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# Double Wells: Nevanlinna Analyticity, Distributional Borel Sum and Asymptotics

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**Abstract:** We consider the energy levels of a Stark family, in the parameter $j$, of quartic double wells with perturbation parameter $g$:

$$H(g,j) = p^2 + x^2(1-gx)^2 - j\left(gx - \frac{1}{2}\right).$$

For non-even $j$ (and for the symmetric case $j = 0$) we prove analyticity in the full Nevanlinna disk $\Re g^{-2} > R^{-1}$ of the $g^2$-plane, as predicted by Crutchfield. By means of an approximation we give a heuristic estimate of the asymptotic small $g$ behaviour, showing the relation between the avoided crossings and the failure of the usual perturbation series.

## 1. Introduction

The eigenvalues of the quartic anharmonic oscillator

$$A(g^2) = p^2 + x^2 + g^2x^4 \tag{1.1}$$

are interesting examples of Borel summability of the Rayleigh Schrödinger perturbation series [Gr-Gr-Si]. The unstable anharmonic oscillator (or "volcano")

$$\tilde{A}(g^2) = p^2 + x^2 - g^2x^4 \tag{1.2}$$

has "resonances" defined as the eigenvalues of the analytic continuations $A[(\pm ig)^2]$ of $A(g^2)$. Such "resonances," as well as the Hydrogen Stark effect resonances, are given by a pair of distributional Borel sums called upper and lower Borel sums (US, LS) [Ca-Gr-Ma1, 2, 4].

The energy levels of the double well Stark family

$$H(g,j) = p^2 + x^2(1-gx)^2 - j\left(gx - \frac{1}{2}\right) \tag{1.3}$$

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are much more difficult to treat. First of all, for $j$ even, infinitely many eigenvalues are unstable at $g = 0$, since a pair of eigenvalues shrinks to one [Re-Si]. Moreover in the case $j = -2$, as discussed by Herbst–Simon [He-Si], the first eigenvalue is stable and positive for $g$ positive, while the perturbation series is identically zero with obviously zero sum. Furthermore it was known that at the top energy $E = (16g^2)^{-1}$ the eigenvalues have very bad analyticity properties due to the crossings [Be-Wu, Re-Si]. Lacking direct information, the same situation was expected for the first eigenvalues. Crutchfield [Cr] showed that actually the first eigenvalues extend analytically to a certain domain of the complex plane of the coupling constant $g^2$. We give here the full proof of the analyticity on a Nevanlinna disk.

For what concerns Borel summability of the perturbation series, the situation is not so simple. Since the Herbst–Simon example [He-Si] shows that at least in one case the perturbation series does not give enough information, we should first extend the perturbation series. More precisely, starting from the usual decomposition $\lambda_n(g,j) = N(g,j)/(D(g,j))$, we work separately on the expansion of $N(g,j)$ and $D(g,j)$. Considering, for instance, $N(g,j)$ $(D(g,j))$ for $j$ real, we extend the standard perturbation series $\Sigma_k a_k g^{2k}$ by the new one $\Sigma_k a_k(g)g^{2k} + i\Sigma_k b_k(g)g^{2k}$, which gives the exact result if summed in the appropriate way, to be described in Sect. 5 of this paper.

This program was initiated in previous papers [Ca-Gr-Ma1, Ca-Gr-Ma3]. Here it is completed by our new general proof of the existence of a Nevanlinna disk of analyticity. We also use our previous results [Ca-Gr-Ma4] on the "resonances" of the "volcano," thanks to their connection with the "resonances" $E_n^{+}(g,j)$ of (1.3) as given by Buslaev–Grecchi [Bu-Gr].

Since the exact expression for the eigenvalues (given by the perturbation series (1.4) and by the appropriate summation method) is not very simple, we give an asymptotic approximation, obtained by taking the limit of our function $h(g,z)$ as $g \to 0$ and $z$ on a fixed path $\Gamma$ surrounding $\lambda_n(0,j)$. Let $F(g,j) = N(g,j)$ (or $D(g,j)$), then we have:

$$F(g,j) \sim \left(1 + \frac{\pi}{2}\frac{\Im E_n^{+}(g,j)}{\sin^2(\pi j/2)}\right) \Sigma_k\, a_k g^{2k} + \frac{i}{2}\cot(j\pi/2)\Delta\, \Sigma_k\, a_k g^{2k} \quad \text{as } g \to 0 \tag{1.4}$$

for $j$ not even, where $\Sigma$ means the distributional Borel sum (DBS) of the series and $\Delta\Sigma$ means the Borel discontinuity of the series (DOS). Let us notice that the coefficient of the DBS in (1.4) is the same for $D(g,j)$ and for $N(g,j)$, so that it can be factored out and set equal to 1 at the same approximation level. The coefficient of the "Borel discontinuity" in (1.4), i.e. $\cot(j\frac{\pi}{2})$, is singular and odd near $j = 0$, so that it is a typical avoided crossing term.

Actually the perturbation series of any eigenvalue is analytic in $j$ in a neighbourhood of the real axis for $g$ small, but not the eigenvalue itself. On the other hand it is possible that the direct distributional Borel summability of the perturbation series to the partition function gives the exact result as was originally suggested by 't Hooft ['t] and it is proved for a simple approximation [Ca-Gr-Ma1]. Some of the results of the present paper are announced at the 1993 Conference of the International Euler Institute [Ca-Gr-Ma5].

The paper is organized in the following way. In Sects. 2, 3, 4 we prove analyticity in the Nevanlinna disk, respectively, in the symmetric case, in the stable case, and in the unstable asymmetric case. In Sect. 5 we prove Borel summability and we give the extended perturbation expression of the matrix elements. In the Appendix we state and comment on stability theorems in a form which is convenient in this context.

## 2. Analyticity of Double Well Eigenvalues in the Nevanlinna Disk

Consider the double well oscillator with coupling constant $g = \rho e^{i\theta}$:

$$A = p^2 + x^2(1 - gx)^2 \equiv p^2 + V(x) . \tag{2.1}$$

In order to prove analyticity of the eigenvalues in some Nevanlinna disk $\Re g^{-2} > R^{-1}$, $R > 0$ assume $g$ small and in the boundary, that is $g = \rho e^{-i(\pi/4+\varepsilon/2)}$, with $\sin\varepsilon = R^{-1}\rho^2$. By the usual scaling $x \to xe^{-i\theta}$, $H$ is transformed into

$$A(\theta) = e^{2i\theta}p^2 + e^{-2i\theta}x^2(1 - \rho x)^2 , \tag{2.2}$$

with real part

$$\Re A(\theta) = \sin\varepsilon\{p^2 + x^2(1 - \rho x)^2\} . \tag{2.3}$$

**Definition 1.** *Let $\eta > 0$ be small and fixed. For each $\rho > 0$, $\psi \in L^2(R)$, let $(U\psi)(x) = \psi(\xi_\rho(x))$, where the complex-valued distorsion $\xi_\rho \in C^\infty(R)$ is defined by $\Re\xi_\rho(x) = x$, $\forall x \in R$ and*:

$$\Im\xi_\rho(x) = -\eta \operatorname{atan}\frac{x}{[1+x^2]^{1/4}}, \qquad x \leqq \frac{1}{2\rho} - 2\eta , \tag{2.4a}$$

$$\Im\xi_\rho(x) = 0, \qquad \frac{1}{2\rho} - \eta \leqq x \leqq \frac{1}{2\rho} + \eta , \tag{2.4b}$$

$$\Im\xi_\rho(x) = -\eta \operatorname{atan}\frac{x - \frac{1}{\rho}}{[1 + (x - \frac{1}{\rho})^2]^{1/4}}, \qquad x \geqq \frac{1}{2\rho} + 2\eta , \tag{2.4c}$$

*and elsewhere according to the prescriptions*

(i) $\Im\xi_\rho$ *monotone in each of the two remaining intervals*,

(ii) $\Re\xi_\rho(x) = x$, $\Im\xi_\rho$ *odd w.r.t.* $\dfrac{1}{2\rho}$ .

**Lemma 2.** *Setting $f_\rho(x) = (\xi_\rho'(x))^{-1}$, $H_\rho = UA(\theta)U^{-1}$ the transformed operator*

$$H_\rho := e^{2i\theta}\{pf_\rho^2 p + 4^{-1}(f_\rho^2)''\} + e^{-2i\theta}\xi_\rho(x)^2(1 - \rho\xi_\rho(x))^2 \tag{2.5}$$

*has the same spectrum as $A(\theta)$.*

*Proof.* The distortion preserves the same exponential decay of the solutions of $A(\theta)u = Eu$, since $\xi_\rho(x) \sim x$ as $x \to \pm\infty$. Notice that the notation $H_\rho$ is allowed since $\theta = -\pi/4 + \varepsilon/2$, where $\sin\varepsilon = R^{-1}\rho^2$.

In order to apply Theorem A2, let us define the parity projections,

$$[P^\pm(\rho)u](x) = 2^{-1}[u(x) \pm u(\rho^{-1} - x)] .$$

**Lemma 3.** *If $H_\rho = UA(\theta)U^{-1}$, the parity projections $P_\rho^\pm$ satisfy hypotheses* (a), (b), (c), (d) *of Theorem* A2.

*Proof.* It is sufficient to verify property (c), since the other ones are obvious. Since $V$ is even w.r.t. $(2\rho)^{-1}$, and $\xi_\rho(x) = x + iy(x)$, where $y(x)$ is odd w.r.t. $(2\rho)^{-1}$, we have that $V[\xi_\rho(x)]$ is even with respect to $(2\rho)^{-1}$. Moreover, since $\xi'_\rho$ is even, $f_\rho = (\xi'_\rho)^{-1}$ is even, as well as the second derivative: $(f_\rho^2)''$. Thus the operator (2.5), i.e. $H_\rho$, is "even" in the sense of property (c).

**Lemma 4.**

$$\Re V(\xi_\rho(x)) \geqq \frac{c_1}{R} + c_2, \quad \forall x \notin (-n, n) \cup \left(\frac{1}{\rho} - n, \frac{1}{\rho} + n\right)$$

*for some* $c_1 > 0$, $c_2 \in R$, $\forall n \geqq n_o$, $0 < \rho < \rho_o$.

