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# Critical Circle Maps Near Bifurcation

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**Abstract:** We estimate harmonic scalings in the parameter space of a one-parameter family of critical circle maps. These estimates lead to the conclusion that the Hausdorff dimension of the complement of the frequency-locking set is less than 1 but not less than 1/3. Moreover, the rotation number is a Hölder continuous function of the parameter.

## 1. Preliminaries

*1.1. Introduction.* This paper will present results about circle maps and families of circle maps that we were able to obtain during the past couple of years. We will not discuss diffeomorphisms, which by far and large are the best understood class of circle maps. In the present paper, we will deal with critical homeomorphisms. Some methods and estimates can be carried over to non-invertible maps, but we only mention [4] here.

Let us start by defining the class of maps we consider.

**The objects that we intend to investigate.** Points of the real line can be projected onto the unit circle in the complex plane by means of the map

$$x \longrightarrow \exp(2\pi i x)\,.$$

Maps from the real line project on the circle if they satisfy

$$f(x+1) - f(x) \in \mathbf{Z}$$

for every real $x$. Obviously, for a continuous map this difference must be constant, and is the topological degree of the circle map.

Unless necessary, we will not make a strong distinction between objects that live on the circle and their lifts to the universal cover. Whenever we want to make

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a strong point of something being actually on the circle, we will write (mod 1) near the formula.

If $x$ and $y$ are objects on the circle, $|x - y|$ is supposed to mean the distance in the natural metric.

**Hypotheses.** *We consider a family of circle maps given by*

$$f_t(x) = f_0(x) + t \pmod 1,$$

*where $t$ is a real parameter which ranges on the real line and the projection of $f_0$ on the circle is a degree one circle homeomorphism $F_0$ which in addition satisfies:*

- *It is at least three times continuously differentiable.*
- *The derivative vanishes in exactly one point which is identified with* 0.
- *The function is differentiable enough times so as to satisfy*

$$\frac{d f_0^l}{dx^l}(0) \neq 0$$

*for some $l$, where the $l^{th}$ derivative exists and is continuous everywhere. This number $l$ will also be referred to as* the flatness of the critical point.

In addition, we consider the corresponding family $f_t$ of lifts of maps $F_t$. We denote wth $\rho(t)$ the rotation number of $F_t$.[1]

We state two theorems.

**Theorem A.** *Under the hypotheses listed above, the function $\rho(t)$ is Hölder continuous.*

**Theorem B.** *Consider the set*

$$\Omega' := \rho^{-1}(\mathbf{R} \backslash \mathbf{Q}).$$

*Under our hypotheses the Hausdorff dimension of $\Omega'$ satisfies*

$$1/3 \leqq HD(\Omega') < \alpha < 1;$$

*where a number $\alpha < 1$ depends on $l$, but otherwise is independent of the choice of $F_0$ which determines our family.*

**A comment on the results.** Theorem A is the first result that we know about concerning regularity of the rotation function for critical families. Theorem B is a refinement of the result that the measure of $\Omega'$ is zero and together with the scalings rules stated in Proposition 5 shows that the set $\Omega'$ has a fractal structure. However, it still falls well short of numerically established universality of the Hausdorff dimension (equal to about 0.87 for cubic families). Theorem B has recently appeared in [14]. That paper described the main steps of the argument, which are pretty similar to our approach, but did not give complete proofs.

It is worth noting that the lower estimate in Theorem B contradicts a certain conjecture based on extrapolating numerical data. The work [1] gave an asymptotic formula for the fractal dimension (which is essentially another name for Hausdorff dimension, see [3]) of $\Omega'$ which was expected to tend to 0 as the critical exponent ($l$ in our notations) grew to infinity. This is contradicted by our result. In fact, the behavior of the family at near bifurcation is what one tends to miss when doing

[1] We define the rotation number a little later.

numerics. However, we show how this phenomenon can be well understood using analytic tools.

The results have three roots. One is the Bounded Geometry of critical maps which has been known for awhile. Since complete proofs of the Bounded Geometry have not been available in the literature, we provide them, too.

Another ingredient is much stronger estimates of the geometry near a bifurcation point. Similar estimates were given with ideas of proofs in [14].

Finally, there is a way of establishing similarity between objects in the phase space and in parameters. We show an easy technique to achieve that. Incidentally, this new approach is both simpler and stronger than estimates of this kind used in [17].

An important point is that the estimates of the Hausdorff dimension are universal, that is independent of the family from our class apart from $l$. A heuristic explanation of this phenomena relies on the fact that Hausdorff dimension is an asymptotic quantity only depending on the structure of the set in small scale. Small scale in parameters means considering high iterations in the phase space. For high iterations, our main tools, the Bounded Geometry and the Distortion Lemma, become universal. As far as the Bounded Geometry is concerned, this is not a surprising fact, since it was observed in [16] for interval maps. The reason is that "cross-ratio inequalities" (see [17]) which are the source of the Bounded Geometry estimates become universal if applied to very short intervals. This fact can be seen immediately from the "pure singularity property" of [18], but also follows easily from the much simpler Corollary to Proposition 1. The Distortion Lemma also becomes universal if applied to maps with rotation number of degree sufficiently large since it is derived from cross-ratio inequalities as well.

*1.2. Topological Description of Dynamics.* The *rotation number* of a circle map $F$ is given by

$$\rho(F) = \lim_{n \to \infty} \frac{f^n(x)}{n} \pmod 1 ,$$

where $x$ is any point and $f$ any lift of $F$ to the universal cover. For the maps we consider, the limit always exists and is independent of the choice of $x$ or a particular lift $f$. If the rotation number is irrational, it is a full topological invariant, and even if it is rational, still a lot of information about the underlying dynamics can be read.

**The structure of rotation numbers.** There is no general agreement on what is the best way to organize rotation numbers. One way is to use the so-called *Farey tree*, and another is based on *continued fractions.* We will base our approach on the Farey tree structure and only comment marginally on the connection with continued fractions.

**Farey Trees**

**Definition 1.1.** *We define the structure of a directed graph whose vertices are exactly all rational numbers from* $(0, 1)$.

*By definition, each vertex* $p/q$ *has exactly two outgoing edges. One leads to a smaller number called the "left daughter," and the other to a greater number called the "right daughter."*

*If* $p/q$ *is in the lowest terms, we determine* $u < p/q < v$ *defined as the closest neighbors in* $[0, 1]$ *with denominators not larger than* $q$*. Then, the left daughter is the rational number with the smallest denominator contained in* $(u, p/q)$*, while the* *right daughter is the number with the smallest denominator contained in* $(p/q, v)$ *(they are unique.)*

*It is known that the graph of this relation is a connected binary tree and* $1/2$ *is the root. This tree is called the* Farey tree.

**Coding.** Thus, there is a one-to-one correspondence between a rational number mod 1 and a finite symbolic sequence $(\alpha_i)$ (the symbols are $L$ for the left daughter and $R$ for the right daughter) which tells us how to go to the number from the root.

Also, there is a unique infinite symbolic representation of every irrational number defined by the property that the rationals which correspond to finite initial segments of the code tend to the irrational number. The reader may try to compute the value of the alternating code $LRLR\ldots$ (it equals $(3-\sqrt{5})/2$).

**The degree.** For every itinerary in the Farey tree, we define inductively its *turning points* $m_i$. The first turning point $m_1$ is defined to be the least $i$ so that $a_i \neq a_{i+1}$. If it does not exist, there are no turning points. Once $m_j$ has been found, $m_{j+1}$ is the least $i > m_j + 1$ so that $a_i \neq a_{i+1}$. Again, if it does not exist, the sequence of turning points ends. We also define *fake turning points.* The index value $i$ is a fake turning point if $i-1$ is a turning point and $a_i \neq a_{i+1}$. So, if $a_i \neq a_{i+1}$, then $i$ is either a turning point or a fake turning point, but never both.

*The degree* of a rational number $u$ (denoted $\deg(u)$) is the number of turning points of the itinerary leading to $u$ in the Farey tree, incremented by 1.

The reader may try to determine the degree of the number coded by

$$LRLLLRRLRLR \quad \text{(five)}.$$

**Closest returns.** Consider an infinite symbolic sequence $\mathscr{A}$ which codes an irrational number $\rho$. We consider the sequence $q_i$ defined as denominators of consecutive rationals which correspond to symbolic sequences

$$a_1, \ldots, a_{t_i},$$

where $t_i$ are turning points of $\mathscr{A}$.

This sequence has a transparent interpretation in terms of the dynamics of the rotation by $2\pi\rho$. Namely, fix a point on the circle and consider the sequence of iterates which map this point closer to itself than any previous iterate. This turns out to be exactly the sequence $q_i$ defined above. Because of this interpretation we wil refer to $q_i$ as the sequence of *closest returns* for $\rho$.

**Farey Domains and the Harmonic Subdivision**

**Definition 1.2.** *The interval* $(P/Q, P'/Q') \subset [0,1]$ *is called a* Farey domain *if and only if either there is an edge between* $P/Q$ *and* $P'/Q'$ *in the Farey tree or it is one of the three intervals*: $(0,1), (0,1/2)$, *or* $(1/2,1)$.

**Fact 1.1.** *If* $(P/Q, P'/Q')$ *is a Farey domain, then* $|PQ' - P'Q| = 1$ *and* $1/2 \leqq Q/Q' \leqq 2$.

For every Farey domain we consider a sequence $u_n$, where $n$ ranges over all integers. If $n$ is positive,

$$u_n := \frac{(n+1)P + P'}{(n+1)Q + Q'}.$$

For $n$ non-positive,

$$u_n := \frac{P' + (1-n)P}{Q' + (1-n)P}.$$

It is an elementary check that $(u_n, u_{n+1})$ are all Farey domains. We will call the collection of Farey domains of this form the *harmonic subdivision* of $(P/Q, P'/Q')$. The numbers $u_n$ themselves can be called endpoints of the harmonic subdivision.

We fix our notations so that $(P/Q, P'/Q')$ always denotes a Farey domain. We will always normalize the picture so that $Q < Q'$. Then $q := Q' - Q$ and $p := P' - P$. Occasionally, $r$ defined as the remainder from the division of $Q$ by $q$ and $s$ which is the remainder from the division of $P$ by $p$ will also be considered. If the degree of $P/Q$ is at least 3, then $p/q$ corresponds to the last turning point in the itinerary of $P/Q$, and $r/s$ is the one before it.

**Harmonic coding.** Start with the unit interval and consider its subdivision of level 1, that is its harmonic subdivision. Next, take the harmonic subdivision of every domain of the subdivision of the first level to get the harmonic subdivision of the second level and so on. The Farey domains obtained on the $k^{\text{th}}$ level will be called *fundamental domains* of level $k$. Now, consider a finite sequence of integers $(n_1, \dots, n_k)$ defined by induction. For negative values, $n_r$ refers to the interval $(u_{n_r}, u_{n_r+1})$ of the harmonic subdivision of the fundamental domain defined by the preceding part of the code, while for positive values, to the interval $(u_{n_r-1}, u_{n_r})$ of the same subdivision. So, a fundamental domain of level $k$ can be coded by a sequence of non-zero integers of length $k$. Endpoints of the domains of level $k$ will have by definition the same coding as domains. This sacrifices uniqueness since all rationals have two different harmonic codings. On the other hand, irrational numbers can be uniquely coded by infinite sequences. This coding will be called *harmonic coding* and sometimes is more useful than the Farey coding.

*Farey coding, harmonic coding and continued fractions.* For a given number there is a close correspondence between the symbols of its harmonic code and its continued fraction expansion. To see this we will describe the way harmonic and continued fraction coefficients are built by means of formal sequences $(\alpha_1, \dots, \alpha_k, \dots)$ of symbols $L$ and $R$ which code itineraries in the Farey tree and thus can be identified with numbers in the unit interval $(0,1)$.

The absolute value of the first coefficient in the harmonic coding is equal to $m_i$ and its sign is positive if and only if all $\alpha_{m_1} = L$. For all other coefficients $n_k$, $k > 1$, the absolute value is equal to $m_k - m_{k-1} - 1$. The sign is positive if and only if $\alpha_{m_k} = L$.

Now we pass to the description of the continued fraction coefficients via harmonic ones and itineraries in the Farey tree. Generally, a harmonic coefficient can correspond to one or two continued fraction coefficients (if there are two, the first of them will be always equal to 1) depending on whether between two consecutive turning points there is a fake one or not. Indeed, observe that continued fraction approximants are given by these rationals whose itineraries in the Farey tree end just before ordinary or fake turning points. So given $n_k$, $k > 1$, we find that $a_s = 1$ and $a_{s+1} = |n_k|$, provided there is a fake turning point between $m_{k-1}$ and $m_k$. If not $a_s = |n_k| + 1$. The value of $s$ is equal to $k$ plus the number of the fake turning points up to $m_k$ plus $\varepsilon$, where $\varepsilon$ is equal to the zero if the first symbol $\alpha_1 = L$ or 1 if the opposite. The only remaining point is to define the first or the first two continued fraction coefficients having given $n_1$. We leave the reader with this simple problem.

