W(\psi^4, \beta, \overline{\beta}) + \langle \psi, \beta \rangle \langle \overline{\beta}, \overline{\psi} \rangle W(\psi^4, \alpha, \overline{\alpha}) . \tag{3.13}$$

Written out in components, this reads

$$\partial \Psi^{A_1 A_2 \cdots A_5}_{A'_1} h^{A\,B}_{A'B'} = \varepsilon^{A(A_1} \varepsilon^{|B|A_2} A^{A_3 A_4 A_5)}_{A'_1 A' B'} - \varepsilon^{A(A_1} \varepsilon_{A'(A'_1} W^{A_2 A_3 A_4 A_5)B}_{B')} - \varepsilon^{B(A_1} \varepsilon_{B'(A'_1} W^{A_2 A_3 A_4 A_5)A}_{A')} .$$

With the symmetries as indicated in (3.13), the spinors A and W are uniquely defined by $\partial^3_{\Psi} h$. This result completely solves (3.8).

The next step is to analyze the consequences of setting to zero the terms in (3.4) which are quadratic in the Penrose fields $\Psi^{A'_1}_{A_1 A_2 \cdots A_5}$. To accomplish this we differentiate (3.4) with respect to $\Psi^{C'_1}_{C_1 C_2 \cdots C_5}$ and $\Psi^{D'_1}_{D_1 D_2 \cdots D_5}$ and multiply the result by "symmetrizing" fields $\psi_{C_1} \psi_{C_2} \cdots \psi_{C_5}$, $\chi_{D_1} \chi_{D_2} \cdots \chi_{D_5}$, and $\overline{\psi}^{C'_1} \overline{\chi}^{D'_1}$. Because

$$[\partial \Psi^{C_1 C_2 \cdots C_5}_{C'_1} \Psi^{A'_1}_{A_1 A_2 \cdots A_5}] \psi_{C_1} \psi_{C_2} \cdots \psi_{C_5} \overline{\psi}^{C'_1} = \psi_{A_1} \psi_{A_2} \cdots \psi_{A_5} \overline{\psi}^{A'_1} ,$$

we obtain the equations

$$\begin{aligned} &-2\langle \psi, \chi \rangle \langle \overline{\psi}, \overline{\chi} \rangle (\partial^3_{\Psi} \partial^3_{\Psi} h)(\psi^5, \overline{\psi}; \chi^5, \overline{\chi}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\ &\quad + \langle \psi, \beta \rangle \langle \overline{\psi}, \overline{\beta} \rangle (\partial^3_{\Psi} \partial^3_{\Psi} h)(\psi^5, \overline{\psi}; \chi^5, \overline{\chi}; \alpha, \chi, \overline{\chi}, \overline{\alpha}) \\ &\quad + \langle \chi, \beta \rangle \langle \overline{\chi}, \overline{\beta} \rangle (\partial^3_{\Psi} \partial^3_{\Psi} h)(\psi^5, \overline{\psi}; \chi^5, \overline{\chi}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) \\ &\quad + \langle \psi, \alpha \rangle \langle \overline{\psi}, \overline{\alpha} \rangle (\partial^3_{\Psi} \partial^3_{\Psi} h)(\psi^5, \overline{\psi}; \chi^5, \overline{\chi}; \beta, \chi, \overline{\chi}, \overline{\beta}) \\ &\quad + \langle \chi, \alpha \rangle \langle \overline{\chi}, \overline{\alpha} \rangle (\partial^3_{\Psi} \partial^3_{\Psi} h)(\psi^5, \overline{\psi}; \chi^5, \overline{\chi}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0 . \end{aligned} \tag{3.14}$$

The six terms in this equation (one appears twice) come from each of the 3 terms in (3.4). Each term of (3.4) contributes twice to (3.14) because the coefficient of these terms is quadratic in the Penrose fields $\Psi^{A'_1}_{A_1 A_2 \cdots A_5}$. Note that we have again used semi-colons to separate the arguments of $\partial^3_{\Psi} \partial^3_{\Psi} h$ corresponding to the different partial derivatives.

Using (3.9) we immediately find that all the terms in (3.14) vanish but the first, so that

$$(\partial^3_{\Psi} \partial^3_{\Psi} h)(\psi^5, \overline{\psi}; \chi^5, \overline{\chi}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0 .$$

In components this equation is

$$\partial \Psi^{A_1 A_2 \cdots A_5}_{A'_1} \partial \Psi^{B_1 B_2 \cdots B_5}_{B'_1} h^{A\,B}_{A'B'} = 0 ,$$

and hence $h^{A\,B}_{A'B'}$ *must be linear in the third derivative fields* $\Psi^{A'_1}_{A_1A_2\cdots A_5}$. This then implies, by the uniqueness of the representation (3.13), that the spinors A and W in (3.13) can only depend upon the second derivative Penrose field Ψ_{ABCD}. This exhausts the information arising from the coefficients of the highest derivative term $\Psi^{A'_1A'_2A'_3}_{A_1A_2\cdots A_7}$ and the quadratic terms $\Psi^{A'_1A'_2}_{A_1A_2\cdots A_6}\Psi^{B'_1B'_2}_{B_1B_2\cdots B_6}$ in the linearized equations.

To summarize the results up to this point, we have shown that the generalized symmetry (3.3) must take the form

$$h^{A\,B}_{A'B'} = A^{A_1A_2A_3}_{A'_1A'B'}\Psi^{ABA'_1}_{A_1A_2A_3} + W^{A_1\cdots A_4B}_{(A'}\Psi^{A}_{B')A_1\cdots A_4} + W^{A_1\cdots A_4A}_{(A'}\Psi^{B}_{B')A_1\cdots A_4} + \tilde{h}^{A\,B}_{A'B'}\,,$$

where the spinors A, W and $\tilde{h}$ all depend on Ψ_{ABCD}.

The next step is to examine the terms in the linearized equations (3.4) which involve the spinor

$$\Psi^{A'_1}_{A_1A_2\cdots A_5}\Psi^{B'_1B'_2}_{B_1B_2\cdots B_6}\,.$$

Taking into account the fact that $h^{A\,B}_{A'B'}$ is linear in $\Psi^{A'_1}_{A_1A_2\cdots A_5}$, and the structure equations (2.20) for the derivative $\nabla^C_{C'}\Psi^{A'_1A'_2}_{A_1A_2\cdots A_6}$, we find the relevant terms to be

$$\begin{aligned}\nabla^C_{C'}\nabla^D_{D'}h^{A\,B}_{A'B'} =&(\partial_\Psi{}^{A_1A_2A_3A_4}\partial_{\Psi^{B'_5}}{}^{B_1B_2\cdots B_5}h^{A\,B}_{A'B'})(\Psi^C_{A_1A_2A_3A_4C'}\Psi^D_{B_1B_2\cdots B_5D'}\\ &+\Psi^D_{A_1A_2A_3A_4D'}\Psi^C_{B_1B_2\cdots B_5C'}) + \{\bigstar\}\,.\end{aligned}$$

Thus, when we differentiate (3.4) with respect to $\Psi^{A'_1}_{A_1A_2\cdots A_5}$, $\Psi^{B'_1B'_2}_{B_1B_2\cdots B_6}$ and contract with the fields $\psi_{A_1}\cdots\psi_{A_5}$, $\chi_{B_1}\cdots\chi_{B_6}$, and $\overline{\psi}^{A'_1}\overline{\chi}^{B'_1}\overline{\chi}^{B'_2}$ we conclude, after some simplifications, that the derivatives

$$\begin{aligned}&[\partial^2_\Psi\partial^3_\Psi h](\psi^4;\chi^5,\overline{\chi};\alpha,\beta,\overline{\alpha},\overline{\beta})\\ &\quad= (\partial_\Psi{}^{A_1A_2A_3A_4}\partial_{\Psi^{B'_5}}{}^{B_1B_2\cdots B_5}h^{A\,B}_{A'B'})\psi_{A_1}\psi_{A_2}\psi_{A_3}\psi_{A_4}\chi_{B_1}\chi_{B_2}\cdots\chi_{B_5}\overline{\chi}^{B'_5}\alpha_A\beta_B\overline{\alpha}^{A'}\overline{\beta}^{B'}\end{aligned}$$

satisfy the algebraic conditions

$$\begin{aligned}&-2\langle\psi,\chi\rangle\langle\overline{\psi},\overline{\chi}\rangle[\partial^2_\Psi\partial^3_\Psi h](\chi^4;\psi^5,\overline{\psi};\alpha,\beta,\overline{\alpha},\overline{\beta})\\ &+\langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle[\partial^2_\Psi\partial^3_\Psi h](\psi^4;\chi^5,\overline{\chi};\alpha,\psi,\overline{\psi},\overline{\alpha})\\ &+\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle[\partial^2_\Psi\partial^3_\Psi h](\psi^4;\chi^5,\overline{\chi};\beta,\psi,\overline{\psi},\overline{\beta})\\ &+\langle\chi,\beta\rangle\langle\overline{\chi},\overline{\beta}\rangle[\partial^2_\Psi\partial^3_\Psi h](\chi^4;\psi^5,\overline{\psi};\alpha,\psi,\overline{\psi},\overline{\alpha})\\ &+\langle\chi,\alpha\rangle\langle\overline{\chi},\overline{\alpha}\rangle[\partial^2_\Psi\partial^3_\Psi h](\chi^4;\psi^5,\overline{\psi};\beta,\psi,\overline{\psi},\overline{\beta})\\ &+\langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle[\partial^2_\Psi\partial^3_\Psi h](\chi^4;\psi^5,\overline{\psi};\alpha,\chi,\overline{\chi},\overline{\alpha})\\ &+\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle[\partial^2_\Psi\partial^3_\Psi h](\chi^4;\psi^5,\overline{\psi};\beta,\chi,\overline{\chi},\overline{\beta}) = 0\,.\end{aligned}\tag{3.15}$$

These equations we analyze in 2 steps. First, Eq. (3.9) implies that the coefficients of $\langle\chi,\beta\rangle\langle\overline{\chi},\overline{\beta}\rangle$ and $\langle\chi,\alpha\rangle\langle\overline{\chi},\overline{\alpha}\rangle$ each vanish, and so we can rewrite Eq. (3.15) as

$$\begin{aligned}
&-2\langle\psi,\chi\rangle\,\langle\overline{\psi},\overline{\chi}\rangle\,[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\chi^4;\psi^5,\overline{\psi};\alpha,\beta,\overline{\alpha},\overline{\beta})\\
&\quad+\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle\{[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\psi^4;\chi^5,\overline{\chi};\beta,\psi,\overline{\psi},\overline{\beta})\\
&\qquad\qquad+[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\chi^4;\psi^5,\overline{\psi};\beta,\chi,\overline{\chi},\overline{\beta})\}\\
&\quad+\langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle\{[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\psi^4;\chi^5,\overline{\chi};\alpha,\psi,\overline{\psi},\overline{\alpha})\\
&\qquad\qquad+[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\chi^4;\psi^5,\overline{\psi};\alpha,\chi,\overline{\chi},\overline{\alpha})\}=0\,. \qquad (3.16)
\end{aligned}$$

Setting $\overline{\alpha}=\overline{\beta}=\overline{\psi}$ in Eq. (3.16), we conclude that

$$[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\chi^4;\psi^5,\overline{\psi};\alpha,\beta,\overline{\psi},\overline{\psi})=0\,.$$

In terms of the decomposition (3.13), this implies that

$$[\partial^2_{\Psi}A](\chi^4;\psi^3,\overline{\psi}^3)=0\,, \qquad (3.17)$$

and so A in (3.13) is independent of the spinor Ψ_{ABCD}, i.e., A is independent of the Penrose fields. Together, Eqs. (3.16) and (3.17) show that

$$\begin{aligned}
[\partial^2_{\Psi}\partial^3_{\overline{\Psi}}h](\chi^4;\psi^5,\overline{\psi};\alpha,\beta,\overline{\alpha},\overline{\beta})&=\langle\psi,\alpha\rangle\langle\overline{\alpha},\overline{\psi}\rangle[\partial^2_{\Psi}W](\chi^4;\psi^4,\beta,\overline{\beta})\\
&\quad+\langle\psi,\beta\rangle\langle\overline{\beta},\overline{\psi}\rangle[\partial^2_{\Psi}W](\chi^4;\psi^4,\alpha,\overline{\alpha})\,. \qquad (3.18)
\end{aligned}$$

Next, we set $\alpha=\beta$ and $\overline{\alpha}=\overline{\beta}$ in (3.13), and substitute from (3.18) to arrive at

$$\begin{aligned}
2\langle\psi,\chi\rangle\langle\overline{\psi},\overline{\chi}\rangle[\partial^2_{\Psi}W](\chi^4;\psi^4,\alpha,\overline{\alpha})=&\langle\chi,\alpha\rangle\langle\overline{\chi},\overline{\psi}\rangle[\partial^2_{\Psi}W](\psi^4;\chi^4,\psi,\overline{\alpha})\\
&+\langle\chi,\psi\rangle\langle\overline{\chi},\overline{\alpha}\rangle[\partial^2_{\Psi}W](\psi^4;\chi^4,\alpha,\overline{\psi})\\
&+\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\chi}\rangle[\partial^2_{\Psi}W](\chi^4;\psi^4,\chi,\overline{\alpha})\\
&+\langle\psi,\chi\rangle\langle\overline{\psi},\overline{\alpha}\rangle[\partial^2_{\Psi}W](\chi^4;\psi^4,\alpha,\overline{\chi})\,.
\end{aligned}$$

The right-hand side of this equation is unchanged by the simultaneous interchange of ψ with χ and $\overline{\psi}$ with $\overline{\chi}$, so we conclude

$$[\partial^2_{\Psi}W](\chi^4;\psi^4,\alpha,\overline{\alpha})=[\partial^2_{\Psi}W](\psi^4;\chi^4,\alpha,\overline{\alpha})\,.$$

Written out in full, this equation is the curl condition

$$\frac{\partial W_{A'}^{B_1B_2B_3B_4A}}{\partial\Psi_{A_1A_2A_3A_4}}=\frac{\partial W_{A'}^{A_1A_2A_3A_4A}}{\partial\Psi_{B_1B_2B_3B_4}}\,.$$

We therefore deduce that there are functions

$$X^A_{A'}=X^A_{A'}(\Psi_{A_1A_2A_3A_4})$$

such that

$$W_{A'}^{A_1A_2A_3A_4A}=\frac{\partial X^A_{A'}}{\partial\Psi_{A_1A_2A_3A_4}}\,. \qquad (3.19)$$

Together, Eqs. (3.17) and (3.19) solve (3.15) completely.

Define

$$k_{A'B'}^{A\,B} = h_{A'B'}^{A\,B} - (\nabla_{A'}^{A} X_{B'}^{B} + \nabla_{B'}^{B} X_{A'}^{A}) . \tag{3.20}$$

By Proposition 2.5, $k_{A'B'}^{A\,B}$ is also a generalized symmetry. On account of (3.19), a simple computation shows that

$$[\partial_{\Psi}^{3} k](\psi^5, \overline{\psi}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = \langle \psi, \alpha \rangle \langle \psi, \beta \rangle A(\psi^3, \overline{\psi}\overline{\alpha}\overline{\beta}) . \tag{3.21}$$

This means that $k_{A'B'}^{A\,B}$ takes the form

$$k_{A'B'}^{A\,B} = A_{A_1'A'B'}^{A_1A_2A_3} \Psi_{A_1A_2A_3}^{ABA_1'} + \tilde{h}_{A'B'}^{A\,B} .$$

We have already shown that the spinor A is constant. We now show that this spinor must in fact vanish. To do this we must isolate the terms in (3.4) which are linear in the fourth order Penrose fields $\Psi_{A_1A_2\cdots A_6}^{A_1'A_2'}$. This can be done by simply expanding the total covariant derivatives as we did in (3.5). However, this procedure is somewhat complicated, and does not readily generalize to the higher-order symmetry analysis we shall give in subsequent sections. We therefore introduce an alternative, more powerful, approach to this step in our analysis, one based upon the commutation rules for the total covariant derivative operator $\nabla_{D'}^{D}$ and the partial derivative operator $\partial_{\Psi_{B_1'B_2'}^{B_1B_2\cdots B_6}}$.

Lemma 3.1. *If* $F = F(\Psi_{A_1A_2A_3A_4}, \Psi_{A_1A_2\cdots A_5}^{A_1'}, \Psi_{A_1A_2\cdots A_6}^{A_1'A_2'})$, *then*

$$\partial_{\Psi_{B_1'B_2'}^{B_1B_2\cdots B_6}}(\nabla_{D'}^{D} F) = \nabla_{D'}^{D}(\partial_{\Psi_{B_1'B_2'}^{B_1B_2\cdots B_6}} F) - \varepsilon^{D(B_6}\varepsilon_{D'(B_2'}\partial_{\Psi_{B_1')}^{B_1B_2\cdots B_5)}} F . \tag{3.22}$$

If $F = F(\Psi_{A_1A_2A_3A_4}, \Psi_{A_1A_2\cdots A_5}^{A_1'})$, *then*

$$\partial_{\Psi_{B_1'}^{B_1B_2\cdots B_5}}(\nabla_{D'}^{D} F) = \nabla_{D'}^{D}(\partial_{\Psi_{B_1'}^{B_1B_2\cdots B_5}} F) - \varepsilon^{D(B_5}\varepsilon_{D'B_1'}\partial_{\Psi^{B_1B_2\cdots B_4)}} F . \tag{3.23}$$

Proof. These formulas follow directly from the chain rule (Proposition 2.8) and the structure equations (2.20). □

Note that we can express (3.22) and (3.23) more succinctly as

$$[\partial_{\Psi}^{4}(\nabla_{D'}^{D} F)](\psi^6, \overline{\psi}^2) = \nabla_{D'}^{D}[\partial_{\Psi}^{4} F](\psi^6, \overline{\psi}^2) + \psi^D \overline{\psi}_{D'} [\partial_{\Psi}^{3} F](\psi^5, \overline{\psi})$$

and

$$[\partial_{\Psi}^{3}(\nabla_{D'}^{D} F)](\psi^5, \overline{\psi}) = \nabla_{D'}^{D}[\partial_{\Psi}^{3} F](\psi^5, \overline{\psi}) + \psi^D \overline{\psi}_{D'} [\partial_{\Psi}^{2} F](\psi^4) .$$

We can apply this lemma to the spinor $\nabla_{C'}^{C}\nabla_{D'}^{D} k_{A'B'}^{A\,B}$; we find

$$\begin{aligned}
&\partial_{\Psi}^{4}[\nabla_{C'}^{C}\nabla_{D'}^{D} k_{A'B'}^{A\,B}](\psi^6, \overline{\psi}^2) \\
&\quad = \nabla_{C'}^{C}\{[\partial_{\Psi}^{4}(\nabla_{D'}^{D} k_{A'B'}^{A\,B})]\}(\psi^6, \overline{\psi}^2) + \psi^C \overline{\psi}_{C'} [\partial_{\Psi}^{3}(\nabla_{D'}^{D} k_{A'B'}^{A\,B})](\psi^5, \overline{\psi}) \\
&\quad = \psi^D \overline{\psi}_{D'} \nabla_{C'}^{C} [\partial_{\Psi}^{3} k_{A'B'}^{A\,B}](\psi^5, \overline{\psi}) + \psi^C \overline{\psi}_{C'} \nabla_{D'}^{D} [\partial_{\Psi}^{3} k_{A'B'}^{A\,B}](\psi^5, \overline{\psi}) \\
&\qquad + \psi^C \overline{\psi}_{C'} \psi^D \overline{\psi}_{D'} [\partial_{\Psi}^{2} k_{A'B'}^{A\,B}](\psi^4) .
\end{aligned} \tag{3.24}$$

We now use (3.24) to compute the derivative of (3.4) with respect to $\Psi^{A_1'A_2'}_{A_1A_2\cdots A_6}$. Taking into account (3.9), we find, after some lengthy but straightforward manipulations, that

$$\langle\beta,\psi\rangle\langle\overline{\beta},\overline{\psi}\rangle\{[\partial^2_\Psi k](\psi^4;\alpha,\psi\overline{\psi},\overline{\alpha})+\alpha_A\overline{\alpha}^{B'}[\nabla^{A'}_B\partial^3_\Psi k^{A\,B}_{A'B'}](\psi^5,\overline{\psi})\}$$
$$+\langle\alpha,\psi\rangle\langle\overline{\alpha},\overline{\psi}\rangle\{[\partial^2_\Psi k](\psi^4;\beta,\psi,\overline{\psi},\overline{\beta})+\beta_A\overline{\beta}^{B'}[\nabla^{A'}_B\partial^3_\Psi k^{A\,B}_{A'B'}](\psi^5,\overline{\psi})\}$$
$$+\psi_A\overline{\psi}^{A'}[\nabla^A_{A'}\partial^3_\Psi k](\psi^5,\overline{\psi};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,. \tag{3.25}$$

In (3.25) we set $\overline{\alpha}=\overline{\beta}=\overline{\psi}$ and use (3.21); we find that

$$\psi_A\overline{\psi}^{A'}[\nabla^A_{A'}A](\psi^3,\overline{\psi}^3)=0\,. \tag{3.26}$$

In components (3.26) is the condition

$$\nabla^{(D}_{(D'}A^{A_1A_2A_3)}_{A_1'A_2'A_3')}=0\,.$$

By differentiating this equation with respect to the spin connection coefficients, it is straightforward to show (see Proposition 7.6) that this condition forces

$$A^{A_1A_2A_3}_{A_1'A_2'A_3'}=0\,,$$

that is,

$$\partial_{\Psi^{A_1A_2\cdots A_5}_{A_1'}}k^{A\,B}_{A'B'}=0\,.$$

Therefore, (3.20) becomes

$$h^{A\,B}_{A'B'}=\nabla^A_{A'}X^B_{B'}+\nabla^B_{B'}X^A_{A'}+\widetilde{h}^{A\,B}_{A'B'}(\Psi_{A_1A_2A_3A_4})\,.$$

The spinor $\widetilde{h}^{A\,B}_{A'B'}$ is a second-order generalized symmetry of the Einstein equations. We can analyze its structure by repeating the steps of this section. In particular, the derivative of $h^{AB}_{A'B'}$ with respect to the Penrose field Ψ_{ABCD} has the form

$$[\partial^2_\Psi\widetilde{h}](\psi^4;\alpha,\beta,\overline{\alpha},\overline{\beta})=\langle\psi,\alpha\rangle\langle\psi,\beta\rangle\widetilde{A}(\psi^2,\overline{\alpha},\overline{\beta})\,.$$

The spinor $\widetilde{A}$ is shown to vanish as before. Thus $\widetilde{h}^{A\,B}_{A'B'}$ is seen to be independent of the Penrose field Ψ_{ABCD}. It is straightforward to verify that the only constant solution to the linearized equations (3.4) is the spinor form of a constant times the metric. Thus we have

$$\widetilde{h}^{A\,B}_{A'B'}=c\,\varepsilon^{AB}\varepsilon_{A'B'}\,.$$

This completes the classification of generalized symmetries of the form (3.3) for the Einstein equations. The rest of this paper is devoted to extending the analysis of this section to the general higher-order symmetry. The computations are somewhat more intricate, but the ingredients are much the same as exhibited in this simple example.