*Proof.* In our context we can suppose $n \ll \frac{1}{\rho}$ (see hypothesis (2) in Theorem A1). By the definition of $\xi_\rho(x)$ and by its parity properties it suffices to verify the stated inequality in the points $x = \frac{1}{2\rho} - 2\eta$ and $x = \frac{1}{2\rho} - \eta$, $x = \frac{1}{2\rho}$. By definition of $\xi_\rho$,

$$\xi_\rho\left(\frac{1}{2\rho} - 2\eta\right) \sim \frac{1}{2\rho} - 2\eta - i\eta\frac{\pi}{2} \quad \text{as } \rho \to 0\,, \tag{2.6}$$

so in the first case, as $\rho \to 0$,

$$\Re V\left[\xi_\rho\left(\frac{1}{2\rho} - 2\eta\right)\right] \sim (\sin\varepsilon)\frac{1}{16\rho^2} + \cos\varepsilon\eta^2\pi \geqq \frac{c_1}{R} + c_2 \quad \text{as } \rho \to 0\,, \tag{2.7}$$

where the equality $\sin\varepsilon = R^{-1}\rho^2$ is used. In the second and third case we have $\xi_\rho(x) = x$, so

$$\Re V\left[\xi_\rho\left(\frac{1}{2\rho} - \eta\right)\right] = (\sin\varepsilon)\frac{1}{\rho^2}\left(\frac{1}{4} - \rho^2\eta^2\right)^2 \geqq \frac{c_1}{R}\,, \tag{2.8}$$

and the lemma is proved.

**Lemma 5.** *In the notation of Theorem* A2 (*i.e.* $H_\rho^+ = H_\rho P_\rho^+$),

$$\Re\langle H_\rho^+ u, u\rangle \geqq c_3 \int_{-\infty}^{\frac{1}{2\rho}-2\eta} \frac{(1+x^2)^{1/4}}{x^2 + (1+x^2)^{1/2}} |pu|^2 dx - c_4\|u\|^2\,. \tag{2.9}$$

*Proof.* Since $H_\rho^+ = P_\rho^+ H_\rho P_\rho^+$, it is sufficient to consider expectation values of $H_\rho$ on even vectors (i.e. $u = P_\rho^+ u$):

$$\Re\langle H_\rho u, u\rangle = \Re\int_R \left\{\alpha f_\rho^2 |pu|^2 + \frac{\alpha}{4}(f_\rho^2)''|u|^2 + \alpha^{-1}\xi_\rho^2(1-\rho\xi_\rho)^2|u|^2\right\} dx\,,$$

where $\alpha = e^{i(-\pi/2+\varepsilon)}$, and $f_\rho(x) = (\xi'_\rho(x))^{-1}$. Now in the region $x \leqq (2\rho)^{-1} - 2\eta$ some calculations show

$$\Re f_\rho^2 \geqq \frac{1}{16}\left\{1 - \frac{9}{4}\eta^2 \frac{(1+x^2)^{1/2}}{(x^2 + (1+x^2)^{1/2})^2}\right\}\,, \tag{2.10}$$

and similarly

$$\Im f_\rho^2 \geqq \left(\frac{2\eta}{16}\right) \frac{(1+x^2)^{1/4}}{x^2 + (1+x^2)^{1/2}}\,. \tag{2.11}$$

In the interval $(2\rho)^{-1} - \eta \leqq x \leqq (2\rho)^{-1} + \eta$ we have simply $\xi'_\rho = 1$, so that $\Re(\alpha f_\rho^2) = \sin\varepsilon$. Therefore, since $f_\rho^2$ and $(pu)(\overline{pu})$ are both even with respect to $(2\rho)^{-1}$, by (2.10) and (2.11) we obtain

$$\Re\int_{-\infty}^{+\infty} \alpha(pf_\rho^2 pu)\bar{u}\,dx = \Re\int_{-\infty}^{+\infty} \alpha f_\rho^2(pu)(\overline{pu})\,dx = 2\Re\int_{-\infty}^{\frac{1}{2\rho}} \alpha f_\rho^2|pu|^2 dx$$

$$\geqq c_3 \int_{-\infty}^{\frac{1}{2\rho}-2\eta} \frac{(1+x^2)^{1/4}}{x^2+(1+x^2)^{1/2}}|pu|^2 dx \tag{2.12}$$

for some $c_3 > 0$. Besides, for some $c_4 \in R$,

$$\Re\int_R \left\{\frac{\alpha}{4}(f_\rho^2)''|u|^2 + \alpha^{-1}\xi_\rho^2(1-\rho\xi_\rho)^2|u|^2\right\} dx \geqq -c_4\|u\|^2\,, \tag{2.13}$$

since $|(f_\rho^2)''|$ is bounded and $\Re V(\xi_\rho(x))$ is bounded below.

On the basis of the above estimate, the hypotheses (1′) through (4′) of Theorem A2 are verified in the following lemmas.

**Lemma 6.** *Let*

$$\xi_o(x) = x - i\eta\,\mathrm{atan}\frac{x}{[1+x^2]^{1/4}} \quad \forall x \in R, \qquad f_o(x) = (\xi'_o(x))^{-1} \tag{2.14}$$

*and let*

$$H_0 = e^{-i\pi/2}\left\{pf_o^2 p + \frac{1}{4}(f_o^2)''\right\} + e^{i\pi/2}\xi_o^2 \tag{2.15}$$

*be the dilated harmonic oscillator with eigenvalues* $1, 3, \ldots$. *Then hypotheses* (1′a), (1′b) *of Theorem* A2 *are fulfilled.*

*Proof.* The property (1′a) follows from the fact that $\xi(x) \to \xi_o(x)$, as $\rho \to 0$, uniformly on compacts. As for (1′b), it follows from Lemma 5 that the numerical range of $H_\rho^+$ is contained in some $\rho$-independent right half-plane $S$. Since the spectrum of $H_\rho^+$ consists only of eigenvalues, $\forall z \notin S$ we have $\|(z - H_\rho^+)^{-1}\| \leqq \{\mathrm{dist}(z, S)\}^{-1}$, and thus $z \in \Delta^+$.

**Definition 7.** *Let* $\chi \in C_o^\infty$ *with* $\chi(x) = 1$ *for* $|x| \leqq 1$, $\chi(x) = 0$ *for* $|x| \geqq 2$ *and* $0 \leqq \chi(x) \leqq 1$, $\forall x \in R$. *For* $n \in N$, *let* $\chi_n(x) := \chi(\frac{x}{n})$. *If* $2n < (2\rho)^{-1}$, *define*

$$\chi_n^\rho(x) := \chi_n(x) + \chi_n(\rho^{-1} - x)$$

*and*

$$M_n^\rho(x) := 1 - \chi_n^\rho(x)\,.$$

*Remark.* (R1) The definition of $\chi_n^\rho$ when $2n \geqq (2\rho)^{-1}$ can be given as in [Ca-Gr-Ma3], but the only important case is the above one. The function $\chi_n^\rho \in C_o^\infty$ is even w.r.t. $(2\rho)^{-1}$. $M_n^\rho$ is in the range of $P_\rho^+$, too, and it is supported away from the wells.

**Lemma 8.** *Assumption* (3′) *of Theorem* A2 *is fulfilled by the multiplication operators* $\chi_n^\rho$.

*Proof.* Since ($3'$) is trivial for odd $u$, let $u \in \operatorname{Range} P_\rho^+$ and let $\gamma_{2n}$ be the characteristic function of $[-2n, 2n]$:

$$\|[H_\rho^+, \chi_n^\rho]u\| = \|[H_\rho, \chi_n^\rho]u\| = 2\left(\int_{-\infty}^{(2\rho)^{-1}} \gamma_{2n}|[pf_\rho^2 p, \chi_n]u|^2 dx\right)^{1/2} . \tag{2.16}$$

Now

$$|[pf_\rho^2 p, \chi_n]u| = |(n^{-1}(f_\rho^2)'\chi_n' + n^{-2}f_\rho^2\chi_n'' - 2if_\rho^2\chi_n' p)u| \leqq \frac{k}{n}(|u| + |pu|) , \tag{2.17}$$

since $\chi_n, \chi_n', f^2, (f^2)'$ are all bounded functions. Hence, if $\|u\| = 1$, the preceding expression is estimated by:

$$\leqq c_5 \frac{1}{n}\left\{\left(\int_{-2n}^{2n} |pu|^2 \frac{(1+x^2)^{1/4}}{x^2 + (1+x^2)^{1/2}} \cdot \frac{x^2 + (1+x^2)^{1/2}}{(1+x^2)^{1/4}} dx\right)^{1/2} + 1\right\}$$

$$\leqq c_6 n^{-1/4}\{\Re\langle H_\rho u, u\rangle + c + 1\} ,$$

where the last inequality follows from (2.9). So the lemma is proved.

**Lemma 9.** *Assumption* ($2'$) *of Theorem* A2 *is fulfilled by the multiplication operators* $\chi_n^\rho$.

*Proof.* Given the vectors $u_m$ such that $P_{\rho_m}^+ u_m = u_m$, the superscript "+" can be neglected since $(H_o - H_{\rho_m}^+)u_m = (H_o - H_{\rho_m})u_m$.

Now, let $H_\rho' = \alpha^{-1}H_\rho$ and $\lambda \in C - \sigma(H_o')$ be fixed. Then

$$\|\chi_n^{\rho_m} u_m\|^2 = 2 \int_{-\infty}^{(2\rho_m)^{-1}} |\chi_n(x) u_m(x)|^2 dx = 2\|\chi_n u_m\|^2$$

$$\leqq c\{\|\chi_n R_o'(H_o' - H_{\rho_m}')u_m\|^2 + \|\chi_n R_o'(H_{\rho_m}' - \lambda)u_m\|^2\} . \tag{2.18}$$

Now, if the characteristic function of $[-2n, 2n]$, is denoted by $\gamma_{2n}$, the same arguments of the proof of Lemma 5 in [Ca-Gr-Ma4] allow to conclude that (2.17) tends to 0 as $n \to \infty$. Thus Lemma 9 is proved.