In particular, properties of irrational numbers being of constant or Diophantine type are defined by conditions on the growth of continued fraction coefficients and are equivalent to the same conditions imposed on absolute values of harmonic coefficients.

**Miscellaneous properties.** All numbers in the same fundamental domain of level $k$ have the same closest returns up to $q_{k-1}$.

**What all this means for our family of maps.** Since our assumptions guarantee that the rotation number is a non-decreasing function of the parameter, all those objects from the realm of rotation numbers can be transported back to the parameter space. So we will talk of Farey domains and harmonic subdivisions in the parameter space as well.

There is a caveat, though. The rational numbers form an insignificant countable subset of the set of rotation numbers, but their preimage in the parameter space, called the *frequency-locking* set, is a huge set of full measure, as it was demonstrated in [17]. What happens is that the rotation number does not always grow with the parameter. It makes stops on all rational numbers but never on irrationals.

*1.3. Harmonic Scalings.* With the comments we have made so far, we hope to have explained the main purpose of the paper as stated in the abstract.

The main technical lemma of the paper will concern the harmonic scalings in the parameter space. To explain the notion we have to go back to our construction of the harmonic subdivision. The elements of the subdivision accumulate to the endpoints of the parent Farey domain. Exactly how fast their sizes decrease is the question of harmonic scalings.

It has long been known that decrease is governed by a cubic law. The earliest mention we found in the literature is [11]. The first mathematically rigorous work which established the result was [10]. However, the estimates were non-uniform, i.e. it was proved that the scalings are indeed asymptotically cubic near every frequency-locking interval, but no estimate was given on how long one should wait to see the asymptotics take over in each particular case.

**The saddle-node phenomenon.** To fix the notations, let us concentrate on a Farey domain $(P/Q, P'/Q')$.

**We can assume $Q < Q'$.** Indeed so, because the map is symmetric with respect to the choice of an orientation. More precisely, instead of our family $f_t$ we could consider a family $\phi_t$ given by

$$\phi_t(x) = -f_{-t}(-x) \; .$$

It is easy to check that this operation means changing the direction in the parameter space and the orientation on the circle. The rotation number of $\phi_{-t}$ is going to be equal to $1 - \mathbf{rot}(f_t)$. The new family $\phi_t$ still satisfies our assumptions, but because the rotation numbers have been flipped around one half, so has the Farey tree. So $P/Q < P'/Q'$ and $Q < Q'$ remain our standing assumptions throughout the paper.

**What happens near the lower extreme of $\rho^{-1}(P/Q, P'/Q')$.** Directly below $\rho^{-1}(P/Q, P'/Q')$ there is a frequency-locking interval which belongs to $P/Q$. The most interesting point for us is the upper boundary of this frequency-locking. This parameter value will be denoted by $t_0$. The mapping $f_{t_0}$ is structurally unstable, even within the family. The graph of $f_{t_0}$ is tangent to the diagonal so that all non-periodic orbits are attracted to the neutral orbits. When the parameter value increases, the graph is pushed up and a funnel opens between the graph and the diagonal.

**How to measure the scalings.** Let us take a closer look at the situation for a parameter value $t$ just a little above $t_0$ and the dynamics on the interval between 0 and the nearest critical point of $f_t^Q$ on the right of 0. This critical point must be equal mod 1 to a preimage of 0 and we denote it $F_t^{-q}(0)$, where $q = Q' - Q$ is the previous closest return common to all maps from $\rho^{-1}(P/Q, P'/Q')$.

Since its rotation number is a little greater than $P/Q, F_t^Q$ moves points a little to the right. Thus, the critical point, for example, moves to the right every $Q$ iterate. Finally, it will leave the interval $(0, F_t^{-q}(0))$ and the number of steps it takes tells us exactly which domain of the harmonic subdivision we are in. So one way to determine the scalings could be to measure the interval in the parameter space between $t_k$ for which the image of 0 by $F_{t_k}^Q$ hits $F_{t_k}^{-q}(0)$ after $k$ steps and $t_{k+1}$, where the same requires $k+1$ steps.

**The scalings near the upper endpoint of $\rho^{-1}(P/Q, P'/Q')$ must follow the same rules.** This is a rather trivial reduction. We can consider the Farey domain $((P + P')/(Q + Q'), P'/Q')$ and then flip the Farey tree as described before. What we get is a Farey domain $(1 - P'/Q', 1 - \frac{P+P'}{Q+Q'})$ and now what used to be the scalings near the top of $\rho^{-1}(P/Q, P'/Q')$ now are equal to corresponding scalings at the bottom of the new domain.

So, we will only consider the scalings near the lower extreme of the Farey domain, but the results will automatically extend to the upper scalings as well.

**The crucial role of the funnel.** The key observation made by the authors of the earlier works is that the decisive factor in estimating the scalings is the time it takes the image of 0 to go through the funnel. There are two reasons for that. The first is that the image of 0 spends most of its time in the funnel; the other is that as we consider the scalings in a very close proximity of the end of $\rho^{-1}(P/Q, P'/Q')$, the corresponding changes of the parameter are so tiny that they only bring about minute modifications to the orbit of 0 in the region away from the funnel. The main factor which effects the orbit is the change in the funnel clearance.

To prove what has been said here and study the effect of the funnel clearance on the orbit was the main achievement of both [10] and [5]. The work [10] studied this effect for the critical maps, but at that moment it was very hard technically to get uniform estimates, in particular independent of the degree of the Farey domain $(P/Q, P'/Q')$. In the meanwhile, [5] provided estimates which were uniform in this sense, but only applied to families of diffeomorphisms. Now we are finally able to give uniform estimates for the critical maps as well.

### *1.4. Notations and Technical Propositions*

**Uniform and universal bounds.** Letters $K$ with a subscript will be reserved for "uniform constants." An estimate is *uniform* if it only depends on the family from our class. Estimates are *universal* if they only depend on $l$, but otherwise are independent on a choice of a family from our class.

**Eventually negative Schwarzian derivative.** We do not want to assume that our function $f$ has negative Schwarzian derivative. However, there is a remarkable, though not hard, fact that high iterates of our functions already have negative Schwarzian. We will use this fact in some of our future estimates, which will, therefore, be valid only for a large enough number of iterates. The idea that high iterates become negative Schwarzian maps is certainly not new and has been known to people working in the field. However, we are not aware of any proof in the literature. One reason for that may be that it is unclear how to formulate this result in reasonable generality. Our lemma does not pretend to be general, but the reader will see from the proof that an analogous argument will work in many other situations.

We will finish the discussion by stating a well-known but fundamental inequality about the distortion of the cross-ratio **Cr** by diffeomorphisms with negative Schwarzian derivative (see [17] for more details).

If $a < b < c < d$ or $a > b > c > d$, then define their *cross-ratio* by

$$\mathbf{Cr}(a,b,c,d) := \frac{|b-a||d-c|}{|c-a||d-b|} .$$

Next, if $a, b, c, d$ are on the circle ordered as indicated, their cross-ratio **Cr** $(a,b,c,d)$ is defined by choosing any lift $\pi$ of the arc $(a,d)$ to the universal cover, and defining

$$\mathbf{Cr}(a,b,c,d) := \mathbf{Cr}(\pi(a),\pi(b),\pi(c),\pi(d) .$$

Since this is independent of the lift, the cross-ratio is well-defined four ordered quadruples of points on the circle.

**Fact 1.2.** *For any points $a < b < c < d$ and any diffeomorphism $F$ with negative Schwarzian derivative, the following inequality holds.*

$$\mathbf{Cr}(a,b,c,d) < \mathbf{Cr}(F(a),F(b),F(c),F(d)) .$$

**Rescaling.** There are few uniform estimates on higher order derivatives for high iterates. However, it is often possible to get estimates if the map is properly rescaled. On the formal level that means that we take an arc of the circle and an iterate of the function which maps a part of this arc into the arc. Next, we conjugate it affinely, usually so that the length of the arc becomes one. We will refer to this operation most frequently as to "changing the unit of length."

**Locally Negative Schwarzian**

**Lemma 1.1.** *For every family, from our class, there is a uniform $U > 0$ so that*

$$Sf_t \leqq -1$$

*on $(-U, U)$.*

*Proof.* Denote $A = f_t^{(l)}(0)$. This $A$ is not 0 by assumption. So, by Taylor's theorem, the Schwarzian derivative of $f_t$ near 0 is

$$Sf_t(x) = -\frac{(l-1)(l+1)}{2x^2}(1+o(1)) ,$$

where $o(1)$ is a function of $x$ otherwise depending on the family only, so that $o(1) \to 0$ as $x$ goes to 0. □

**Globally Negative Schwarzian**

**Proposition 1.** *There exists a uniform bound $n(F_0)$ with the following property. Assume that the degree of $P/Q$ is larger than $n(F_0)$. Consider a parameter value $t$ from the Farey domain $u(P/Q, P'/Q')$ and let $\Gamma$ be the affine transformation of $(0, f_t^{-q}(0) + p)$ onto $(0, 1)$.*

*Then,* $S(f_t^Q \circ \Gamma^{-1}) \leqq -K$ *on*

$$\Gamma((f_t^{-Q}(0) + P, F_t^{-q}(0) + p)\setminus\{0\}),$$

*where* $K > 0$ *is an universal constant. Moreover, if*

$$x, y \in (f_t^Q(0) - P, f_t^{-Q-q}(0) + P + p),$$

*then*

$$(K')^{-1} \leqq \frac{Sf_t^Q(x)}{Sf_t^Q(y)} \leqq K'$$

*for another universal constant* $K'$.

**Better distortion estimates.** For a function $h$, we introduce a quantity $\mathcal{N}h(x) := h''/h'(x)$, also called the *nonlinearity* of $h$.

**Corollary to Proposition 1**

**Fact 1.3.** *Consider a map* $F_t$ *whose rotation number belongs to a fundamental domain of degree* $l$. *Assume that for some* $i \leqq q_l$, *the iterate* $f_t^i$ *does not have critical points on an open interval* $J$. *Then, a uniform bound* $K$ *exists so that* $f_t^i$ *on* $J$ *can be represented as*

$$f_t^i = \chi_1 \circ g \circ \chi_2$$

*with the following properties*:

- mappings $\chi_1, \chi_2$ and $g$ are iterations of $f_t$,
- the nonlinearities of $\chi_i$ are bounded as follows:

$$\mathcal{N}\chi_1 \cdot |g \circ \chi_2(J)| \leqq K,$$

$$\mathcal{N}\chi_2 \cdot |J| \leqq K,$$

- the Schwarzian of $g$ is negative.

*Proof.* Choose $k$ so that the degree of $p_k/q_k$ is greater than $n(F_0)$ specified by Proposition 1 and $p_{k-1}/q_{k-1} > p_k/q_k$. Then apply this proposition to the Farey domain

$$\left(\frac{p_k}{q_k}, \frac{p_{k-1} + p_k}{q_{k-1} + q_k}\right).$$

To preserve familiar notations, denote $Q = q_{k+1}$ and $q = q_k$. Then, we consider intervals $I_1 = (F_t^{-Q}(0), F_t^{-q-Q}(0))$ and $I_2 = (F_t^{-q-Q}(0), F_t^{-q}(0))$. On $I_1 \cup I_2$, the first return time of $F_t$ is either the $Q$ on $I_1$, or $q$ on $I_2$.

We want to pick $i_1$ and $i_2$ so that $\chi_1 = f_t^{i_1}$ and $\chi_2 = f_t^{i_2}$. We pick $i_1$ as the first moment when the image of $J$ hits $I_1 \cup I_2$. If this never happens, we get $i < Q$, so we can put $i_1 = i, i_2 = 0$ and $g$ the identity. If $i_1$ was properly chosen, then $i_2$ is the smallest so that $F_t^{i-i_2}(J)$ contains one of the following points: 0, an endpoint of $I_1$, or an endpoint of $I_2$. Then, all other iterates can be accounted for by composing the pieces of the first return map on $I_1 \cup I_2$, whose Schwarzian is negative by Proposition 1. On the other hand, $i_1$ and $i_2$ are uniformly bounded, and were chosen so that the intermediate images of $J$ and $g \circ \chi_2(J)$ by $f_t^{i_1}$ and $f_t^{i_2}$ respectively avoid $I_1 \cup I_2$. So the rescaled nonlinearity of $\chi_1$ and $\chi_2$ is bounded. □

**Proposition 2.** *Choose a parameter $t$ from the Farey domain and change the unit of length by applying the same transformation $\Gamma$ as in Proposition 1. Then, we obtain the following estimate on the rescaled map:*

$$\mathcal{N} f_t^Q \circ \Gamma^{-1} < K_1$$

*on $\Gamma(F_t^Q(0), F_t^{-Q-q}(0))$.*

*Also, there is a uniform bound $n(F_0)$ so that if the degree of $P/Q$ exceeds $n(F_0)$, we get two more estimates:*

- $$S(f_t^Q \circ \Gamma^{-1}) > -K_2, \quad K_2 > 0$$

*on the same interval,*

- *If $|F_t^Q(\Gamma^{-1}(x)) - \Gamma^{-1}(x)| < K_3|f_t^{-q}(0) + p|$, then*

$$\mathcal{N}(f_t^Q \circ \Gamma^{-1})(x) > K_2 > 0 .$$

*The bound $K_1$ is uniform. Moreover, if the degree of $P/Q$ is greater than $n(F_0)$, all three bounds are universal.*

### The Rate of Change of $f^Q$ Depending on the Parameter

**Proposition 3.** *We define a set $S \subset T \times \mathbf{S}^1$ as follows. The $t$ component ranges between the lower extreme of the frequency-locking $P/Q$ and the lower endpoint of the frequency-locking of $P'/Q'$. If a pair $(t,x)$ belongs to $S$, then $x$ must be in $[0, F_t^{-q-Q}(0)]$.*

*We claim that if $(t_1,x_1)$ and $(t_2,x_2)$ belong to $S$, then*

$$K^{-1} \leqq \frac{\frac{\partial f^Q}{\partial \tau}(t_1,x_1)}{\frac{\partial f^Q}{\partial \tau}(t_2,x_2)} \leqq K$$

*with $K \geqq 1$ uniform. Moreover, a uniform number $n(F_0)$ exists so that if $\deg(P/Q) > n(F_t)$, the bound $K$ becomes universal.*

*Comment.* This is a new estimate. It was apparently unknown when [17] was written, as that paper instead uses a very complicated and round-about method in order to obtain inequalities in the parameter space.