4. Natural Generalized Symmetries of the Vacuum Einstein Equations

In this section we obtain a complete classification of all natural generalized symmetries of the vacuum Einstein equations, that is, we find all solutions to the linearized equations

$$[-\varepsilon_{CD}\varepsilon^{C'D'}\alpha_A\beta_B\overline{\alpha}^{A'}\overline{\beta}^{B'} + \varepsilon_{BC}\varepsilon^{A'C'}\alpha_A\beta_D\overline{\alpha}^{B'}\overline{\beta}^{D'} + \varepsilon_{BC}\varepsilon^{A'C'}\alpha_D\beta_A\overline{\alpha}^{D'}\overline{\beta}^{B'}]\nabla^C_{C'}\nabla^D_{D'}h^{AB}_{A'B'} = 0\,, \tag{4.1}$$

where

$$h^{AB}_{A'B'} = h^{AB}_{A'B'}(\Psi^2,\overline{\Psi}^2,\Psi^3,\overline{\Psi}^3,\dots,\Psi^k,\overline{\Psi}^k)$$

is a natural spinor depending upon the Penrose fields to order k. Equation (4.1) and all subsequent equations in this section hold by virtue of the Einstein equations and their derivatives.

Before beginning the detailed analysis of (4.1), let us review the principal steps. Since $h^{AB}_{A'B'}$ is assumed to be of order k, the linearized equation is an identity to order $k+2$ in the Penrose fields. It is easy to see that this identity can be written symbolically as

$$\alpha\Psi^{k+2} + \beta\overline{\Psi}^{k+2} + \gamma\Psi^{k+1}\Psi^{k+1} + \delta\Psi^{k+1}\overline{\Psi}^{k+1} + \varepsilon\overline{\Psi}^{k+1}\overline{\Psi}^{k+1} + \rho\Psi^{k+1} + \tau\overline{\Psi}^{k+1} + \upsilon = 0\,, \tag{4.2}$$

where the coefficients $\alpha,\beta,\dots,\upsilon$ are complicated expressions of order k involving $h^{AB}_{A'B'}$ and its repeated derivatives with respect to $\Psi^2,\overline{\Psi}^2,\dots\Psi^k,\overline{\Psi}^k$. Each of the coefficients $\alpha,\beta,\dots,\upsilon$ must vanish identically because the fields Ψ^{k+2}, $\overline{\Psi}^{k+2},\Psi^{k+1},\overline{\Psi}^{k+1}$ may be freely specified on $\mathscr{E}^{k+2}$. As is standard practice in symmetry group analysis, we analyze this complicated identity beginning with the highest-order conditions $\alpha=0$ and $\beta=0$.

Let $\partial^k_\Psi h$ and $\partial^k_{\bar\Psi} h$ denote the partial derivatives of $h^{AB}_{A'B'}$ with respect to Ψ^k and $\overline{\Psi}^k$. The conditions $\alpha=0$ and $\beta=0$ impose certain algebraic conditions on the spinors $\partial^k_\Psi h$ and $\partial^k_{\bar\Psi} h$ which, when carefully analyzed, lead to unique spinor decompositions that we shall write symbolically as

$$\partial^k_\Psi h = A + B + W \quad\text{and}\quad \partial^k_{\bar\Psi} h = D + E + U\,. \tag{4.3}$$

This we do in Sect. 4A; see Propositions 4.3 and 4.4. Each term $A,B,\dots,U$ in these decompositions separately satisfies the algebraic conditions arising from $\alpha=0$ and $\beta=0$. In Sect. 4B we show that the vanishing of the coefficients $\gamma,\delta,\varepsilon$ force $h^{AB}_{A'B'}$ to be linear in the highest-order Penrose fields Ψ^k and $\overline{\Psi}^k$, so that the spinors $A,B,\dots,U$ in the representation (4.3) are all at most of order $k-1$. The analysis of the conditions $\rho=0$ and $\tau=0$ is accomplished in two steps. In Sect. 4C we prove that A,B,D,E must actually be of order $k-2$, and that there is a generalized natural vector field

$$X^A_{A'} = X^A_{A'}(\Psi^2,\overline{\Psi}^2,\dots,\Psi^{k-1},\overline{\Psi}^{k-1})$$

such that

$$W = \partial^{k-1}_\Psi X \quad\text{and}\quad U = \partial^{k-1}_{\bar\Psi} X\,.$$

We let

$$l^{AB}_{A'B'} = h^{AB}_{A'B'} - (\nabla^A_{A'} X^B_{B'} + \nabla^B_{B'} X^A_{A'}) .$$

Then $l^{AB}_{A'B'}$ satisfies (4.2) and (4.3) with $W = 0$ and $U = 0$. In Sect. 4D we find that the remaining coefficients A, B, D, E in (4.3) now satisfy certain covariant constancy conditions, from which it readily follows that $A = B = D = E = 0$. The classification of the natural generalized symmetries of the Einstein equations is then completed by a simple induction argument. Note that our analysis of natural symmetries completely parallels that of Sect. 3.

We begin by fixing some notation. If

$$T^{C_1\cdots C_p}_{C'_1\cdots C'_q} = T^{C_1\cdots C_p}_{C'_1\cdots C'_q}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k)$$

is a natural spinor of type (p, q) and order k, then the partial derivative of $T^{C_1\cdots C_p}_{C'_1\cdots C'_q}$ with respect to Ψ^l is a natural spinor of type $(p + l + 2, q + l - 2)$. We shall write

$$\begin{aligned}
&[\partial^l_\Psi T^{C_1\cdots C_p}_{C'_1\cdots C'_q}](\psi^1 \cdots \psi^{l+2}, \overline{\psi}_1 \cdots \overline{\psi}_{l-2}) \\
&\quad = [\partial \Psi^{A_1\cdots A_{l+2}}_{A'_1\cdots A'_{l-2}} T^{C_1\cdots C_p}_{C'_1\cdots C'_q}]\psi^1_{A_1} \cdots \psi^{l+2}_{A_{l+2}} \overline{\psi}^{A'_1}_1 \cdots \overline{\psi}^{A'_{l-2}}_{l-2} .
\end{aligned} \tag{4.4}$$

Further, let $\phi^1, \ldots, \phi^p$ and $\overline{\phi}_1, \ldots, \overline{\phi}_q$ be arbitrary spinors; we shall write

$$\begin{aligned}
&[\partial^l_\Psi T](\psi^{l+2}, \overline{\psi}^{l-2}; \phi^1, \ldots, \phi^p, \overline{\phi}_1, \ldots, \overline{\phi}_q) \\
&\quad = [\partial^l_\Psi T^{C_1\cdots C_p}_{C'_1\cdots C'_q}](\psi^{l+2}, \overline{\psi}^{l-2})\phi^1_{C_1} \cdots \phi^p_{C_p} \overline{\phi}^{C'_1}_1 \cdots \overline{\phi}^{C'_q}_q .
\end{aligned}$$

A semi-colon will always be used to separate arguments corresponding to derivatives with respect to the coordinates (2.19). Partial derivatives with respect to $\overline{\Psi}^{A_1\cdots A_{l-2}}_{A'_1\cdots A'_{l+2}}$ will be similarly denoted. Examples of this notation can be found in the previous section.

We shall repeatedly need certain commutation relations between the partial derivative operators $\partial_{\Psi}{}^{A_1\cdots A_{m+2}}_{A'_1\cdots A'_{m-2}}$ and $\partial_{\overline{\Psi}}{}^{A_1\cdots A_{m-2}}_{A'_1\cdots A'_{m+2}}$ and the covariant derivative operator $\nabla^C_{C'}$.

Proposition 4.1. *Let*

$$T^{\cdots}_{\cdots} = T^{\cdots}_{\cdots}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^m, \overline{\Psi}^m)$$

be a natural spinor of order m. Then

$$[\partial^{m+1}_\Psi \nabla^C_{C'} T^{\cdots}_{\cdots}](\psi^{m+3}, \overline{\psi}^{m-1}) = \psi^C \overline{\psi}_{C'} [\partial^m_\Psi T^{\cdots}_{\cdots}](\psi^{m+2}, \overline{\psi}^{m-2}) , \tag{4.5}$$

and

$$\begin{aligned}
&[\partial^m_\Psi \nabla^C_{C'} T^{\cdots}_{\cdots}](\psi^{m+2}, \overline{\psi}^{m-2}) \\
&\quad = [\nabla^C_{C'} \partial^m_\Psi T^{\cdots}_{\cdots}](\psi^{m+2}, \overline{\psi}^{m-2}) + \psi^C \overline{\psi}_{C'} [\partial^{m-1}_\Psi T^{\cdots}_{\cdots}](\psi^{m+1}, \overline{\psi}^{m-3}) ,
\end{aligned} \tag{4.6}$$

and similarly,

$$[\partial^{m+1}_{\overline{\Psi}} \nabla^C_{C'} T^{\cdots}_{\cdots}](\psi^{m-1}, \overline{\psi}^{m+3}) = \psi^C \overline{\psi}_{C'} [\partial^m_{\overline{\Psi}} T^{\cdots}_{\cdots}](\psi^{m-2}, \overline{\psi}^{m+2}) , \tag{4.7}$$

and

$$[\partial^m_{\overline{\psi}} \nabla^C_{C'} T^{\cdots}_{\cdots}](\psi^{m-2}, \overline{\psi}^{m+2}) = [\nabla^C_{C'} \partial^m_{\overline{\psi}} T^{\cdots}_{\cdots})](\psi^{m-2}, \overline{\psi}^{m+2}) + \psi^C \overline{\psi}_{C'} [\partial^{m-1}_{\overline{\psi}} T^{\cdots}_{\cdots}](\psi^{m-3}, \overline{\psi}^{m+1}) . \qquad (4.8)$$

Proof. These formulas follow directly from Proposition 2.8 and the structure equations (2.20). Examples of these formulas can be found in Sect. 3. □

4A. The Ψ^{k+2} and $\overline{\Psi}^{k+2}$ Analysis. We suppose that $h^{AB}_{A'B'}$ is a natural generalized symmetry of the vacuum Einstein equations of order k:

$$h^{AB}_{A'B'} = h^{AB}_{A'B'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k) .$$

In this section we derive necessary and sufficient conditions for the vanishing of the coefficients α and β in (4.2), and we analyze these conditions in detail.

We have, by two applications of (4.5),

$$[\partial^{k+2}_{\Psi} \nabla^C_{C'} \nabla^D_{D'} h^{AB}_{A'B'}](\psi^{k+4}, \overline{\psi}^k) = \psi^C \overline{\psi}_{C'} [\partial^{k+1}_{\Psi} \nabla^D_{D'} h^{AB}_{A'B'}](\psi^{k+3}, \overline{\psi}^{k-1})$$
$$= \psi^C \psi^D \overline{\psi}_{C'} \overline{\psi}_{D'} [\partial^k_{\Psi} h^{AB}_{A'B'}](\psi^{k+2}, \overline{\psi}^{k-2}) .$$

Therefore, if we differentiate Eq. (4.1) with respect to Ψ^{k+2} it follows that

$$\langle \beta, \psi \rangle \langle \overline{\beta}, \overline{\psi} \rangle [\partial^k_{\Psi} h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) + \langle \alpha, \psi \rangle \langle \overline{\alpha}, \overline{\psi} \rangle [\partial^k_{\Psi} h](\psi^{k+2}, \overline{\psi}^{k-2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0 . \qquad (4.9)$$

When $k = 3$, this is exactly Eq. (3.8) obtained in our model problem.

Similarly, we differentiate the linearized equations (4.1) with respect to $\overline{\Psi}^{k+2}$ and use (4.7) to find

$$\langle \beta, \psi \rangle \langle \overline{\beta}, \overline{\psi} \rangle [\partial^k_{\overline{\Psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) + \langle \alpha, \psi \rangle \langle \overline{\alpha}, \overline{\psi} \rangle [\partial^k_{\overline{\Psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0 . \qquad (4.10)$$

Proposition 4.2. *If $h^{AB}_{A'B'}$ is a natural generalized symmetry of order k for the vacuum Einstein equations, then*

$$[\partial^k_{\Psi} h](\psi^{k+2}, \overline{\psi}^{k-2}; \psi, \alpha, \overline{\alpha}, \overline{\psi}) = 0 \qquad (4.11)$$

and

$$[\partial^k_{\overline{\Psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \psi, \alpha, \overline{\alpha}, \overline{\psi}) = 0 . \qquad (4.12)$$

Proof. In Eq. (4.9) we set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$ to deduce that

$$[\partial^k_{\Psi} h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) = 0 .$$

The symmetry $h_{ABA'B'} = h_{BAB'A'}$ then leads to (4.11). In Eq. (4.10) we set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$, and then use the symmetry of $h_{ABA'B'}$ to arrive at (4.12). Note that (4.11) and (4.12) are necessary and sufficient for (4.9) and (4.10) to hold respectively. □

Theorem 7.7 allows us to explicitly characterize all natural spinors that satisfy (4.11) and (4.12).

Proposition 4.3. *The spinor* $[\partial^k_\Psi h](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})$ *satisfies the symmetry conditions* (4.11) *if and only if there are natural spinors,*

$$A = A(\psi^k, \overline{\psi}^k), \qquad B = B(\psi^{k+4}, \overline{\psi}^{k-4}), \qquad W = W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}), \qquad (4.13)$$

such that

$$\begin{aligned}\partial^k_\Psi h(\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\ = \langle\psi, \alpha\rangle\langle\psi, \beta\rangle A(\psi^k, \overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi}, \overline{\alpha}\rangle\langle\overline{\psi}, \overline{\beta}\rangle B(\psi^{k+2}\alpha\beta, \overline{\psi}^{k-4}) \\ + \langle\psi, \alpha\rangle\langle\overline{\alpha}, \overline{\psi}\rangle W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) + \langle\psi, \beta\rangle\langle\overline{\beta}, \overline{\psi}\rangle W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) . \quad (4.14)\end{aligned}$$

The spinor A is symmetric in its first k and last k arguments; the spinor B is symmetric in its first $k+4$ and last $k-4$ arguments; and the spinor W is symmetric in its first $k+1$ and following $k-3$ arguments. With these symmetries, the spinors A, B, W are uniquely determined by $\partial^k_\Psi h$. When $k = 3$, (4.14) *is valid with $B = 0$ and $W = W(\psi^4, \alpha, \overline{\alpha})$. When $k = 2$,* (4.14) *holds with $B = 0$ and $W = 0$.*

We note that the case $k = 3$ is treated in Sect. 3.

Let us remark that (4.14) contains the algebraic form of the generalized diffeomorphism symmetry. Indeed, if

$$X^A_{A'} = X^A_{A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1})$$

is the spinor form of a natural vector field of order $k-1$, and we let

$$d^{AB}_{A'B'} = \nabla^A_{A'} X^B_{B'} + \nabla^B_{B'} X^A_{A'} ,$$

then, by (4.5),

$$\begin{aligned}[\partial^k_\Psi d](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = \langle\psi, \alpha\rangle\langle\overline{\alpha}, \overline{\psi}\rangle[\partial^{k-1}_\Psi X](\psi^{k+1}, \overline{\psi}^{k-3}; \beta, \overline{\beta}) \\ + \langle\psi, \beta\rangle\langle\overline{\beta}, \overline{\psi}\rangle[\partial^{k-1}_\Psi X](\psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}) . \quad (4.15)\end{aligned}$$

We observe that with $W = \partial^{k-1}_\Psi X$ the right-hand side of (4.15) coincides with the expression involving W in (4.14). In Sect. 4C we shall prove W satisfies integrability conditions that imply $W = \partial^{k-1}_\Psi X$.

There is an analogous decomposition for $\partial^k_{\overline{\Psi}} h$.

Proposition 4.4. *The spinor* $[\partial^k_{\overline{\Psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})$ *satisfies the symmetry conditions* (4.12) *if and only if there are natural spinors,*

$$D = D(\overline{\psi}^k, \psi^k), \qquad E = E(\overline{\psi}^{k+4}, \psi^{k-4}), \qquad U = U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}), \qquad (4.16)$$

such that

$$
\begin{aligned}
&[\partial^k_{\overline{\Psi}} h](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\
&\quad = \langle \overline{\psi}, \overline{\alpha}\rangle \langle \overline{\psi}, \overline{\beta}\rangle D(\overline{\psi}^k, \psi^{k-2}\alpha\beta) + \langle \psi, \alpha\rangle \langle \psi, \beta\rangle E(\overline{\psi}^{k+2}\overline{\alpha}\overline{\beta}, \psi^{k-4}) \\
&\qquad + \langle \overline{\psi}, \overline{\alpha}\rangle \langle \alpha, \psi\rangle U(\overline{\psi}^{k+1}, \psi^{k-3}, \beta, \overline{\beta}) + \langle \overline{\psi}, \overline{\beta}\rangle \langle \beta, \psi\rangle U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}) \,. \quad (4.17)
\end{aligned}
$$

The spinor D is symmetric in its first k and last k arguments; the spinor E is symmetric in its first $k+4$ and last $k-4$ arguments; and the spinor U is symmetric in its first $k+1$ and following $k-3$ arguments. With these symmetries the spinors D, E, U are unique. When $k=3$, (4.17) is valid with $E=0$ and $U = U(\overline{\psi}^4, \alpha, \overline{\alpha})$. When $k=2$, (4.17) holds with $E=0$ and $U=0$.

4B. The $\Psi^{k+1}\Psi^{k+1}$, $\Psi^{k+1}\overline{\Psi}^{k+1}$, and $\overline{\Psi}^{k+1}\overline{\Psi}^{k+1}$ Analysis. In this step we prove that if $h^{AB}_{A'B'}$ is a natural generalized symmetry of order k, then $h^{AB}_{A'B'}$ must be linear in the highest derivatives Ψ^k and $\overline{\Psi}^k$. To begin, we use the commutation rules (4.5) and (4.6) to find that

$$
\begin{aligned}
&(\partial^{k+1}_{\Psi} \partial^{k+1}_{\Psi} \nabla^C_{C'} \nabla^D_{D'} h^{AB}_{A'B'})(\chi^{k+3}, \overline{\chi}^{k-1}; \psi^{k+3}, \overline{\psi}^{k-1}) \\
&\quad = \big[\partial^{k+1}_{\Psi} \{\psi^C \overline{\psi}_{C'} (\partial^k_{\Psi} \nabla^D_{D'} h^{AB}_{A'B'})(\psi^{k+2}, \overline{\psi}^{k-2}) \\
&\qquad + \nabla^C_{C'} (\partial^{k+1}_{\Psi} \nabla^D_{D'} h^{AB}_{A'B'})(\psi^{k+3}, \overline{\psi}^{k-1})\}\big](\chi^{k+3}, \overline{\chi}^{k-1}) \\
&\quad = [\partial^{k+1}_{\Psi} \{\psi^C \overline{\psi}_{D'} (\partial^k_{\Psi} \nabla^D_{D'} h^{AB}_{A'B'})(\psi^{k+2}, \overline{\psi}^{k-2}) \\
&\qquad + \psi^D \overline{\psi}_{D'} \nabla^C_{C'} (\partial^k_{\Psi} h^{AB}_{A'B'})(\psi^{k+2}, \overline{\psi}^{k-2})\}](\chi^{k+3}, \overline{\chi}^{k-1}) \\
&\quad = (\psi^C \overline{\psi}_{C'} \chi^D \overline{\chi}_{D'} + \psi^D \overline{\psi}_{D'} \chi^C \overline{\chi}_{C'})(\partial^k_{\Psi} \partial^k_{\Psi} h^{AB}_{A'B'})(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}) \,. \quad (4.18)
\end{aligned}
$$

We differentiate the symmetry equation (4.1) twice with respect to Ψ^{k+1} and use (4.18); after some elementary simplifications we obtain

$$
\begin{aligned}
&-2\langle \psi, \chi\rangle \langle \overline{\psi}, \overline{\chi}\rangle (\partial^k_{\Psi} \partial^k_{\Psi} h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\
&\quad + \langle \psi, \beta\rangle \langle \overline{\psi}, \overline{\beta}\rangle (\partial^k_{\Psi} \partial^k_{\Psi} h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \chi, \overline{\chi}, \overline{\alpha}) \\
&\quad + \langle \chi, \beta\rangle \langle \overline{\chi}, \overline{\beta}\rangle (\partial^k_{\Psi} \partial^k_{\Psi} h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) \\
&\quad + \langle \psi, \alpha\rangle \langle \overline{\psi}, \overline{\alpha}\rangle (\partial^k_{\Psi} \partial^k_{\Psi} h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}; \beta, \chi, \overline{\chi}, \overline{\beta}) \\
&\quad + \langle \chi, \alpha\rangle \langle \overline{\chi}, \overline{\alpha}\rangle (\partial^k_{\Psi} \partial^k_{\Psi} h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}; \beta, \psi, \overline{\psi}, \overline{\beta}) = 0 \,.
\end{aligned}
$$

In the notation of Eq. (4.2) this is the condition $\gamma = 0$. Using Proposition 4.2, we immediately find that this equation simplifies to

$$
(\partial^k_{\Psi} \partial^k_{\Psi} h)(\psi^{k+2}, \overline{\psi}^{k-2}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) = 0 \,. \quad (4.19)
$$

This proves that $h^{AB}_{A'B'}$ is at most linear in the variables Ψ^k. Likewise, if we take the second derivative of the linearized equations (4.1) with respect to $\bar{\Psi}^{k+1}$ and use Proposition 4.2, we obtain

$$(\partial^k_{\bar{\Psi}}\partial^k_{\bar{\Psi}}h)(\psi^{k-2},\overline{\psi}^{k+2};\chi^{k-2},\overline{\chi}^{k+2};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,, \tag{4.20}$$

which implies that $h^{AB}_{A'B'}$ is linear in the variables $\overline{\Psi}^k$. Finally, differentiation of the symmetry condition (4.1) with respect to $\overline{\Psi}^{k+1}$ and Ψ^{k+1}, followed by use of Proposition 4.2, leads to

$$(\partial^k_{\bar{\Psi}}\partial^k_{\Psi}h)(\psi^{k-2},\overline{\psi}^{k+2};\chi^{k+2},\overline{\chi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,. \tag{4.21}$$

Together, Eqs. (4.19), (4.20), and (4.21), which follow from setting the coefficients γ, δ, and ε in (4.2) to zero, prove the following proposition.