**Lemma 10.** *Any fixed eigenvalue of the harmonic oscillator satisfies hypothesis* ($4'$) *of Theorem* A2, *i.e. it has positive distance from the asymptotic numerical range relative to* $H_\rho^+$ *and to the multiplication operators* $M_n^\rho$.

*Proof.* By Lemma 4,

$$\Re\langle V(\xi(x))M_n^\rho u, M_n^\rho u\rangle \geqq \frac{c_1}{R} + c_2 > d > 0$$

for any $u$ with $\|M_n^\rho u\| = 1$, if $R$ is chosen sufficiently small. The kinetic part of $H_\rho$ is bounded from below by the proof of Lemma 5, therefore the lemma is proved.

**Theorem 11.** *Any eigenvalue* $\lambda_0$ *of the harmonic oscillator* $H_o$ *is stable in the sense of Kato with respect to the family* $K_\rho^+$ *as* $\rho \to 0$ (*and, similarly, eigenvalue stability holds with respect to the odd version of the double well operator* $K_\rho^-$).

*Proof.* A direct application of Theorem A2, whose hypotheses are verified by the above lemmas.

**Theorem 12.** *Let* $E_n = 2n + 1$ *be an eigenvalue of the harmonic oscillator, for fixed* $n \in \mathbf{N}$. *There is* $R_n > 0$ *such that two distinct simple eigenvalues* $E^{\pm}(g)$ *of the symmetric double well operator* (2.1) *exist and are analytic in the region*

$$D_{R_n} = \{g \in C: \Re g^{-2} > R_n^{-1}\}$$

(*a Nevanlinna disk in the* $g^2$*-plane*).

*Proof.* In Sect. 2 of [Ca-Gr-Ma3] such analyticity was proved in regions

$$\{g \in C: |\arg(g)| < \pi/4 - \varepsilon, \ |g| < k(\varepsilon)\} ,$$

where the dependence $k(\varepsilon)$ was unknown. Theorem 11 ensures stability and hence the absence of level crossings for $g^2$ near the origin in the boundary of some Nevanlinna disk, and this completes the proof of analyticity in the whole region.

## 3. Asymmetric Double Well Eigenvalues: The Case $p^2 + x^2(1 - gx)^2 - j(gx - \frac{1}{2})$, Where $j$ is not Even

In this section we consider an operator family in which both stable eigenvalues and "dying" eigenvalues are expected. To prove stability in these cases, Theorem A1 is not sufficient, because its original proof [Vo-Hu] uses the absence (i) of dying eigenvalues as a step towards the stability property (ii).

**Theorem 13.** *Let* $\Omega$ *be an open subset of* $\mathbf{C}$ *and let* $\{H_\rho\}_{\rho \geqq 0}, \{K_\rho\}_{\rho \geqq 0}$ *be two families of Schrödinger type operators with a common core* $C_o^\infty(R)$ *and*

$$\sigma_{\text{ess}}(H_\rho) \cap \Omega = \sigma_{\text{ess}}(K_\rho) \cap \Omega = \emptyset ,$$

$$\sigma(H_o) \cap \sigma(K_o) \cap \Omega = \emptyset . \tag{3.1}$$

*Moreover, let the following conditions be satisfied*:

$$K_\rho = U_\rho H_\rho U_\rho^{-1}, \quad \textit{for some unitary operator } U_\rho, \ \rho > 0 , \tag{3.2}$$

*and let there exist, in* $\Omega$, $\lambda_o \in \sigma(H_o) - \sigma(K_o)$, $\lambda_o' \in \sigma(K_o) - \sigma(H_o)$.
*Moreover*:

1) $H_\rho u \to H_o u$, $K_\rho u \to K_o u$ *as* $\rho \to 0$, $\forall u \in C_o^\infty$ *and similarly for the adjoints* $H_\rho^*, K_\rho^*$;

2) *there are bounded multiplication operators* $\chi_n^\rho$ *such that*

$$\begin{aligned}(\rho_m \to 0^+, \ u_m \in C_o^\infty, \ \|u_m\| \to 1, \ u_m \to 0 \textit{ weakly} ,\\ U_{\rho_m} u_m \to 0 \textit{ weakly}, \ \|H_{\rho_m} u_m\| \leqq C)\\ \Rightarrow (\exists m = m(n) : \lim_{n \to \infty} \|\chi_n^{\rho_m} u_m\| = 0) ;\end{aligned} \tag{3.3}$$

3)

$$\exists \varepsilon_n \to 0 : \|[H_\rho, \chi_n^\rho]u\| \leqq \varepsilon_n(\|H_\rho u\| + \|u\|), \quad \forall u \in C_o^\infty, \ 0 \leqq \rho \leqq \rho_o \tag{3.4}$$

*and the analogous commutator estimate holds for* $H_\rho^*$;

4) *denoting* $M_n^\rho = 1 - \chi_n^\rho$, *for all* $\lambda \in \Omega$ *we have*

$$\operatorname{dist}(\lambda, \langle H_\rho M_n^\rho u, M_n^\rho u\rangle) \geqq d > 0, \quad \forall n \geqq n_0,\ 0 < \rho \leqq \rho_o\,, \tag{3.5}$$

$\forall u \in C_o^\infty$ *such that* $\|M_n^\rho u\| = 1$;

5) $U_\rho \to 0$ *weakly as* $\rho \to 0$, $M_n^\rho \to 0$ *strongly as* $n \to \infty$.

*Then*, $\forall \lambda \in \Omega$,

i) $\lambda \notin \sigma(H_o) \cup \sigma(K_o) \Rightarrow \lambda \in \Delta$, *where* $\Delta$ *appears in* (3.6),

ii) $\lambda \in \sigma(H_o) \Rightarrow \lambda$ *is a stable eigenvalue w.r.t.* $H_\rho$, *as* $\rho \to 0^+$,

iii) $\lambda \in \sigma(K_o) \Rightarrow \lambda$ *is a stable eigenvalue w.r.t.* $K_\rho$, *as* $\rho \to 0^+$.

*In particular if* $\lambda_o \in \sigma(H_o)$ *and* $\lambda'_o \in \sigma(K_o)$, *then* $H_\rho$ *admits two distinct families of eigenvalues* $\lambda_\rho \to \lambda_o$ *and* $\lambda'_\rho \to \lambda'_o$ *as* $\rho \to 0^+$.

*Proof.* Let

$$\Delta = \{\lambda \in \mathbf{C}\colon (\lambda - H_\rho)^{-1} \text{ exists and is uniformly bounded as } \rho \to 0^+\}\,. \tag{3.6}$$

The proof consists of the following steps.

(a) Let $\lambda \in \Omega - (\sigma(H_o) \cup \sigma(K_o))$; then $\lambda \in \Delta$ unless there exist two sequences $\rho_m$, $u_m$ such that

$$\rho_m \to 0^+,\ u_m \in D(H_{\rho_m}),\ \|u_m\| \to 1,\ u_m \to 0 \text{ weakly},\ v_m \equiv U_{\rho_m} u_m \to 0 \\ \text{weakly},\ \|(\lambda - H_{\rho_m})u_m\| \to 0,\ \|(\lambda - K_{\rho_m})v_m\| \to 0\,. \tag{3.7}$$

To prove this fact one can proceed as in Lemma 5.1 of [Vo-Hu]: we only have to verify that $\|(\lambda - K_{\rho_m})v_m\| \to 0$ (a consequence of $\|(\lambda - H_{\rho_m})u_m\| \to 0$), and that $v_m \to 0$ weakly.

By passing, if necessary, to a subsequence, assume that $v_m \to v$ weakly, so that, $\forall \psi \in C_o^\infty$,

$$0 = \lim_m \langle \psi, (\lambda - K_{\rho_m})v_m\rangle = \langle (\lambda - K_o)^* \psi, v\rangle\,. \tag{3.8}$$

This implies $v = 0$ and the assertion (a) is proved.

(b) $\Omega - (\sigma(H_o) \cup \sigma(K_o)) \subset \Delta$. Indeed, if $\lambda \notin \sigma(H_o) \cup \sigma(K_o)$ and $\lambda \notin \Delta$, there exist two sequences $\rho_m$, $u_m$ satisfying the properties asserted in step (a). Now we prove that the same properties are satisfied by the sequences $\rho_m$, $M_n^{\rho_m} u_m$, by a suitable choice of $m(n)$, as $n \to +\infty$.

By hypothesis (2), for any $n$, $\lim_m \|M_n^{\rho_m} u_m\| = 1$. Thus (a) holds for the two modified sequences if we can verify the weak convergence to zero of both $M_n^{\rho_m} u_m$ and $U_{\rho_m} M_n^{\rho_m} u_m$ (with $m = m(n)$) as $n \to \infty$. Such weak convergence takes place because, by hypothesis (5), both $M_n^\rho$ and $\tilde{M}_n^\rho \equiv U_\rho M_n^\rho$ tend strongly to 0, as $n \to \infty$, uniformly in $\rho$.

Therefore hypothesis (4) is contradicted and necessarily $\lambda \in \Delta$.

(c) If $\lambda \in \sigma_d(H_o)$ then it is a stable eigenvalue with respect to the family $H_\rho$ as $\rho \to 0^+$. To prove this stability, one can proceed in analogy with Theorem 5.4 of [Vo-Hu], except for the following modifications. For $\rho \geqq 0$ let $P(\rho) = \int_{|z - \lambda_o| = r} (z - H_\rho)^{-1} dz$, where $r$ is small enough so that the integration cycle is contained in $\Delta$ (this is possible by step (b)), and so that the only point of the spectrum contained in it is $\lambda_0$. Assuming ab absurdo that there exist two sequences $\rho_m \to 0^+$, $u_m \in L^2(\mathbf{R})$ such that

$$\|u_m\| = 1, \qquad P(\rho_m)u_m = u_m, \qquad P(0)u_m = 0\,, \tag{3.9}$$

the contradiction is obtained in analogy with the cited proof in [Vo-Hu 1982], on condition that $v_m \equiv U_{\rho_m}u_m \to 0$ weakly. Setting $Q(\rho) = \int_{|z-\lambda_o|=r}(z - K_\rho)^{-1}dz$, it follows from (3.9) that $Q(\rho_m)v_m = v_m$. By passing to a subsequence if necessary, one has $v_m \to v$ weakly, and hence $Q(0)v = v$, because $Q(\rho) \to Q(0)$ strongly. On the other hand $Q(0)$ is the null operator on condition that $r$ is chosen so that the integration cycle does not encircle any eigenvalue of $K_0$: this is possible because $\lambda_o \in \sigma_d(H_o) - \sigma(K_o)$. Therefore $v = 0$ and the assertion (c) is proved.