### The Bounded Geometry

**Proposition 4.** *Let $F$ be a map from our family with the rotation number in $(P/Q, P'/Q')$ so that $0 < q = Q' - Q$. There are uniform positive constants $K_1$ and $K_2$ for which the following estimates hold:*

- *For any $x$,*

$$K^{-1} < \frac{|x - F^Q(x) - P|}{|x - F^{-Q}(x) + P|} < K_1 ,$$

*and the same holds with $Q$ replaced by $Q'$ and $P$ substituted with $P'$ respectively,*

- *For $x = F^i(0)$, $i = -Q, \ldots, Q$,*

$$K_2^{-1} < \frac{|x - F^Q(x) - P|}{|x - F^q(x) - p|} < K_2 .$$

**The universal Bounded Geometry.** *There is a uniform number $n(F_0)$ so that if* $\deg(P/Q) > n(F)$, *then the bounds $K_1$ and $K_2$ of Proposition* 4 *become universal.*

*Comment.* Even though the Bounded Geometry is known to the experts in the field, there is no proof in literature. If the orbit of the critical point is periodic, Proposition 4 was proved in [17]. Then, it was shown (see [9]) how a similar argument can be developed to prove bounded geometry estimates for mappings with irrational rotation number. Our proof, however, is different.

**Basic techniques for estimating distortion.** The Schwarzian derivative of a $C^3$ local diffeomorphism is given by:

$$Sf := (f''/f')' - \frac{1}{2}(f''/f')^2 .$$

There is this remarkable formula for the Schwarzian of a composition:

$$S(f \circ g) = Sf \circ g \cdot (g')^2 + Sg ,$$

which for iterates of $f$ becomes:

$$Sf^n = \sum_{i=0}^{n-1} (Sf \circ f^i) \cdot ((f^i)')^2 .$$

**The Real Köbe Lemma.** Consider a diffeomorphism $h$ onto its image $(b, c)$. Suppose that it has an extension $\tilde{h}$ onto a larger image $(a, d)$ which is still a diffeomorphism. If $\tilde{h}$ has negative Schwarzian derivative, and $\mathbf{Cr}(a, b, c, d) \geqq \varepsilon$, we will say that $h$ is *$\varepsilon$-extendible.* The following holds for $\varepsilon$-extendible maps:

**Fact 1.4.** *There is a function $C$ of $\varepsilon$ only so that $C(\varepsilon) \to 0$ as $\varepsilon \to 1$ and for every $h$ defined on an interval $I$ and $\varepsilon$-extendible,*

$$|\mathcal{N}h| \cdot |I| \leqq C(\varepsilon) .$$

*Proof.* Apart from the limit behavior as $\varepsilon$ goes to 1, this fact is proved in [13], Theorem IV.1.2. The asymptotic behavior can be obtained from Lemma 1 of [7] which says that if $\tilde{h}$ maps the unit interval into itself, then

$$\mathcal{N}h(x) \leqq \frac{2h'(x)}{\operatorname{dist}(\{0,1\}, h(x))} . \tag{1}$$

The normalization condition can be satisfied by pre- and post-composing $\tilde{h}$ with affine maps. This will not change $\mathcal{N}\tilde{h} \cdot |I|$, so we just assume that $\tilde{h}$ is normalized. Since we are interested in $\varepsilon$ close to 1, the denominator of (1) is large and $h'(x)$ is no more than

$$\exp\left[C\left(\frac{1}{2}\right)\right] \frac{|h(I)|}{|I|} .$$

As $|h(I)|$ goes to 0 with $\varepsilon$ growing to 1, we are done. □

**The Distortion Lemma.** For iterations of $f$ with Schwarzian derivative not necessarily negative, we have this tool.

**Lemma 1.2.** *Suppose that a chain of the intervals*

$$(a, d), (F(a), F(d)), \ldots, (F^m(a), F^m(d))$$

*is disjoint with the critical point and covers each point of the circle at most k times. There is a uniform constant* $K_{[k]}$ *such that for any two points* $x, y \in (a,d)$ *the estimate*

$$|\log(f^m)'(x) - \log(f^m)'(y)| \leqq K_{[k]} - 2\log \mathbf{Cr}(f^m(a), f^m(x), f^m(y), f^m(d))$$

*holds. Moreover, there is a uniform bound* $n(F_0, k)$ *so that* $K_{[k]}$ *become universal if only* $(a,d)C(a, F^{n(F_0,k)}(a))$.

*Comment.* If we drop the part of the claim concerning the universality of estimates, then Lemma 1.2 is well-known (see [13]). The universality of the constants $K_{[k]}$ will be obtained by showing that they tend exponentially fast to 0 with growing dynamical size of $(a,d)$.

We have to postpone the proof of the Distortion Lemma until after we have proved the Bounded Geometry.

## 2. Proofs of Technical Propositions

### 2.1. Proof of the Bounded Geometry

*2.1.1. Maps with Rational Rotation Numbers.* We prove the Bounded Geometry for critical homeomorphisms with rational rotation numbers. The proof is practically the same as in [17] but we emphasize the universal character of estimates. The core technical observation made in [17] was that the distortion of the cross-ratio **Cr** is bounded on chains of intervals which cover the critical point finitely many times.

Here is the precise formulation of the cross-ratio inequality of [17].

**The Cross-Ratio Inequality**

**Fact 2.1.** *Let f be a lift of a circle homeomorphism which belongs to a family of our class. Suppose that intervals* $(a,d), \ldots, (f^{m-1}(a), f^{m-1}(d)) \pmod 1$ *cover each point of the circle at most k times. Then*

$$\mathbf{Cr}(a,b,c,d) \leqq K_{[k]}\mathbf{Cr}(f^m(a), f^m(b), f^m(c), f^m(d)),$$

*where* $K_{[k]}$ *is a constant independent from the choice of* $(a,d)$. *We do not require f to be a diffeomorphism on the interval* $(a,d)$.

Let $F := F_t$ be a map with a periodic point $z$ and the rotation number $P/Q$. Set $Q = nq + s$ and let $p/q$ be the nearest neighbor of $P/Q$ among fractions whose denominators are bounded by $q$. The orbit of $z$ cuts the circle into $Q$ disjoint intervals. The collection of these intervals will be denoted by $\mathscr{A}(z)$.

**Lemma 2.1.** *For any point z and a rational P/Q there is exactly one parameter value t so that x is periodic with rotation number P/Q.*

*Proof.* Consider the function $t \to f_t^Q(x) - P$, which is negative for $t$ close to $-\infty$, positive for $t$ large and increasing. □

**Lemma 2.2.** *The lengths of any two intervals from* $\mathscr{A}(z)$ *are uniformly comparable and the bounds depend solely on* $K_{2.1[2]}$.

*Proof.* Let $I$ be the shortest interval in $\mathscr{A}(z)$. Build up cross-ratios by taking four endpoints of any three consecutive intervals from $\mathscr{A}(z)$. By appropriate iterates of $F$ we can transport these cross-ratios so that the middle intervals are mapped on

$I$ and each of the resulting chains of intervals cover the circle at most twice. Since the final cross-ratio is greater than 1/4, the lemma follows by Fact 2.1. □

Assume now that $F$ is chosen so that the critical point 0 is periodic with rotation number $P/Q$ and as before $Q = nq + s$. For simplicity of notation we will write $|x|$ for the distance $|x - 0|$ between $x$ and 0 in the natural metric on the circle.

**Lemma 2.3.** *Suppose that $F_t^Q(0) = 0$ and denote $F := F_t$. We claim that there is a uniform constant $K \geqq 1$ depending solely on $K_{[0]}$ and the flatness of the critical point of $F$ so that*

$$\frac{1}{K} \leqq \frac{|F_t^q(0)|}{|F_t^s(0)|} \leqq K .$$

*Proof.* Set $x := |F^{-q}(0)|$, $y := |F^s(0)|$, and $|F^{-q+s}(0)| = z$. If $x = y$ then the lemma is reduced to Lemma 2.2, so we assume that $x < y$. Consider the cross-ratio $\mathbf{Cr}(F^{-q+s}(0), F^s(0), F^{-q}(0), 0)$. Map these points forward by $F$ and use the cross-ratio inequality (Fact 2.1) for the resulting cross-ratio and the number of iterates equal to $q - 1$. We obtain that

$$\frac{x^l z^l - y^l}{y^l z^l - x^l} \geqq K_1 \mathbf{Cr}(F^s(0), F^{s+q}(0), 0, F^q(0)) . \tag{2}$$

Next, there is a uniform constant $K_2$ so that

$$\mathbf{Cr}(F^s(0), F^{s+q}(0), 0, F^q(0)) > K_2 \, \mathbf{Cr}(F^{-q+s}(0), F^s(0), F^{-q}(0), 0) . \tag{3}$$

Actually, these two cross-ratios are comparable. To see this observe that the corresponding intervals of the triples involved in their definitions are shifted with respect to each other by a one interval from $\mathscr{A}(0)$. The margin intervals are thus uniformly comparable by Lemma 2.2. The central intervals, which are of the form $(F^s(w), w)$, where $w$ is a point of the orbit of 0, are of smaller dynamical size than the intervals from $\mathscr{A}(0)$ and thus contain at least two of them. Consequently, they are uniformly comparable again by Lemma 2.2. After combining the inequalities (2) and (3) we get

$$\frac{x^{l-1}}{y^{l-1}} \geqq K_3 \frac{z^l - x^l}{z^l - y^l} \frac{z - y}{z - x} \geqq K_3 \frac{z^{l-1} + z^{l-2}x + \cdots + zx^{l-2} + x^{l-1}}{z^{l-1} + z^{l-2}y + \cdots + zy^{l-2} + y^{l-1}} \geqq \frac{K_5}{l} .$$

The intervals $(F^{-q}(0), 0)$ and $(F^q(0), 0)$ belong to $\mathscr{A}(0)$ and thus they are uniformly comparable. This concludes the proof. □

Observe that the estimate of Lemma 2.2 does not involve the flatness of the critical point and holds for diffeomorphisms too. The estimate of Lemma 2.3 requires the existence of the critical point even though its flatness can be arbitrary small. This shows that the bifurcation exponent at which a sudden pass from a non-uniform geometry of diffeomorphisms to bounded geometry of critical maps takes place is equal to 1. Compare this phenomena with the results of [12] and [6] where the bifurcation exponent was found to be equal to 2, respectively of Fibonacci maps and critical circle maps with a flat spot.

*2.1.2. A map embedded in a one parameter family.* We will prove the first inequality of Bounded Geometry for $F$ from a family from our class. The rotation number of $F$ is an arbitrary $\rho \in (P/Q, P'/Q')$. The map $F$ is naturally embedded in the family $F_t$ with $F_0 := F$. Without loss of generality we may assume (see Subsect. 1.2) that $Q' > Q$.

**Proof of the first inequality of Bounded Geometry.** If $F$ is a map with a periodic point $z$ and the rotation number $\rho$,

$$\rho = u_n = \frac{Pn + P'}{Qn + Q'}, \quad n > 0\,,$$

then we are done by Lemma 2.2. Hence, assume that this is not the case. Let $F_{t_z(n)}$ be a map with the rotation number $u_n, n > 0$ defined by the following conditions:

1. $z$ is a periodic point of $F_{t_z(n)}$,
2. $u_n$ is the largest number in the set $\{u_i : i > 0\}$ less than $\rho$.