Proposition 4.5. *Let*

$$h^{AB}_{A'B'}=h^{AB}_{A'B'}(\Psi^2,\overline{\Psi}^2,\ldots,\Psi^k,\overline{\Psi}^k)$$

be a generalized symmetry of the vacuum Einstein equations. Then $h^{AB}_{A'B'}$ *is at most linear in the top-order Penrose fields* Ψ^k *and* $\overline{\Psi}^k$.

Corollary 4.6. *The spinors* A,B,W *and* D,E,U *in Eqs.* (4.14) *and* (4.17) *are at most of order* $k-1$.

Proof. This corollary follows from Proposition 4.5 and the fact that the spinors A,B,W and D,E,U in the decompositions (4.14) and (4.17) are unique. □

At this point we are able to prove that there are no natural generalized symmetries of the Einstein equations of differential order two in the metric, aside from the scaling symmetry (2.11).

Corollary 4.7. *Let* $h^{AB}_{A'B'}(\Psi^2,\overline{\Psi}^2)$ *be a natural generalized symmetry of the vacuum Einstein equations of order* 2. *Then*

$$h^{AB}_{A'B'}=c\varepsilon_{A'B'}\varepsilon^{AB}\,,$$

where c *is a constant.*

Proof. According to Proposition 4.3 and Proposition 4.4, we have that

$$[\partial^2_{\Psi}h](\psi^4;\alpha,\beta,\overline{\alpha},\overline{\beta})=\langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^2,\overline{\alpha}\overline{\beta})\,,$$

and

$$[\partial^2_{\bar{\Psi}}h](\overline{\psi}^4;\alpha,\beta,\overline{\alpha},\overline{\beta})=\langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle D(\overline{\psi}^2,\alpha\beta)\,.$$

Proposition 4.5 implies that the spinors A and D are independent of the Penrose fields Ψ^2 and $\overline{\Psi}^2$. Because h is $SL(2,\mathbf{C})$ invariant, A and D are $SL(2,\mathbf{C})$ invariant, and consequently they are constructed solely from the ε-spinors. It is easy to check that there are no spinors with the rank and symmetries of A and D built solely from the ε-spinors. Therefore $A=D=0$. This implies that $h^{AB}_{A'B'}$ is constructed only from the ε-spinors from which the corollary follows. □

4C. The $\Psi^k\Psi^{k+1}$, $\overline{\Psi}^k\Psi^{k+1}$, $\Psi^k\overline{\Psi}^{k+1}$, and $\overline{\Psi}^k\overline{\Psi}^{k+1}$ Analysis. In this section we shall prove that the spinors A, B, D and E must be of order $k-2$, and that there exists a natural type $(1,1)$ spinor X of order $k-1$,

$$X^A_{A'} = X^A_{A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}), \tag{4.22}$$

such that

$$W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) = [\partial^{k-1}_{\overline{\Psi}} X](\psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}), \tag{4.23}$$

and

$$U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}) = [\partial^{k-1}_{\overline{\psi}} X](\psi^{k-3}, \overline{\psi}^{k+1}; \alpha, \overline{\alpha}). \tag{4.24}$$

We obtain these results by analyzing the equations arising from the coefficients of $\Psi^k\Psi^{k+1}$, $\Psi^k\overline{\Psi}^{k+1}$, $\overline{\Psi}^k\Psi^{k+1}$, and $\overline{\Psi}^k\overline{\Psi}^{k+1}$ in the linearized equations (4.1).

We begin with the $\Psi^k\Psi^{k+1}$ terms. Because $h^{AB}_{A'B'}$ is linear in the Penrose fields $\Psi^k, \overline{\Psi}^k$, we can use the commutation rules in Proposition 4.1 to deduce that

$$\begin{aligned}
&[\partial^k_{\Psi}\partial^{k+1}_{\Psi}\nabla^C_{C'}\nabla^D_{D'}h^{AB}_{A'B'}](\chi^{k+2}, \overline{\chi}^{k-2}; \psi^{k+3}, \overline{\psi}^{k-1}) \\
&\quad = \psi^C\psi^D\overline{\psi}_{C'}\overline{\psi}_{D'}[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h^{AB}_{A'B'}](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k-2}) \\
&\qquad + \psi^C\chi^D\overline{\psi}_{C'}\overline{\chi}_{D'}[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h^{AB}_{A'B'}](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}) \\
&\qquad + \chi^C\psi^D\overline{\chi}_{C'}\overline{\psi}_{D'}[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h^{AB}_{A'B'}](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}).
\end{aligned} \tag{4.25}$$

We now apply the operator $\partial^k_{\Psi}\partial^{k+1}_{\Psi}$ to the linearized equations (4.1) to find, after substituting from (4.25) and simplifying, that

$$\begin{aligned}
&-2\langle\psi,\chi\rangle\langle\overline{\psi},\overline{\chi}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\
&+ \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) \\
&+ \langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k-2}; \beta, \psi, \overline{\psi}, \overline{\beta}) \\
&+ \langle\chi,\beta\rangle\langle\overline{\chi},\overline{\beta}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) \\
&+ \langle\chi,\alpha\rangle\langle\overline{\chi},\overline{\alpha}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \beta, \psi, \overline{\psi}, \overline{\beta}) \\
&+ \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \chi, \overline{\chi}, \overline{\alpha}) \\
&+ \langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \beta, \chi, \overline{\chi}, \overline{\beta}) = 0.
\end{aligned} \tag{4.26}$$

The symmetry condition (4.11) implies that the coefficients of $\langle\chi,\beta\rangle\langle\overline{\chi},\overline{\beta}\rangle$ and $\langle\chi,\alpha\rangle\langle\overline{\chi},\overline{\alpha}\rangle$ each vanish, and so we can rewrite Eq. (4.26) as

$$\begin{aligned}
&-2\langle\psi,\chi\rangle\ \langle\overline{\psi},\overline{\chi}\rangle\ [\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\
&+\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle\{[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k-2}; \beta, \psi, \overline{\psi}, \overline{\beta}) \\
&\qquad + [\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \beta, \chi, \overline{\chi}, \overline{\beta})\} \\
&+\langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle\{[\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) \\
&\qquad + [\partial^{k-1}_{\Psi}\partial^k_{\Psi}h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \chi, \overline{\chi}, \overline{\alpha})\} = 0.
\end{aligned} \tag{4.27}$$

In this equation we set $\alpha = \beta = \psi$ to arrive at

$$[\partial_\Psi^{k-1}\partial_\Psi^k h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \psi, \psi, \overline{\alpha}, \overline{\beta}) = 0 .$$

In terms of the decomposition (4.14) we have that

$$[\partial_\Psi^k h](\psi^{k+2}, \overline{\psi}^{k-2}; \psi, \psi, \alpha, \overline{\beta}) = \langle \overline{\psi}, \overline{\alpha}\rangle \langle \overline{\psi}, \overline{\beta}\rangle B(\psi^{k+4}, \overline{\psi}^{k-4}) ,$$

and so this equation implies that

$$[\partial_\Psi^{k-1} B](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+4}, \overline{\psi}^{k-4}) = 0 . \tag{4.28}$$

In other words, B is independent of the spinor Ψ^{k-1}. Likewise, by setting $\overline{\alpha} = \overline{\beta} = \overline{\psi}$ in Eq. (4.27), we conclude that

$$[\partial_\Psi^{k-1} A](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^k, \overline{\psi}^k) = 0 , \tag{4.29}$$

and so A is independent of the spinor Ψ^{k-1}. Together, Eqs. (4.14), (4.28), and (4.29) show that

$$\begin{aligned}
&[\partial_\Psi^{k-1}\partial_\Psi^k h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\
&= \langle \psi, \alpha\rangle \langle \overline{\alpha}, \overline{\psi}\rangle [\partial_\Psi^{k-1} W](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) \\
&\quad + \langle \psi, \beta\rangle \langle \overline{\beta}, \overline{\psi}\rangle [\partial_\Psi^{k-1} W](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) .
\end{aligned} \tag{4.30}$$

We next set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$ in (4.27), and substitute from (4.30) to arrive at

$$\begin{aligned}
&2\langle \psi, \chi\rangle \langle \overline{\psi}, \overline{\chi}\rangle [\partial_\Psi^{k-1} W](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) \\
&= \langle \chi, \alpha\rangle \langle \overline{\chi}, \overline{\psi}\rangle [\partial_\Psi^{k-1} W](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+1}, \overline{\chi}^{k-3}, \psi, \overline{\alpha}) \\
&\quad + \langle \chi, \psi\rangle \langle \overline{\chi}, \overline{\alpha}\rangle [\partial_\Psi^{k-1} W](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+1}, \overline{\chi}^{k-3}, \alpha, \overline{\psi}) \\
&\quad + \langle \psi, \alpha\rangle \langle \overline{\psi}, \overline{\chi}\rangle [\partial_\Psi^{k-1} W](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}, \chi, \overline{\alpha}) \\
&\quad + \langle \psi, \chi\rangle \langle \overline{\psi}, \overline{\alpha}\rangle [\partial_\Psi^{k-1} W](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\chi}) .
\end{aligned} \tag{4.31}$$

The right-hand side of this equation is unchanged by the simultaneous interchange of ψ with χ and $\overline{\psi}$ with $\overline{\chi}$ so we conclude

$$[\partial_\Psi^{k-1} W](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) = [\partial_\Psi^{k-1} W](\psi^{k+1}, \overline{\psi}^{k-3}; \chi^{k+1}, \overline{\chi}^{k-3}, \alpha, \overline{\alpha}) . \tag{4.32}$$

Equation (4.32) is necessary and sufficient for Eq. (4.31) to hold, and is one of the integrability conditions needed to establish Eq. (4.23).

In exactly the same fashion we can apply the operator $\partial_{\bar{\Psi}}^k \partial_{\bar{\Psi}}^{k+1}$ to the linearized equations (4.1) to show that

$$[\partial_{\bar{\Psi}}^{k-1} D](\psi^{k-3}, \overline{\psi}^{k+1}; \overline{\chi}^k, \chi^k) = 0 , \tag{4.33}$$

$$[\partial_{\bar{\Psi}}^{k-1} E](\psi^{k-3}, \overline{\psi}^{k+1}; \overline{\chi}^{k+4}, \chi^{k-4}) = 0 . \tag{4.34}$$

Moreover, we have that

$$
\begin{aligned}
&[\partial_{\bar{\Psi}}^{k-1}U](\psi^{k-3},\overline{\psi}^{k+1};\overline{\chi}^{k+1},\chi^{k-3},\alpha,\overline{\alpha}) \\
&\quad=[\partial_{\bar{\Psi}}^{k-1}U](\chi^{k-3},\overline{\chi}^{k+1};\overline{\psi}^{k+1},\psi^{k-3},\alpha,\overline{\alpha})\,. \qquad (4.35)
\end{aligned}
$$

Before applying the operator $\partial_{\bar{\Psi}}^{k}\,\partial_{\Psi}^{k+1}$ to the linearized equations, we first use the commutation rules of Proposition 4.1 and the fact that $h^{AB}_{A'B'}$ is linear in Ψ^{k} and $\overline{\Psi}^{k}$ to deduce that

$$
\begin{aligned}
&[\partial_{\bar{\Psi}}^{k}\,\partial_{\Psi}^{k+1}\nabla^{C}_{C'}\nabla^{D}_{D'}h^{AB}_{A'B'}](\chi^{k-2},\overline{\chi}^{k+2};\psi^{k+3},\overline{\psi}^{k-1}) \\
&\quad=\psi^{C}\psi^{D}\overline{\psi}_{C'}\overline{\psi}_{D'}[\partial_{\bar{\Psi}}^{k}\,\partial_{\Psi}^{k-1}h^{AB}_{A'B'}](\chi^{k-2},\overline{\chi}^{k+2};\psi^{k+1},\overline{\psi}^{k-3}) \\
&\qquad+\psi^{C}\chi^{D}\overline{\psi}_{C'}\overline{\chi}_{D'}[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h^{AB}_{A'B'}](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2}) \\
&\qquad+\chi^{C}\psi^{D}\overline{\chi}_{C'}\overline{\psi}_{D'}[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h^{AB}_{A'B'}](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2})\,.
\end{aligned}
$$

Using this result, if we differentiate (4.1) with respect to $\overline{\Psi}^{k}$ and Ψ^{k+1} and take into account the leading order symmetry conditions of Proposition (4.2), we have

$$
\begin{aligned}
&-2\langle\psi,\chi\rangle\;\langle\overline{\psi},\overline{\chi}\rangle\;[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta}) \\
&+\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle\{[\partial_{\Psi}^{k-1}\partial_{\bar{\Psi}}^{k}h](\psi^{k+1},\overline{\psi}^{k-3};\chi^{k-2},\overline{\chi}^{k+2};\beta,\psi,\overline{\psi},\overline{\beta}) \\
&\qquad+[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2};\beta,\chi,\overline{\chi},\overline{\beta})\} \\
&+\langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle\{[\partial_{\Psi}^{k-1}\partial_{\bar{\Psi}}^{k}h](\psi^{k+1},\overline{\psi}^{k-3};\chi^{k-2},\overline{\chi}^{k+2};\alpha,\psi,\overline{\psi},\overline{\alpha}) \\
&\qquad+[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\chi,\overline{\chi},\overline{\alpha})\}=0\,. \qquad (4.36)
\end{aligned}
$$

With $\alpha=\beta=\psi$, and then with $\overline{\alpha}=\overline{\beta}=\overline{\psi}$, Eq. (4.36) implies

$$
[\partial_{\bar{\Psi}}^{k-1}B](\overline{\chi}^{k-3},\chi^{k+1};\psi^{k+4},\overline{\psi}^{k-4})=0 \qquad (4.37)
$$

and

$$
[\partial_{\bar{\Psi}}^{k-1}A](\overline{\chi}^{k-3},\chi^{k+1};\psi^{k},\overline{\psi}^{k})=0\,. \qquad (4.38)
$$

We set $\alpha=\beta$ and $\overline{\alpha}=\overline{\beta}$ in (4.36) to find

$$
\begin{aligned}
&\langle\psi,\chi\rangle\;\langle\overline{\psi},\overline{\chi}\rangle\{[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\alpha,\overline{\alpha},\overline{\alpha}) \\
&\quad=\langle\psi,\alpha\rangle\;\langle\overline{\psi},\overline{\alpha}\rangle\{[\partial_{\Psi}^{k-1}\partial_{\bar{\Psi}}^{k}h](\psi^{k+1},\overline{\psi}^{k-3};\chi^{k-2},\overline{\chi}^{k+2};\alpha,\psi,\overline{\psi},\overline{\alpha}) \\
&\qquad+[\partial_{\bar{\Psi}}^{k-1}\partial_{\Psi}^{k}h](\chi^{k-3},\overline{\chi}^{k+1};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\chi,\overline{\chi},\overline{\alpha})\}\,. \qquad (4.39)
\end{aligned}
$$

Again, in exactly the same manner, the $\partial^k_\psi \partial^{k+1}_{\bar\psi}$ derivative of the linearized equation (4.1) yields

$$[\partial^{k-1}_{\bar\psi} D](\psi^{k+1}, \overline{\psi}^{k-3}; \overline{\chi}^k, \chi^k) = 0 , \tag{4.40}$$

$$[\partial^{k-1}_{\bar\psi} E](\psi^{k+1}, \overline{\psi}^{k-3}; \overline{\chi}^{k+4}, \chi^{k-4}) = 0 , \tag{4.41}$$

as well as

$$\begin{aligned}\langle\psi, \chi\rangle\ \langle\overline{\psi}, \overline{\chi}\rangle\ &[\partial^{k-1}_{\psi} \partial^k_{\bar\psi} h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \alpha, \overline{\alpha}, \overline{\alpha}) \\ = \langle\psi, \alpha\rangle\ \langle\overline{\psi}, \overline{\alpha}\rangle &\{[\partial^{k-1}_{\bar\psi} \partial^k_{\psi} h](\psi^{k-3}, \overline{\psi}^{k+1}; \chi^{k+2}, \overline{\chi}^{k-2}; \alpha, \psi, \overline{\psi}, \overline{\alpha}) \\ &+ [\partial^{k-1}_{\psi} \partial^k_{\bar\psi} h](\chi^{k+1}, \overline{\chi}^{k-3}; \psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \chi, \overline{\chi}, \overline{\alpha})\} . \end{aligned}\tag{4.42}$$

Equations (4.28), (4.29), (4.33), (4.34), (4.37), (4.38), (4.40), and (4.41) prove the following proposition.

Proposition 4.8. *Let $h^{AB}_{A'B'}$ be a natural generalized symmetry of order k. Then the spinors A, B, D, E appearing in the decompositions* (4.14) *and* (4.17) *are at most of order* $k-2$.

On taking Proposition (4.8) into account, the substitution of (4.14) and (4.17) into (4.39) and (4.42) gives rise to

$$\begin{aligned}2\langle\psi, \chi\rangle\langle\overline{\psi}, \overline{\chi}\rangle&[\partial^{k-1}_{\bar\psi} W](\chi^{k-3}, \overline{\chi}^{k+1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) \\ = \langle\chi, \psi\rangle\langle\overline{\chi}, \overline{\alpha}\rangle&[\partial^{k-1}_{\psi} U](\psi^{k+1}, \overline{\psi}^{k-3}; \overline{\chi}^{k+1}, \chi^{k-3}, \alpha, \overline{\psi}) \\ + \langle\chi, \alpha\rangle\langle\overline{\chi}, \overline{\psi}\rangle&[\partial^{k-1}_{\psi} U](\psi^{k+1}, \overline{\psi}^{k-3}; \overline{\chi}^{k+1}, \chi^{k-3}, \psi, \overline{\alpha}) \\ + \langle\psi, \chi\rangle\langle\overline{\psi}, \overline{\alpha}\rangle&[\partial^{k-1}_{\bar\psi} W](\chi^{k-3}, \overline{\chi}^{k+1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\chi}) \\ + \langle\psi, \alpha\rangle\langle\overline{\psi}, \overline{\chi}\rangle&[\partial^{k-1}_{\bar\psi} W](\chi^{k-3}, \overline{\chi}^{k+1}; \psi^{k+1}, \overline{\psi}^{k-3}, \chi, \overline{\alpha}) , \end{aligned}\tag{4.43}$$

along with

$$\begin{aligned}2\langle\psi, \chi\rangle\langle\overline{\psi}, \overline{\chi}\rangle&[\partial^{k-1}_{\psi} U](\chi^{k+1}, \overline{\chi}^{k-3}; \overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}) \\ = \langle\chi, \alpha\rangle\langle\overline{\chi}, \overline{\psi}\rangle&[\partial^{k-1}_{\bar\psi} W](\psi^{k-3}, \overline{\psi}^{k+1}; \chi^{k+1}, \overline{\chi}^{k-3}, \psi, \overline{\alpha}) \\ + \langle\chi, \psi\rangle\langle\overline{\chi}, \overline{\alpha}\rangle&[\partial^{k-1}_{\bar\psi} W](\psi^{k-3}, \overline{\psi}^{k+1}; \chi^{k+1}, \overline{\chi}^{k-3}, \alpha, \overline{\psi}) \\ + \langle\psi, \chi\rangle\langle\overline{\psi}, \overline{\alpha}\rangle&[\partial^{k-1}_{\psi} U](\chi^{k+1}, \overline{\chi}^{k-3}; \overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\chi}) \\ + \langle\psi, \alpha\rangle\langle\overline{\psi}, \overline{\chi}\rangle&[\partial^{k-1}_{\psi} U](\chi^{k+1}, \overline{\chi}^{k-3}; \overline{\psi}^{k+1}, \psi^{k-3}, \chi, \overline{\alpha}) . \end{aligned}\tag{4.44}$$

In this last equation, we simultaneously interchange ψ with χ and $\overline{\psi}$ with $\overline{\chi}$; a comparison with (4.43) allows us to deduce that

$$[\partial^{k-1}_{\bar\psi} W](\chi^{k-3}, \overline{\chi}^{k+1}; \psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}) = [\partial^{k-1}_{\psi} U](\psi^{k+1}, \overline{\psi}^{k-3}; \overline{\chi}^{k+1}, \chi^{k-3}, \alpha, \overline{\alpha}) . \tag{4.45}$$

Equations (4.32), (4.35), and (4.45) are the integrability conditions for (4.23) and (4.24).