(d) If $\lambda' \in \sigma_d(K_o)$ then it is a stable eigenvalue with respect to the family $K_\rho$ as $\rho \to 0^+$. The proof of (d) proceeds in full analogy with steps (a)–(c) by interchanging $H_\rho$ with $K_\rho$ and $U_\rho$ with $U_\rho^{-1}$, on condition that properties analogous to (2), (3) are proved for what concerns $K_\rho$:

2′) the functions $\tilde{\chi}_n^\rho \equiv U_\rho \chi_n^\rho$, considered as multiplication operators are such that

$$(\rho_m \to 0^+,\ v_m \in C_o^\infty,\ \|v_m\| \to 1,\ v_m \to 0 \text{ weakly},$$

$$U_{\rho_m}^{-1} v_m \to 0 \text{ weakly},\ \|K_{\rho_m} v_m\| \leqq C)$$

$$\Rightarrow (\exists m = m(n)\colon \lim_{n\to\infty} \|\tilde{\chi}_n^{\rho_m} v_m\| = 0)\,;$$

3′)

$$\exists \varepsilon_n \to 0\colon \|[K_\rho, \tilde{\chi}_n^\rho]u\| \leqq \varepsilon_n(\|K_\rho u\| + \|u\|), \quad \forall u \in C_o^\infty,\ 0 < \rho \leqq \rho_o\,,$$

and the analogous commutator estimate holds for $K_\rho^*$.

Now (2′) and (3′) can be drawn from (2) and (3) by using the unitary transform $U_\rho$. Hence the assertion (d) follows and the theorem is proved.

*Remark.* (R2) From now on, let $2\mathbf{Z}$ be the set of even integers. By this theorem the spectrum of

$$H_g = p^2 + x^2(1 - gx)^2 - j\left(gx - \frac{1}{2}\right), \quad g > 0,\ j \in \mathbf{R} - 2\mathbf{Z}\,, \tag{3.10}$$

which is equivalent through the unitary translation operator $U_g u(x) = U(x + g^{-1})$ to

$$K_g = p^2 + x^2(1 + gx)^2 - j\left(gx + \frac{1}{2}\right), \tag{3.11}$$

is the union of two families of eigenvalues: $\lambda_n(g)$ and $\lambda_n'(g)$ such that $\lambda_n(g) \to 2n + 1 + j/2$, $\lambda_n'(g) \to 2n + 1 - j/2$. In other words the two distinct limiting operators $p^2 + x^2 + j/2$ and $p^2 + x^2 - j/2$ are used to display such families. In particular there exist simple isolated "dying eigenvalues" $\lambda_n'(g)$ as $g \to 0^+$.

When $2n + 1 = 2m + 1 - j$ for some $n, m$ (case $j \in 2\mathbf{Z}$), an asymptotic degeneration is expected analogous to the case of the symmetric double well $j = 0$.

Now, by Theorem 13, we can treat the operator (1.3) for non-even parameter $j$: in particular we shall extend the results quoted in Remark (R2) from real $g \to 0^+$ to complex-valued $g \to 0$ in some Nevanlinna domain $D_R = \{g\colon \Re g^{-2} > R^{-1}\}$.

If $g = \rho e^{i\theta}$, by the usual scaling $x \to xe^{-i\theta}$, the operator (1.3) is transformed into $e^{2i\theta}p^2 + e^{-2i\theta}x^2(1-\rho x)^2 - j(\rho x + \frac{1}{2})$.

To study such an operator on the boundary of the above Nevanlinna disk, it is sufficient to set $\theta = -\pi/4 + \varepsilon/2$ with $\sin\varepsilon = R^{-1}\rho^2$, for fixed $R > 0$. Then

$$H_\rho = e^{2i\theta}p^2 + e^{-i\theta}x^2(1-\rho x)^2 - j\left(\rho x - \frac{1}{2}\right) \tag{3.12}$$

has real part

$$\Re H_\rho = \sin\varepsilon\{p^2 + x^2(1-\rho x)^2\} - j\left(\rho x - \frac{1}{2}\right) . \tag{3.13}$$

In the preceding section the distorsions $x \to \xi_\rho(x)$, for $\rho \geqq 0$, were introduced, setting $f_\rho(x) = (\xi'_\rho(x))^{-1}$. Now we introduce the analogous distorsion $\zeta_\rho$ so that the role of $x = 0$, $x = \frac{1}{\rho}$ be played by $x = -\frac{1}{\rho}$, $x = 0$ respectively:

**Definition 14.** *Define*

$$\zeta_\rho(x) := \xi_\rho\left(x + \frac{1}{\rho}\right) - \frac{1}{\rho} \quad \textit{and} \quad g_\rho(x) := (\zeta'_\rho(x))^{-1} \tag{3.14}$$

*for* $\rho > 0$. *Similarly, let*

$$\zeta_o(x) := \lim_{\rho\to 0^+} \zeta_\rho(x) = x - i\eta\,\mathrm{atan}\frac{x}{[1+x^2]^{1/4}}, \quad g_o(x) := (\zeta'_o(x))^{-1} . \tag{3.15}$$

As a consequence, $\Re\zeta_\rho(x) \equiv x$ and $\Im\zeta_\rho(x)$ is odd with respect to $-\frac{1}{2\rho}$.

**Lemma 15.** *Let* $j \in \mathbf{R} - 2\mathbf{Z}$. *If* $\rho > 0$, *the two operators defined on* $D(p^2) \cap D(x^4)$,

$$H_\rho := pf_\rho^2 p + \frac{1}{4}(f_\rho^2)'' + \xi_\rho(x)^2[1 - \rho\xi_\rho(x)]^2 - j\left[\rho\xi_\rho(x) - \frac{1}{2}\right] , \tag{3.16}$$

$$K_\rho := pg_\rho^2 p + \frac{1}{4}(g_\rho^2)'' + \zeta_\rho(x)^2[1 + \rho\zeta_\rho(x)]^2 - j\left[\rho\zeta_\rho(x) + \frac{1}{2}\right] , \tag{3.17}$$

*are unitarily equivalent via the translation* $T_{\frac{1}{\rho}}u(x) = u(x + \frac{1}{\rho})$. *Their two distinct (strong resolvent) limits, as* $\rho \to 0$, *are*

$$H_o = pf_o^2 p + \frac{1}{4}(f_o^2)'' + \xi_o^2 + j/2, \qquad K_o := pf_o^2 p + \frac{1}{4}(f_o^2)'' + \xi_o(x)^2 - j/2$$

*with eigenvalues* $\{2n + 1 + j/2\}_{n\in\mathbf{N}}$ *and* $\{2n + 1 - j/2\}_{n\in\mathbf{N}}$ *respectively.*

*Proof.* Dropping for simplicity the subscript $\rho$, we have $\xi(x + \frac{1}{\rho}) = \zeta(x) + \frac{1}{\rho}$. Thus $f(x + \rho^{-1}) = [\xi'(x + \rho^{-1})]^{-1} = [\zeta'(x)]^{-1} = g(x)$ and

$$\begin{aligned} &\xi(x+\rho^{-1})^2[1 - \rho\xi(x+\rho^{-1})]^2 - j\left[\rho\xi(x+\rho^{-1}) - \frac{1}{2}\right] \\ &= \zeta(x)^2[1 + \rho\zeta(x)]^2 - j\left[\rho\zeta(x) + \frac{1}{2}\right] . \end{aligned} \tag{3.18}$$

Therefore $T_{\frac{1}{\rho}} H_\rho T_{-\frac{1}{\rho}} = K_\rho$. As for the limits as $\rho \to 0$, notice that $\zeta_o(x) = \xi_o(x)$, and $f_o(x) = g_o(x)$ $\forall x$. Thus the lemma is proved.

**Lemma 16.** *Let*

$$V(x) = x^2(1-\rho x)^2 - j\left(\rho x - \frac{1}{2}\right), \qquad V_K(x) = x^2(1+\rho x)^2 - j\left(\rho x + \frac{1}{2}\right). \tag{3.19}$$

*For some* $c_1 > 0$, $c_2$, $c \in \mathbf{R}$, $\forall u, v$ *with* $\|u\| = \|v\| = 1$,

$$\Re V[\xi(x)] \geqq -c, \qquad \Re V_K[\zeta(x)] \geqq -c, \tag{3.20}$$

$$\Re\langle V[\xi(x)]u, u\rangle \geqq \frac{c_1}{R} + c_2, \qquad \Re\langle V_K[\zeta(x)]v, v\rangle \geqq \frac{c_1}{R} + c_2, \tag{3.21}$$

*if* $\operatorname{supp} u \cap [(-n,n) \cup (\frac{1}{\rho} - n, \frac{1}{\rho} + n)] = \emptyset$ *and* $\operatorname{supp} v \cap [(-\frac{1}{\rho} - n, -\frac{1}{\rho} + n) \cup (-n,n)] = \emptyset$.

*Proof.* The first inequality in (3.21) is an easy variant of Lemma 4. The second estimate can be reduced to the first one by using the identity $V_K[\zeta_\rho(x)] = V[\xi(x + \rho^{-1})]$ and by unitarity of $T_{\frac{1}{\rho}}$. Moreover the potential is globally bounded from below, thus the lemma is proved.