**Lemma 2.4.** *The following chains of inequalities show the order of the points near* $z$:

$$z < f^{Q}_{t_z(n)}(z) - P < f^{Q}(z) - P < F^{2Q}_{t_z(n)}(z) - 2P$$

*and*

$$f^{-2Q}_{t_z(n)} + 2P < f^{-Q}(z) + P < f^{-Q}_{t_z(n)}(z) + P < z\,.$$

*Proof.* We will prove only the first chain of the inequalities since the other can be obtained by replacing $F$ by $F^{-1}$ and reversing all the inequalities in the reasoning below.

The family $f_t$ is increasing. Thus, by the choice of $t_z(n)$ we have that

$$z < f^{Q}_{t_z(n)}(z) - P < f^{Q}(z) - P\,.$$

To prove the remaining part of the first chain suppose, on the contrary, that $f^{2Q}_{t_z(n)}(z) - 2P < f^{Q}(z) - P$. Iterating these two points by $f^{(n-1)Q+q}_{t_z(n)}$ and $f^{(n-1)Q+q}$ respectively, we obtain

$$z = f^{nQ+Q'}_{t_z(n)}(z) - nP - P' < F^{(n-1)Q+Q'}(z) - (n-1)P - P'\,. \tag{4}$$

By the choice of $t_z(n)$, the rotation number $\rho$ of $F$ is contained in $(u_n, u_{n-1})$. Hence the point $f^{(n-1)Q+Q'}(z) < z + (n-1)P + P'$, which contradicts the inequality 4. □

From Lemmas 2.4 and 2.2 we infer the first inequality of the Bounded Geometry.

**Proof of the second inequality.** We preserve notation and assumptions from the previous subparagraph. We prove first, that the claim of Lemma 2.3 holds even if we drop the assumption of the lemma about the rotation number. Let us concentrate on the lower bound since the reasoning which gives the upper bound is very much the same. We observe that

$$|F^{Q}(0)| > |F^{Q}_{t_0(n)}(0)| \geqq K|F^{q}_{t_0(n)}(0)|\,.$$

The first inequality follows from Lemma 2.4. The second one can be derived from Lemma 2.3 by substituting $q := Q$, $s := q$ and $t := t_0(n)$. Furthermore,

$$K|F^{q}_{t_0(n)}(0) \geqq K_1|F^{2q}_{t_0(n)}(0)| > K_1|F^{q}(0)|\,,$$

where the first inequality follows from the first estimated of the Bounded Geometry applied with $Q := q$, $x := F^q_{t_0(n)}(0)$, and $F := F_{t_0(n)}$, while the second one is a direct consequence of Lemma 2.4.

Thus, the second inequality of Bounded Geometry holds for $z = 0$. We will propagate it along the orbit of 0 using the cross-ratio technique. Let $z$ be a point of the orbit $\{F^i(0) : -Q \leqq i \leqq Q\}$. Obviously,

$$\frac{|f^Q(z) - z - P|}{|f^q(z) - z - p|} > \mathbf{Cr}(F^{2q}(z), F^q(z), z, F^Q(z)) \, .$$

Map these points by $F^{Q+i}$. Since the intermediate images do not intersect more than 6 at one point, this can only expand the cross-ratio by a uniform factor. Therefore,

$$\mathbf{Cr}(F^{2q+Q}(0), F^{q+Q}(0), F^Q(0), F^{2Q}(0)) \leqq K_{[6]}\, \mathbf{Cr}(F^{2q}(z), F^q(z), z, F^Q(z)) \, .$$

We will show that the left-hand side of this inequality is comparable to $\mathbf{Cr}(F^{2q}(0), F^q(0), 0, F^Q(0))$ within a uniform multiplicative factor. Indeed, the intervals involved in the definitions of these cross-ratios are shifted with respect to each other by a one interval of the form $(F^Q(x), x)$. This changes the lengths of marginal intervals only by a uniform factor from the first inequality of Bounded Geometry. The denominator of the smallest rational number in the Farey domain $(P/Q, P'/Q')$ is equal to $2Q + q$. Hence, the interval $(F^{q+Q}(0), 0)$ contains the point $F^{q+2Q}(0)$ and consequently the middle intervals are also uniformly comparable by the first inequality of Bounded Geometry. Since we already proved the second estimate of the bounded geometry for $z = 0$, it follows that $\mathbf{Cr}(F^{2q}(0), F^q(0), 0, F^Q(0))$ is uniformly bounded away from zero which completes the proof.

### *2.2. Proof of the Universal Bounded Geometry*

**Dynamical size versus length.** Bounded Geometry enables us to state an important corollary.

**Fact 2.2.** *Let $I$ and $J : I \subset J$ be the intervals of dynamical size equal to $i$ and $j$ respectively. Then*

$$|\log(|I|/|J|)| \geqq K|i - j| \, ,$$

*where $K$ is a uniform constant. We say that $I$ has dynamical size $i$ if $I$ is enclosed in some interval bounded by $x$ and $f^{q_i}(x)$, but never in an interval between $x$ and $f^{q_{i+1}}(x)$.*

So the geometric size of an interval goes down exponentially fast as its dynamical size grows.

*2.2.1. For $P/Q$ of large degree the bounds are universal.* In the proof of Bounded Geometry analytic properties of $F$ are used only twice: in the proof of the cross-ratio inequality (see [17]) and in the proof of Lemma 2.3. We want to show that for $P/Q$ of large degree only the flatness of the critical point matters in these proofs. To see this consider first, as in the proof of Lemma 2.3, the ratio of two intervals with the common endpoint at 0. Push forward the ratio by $Ax^l$ instead of $F$. The distortion then depends solely on $l$ and the relative error due to the replacement is of the same order as the lengths of the intervals.

Now we pass to the study of the cross-ratio inequality in a small scale.

**The universal cross-ratio inequality.** *A uniform bound $n(F_0)$ exists so that the constants $K_{[k]}$ in the cross-ratio inequality depend solely on $k$ and $l$ provided that the dynamical size of the interval $(a,d)$ is larger than $n(F_0)$.*

*Proof.* According to Lemma 1.1, the Schwarzian derivative is negative on a symmetric neighborhood $U$ of 0. Outside this interval $F$ is a diffeomorphism. We split the images of $(a,d)$ into three classes $M_i, i = 1,2,3$. The first class consists of all intervals which are not contained in $U$. The second one comprises all that are contained in $U$ but do not cover the critical point, and finally the remaining intervals belong to the third class. Suppose now that the dynamical size of $(a,d)$ is large. Then the intervals from $M_3$ cover the critical point and are contained in $U$, while these from $M_1$ are in the distance at least $|U|/4$ away from 0. By a compactness argument, the derivative of $F_t$ is bounded away from 0 on the union of intervals which form $M_1$. However, this bound is only uniform.

We calculate separately the distortion of the cross-ratio on the intervals from these three groups. First, the total distortion over the intervals of $M_2$ is less than 1 because of the negative Schwarzian derivative (see Fact 1.2). The distortion over the intervals of $M_3$, there are at most $k$ of them, is calculated for $Ax^l$ instead of $F$. An error due to this substitution is exponentially diminishing in terms of the dynamical size of $(a,d)$. Detailed computations are done in the proof of the cross-ratio inequality in [17]. So, the product of increases of the cross-ratio over the intervals from $M_2$ and $M_3$ is universally bounded, provided that the dynamical size of $(a,d)$ is large enough. To deal with the increases over the intervals from $M_1$, we will prove that the cross-ratio is the quantity of the second order and the Schwarzian is the right object to control its distortion.

**Lemma 2.5.** *Let $f$ be a $C^3$ diffeomorphism. Then*

$$\log \frac{\mathbf{Cr}(f(a),f(b),f(c),f(d))}{\mathbf{Cr}(a,b,c,d)} = \frac{1}{6} Sf(x)|a-d||b-c| + o(|a-d||b-c|)\,,$$

*where $x$ is any point in $(b,c)$ and $o(z)$ is a function which depends on $f$ but not on $a,b,c,d$ so that*

$$\lim_{z\to 0} \frac{o(z)}{z} = 0\,.$$

*Proof.* Let us recall that $-\log \mathbf{Cr}(a,b,c,d)$ is equal to the hyperbolic length of the interval $(b,c)$ in the disk of the diameter $(a,d)$. Therefore,

$$-\log \mathbf{Cr}(a,b,c,d) = \int_b^c 1/(x-a) + 1/(d-x)dx\,,$$

and consequently

$$-\log \frac{\mathbf{Cr}(F(a),(b),(c),(d))}{\mathbf{Cr}(a,b,c,d)} = \int_b^c \frac{f'(x)}{f(x)-f(a)} - \frac{1}{x-a}dx$$
$$+ \int_b^c \frac{f'(x)}{f(d)-f(x)} - \frac{1}{d-x}dx\,.$$

We approximate the denominator by the Taylor polynomials of the third degree

$$f(x)-f(a)=f'(x)(x-a)-\frac{1}{2}f''(x)(x-a)^2+\frac{1}{6}f'''(x)(x-a)^3+o(|x-a|^3)\,,$$

$$f(d)-f(x)=f'(x)(d-x)+\frac{1}{2}f''(x)(d-x)^2+\frac{1}{6}f'''(x)(d-x)^3+o(|d-x|^3)\,.$$

The integrand can be thus rewritten in the form

$$\frac{\frac{1}{2}f''(x)-\frac{1}{6}f'''(x)(x-a)+o(|x-a|)}{f'(x)-\frac{1}{2}f''(x)(x-a)+\frac{1}{3}f'''(x)(x-a)^2+o(|x-a|^2)}$$

$$+\frac{-\frac{1}{2}f''(x)+\frac{1}{6}f'''(x)(d-x)+o(|d-x|)}{f'(x)+\frac{1}{2}f''(x)(d-x)+\frac{1}{6}f'''(x)(d-x)^2+o(|d-x|^2)}\,.$$

Write the fractions above in the common denominator neglecting all terms of order higher than 2. We obtain

$$\frac{-\frac{1}{4}(f''(x))^2(d-a)+\frac{1}{6}f'''(x)f'(x)(d-a)+o(f'(x)|d-a|)}{(f'(x))^2+o(f'(x)|d-a|)}\,,$$

which is equal to $\frac{1}{6}Sf(x)(d-a)+o(|d-a|)$. □

From the lemma it follows that the total distortion of the cross-ratios over intervals of $M_1$ is bounded from above by

$$K\exp\left(\frac{1}{6}\sum_{I\in M_1}Sf|I|^2\right)\,. \tag{5}$$

It remains to observe that the expression (5) tends to 1 as the dynamical size of $(a,d)$ grows. Indeed, since $\sum_{I\in M_1}|I|\leqq k$, the sum of squares of the lengths goes to 0 with the biggest length. The quantity $K$ comes from multiplying the terms

$$\exp(o((|f^i(d)-f^i(a)|)^2))$$

and is likewise decreasing with the length of the biggest interval. This completes the proof of the universal version of the cross-ratio inequality. The universal cross-ratio inequality implies the Uniform Bounded Geometry. □

*Proof of the Distortion Lemma.* For a quadruple $(a,b,c,d)$ define its cross-ratio **Po** by the formula

$$\mathbf{Po}(a,b,c,d):=\frac{|a-d||b-c|}{|a-b||c-d|}\,.$$

**Fact 2.3.** *Under the assumptions of the Distortion Lemma we have that*

$$\mathbf{Po}(f^m(a),f^m(b),f^m(c),f^m(d)\geqq K_{[k]}\mathbf{Po}(a,b,c,d)\,,$$

*where each $K_{[k]}$ depends only on the family $F_t$ and the dynamical size of $(a,d)$, and each one goes to 1 as this dynamical size grows to infinity.*

*Proof.* This follows from the Bounded Geometry and Theorem IV.2.1, part 1, of [13]. □

In fact we will prove a little more than we promised. We will show that if there is extendibility only on the one side of the interval $(x, y)$ then $f'(x)/f'(y)$ stays bounded away either from zero or infinity in dependence on the order of points $x$ and $y$.

Let $a < x < y < d$. Take a point $z \in (x, y)$. Then by Fact 2.3,

$$\frac{\mathbf{Po}(f^m(a), f^m(x), f^m(z), f^m(y))}{\mathbf{Po}(a, x, z, y)} \geqq K_{2.3,[k]}\,.$$

Passing with $z$ to the limits, $x$ and $y$, we obtain two infinitesimal versions of the above inequality. By algebra, we can rewrite them in the form:

$$f'(x)\frac{|f^m(a) - f^m(y)|}{|a - y|} \geqq K_{2.3,[k]}\frac{|f^m(a) - f^m(x)|}{|a - x|}\frac{|f^m(x) - f^m(y)|}{|x - y|}\,.$$

and

$$\frac{1}{f'(y)}\frac{|f^m(a) - f^m(y)|}{|a - y|} \geqq K_{2.3,[k]}\frac{|f^m(a) - f^m(x)|}{|a - x|}\frac{|x - y|}{|f^m(x) - f^m(y)|}\,.$$

The above inequalities remain true if we drop $|a - y|$ and $|a - x|$ on both sides. Multiplying them by sides, we obtain that

$$\frac{f'(x)}{f'(y)} \geqq K_{1,[k]}\left(\frac{|f^m(a) - f^m(x)|}{|f^m(a) - f^m(y)|}\right)^2\,. \tag{6}$$

The right-hand side of (6) is greater than the square of

$$\mathbf{Cr}(f^m(a), f^m(x), f^m(y), f^m(d))\,.$$

To obtain the lower bound of $f'(x)/f'(y)$ replace in the reasoning above the triple $\{a, x, y\}$ by $\{x, y, d\}$. The Distortion Lemma follows.

