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Supercoherent States, Super-Kähler Geometry and Geometric Quantization

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Abstract: Generalized coherent states provide a means of connecting square integrable representations of a semi-simple Lie group with the symplectic geometry of some of its homogeneous spaces. In the first part of the present work this point of view is extended to the supersymmetric context through the study of the OSp(2/2) coherent states. These are explicitly constructed starting from the known abstract typical and atypical representations of osp(2/2). Their underlying geometries turn out to be those of supersymplectic OSp(2/2)-homogeneous spaces. Moment maps identifying the latter with coadjoint orbits of OSp(2/2) are exhibited via Berezin's symbols. When considered within Rothstein's general paradigm, these results lead to a natural general definition of a super-Kähler supermanifold, the supergeometry of which is determined in terms of the usual geometry of holomorphic Hermitian vector bundles over Kähler manifolds. In particular, the supergeometry of the above orbits is interpreted in terms of the geometry of Einstein–Hermitian vector bundles. In the second part, an extension of the full geometric quantization procedure is applied to the same coadjoint orbits. Thanks to the super-Kähler character of the latter, this procedure leads to explicit super-unitary irreducible representations of osp(2/2) in super-Hilbert spaces of superholomorphic square-integrable sections of prequantum bundles of the Kostant type. This work lays the foundations of a program aimed at classifying Lie supergroups' coadjoint orbits and their associated irreducible representations, ultimately leading to harmonic superanalysis. For this purpose a set of consistent conventions is exhibited.

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1. Introduction

1.1. Coherent states were originally those very special quantum states of the harmonic oscillator first introduced by Schrödinger [1] and studied further by Glauber [2]. Their special character stems from their property of being the closest possible states to the classical theory. This is reflected in the fact that they minimize Heisenberg's uncertainty relations. Moreover, they form an overcomplete basis of the Hilbert space of quantum states. Since Glauber's contribution, the concept of coherent states has evolved very rapidly. The key step of this evolution is without any doubt Perelemov's group theoretical generalization [3]. Nowadays coherent states find applications in several areas of physics and mathematics. Let us mention, for instance, their occurrence in quantum optics [4], in signal analysis (where they are called wavelets) [5], and in mathematical physics (in connection with the quantization-versus-classical-limit procedures) [6, 7] (see also [3] and references therein). This last application constitutes the main interest of the present paper. More precisely, here we extend to the supersymmetric context both the coherent states approach to the evaluation of the classical limit, and the geometric quantization of coadjoint orbits. Even though these two points have already been studied in the case of the OSp(1/2) Lie supergroup in [8, 9], the analysis of a richer example, such as the OSp(2/2) Lie supergroup considered here, brings new insights that improve our understanding of the very interesting supergeometric structures underlying the so-called supercoherent states. Let us now present an overview of the subject. More details and references about coherent states methods can be found in [3, 4, 10].

1.2. The harmonic oscillator coherent states admit a group theoretical construction based on the Weyl–Heisenberg group underlying this physical system. By extending this construction to general Lie groups, Perelomov introduced the notion of generalized coherent states [3]. For a given Lie group G and a given unitary irreducible representation U of G in some Hilbert space $\mathscr{H}$, the generalized coherent states are the states belonging to a $U(G)$-orbit in $\mathscr{H}$ through a chosen initial state, usually called the fiducial state. The generalized coherent states so defined are too general to share the interesting properties of the harmonic oscillator ones. Indeed, the over-completeness property holds only when U is a square integrable representation, and the uncertainty relations associated with the commutation relations of the Lie algebra $\mathfrak{g}$ of G are minimized only when the fiducial state is an extremal-weight state [3, 11, 12]. In what follows we will only consider this type of generalized coherent states and we will simply designate them by coherent states (CS) or G-coherent states (G-CS).

1.3. Not all Lie groups possess square integrable representations. Unfortunately, most physically interesting groups, such as the Poincaré group, do not possess such representations. There is however enough room to actually take advantage of the particularly rich properties of CS. Indeed, all representations of compact semi-simple Lie groups and all discrete series representations of the non-compact semi-simple Lie groups are square integrable. Moreover, physical interpretations can be attached to the associated CS. For instance, SU(2)-CS [13] allow a semi-classical description of spin and the SU(1, 1)-CS [12] are optimally localized states for the quantum mechanics of a free particle on the $(1+1)$-dimensional anti-de Sitter spacetime. It is worth mentioning that the square integrability condition has recently been replaced by the less restrictive notion of square integrability modulo a subgroup which allows the construction of (quasi-)coherent states for the Poincaré group in $(1+1)$-dimensions [14]. A more general framework is described in [15].

1.4. Let us now discuss the relevance of CS to the quantization-versus-classical-limit procedures. A classical G-elementary system is generally described by a coadjoint orbit of G, which is a symplectic G-homogeneous space $(G/H, \omega)$, where H is a closed subgroup of G, and ω is a G-invariant, closed and non-degenerate 2-form on G/H. On the other hand, a quantum G-elementary system is described by a pair $(U, \mathscr{H})$, where U is a Unitary Irreducible Representation (UIR) of G in a Hilbert space $\mathscr{H}$. Classical and quantum G-elementary systems are related to each other by on the one hand the quantization methods, such as the Kirillov–Kostant–Souriau geometric quantization (also known as the orbit method) [16, 17, 18], or Berezin's quantization [6], or the deformation quantization [19], and on the other hand the classical limit procedures, such as the CS-inspired one described by Onofri [7]. Since at different stages of this work we will be dealing with these two kinds of procedures, we now briefly hint at their intuitive content.