**Lemma 17.** *For the expectation values of the kinetic part we have*

$$\begin{aligned}\Re\langle (H_\rho - V[\xi(x)])u, u\rangle &\geqq c_3 \left\{ \int_{-\infty}^{\frac{1}{2\rho}-2\eta} \frac{(1+x^2)^{\frac{1}{4}}}{x^2 + (1+x^2)^{\frac{1}{2}}} |pu|^2 dx + \varepsilon \int_{\frac{1}{2\rho}-\eta}^{\frac{1}{2\rho}+\eta} |pu|^2 dx \right. \\ &\left. + \int_{\frac{1}{2\rho}+2\eta}^{+\infty} \frac{[1+(x-\frac{1}{\rho})^2]^{\frac{1}{4}}}{(x-\frac{1}{\rho})^2 + [1+(x-\frac{1}{\rho})^2]^{\frac{1}{2}}} |pu|^2 dx \right\} - c_4 \|u\|^2 .\end{aligned} \tag{3.22}$$

*Proof.* One can proceed like in the proof of Lemma 5 for what concerns the intervals $(-\infty, \frac{1}{2\rho} - 2\eta)$ and $(\frac{1}{2\rho} - \eta, \frac{1}{2\rho} + \eta)$. The difference from Lemma 6 is that $H_\rho^+$ (the even version of $H_\rho$ with respect to $\frac{1}{2\rho}$) is now replaced by $H_\rho$ itself. Therefore the interval $(\frac{1}{2\rho} + 2\eta, +\infty)$ does not simply double the previous contribution, but gives rise to a similar integral, with $x - \frac{1}{\rho}$ in place of $x$. So the lemma is proved.

**Lemma 18.**

$$\begin{aligned}\Re\langle K_\rho u, u\rangle &\geqq -c_6\|u\|^2 + c_5 \left\{ \int_{-\infty}^{-\frac{1}{2\rho}-2\eta} \frac{[1+(y+\frac{1}{\rho})^2]^{\frac{1}{4}}}{(y+\frac{1}{\rho})^2 + [1+(y+\frac{1}{\rho})^2]^{\frac{1}{2}}} |pu(y)|^2 dy \right. \\ &\left. + \varepsilon \int_{-\frac{1}{2\rho}-\eta}^{-\frac{1}{2\rho}+\eta} |pu(y)|^2 dy + \int_{-\frac{1}{2\rho}+2\eta}^{+\infty} \frac{[1+y^2]^{\frac{1}{4}}}{y^2 + [1+y^2]^{\frac{1}{2}}} |pu(y)|^2 dy \right\}.\end{aligned} \tag{3.23}$$

*Proof.* The left-hand side of the inequality can be written

$$\Re\langle T_{\frac{1}{\rho}} H_\rho (T_{\frac{1}{\rho}})^{-1} u\rangle = \Re\langle H_\rho (T_{\frac{1}{\rho}})^{-1} u, (T_{\frac{1}{\rho}})^{-1} u\rangle \,. \tag{3.24}$$

So the proof is reduced to Lemmas 16, 17 if we set $(T_{\frac{1}{\rho}})^{-1}u(x) = u(x - \frac{1}{\rho}) = u(y)$ in place of $u(x)$.

**Lemma 19.** *Let* $\chi_n^\rho$ *be the multiplication operators of Definition* 8. *Then*

$$\|[H_\rho, \chi_n^\rho]u\| \leqq cn^{-\frac{1}{4}}\{\|H_\rho u\| + \|u\|\} \,. \tag{3.25}$$

*Proof.* Calling $\gamma_{2n}^\rho$ the characteristic function of $[-2n, 2n] \cup [\frac{1}{\rho} - 2n, \frac{1}{\rho} + 2n]$, we proceed in analogy with Lemma 8. Since $\operatorname{supp} \chi_n^\rho \subset [-2n, 2n] \cup [\frac{1}{\rho} - 2n, \frac{1}{\rho} + 2n]$, by (2.17) we have

$$\|[H_\rho, \chi_n^\rho]u\| \leqq \frac{c}{n}\left[\left\{\int_{-2n}^{2n} |pu|^2 dx + \int_{\frac{1}{\rho}-2n}^{\frac{1}{\rho}+2n} |pu|^2 dx\right\}^{\frac{1}{2}} + 1\right]$$

$$\leqq cn^{-1/4}\left[\left\{\int_{-2n}^{2n} \frac{(1+x^2)^{1/4}}{x^2 + (1+x^2)^{\frac{1}{2}}} |pu|^2 dx\right.\right.$$

$$\left.\left. + \int_{\frac{1}{\rho}-2n}^{\frac{1}{\rho}+2n} \frac{[1 + (x - \frac{1}{\rho})^2]^{1/4}}{(x - \frac{1}{\rho})^2 + [1 + (x - \frac{1}{\rho})^2]^{1/2}} |pu|^2 dx\right\}^{1/2} + 1\right]$$

$$\leqq cn^{-1/4}[\Re\langle H_\rho u, u\rangle + c' + 1] \,. \tag{3.26}$$

Indeed, to obtain the last inequality, we have used (3.22) and Lemma 16.

**Lemma 20.** *If* $\chi_n^\rho$, *are the multiplication operators of Definition* 8,

$$(u_m \to 0 \text{ weakly},\ \|u_m\| \to 1,\ \rho_m \to 0,\ U_{\rho_m} u_m \to 0 \text{ weakly},\ \|H_{\rho_m} u_m\| \leqq c)$$

$$\Rightarrow (\exists m = m(n)\colon \|\chi_n^{\rho_m} u_m\| \to 0,\ \text{as } n \to \infty) \,. \tag{3.27}$$

*Proof.* Without loss of generality, let us consider $\rho$ small enough so that $\chi_n^\rho(x) = \chi_n(x) + \chi_n(x - \rho^{-1})$. Let $v_m(x) = u_m(x + \rho^{-1})$. Since $\operatorname{supp}(\chi_n) = [-2n, 2n]$,

$$\|\chi_n^\rho u_m\|^2 \leqq \|\chi_n u_m\|^2 + \|\chi_n v_m\|^2 \,. \tag{3.28}$$

As for the first summand, setting $R_\rho = (H_\rho - \lambda)^{-1}$, for $\rho \geqq 0$ as in Lemma 9, we have

$$\|\chi_n u_m\|^2 \leqq 2\{\|\chi_n R_o (H_o - H_{\rho_m}) u_m\|^2 + \|\chi_n R_o (H_{\rho_m} - \lambda) u_m\|^2\} \,. \tag{3.29}$$

From here on one can proceed as above in the proof of Lemma 9. Indeed, the required inequality on commutators (Lemma 19) is still valid if $\chi_n^\rho$ is replaced simply by $\chi_n$. Therefore $\|\chi_n u_m\| \to 0$ for some suitable sequence $m = m(n)$, as $n \to \infty$. To show that $\|\chi_n v_m\| \to 0$ the procedure is analogous and the theorem is proved.

**Theorem 21.** *Let $\{H_\rho\}_{\rho \geqq 0}$ and $\{K_\rho\}_{\rho \geqq 0}$ be the operator families of Lemma* 15 *(i.e. j not even). Then the eigenvalues $\lambda_n(0) = 2n + 1 + j/2$ of $H_o$ are stable with respect to $H_\rho$ as $\rho \to 0$. Moreover, for small $\rho$, $H_\rho$ admits further eigenvalues $\lambda_n'(\rho)$ which tend to the eigenvalues of $K_o$ as $\rho \to 0$.*

*Proof.* The assertion follows from Theorem 13 because all its hypotheses are fullfilled, choosing $(U_\rho\psi)(x) = \psi(x + \rho^{-1})$. Indeed conditions (1), (2), (3) are verified by Lemmas 15, 20 and 19, respectively, while hypotheses (4), (4′) follow from Lemma 16. Thus the theorem is proved.

**Theorem 22.** *Let*

$$H(g,j) = p^2 + x^2(1 - gx)^2 - j\left(gx - \frac{1}{2}\right), \quad \textit{for } j \in \mathbf{R} - 2\mathbf{Z}\,. \tag{3.30}$$

*For any $n \in \mathbf{N}$, there is $R_n > 0$ such that two distinct families of eigenvalues $\lambda_n(g), \lambda_n'(g)$ of* (3.30) *exist and are analytic in the region $D_{R_n} = \{g \in \mathbf{C}\colon \Re g^{-2} > R_n^{-1}\}$. They are convergent to $2n + 1 + j/2$ and to $2n + 1 - j/2$ respectively as $g \to 0$ in such a domain.*

*Proof.* By Theorem 21 stability (and hence the absence of crossing) is established as $g \to 0$ along the boundary of $D_R$. Now, the estimates of all the preceding propositions are uniform with respect to $R$, for $R$ sufficiently small. Since $D_R = \bigcup_{0<r<R} \partial D_r$, analyticity of eigenvalues is verified in the whole stated domain.

## 4. Asymmetric Double Well Eigenvalues: The Case $p^2 + x^2(1 - gx)^2 - j(gx - 1/2)$ with Even $j \neq 0$

In the case of asymmetric double well with $j \in 2\mathbf{Z} - \{0\}$, the first levels, which are isolated uniformly as $g \to 0$, can be proved to be analytic in $D_R$ as in Theorem 22. As for the remaining eigenvalues, we can prove the following proposition.

**Theorem 23.** *For small $|g|$, for any pair of $n, m \in \mathbf{N}$ such that $2n + 1 = 2m + 1 - j$ the asymmetric double well oscillator*

$$H(g,j) = p^2 + x^2(1 - gx)^2 - j\left(gx - \frac{1}{2}\right), \quad \textit{for } j \in 2\mathbf{Z} - \{0\}\,, \tag{4.1}$$

*admits a family of projections $P(g,j) = \int_{|2n+1-z|=1} [z - H_g(j)]^{-1} dz$ of dimension* 2, *with analytic continuation to some Nevanlinna domain $D_R = \{g \in \mathbf{C}\colon \Re g^{-2} > R^{-1}\}$.*

*Proof.* Setting, for $\rho > 0$, $U_\rho\phi(x) = \phi(x + \rho^{-1})$,

$$H(\rho,j) = pf_\rho^2 p + \frac{1}{4}(f_\rho^2)'' + \xi_\rho(x)^2 \left[1 - \rho\xi_\rho(x)\right]^2 - j\left[\rho\xi_\rho(x) - \frac{1}{2}\right], \tag{4.2}$$

$K(\rho,j) = U_\rho H_\rho U_\rho^{-1}$, we can consider the two formal limits $H(0,j) := pf_o^2p + \frac{1}{4}(f_o^2)'' + \xi_o^2 + j/2$, $K(0,j) := H(0,j) - j$, as in the above definitions of Lemma 15.