### 2.3. *Proof of Proposition 1*

**Vanishing positive Schwarzian.** Consider an iterate $f^m := f_t^m$ on an interval $(a, b)$. Choose an open neighborhood of 0 *on the circle*. For any point $x \in (a, b)$ we look at $S_U f^m(x)$ defined to be

$$S_U f^m := \sum_{i=0}^{m-1} Sf(f^i(x)) \cdot ((f^i)')^2 \cdot (1 - \chi_U(f^i(x)(\mathrm{mod}\, 1)))\,, \tag{7}$$

where $\chi_U$ is the set function of $U$. In this situation we have a lemma:

**Lemma 2.6.** *If the following conditions are satisfied*:

- *there is a larger interval* $(a', b') \supset (a, b)$ *so that the derivative of* $F^m$ *does not vanish on* $(a', b')$,
- $\mathbf{Cr}(f^m(a'), f^m(a), f^m(b), f^m(b')) \geqq \varepsilon$,
- *all intervals* $(a, b), \ldots, F^{n-1}(a, b)$ *are disjoint on the circle, then for every* $U$, *every family* $F_t$ *and every* $\varepsilon > 0$ *there is a constant* $K$ *so that for every* $x \in (a, b)$

$$S_U f^m \cdot |b - a|^2 \leqq K \max\{|f^i(a, b)| : 0 \leqq i \leqq m - 1\}\,.$$

*Proof.* By the Distortion Lemma, each derivative $(f^i)'(x)$ in the formula (7) is comparable to the ratio $|f^i(a,b)|/|(a,b)|$ within factors depending on $\varepsilon$ and the family. On the other hand, the Schwarzian derivative of $f$ on the complement of $U$ is bounded. Since the sum of lengths of all images of $(a,b)$ is less than 1, the lemma follows. □

*Conclusion of the proof.* Let us denote

$$W := \max\{|f_t^i(f_t^{-Q}(0)+P, f_t^{-q}(0)+p)| : i = 0,\dots,Q-1\}$$

and $w = |f_t^{-Q}(0)+P, f_t^{-q}(0)+p)|$. For every family, we can find a neighborhood $U$ of 0 on the circle on which the Schwarzian derivative can bounded

$$C_1^{-1} \leqq \frac{-Sf_t(x)}{|x|^2} \leqq C_1$$

for $C_1$ universal. This simply follows from looking at the Taylor expansion of $f$ at 0, in fact like in Lemma 1.1. The bound $n(F_0)$ can be chosen so that $(F_t^{-Q}(0), F_t^{-q}(0))$ is contained in $U$ for every $t$ with $\rho(t)$ of degree more than $n(F_0)$. This $n(F_0)$ exists and is uniform by the Bounded Geometry. Then we split the set $0,\dots,n-1$ into $M_1, M_2$ and $M_3$ depending on whether $f_t^i(f_t^{-Q}(0)+P, f_t^{-q}(0)+p) \pmod 1$ is contained in $U$, contained in the complement of $U$, or overlaps. Accordingly, we can split

$$Sf_t^Q(x) = \sum_{i\in M_1} Sf_t(f^i(x)) \cdot ((f^{i-1})'(x))^2 + \sum_{i\in M_2} Sf_t(f^i(x)) \cdot ((f^{i-1})'(x))^2$$
$$+ \sum_{i\in M_3} Sf_t(f^i(x)) \cdot ((f^{i-1})'(x))^2 .$$

The set $M_3$ contains at most two iterates. By possibly making $n(F_0)$ even larger we can make sure that these intervals are bounded away from 0 by a universal distance. The bound

$$\left|\sum_{i\in M_3} Sf_t(f^i(x)) \cdot ((f^{i-1})'(x))^2\right| \leqq \frac{C_2 W^2}{w^2}$$

follows directly. The bound

$$\left|\sum_{i\in M_2} Sf_t(f^i(x)) \cdot ((f^{i-1})'(x))^2\right| \leqq \frac{C_2 W}{w^2}$$

is a direct consequence of Lemma 2.6.

To consider the first sum, we again replace the derivatives with ratios of intervals. The Bounded Geometry and the Distortion Lemma imply that this contributes a universally bounded error provided $n(F_0)$ was chosen large enough,

$$\sum_{i\in M_1} Sf_t(f^i(x)) \cdot ((f^{i-1})'(x))^2 \sim \sum_{i\in M_1} \frac{|f_t^{i+Q}(0) - f_t^{i-q-Q}(0) + P + p|^2}{|F_t^i(x)|^2 |w^2} . \quad (8)$$

The term for $i = 0$ is of the order of $w^{-2}$, so it majorizes the two remaining sums for $P/Q$ of large degree because they are of the order $W \cdot w^{-2}$. This proves that the Schwarzian derivative is bounded away from 0 as is the claim of Proposition 1. To prove the second claim, we allow $x$ to range over the interval $(f_t^Q(0) - P, f_t^{-Q-q}(0) + P + p)$. By the Bounded Geometry the corresponding terms in the sums (8) are universally comparable, then so are the sums themselves.

*2.4. Proof of Proposition 3.* The key tool that we will use more than once in this paper is an approximate representation of the parameter derivative in terms of the lengths of dynamically defined intervals in the phase space.

**Lemma 2.7.** *If* $(t,x) \in S$, (*the set* $S$ *is defined in the statement of Proposition* 3), *then*

$$\left|\log\left(\frac{\partial f_t^Q}{\partial t}(x)\right) - \log\left(\sum_{i=1}^{Q}\frac{|f_t^{q+Q}(x) - f_t^Q(x) - p|}{|f_t^{i+q}(x) - f_t^i(x) - p|}\right)\right| \leqq K_1 .$$

*The bound* $K_1$ *is uniform, but it also becomes universal for a large enough degree of* $P/Q$.

*Proof.* The parameter derivative has the form

$$\sum_{i=1}^{Q}\frac{df^{Q-i}}{dx}(f^i(x)) .$$

For a given $i$ consider the chain $\{(f^{i+j+q}(x) - p, f^{i+j}(x)),\ j = 0,\dots,Q-i\}$. We check easily that the chain is disjoint with the critical point. By the Bounded Geometry $f^{Q-i}$ can be extended as a diffeomorphism onto a larger interval, namely because the critical values of $f^{Q-i}$ are outside the neighborhood of 0 delimited by $f^q(0) - p$ on one side and $f^{Q-q}(0) - P + p$ on the other. Therefore, the Distortion Lemma allows us to replace the derivatives of $f^{Q-i}$ by the ratios

$$\frac{|f^{q+Q}(x) - f^Q(x) - p|}{|f^{i+q}(x) - f^i(x) - p|}$$

with a bounded error and the estimate of the lemma follows. Because the estimates of the Bounded Geometry are eventually universal, and so is the Distortion Lemma, the universality of $K_1$ for large degrees of $P/Q$ also follows. □

**Lemma 2.8.** *Let* $t_1$ *and* $t_2$ *belong to* $\rho^{-1}[P/Q, P'/Q']$, *where* $P/Q$ *is a Farey domain, and let* $x$ *be any point. Then*

$$K_2^{-1} < \frac{|f_{t_1}^q(x) - x - p|}{|f_{t_2}^q(x) - x - p|} < K_2 .$$

*The bound* $K_2$ *is universal provided that the degree of* $P/Q$ *is large enough.*

*Proof.* Let us assume that $t_1 < t_2$. We fixed the configuration so that the $f_t^q - q^{\text{th}}$ iterate moves points to the left for $t$ below $\rho^{-1}(p/q)$. Thus, the estimate from below is obvious. Note that $f_{t_1}^q(x) > f_{t_2}^{3q}(x) - 2p$. For a contradiction, assume $f_{t_1}^q(x) \leqq f_{t_2}^{3q}(X) - 2p$. Let $Q = mq + s$ and apply $f_{t_1}^{(m-2)q}$ to the left-hand side, and $f_{t_2}^{(m-2)q}$ to the right-hand side of this inequality. This gives

$$f_{t_1}^{(m-1)q}(x) \leqq f_{t_2}^{(m+1)q}(x) - 2p .$$

However,

$$f_{t_1}^{(m-1)q}(x) - (m-1)p > f_{t_1}^{-s}(x) + s > f_{t_2}^{-s}(x) + s > f_{t_2}^{(m+1)q}(x) - (m+1)p$$

by the ordering implied by the rotation number. This is a contradiction, so $f_{t_1}^{q}(x) > f_{t_2}^{3q}(x) - 2p$ as claimed. The first inequality of the Bounded Geometry allows us to conclude the proof. □

**Lemma 2.9.** *Let $t$ belong to $\rho^{-1}(P/Q, P'/Q') \cup \rho^{-1}(P'/Q')$. Choose $x$ and $y$ so that*

$$y \in [x, F_t^Q(x)] .$$

*Then,*

$$K_3^{-1} \leqq \frac{|f_t^Q(x) - x - P|}{|f_t^Q(y) - y - P|} \leqq K_3 .$$

*The bound $K_3$ is universal provided that the degree of $P/Q$ is sufficiently large.*

*Proof.* This follows by the first statement of Bounded Geometry. We observe the ordering of points $F_t^{-Q}(x), F_t^{-Q}(y), x, y, F_t^Q(x), F_t^Q(y)$ on the circle. Thus, the arc $(x, F_t^Q(x))$ is contained in $(F_t^{-Q}(y), F_t^Q(y))$. Hence, its length is bounded by $L_3|F_t^Q(y) - y - P|$ by the Bounded Geometry. The other inequality follows in a symmetrical way. □

Finally, we get this estimate:

**Lemma 2.10.** *Let $t_1$ and $t_2$ be contained in $\rho^{-1}(P/Q) \cup \rho^{-1}(P/Q, P'/Q')$. Then, the time derivatives of $f_{t_1}^Q(0)$ and $f_{t_2}^Q(0)$ are comparable within uniform constants.*

*Proof.* We first use Lemma 2.7 to convert the parameter derivatives to sums of ratios of intervals. Next, we need to show that any the lengths of any two corresponding intervals in both sums are comparable within uniform constants.

Consider $(F_{t_1}^i(0), F_{t_1}^{i+q}(0))$ and $(F_{t_2}^i(0), F_{t_2}^{i+q}(0))$. By Lemma 2.8 $|F_{t_2}^i(0) - F_{t_2}^{i+q}(0))|$ and $|F_{t_2}^i(0) - F_{t_1}^q(F_{t_2}^i(0))|$ are uniformly comparable. Observe that $F_{t_1}^i(0)$ belongs to the interval $(F_{t_1}^q(F_{t_2}^i(0)), F_{t_2}^i(0))$. Therefore, Lemma 2.9 stated for the Farey domain $[p'/q', p/q)$, which contains $[P/Q, P'/Q']$ (in the worst case $p'/q' = P/Q$), concludes the proof. □

*The Proof of Proposition 3.* In view of Lemma 2.10, the only thing that remains to be shown is the uniformity of change with respect to $x$ with $t$ fixed. By Lemma 2.7 this comes down to estimating the ratios of

$$\frac{|F_t^{i+q}(x) - F_t^i(x)|}{|F_t^{i+q}(0) - F_t^i(0)|} .$$

We need $t$ in the preimage of a Farey domain bounded by $p/q$ to conclude the argument by Lemma 2.9. Since the left daughter of $p/q$ is at most $P/Q$ such a $t$ can be found.

*2.5. Proof of Proposition 2.* To see the first statement, observe that the critical values of $F_t^Q$ closest to 0 are $F_t^{Q-q}(0)$ and $F_t^{Q-s}(0)$. By Bounded Geometry this means extendibility of $f_t^Q$ as a diffeomorphism onto a larger image. Then we apply the Corollary to Proposition 1 with $i := Q$. We see that $\chi_1^{-1}$ preserves this extendibility. Thus, the nonlinearity of rescaled $g$ is bounded by the Real Köbe Lemma, and the first claim of Proposition 2 follows.

The key to the other two statements follow from the differential equation

$$D\mathcal{N}g = Sg + 1/2(\mathcal{N}g)^2 , \tag{9}$$

which is satisfied by every $C^3$ diffeomorphism $g$ (for a direct check, also see [13] p. 56.) For the second claim, choose $n(F_0)$ as given by Proposition 1. Let $L$ denote the maximum of $S(f_t^Q \circ \Gamma)$ on the unit interval, and $K_1$ be the upper bound on $|\mathcal{N}(f_t^Q \circ \Gamma)|$ ($\Gamma$ is affine mapping the unit interval onto $(0, f_t^{-q} + p)$) obtained in the first part of Proposition 2. We see that $L \geqq -3K_1 - K_1^2/2$ or the differential equation (9) would clash with the bound on the nonlinearity. By the second claim of Proposition 1 the Schwarzian derivative is comparable at various points of the unit interval, so the bound from below follows.