Proposition 4.9. *Let $h^{AB}_{A'B'}$ be a generalized symmetry of order k. Then there is a natural vector field of order* $k-1$,

$$X^A_{A'} = X^A_{A'}(\Psi^2, \bar{\Psi}^2, \ldots, \Psi^{k-1}, \bar{\Psi}^{k-1}),$$

such that the spinors W and U in (4.14) *and* (4.17) *are the gradients*

$$[\partial^{k-1}_{\Psi} X](\psi^{k+1}, \overline{\psi}^{k-3}; \alpha, \overline{\alpha}) = W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}), \tag{4.46}$$

and

$$[\partial^{k-1}_{\bar{\Psi}} X](\psi^{k-3}, \overline{\psi}^{k+1}; \alpha, \overline{\alpha}) = U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}). \tag{4.47}$$

Proof. We have already seen that the linearized equations (4.1) imply the integrability conditions for Eqs. (4.46) and (4.47) are satisfied. It is easy to check that

$$X^A_{A'} = \int_0^1 dt \Psi^{B'_1 \cdots B'_{k-3}}_{B_1 \cdots B_{k-3} \cdots B_{k+1}} W^{B_1 \cdots B_{k-3} \cdots B_{k+1} A}_{B'_1 \cdots B'_{k-3} A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-2}, \overline{\Psi}^{k-2}, t\Psi^{k-1}, t\overline{\Psi}^{k-1})$$

$$+ \int_0^1 dt \bar{\Psi}^{B'_1 \cdots B'_{k-3} \cdots B'_{k+1}}_{B_1 \cdots B_{k-3}} U^{B_1 \cdots B_{k-3} A}_{B'_1 \cdots B'_{k-3} \cdots B'_{k+1} A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-2}, \overline{\Psi}^{k-2}, t\Psi^{k-1}, t\overline{\Psi}^{k-1})$$

defines a real, natural vector field that satisfies Eqs. (4.46) and (4.47). □

4D. Reduction in Order of $h^{AB}_{A'B'}$. Let us set

$$d^{AB}_{A'B'} = \nabla^A_{A'} X^B_{B'} + \nabla^B_{B'} X^A_{A'},$$

where $X^A_{A'}$ is defined in Proposition 4.9. By Proposition 2.5, we know that $d^{AB}_{A'B'}$ is a solution to the linearized equations (4.1) and so defines a generalized symmetry of the vacuum Einstein equations. Therefore

$$l^{AB}_{A'B'} = h^{AB}_{A'B'} - d^{AB}_{A'B'}$$

is also a generalized symmetry. Since

$$[\partial^k_{\Psi} d](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})$$
$$= \langle \psi, \alpha \rangle \langle \overline{\alpha}, \overline{\psi} \rangle W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) + \langle \psi, \beta \rangle \langle \overline{\beta}, \overline{\psi} \rangle W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha})$$

and

$$[\partial^k_{\bar{\Psi}} d](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha, \beta, \overline{\alpha}, \overline{\beta})$$
$$= \langle \overline{\psi}, \overline{\alpha} \rangle \langle \alpha, \psi \rangle U(\overline{\psi}^{k+1}, \psi^{k-3}, \beta, \overline{\beta}) + \langle \overline{\psi}, \overline{\beta} \rangle \langle \beta, \psi \rangle U(\overline{\psi}^{k+1}, \psi^{k-3}, \alpha, \overline{\alpha}),$$

we have, from our basic decomposition (4.14) and (4.17),

$$[\partial^k_\Psi l](\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})$$
$$=\langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^k,\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta})+\langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4}) \tag{4.48}$$

and

$$[\partial^k_{\overline{\Psi}} l](\psi^{k-2},\overline{\psi}^{k+2};\alpha,\beta,\overline{\alpha},\overline{\beta})$$
$$=\langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle D(\overline{\psi}^k,\psi^{k-2}\alpha\beta)+\langle\psi,\alpha\rangle\langle\psi,\beta\rangle E(\overline{\psi}^{k+2}\overline{\alpha}\overline{\beta},\psi^{k-4})\,. \tag{4.49}$$

As in Sect. 3, we now show that the linearized equations (4.1) force

$$A=B=D=E=0\,, \tag{4.50}$$

and hence

$$h^{AB}_{A'B'}=\nabla^A_{A'}X^B_{B'}+\nabla^B_{B'}X^A_{A'}+l^{AB}_{A'B'}\,, \tag{4.51}$$

where $l^{AB}_{A'B'}$ is now of order $k-1$.

To prove (4.50) we differentiate Eq. (4.1) one final time with respect to Ψ^{k+1} and use the leading order symmetry condition satisfied by $l^{AB}_{A'B'}$, namely

$$[\partial^k_\Psi l](\psi^{k+2},\overline{\psi}^{k-2};\psi,\alpha,\overline{\alpha},\overline{\psi})=0\,,$$

to arrive at

$$\langle\beta,\psi\rangle\langle\overline{\beta},\overline{\psi}\rangle\{[\partial^{k-1}_\Psi l](\psi^{k+1},\overline{\psi}^{k-3};\alpha,\psi,\overline{\psi},\overline{\alpha})+[\mathrm{Div}\ \partial^k_\Psi l](\psi^{k+2},\overline{\psi}^{k-2};\alpha,\overline{\alpha})\}$$
$$+\langle\alpha,\psi\rangle\langle\overline{\alpha},\overline{\psi}\rangle\{[\partial^{k-1}_\Psi l](\psi^{k+1},\overline{\psi}^{k-3};\beta,\psi,\overline{\psi},\overline{\beta})+[\mathrm{Div}\ \partial^k_\Psi l](\psi^{k+2},\overline{\psi}^{k-1};\beta,\overline{\beta})\}$$
$$+[\mathrm{Grad}\ \partial^k_\Psi l](\psi,\overline{\psi};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,, \tag{4.52}$$

where Grad is defined in (7.15) and

$$[\mathrm{Div}\ \partial^k_\Psi l](\psi^{k+2},\overline{\psi}^{k-2};\alpha,\overline{\alpha})=\alpha_A\overline{\alpha}^{B'}[\nabla^{A'}_B\partial^k_\Psi l^{AB}_{A'B'}](\psi^{k+2},\overline{\psi}^{k-2})\,. \tag{4.53}$$

In (4.52) we now set $\alpha=\beta=\psi$; by virtue of Eq. (4.48) we then find

$$[\mathrm{Grad}\ B](\psi,\overline{\psi};\psi^{k+4},\overline{\psi}^{k-4})=0\,. \tag{4.54}$$

Similarly, if we set $\overline{\alpha}=\overline{\beta}=\overline{\psi}$ in (4.52) and use (4.49) we find that

$$[\mathrm{Grad}\ A](\psi,\overline{\psi};\psi^k,\overline{\psi}^k)=0\,. \tag{4.55}$$

Proposition 7.6 implies that $A=0$ and $B=0$.

We have thus found that

$$[\partial^k_\Psi l](\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,.$$

Likewise, by differentiating the linearized equations (4.1) with respect to $\bar{\Psi}^{k+1}$ we can show that $D=0$ and $E=0$ so that

$$[\partial^k_{\overline{\Psi}}\, l](\psi^{k-2},\overline{\psi}^{k+2}\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,.$$

These last two equations prove (4.50).

Theorem 4.10. *Let*

$$h^{AB}_{A'B'} = h^{AB}_{A'B'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k)$$

be a natural generalized symmetry of the vacuum Einstein equations of order k. Then there exists a natural vector

$$X^A_{A'} = X^A_{A'}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1})$$

of order $k-1$ and a constant c, such that

$$h^{AB}_{A'B'} = c\ \varepsilon^{AB}\varepsilon_{A'B'} + \nabla^A_{A'}X^B_{B'} + \nabla^B_{B'}X^A_{A'} \quad \text{on } \mathscr{E}^k .$$

Proof. If $k = 2$ this theorem reduces to Corollary 4.7. Let $k > 2$. We have shown that

$$h^{AB}_{A'B'} = \nabla^A_{A'}X^B_{B'} + \nabla^B_{B'}X^A_{A'} + l^{AB}_{A'B'},$$

where $l^{AB}_{A'B'}$ is a natural spinor of order $k-1$. A straightforward induction argument now shows that $l^{AB}_{A'B'}$ can be reduced to a function of the Penrose fields Ψ^2, $\overline{\Psi}^2$ at the expense of changing the vector field $X^A_{A'}$. We apply Corollary 4.7 to the natural generalized symmetry $l^{AB}_{A'B'}$ to show that

$$l^{AB}_{A'B'} = c\ \varepsilon^{AB}\varepsilon_{A'B'} ,$$

and our classification of the natural generalized symmetries of the vacuum Einstein equations is complete. □

5. First-Order Generalized Symmetries

In this section we begin our classification of *all* generalized symmetries of the vacuum Einstein equations by determining all first-order generalized symmetries. As mentioned in the introduction, the calculation of the higher-order generalized symmetries reduces to that of the first-order generalized symmetries. While the analysis of the higher-order symmetries is similar in spirit to that of the natural symmetries, as presented in the previous section, the analysis of the first-order symmetries is rather more complex and merits a separate presentation.

To begin, let

$$h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,k})$$

be the components of a first-order generalized symmetry. We emphasize that the functions h_{ab} are no longer assumed to be the components of a natural tensor and hence may depend explicitly upon the coordinates x^i and the first derivatives of the metric $g_{ij,k}$. The linearized equations

$$[-g^{cd}\delta^a_i\delta^b_j - g^{ab}\delta^c_i\delta^d_j + g^{ac}(\delta^b_i\delta^d_j + \delta^b_j\delta^d_i)]\nabla_c\nabla_d h_{ab} = 0 \tag{5.1}$$

involve the metric and its first 3 derivatives, and must be satisfied when the Einstein equations

$$R_{ab} = 0 \quad \text{and} \quad \nabla_c R_{ab} = 0 \tag{5.2}$$

are satisfied. In accordance with the results of Sect. 2, we write h_{ab} as a new function

$$h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^i_{jk})$$

and express the linearized equations in terms of the jet coordinates

$$\{x^i, g_{ij}, \Gamma^i_{jk}, \Gamma^i_{jhk}, \Gamma^i_{jhkl}, Q_{ij,kl}, Q_{ij,klm}\} \tag{5.3}$$

for $J^3(\mathscr{G})$, which were introduced in Sect. 2 (see (2.12) and (2.13)). The Einstein equations (5.2) hold if and only if the variables $Q_{ij,kl}$ and $Q_{ij,klm}$ are completely trace-free. Consequently, the linearized equations (5.1) for the first-order generalized symmetry must hold identically for all values of

$$\{x^i, g_{ij}, \Gamma^i_{jk}, \Gamma^i_{jhk}, \Gamma^i_{jhkl}, [Q_{ij,kl}]_{\text{tracefree}}, [Q_{ij,klm}]_{\text{tracefree}}\}\,.$$

In order to determine the dependence of the linearized equations on these adapted jet coordinates we will need the following structure equations for the coordinates (5.3):

$$D_i g_{jk} = g_{jl}\Gamma^l_{ik} + g_{kl}\Gamma^l_{ij}\,, \tag{5.4}$$

$$D_k\Gamma^h_{ij} = \Gamma^h_{ijk} + \tfrac{2}{3}Q^h_{k,ij} + \Gamma^h_{mi}\Gamma^m_{jk} + \Gamma^h_{mj}\Gamma^m_{ik}\,, \tag{5.5}$$

$$D_l\Gamma^h_{ijk} = \Gamma^h_{ijkl} + \tfrac{1}{2}Q^h_{l,ijk} - \tfrac{2}{3}Q^m_{l,(ij}\Gamma^h_{k)m} + \tfrac{4}{3}\Gamma^m_{(ik}R_{j)}{}^h{}_{lm} - 3\Gamma^m_{(ik}\Gamma^h_{j)ml}\,, \tag{5.6}$$

and

$$\nabla_m Q_{ij,kl} = Q_{ij,klm} + \tfrac{1}{2}(Q_{m(i,j)kl} + Q_{kl,ijm})\,. \tag{5.7}$$

We will use the following notation. The derivatives of h_{ab} with respect to the metric g_{rs} and connection variables Γ^t_{rs} will be denoted by

$$\partial^{rs}h_{ab} = \frac{\partial h_{ab}}{\partial g_{rs}} \quad \text{and} \quad \partial^{rs}_t h_{ab} = \frac{\partial h_{ab}}{\partial \Gamma^t_{rs}}\,.$$

Note that these quantities are symmetric in the indices rs and ab. If

$$X = X^a\frac{\partial}{\partial x^a}, \qquad Y = Y^a\frac{\partial}{\partial x^a}, \quad \text{and} \quad \alpha = \alpha_r dx^r\,,$$

we let

$$[\partial_g h](\alpha\alpha; XX) = \alpha_r\alpha_s X^a X^b(\partial^{rs}h_{ab})$$

and

$$[\partial_\Gamma h](\alpha\alpha, Y; XX) = \alpha_r\alpha_s Y^t X^a X^b(\partial^{rs}_t h_{ab})\,.$$

We denote by $\alpha^\sharp$ the vector field obtained from the 1-form α by "raising the index" with the metric,

$$\alpha^\sharp = g^{rs}\alpha_s\frac{\partial}{\partial x^r}\,,$$

and we denote by $X^\flat$ the 1-form obtained from the vector X by "lowering the index" with the metric,

$$X^\flat = g_{ij}X^i dx^j\,.$$

The natural pairing of X and α is

$$\langle X, \alpha\rangle = X^i\alpha_i\,.$$

Proposition 5.1. *Let $h_{ab} = h_{ab}(x^l, g_{ij}, \Gamma^i_{jk})$ be a first-order generalized symmetry for the vacuum Einstein equations. Then there are zeroth-order quantities*

$$M^s_{bt} = M^s_{bt}(x^l, g_{ij})$$

such that

$$\partial^{rs}_t h_{ab} = \delta^{(r}_{(a} M^{s)}_{b)t} \,. \tag{5.8}$$

Proof. Since

$$\begin{aligned}\nabla_d h_{ab} &= D_d h_{ab} - \Gamma^l_{ad} h_{ib} - \Gamma^l_{bd} h_{ai} \\ &= (\partial^{rs}_t h_{ab}) \Gamma^t_{rsd} + \{\bigstar\} \,,\end{aligned}$$

where $\{\bigstar\}$ denotes terms involving the variables x^i, g_{ij}, Γ^i_{jh}, $Q^j_{i,hk}$, we conclude using Eqs. (5.6) and (5.7) that

$$\nabla_c \nabla_d h_{ab} = (\partial^{rs}_t h_{ab}) \Gamma^t_{rscd} + \{\bigstar\bigstar\} \,,$$

where $\{\bigstar\bigstar\}$ denotes terms involving the variables x^i, g_{ij}, Γ^i_{jh}, Γ^i_{hjk}, $Q^j_{i,hk}$, $Q^j_{i,hkl}$. Hence, by differentiating the linearized equations (5.1) with respect to Γ^t_{rscd} and contracting the result with $X^i X^j Y^t \alpha_r \alpha_s \alpha_c \alpha_d$, we arrive at

$$\begin{aligned}&\langle \alpha^\sharp, \alpha \rangle [\partial_\Gamma h](\alpha\alpha, Y; XX) \\ &\quad = \langle X, \alpha \rangle \{ -\langle X, \alpha \rangle [\partial_\Gamma \mathrm{tr}\, h](\alpha\alpha, Y) + 2\, [\partial_\Gamma h](\alpha\alpha, Y; \alpha^\sharp X) \} \,. \end{aligned} \tag{5.9}$$

Here we have defined the trace of h_{ab} in the usual way:

$$\mathrm{tr}\, h = g^{ab} h_{ab} \,.$$

When α is a null 1-form, the expression in brackets on the right-hand side of (5.9) must vanish. By Proposition 7.4, this implies that there are quantities M^s_{bt} such that

$$-\langle X, \alpha \rangle [\partial_\Gamma \mathrm{tr}\, h](\alpha\alpha, Y) + 2\, [\partial_\Gamma h](\alpha\alpha, Y; \alpha^\sharp X) = \langle \alpha^\sharp, \alpha \rangle M(X, Y, \alpha) \,,$$

where

$$M(X, Y, \alpha) = M^s_{bt} X^b Y^t \alpha_s \,.$$

Thus (5.9) reduces to

$$[\partial_\Gamma h](\alpha\alpha, Y; XX) = \langle X, \alpha \rangle M(X, Y, \alpha) \,. \tag{5.10}$$

We have shown that Eq. (5.10) is necessary for (5.9) to hold. It is also sufficient. This is easily verified if we observe that (5.10) implies

$$[\partial_\Gamma h](\alpha\alpha, Y; \alpha^\sharp X) = \tfrac{1}{2}(\langle \alpha^\sharp, \alpha \rangle M(X, Y, \alpha) + \langle X, \alpha \rangle M(\alpha^\sharp, Y, \alpha))$$

and

$$[\partial_\Gamma \mathrm{tr}\, h](\alpha\alpha, Y) = M(\alpha^\sharp, Y, \alpha) \,.$$

It remains to be shown that M^s_{bt} is independent of the connection variables Γ^i_{jk}. To this end we first differentiate Eq. (5.10) with respect to Γ^i_{jk} to obtain

$$[\partial_\Gamma \partial_\Gamma h](\beta\beta, Z; \alpha\alpha, Y; XX) = \langle X, \alpha \rangle [\partial_\Gamma M](\beta\beta, Z; X, Y, \alpha) \,. \tag{5.11}$$

The left-hand side of this equation is symmetric under interchange of (β, Z) with (α, Y), and therefore

$$\langle X, \alpha\rangle [\partial_\Gamma M](\beta\beta, Z; X, Y, \alpha) = \langle X, \beta\rangle [\partial_\Gamma M](\alpha\alpha, Y; X, Z, \beta)\,.$$

Using Proposition 7.5 we conclude that $[\partial_\Gamma M]$ takes the form

$$[\partial_\Gamma M](\beta\beta, Z; X, Y, \alpha) = \langle X, \beta\rangle W(\alpha, \beta, Y, Z)\,, \tag{5.12}$$

where W has the symmetry property

$$W(\alpha, \beta, Y, Z) = W(\beta, \alpha, Z, Y)\,.$$

Equation (5.11) becomes

$$[\partial_\Gamma \partial_\Gamma h](\beta\beta, Z; \alpha\alpha, Y; XX) = \langle X, \alpha\rangle \langle X, \beta\rangle W(\alpha, \beta, Y, Z)\,. \tag{5.13}$$

Next we observe that the structure equations (5.4)–(5.7) imply

$$\nabla_c \nabla_d h_{ab} = (\partial^{uv}_w \partial^{rs}_t h_{ab}) \Gamma^t_{rsd} \Gamma^w_{uvc} + \{\bigstar\}\,,$$

where $\{\bigstar\}$ denotes terms that are at most linear in the coordinates Γ^i_{jhk}. Using this equation, we now differentiate the linearized equations with respect to Γ^t_{rsd} and Γ^w_{uvc} to find that

$$\begin{aligned}\langle \beta^\sharp, \alpha\rangle [\partial_\Gamma \partial_\Gamma h](\beta\beta, Z; \alpha\alpha, Y; XX) + \langle X, \beta\rangle\langle X, \alpha\rangle [\partial_\Gamma \partial_\Gamma \mathrm{tr}\, h](\beta\beta, Z; \alpha\alpha, Y)\\ = \langle X, \beta\rangle [\partial_\Gamma \partial_\Gamma h](\beta\beta, Z; \alpha\alpha, Y; X\alpha^\sharp) + \langle X, \alpha\rangle [\partial_\Gamma \partial_\Gamma h](\beta\beta, Z; \alpha\alpha, Y; X\beta^\sharp)\,.\end{aligned}$$

Into this equation we substitute from Eq. (5.13) to deduce that

$$\begin{aligned}&\left[\langle \beta^\sharp, \alpha\rangle \langle X, \alpha\rangle\langle X, \beta\rangle - \tfrac{1}{2}\langle X, \beta\rangle^2 \langle \alpha^\sharp, \alpha\rangle - \tfrac{1}{2}\langle X, \alpha\rangle^2\langle \beta^\sharp, \beta\rangle\right]\\ &\quad\times W(\alpha, \beta, Y, Z) = 0\,.\end{aligned}$$

Because the expression in square brackets is not identically zero, this equation implies that $W = 0$ and therefore $\partial_\Gamma M = 0$, as claimed. □

Next we turn to an analysis of the terms involving $Q_{ij,hkl}$ in the linearized equations (5.1). In the following proposition we let

$$M^{sr}_a = M^s_{at} g^{rt} \quad \text{and} \quad M^{asr} = g^{ab} M^s_{bt} g^{rt}\,,$$

and we let $\varepsilon_{ijhk} = \pm 1$ denote the usual totally antisymmetric tensor density.