(a) First one can reproduce the steps (a), (b) of the proof of Theorem 13, to conclude that $\mathbf{C} - [\sigma(K(0,j)) \cup \sigma(H(0,j))] \subset \Delta$, where $\Delta$ is the set of uniform boundedness of the resolvents $[z - H(\rho,j)]^{-1}$, as $\rho \to 0$. Notice that such uniform bounds occur for all $j$, whether or not $j \in \mathbf{R} - 2\mathbf{Z}$.

(b) Let us consider $j \in 2\mathbf{Z}$ as a limiting case of non-integer $j + \delta$ as $\delta \to 0$. Now, the multiplication operator $\rho x$, as well as $\rho\xi_\rho(x)$, is relatively bounded with respect to $H(\rho,j)$ uniformly for $0 \leqq \rho \leqq \rho_o$.

This can be proved by standard quadratic estimates.

(c) Setting $P(\rho,j+\delta) = \int_{|z-(2n+1+j/2)|=1} [H(\rho,j+\delta) - z]^{-1} dz$, we have

$$\dim P(\rho,j) = 2, \quad j \in 2\mathbf{Z} \text{ for small } \rho > 0 . \tag{4.3}$$

Indeed $P(\rho,j+\delta) \to P(\rho,j)$ in norm as $\delta \to 0$, uniformly with respect to $\rho$. This is a consequence of (a) and (b). Hence the projections have the same dimension, which is 2 by Theorem 22.

(d) If $\psi_n, \psi_m$ are the eigenfunctions such that $H(0,j)\psi_n = (2n+1+j/2)\psi_n$, and $H(0,j)\psi_m = (2m+1+j/2)\psi_m$, we set

$$\phi_1(\rho,j+\delta) = P(\rho,j+\delta)\psi_n, \qquad \phi_2(\rho,j+\delta) = P(\rho,j+\delta)[U_\rho\psi_m] , \tag{4.4}$$

where $U_\rho\psi_m(x) = \psi_m(x+\rho^{-1})$. Then $\phi_1, \phi_2$ are a base of Range $P(\rho,j+\delta)$ and

$$\langle \phi_1(\rho,j+\delta), H(\rho,j+\delta)\phi_2(\rho,j+\delta)\rangle \to \langle \phi_1(\rho,j), H(\rho,j)\phi_2(\rho,j)\rangle$$

as $\delta \to 0$, uniformly for small $\rho$. This convergence is a consequence of the preceding steps. An analogous convergence takes place for the couples $(\phi_1, \phi_1)$, etc.

(e) Finally, the projection $P(g) = \int_{|2n+1-z|=1} [z - H_g(j)]^{-1} dz$ has an analytic 2-dimensional continuation to the Nevanlinna domain $D_R$ for some $R > 0$.

Indeed, by step (c) dim $P(g) = 2$ if $g$ lies in the boundary of some $D_R$: this is due to the above choice of the function $\theta(\rho)$ which is the phase of $g \equiv \rho e^{i\theta}$ (see the beginning of Sect. 3). Now, since $D_R = \bigcup_{0<r<R} \partial D_r$ and since all estimates are uniform for $R$ small, $\dim[P(g)] = 2$ for $g \in D_R$.

Analyticity follows from (d) and from analyticity of projections for $j \notin 2\mathbf{Z}$ (Theorem 22). Indeed the matrix elements of $P(g)$ turn out to be the limits, as $\delta \to 0$, of analytic functions, with uniformity in $D_R$. The theorem is thus proved.

## 5. Distributional Borel Sum

In this section we are going to apply the Distributional Borel Sum (DBS from now on) to the double well problem.

Following G.'t Hooft ['t] in [Ca-Gr-Ma1, 2] a definition and a criterion were given for a DBS of a series which extends the original Borel one to critical cases. Actually the summability criterion we gave defines directly a pair of complex conjugate sums, called upper and lower Borel sums (US, LS): $\Phi(z) = \Sigma^+$, $\bar{\Phi}(\bar{z}) = \Sigma^-$, whose difference is called the discontinuity of the Borel sum (DOS): $d(z) = \Sigma^+ - \Sigma^-$, and whose mean is the DBS itself.

Before the introduction of the DBS, the limit of the usual Borel sum to the critical direction was used in various problems. For example, for the Stark effect resonances we have proved [Ca-Gr-Ma 4] that the limit from above (below) coincides with the US (LS) given by the criterion of summability proposed. It is clear that the proof of the DB summability is a stronger result than the proof of the simple existence of upper and lower limits of Borel sums. In particular it allows us to connect directly the asymptotics of the perturbation series with the asymptotics of the imaginary part and the nature of the first singularities of the Borel transform on $\mathbf{R}^+$ [Ca-Gr-Ma 4].

The problem of a DBS for the double well eigenvalues needs to be handled by considering, in the usual expression which defines the eigenvalue, a $g$-dependent test vector with a definite parity with respect to $\frac{1}{2g}$. The procedure is described in [Ca-Gr-Ma 3], and it receives its full meaning from the analyticity results of Sects. 3, 4 and 5.

1) *DBS for the symmetric double well.* The Green function of $H(g) \equiv p^2 + x^2(1 - gx)^2$ can be written as a combination:

$$G_{3,0}(x,y) = d_+ G_{3,1}(x,y) + d_- G_{3,-1}(x,y)$$
$$= \frac{1}{2}(1 + ih)G_{3,1}(x,y) + \frac{1}{2}(1 - ih)G_{3,-1}(x,y)\,. \tag{5.1}$$

Here $G_{3,1}, G_{3,-1}$ denote the Green functions of the "resonance" operators defined in [Gr-Gr] and [Ca-Gr-Ma 3],

$$Q^{\pm}(g) = e^{-i(\pm\pi - 2\theta)/3}\{p^2 + x^2(e^{i(\pm\pi-2\theta)/3} - |g|e^{\pm i\pi/2}x)\}, \quad g = |g|e^{i\theta}\,. \tag{5.2}$$

We refer to [Ca-Gr-Ma 3] for the expressions of $h(g,z)$ and $k(g,z)$ in terms of Wronskians, with the relation $h = -i(1+k)/(1-k)$. We have

**Lemma 24.** *For any eigenvalue $\lambda(0)$ of the harmonic oscillator there is $R > 0$ such that the corresponding eigenvalues $E^{\pm}(g)$ of the resonance operators $Q^+(g), Q^-(g)$ exist and are analytic for $g \in D_R \equiv \{g\colon \Re g^{-2} > R^{-1}\}$.*

*Proof.* As recently proved by Buslaev and Grecchi ([Bu-Gr], Corollary 4) the "resonances," i.e. the eigenvalues of non-modal operators $Q^{\pm}(g)$, coincide with the eigenvalues of the operator

$$A(g) = e^{i(-\pi/2+2\varepsilon/3)}\left\{p^2 + \frac{(j^2-1)}{4r^2}\right\} + e^{i(\pi/2-2\varepsilon/3)}r^2 + \rho^2 e^{i(-\pi/2+2\varepsilon/3)}r^4 \tag{5.3}$$

(where $g = \rho e^{i\theta}$, $\theta = -3\pi/4 + \varepsilon$) which represents the radial part of the $d$-dimensional quartic oscillator (with $j = j(d)$, see [Ca-Gr-Ma 4]). These eigenvalues, in turn, are analytic in the stated region by Theorem 1 of [Ca-Gr-Ma 4]. Thus the lemma is proved.

**Theorem 25.** *Let $0 < g < R^{1/2}$ and let $\lambda(g) = \lambda^{\pm}(g)$ be a double well eigenvalue (for a fixed choice of parity) admitting analytic extension to $D_R$ for some $R > 0$ as in Theorem* 12. *Let*

$$P^{\pm}(g)v(x) = 2^{-1}[v(x) \pm v(g^{-1} - x)]\,, \tag{5.4}$$

*and let*

$$\psi(g) = P^{\pm}(g)\psi\,, \qquad \psi_\alpha(g)[x] = \psi(g)[xe^{i\alpha}]\,, \tag{5.5}$$

*with the same choice of parity, + or −, with respect to* $(2g)^{-1}$. *Let* $R^{\pm}(g) = [Q^{\pm}(g) - z]^{-1}$, *denote the resolvents of the above two "resonance" operators and let* $H(g), R(g)$ *be the symmetric double well operator and its resolvent. Then*

(i) $$\lambda(g) = \frac{N(g)}{D(g)} \quad \textit{with } N(g) = F_1(g,g),\ D(g) = F_o(g,g)\,, \tag{5.6}$$

*where (for* $l = 0, 1$)

$$F_l(g,\gamma) = [\Phi_l(g,\gamma) + \bar{\Phi}_l(g,\bar{\gamma})]/2\,,$$

$$\Phi_l(g,\gamma) = (2\pi i)^{-1}\int_\Gamma z^l(1 + ih(g,z))\{\langle\psi_{-\alpha}(g), R^+(\gamma,z)\psi_\alpha(0)\rangle$$
$$+ \overline{\langle\psi_\alpha(g), R^-(\bar{\gamma},\bar{z})\psi_\alpha(0)\rangle}\}dz\,. \tag{5.7}$$

*In these expressions* $\gamma \in D_R$, $\Gamma$ *is a circle surrounding* $E(0)$ *at distance* 1, *and* $\alpha = \pi/6 - \arg(\gamma)/3$.

(ii) $$F_l(g,\gamma) = F_l^R(g,\gamma) + \frac{i}{2}d_l^I(g,\gamma)\,,$$

*where* $F_l^R(g,\gamma)$ *is the* DB *sum of* $\sum_{k=0}^{\infty} a_{lk}(g)\gamma^k$ *and* $d_l^I(g,\gamma)$ *is the Borel discontinuity of* $\sum_{k=0}^{\infty} b_{lk}(g)\gamma^k$, *for* $0 < \gamma < R^{1/2}$ *(in particular for* $\gamma = g$). *Here the coefficients* $a_{lk}(g), b_{lk}(g)$ *are*:

$$a_{lk}(g) = (2\pi i)^{-1}\int_\Gamma z^l[A_k(g,z) + \overline{A_k(g,\bar{z})}]/2\,dz\,, \tag{5.8}$$

$$b_{lk}(g) = (2\pi i)^{-1}\int_\Gamma h(g,z)z^l[A_k(g,z) + \overline{A_k(g,\bar{z})}]/2\,dz\,, \tag{5.9}$$

*where* $A_k(g,z)$ *is given by*

$$\left\langle \psi(g), R(0,z) \sum_{m=[k/2]}^{k} \binom{k}{k-m} [2x^3R(0,z)]^{2m-k}[-x^4R(0,z)]^{k-m}\psi(0) \right\rangle .$$

*Remark.* $\Phi_l(g,\gamma)$, for fixed $g$, is a Distributional Borel Upper Sum of its expansion.