To prove the last claim, consider an abstract class of diffeomorphisms $\mathcal{G}(w, L)$ defined as the set of functions defined in a neighborhood of 0 and having the following properties:

1. Their Schwarzian derivatives are negative and bounded away from 0 by some $-\beta$.
2. For any $g \in \mathcal{G}$, $g(0) = w$.
3. There is no $x \in (-L, L)$ where $g$ is defined and $g(x) \leqq x$.

Observe that for $x \in \Gamma^{-1}(f_t^Q - P, f_t^q - p)$, the function

$$g_0(y) = f_t^Q(\Gamma(y - x)) - P$$

belongs to $\mathcal{G}(w, L)$ with

$$w := \frac{|f_t^Q(x) - x - P|}{|f^{-q}(0) + p|}$$

and $L$ universal determined by the Bounded Geometry. So, it suffices to show that for every $L > 0$ there is a $K_3 > 0$ universal so that $w < K_3$ implies $\mathcal{N}g(0) > K_2 > 0$.

We observe that every function from $g \in \mathcal{G}(w, L)$ is uniquely determined by three parameters: a continuous function $\psi = Sg$ and two numbers $v$ and $\mu$ equal to $\mathcal{N}g(0)$ and $g'(0)$ respectively. Indeed, given $\psi$ and $v$, $\mathcal{N}g$ is uniquely determined by the differential equation (9), this together with $\mu$ determines $g'$, and finally $g$ is also defined by $w$. Observe that with $\mu$ fixed, $g$ is an increasing function of $\psi$ and $v$. Indeed, a look at Eq. (9) reveals that if $\psi_1 \geqq \psi_2$ with the same $v$, then the solution $\mathcal{N}g_1(x) \geqq \mathcal{N}g_2(x)$ for $x \geqq 0$, while $\mathcal{N}g_1 \leqq \mathcal{N}g_2$ for $x \leqq 0$. This is immediate if $\psi_1 > \psi_2$, since we see that at every point where the solutions cross $\mathcal{N}g_1$ is bigger on a right neighborhood and less on a left neighborhood. Then we treat $\psi_1 \geqq \psi_2$ by studying $\psi' = \psi_1 + c$, where $c$ is a positive parameter and using continuous dependence on parameters. As $g'$ is clearly an increasing function of $\mathcal{N}g$ and $v$, the monotonicity with respect to $\psi$ and $v$ follows. So, if we can show that for some $\tilde{\psi}$ and $\tilde{v}$ and every $\mu$ the condition $g(x) \geqq x$ is violated on $(-L, L)$, it follows that there exists $\delta > 0$ so that for every $\psi \leqq \tilde{\psi}$, we must have $v > \tilde{v} + \varepsilon$ if the function is in $\mathcal{G}(w, L)$.

Pick $\tilde{\psi} = -\beta$ and $v = 0$. The problem becomes quite explicit. From another well-known differential formula $u'' = Sg \cdot u$ satisfied by $u = 1/\sqrt{g'}$ we find

$$g'(x) = \frac{\mu}{\sqrt{\cosh \beta x}} \ .$$

Let $w_n \to 0$ and pick $\mu_n$ so that the corresponding $g$ satisfies $g(x) \geqq x$, or escapes to $+\infty$, on $(-L, L)$. Observe that $\mu_n$ must be a bounded sequence, since we have $g(x) \leqq \frac{\mu}{C(L,\beta)} x + w$ for $x < 0$, where $C(L, \beta)$ is the upper bound of $\cosh \beta x$ on $[-L, 0]$. Thus if such a sequence existed, we could take a limit parameter $\mu_\infty$ which would preserve $g(x) \geqq x$ even for $w = 0$, and this cannot be.

The third claim of Proposition 2 follows.

## 3. Scaling Rules

*3.1. Notations and the main result.* Throughout this chapter, we fix a Farey domain $(P/Q, P'/Q')$. As explained in the Introduction, we may adopt the convention $0 < q = Q' - Q$. The frequency-locking intervals which bound $\rho^{-1}(P/Q, P'/Q')$ from below and from above are $\rho^{-1}(P/Q)$ and $\rho^{-1}(P'/Q')$. For every rational $u$, within $\rho^{-1}(u)$ there is a unique point $c(u)$, called the *center* of the corresponding frequency-locking, and characterized by the property that the critical point is periodic.

Next, we consider the sequence $u_n$ of endpoints of the harmonic subdivision of $(P/Q, P'/Q')$.

We define

$$J_n := (c(u_{n+1}), c(u_n)),$$

and

$$J := (c(P/Q), c(P'/Q')) \ .$$

**Definition 3.1.** *Harmonic scalings $h_n$ are defined as the ratios*

$$h_n := \frac{|J_n|}{|J|}$$

*for $n \in \mathbf{Z}$.*

Our main result about harmonic scalings is contained in the following proposition:

**Proposition 5.** *Harmonic scalings decrease no faster than according to a uniform cubic law, i.e.*

$$h_n \geqq \frac{K_1}{|n|^3} \ ,$$

*where $K_1$ is uniform.*

*If, in addition, for all $t \in J$, $Sf_t^Q < -L$ with $L > 0$ on $(f_t^{-Q}(0) + P,\ f^{-q}(0) + p)$ with 0 removed, then*

$$h_n \leqq \frac{K_2}{|n|^3} \ .$$

*Here $K_1$ is a uniform constant, while $K_2$ is a uniform function of $L$ only. $K_1$ and $K_2$ become universal if the degree of $P/Q$ is sufficiently large.*

The rest of this section will be devoted to the proof of Proposition 5.

*3.2. First estimates.* As it was noticed in Sect. 1, it is enough to estimate $h_n$ for $n$ positive, since we can use the Farey domain $((P+P')/(Q+Q'), P'/Q')$ and then flip the Farey tree. We define

$$t_n := c(u_n)$$

and $t_\infty = c(P/Q)$, $t_{-\infty} = c(P'/Q')$.

As $t$ moves from $t_{n+1}$ to $t_n$, $F_t^{(n+1)Q}(0)$ travels from $F_{t_{n+1}}^{-q-Q}(0)$ to $F_{t_n}^{-q}(0)$. Thus,

$$|J_n| \frac{df^{(n+1)Q}_{t|t=\xi}(0)}{dt} = |F_{t_{n+1}}^{-q-Q}(0) - F_{t_n}^{-q}(0)|\,,$$

where $\xi$ is given by the Mean Value Theorem, thus $\xi \in (t_{n+1}, t_n)$. An analogous argument shows that

$$|J| \frac{df^{Q}_{t|t=\eta}(0)}{dt} = |F_{t_{-\infty}}^{-q}(0) - 0|\,,$$

this time with $\eta \in \rho^{-1}[P/Q, P'/Q']$.

These equalities allow us to express the harmonic scalings in terms of ratios of time derivatives and lengths of relevant intervals. We will now work to make this relation as simple as possible.

**Lemma 3.1.** *A number $K > 0$ exists so that*

$$K^{-1} \leqq \frac{|f_{t_{n+1}}^{-q-Q}(0) - f_{t_n}^{-q}(0) + P|}{|f_{t_{-\infty}}^{-q}(0) + p|} \leqq K\,.$$

*This K is universal provided that the degree of $P/Q$ is sufficiently large.*

*Proof.* By definition $t_{n+1}$ and $t_n$ are in the frequency-locking intervals adjacent to the Farey domain $(u_{n+1}, u_n)$. Since this Farey domain belongs to the harmonic subdivision of $(P/Q, P'/Q')$, Lemma 2.8 can be applied with $q := Q$ and $Q := nQ + Q'$ to see that $|f_{t_{n+1}}^{-q-Q}(0) - f_{t_n}^{-q}(0) + P|$ is uniformly comparable to $|f_{t_n}^{-q-Q}(0) - f_{t_n}^{-q}(0) + P|$. By the Bounded Geometry, that is in a uniform ratio with $|f_{t_n}^{-q}(0) + p|$, and another application of Lemma 2.8 with $q := q$, $t_1 := t_n$ and $t_2 := t_{-\infty}$ gives the claim. $\square$

In view of Lemma 3.1,

$$K^{-1} h_n \leqq \frac{\frac{df^{Q}_{t|t=\eta}(0)}{dt}}{\frac{df^{(n+1)Q}_{t|t=\xi}(0)}{dt}} \leqq K h_n\,. \tag{10}$$

We now concentrate on estimating the ratio in (10).

**Lemma 3.2.** *In the situation of Proposition 5 and for $n$ positive, there is a uniform bound $K$ so that*

$$\frac{K^{-1}}{\sum_{k=0}^{n-1} \frac{|F_\xi^{-q}(0) - 0|}{|F_\xi^{kQ}(0) - F_\xi^{(k+1)Q}(0)|}} \leqq \frac{\frac{df^{Q}_{t|t=\eta}(0)}{dt}}{\frac{df^{(n+1)Q}_{t|t=\xi}(0)}{dt}} \leqq \frac{K}{\sum_{k=0}^{n-1} \frac{|F_\xi^{-q}(0) - 0|}{|F_\xi^{kQ}(0) - F_\xi^{(k+1)Q}(0)|}}\,.$$

*The bound K becomes universal if the degree of $P/Q$ is sufficiently large.*

*Proof.* In this proof the notation $x \sim y$ means that the ratio $x/y$ is bounded from both sides by positive constants that become universal for large degrees of $P/Q$. By the Bounded Geometry and the Distortion Lemma, we see that

$$\frac{df^{(n+1)Q}_{t|t=\xi}(0)}{dt} \sim \sum_{k=0}^{n-1} \frac{\partial f^Q}{\partial t}(\xi, f_\xi^{kQ}(0)) \frac{|F_\xi^{nQ}(0) - F_\xi^{(n+1)Q}(0)|}{|F_\xi^{kQ}(0) - F_\xi^{(k+1)Q}(0)|} .$$

Here, we replaced the spatial derivatives with ratios of intervals like in the proof of Lemma 2.7. Next, we use Proposition 3 to replace all parameter derivatives in this expression by

$$\frac{\partial f^Q}{\partial t}(\xi, 0) ,$$

again preserving comparability by uniform constants.

We see that

$$\frac{\frac{df^Q_{t|t=\eta}(0)}{dt}}{\frac{df^{(n+1)Q}_{t|t=\xi}(0)}{dt}} \sim \frac{\frac{\partial f^Q}{\partial t}(\eta,0)}{\frac{\partial f^Q}{\partial t}(\xi,0)} \frac{1}{\sum_{k=0}^{n-1} \frac{|F_\xi^{nQ}(0)-F_\xi^{(n+1)Q}(0)|}{|F_\xi^{kQ}(0)-F_\xi^{(k+1)Q}(0)|}} . \tag{11}$$

It suffices to show that

$$\frac{\frac{\partial f^Q}{\partial t}(\eta,0)}{\frac{\partial f^Q}{\partial t}(\xi,0)} \frac{1}{\sum_{k=0}^{n-1} \frac{|F_\xi^{nQ}(0)-F^{(n+1)Q}(0)|}{|F_\xi^{kQ}(0)-F_\xi^{(k+1)Q}(0)|}} \sim \frac{1}{\sum_{k=0}^{n-1} \frac{|f_\xi^{-q}(0)+p|}{|f_\xi^{kQ}(0)-f_\xi^{(k+1)Q}(0)+P|}} .$$

We have

$$\frac{\frac{\partial f^Q}{\partial t}(\eta,0)}{\frac{\partial f^Q}{\partial t}(\xi,0)} \sim 1$$

as a consequence of Proposition 3. So, we only need to show that

$$|f_\xi^{nQ}(0) - f_\xi^{(n+1)Q}(0) + P| \sim |f_\xi^{-q}(0) + p| .$$

First,

$$|f_\xi^{nQ}(0) - f_\xi^{(n+1)Q}(0) + P| \sim |f_\xi^{-Q-q}(0) - f_\xi^{-q}(0) + P|$$

by Lemma 2.9 applied with $x := f^{(n+1)Q_\xi(0)}$ and $y := f_\xi^{-q}(0)$ and the first estimate of the Bounded Geometry. Finally,

$$|f_\xi^q(0) - p| \sim |f_\xi^{-Q-q}(0) - f_\xi^{-q}(0) + P|$$

by the second estimate of the Bounded Geometry. Lemma 3.2 follows. □

*3.3. Saddle-node estimates.* In this section we derive estimates for harmonic scalings as uniform functions of $n$. To this end, we will use Lemma 3.2 in conjunction with estimate (10) which makes this task equivalent to estimating

$$\sum_{k=0}^{n-1} \frac{|f_\xi^{-q}(0) + p|}{|f_\xi^{kQ}(0) - f_\xi^{(k+1)Q}(0) + P|} . \tag{12}$$

**It is sufficient to consider $h_n$ with $n$ large.** By the Bounded Geometry, the expression (12) gives values bounded away from 0 and infinity in a uniform fashion for any bounded $n$. Thus, the claim of Proposition 5 can be satisfied by choosing the uniform constants appropriately.