Proposition 5.2. *If $h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^i_{hk})$ is a first-order generalized symmetry of the vacuum Einstein equations, then there are quantities*

$$V^a = V^a(x^i, g_{ij}) \quad \text{and} \quad W^a = W^a(x^i, g_{ij})$$

such that

$$M^{[sr]}_a = \delta^{[s}_a V^{r]} + g^{sp} g^{rq} \varepsilon_{apql} W^l\,. \tag{5.14}$$

Proof. Because

$$\nabla_d h_{ab} = \tfrac{2}{3}(\partial^{rs}_t h_{ab}) Q^t_{d,rs} + (\partial^{rs}_t h_{ab}) \Gamma^t_{rsd} + \{\bigstar\}\,,$$

where $\{\bigstar\}$ denotes terms involving the variables x^i, g_{ij}, Γ^i_{jk}, we can show

$$\nabla_c \nabla_d h_{ab} = \tfrac{2}{3}(\partial^{rs}_h h_{ab}) Q^h_{d,rs|c} + \tfrac{1}{2}(\partial^{rs}_h h_{ab}) Q^h_{c,rsd} + \{\bigstar\bigstar\},$$

where $\{\bigstar\bigstar\}$ now indicates terms involving the variables x^i, g_{ij}, Γ^k_{ij}, Γ^k_{ijh}, Γ^k_{ijhl}, $Q_{ij,kl}$. Therefore, for the linearized equations to hold we must have that

$$[-g^{cd}\delta^a_i\delta^b_j - g^{ab}\delta^c_i\delta^d_j + g^{ac}\delta^b_i\delta^d_j + g^{bc}\delta^a_j\delta^d_i]\left[\tfrac{2}{3}(\partial^{rs}_h h_{ab}) Q^h_{d,rs|c} + \tfrac{1}{2}(\partial^{rs}_h h_{ab}) Q^h_{c,rsd}\right] = 0 \tag{5.15}$$

for all $Q^h_{c,rs|c}$ and $Q^h_{c,rsd}$ that are completely trace-free. We multiply (5.15) by X^iX^j and substitute for $\partial^{rs}_h h_{ab}$ from Proposition 5.1 and for $Q^h_{c,rsd}$ and $Q^h_{d,rs}$, from (2.13) to obtain

$$\begin{aligned}&[-M^{bsh}X^cX^d + M^{csh}X^bX^d]\\ &\quad\times\Bigg[\tfrac{1}{12}(R_{bhcs|d} + R_{dhcb|s} + R_{shcd|b}\\ &\quad + R_{bhcd|s} + R_{shcb|d} + R_{dhcs|b}) + \tfrac{1}{3}(R_{bhds|c} + R_{shdb|c})\Bigg] = 0\,.\end{aligned}$$

By using the algebraic curvature symmetries and the Bianchi identities, every term in this equation may be expressed as either a multiple of $M^{bsh}X^cX^dR_{dhbc|s}$ or $M^{bsh}X^cX^dR_{shbc|d}$. The coefficient of the former term vanishes, while that of the latter term is one. Thus (5.15) holds if and only if

$$M^{bsh}X^cX^d[R_{shbc|d}]_{\text{tracefree}} = 0\,. \tag{5.16}$$

To analyze this condition it is convenient to revert to spinors. We set

$$M^{BB'AA'HH'} = M^{bst}\sigma^{BB'}_b\sigma^{AA'}_s\sigma^{HH'}_t\,,$$

and use (2.17) and (2.20) to write

$$[R_{shbc|d}]_{\text{tracefree}} \longleftrightarrow \varepsilon_{SH}\varepsilon_{BC}\overline{\Psi}_{S'H'B'C'D'D} + \varepsilon_{S'H'}\varepsilon_{B'C'}\Psi_{SHBCDD'}\,,$$

so that the condition (5.16) is equivalent to

$$X^{CC'}X^{DD'}M^{BB'SS'HH'}[\varepsilon_{SH}\varepsilon_{BC}\overline{\Psi}_{S'H'B'C'D'D} + \varepsilon_{S'H'}\varepsilon_{B'C'}\Psi_{SHBCDD'}] = 0 \tag{5.17}$$

for all Penrose fields Ψ^3 and $\overline{\Psi}^3$. We differentiate this expression with respect to $\Psi_{SHBCDD'}$ and multiply the resulting equation by $\psi_S\psi_H\psi_B\psi_C\psi_D\overline{\psi}_{D'}$ to conclude

$$\varepsilon_{A'H'}\psi_A\psi_H\psi_BM^{BB'AA'HH'} = 0\,. \tag{5.18}$$

Similarly, differentiation of (5.17) with respect to $\overline{\Psi}_{S'H'B'C'D'D}$ leads to

$$\varepsilon_{AH}\overline{\psi}_{A'}\overline{\psi}_{H'}\overline{\psi}_{B'}M^{BB'AA'HH'} = 0\,. \tag{5.19}$$

To solve Eqs. (5.18) and (5.19) we decompose M as

$$M^{BB'AA'HH'} = P^{BB'AA'HH'} + S^{BB'}\varepsilon^{AH}\varepsilon^{A'H'} + T^{BB'A'H'}\varepsilon^{AH} + \overline{T}^{BB'AH}\varepsilon^{A'H'}\,, \tag{5.20}$$

where the spinors $P, T, \overline{T}$ are each symmetric in the indices AH and $A'H'$. Note that the spinors T and $\overline{T}$ correspond to the skew symmetric part of M in (5.14). Equations (5.18) and (5.19) now imply that

$$\psi_A\psi_H\psi_B\overline{T}^{BB'AH} = 0$$

and

$$\overline{\psi}_{A'}\overline{\psi}_{H'}\overline{\psi}_{B'}T^{BB'A'H'} = 0\,.$$

These equations can be analyzed using Proposition 7.2; we find that there must exist quantities $Z^{AA'}$ such that

$$T^{BB'A'H'} = \varepsilon^{A'B'}Z^{BH'} + \varepsilon^{H'B'}Z^{BA'}\,, \tag{5.21}$$

and

$$\overline{T}^{BB'AH} = \varepsilon^{AB}\overline{Z}^{B'H} + \varepsilon^{HB}\overline{Z}^{B'A}\,. \tag{5.22}$$

We insert (5.21) and (5.22) into (5.20). We then write the resulting equation in tensor form to complete the proof. □

We now turn to an analysis of the conditions arising from the $\Gamma^r_{stu}\Gamma^m_{pq}$ terms in the linearized equation. This analysis will enable us to prove that every first-order generalized symmetry is, modulo a generalized diffeomorphism symmetry, an evolutionary zeroth-order symmetry.

Proposition 5.3. *Let $h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^k_{ij})$ be a first-order generalized symmetry of the vacuum Einstein equations. Then there are zeroth-order quantities $V_i = V_i(x^i, g_{ij})$ and $\widehat{h}_{ab} = \widehat{h}_{ab}(x^i, g_{ij})$ such that*

$$h_{ab} = \widehat{h}_{ab} + \nabla_a V_b + \nabla_b V_a\,.$$

Proof. Let

$$\widehat{h}_{ab} = h_{ab} - (\nabla_a V_b + \nabla_b V_a)\,,$$

where $V_a = g_{ab}V^b$ is defined by Proposition 5.2. Then $\widehat{h}_{ab}$ is a first-order generalized symmetry and therefore, by Proposition 5.1, there exist zeroth-order quantities $\widehat{M}^s_{at} = \widehat{M}^s_{at}(x^i, g_{ij})$ such that

$$\partial^{rs}_t\widehat{h}_{ab} = \delta^{(r}_{(a}\widehat{M}^{s)}_{b)t}\,. \tag{5.23}$$

Moreover, by construction, $\widehat{M}$ will satisfy Proposition 5.2 with $V^i = 0$, and hence

$$\widehat{M}^{st}_a = \widehat{M}^{(st)}_a + g^{sp}g^{tq}\varepsilon_{apql}W^l\,. \tag{5.24}$$

This decomposition will allow us to prove, from the coefficient of $\Gamma^r_{stu}\Gamma^m_{pq}$ in the linearized equations, that $\widehat{M}^{st}_a = 0$, that is,

$$\widehat{h}_{ab} = \widehat{h}_{ab}(x^i, g_{ij})\,.$$

The derivation of the condition arising from the coefficient of $\Gamma^r_{stu}\Gamma^m_{pq}$ in the linearized equations is the longest single calculation in this paper. To begin we first compute

$$\alpha_s\alpha_t\alpha_u\partial_r^{stu}(\nabla_c\nabla_d\widehat{h}_{ab}) = \alpha_s\alpha_t\alpha_u[D_c(\partial_r^{st}\widehat{h}_{ab})\delta_d^u + \delta_c^u\partial_r^{st}\nabla_d\widehat{h}_{ab} - 3\delta_c^u\Gamma_{ij}^t\delta_d^{(s}(\partial_r^{ij)}\widehat{h}_{ab}) \\ - \Gamma_{cd}^s(\partial_r^{tu}\widehat{h}_{ab}) - \Gamma_{ac}^l\delta_d^s(\partial_r^{tu}\widehat{h}_{lb}) - \Gamma_{bc}^l\delta_d^s(\partial_r^{tu}\widehat{h}_{la})]\,. \tag{5.25}$$

The second term on the right-hand side of this equation is found to be

$$\alpha_s\alpha_t\partial_r^{st}\nabla_d\widehat{h}_{ab} = \alpha_s\alpha_t[D_d(\partial_r^{st}\widehat{h}_{ab}) + 2g_{jr}\delta_d^t(\partial^{sj}\widehat{h}_{ab}) + 2\Gamma_{jd}^t(\partial_r^{sj}\widehat{h}_{ab}) + 2\delta_d^t\Gamma_{ri}^h(\partial_h^{is}\widehat{h}_{ab}) \\ - \Gamma_{ad}^l(\partial_r^{st}\widehat{h}_{lb}) - \Gamma_{bd}^l(\partial_r^{st}\widehat{h}_{la}) - \delta_d^t\delta_a^s\widehat{h}_{rb} - \delta_d^t\delta_b^s\widehat{h}_{ra}]\,. \tag{5.26}$$

Together, Eqs. (5.25) and (5.26) imply that

$$X^rY^m\alpha_s\alpha_t\alpha_u\beta_p\beta_q[\partial_r^{stu}\partial_m^{pq}(\nabla_c\nabla_d\widehat{h}_{ab})] \\ = 4\beta_{(c}\alpha_{d)}[\partial_g\partial_\Gamma\widehat{h}_{ab}](\beta Y^\flat;\alpha\alpha,X) + 2\alpha_c\alpha_d[\partial_g\partial_\Gamma\widehat{h}_{ab}](\alpha X^\flat;\beta\beta,Y) \\ -\alpha_c\beta_a\beta_dY^m[\partial_\Gamma\widehat{h}_{mb}](\alpha\alpha,X) - \alpha_c\beta_b\beta_dY^m[\partial_\Gamma\widehat{h}_{ma}](\alpha\alpha,X) - \alpha_a\alpha_c\alpha_dX^m[\partial_\Gamma\widehat{h}_{mb}](\beta\beta,Y) \\ -\alpha_b\alpha_c\alpha_dX^m[\partial_\Gamma\widehat{h}_{ma}](\beta\beta,Y) - \beta_a\beta_c\alpha_dY^m[\partial_\Gamma\widehat{h}_{mb}](\alpha\alpha,X) - \beta_b\beta_c\alpha_dY^m[\partial_\Gamma\widehat{h}_{ma}](\alpha\alpha,X) \\ + 2\alpha_c\alpha_d\langle X,\beta\rangle[\partial_\Gamma\widehat{h}_{ab}](\alpha\beta,Y) - \alpha_c\alpha_d\langle Y,\alpha\rangle[\partial_\Gamma\widehat{h}_{ab}](\beta\beta,X) \\ - \beta_c\beta_d\langle Y,\alpha\rangle[\partial_\Gamma\widehat{h}_{ab}](\alpha\alpha,X)\,.$$

We substititute this equation into the linearized equations (5.1) multiplied by Z^iZ^j and use (5.23) to obtain, after considerable algebraic simplifications,

$$2\langle Z,\alpha\rangle^2\{[\partial_g\widehat{M}](\beta Y^\flat;\beta^\sharp,X,\alpha) - \partial_g\widehat{M}(\alpha X^\flat;\beta^\sharp,Y,\beta)\} \\ + 2\langle Z,\alpha\rangle\langle Z,\beta\rangle\ \{[\partial_g\widehat{M}](\alpha X^\flat;\alpha^\sharp,Y,\beta) - \partial_g\widehat{M}(\beta Y^\flat;\alpha^\sharp,X,\alpha)\} \\ + 2\langle Z,\alpha\rangle\langle\alpha^\sharp,\beta\rangle\{[\partial_g\widehat{M}](\alpha X^\flat;Z,Y,\beta) - \partial_g\widehat{M}(\beta Y^\flat;Z,X,\alpha)\} \\ + 2\langle\alpha^\sharp,\alpha\rangle\langle Z,\beta\rangle\{[\partial_g\widehat{M}](\beta Y^\flat;Z,X,\alpha) - \partial_g\widehat{M}(\alpha X^\flat;Z,Y,\beta)\} \\ - \langle Z,\alpha\rangle^2\langle\beta^\sharp,\beta\rangle\widehat{M}(Y,X,\alpha) - \langle Z,\beta\rangle^2\langle\alpha^\sharp,\alpha\rangle\widehat{M}(Y,X,\alpha) \\ + \langle Z,\alpha\rangle^2\langle\alpha,Y\rangle\widehat{M}(\beta^\sharp,X,\beta) - \langle Z,\alpha\rangle^2\langle X,\beta\rangle\widehat{M}(\beta^\sharp,Y,\alpha) \\ + [\langle\alpha^\sharp,\alpha\rangle\langle Y,\alpha\rangle\langle Z,\beta\rangle - \langle Z,\alpha\rangle\langle Y,\alpha\rangle\langle\alpha^\sharp,\beta\rangle]\widehat{M}(Z,X,\beta) \\ + [\langle Z,\alpha\rangle\langle X,\beta\rangle\langle\alpha^\sharp,\beta\rangle - \langle\alpha^\sharp,\alpha\rangle\langle X,\beta\rangle\langle Z,\beta\rangle]\widehat{M}(Z,Y,\alpha) \\ - \langle Z,\alpha\rangle\langle Y,\alpha\rangle\langle Z,\beta\rangle\widehat{M}(\alpha^\sharp,X,\beta) + \langle Z,\alpha\rangle\langle X,\beta\rangle\langle Z,\beta\rangle\widehat{M}(\alpha^\sharp,Y,\alpha) \\ + 2\langle Z,\alpha\rangle\langle Z,\beta\rangle\langle\alpha^\sharp,\beta\rangle\widehat{M}(Y,X,\alpha) = 0\,. \tag{5.27}$$

As a check of the accuracy of this equation, we used *Maple* to verify that the diffeomorphism symmetry, for which

$$\widehat{M}(X,Z,\alpha) = 2[\partial_gV](Z^\flat\alpha;X) - \langle X,\alpha\rangle V(Z)\,,$$

and $V_i = V_i(x^i, g_{kl})$, provides a solution to (5.27).

In order to simplify Eq. (5.27) using (5.24) we set

$$\widehat{N}_a^{sr} = \tfrac{1}{2}(M_{at}^s g^{rt} + M_{at}^r g^{st}),$$

$$\widehat{N}(Z,\beta,\alpha) = \widehat{N}_a^{sr} Z^a \beta_s \alpha_r \quad \text{and} \quad det(X,Y,Z,U) = \varepsilon_{abcd} X^a Y^b Z^c U^d,$$

and observe that

$$[\partial_g \widehat{M}](\beta\gamma; Z,X,\alpha) = [\partial_g \widehat{N}](\beta\gamma; Z, X^\flat, \alpha) + det(Z,X,\alpha^\sharp,[\partial_g W](\beta\gamma)) + \tfrac{1}{2}\langle X,\beta\rangle \widehat{N}(Z,\gamma,\alpha)$$
$$+ \tfrac{1}{2}\langle X,\gamma\rangle \widehat{N}(Z,\beta,\alpha) - \tfrac{1}{2}\langle \alpha^\sharp,\beta\rangle det(Z,\gamma^\sharp,X,W) - \tfrac{1}{2}\langle \beta^\sharp,\gamma\rangle det(Z,\alpha^\sharp,X,W)\,.$$

We substitute this equation into (5.27) and use the fact that

$$\widehat{N}(Z,\alpha,\beta) = \widehat{N}(Z,\beta,\alpha) \tag{5.28}$$

to deduce, again after lengthy algebraic simplifications, that

$$\langle Z,\alpha\rangle^2 K(\beta,Y,\beta^\sharp,\alpha,X) + \langle Z,\alpha\rangle\langle Z,\beta\rangle K(\alpha,X,\alpha^\sharp,\beta,Y)$$
$$+ [\langle Z,\alpha\rangle\langle\alpha^\sharp,\beta\rangle - \langle Z,\beta\rangle\langle\alpha^\sharp,\alpha\rangle] K(\alpha,X,Z,\beta,Y) = 0\,, \tag{5.29}$$

where

$$K(\alpha,X,Z,\beta,Y) = [\partial_g \widehat{N}](\alpha X^\flat; Z, Y^\flat, \beta) - [\partial_g \widehat{N}](\beta Y^\flat; Z, X^\flat, \alpha)$$
$$+ det(Z,\beta^\sharp,Y,[\partial_g W](\alpha X^\flat))$$
$$- det(Z,\alpha^\sharp,X,[\partial_g W](\beta Y^\flat)) + \tfrac{1}{2}\langle Z,\beta\rangle \widehat{N}(Y,\alpha,X^\flat) - \tfrac{1}{2}\langle Z,\alpha\rangle \widehat{N}(X,\beta,Y^\flat)$$
$$+ \tfrac{1}{2}\langle Z,\beta\rangle det(Y,\alpha^\sharp,X,W) + \langle\alpha^\sharp,\beta\rangle det(Z,Y,X,W) + \tfrac{1}{2}\langle Z,\alpha\rangle det(\beta^\sharp,X,Y,W). \tag{5.30}$$

Equation (5.29) implies that $K(\alpha,X,Z,\beta,Y) = 0$ whenever $\langle Z,\alpha\rangle = 0$. Therefore, by Proposition 7.5, there exist quantities L such that

$$K(\alpha,X,Z,\beta,Y) = \langle Z,\alpha\rangle L(X,\beta,Y)\,.$$

Substituting this expression back into (5.29) and simplifying the result, we find

$$\langle\beta^\sharp,\beta\rangle L(Y,\alpha,X) + \langle\alpha^\sharp,\beta\rangle L(X,\beta,Y) = 0\,.$$

In this equation we set $\alpha = \beta$ to conclude that $L = 0$ and hence $K = 0$.

In the equation

$$K(\alpha,X,Z,\beta,Y) - K(X^\flat,\alpha^\sharp,Z,\beta,Y) = 0 \tag{5.31}$$

we put $Y = \beta^\sharp$ and $Z = \alpha^\sharp$ to deduce that $\widehat{N} = 0$. We then substitute this result in (5.31) with $Z = \alpha^\sharp$ to get $W = 0$. □

We are now ready to complete our classification of first-order generalized symmetries.

Theorem 5.4. *Let $h_{ab} = h_{ab}(x^i, g_{ij}, \Gamma^k_{ij})$ be a first-order generalized symmetry of the vacuum Einstein equations. Then there is a constant c and zeroth-order quantities $V_i = V_i(x^i, g_{ij})$ such that*

$$h_{ab} = c g_{ab} + \nabla_a V_b + \nabla_b V_a\,.$$

Proof. Proposition 5.3 reduces the proof to showing that the zeroth-order symmetry $\widehat{h}_{ab}$ is in fact a constant times the metric. This follows from the classification of the point symmetries of the Einstein equations [24]. We include the proof here for completeness.

Let us begin with the conditions placed on $\widehat{h}_{ab}$ by the vanishing of the terms in the linearized equations involving Γ^a_{bcd}. From the structure equations (5.4)–(5.6) it is a straightforward matter to show that

$$\nabla_c\nabla_d\widehat{h}_{ab} = 2\frac{\partial\widehat{h}_{ab}}{\partial g_{mn}}g_{mp}\left[\Gamma^p_{ncd} + \tfrac{2}{3}Q^p_{c,nd}\right] - \widehat{h}_{pa}\left[\Gamma^p_{bdc} + \tfrac{2}{3}Q^p_{c,db}\right]$$
$$- \widehat{h}_{pb}\left[\Gamma^p_{adc} + \tfrac{2}{3}Q^p_{c,da}\right] + \{\bigstar\}\,, \tag{5.32}$$

where $\{\bigstar\}$ denotes terms depending only on the variables x^i, g_{ij}, Γ^k_{ij}. We multiply the linearized equations by X^iX^j and differentiate them with respect to Γ^a_{bcd}. The result, after multiplying by $\alpha_b\alpha_c\alpha_d Z^a$ and simplifying, is given by

$$\langle\alpha^\sharp,\alpha\rangle[\partial_g\widehat{h}](Z^\flat\alpha;XX) = \langle\alpha,X\rangle\{2[\partial_g\widehat{h}](Z^\flat\alpha;\alpha^\sharp X) - \langle\alpha,X\rangle[\partial_g\mathrm{tr}\,\widehat{h}](Z^\flat\alpha)\}\,. \tag{5.33}$$

Proposition 7.5 now implies that there exist zeroth-order quantities A such that

$$[\partial_g\widehat{h}](Z^\flat\alpha;XX) = \langle\alpha,X\rangle A(Z^\flat,X)\,.$$

The symmetry of $(\partial_g\widehat{h})$ in $Z^\flat\alpha$ implies that

$$\langle\alpha,X\rangle A(Z^\flat,X) = \langle Z^\flat,X\rangle A(\alpha,X)\,,$$

and therefore, by Proposition 7.5, there exists a zeroth-order function $F = F(x^i, g_{ij})$ such that

$$A(\alpha,X) = \langle\alpha,X\rangle F\,.$$

We have therefore found that

$$[\partial_g\widehat{h}](\alpha\alpha;XX) = \langle\alpha,X\rangle^2 F\,. \tag{5.34}$$

It is easily verified that this equation is necessary and sufficient for (5.33) to hold. Next, we differentiate (5.34) with respect to g_{ij} to obtain

$$[\partial_g\partial_g\widehat{h}](\beta\beta;\alpha\alpha;XX) = \langle\alpha,X\rangle^2[\partial_g F](\beta\beta)\,.$$

The left-hand side of this equation is symmetric under interchange of α and β, and we therefore have

$$\langle\alpha,X\rangle^2[\partial_g F](\beta\beta) = \langle\beta,X\rangle^2[\partial_g F](\alpha\alpha)\,.$$

From Proposition 7.5 it is easily seen that this equation implies

$$[\partial_g F](\alpha\alpha) = 0\,. \tag{5.35}$$

Equations (5.34), (5.35) imply that $\widehat{h}_{ab}$ is of the form

$$\widehat{h}_{ab} = F(x^i)g_{ab} + k_{ab}(x^i)\,. \tag{5.36}$$

Now we turn to the conditions on $\widehat{h}_{ab}$ arising from the terms in the linearized equations depending on $Q_{ab,cd}$. It is straightforward to show, using (5.32), that this condition takes the form

$$Q_{ij,kl}\left[2X^iX^cg^{rk}\frac{\partial\widehat{h}_{rc}}{\partial g_{jl}} - X^iX^kg^{bc}\frac{\partial\widehat{h}_{bc}}{\partial g_{jl}} - \frac{3}{2}X^iX^j\widehat{h}^{kl}\right] = 0\,,$$

when $Q_{ij,kl}$ is completely trace-free. If we substitute from (5.36) the first and second terms vanish leaving us with

$$X^iX^jk_{ab}g^{ak}g^{bl}[Q_{ij,kl}]_{\text{tracefree}} = 0\,.$$

Because k_{ab} is independent of the metric, this equation implies that $k_{ab} = 0$.