*Proof.* Part (i) is the extension to the whole disk $D_R$ of the representation formulas already obtained in [Ca-Gr-Ma 3]. As for (ii), it is enough to note that $\Phi_l(g,\gamma)$ (where $g$ is fixed, $l = 0$ or 1) has the same analyticity properties of the eigenvalues of $Q^{\pm}(\gamma)$. Indeed, $R^{\pm}(\gamma,z)$ is analytic for $\gamma \in D_R$ by Lemma 24. Moreover $h(g,z)$ ($g > 0,\ z \in \Gamma$) is uniformly bounded on $\Gamma$ for small $g$, since it is convergent as $g \to 0^+$ by Theorem 3.6 of [Ca-Gr-Ma 3]. In particular we have $k(g,z) \to \exp(i(z+1)\pi)$, so that $h(g,z) \to \mathrm{cotan}((z+1)\pi/2)$ uniformly on the compact set $\Gamma$ which does not contain the singular points $z = 2n+1, n = 0, 1, \ldots$. Moreover notice that $\psi(g) \to \psi(0)$ weakly as $g \to 0$.

Finally formulas (5.8) and (5.9), which are the same as in Theorem 4.4 of [Ca-Gr-Ma 3], are now valid $\forall\gamma \in D_R$. Notice that the coefficients $a_{lk}(g)$ are directly computable, while in the expression of $b_{lk}(g)$ the factor $h(g,z)$ can be replaced by $h(0,z)$, or by any better semiclassical approximation, without destroying the summability properties discussed above.

2) *DBS for the asymmetric double well*: $j \notin 2\mathbf{Z}$. By Theorem 22 we can apply the perturbation theory for an isolated stable eigenvalue as $g \to 0^+ : \lambda_n(g) = N(g)/D(g)$, where

$$N(g) = (2\pi i)^{-1} \int_\Gamma z\langle\psi_1, R(g,z)\psi_2\rangle\, dz\,, \tag{5.11}$$

$$D(g) = (2\pi i)^{-1} \int_\Gamma \langle\psi_1, R(g,z)\psi_2\rangle\, dz \neq 0\,, \tag{5.12}$$

and $\Gamma = \{z : |\lambda_n(0) - z| = \varepsilon\}$, $0 < \varepsilon < 1$. As for the vectors, unlike the symmetric case, we can simply choose

$$\psi_1 = \psi_2 = \psi : \ \psi(x) = H_n(x)e^{-x^2/2}\,, \tag{5.13}$$

where $H_n(x)$ are the Hermite polynomials, without any dependence of $g$ and $\gamma$. Then, in analogy with the above notations, writing

$$\Phi_l(g,\gamma) = (2\pi i)^{-1} \int_\Gamma z^l(1 + ih(g,z))\langle\psi_{-\alpha}(\bar{\gamma}), R^+(\gamma,z)\psi_\alpha(\gamma)\rangle\, dz\,,$$

($l = 0, 1$; $\alpha = \pi/6 - \arg(\gamma)/3$), the following holds:

**Theorem 26.** *Let* $j \notin 2\mathbf{Z}$ *and let* $\lambda_n(g)$, $(n = 0, 1, \ldots)$ *be those eigenvalues of* $H(g,j) = p^2 + x^2(1 - gx)^2 - j(gx - 1/2)$, *as in Theorem* 22, *which are convergent to* $2n + 1 + j/2$ *as* $g \to 0^+$. *Then*

(i) *For each* $n$ *there is* $R > 0$ *such that* $\lambda_n(g) = N(g)/D(g)$ *with* $N(g) = F_1(g,g)$, $D(g) = F_o(g,g)$, *and*

$$F_l(g,\gamma) = (\Phi_l(g,\gamma) + \overline{\Phi_l(g,\bar{\gamma})})/2 \quad l = 0, 1\,, \tag{5.14}$$

*where* $\Phi_l(g,\gamma)$, *for fixed* $g$, $0 < g < R^{1/2}$, *is the Distributional Borel Upper Sum of its asymptotic expansion* $\sum_{k=0}^{\infty}[a_{l,k} + ib_{l,k}(g)]\gamma^{2k}$ *in the domain* $D_R$.

(ii) $F_l(g,\gamma)$ *can be decomposed in two terms*

$$F_l(g,\gamma) = F^R(\gamma) + \frac{1}{2} i\, d_l(g,\gamma)\,, \tag{5.15}$$

*where* $F_l^R(\gamma) = (\Phi_l^R(\gamma) + \overline{\Phi_l^R(g,\bar{\gamma})})/2$ *and* $d_l(g,\gamma) = \Phi_l^I(g,\gamma) - \overline{\Phi_l^I(g,\bar{\gamma})}$ *are the* DBS *and the* DOS *of the series* $\sum_{k=0}^{\infty} a_{l,k}\gamma^{2k}$, $\sum_{k=0}^{\infty} b_{l,k}(g)\gamma^{2k}$, *respectively. A similar result holds for* $\lambda_n'(g) \to 2n + 1 - j/2$, *using the operator* $K(g,j)$, *or equivalently* $H(g,-j)$.

*Proof.* As in Theorem 4.2 of [Ca-Gr-Ma 3].

*Remark.* (R3) As for the $b_n(g)'s$, we should recall that they depend on $h(g,z)$, which can be computed by the complex WKB method and DB sums, or may be approximated uniformly on the integration path by $h \simeq \cot((z - \lambda_n'(0))\pi/2)$ for $g$ small. Fixing $z$ as the value of the unperturbed pole, we obtain the simple approximation:

$$F_l(g,g) \simeq \left(1 + \frac{\pi}{2}\Im E^+(g)/\sin^2(\pi j/2)\right) \Sigma a_{l,k}g^{2k} + \frac{i}{2}\cot(j\pi/2)(\Delta\Sigma a_{l,k}g^{2k})\,, \tag{5.16}$$

where the second term is proportional to the imaginary part of the "resonance," which is of the order of the probability of tunneling through the whole barrier, i.e. of order $O(\exp(-2S))$, and where $S$ is the absolute value of the classical action on the barrier.

*Remark.* (R4) The complex WKB method (see [Vo]) suggests better approximations. Actually it is possible to increase the barrier by a positive $C_0^\infty$ function with support near the left well, with $\mu$ as a coefficient. In the limit as $\mu \to 0$ we get a Dirichlet problem on a half-line $[M, +\infty]$, $M \gg 0$. In particular if we define $h_D(g,z)$ for such a Dirichlet operator in the same way as $h(g,z)$, we have

$$|h_D(g,z) - h(g,z)| = O(e^{-2S})$$

for small $g$ and uniformly in $z$ on a fixed path surrounding $\lambda_n(0)$, sufficiently close to $\lambda_n(0)$ and contained in a domain of regularity of $h(0,z)$.

So we can improve the approximation (5.16):

$$F_l(g,g) \simeq (1 - h'_D[\Re E^+(g)]\Im E^+(g))\Sigma a_{l,k} g^{2k} + \frac{i}{2} h_D[\Re E^+(g)] \cdot \Delta \Sigma a_{l,k} g^{2k} . \quad (5.17)$$

3) *DBS for the asymmetric double well*: $j \in 2\mathbf{Z} - \{0\}$. By Theorem 23, for each pair $n, m$ such that $2n + 1 = 2m + 1 - j$ there are two eigenfunctions, say $\psi_n, \psi_m$ corresponding to one eigenvalue $2n + 1 + j/2$. Therefore the two perturbed eigenvalues cannot be recovered simply from a ratio of Borel sums, but from two 2 by 2 matrices depending on both $\psi_n$ and $\psi_m$:

$$\begin{pmatrix} F_{l,n,n} & F_{l,n,m} \\ F_{l,m,n} & F_{l,m,m} \end{pmatrix}, \quad l = 0, 1 . \quad (5.18)$$

Here, for example

$$F_{l,n,m}(g,\gamma) = (\Phi_l(g,\gamma) + \overline{\Phi_l(g,\overline{\gamma})})/2 , \quad (5.19)$$

where

$$\Phi_{l,n,m}(g,\gamma) = (2\pi i)^{-1} \int_\Gamma z^l (1 + ih(g,z)) \langle (\psi_n)_{-\alpha}, R^+(\gamma,z)(\psi_m)_\alpha \rangle dz$$

($\alpha = \pi/6 - \arg(\gamma)/3$).

Thus, on the basis of Theorem 23 we can state the result:

**Theorem 27.** *Under the hypothesis of Theorem* 23, *any perturbed eigenvalue* $\lambda(g)$, *for fixed and small* $g > 0$, *of the asymmetric double well operator is a solution of*

$$\det(F_{1,\cdot,\cdot}(g,g) - \lambda F_{0,\cdot,\cdot}(g,g)) = 0 , \quad (5.20)$$

*where each matrix element satisfies a decomposition of the type*:

$$F_{l,n,m}(g,\gamma) = F^R_{l,n,m}(\gamma) + \frac{1}{2} i\, d^l_{l,n,m}(g,\gamma) , \quad (5.21)$$

*i.e. it is a* DBS *of a series and a* DOS *of an imaginary series, for* $\gamma \in D_R$, *in analogy with Theorems* 25 *and* 26.

## Appendix: Stability Theorems

In the context of Schrödinger eigenvalue problems, we recall the following stability criteria in a form which is useful for both the models in this paper, and for more general applications.