**Normalization.** Consider $F_\xi^Q$ with $\xi \in J_n$ as in Proposition 5. Consider $\chi \in (0, F_\xi^{-q}(0))$, which does not have to be unique, so that

$$|F_\xi^Q(\chi) - \chi| = \inf\{|F_\xi^Q(x) - x| : x \in (0, F_\xi^{-q}(0))\} .$$

For $n$ large, $\chi \in (f_\xi^Q(0) - P, f_\xi^{-q-Q}(0) - p - P)$. This follows from the Bounded Geometry and Lemma 2.9. We change the coordinates by an affine map so that $\chi$ goes to 0 and $F_\xi^{-q}(0)$ goes to 1. The critical point is at some point $c$ whose distance from 0 is uniformly bounded and bounded away from 0. In these coordinates $F_\xi^Q$ becomes a map which we denote by $\phi$. By the first claim of Proposition 2, the second derivative of $\phi$ is bounded on some $(-C_1, C_1)$ with $C_1 > 0$ uniform and universal when the degree of $P/Q$ is large.

**Approximation rules.** We say that $\phi$ satisfies the $(\alpha, \kappa)$ *upper approximation rule* if $\phi$ is not greater than the map

$$x \to x + \alpha x^2 + \phi(0)$$

on some interval $(-\kappa, \kappa)$.

Analogously, $\phi$ satisfies the $(\alpha, \kappa)$ *lower approximation rule* if there is the converse inequality.

Since the second derivative of $\phi$ is bounded on a uniform neighborhood of 0, there is a uniform choice of $(\alpha, \kappa)$ so that the upper approximation rule is always satisfied. In this sense we will say that $\phi$ satisfies the uniform upper approximation rule. Then we define $\phi_u := x + \alpha x^2 + \phi(0)$ with this uniform $\alpha$. We will also show that if the degree of $P/Q$ is large, a uniform lower approximation rule also holds. So, we define $\phi_l$ in the analogous way.

The advantage of approximation rules is the orbits under quadratic maps can be examined more or less explicitly and the interesting quantities simply calculated. This was basically the idea of the authors who previously contributed to the subject (see [11], also [10, 5], which by no means exhaust the list, as trick was discovered independently a couple of times).

**Lemmas about quadratic maps.** Consider a function $\Phi$

$$x \to x + \alpha x^2 + \varepsilon$$

defined on $(-\kappa, \kappa)$. Also assume that $\alpha > \beta > 0$ for some $\beta$, while $\kappa > \gamma > 0$ for some $\gamma$, and $\varepsilon > 0$. A *maximal orbit* is a sequence $(y_i)_{0 \leqq i \leqq l}$ for which $y_{i+1} = \Phi(y_i)$ and $y_0$ has no preimage while $y_l$ is no longer in the domain. We have two facts about maximal sequences which are proved, though not explicitly stated, in [5].

**Fact 3.1.** *The length of a maximal sequence $l$ and $\varepsilon$ are related by*

$$K_1(\beta, \gamma)^{-1}\varepsilon \leqq \frac{\alpha}{l^2} \leqq K_1(\beta, \gamma)\varepsilon ,$$

*where $K_1(\,\cdot\,,\,\cdot\,)$ is a positive function of $\beta, \gamma$ only.*

**Fact 3.2.** *Let $l_0$ be the number of points $y_i$ from the maximal orbit which satisfy $y_{i+1} - y_i < 2\varepsilon$. Then*

$$\frac{l_0}{l} > K_2(\beta)\,,$$

*where again $K_2(\,\cdot\,)$ is a positive function of $\beta$ only.*

**Relating $\phi(0)$ and $n$**

**Lemma 3.3.** *If $\phi$ satisfies the uniform upper approximation rule,*

$$\frac{K_1^{-1}}{n^2} \leqq \phi(0)\,.$$

*Proof.* The number $n$ is at least the length of a maximal orbit by $\phi_u$. The claim then follows by Fact 3.1. □

*Proof of the first estimate of Proposition 5.* There are $n$ terms in formula (12). By Lemma 3.3, each is at most of the order of $n^2$. The estimate on $h_n$ from below follows from Lemma 3.2 and formula (10).

**The negative Schwarzian case.** The other estimate of Proposition 5 is harder, but since we only claim it in the negative Schwarzian case, we can be aided by the strong claim of Proposition 2. Among other things, we now know that $\phi$ satisfies both uniform approximation rules. Also, $\phi'$ has exactly one minimum which must be attained on the right of 0. Consider $(-\kappa,\kappa)$ so that both uniform approximation rules hold.

**Only what's inside $(-\kappa,\kappa)$ counts.** We prove this lemma:

**Lemma 3.4.** *The number of iterates $k$ so the $0 < k < n$ and $\phi^k(c) \notin (-\kappa,\kappa)$ is uniformly bounded. Moreover, for all such values of $k$, $\phi^{k+1}(c) - \phi^k(c)$ is uniformly bounded away from 0.*

*Proof.* Actually, the first part of the claim obviously follows from the second. This is only a problem if there are many iterates $k$ so that $\phi^k(c) \in (-\kappa,\kappa)$, otherwise $\phi(0)$ is bounded away from 0. Thus, by the approximation rules, if $k_1$ and $k_2$ are the smallest and the largest $k$ so that $\phi^k(c) \in (-\kappa,\kappa)$, the distances $\phi_{k_1+1}(c) - \phi_{k_1}(c)$ and $\phi_{k_2+1}(c) - \phi_{k_2}(c)$ are uniformly large. But they are still smaller than the analogous distances for the values of $k$ for which $\phi^k(c) \notin (-\kappa,\kappa)$ by the negative Schwarzian property. □

That means that in the sum (12) the contribution from the terms corresponding to values of $k$ with the property that $\phi^k(c) \notin (-\kappa,\kappa)$ is uniformly bounded. On the other hand, it is clear that the whole expression grows at least as $n^2$. Thus, for values of $n$ uniformly sufficiently large, only the points of the orbit which are contained in $(-\kappa,\kappa)$ can be considered, and the result will approximate the whole sum up to a uniform multiplicative factor.

**Essential estimates.** Consider the smallest $k$ so that $\phi^k(c) \in (-\kappa,\kappa)$, and denote this point with $x_0$. Correspondingly, let $x_l$ be the last point of the orbit still in $(-\kappa,\kappa)$, and in between we get a sequence which satisfies $x_{i+1} = \phi(x_i)$ for $i = 0,\ldots,l-1$. We are interested in estimating the sum

$$\sum_{i=0}^{l-1} \frac{1}{x_{i+1} - x_i} \tag{13}$$

from below.

**Lemma 3.5.** *In our situation,*

$$\frac{K_1^{-1}}{n^2} \leqq \phi(0) \leqq \frac{K_1}{n^2}.$$

*Proof.* This is a stronger version of Lemma 3.3 under stronger assumptions, and the proof is very much the same. □

**Estimating the sum (13) from below.** Let $y$ be the largest point of the orbit of $c$ by $\phi$ still negative. Then $(y_i^l)$ and $(y_i^u)$ denote maximal orbits for $\phi_l$ and $\phi_u$ respectively which also contain $y$. The sum given by (13) is larger than the corresponding sum for $(y_i^u)$, as the intervals occurring in the latter are fewer and longer. An analogous argument shows that the sum for $(y_i^l)$ bounds the interesting expression from above.

To estimate

$$\sum_i \frac{1}{y_{i+1}^u - y_i^u}$$

from below we have to use Lemma 3.5 which then asserts simply that $\phi(0)$ is comparable to $n^{-2}$. Then, Fact 3.1 implies that $n$ is comparable to the length of the maximal orbit of both $\phi_u$ and $\phi_l$.

Then, by Fact 3.2 we see that the number of intervals for $\phi_u$ with lengths not greater than $2\phi(0)$ is still comparable with the length of the maximal orbit, thus with $n$.

Thus, we get a uniform cubic estimate from below. Then Lemma 3.2 with estimate (10) enable us to derive the second claim of Proposition 5.

## 4. Hölder Continuity of the Rotation Number

In this section, we will prove Theorem A announced in the Introduction.

*4.1. Some consequences of scaling rules.* We will derive certain "Hölder type" estimates as consequences of Proposition 5. The reasoning is actually a repetition of that used in ([5]) to prove Hölder continuity of the rotation number for families of diffeomorphisms. We want to emphasize that we need estimates everywhere on the parameter space and we are unwilling to assume that the denominators of Farey domains that we work with are large enough. So, only the first claim of Proposition 5 holds, i.e.

$$h_n \geqq \frac{K_{1,P.5}}{|n|^3}.$$

**Two estimates.** We fix our attention on a Farey domain $(P/Q, P'/Q')$ subject to our usual convention $0 < q = Q' - Q$. We consider the harmonic subdivision by points $u_n$. Also, the centers of mode-locking intervals are denoted with $t_n$ ($n$ may be infinite) as in the Scaling Rules section.

**Lemma 4.1.** *Let* $u = u_n$, $v = u_{n+1}$. *Then,*

$$|u - w|\,|c(u) - c(w)|^{-\alpha} \leqq \beta |P/Q - P'/Q'|\,|c(P/Q) - c(P'/Q')|^{-\alpha}$$

*with uniform* $0 < \alpha, \beta < 1$.

*Proof.* We have the following estimate as a consequence of Proposition 5:

$$\frac{|\rho(t_{n+1})-\rho(t_n)|}{|\rho(t_{-\infty})-\rho(t_\infty)|}\frac{|t_{-\infty}-t_\infty|^\alpha}{|t_{n+1}-t_n|^\alpha} \leqq \frac{n^{3\alpha}QQ'}{K_{1,P.5}^\alpha(nQ+Q')((n+1)Q+Q')}$$

(recall that $|QP'-PQ'|=1$ as Farey neighbors).

Consequently,

$$\frac{|\rho(t_{n+1})-\rho(t_n)|}{|\rho(t_{-\infty})-\rho(t_\infty)|}\frac{|t_{-\infty}-t_\infty|^\alpha}{|t_{n+1}-t_n|^\alpha} < \frac{2(n+2)^{3\alpha-2}}{K_{1,P.5}^\alpha},$$

where we used $Q<Q'<2Q$. By choosing $\alpha$ sufficiently close to 0 we can get the ratio on the right-side smaller than some number less than 1, say $\beta$. □

Next, we want to generalize Lemma 4.1 to $u,v$ arbitrary endpoints of the harmonic subdivision of $(P/Q,P'/Q')$.

**Lemma 4.2.** *Let $u,v$ be arbitrary two endpoints of the harmonic subdivision of $(P/Q,P'/Q')$, or $P/Q$ or $P'/Q'$. Then,*

$$|u-w|\,|c(u)-c(w)|^{-\alpha} \leqq K_1|P/Q-P'/Q'||c(P/Q)-c(P'/Q')|^{-\alpha}$$

*with $\alpha$ uniform and positive.*

*Proof.* First, we note that it suffices to prove the lemma when $u=u_n$ and $w=u_{n'}$ with $n\cdot n'\geqq 0$. Indeed, in the situation when both $u$ and $v$ are endpoints, but $n\cdot n'<0$, we can consider $n$ and 0 as well as $n'$ and 0 separately. Then, if we sum up resulting inequalities and use convexity of $\frac{1}{x^\alpha}$, we can infer the claim of the lemma.

If, for example, $u$ is $P/Q$, we can take the limit with $u_n$, where $n$ tends to $+\infty$. By continuity of the rotation function, this would give us almost the estimate of the lemma, except that on the left-hand side $c(u)$ is replaced with $t_0$ which is the upper endpoint of $\rho^{-1}(P/Q)$. However, this is stronger than the estimate claimed by the lemma.

Furthermore, we can restrict our attention to $n,n'\geqq 0$. We define $m$ by $0<m=n'-n$.

Again, we use Proposition 5:

$$\frac{|\rho(t_{n+m})-\rho(t_n)|}{|\rho(t_{-\infty})-\rho(t_\infty)|}\frac{|t_{-\infty}-t_\infty|^\alpha}{|t_{n+m}-t_n|^\alpha}$$

$$\leqq \frac{n^{3\alpha}QQ'}{K_{1,P.5}^\alpha(nQ+Q')((n+m)Q+Q')}\left(\frac{1}{\sum_{k=n}^{n+m-1}1/k^3}\right)^\alpha$$

$$\leqq \frac{2}{K_{1,P.5}^\alpha}\frac{m}{(m+n+1)(n+1)}\left(\frac{1}{1/n^2-1/(m+n)^2}\right)^\alpha$$

$$\leqq \frac{2}{nK_{1,P.5}^\alpha}\frac{n^{2\alpha}(n+m)^{2\alpha}}{m^\alpha(2n+m)^\alpha} \leqq \frac{2^{2\alpha+1}}{K_{1,P.5}^\alpha}n^{2\alpha-1}.$$

This expression is bounded by some $K_1$ if $\alpha\leqq 1/2$. □

*4.2. Global estimates.* Then, we let $u<w$ be arbitrary rational numbers from the unit interval. There is a unique simple path in the Farey tree from $u$ to $w$. It contains

the "highest" node $V$. This splits the path into two parts: from $u$ to $V$ and from $V$ to $w$.