We have reduced $\widehat{h}_{ab}$ to the form

$$\widehat{h}_{ab} = F(x^i)g_{ab}\,.$$

We now substitute this equation for $\widehat{h}_{ab}$ into the linearized equations to find

$$-g_{ij}\nabla^a\nabla_aF - 2\nabla_i\nabla_jF = 0\,.$$

We differentiate this equation with respect to Γ^r_{st} and obtain

$$[g_{ij}g^{st} + 2\delta^{(s}_{(i}\delta^{t)}_{j)}]\frac{\partial F}{\partial x^r} = 0\,,$$

which implies that $\dfrac{\partial F}{\partial x^r} = 0$, and thus F is a constant. □

6. Complete Classification of Generalized Symmetries of the Vacuum Einstein Equations.

We now turn to the computation of all generalized symmetries of the Einstein equations. Let

$$h^{AB}_{A'B'} = h^{AB}_{A'B'}(x, \sigma, \Gamma^1, \Gamma^2, \Psi^2, \overline{\Psi}^2, \ldots, \Gamma^l, \Psi^k, \overline{\Psi}^k) \tag{6.1}$$

be the components of a generalized symmetry of the Einstein equations. Initially, we have $l = k$, so the generalized symmetry is of order k. The repeated covariant derivative of $h^{AB}_{A'B'}$ can be given schematically by

$$\nabla\nabla h = DDh + \gamma\cdot Dh + (D\gamma)\cdot h + \gamma\cdot\gamma\cdot h\,,$$

where $\gamma\cdot Dh$ is a sum of products of spin connections $\gamma^{BC}_{AA'}$ and $\overline{\gamma}^{B'C'}_{AA'}$ and total derivatives $D^C_{C'}h^{AB}_{A'B'}$, and so on. The linearized equation,

$$\begin{aligned}[][-\varepsilon_{CD}\varepsilon^{C'D'}\alpha_A\beta_B\overline{\alpha}^{A'}\overline{\beta}^{B'} + \varepsilon_{BC}\varepsilon^{A'C'}\alpha_A\beta_D\overline{\alpha}^{B'}\overline{\beta}^{D'}\\ + \varepsilon_{BC}\varepsilon^{A'C'}\alpha_D\beta_A\overline{\alpha}^{D'}\overline{\beta}^{B'}]\nabla^C_{C'}\nabla^D_{D'}h^{AB}_{A'B'} = 0 \quad \text{on } \mathcal{E}^{k+2}\,,\end{aligned} \tag{6.2}$$

is an $SL(2,\mathbf{C})$ invariant identity depending on the variables x^i, $\sigma_{aAA'}$, $\sigma_{aAA',b}$, $\sigma_{aAA',bc}$, Γ^1, Γ^2, Ψ^2, $\overline{\Psi}^2$, …, Γ^{l+2}, Ψ^{k+2}, $\overline{\Psi}^{k+2}$. On the Einstein equation manifold $\mathcal{E}^{k+2}$ there are relationships between $\sigma_{aAA',bc}$ and Γ^2, Ψ^2, $\overline{\Psi}^2$, but in what follows we are careful only to consider terms involving Ψ^l and $\overline{\Psi}^l$ for $l \geqq 3$. The rather complicated lower-derivative analysis was performed in Sect. 5.

In order to analyze the dependence of this equation on our adapted jet coordinates, we need the following structure equations on $\mathscr{E}^{k+1}$:

$$\begin{aligned} D_{j_{k+1}}\Gamma^i_{j_0j_1\cdots j_k} = \Gamma^i_{j_0j_1\cdots j_{k+1}} &+ A^i_{j_0j_1\cdots j_{k+1}}(\sigma, \Psi^{k+1}, \overline{\Psi}^{k+1}) + B^i_{j_0j_1\cdots j_{k+1}}(\Gamma^1, \Gamma^k) \\ &+ C^i_{j_0j_1\cdots j_{k+1}}(\sigma, \Gamma^1, \Psi^k, \overline{\Psi}^k) \\ &+ E^i_{j_0j_1\cdots j_{k+1}}(\sigma, \Gamma^1, \ldots, \Gamma^{k-1}, \Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}). \end{aligned} \tag{6.3}$$

Here $A^{\cdots}_{\cdots}$ is linear in Ψ^k and $\overline{\Psi}^k$, $B^{\cdots}_{\cdots}$ is bilinear in its arguments, $C^{\cdots}_{\cdots}$ is linear in Ψ^k and $\overline{\Psi}^k$ with coefficients depending on σ and Γ^1.

We also have (see (2.20))

$$\begin{aligned} D^{A'}_{A}\Psi^{J'_1\cdots J'_{k-2}}_{J_1\cdots J_{k+2}} = \Psi^{A'J'_1\cdots J'_{k-2}}_{AJ_1\cdots J_{k+2}} &+ M^{A'J'_1\cdots J'_{k-2}}_{AJ_1\cdots J_{k+2}}(\gamma, \overline{\gamma}, \Psi^k) \\ &+ N^{A'J'_1\cdots J'_{k-2}}_{A\ J_1\cdots J_{k+2}}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^{k-1}, \overline{\Psi}^{k-1}), \end{aligned} \tag{6.4}$$

where $M^{\cdots}_{\cdots}$ is linear in Ψ^k. There is an analogous formula for the total derivative of $\overline{\Psi}^k$.

Let

$$f(x^i, \sigma, \Gamma^1, \Gamma^2, \Psi^2, \overline{\Psi}^2, \ldots, \Gamma^l, \Psi^k, \overline{\Psi}^k)$$

be a smooth function. We retain the notation

$$[\partial^m_\Psi f](\psi^{m+2}, \overline{\psi}^{m-2}) \quad \text{and} \quad [\partial^m_{\overline{\Psi}} f](\psi^{m-2}, \overline{\psi}^{m+2})$$

introduced in Sect. 4 for the derivatives of f with respect to Ψ^m and $\overline{\Psi}^m$, and we define

$$[\partial^m_\Gamma f](Y, \omega^{m+1}) = \frac{\partial f}{\partial \Gamma^i_{j_0j_1\cdots j_m}} Y^i \omega_{j_0}\omega_{j_1}\cdots\omega_{j_m}.$$

In many of our subsequent formulas the spinor components

$$\omega^A_{A'} = \sigma^{jA}_{A'}\omega_j$$

of the covector ω will appear. In addition, we will use ω as a bilinear map

$$\omega(\alpha, \overline{\beta}) = \omega^A_{A'}\alpha_A\overline{\beta}^{A'}.$$

Finally, we write

$$h(\alpha, \omega, \overline{\alpha}) = h^{AB}_{A'B'}\alpha_A\omega^{A'}_B\overline{\alpha}^{B'}.$$

From the structure equations (6.3)–(6.4) we readily derive the following commutation rules. For $l \geqq 2$ we have

$$[\partial^{l+1}_\Gamma D^A_{A'} f](Y, \omega^{l+2}) = \omega^A_{A'}[\partial^l_\Gamma f](Y, \omega^{l+1}) \tag{6.5}$$

and

$$\begin{aligned} [\partial^l_\Gamma D^A_{A'} f](Y, \omega^{l+1}) = \omega^A_{A'}[\partial^{l-1}_\Gamma f](Y, \omega^l) &+ (D^A_{A'}[\partial^l_\Gamma f])(Y, \omega^{l+1}) \\ &+ [\Gamma^1 \cdot \partial^l_\Gamma f]^A_{A'}(Y, \omega^{l+1}), \end{aligned} \tag{6.6}$$

while for $l < k$ we find that

$$[\partial_\Psi^{k+1} D^A_{A'} f](\psi^{k+3}, \overline{\psi}^{k-1}) = \psi^A \overline{\psi}_{A'} [\partial_\Psi^k f](\psi^{k+2}, \overline{\psi}^{k-2}) \tag{6.7}$$

and

$$[\partial_\Psi^k D^A_{A'} f](\psi^{k+2}, \overline{\psi}^{k-2}) = \psi^A \overline{\psi}_{A'} [\partial_\Psi^{k-1} f](\psi^{k+1}, \overline{\psi}^{k-3}) + (D^A_{A'}[\partial_\Psi^k f])(\psi^{k+2}, \overline{\psi}^{k-2})$$
$$+ [\gamma^1 \cdot \partial_\Psi^k f]^A_{A'}(\psi^{k+2}, \overline{\psi}^{k-2}) + [\partial_\Gamma^{k-1} f]^A_{A'}(\psi^{k+2}, \overline{\psi}^{k-2})\,. \tag{6.8}$$

The terms in (6.6) and (6.8) involving $[\Gamma^1 \cdot]$ and $[\gamma^1 \cdot]$ denote a sum of terms each linear and homogeneous in the spin-connections.

The analysis of (6.2) now proceeds along lines very similar to those presented in Sect. 4. As in that section, the linearized equations are viewed as identities in our adapted jet coordinates. Starting at the highest derivative order, the linearized equations are differentiated with respect to the various coordinates on $\mathscr{R}^{k+2}$. Accordingly, we shall not provide all the details of the many calculations involved in the lengthy analysis, but rather simply list the various steps and the conclusions obtained in each.

6A. The Γ^{l+2} Analysis, $l \geqq k-1$, $k \geqq 2$. When we differentiate (6.2) with respect to Γ^{l+2}, we find that

$$\langle\omega,\omega\rangle[\partial_\Gamma^l h](Y,\omega^{l+1};\alpha,\overline{\alpha},\beta,\overline{\beta}) + \omega(\beta,\overline{\beta})[\partial_\Gamma^l h](Y,\omega^{l+1};\alpha,\omega,\overline{\alpha})$$
$$+ \omega(\alpha,\overline{\alpha})[\partial_\Gamma^l h](Y,\omega^{l+1};\beta,\omega,\overline{\beta}) = 0\,. \tag{6.9}$$

In this equation, set $\omega^A_{A'} = \psi^A \overline{\psi}_{A'}$ to conclude that

$$[\partial_\Gamma^l h](Y,\omega^{l+1};\alpha,\omega,\overline{\alpha}) = 0$$

whenever ω is a null vector. By Proposition 7.4 this implies there is a real spinor

$$P = P(Y,\omega^l,\alpha,\overline{\alpha})$$

such that

$$[\partial_\Gamma^l h](Y,\omega^{l+1};\alpha,\omega,\overline{\alpha}) = -\tfrac{1}{2}\langle\omega,\omega\rangle P(Y,\omega^l,\alpha,\overline{\alpha})\,.$$

This fact allows us to use (6.9) to show that the highest Γ derivative of h has the algebraic form

$$[\partial_\Gamma^l h](Y,\omega^{l+1};\alpha,\overline{\alpha},\beta,\overline{\beta}) = \tfrac{1}{2}\omega(\alpha,\overline{\alpha})P(Y,\omega^l,\beta,\overline{\beta}) + \tfrac{1}{2}\omega(\beta,\overline{\beta})P(Y,\omega^l,\alpha,\overline{\alpha})\,. \tag{6.10}$$

Note that the commutativity of the partial derivatives $\partial_\Gamma^l \partial_\Gamma^l$ implies, using Eq. (6.10) with $\beta = \alpha$ and $\overline{\beta} = \overline{\alpha}$, that

$$\omega(\alpha,\overline{\alpha})[\partial_\Gamma^l P](Z,\eta^{l+1};Y,\omega^l,\alpha,\overline{\alpha}) = \eta(\alpha,\overline{\alpha})[\partial_\Gamma^l P](Y,\omega^{l+1};Z,\eta^l,\alpha,\overline{\alpha})\,. \tag{6.11}$$

6B. The $\Gamma^{l+1}\Gamma^{l+1}$ Analysis, $l \geqq k-1$, $k \geqq 2$. The repeated derivative of (6.2) with respect to Γ^{l+1} becomes, with $\beta = \alpha$ and $\overline{\beta} = \overline{\alpha}$,

$$\begin{aligned}\langle\omega,\eta\rangle&[\partial^l_\Gamma\partial^l_\Gamma h](Y,\omega^{l+1};Z,\eta^{l+1};\alpha,\alpha,\overline{\alpha},\overline{\alpha})\\&+\eta(\alpha,\overline{\alpha})[\partial^l_\Gamma\partial^l_\Gamma h](Y,\omega^{l+1};Z,\eta^{l+1};\alpha,\omega,\overline{\alpha})\\&+\omega(\alpha,\overline{\alpha})[\partial^l_\Gamma\partial^l_\Gamma h](Y,\omega^{l+1};Z,\eta^{l+1};\alpha,\eta,\overline{\alpha})=0\,.\end{aligned}\tag{6.12}$$

We now substitute into (6.12) from (6.10), multiply by $\eta(\alpha,\overline{\alpha})$, and use (6.11) to deduce that

$$\begin{aligned}[\langle\omega,\omega\rangle\eta^2(\alpha,\overline{\alpha})+\langle\eta,\eta\rangle\omega^2(\alpha,\overline{\alpha})-2\langle\omega,\eta\rangle\omega(\alpha,\overline{\alpha})\eta(\alpha,\overline{\alpha})]\\\times[\partial^l_\Gamma P](Z,\eta^{l+1};Y,\omega^l,\alpha,\overline{\alpha})=0\,.\end{aligned}$$

Because the first spinor in brackets is not identically zero, we find that

$$[\partial^l_\Gamma P](Z,\eta^{l+1};Y,\omega^l,\alpha,\overline{\alpha})=0\,,\tag{6.13}$$

and thus $h^{AB}_{A'B'}$ is at most linear in the variables Γ^l.

6C. The $\Psi^{k+2}\Gamma^l$ and $\overline{\Psi}^{k+2}\Gamma^l$ Analysis, $l \geqq k-1$, $k \geqq 2$. The commutation rules (6.5)–(6.8) do not allow us to immediately differentiate with respect to Ψ^{k+2} and $\overline{\Psi}^{k+2}$ to arrive at the Eqs. (4.11) and (4.12), which were the basic starting equations for the analysis of natural generalized symmetries. Nevertheless, if we use the linearity of $h^{AB}_{A'B'}$ in the variables Γ^l, we can differentiate (6.2) with respect to Ψ^{k+2} and Γ^l to find that

$$[\partial^l_\Gamma\partial^k_\Psi h](Y,\omega^{l+1};\psi^{k+2},\overline{\psi}^{k-2};\psi,\alpha,\overline{\alpha},\overline{\psi})=0\,,\tag{6.14}$$

and

$$[\partial^l_\Gamma\partial^k_{\overline{\Psi}} h](Y,\omega^{l+1};\psi^{k-2},\overline{\psi}^{k+2};\psi,\alpha,\overline{\alpha},\overline{\psi})=0\,.\tag{6.15}$$

6D. The $\Gamma^{l+1}\Psi^{k+1}$, $\Gamma^{l+1}\overline{\Psi}^{k+1}$ Analysis, $l \geqq k-1$, $k \geqq 2$. Here we find, in a very straightforward manner, that

$$[\partial^k_\Psi\partial^l_\Gamma h](\psi^{k+2},\overline{\psi}^{k-2};Y,\omega^{l+1};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,,\tag{6.16}$$

and

$$[\partial^k_{\overline{\Psi}}\partial^l_\Gamma h](\psi^{k-2},\overline{\psi}^{k+2};Y,\omega^{l+1};\alpha,\beta,\overline{\alpha},\overline{\beta})=0\,.\tag{6.17}$$

In deriving these equations we used (6.14) and (6.15).

6E. The $\Gamma^{l+1}\Gamma^l$ Analysis, $l \geqq k-1$, $k \geqq 3$ and $l=2$, $k=2$. We differentiate (6.2) with respect to Γ^l and Γ^{l+1}. In the resulting equation we set $\beta=\alpha$, $\overline{\beta}=\overline{\alpha}$ and substitute from (6.10) to obtain

$$\begin{aligned}\langle\omega,\omega\rangle\eta(\alpha,\overline{\alpha})&\{[\partial^{l-1}_\Gamma P](Y,\omega^l;Z,\eta^l,\alpha,\overline{\alpha})-[\partial^{l-1}_\Gamma P](Z,\eta^l;Y,\omega^l,\alpha,\overline{\alpha})\}\\&+2\omega(\alpha,\overline{\alpha})\{\langle\omega,\eta\rangle[\partial^{l-1}_\Gamma P](Z,\eta^l;Y,\omega^l,\alpha,\overline{\alpha})+[\partial^l_\Gamma\partial^{l-1}_\Gamma h](Z,\eta^{l+1};Y,\omega^l;\alpha,\omega,\overline{\alpha})\\&+[\partial^{l-1}_\Gamma\partial^l_\Gamma h](Z,\eta^l;Y,\omega^{l+1};\alpha,\eta,\overline{\alpha})\}=0\,.\end{aligned}$$

We multiply this equation by $\eta(\alpha,\overline{\alpha})$ and subtract from it the product of $\omega(\alpha,\overline{\alpha})$ with the result of interchanging (Z,η) with (Y,ω) to deduce that

$$[\partial_{\Gamma}^{l-1}P](Z,\eta^{l};Y,\omega^{l},\alpha,\overline{\alpha}) = [\partial_{\Gamma}^{l-1}P](Y,\omega^{l};Z,\eta^{l},\alpha,\overline{\alpha})\,. \tag{6.18}$$

6F. A Partial Reduction in Order. Equations (6.13), (6.16), (6.17), and (6.18) show that there is a vector field

$$X_{A'}^{A} = X_{A'}^{A}(x,\sigma,\Gamma^{1},\ldots,\Gamma^{l-1},\Psi^{k-1},\overline{\Psi}^{k-1})$$

such that

$$[\partial_{\Gamma}^{l-1}X](Y,\omega^{l};\alpha,\overline{\alpha}) = \tfrac{1}{2}P(Y,\omega^{l};\alpha,\overline{\alpha})\,.$$

Hence the generalized symmetry

$$\widetilde{h}_{A'B'}^{AB} = h_{A'B'}^{AB} - (\nabla_{A'}^{A}X_{B'}^{B} + \nabla_{B'}^{B}X_{A'}^{A})$$

is independent of the variables Γ^{l}, and accordingly we may now assume that the original generalized symmetry (6.1) is of the type

$$h_{A'B'}^{AB} = h_{A'B'}^{AB}(x,\sigma,\Gamma^{1},\Gamma^{2},\Psi^{2},\overline{\Psi}^{2},\ldots,\Gamma^{k-1},\Psi^{k},\overline{\Psi}^{k})\,. \tag{6.19}$$

This partial reduction in the order of $h_{A'B'}^{AB}$ is important because it enables us to repeat, almost without modification, the arguments of Sect. 4.

6G. Repetition of Steps A through E and the Natural Symmetry Analysis, $l = k-1$, $k \geqq 3$. We now repeat steps A through E assuming $h_{A'B'}^{AB}$ to be of the form (6.19), that is, with the Γ derivative-dependence reduced by one order. We can also repeat steps A and B of Sect. 4 to conclude that now

$$\begin{aligned}
&[\partial_{\Psi}^{k}h](\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})\\
&\quad= \langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4})\\
&\quad\quad+ \langle\psi,\alpha\rangle\langle\overline{\alpha},\overline{\psi}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\beta,\overline{\beta}) + \langle\psi,\beta\rangle\langle\overline{\beta},\overline{\psi}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha})\,,
\end{aligned} \tag{6.20}$$

$$\begin{aligned}
&[\partial_{\overline{\Psi}}^{k}h](\psi^{k-2},\overline{\psi}^{k+2};\alpha,\beta,\overline{\alpha},\overline{\beta})\\
&\quad= \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle D(\overline{\psi}^{k},\psi^{k-2}\alpha\beta) + \langle\psi,\alpha\rangle\langle\psi,\beta\rangle E(\overline{\psi}^{k+2}\overline{\alpha}\overline{\beta},\psi^{k-4})\\
&\quad\quad+ \langle\overline{\psi},\overline{\alpha}\rangle\langle\alpha,\psi\rangle U(\overline{\psi}^{k+1},\psi^{k-3},\beta,\overline{\beta}) + \langle\overline{\psi},\overline{\beta}\rangle\langle\beta,\psi\rangle U(\overline{\psi}^{k+1},\psi^{k-3},\alpha,\overline{\alpha})\,,
\end{aligned} \tag{6.21}$$

and

$$[\partial_{\Gamma}^{k-1}h](Y,\omega^{k};\alpha,\omega,\overline{\alpha}) = -\tfrac{1}{2}\langle\omega,\omega\rangle P(Y,\omega^{k-1},\alpha,\overline{\alpha})\,. \tag{6.22}$$

The coefficients A, B, W, D, E, U, and P are functions of the variables $x, \sigma, \ldots, \Gamma^{k-2}, \Psi^{k-1}, \overline{\Psi}^{k-1}$. Note that steps A and B of Sect. 4 are valid even when $k = 2$.

Next we repeat step C of Sect. 4 to find that A, B, D, E are independent of the variables Ψ^{k-1} and $\overline{\Psi}^{k-1}$. We also arrive at the integrability conditions (4.32), (4.35) and (4.45). Note that Sect. 4C is valid even when $k = 2$.