**Theorem A.1.** *Let* $\Omega$ *be an open subset of* $\mathbf{C}$ *and let* $\{H_\rho\}_{\rho \geqq 0} = pf_\rho^2 p + \frac{1}{4}(f_\rho^2)'' + V_\rho(\xi_\rho(x))$ *(with* $f_\rho(x) = (\xi'_\rho(x))^{-1}$ *for some* $C^\infty$ *function* $\xi_\rho(x)$*) be an operator family in* $L^2(R)$ *for which* $C_o^\infty$ *is a core and*

$$\sigma_{\text{ess}}(H_\rho) \cap \Omega = \emptyset \,.$$

*Moreover:*

1)
$$H_\rho u \to H_o u, \quad H_\rho^* u \to H_o^* u \text{ as } \rho \to 0, \; \forall u \in C_o^\infty \,;$$

2) *there exist multiplication operators* $\chi_n^\rho$ *such that*

$$(u_m \to 0 \text{ weakly}, \; \rho_m \to 0, \; \|H_{\rho_m} u_m\| \leqq c)$$
$$\Rightarrow (\exists m = m(n) \colon \|\chi_n^{\rho_m} u_m\| \to 0, \text{ as } n \to \infty) \,;$$

3) *there is* $\{\varepsilon_n\} \to 0$ *such that*

$$\|[H_\rho, \chi_n^\rho] u\| \leqq \varepsilon_n (\|H_\rho u\| + \|u\|)$$

*and the analogous commutator estimate holds for* $H_\rho^*$ *uniformly in* $\rho$;

4) *setting* $M_n^\rho = 1 - \chi_n^\rho$, *any* $\lambda \in \Omega$ *satisfies*

$$\text{dist}(\lambda, \langle H_\rho M_n^\rho u, \; M_n^\rho u \rangle) \geqq d > 0, \quad \forall n \geqq n_o, \; 0 < \rho \leqq \rho_o$$

$\forall u \in C_o^\infty$ *such that* $\|M_n^\rho u\| = 1$.
*Then*

(i) $\lambda \notin \sigma_d(H_o) \cap \Omega \Rightarrow (H_\rho - \lambda)^{-1}$ *is uniformly bounded as* $\rho \to 0$ ,
(ii) $\lambda \in \sigma_d(H_0) \cap \Omega \Rightarrow \lambda$ *is a stable eigenvalue with respect to* $H_\rho$ .

*Proof.* It is not difficult to verify the hypotheses of Theorem 5.5 in [Vo-Hu], where they are formulated in a slightly more abstract way.

*Remarks.*

(R5) The above formulation of the theorem by Vock and Hunziker explicitly indicates how to work to prove stability in wide classes of actual problems ([Ca-Ma], [Ca-Gr-Ma3], [Ma-Sa], [Gr-Ma-Sa], [Ca-Gr-Ma 4]).

(R6) Conditions (1), (2), (3), (4) have a simple intuitive interpretation as follows: Condition (1) implies that

$$\dim P_\rho \geqq \dim P_o, \quad \text{for small } \rho > 0 \,,$$

where $P_\rho$ and $P_o$ are, respectively, the perturbed and the unperturbed eigenprojection corresponding to an eigenvalue $\lambda_o$. Conditions (2), (3), (4) are needed to prove the opposite inequality $\dim P_\rho \leqq \dim P_o$, for small $\rho > 0$, e.g. the absence of any further eigenfunction with eigenvalue in a small neighbourhood of $\lambda_o$.

In particular, as regards (2), the multiplication operators $\chi_n$ are usually $C_o^\infty$ functions supported in intervals $(-kn, kn)$, where any perturbed eigenfunction is expected to be concentrated (the "well"). Condition (2) roughly says that any possible further eigenfunction must be supported far away from the well. To prove this fact hypothesis (3) is also needed, due to the commutator of the $\chi_n$ with $H_\rho$.

Condition (4) says that the $\lambda$'s have positive distance from the asymptotic numerical range $\{\langle H_\rho M_n u, M_n u\rangle : n \geqq n_o,\ 0 < \rho < \rho_o,\ \|M_n u\| = 1\}$: by the meaning of condition (2), this means that there are no dying eigenvalues of $H_\rho$ as $\rho \to 0$.

(R7) The selfadjoint double well operator $H_\rho = p^2 + x^2(1 - \rho x)^2$, which provides the typical example of instability of eigenvalues as $\rho \to 0$, ([Re-Si]) fails to satisfy condition (4). Indeed, setting for example $\chi_n(x) = \chi(x/n)$, where $0 \leqq \chi \leqq 1$ and $\chi \in C_o^\infty$, *no* eigenvalue $\lambda_o$ of the harmonic oscillator $p^2 + x^2$ has positive distance from the asymptotic numerical range uniformly for $\rho$ small: there is *no* $d > 0$ such that

$$\operatorname{dist}(\lambda_o, \{\langle H_\rho M_n u, M_n u\rangle\}) \geqq d, \quad n \geqq n_o,\ 0 < \rho \leqq \rho_o .$$

**Theorem A.2.** *Let $\Omega$ be an open subset of* **C** *and let $\{H_\rho\}_{\rho \geqq 0}$ be an operator family in $L^2(R)$ for which $C_o^\infty$ is a core and*

$$\sigma_{\text{ess}}(H_\rho) \cap \Omega = \emptyset .$$

*Let orthogonal projections $P^\pm(\rho)$ exist with the following properties*:

a) $P^+(\rho) + P^-(\rho) = I,\ P^+(\rho)P^-(\rho) = P^-(\rho)P^+(\rho) = 0$;

b) $\|P^\pm(\rho)\| = 1$;

c) $P^\pm(\rho)H_\rho u = H_\rho P^\pm(\rho)u,\ \forall u \in D(H_\rho)$;

d) $\langle (P_\rho^+ - P_\rho^-)u, v\rangle \to 0,$ *as* $\rho \to 0,\ \forall u, v \in L^2$.

*Moreover, defining the operators*

$$H_\rho^\pm = H_\rho P_\rho^\pm, \qquad D(H_\rho^\pm) = D(H_\rho),$$

*assume*:

1′a) $H_\rho u \to H_o u,$ *as* $\rho \to 0,\ \forall u \in C_o^\infty$;

1′b) $\Delta^+ \neq \emptyset$, *where*

$$\Delta^+ = \{z \in C\colon [H_\rho^+ - z]^{-1} \text{ exists and is uniformly bounded as } \rho \to 0\},$$

2′) *there exist multiplication operators $\chi_n^\rho$ with $P^+(\rho)\chi_n^\rho = \chi_n^\rho$ such that*

$$(u_m \to 0 \text{ weakly},\ \rho_m \to 0,\ P_{\rho m}^+ u_m = u_m,\ \|H_{\rho m}^+ u_m\| \leqq c)$$

$$\Rightarrow (\exists m = m(n)\colon \|\chi_n^{\rho m} u_m\| \to 0, \text{ as } n \to \infty);$$

3′) *there is $\{\varepsilon_n\} \to 0$ such that*

$$\|[H_\rho^+, \chi_n^\rho]u\| \leqq \varepsilon_n(\|H_\rho^+ u\| + \|u\|)$$

*and the analogous commutator estimate holds for the adjoint operator, uniformly for small $\rho$*;

4′) *setting $M_n^\rho = 1 - \chi_n^\rho$, any $\lambda \in \Omega - \{0\}$ satisfies*

$$\operatorname{dist}(\lambda, \langle H_\rho^+ M_n^\rho u, M_n^\rho u\rangle) \geqq d > 0, \quad \forall n \geqq n_o,\ 0 < \rho \leqq \rho_o$$

$\forall u \in C_o^\infty$ *such that* $\|M_n^\rho u\| = 1$.

*Then*

$$\text{(i) } \lambda \notin \sigma_d(H_o) \Rightarrow (H_\rho^+ - \lambda)^{-1} \textit{is uniformly bounded as } \rho \to 0 ,$$
$$\text{(ii) } \lambda \in \sigma_d(H_o) \Rightarrow \lambda \textit{ is a stable eigenvalue with respect to } H_\rho^+ .$$

*The eigenvalue* $\lambda$ *is stable with respect to* $H_\rho^-$, *too, if hypotheses analogous to* (1′), (2′), (3′), (4′) *are satisfied by some other multiplication operators suited for* $H_\rho^-$.

*Remarks.*

(R8) The stated theorem is essentially proved in Sect. 2 of [Ca-Gr-Ma 3]. By the procedure there performed, any eigenvalue of the harmonic oscillator $H_o = p^2 + x^2$ turns out to be stable separately with respect to the odd and the even versions of the double well operator $H_\rho = p^2 + x^2(1 - \rho x)^2$:

$$H_\rho^\pm = H_\rho P^\pm(\rho), \qquad [P^\pm(\rho)u](x) = 2^{-1}[u(x) \pm u(\rho^{-1} - x)] ,$$

where the parity is with respect to the point of "barrier" $x = (2\rho)^{-1}$.

Actually in Sect. 2 of [Ca-Gr-Ma3] such stability is proved with respect to the odd and even versions of

$$H(g) = p^2 + x^2(1 - gx)^2, \quad \text{for } |\arg(g)| < \pi/4 - \varepsilon$$

for any fixed $\varepsilon > 0$. This implies analyticity of double well eigenvalues in regions

$$\{g \in C : |\arg(g)| < \pi/4 - \varepsilon, \ |g| < k(\varepsilon)\} ,$$

where the dependence $k(\varepsilon)$ is unknown. In this paper Theorem A2 will be used to prove stability and hence analyticity in a Nevanlinna disk $\Re g^{-2} > R^{-1}$, which is tangent in $g^2 = 0$ to the imaginary axis in the $g^2$-plane, for some radius $R > 0$.

(R9) Conditions (a), (b), (c), (d) are hypotheses about the symmetry of the problem: they are useful to prove stability separately with respect to $H_\rho^+$, $H_\rho^-$, just when the symmetry itself prevents stability with respect to $H_\rho$.

Hypotheses (a), (c) imply that $H_\rho^\pm u = 0$, $\forall u \in \text{Range}(P^\mp(\rho))$, i.e. 0 is an eigenvalue with infinite multiplicity. However the requirement $\lambda \neq 0$ in (4′) does not restrict the final information on the eigenvalues of $H_\rho$: by redefining the energy, the statement holds for the redefined auxiliary operators $H_\rho^\pm$.

(R10) The situation of degeneracy of this theorem is rather special due to symmetry. The generic analogous situation (i.e. without symmetry) is solved by Theorem 23.

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