Then, we define maps $\mu$ and $\nu$ on the Farey tree. Given a rational number $v, \mu(v)$ is the rational number that corresponds to the initial segment of the symbolic code of $v$ cut off at the last turning point (i.e. if the turning points are $m_1, \dots, m_k$, the symbolic sequence of $\mu(v)$ is $a_1, \dots, a_m$).

Then, $\nu(v)$ is the mother of $\mu(v)$.

Clearly,

$$\deg(\mu(v)) = \deg(\nu(v)) = \deg(v) - 1 .$$

Furthermore, $v$ lies strictly between $\mu(v)$ and $\nu(v)$, and $v$ is in fact an end-point of the harmonic subdivision of $(\nu(v), \mu(v))$.[2]

**The function $\zeta$.** If $x$ and $y$ are two rationals from the unit interval, we define

$$\zeta(x, y) := |x - y| \cdot |c(x) - c(y)|^{-\alpha} .$$

Lemma 4.2 gives us a fundamental estimate

$$\zeta(x, y) \leqq K_1 \zeta(\mu(x), \nu(x)) , \tag{14}$$

provided $\mu(x) = \mu(y)$.

If we iterate $\mu$ and $\nu$, we get a nested sequence of growing fundamental domains bounded by $\mu^i(x)$ and $\nu^i(x)$. For any $i$ we have

$$\zeta(\mu^i(x), \nu^i(x)) \leqq \beta \zeta(\mu^{i+1}(x), \nu^{i+1}(x)) \tag{15}$$

by Lemma 4.1.

Finally, we define a sequence $u_0 = v$ and $u_i$ is equal to the greater of $\mu^i(v)$ and $\nu^i(v)$. Similarly, we define the sequence $w_i$ so that $w_0 = w$ and $w_i$ is the minimum of $\mu_i(w)$ and $\nu_i(w)$. Also, let $l$ be the largest so that $w_l \geqq V$.

We want to bound $\zeta(u, u_k)$. As $\zeta(v, v + v')$ is convex as a function of $v'$, and the sequence $u_i$ is growing, clearly

$$\zeta(u, u_k) \leqq \sum_{i=1}^{k} \zeta(u_{i-1}, u_i) . \tag{16}$$

Since $u_i$ and $u_{i+1}$ are contained between $\mu(u_{i+1})$ and $\nu(u_{i+1})$, we can estimate

$$\zeta(u_i, u_{i+1}) \leqq K_1 \zeta(\mu(u_{i+1}), \nu(u_{i+1}))$$

by inequality (14). Then we iterate $\mu$ and $\nu$ on $\mu(u_{i+1})$ as many times as possible, which at least $\max(k - i - 1, 0)$. By the repeated use of inequality (15), we get

$$\zeta(\mu(u_{i+1}), \nu(u_{i+1})) \leqq \beta^{\max(0, k-i-1)} \zeta(0, 1) .$$

Since $\zeta(0, 1) = 1$, we can finally get from estimate (16),

$$\zeta(u, u_k) \leqq K_2 ,$$

by simply adding up a geometric progression.

The same argument shows that

$$\zeta(w_l, w) \leqq K_2 .$$

[2] In this section, if we write $(a, b)$, we perhaps mean $(b, a)$ when $a > b$.

Finally, by their definition $\mu(w_l) = \mu(u_k)$ so that they both are in the fundamental domain bounded by $\mu(u_k)$ and $\nu(u_k)$.

Again, we get

$$\zeta(u_k, w_l) \leqq K_1 \zeta(\mu(u_k), \nu(u_k))$$

from estimate (14), and

$$\zeta(\mu(u_k), \nu(u_k)) \leqq 1$$

by the repeated use of estimate (15).

Since

$$\zeta(u, w) \leqq \zeta(u, u_k) + \zeta(u_k, w_l) + \zeta(w_l, w)$$

by convexity, we have proved that

$$\zeta(u, w) \leqq K_3$$

with uniform $K_3$, which means the Hölder estimate. That is, we have proven that

$$|\rho(x) - \rho(y)| \leqq K_3 |x - y|^\alpha ,$$

provided that $x$ and $y$ are both centers of frequency-lockings.

**The general Hölder estimate.** By continuity, we also get the same Hölder estimate if $x$ and $y$ belong to the closure of the set of centers, i.e. to the complement of the union of interiors of all frequency-locking intervals. If $x < y$ are arbitrary, we consider $x'$ which is the infimum of the set of centers which are between $x$, and $y'$ is the supremum of the same set. They are well-defined unless $x$ and $y$ are in the same frequency-locking interval, in which case the estimate is evident. The Hölder estimate holds for $x'$ and $y'$. Moreover, $\rho(x') = \rho(x)$ and $\rho(y') = \rho(y)$, while $|x' - y'| \leqq |x - y|$. The Hölder estimate follows again.

## 5. Hausdorff Dimension

We will prove Theorem B. In this section, we will use the harmonic formalism of explained in the Introduction section. That is, we have a one-to-one coding of fundamental domains in the parameter space by finite sequences of $\mathbb{Z}$-type symbols. The length of the code will be called the *degree* of the corresponding fundamental domain.

The fundamental domain which corresponds to the code $(n_1, \ldots, n_r)$ will be denoted with $\mathscr{D}(n_1, \ldots, n_r)$.

Let $\Omega'$ be $T \backslash \bigcup_{w \in \mathbf{Q}} \rho^{-1}(w)$.

It is sufficient to prove that the Hausdorff dimension of

$$\Omega' \cap \mathscr{D}(n_1, \ldots, n_r)$$

satisfies our bounds for $r$ large enough. That means that we can assume that Proposition 1 holds and, consequently, use both claims of Proposition 5. Moreover, both constants from Proposition 5 are universal, i.e. they don't depend on the family.

*5.1. The estimate of $HD(\Omega')$ from above.* Take a cover of $\Omega'$ which consists of all fundamental domains of degree $r$.

Then

$$\sum_{n_1, \ldots, n_r \in \mathbf{Z}, |n_r| > k} |\mathscr{D}(n_1, \ldots, n_r)|^\beta \leqq K_2(k) \sum_{n_1, \ldots, n_{r-1}} |D(n_1, \ldots, n_{r-1})|^\beta ,$$

if $\beta > 1/3$ as a consequence of scalings rules (Proposition 5.) Here, $K_2(\cdot)$ is a uniform positive function with limit 0 at infinity.

Since for $|n_r| \leqq k$,

$$K_1 < \frac{|\mathcal{D}(n_1,\ldots,n_r)|}{|\mathcal{D}(n_1,\ldots,n_{r-1})|}$$

by Proposition 5, by Young's inequality

$$\sum_{n_1,\ldots,n_r \in \mathbf{Z}, |n_r| \leqq k} |\mathcal{D}(n_1,\ldots,n_r)|^\beta \leqq k^{1-\beta} K_1^\beta \sum_{n_1,\ldots,n_{r-1}} |D(n_1,\ldots,n_{r-1})|^\beta .$$

Thus,

$$\sum_{n_1,\ldots,n_r \in \mathbf{Z}} |\mathcal{D}(n_1,\ldots,n_r)|^\beta \leqq (k^{1-\beta} K_1^\beta + K_2(k)) \sum_{n_1,\ldots,n_{r-1}} |D(n_1,\ldots,n_{r-1})|^\beta .$$

We claim that $k^{1-\beta} K_1^\beta + K_2(k)$ can be made less than 1 by choosing $\beta$ sufficiently close to, but less than, one. Indeed, remember that $K_1 < 1$. So, we first choose $k$ so large that $K_2(k)$ is less than $0.5(1 - K_1)$. Then, by adjusting $\beta$ we can make $k^{1-\beta} K_1^\beta$ arbitrarily close to $K_1$.

Since the diameters of the fundamental domains tend to 0 with the degree, this $\beta$ is not larger than the Hausdorff dimension of $\Omega'$.

*5.2. Estimate from below by 1/3.* The proof is based on the following Frostman's Lemma, which we borrowed from [15]:

**Fact 5.1.** *Suppose that $\mu$ is a probabilistic Borel measure on the interval and that for $\mu$-a.e. $x$*

$$\liminf_{\varepsilon \to 0} \log(\mu(x-\varepsilon, x+\varepsilon))/\log(\varepsilon) \geqq \lambda .$$

*Then the Hausdorff dimension of $\mu$ is not less than $\lambda$.*[3]

Take $\eta < 1/3$. By the scalings rules it is clear that a number $k$ can be found independently of the fundamental domain $\mathcal{D}(n_1,\ldots,n_r)$ so that

$$\sum_{n_1,\ldots,n_r \in \mathbf{Z}, |n_r| \leqq k} |\mathcal{D}(n_1,\ldots,n_r)|^\eta \geqq |D(n_1,\ldots,n_{r-1})|^\eta .$$

We now define $\mu$ as a limit of probabilistic measures. The measure $\mu_0$ is just the Lebesgue measure on $\rho^{-1}(0,1)$ properly scaled. To obtain $\mu_{i+1}$, we consider all fundamental domains of degree $i$. If $\mathcal{D}(n_1,\ldots,n_i)$ is one of those, the density of $\mu_{i+1}$ with respect to $\mu_i$ on $\mathcal{D}(n_1,\ldots,n_i)$ equals

$$\frac{|\mathcal{D}(n_1,\ldots,n_{i+1})|^{\eta-1}}{\sum_{n_{i+1}=-k}^{n_{i+1}=k} |\mathcal{D}(n_1,\ldots,n_{i+1})|^\eta}$$

on fundamental domains of the harmonic subdivision with $|n_{j+1}| \leqq k$, and is zero in the mode-locking intervals which belong to the endpoints of the subdivision. The sequence has a limit which is supported on a set $\Omega_0$ contained in $\Omega'$. The set $\Omega_0$ consists of the preimages in the parameter space of all irrationals with harmonic code symbols bounded by $k$ as to absolute value. Moreover, by our choice of $\eta$ we

[3] Frostman's Lemma remains true if the $\geqq$ signs are replaced with $\leqq$ signs.

see that the denominator is not less than $|\mathscr{D}(n_1,\dots,n_{i-1})|^\eta$, thus the density is not greater than

$$\frac{|\mathscr{D}(n_1,\dots,n_{i+1})|^{\eta-1}}{|\mathscr{D}(n_1,\dots,n_i)|^\eta},$$

therefore

$$\frac{\mu_{i+1}(\mathscr{D}(n_1,\dots,n_{i+1}))}{\mu_i(\mathscr{D}(n_1,\dots,n_i))} \leqq \frac{|\mathscr{D}(n_1,\dots,n_{i+1})|^\eta}{|\mathscr{D}(n_1,\dots,n_i)|^\eta}.$$

So, by induction,

$$\mu_{i+1}(\mathscr{D}(n_1,\dots,n_{i+1})) \leqq |\mathscr{D}(n_1,\dots,n_{i+1})|^\eta,$$

and since clearly

$$\mu(\mathscr{D}(n_1,\dots,n_{i+1})) = \mu_{i+1}(\mathscr{D}(n_1,\dots,n_{i+1})),$$

the same estimate holds for $\mu$ itself.

Take any small $\varepsilon > 0$ and an $x \in \Omega_0$ and look for the largest $r$ so that

$$\mu((x-\varepsilon, x+\varepsilon)\backslash\mathscr{D}(n_1,\dots,n_r)) = 0$$

for some $n_1,\dots,n_r$. Note that a finite $r$ with this property always exists by topology.

Then

$$\mu((x-\varepsilon, x+\varepsilon)) \leqq |\mathscr{D}(n_1,\dots,n_r)|^\eta$$

and $2\varepsilon$ is greater than the length of the same gap between domains of the harmonic subdivision of $\mathscr{D}(n_1,\dots,n_r)$ which have non-zero measure. Indeed, by the definition of $r$, the interval $(x-\varepsilon, x+\varepsilon)$ must be straddled between at least two such domains. If the size of the gap is denoted with $\gamma$, we get

$$\frac{\log(\mu(x-\varepsilon, x+\varepsilon))}{\log(\varepsilon)} \geqq \eta\frac{\log(|\mathscr{D}(n_1,\dots,n_r)|)}{\log(\gamma)}.$$

By Proposition 5, the lengths of both domains and $|\mathscr{D}(n_1,\dots,n_r)|$ are all related by uniform constants. But $\gamma$ is not much smaller than either of them as a result of the estimates of [17]. So the logarithms differ by a bounded amount and their ratio tends to 1 as $\varepsilon$ shrinks to 0.

So, if we pass to the limit with $\varepsilon \to 0$ we see that the assumptions of Frostman's Lemma are satisfied, therefore the Hausdorff dimension of $\Omega_0$, which is larger than $\Omega'$, is at least $\eta$. But $\eta$ could have be chosen anything less than 1/3. So, the estimate follows.

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