6H. The $\Gamma^{k-1}\Psi^{k+1}$, $\Gamma^{k-1}\overline{\Psi}^{k+1}$, $\Gamma^k\overline{\Psi}^k$ and $\Gamma^k\Psi^k$ Analysis, $k \geqq 3$. The derivative of the linearized equation with respect to Γ^{k-1} and Ψ^{k+1} gives, after taking into account (4.11),

$$\begin{aligned}
&2\omega(\psi,\overline{\psi})[\partial_\Gamma^{k-2}\partial_\Psi^k h](Y,\omega^{k-1};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})\\
&\quad+\langle\alpha,\psi\rangle\langle\overline{\alpha},\overline{\psi}\rangle[\partial_\Gamma^{k-2}\partial_\Psi^k h](Y,\omega^{k-1};\psi^{k+2},\overline{\psi}^{k-2};\beta,\omega,\overline{\beta})\\
&\quad+\langle\beta,\psi\rangle\langle\overline{\beta},\overline{\psi}\rangle[\partial_\Gamma^{k-2}\partial_\Psi^k h](Y,\omega^{k-1};\psi^{k+2},\overline{\psi}^{k-2};\alpha,\omega,\overline{\alpha})\\
&\quad+\langle\alpha,\psi\rangle\langle\overline{\alpha},\overline{\psi}\rangle[\partial_\Psi^{k-1}\partial_\Gamma^{k-1}h](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^k;\beta,\psi,\overline{\psi},\overline{\beta})\\
&\quad+\langle\beta,\psi\rangle\langle\overline{\beta},\overline{\psi}\rangle[\partial_\Psi^{k-1}\partial_\Gamma^{k-1}h](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^k;\alpha,\psi,\overline{\psi},\overline{\alpha})=0\,. \qquad (6.23)
\end{aligned}$$

In this equation we set $\alpha=\beta=\psi$ and then $\overline{\alpha}=\overline{\beta}=\overline{\psi}$ to deduce, in light of (6.20), that

$$[\partial_\Gamma^{k-2}B](Y,\omega^{k-1};\psi^{k+4},\overline{\psi}^{k-4})=0 \quad\text{and}\quad [\partial_\Gamma^{k-2}A](Y,\omega^{k-1};\psi^k,\overline{\psi}^k)=0\,. \qquad (6.24)$$

Now we set $\beta=\alpha$ and $\overline{\beta}=\overline{\alpha}$ in (6.23); after substituting from (6.20) and (6.22) we find that

$$\begin{aligned}
&\tfrac{1}{2}\omega(\alpha,\overline{\psi})[\partial_\Psi^{k-1}P](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^{k-1},\psi,\overline{\alpha})\\
&\quad+\tfrac{1}{2}\omega(\psi,\overline{\alpha})[\partial_\Psi^{k-1}P](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^{k-1},\alpha,\overline{\psi})\\
&\quad-\langle\psi,\alpha\rangle[\partial_\Gamma^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\overline{\psi}\cdot\omega,\overline{\alpha})\\
&\quad-\langle\overline{\psi},\overline{\alpha}\rangle[\partial_\Gamma^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\alpha,\psi\cdot\omega)\\
&\quad=2\omega(\psi,\overline{\psi})[\partial_\Gamma^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha})\,. \qquad (6.25)
\end{aligned}$$

In this equation we have defined

$$(\psi\cdot\omega)^{A'}=\omega_A^{A'}\psi^A \quad\text{and}\quad (\overline{\psi}\cdot\omega)_A=\omega_A^{A'}\overline{\psi}_{A'}\,.$$

Next we differentiate the linearized equation with respect to Γ^k and Ψ^k, then set $\alpha=\beta$ and $\overline{\alpha}=\overline{\beta}$, and substitute from (6.20) and (6.22) to find

$$\begin{aligned}
&\left\{\omega(\psi,\overline{\psi})\omega(\alpha,\overline{\alpha})-\tfrac{1}{2}\langle\omega,\omega\rangle\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle\right\}[\partial_\Psi^{k-1}P](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^{k-1},\alpha,\overline{\alpha})\\
&\quad+\langle\omega,\omega\rangle\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle[\partial_\Gamma^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha})\\
&\quad-\omega(\alpha,\overline{\alpha})\Big\{\tfrac{1}{2}\omega(\alpha,\overline{\psi})[\partial_\Psi^{k-1}P](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^{k-1},\psi,\overline{\alpha})\\
&\quad+\tfrac{1}{2}\omega(\psi,\overline{\alpha})[\partial_\Psi^{k-1}P](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^{k-1},\alpha,\overline{\psi})\\
&\quad-\langle\psi,\alpha\rangle[\partial_\Gamma^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\overline{\psi}\cdot\omega,\overline{\alpha})\\
&\quad-\langle\overline{\psi},\overline{\alpha}\rangle[\partial_\Gamma^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\alpha,\psi\cdot\omega)\Big\}=0\,. \qquad (6.26)
\end{aligned}$$

The last four terms in this equation are precisely the four terms on the left-hand side of (6.25). Therefore, Eqs. (6.25) and (6.26) lead to the integrability condition

$$\tfrac{1}{2}[\partial_{\Psi}^{k-1}P](\psi^{k+1},\overline{\psi}^{k-3};Y,\omega^{k-1},\alpha,\overline{\alpha}) = [\partial_{\Gamma}^{k-2}W](Y,\omega^{k-1};\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha})\,. \quad (6.27)$$

Similarly, an analysis of the $\Gamma^{k-1}\overline{\Psi}^{k-1}$ and $\Gamma^{k}\overline{\Psi}^{k}$ conditions proves that

$$[\partial_{\Gamma}^{k-2}D](Y,\omega^{k-1};\overline{\psi}^{k},\psi^{k}) = 0 \quad \text{and} \quad [\partial_{\Gamma}^{k-2}E](Y,\omega^{k-1};\overline{\psi}^{k+4},\psi^{k-4}) = 0\,, \quad (6.28)$$

and

$$\tfrac{1}{2}[\partial_{\bar{\Psi}}^{k-1}P](\psi^{k-3},\overline{\psi}^{k+1};Y,\omega^{k-1},\alpha,\overline{\alpha}) = [\partial_{\Gamma}^{k-2}U](Y,\omega^{k-1};\overline{\psi}^{k+1},\psi^{k-3},\alpha,\overline{\alpha})\,. \quad (6.29)$$

6I. Reduction in Order, $k \geqq 3$. The integrability conditions (4.32), (4.35), (4.45), (6.18), (6.27), and (6.29) show that there is a real vector field

$$X^{A}_{A'} = X^{A}_{A'}(x,\sigma,\ldots,\Gamma^{k-2},\Psi^{k-1},\overline{\Psi}^{k-1})$$

such that

$$W(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha}) = [\partial_{\Psi}^{k-1}X](\psi^{k+1},\overline{\psi}^{k-3};\alpha,\overline{\alpha})\,,$$

$$U(\overline{\psi}^{k+1},\psi^{k-3},\alpha,\overline{\alpha}) = [\partial_{\bar{\Psi}}^{k-1}X](\psi^{k-3},\overline{\psi}^{k+1};\alpha,\overline{\alpha})\,,$$

$$\tfrac{1}{2}P(Y,\omega^{k-1},\alpha,\overline{\alpha}) = [\partial_{\Gamma}^{k-2}X](Y,\omega^{k-1};\alpha,\overline{\alpha})\,.$$

Just as in Sect. 4, we set

$$l^{AB}_{A'B'} = h^{AB}_{A'B'} - (\nabla^{A}_{A'}X^{B}_{B'} + \nabla^{B}_{B'}X^{A}_{A'})\,. \quad (6.30)$$

Then

$$l^{AB}_{A'B'} = l^{AB}_{A'B'}(x,\sigma,\Gamma^{1},\Gamma^{2},\Psi^{2},\overline{\Psi}^{2},\ldots,\Gamma^{k-2},\Psi^{k},\overline{\Psi}^{k})\,,$$

and

$$[\partial_{\Psi}^{k}l](\psi^{k+2},\overline{\psi}^{k-2};\alpha,\beta,\overline{\alpha},\overline{\beta})$$
$$= \langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4}) \quad (6.31)$$

$$[\partial_{\bar{\Psi}}^{k}l](\psi^{k-2},\overline{\psi}^{k+2};\alpha,\beta,\overline{\alpha},\overline{\beta})$$
$$= \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle D(\overline{\psi}^{k},\psi^{k-2}\alpha\beta) + \langle\psi,\alpha\rangle\langle\psi,\beta\rangle E(\overline{\psi}^{k+2}\overline{\alpha}\overline{\beta},\psi^{k-4})\,. \quad (6.32)$$

Finally, we analyze the terms in the linearized equations involving Ψ^{k+1} and $\overline{\Psi}^{k+1}$. To this end, it is convenient to set

$$R(\psi^{k+2},\overline{\psi}^{k-2},\alpha,\beta,\overline{\alpha},\overline{\beta})$$
$$= \langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4})\,,$$

and

$$S(\psi^{k-2}, \overline{\psi}^{k+2}, \alpha, \beta, \overline{\alpha}, \overline{\beta}) = \langle \overline{\psi}, \overline{\alpha} \rangle \langle \overline{\psi}, \overline{\beta} \rangle D(\overline{\psi}^{k}, \psi^{k-2}\alpha\beta) + \langle \psi, \alpha \rangle \langle \psi, \beta \rangle E(\overline{\psi}^{k+2}\overline{\alpha}\overline{\beta}, \psi^{k-4}) .$$

Then Eqs. (6.30)–(6.32) imply that

$$l = R \cdot \Psi^k + S \cdot \overline{\Psi}^k + \widetilde{l} ,$$

where

$$\widetilde{l} = \widetilde{l}(x, \sigma, \dots, \Gamma^{k-2}, \Psi^{k-1}, \overline{\Psi}^{k-1}) .$$

The repeated covariant derivative of l thus takes the form

$$\begin{aligned}\nabla^{A}_{A'} \nabla^{B}_{B'} l =& (\nabla^{A}_{A'} \nabla^{B}_{B'} R) \cdot \Psi^k + [(\nabla^{A}_{A'} R) \cdot \nabla^{B}_{B'} \Psi^k + (\nabla^{B}_{B'} R) \cdot \nabla^{A}_{A'} \Psi^l] \\ &+ R \cdot (\nabla^{A}_{A'} \nabla^{B}_{B'} \Psi^k) + \nabla^{A}_{A'} \nabla^{B}_{B'} \widetilde{l} + \{\bigstar\} ,\end{aligned}$$

where $\{\bigstar\}$ denotes similar terms derived from $S \cdot \overline{\Psi}^k$. By (6.24) and (6.28), R and S depend upon $x, \sigma, \dots, \Gamma^{k-3}, \Psi^{k-2}, \overline{\Psi}^{k-2}$, and hence the derivatives $\nabla^{A}_{A'} \nabla^{B}_{B'} R$ and $\nabla^{A}_{A'} \nabla^{B}_{B'} S$ are independent of the variables Ψ^{k+1} and $\overline{\Psi}^{k+1}$. Moreover, we have that

$$R \cdot \nabla^{A}_{A'} \nabla^{B}_{B'} \Psi^k = R \cdot \Psi^{k+2} + \{\bigstar\bigstar\} ,$$

where $\{\bigstar\bigstar\}$ denotes terms of order k in the Penrose fields. Hence $R \cdot \nabla^{A}_{A'} \nabla^{B}_{B'} \Psi^k$ does not contain Ψ^{k+1} and $\overline{\Psi}^{k+1}$. Consequently, if we differentiate the linearized equations for $l^{AB}_{A'B'}$ with respect to Ψ^{k+1} and set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$, we obtain

$$\begin{aligned}&(\mathrm{Grad}\, R)(\psi, \overline{\psi}; \psi^{k+2}, \overline{\psi}^{k-2}; \alpha, \alpha, \overline{\alpha}, \overline{\alpha}) + 2\langle \alpha, \psi \rangle \langle \overline{\alpha}, \overline{\psi} \rangle [(\mathrm{Div}\, R)(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \overline{\alpha}) \\ &+ (\partial^{k-1}_{\Psi} \widetilde{l})(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \psi, \overline{\psi}, \overline{\alpha})] = 0 ,\end{aligned} \tag{6.33}$$

where the covariant derivative operators Grad and Div are given by (7.15) and (4.53). With $\alpha = \psi$ and $\overline{\alpha} = \overline{\psi}$, we deduce from this equation the covariant constancy conditions

$$(\mathrm{Grad}\, A)(\psi, \overline{\psi}; \psi^{k}, \overline{\psi}^{k}) = 0 , \tag{6.34}$$

and

$$(\mathrm{Grad}\, B)(\psi, \overline{\psi}; \psi^{k+4}, \overline{\psi}^{k-4}) = 0 . \tag{6.35}$$

Just as in Proposition 7.6, Eq. (6.34) implies that A is independent of all the Γ, Ψ, and $\overline{\Psi}$ variables, that is,

$$A = A(x, \sigma) .$$

But now, the covariant derivative of A takes the general form

$$\nabla^{C}_{C'} A^{\cdots}_{\cdots} = D^{C}_{C'} A^{\cdots}_{\cdots} + \gamma^{C\cdots}_{C'\dots} A^{\cdots}_{\cdots} = \sigma^{aC}_{C'} \left(\frac{\partial A^{\cdots}_{\cdots}}{\partial x^a} + \frac{\partial A^{\cdots}_{\cdots}}{\partial \sigma_{bBB'}} \sigma_{bBB',a} \right) + \gamma^{C\cdots}_{C'\dots} A^{\cdots}_{\cdots} .$$

Since

$$\sigma_{bBB',a} = \Gamma^{e}_{ba} \sigma_{eBB'} + \gamma^{C}_{Ba} \sigma_{bCB'} + \overline{\gamma}^{C'}_{B'a} \sigma_{bBC'} ,$$

we find that

$$\nabla^{C}_{C'}A^{\cdots}_{\cdots} = \Gamma^{e}_{ba}\left(\frac{\partial A^{\cdots}_{\cdots}}{\partial \sigma_{bBB'}}\sigma_{eBB'}\sigma^{aC}_{\ \ C'}\right) + \{\bigstar\}\,,$$

where $\{\bigstar\}$ indicates terms involving x, σ, and the spin connections γ and $\bar{\gamma}$. It is now a simple matter to differentiate (6.34) with respect to Γ^i_{jk}, keeping in mind that Γ^i_{jk} is independent of the spin connections, to arrive at

$$\frac{\partial A}{\partial \sigma_{bBB'}} = 0\,.$$

At this point we can continue, as in the proof of Proposition 7.6, to deduce that $A = 0$. Similarly, B, D, and E satisfy covariant constancy conditions that imply they too vanish.

We have now shown that a generalized symmetry of order $k \geqq 3$ is equivalent, up to a generalized diffeomorphism symmetry, to a generalized symmetry of order $k-1$ depending on x, σ, Γ^i, $i = 1,\ldots,k-2$ and Ψ^j, $\overline{\Psi}^j$, $j = 2,\ldots,k-1$. A straightforward induction argument then implies that any generalized symmetry of order $k \geqq 3$ is, up to a generalized diffeomorphism symmetry, given by a generalized symmetry of order 2 depending on x, σ, Γ^1, Ψ^2, and $\overline{\Psi}^2$. If the order of the original symmetry is $k = 2$, then by repeating steps Sects. 6A through 6F the symmetry is again equivalent, modulo a diffeomorphism symmetry, to a symmetry of order 2 depending on x, σ, Γ^1, Ψ^2, and $\overline{\Psi}^2$.

6J. Reduction to First-Order Generalized Symmetries. The induction argument of Sect. 6I shows that, modulo the generalized diffeomorphism symmetry, any generalized symmetry of order $k \geqq 2$ is equivalent to a symmetry h with the functional dependence

$$h = h(x, \sigma, \Gamma^1, \Psi^2, \overline{\Psi}^2)\,.$$

Sects. 6A through 6D, with $l = 1$ and $k = 2$, show that h takes the schematic form

$$h = P(x,\sigma)\cdot\Gamma^1 + h_0(x,\sigma,\Psi^2,\overline{\Psi}^2)\,.$$

Sects. 4A, 4B, and 4C show that

$$h = P(x,\sigma)\cdot\Gamma^1 + A(x,\sigma)\cdot\Psi^2 + D(x,\sigma)\cdot\overline{\Psi}^2 + l(x,\sigma)\,.$$

The derivative of the linearized equations with respect to Ψ^3 gives an equation similar to (6.33), which we write symbolically as

$$\operatorname{Grad} R + \operatorname{Div} R + \mathcal{O}(x,\sigma) = 0\,.$$

We can then repeat the arguments at the end of Sect. 6I to conclude that $A = 0$. A similar analysis of the terms involving $\overline{\Psi}^3$ in the linearized equations leads to $D = 0$. Thus we reduce our analysis to first-order generalized symmetries, which were classified in Sect. 5 (see Theorem 5.4). We have now proven our main result.

Theorem 6.1. *Let*

$$h_{ab} = h_{ab}(x^i, g_{ij}, g_{ij,h_1},\ldots, g_{ij,h_1\cdots h_k})$$

be the components of a k^{th}-order generalized symmetry of the vacuum Einstein equations $R_{ij} = 0$ in four spacetime dimensions. Then there is a constant c and a generalized vector field

$$X^a = X^a(x^l, g_{ij}, g_{ij,h_1}, \dots, g_{ij,h_1 \cdots h_{k-1}})$$

such that, modulo the Einstein equations,

$$h_{ab} = c g_{ab} + \nabla_a X_b + \nabla_b X_a .$$

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7. Appendix: Results from Tensor and Spinor Analysis

Here we gather together a number of key results which we shall use repeatedly in our study of the generalized symmetries of the Einstein equations. Following the standard algebraic treatment of tensors, we consider spinors as multi-linear maps on complex 2-dimensional vector spaces. For notational convenience, we separate groups of symmetric spinor (or tensor) arguments with a comma and we use no delimiters between arguments within a symmetric set. As an example, if $\alpha, \beta, \gamma, \delta$ are rank 1 spinors, then $T(\alpha\,\beta, \gamma, \delta)$ denotes a rank 4 spinor that is symmetric in α and β,

$$T(\alpha\,\beta, \gamma, \delta) = T(\beta\,\alpha, \gamma, \delta) ,$$

but otherwise has no symmetries. Repeated symmetric arguments of a spinor (or tensor) will be abbreviated using an exponential notation. For example, if T is a spinor of rank $(k+1)$ that is totally symmetric in its first k arguments, we will write

$$T(\psi^k, \bar{\alpha}) = T(\underbrace{\psi, \dots, \psi}_{k \text{ times}}, \bar{\alpha}) .$$

It is important to note that the values of $T(\psi_1 \psi_2 \cdots \psi_k, \bar{\alpha})$, where ψ_1, ψ_2, $\dots$, ψ_k are arbitrary spinors, are completely determined by the values of $T(\psi^k, \bar{\alpha})$.

Our conventions for raising and lowering spinor indices are

$$\beta_B = \varepsilon_{AB} \beta^A \quad \text{and} \quad \alpha^A = \varepsilon^{AB} \alpha_B .$$

The skew-symmetric inner product between α_B and β_A is given by

$$\langle \alpha, \beta \rangle = \alpha_A \beta^A = \varepsilon^{AB} \alpha_A \beta_B = -\langle \beta, \alpha \rangle .$$

We denote by $\langle X, Y \rangle$ the metric inner product between two vectors X and Y.

The following propositions are all elementary facts which we shall use repeatedly [19].

Proposition 7.1. *Let $P = P(\psi^k, \alpha)$ be a rank $(k+1)$ spinor that is symmetric in its first k arguments. Then there are unique, totally symmetric spinors P^* and Q, of rank $k+1$ and $k-1$ respectively, such that*

$$P(\psi^k, \alpha) = P^*(\psi^k \alpha) + \langle \psi, \alpha \rangle Q(\psi^{k-1}) . \tag{7.1}$$

If P is a natural spinor of the Penrose fields $\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k$, *then so are* P^* *and* Q.

Proof. If we define P^* by

$$P^*(\psi^{k+1}) = P(\psi^k, \psi),$$

and Q by

$$\langle \beta, \alpha \rangle Q(\psi^{k-1}) = \frac{k}{k+1}[P(\psi^{k-1}\beta, \alpha) - P(\psi^{k-1}\alpha, \beta)],$$

then we find that

$$\begin{aligned} P(\psi^k, \alpha) - P^*(\psi^k \alpha) &= P(\psi^k, \alpha) - \frac{1}{k+1}P(\psi^k, \alpha) - \frac{k}{k+1}P(\psi^{k-1}\alpha, \psi) \\ &= \frac{k}{k+1}\left[P(\psi^k, \alpha) - P(\psi^{k-1}\alpha, \psi)\right] \\ &= \langle \psi, \alpha \rangle Q(\psi^{k-1}). \end{aligned}$$

The uniqueness of P^* and Q is established by showing that P vanishes if and only if P^* and Q each vanish. To show this, we set $P = 0$ in (7.1):

$$P^*(\psi^k \alpha) + \langle \psi, \alpha \rangle Q(\psi^{k-1}) = 0. \tag{7.2}$$

If we set $\alpha = \psi$ in (7.2), we conclude that $P^* = 0$; substituting this result into (7.2) then shows that $Q = 0$. □

Proposition 7.2. *Let* $P = P(\psi^k, \alpha)$ *be a rank* $(k+1)$ *spinor that is symmetric in its first k arguments. If* $P(\psi^k, \alpha)$ *satisfies*

$$P(\psi^k, \psi) = 0, \tag{7.3}$$

then there is a totally symmetric spinor $Q = Q(\psi^{k-1})$ *such that*

$$P(\psi^k, \alpha) = \langle \psi, \alpha \rangle Q(\psi^{k-1}). \tag{7.4}$$

If P is a natural spinor, then so is Q.

Proof. We put $\alpha = \psi$ in (7.1), and use (7.3) to conclude that $P^* = 0$. □

We note for future use that (7.4) is equivalent to

$$P(\psi^1 \cdots \psi^k, \alpha) = \frac{1}{k}\sum_{i=1}^{k} \langle \psi^i, \alpha \rangle Q(\psi^1 \cdots \psi^{i-1}\psi^{i+1} \cdots \psi^k). \tag{7.5}$$

Proposition 7.3. *Let* $P = P(\psi^k, \alpha)$ *be a rank* $(k+1)$ *spinor that is symmetric in its first k arguments. If* $P(\psi^k, \alpha)$ *satisfies*

$$\langle \psi, \alpha \rangle P(\psi^k, \beta) = \langle \psi, \beta \rangle P(\psi^k, \alpha), \tag{7.6}$$

then there is a unique totally symmetric spinor Q of rank $k-1$ *such that*

$$P(\psi^k, \alpha) = \langle \psi, \alpha \rangle Q(\psi^{k-1}). \tag{7.7}$$

The spinor Q is natural if P is natural. If, in place of (7.6), $P(\psi^k,\alpha)$ satisfies

$$\langle\psi,\alpha\rangle P(\psi^k,\beta) = -\langle\psi,\beta\rangle P(\psi^k,\alpha)\,, \tag{7.8}$$

then $P = 0$.

Proof. Both of these results are proved by setting $\alpha = \psi$ in (7.6) and (7.8) and using Proposition 7.2. □

Proposition 7.4. *Let T be a symmetric rank-k tensor, and suppose that*

$$T(X^k) = 0$$

whenever X is a null vector. Then there exists a unique symmetric tensor P of rank $k-2$ such that, for any vector X,

$$T(X^k) = \langle X,X\rangle P(X^{k-2})\,. \tag{7.9}$$

Proof. The tensor T may be decomposed into a sum of products of metric tensors and trace-free tensors. Thus we can write T as

$$T(X^k) = T_0(X^k) + \langle X,X\rangle P(X^{k-2})\,, \tag{7.10}$$

where T_0 is trace-free and symmetric. The tensor P need not be trace-free. The spinor representation of T_0 is

$$(T_0)_{a_1\cdots a_k} \longleftrightarrow (T_0)^{A'_1\cdots A'_k}_{A_1\cdots A_k}\,,$$

where $(T_0)^{A'_1\cdots A'_k}_{A_1\cdots A_k}$ is completely symmetric in its primed and unprimed indices. With

$$X^a = \sigma^{aA}_{A'}\psi_A\overline{\psi}^{A'}\,,$$

we now find that

$$T(X^k) = T_0(X^k) = (T_0)^{A'_1\cdots A'_k}_{A_1\cdots A_k}\overline{\psi}_{A'_1}\cdots\overline{\psi}_{A'_k}\psi^{A_1}\cdots\psi^{A_k} = 0\,.$$

Because this must hold for all ψ and $\overline{\psi}$, we have that $T_0 = 0$ and (7.10) reduces to (7.9). □

Proposition 7.5. *Let $T(Y^p,X)$ be a tensor that vanishes whenever $\langle Y,X\rangle = 0$. Then there is a unique tensor $U(Y^{p-1})$ such that*

$$T(Y^p,X) = \langle Y,X\rangle U(Y^{p-1})\,. \tag{7.11}$$

Proof. Since

$$\widehat{X} = \langle Y,Y\rangle X - \langle Y,X\rangle Y$$

is always orthogonal to Y we have that

$$T(Y^p,\langle Y,Y\rangle X - \langle Y,X\rangle Y) = 0\,,$$

and so

$$\langle Y,Y\rangle T(Y^p,X) = \langle Y,X\rangle T(Y^p,Y)\,. \tag{7.12}$$

But then $T(Y^p, Y) = 0$ when $\langle Y, Y\rangle = 0$ and so by Proposition 7.4,

$$T(Y^p, Y) = \langle Y, Y\rangle U(Y^{p-1}) .$$

We substitute this result into (7.12) and (7.11) follows. □

Proposition 7.6. *Let*

$$P^{A_1\cdots A_r}_{B'_1\cdots B'_s} = P^{A_1\cdots A_r}_{B'_1\cdots B'_s}(\Psi^2, \overline{\Psi}^2, \ldots, \Psi^k, \overline{\Psi}^k)$$

be a natural spinor that is completely symmetric in the indices $A_1 \cdots A_r$ and $B'_1 \cdots B'_s$. If

$$\nabla^{(C}_{(C'} P^{A_1\cdots A_r)}_{B'_1\cdots B'_s)} = 0 \quad \text{on } \mathscr{E}^{k+1} , \tag{7.13}$$

where $\mathscr{E}^{k+1}$ is the prolonged Einstein equation manifold, then P vanishes.

Proof. Equation (7.13) is equivalent to

$$[\mathrm{Grad}\, P](\alpha, \overline{\alpha}; \alpha^r, \overline{\alpha}^s) = 0 , \tag{7.14}$$

where we have introduced the notation

$$[\mathrm{Grad}\, P](\beta, \overline{\beta}; \alpha^r, \overline{\alpha}^s) = \beta_A \overline{\beta}^{A'} [\nabla^A_{A'} P](\alpha^r, \overline{\alpha}^s) . \tag{7.15}$$

We differentiate (7.14) with respect to Ψ^{k+1} and use the commutation relation (4.5) to deduce that

$$[\partial^k_{\Psi} P](\psi^{k+2}, \overline{\psi}^{k-2}; \alpha^r, \overline{\alpha}^s) = 0 . \tag{7.16}$$

Similarly, if we differentiate with respect to $\overline{\Psi}^{k+1}$ we find that

$$[\partial^k_{\overline{\Psi}} P](\psi^{k-2}, \overline{\psi}^{k+2}; \alpha^r, \overline{\alpha}^s) = 0 . \tag{7.17}$$

Equations (7.16) and (7.17) show P to be independent of Ψ^k and $\overline{\Psi}^k$. A simple induction argument proves that P is independent of all the Penrose fields $\Psi^k, \overline{\Psi}^k, \ldots, \Psi^2, \overline{\Psi}^2$.

The expansion of (7.13) in terms of the spinor connection coefficients $\gamma^{CA}_{C'B}$ and $\gamma^{CA'}_{C'B'}$ now leads to

$$\gamma^{(CA_1}_{(C'|D|} P^{|D|A_2\cdots A_r)}_{B'_1 B'_2\cdots B'_s)} - \overline{\gamma}^{(C|D'|}_{(C'B'_1} P^{A_1A_2\cdots A_r)}_{|D'|B'_2\cdots B'_s)} = 0 .$$

This is an identity that must hold for all spinor connection coefficients and therefore, taking into account the identity

$$\gamma^{CA}_{C'D}\varepsilon_{AB} + \gamma^{CA}_{C'B}\varepsilon_{DA} = 0 ,$$

we conclude that

$$\langle \alpha, \beta\rangle P(\gamma\alpha^{r-1}, \overline{\alpha}^s) + \langle \alpha, \gamma\rangle P(\beta\alpha^{r-1}, \overline{\alpha}^s) = 0 .$$

Setting $\beta = \gamma$ we conclude that

$$P(\alpha^r, \overline{\alpha}^s) = 0 .$$

Alternatively, one may conclude that $P = 0$ from the fact that there are no completely symmetric natural spinors of order zero. □

We close this section with a characterization of spinors with certain symmetries which arise in our symmetry analysis of the Einstein equations.

Theorem 7.7. *Let* $P(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})$ *be a spinor that is symmetric in its first* $k+2$ *and next* $k-2$ *arguments. The spinor* $P(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})$ *enjoys the two symmetry properties*

$$P(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta}) = P(\psi^{k+2}, \overline{\psi}^{k-2}, \beta, \alpha, \overline{\beta}, \overline{\alpha}) \tag{7.18}$$

and

$$P(\psi^{k+2}, \overline{\psi}^{k-2}, \psi, \alpha, \overline{\beta}, \overline{\psi}) = 0 \tag{7.19}$$

if and only if there are spinors,

$$A = A(\psi^{k}, \overline{\psi}^{k}), \qquad B = B(\psi^{k+4}, \overline{\psi}^{k-4}), \qquad W = W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}), \tag{7.20}$$

such that

$$\begin{aligned} &P(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta}) \\ &= \langle\psi, \alpha\rangle\langle\psi, \beta\rangle A(\psi^{k}, \overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi}, \overline{\alpha}\rangle\langle\overline{\psi}, \overline{\beta}\rangle B(\psi^{k+2}\alpha\beta, \overline{\psi}^{k-4}) \\ &\quad + \langle\psi, \alpha\rangle\langle\overline{\alpha}, \overline{\psi}\rangle W(\psi^{k+1}, \overline{\psi}^{k-3}, \beta, \overline{\beta}) + \langle\psi, \beta\rangle\langle\overline{\beta}, \overline{\psi}\rangle W(\psi^{k+1}, \overline{\psi}^{k-3}, \alpha, \overline{\alpha}). \end{aligned} \tag{7.21}$$

The spinor A *is symmetric in its first* k *and last* k *arguments; the spinor* B *is symmetric in its first* $k+4$ *and last* $k-4$ *arguments; and the spinor* W *is symmetric in its first* $k+1$ *and following* $k-3$ *arguments. With these symmetries, the spinors* A, B, W *are uniquely determined by* P*. When* $k = 3$*, (7.21) is valid with* $B = 0$ *and* $W = W(\psi^4, \alpha, \overline{\alpha})$*. When* $k = 2$*, (7.21) holds with* $B = 0$ *and* $W = 0$*.*

Proof. We begin by applying Proposition 7.1 to the arguments $(\overline{\psi}^{k-2}, \overline{\beta})$ of $P(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta})$ to find that

$$P(\psi^{k+2}, \overline{\psi}^{k-2}, \alpha, \beta, \overline{\alpha}, \overline{\beta}) = H(\psi^{k+2}, \overline{\psi}^{k-2}\overline{\beta}, \alpha, \beta, \overline{\alpha}) + \langle\overline{\psi}, \overline{\beta}\rangle T(\psi^{k+2}, \overline{\psi}^{k-3}, \alpha, \beta, \overline{\alpha}), \tag{7.22}$$

where H is symmetric in the arguments $(\overline{\psi}^{k-2}\overline{\beta})$. Applying Proposition 7.1 to the arguments (ψ^{k+2}, α) of H, we obtain

$$H(\psi^{k+2}, \overline{\psi}^{k-1}, \alpha, \beta, \overline{\alpha}) = \widetilde{H}(\psi^{k+2}\alpha, \overline{\psi}^{k-1}, \beta, \overline{\alpha}) + \langle\psi, \alpha\rangle S(\psi^{k+1}, \overline{\psi}^{k-1}, \beta, \overline{\alpha}), \tag{7.23}$$

where $\widetilde{H}$ is symmetric in the arguments $(\psi^{k+2}\alpha)$. Because

$$P(\psi^{k+2}, \overline{\psi}^{k-2}, \psi, \beta, \overline{\alpha}, \overline{\psi}) = H(\psi^{k+2}, \overline{\psi}^{k-1}, \psi, \beta, \overline{\alpha}) = \widetilde{H}(\psi^{k+3}, \overline{\psi}^{k-1}, \beta, \overline{\alpha}),$$

the condition (7.19) implies that the spinor $\widetilde{H}$ is identically zero. The combination of (7.22) and (7.23) now yields

$$P(\psi^{k+2},\overline{\psi}^{k-2},\alpha,\beta,\overline{\alpha},\overline{\beta}) = \langle\psi,\alpha\rangle S(\psi^{k+1},\overline{\psi}^{k-2}\overline{\beta},\beta,\overline{\alpha}) + \langle\overline{\psi},\overline{\beta}\rangle T(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha})\,. \tag{7.24}$$

This form of P satisfies (7.19), but (7.18) does not hold. The key to establishing the decomposition (7.21) is to satisfy *both* (7.19) and (7.18) simultaneously. The condition (7.18) leads to

$$\langle\psi,\alpha\rangle S(\psi^{k+1},\overline{\psi}^{k-2}\overline{\beta},\beta,\overline{\alpha}) + \langle\overline{\psi},\overline{\beta}\rangle T(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha}) = \langle\psi,\beta\rangle S(\psi^{k+1},\overline{\psi}^{k-2}\overline{\alpha},\alpha,\overline{\beta}) + \langle\overline{\psi},\overline{\alpha}\rangle T(\psi^{k+2},\overline{\psi}^{k-3},\beta,\alpha,\overline{\beta})\,. \tag{7.25}$$

In this equation we set $\overline{\alpha} = \overline{\beta} = \overline{\psi}$ to find that

$$\langle\psi,\alpha\rangle S(\psi^{k+1},\overline{\psi}^{k-1},\beta,\overline{\psi}) = \langle\psi,\beta\rangle S(\psi^{k+1},\overline{\psi}^{k-1},\alpha,\overline{\psi})\,,$$

and hence, by Proposition 7.3, there is a spinor A such that

$$S(\psi^{k+1},\overline{\psi}^{k-1},\beta,\overline{\psi}) = \langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k})\,. \tag{7.26}$$

Note that A is totally symmetric.

If we now define a spinor S_1 by

$$S_1(\psi^{k+1},\overline{\psi}^{k-1},\beta,\overline{\alpha}) = S(\psi^{k+1},\overline{\psi}^{k-1},\beta,\overline{\alpha}) - \langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-1}\overline{\alpha})\,, \tag{7.27}$$

then Eq. (7.26) implies that

$$S_1(\psi^{k+1},\overline{\psi}^{k-1},\beta,\overline{\psi}) = 0\,.$$

We can use Proposition 7.2 to conclude that

$$S_1(\psi^{k+1},\overline{\psi}^{k-1},\beta,\overline{\alpha}) = \langle\overline{\psi},\overline{\alpha}\rangle S_2(\psi^{k+1},\overline{\psi}^{k-2},\beta)\,,$$

and therefore, by (7.5),

$$S_1(\psi^{k+1},\overline{\psi}^{k-2}\overline{\beta},\beta,\overline{\alpha}) = \frac{k-2}{k-1}\langle\overline{\psi},\overline{\alpha}\rangle S_2(\psi^{k+1},\overline{\psi}^{k-3}\overline{\beta},\beta) + \frac{1}{k-1}\langle\overline{\beta},\overline{\alpha}\rangle S_2(\psi^{k+1},\overline{\psi}^{k-2},\beta)\,. \tag{7.28}$$

We replace one of the arguments $\overline{\psi}$ in (7.27) by $\overline{\beta}$ and substitute from (7.28) to deduce that

$$S(\psi^{k+1},\overline{\psi}^{k-2}\overline{\beta},\beta,\overline{\alpha}) = \langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \frac{k-2}{k-1}\langle\overline{\psi},\overline{\alpha}\rangle S_2(\psi^{k+1},\overline{\psi}^{k-3}\overline{\beta},\beta) + \frac{1}{k-1}\langle\overline{\beta},\overline{\alpha}\rangle S_2(\psi^{k+1},\overline{\psi}^{k-2},\beta)\,. \tag{7.29}$$

We next derive an equation for the spinor T appearing in (7.24) that is similar to Eq. (7.29) for S. In (7.24) we set $\alpha = \beta = \psi$ and use Proposition 7.3 to show that there is a totally symmetric spinor B such that

$$T(\psi^{k+2},\overline{\psi}^{k-3},\psi,\psi,\overline{\alpha}) = \langle\overline{\psi},\overline{\alpha}\rangle B(\psi^{k+4},\overline{\psi}^{k-4}) \,. \tag{7.30}$$

Let

$$T_1(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha}) = T(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha}) - \langle\overline{\psi},\overline{\alpha}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4}) \,, \tag{7.31}$$

so that, by (7.30), T_1 satisfies

$$T_1(\psi^{k+2},\overline{\psi}^{k-3},\psi,\psi,\overline{\alpha}) = 0 \,. \tag{7.32}$$

We apply Proposition 7.1 to $T_1(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha})$ with respect to the arguments (ψ^{k+2},β) to arrive at

$$T_1(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha}) = \widetilde{T}_1(\psi^{k+2}\beta,\overline{\psi}^{k-3},\alpha,\overline{\alpha}) + \langle\psi,\beta\rangle T_2(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha}) \,, \tag{7.33}$$

where $\widetilde{T}_1$ is symmetric in its first group of arguments $(\psi^{k+2}\beta)$. On account of (7.32), $\widetilde{T}_1$ satisfies

$$\widetilde{T}_1(\psi^{k+3},\overline{\psi}^{k-3},\psi,\overline{\alpha}) = 0 \,,$$

and therefore, by Proposition 7.2,

$$\widetilde{T}_1(\psi^{k+3},\overline{\psi}^{k-3},\alpha,\overline{\alpha}) = \langle\psi,\alpha\rangle T_3(\psi^{k+2},\overline{\psi}^{k-3},\overline{\alpha}) \,.$$

In this equation we replace one of the arguments ψ by β to arrive at

$$\begin{aligned}\widetilde{T}_1(\psi^{k+2}\beta,\overline{\psi}^{k-3},\alpha,\overline{\alpha}) &= \frac{k+2}{k+3}\langle\psi,\alpha\rangle T_3(\psi^{k+1}\beta,\overline{\psi}^{k-3},\overline{\alpha}) \\ &\quad + \frac{1}{k+3}\langle\beta,\alpha\rangle T_3(\psi^{k+2},\overline{\psi}^{k-3},\overline{\alpha}) \,.\end{aligned} \tag{7.34}$$

Finally, the combination of (7.31), (7.33), and (7.34) leads to

$$\begin{aligned}T(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha}) &= \langle\overline{\psi},\overline{\alpha}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4}) + \langle\psi,\beta\rangle T_2(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha}) \\ &\quad + \frac{k+2}{k+3}\langle\psi,\alpha\rangle T_3(\psi^{k+1}\beta,\overline{\psi}^{k-3},\overline{\alpha}) \\ &\quad + \frac{1}{k+3}\langle\beta,\alpha\rangle T_3(\psi^{k+2},\overline{\psi}^{k-3},\overline{\alpha}) \,.\end{aligned} \tag{7.35}$$

The symmetry (7.18) of the spinor P and our initial decomposition (7.24) now imply that

$$\begin{aligned}&P(\psi^{k+2},\overline{\psi}^{k-2},\alpha,\beta,\overline{\alpha},\overline{\beta}) \\ &\quad = \tfrac{1}{2}[P(\psi^{k+2},\overline{\psi}^{k-2},\alpha,\beta,\overline{\alpha},\overline{\beta}) + P(\psi^{k+2},\overline{\psi}^{k-2},\beta,\alpha,\overline{\beta},\overline{\alpha})] \\ &\quad = \tfrac{1}{2}[\langle\psi,\alpha\rangle S(\psi^{k+1},\overline{\psi}^{k-2}\overline{\beta},\beta,\overline{\alpha}) + \langle\psi,\beta\rangle S(\psi^{k+1},\overline{\psi}^{k-2}\overline{\alpha},\alpha,\overline{\beta}) \\ &\quad\quad + \langle\overline{\psi},\overline{\beta}\rangle T(\psi^{k+2},\overline{\psi}^{k-3},\alpha,\beta,\overline{\alpha}) + \langle\overline{\psi},\overline{\alpha}\rangle T(\psi^{k+2},\overline{\psi}^{k-3},\beta,\alpha,\overline{\beta})] \,.\end{aligned}$$

Into this equation we substitute from (7.29) and (7.35). After combining like terms, and using the spinor identity

$$\langle\psi,\alpha\rangle P(\psi^{k-2},\beta) - \langle\psi,\beta\rangle P(\psi^{k-2},\alpha) = \langle\alpha,\beta\rangle P(\psi^{k-2},\psi)\,,$$

we arrive at

$$\begin{aligned}
&P(\psi^{k+2},\overline{\psi}^{k-2},\alpha,\beta,\overline{\alpha},\overline{\beta})\\
&\quad= \langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4})\\
&\quad+ \langle\psi,\alpha\rangle\langle\overline{\alpha},\overline{\psi}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\beta,\overline{\beta}) + \langle\psi,\beta\rangle\langle\overline{\beta},\overline{\psi}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha})\\
&\quad+ \langle\alpha,\beta\rangle\langle\overline{\alpha},\overline{\beta}\rangle S_3(\psi^{k+2},\overline{\psi}^{k-2}) + \langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\beta}\rangle T_4(\psi^{k+1}\beta,\overline{\psi}^{k-3},\overline{\alpha})\\
&\quad+ \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\alpha}\rangle T_4(\psi^{k+1}\alpha,\overline{\psi}^{k-3},\overline{\beta})\,.
\end{aligned}\tag{7.36}$$

In (7.36) we have defined

$$W(\psi^{k+1},\overline{\psi}^{k-3},\overline{\alpha},\alpha) = -\frac{k-2}{2(k-1)}S_2(\psi^{k+1},\overline{\psi}^{k-3}\overline{\alpha},\alpha) - \tfrac{1}{2}T_2(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha})\,,$$

$$S_3(\psi^{k+2},\overline{\psi}^{k-2}) = -\frac{1}{2(k-1)}S_2(\psi^{k+1},\overline{\psi}^{k-2},\psi) + \frac{1}{2(k+3)}T_3(\psi^{k+2},\overline{\psi}^{k-3},\overline{\psi})\,,$$

and

$$T_4(\psi^{k+2},\overline{\psi}^{k-3},\overline{\alpha}) = \frac{k+2}{2(k+3)}T_3(\psi^{k+2},\overline{\psi}^{k-3},\overline{\alpha})\,.$$

The terms involving A, B, W in (7.36) give the required form (7.21) for P, and satisfy both the requirement (7.18) and the condition (7.19). The terms involving S_3 and T_4 satisfy (7.18) but now are subject to (7.19). If we set $\alpha = \psi$ and $\overline{\beta} = \overline{\psi}$ in (7.36), then (7.19) implies that

$$\langle\psi,\beta\rangle\langle\overline{\alpha},\overline{\psi}\rangle S_3(\psi^{k+2},\overline{\psi}^{k-2}) + \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\alpha}\rangle T_4(\psi^{k+2},\overline{\psi}^{k-3},\overline{\psi}) = 0\,,$$

and so

$$S_3(\psi^{k+2},\overline{\psi}^{k-2}) = T_4(\psi^{k+2},\overline{\psi}^{k-3},\overline{\psi})\,.$$

Therefore, the terms involving the spinors S_3 and T_4 in (7.36) become

$$\begin{aligned}
&\langle\alpha,\beta\rangle\langle\overline{\alpha},\overline{\beta}\rangle S_3(\psi^{k+2},\overline{\psi}^{k-2}) + \langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\beta}\rangle T_4(\psi^{k+1}\beta,\overline{\psi}^{k-3},\overline{\alpha})\\
&\quad+ \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\alpha}\rangle T_4(\psi^{k+1}\alpha,\overline{\psi}^{k-3},\overline{\beta})\\
&\quad= \langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle T_4(\psi^{k+1}\beta,\overline{\psi}^{k-3},\overline{\beta}) + \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle T_4(\psi^{k+1}\alpha,\overline{\psi}^{k-3},\overline{\alpha})\,.
\end{aligned}$$

This equality follows from the cyclic permutation of $\overline{\psi},\overline{\beta},\overline{\alpha}$ in the second and third terms on the left-hand side. We can thus absorb the S_3 and T_4 terms in (7.36) into a redefinition of W, and this proves the decomposition (7.21).

To prove the uniqueness of the decomposition (7.21) it suffices to show that if

$$\begin{aligned}
&\langle\psi,\alpha\rangle\langle\psi,\beta\rangle A(\psi^{k},\overline{\psi}^{k-2}\overline{\alpha}\overline{\beta}) + \langle\overline{\psi},\overline{\alpha}\rangle\langle\overline{\psi},\overline{\beta}\rangle B(\psi^{k+2}\alpha\beta,\overline{\psi}^{k-4})\\
&\quad+ \langle\psi,\alpha\rangle\langle\overline{\alpha},\overline{\psi}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\beta,\overline{\beta}) + \langle\psi,\beta\rangle\langle\overline{\beta},\overline{\psi}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha}) = 0\,,
\end{aligned}\tag{7.37}$$

then A, B, W each vanish. To verify this, we put $\overline{\alpha} = \overline{\beta} = \overline{\psi}$ in (7.37) to arrive at

$$A(\psi^k, \overline{\psi}^k) = 0\,.$$

Because of the symmetry of A, this implies $A = 0$. Similarly, we can set $\alpha = \beta = \psi$ in (7.37) to deduce that $B = 0$. Equation (7.37) reduces to

$$\langle\psi,\alpha\rangle\langle\overline{\psi},\overline{\alpha}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\beta,\overline{\beta}) + \langle\psi,\beta\rangle\langle\overline{\psi},\overline{\beta}\rangle W(\psi^{k+1},\overline{\psi}^{k-3},\alpha,\overline{\alpha}) = 0\,. \quad (7.38)$$

We set $\alpha = \beta$ and $\overline{\alpha} = \overline{\beta}$ to obtain

$$W(\psi^{k+1},\overline{\psi}^{k-3},\beta,\overline{\beta}) = 0\,. \qquad \square$$

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