1.5. Whenever U is a square integrable UIR of G, one can construct a family of CS parametrized by G/H, where H is the closed subgroup of G which leaves invariant, up to a phase, the fiducial state. By construction, this family bears in its very structure enough information to allow one to equip G/H with a G-invariant symplectic form ω, which makes $(G/H, \omega)$ into the classical G-elementary system describing the classical limit of the quantum G-elementary system $(U, \mathscr{H})$. More precisely, this explicit construction leads to a Kähler G-homogeneous space [7]. Combining this derivation with the evaluation of Berezin's covariant symbols [6] provides one with a moment map that identifies $(G/H, \omega)$ with a coadjoint orbit of G. Berezin's symbols are the mean values of the quantum representatives of the generators of $\mathfrak{g}$ in the CS.

1.6. Obviously, evaluating the classical limit is more natural than quantizing. However, because of their better understanding of classical theories, mathematical-physicists are very much interested in devising a quantization procedure that would allow the translation of some of the well established classical physical understanding to a quantum counterpart. Several quantization procedures are now available. We have already mentioned three of them (see 1.4 above). Geometric quantization is the method that we will be dealing with in this paper. In its simplest version, this technique associates a quantum G-elementary system to a classical one $(G/H, \omega)$, provided that $[\omega]$ is an integral cohomology class and that G/H admits an invariant polarization.

1.7. The very natural question we address in this work can be formulated as follows: *How do the quantization-versus-classical-limit procedures depicted above extend to the supersymmetric context*? We provide an answer to this, by studying very specific though non-trivial examples, namely those of the typical and the atypical OSp(2/2)-elementary systems. To shed more light on our motivations, we now situate our present contribution within the framework of the fast developing field of supermathematics.

1.8. Supermathematics is the collection of mathematical tools developed during the last thirty years in order to provide physicists with a rigorous framework for the study of the so-called supersymmetric theories, such as supergravity and superstrings. These are theories that possess symmetries which mix their bosonic and fermionic degrees of freedom. Some of the tools were already available before these theories really triggered the interest of a large number of researchers (see [20], pp. 26–28). A description of the super extensions of the usual analytic, algebraic and geometric concepts that have so far been obtained would unfortunately lead us too far from the subject of this paper; we refer the interested reader to the existing literature [21–31] and confine our description to those super ingredients that are crucial for answering the question raised in 1.7.

1.9. Contrary to Lie superalgebras, there exist several notions of Lie supergroups which correspond to the different definitions of a supermanifold (see [30] for more details). Concerning the representation theory, only abstract representations (i.e. abstract algebraic modules) of some simple Lie superalgebras are known [22, 24, 25, 32, 33]. Using these representations, supercoherent states have recently been explicitly constructed [34–36]. On the other hand, supergeometric concepts such as supersymplectic supermanifolds and super coadjoint orbits have been understood since the end of the 70s [22, 28] (see [37] for a recent application in physics). A very nice characterization of the latter in terms of usual geometric objects has been obtained by Rothstein [38] (see also [39]). Hence, all the ingredients needed for answering the question formulated in 1.7 are available. It remains to extend to the super context the methods described in 1.5 and 1.6. This is explicitly carried out here for the case of the typical and atypical OSp(2/2)-CS.

1.10. The paper is organized as follows: In Sect. 2 the so-called typical and atypical super-unitary irreducible representations of the Lie superalgebra osp(2/2) which super extend the discrete series representations of its subalgebra $su(1,1)$ are described. Then, the typical coherent states are constructed. By doing this we reproduce the construction of the OSp(2/2)-CS given in [35], and we cure some of the discrepancies that appear there. In Sect. 3 a super extension of the methods depicted in 1.5 is then applied to these CS. Hence, the OSp(2/2)-homogeneous superspace parametrizing the latter which we denote by $\mathscr{D}^{(1|2)}$ is explicitly equipped with an OSp(2/2)-invariant supersymplectic structure ω. Moreover, a momentum map that identifies $(\mathscr{D}^{(1|2)}, \omega)$ with an OSp(2/2)-coadjoint orbit is exhibited through the evaluation of Berezin's covariant symbols. Finally, the invariant super-measure on $\mathscr{D}^{(1|2)}$ obtained in [35] is recovered through a much simpler computation. In Sect. 4, after an introduction to supermanifolds, the supergeometry of $(\mathscr{D}^{(1|2)}, \omega)$ is studied further. As a result, $(\mathscr{D}^{(1|2)}, \omega)$ is shown to be not only a non-trivial example of Rothstein's general supersymplectic supermanifolds [38] but also a non-trivial example of the notion of a super-Kähler supermanifold already discussed in [8, 9]. In this particular setting, we show how Rothstein's characterization can be improved. In Sect. 5, after a brief introduction to the usual geometric quantization procedure, a super extension of it is applied to $(\mathscr{D}^{(1|2)}, \omega)$. General superprequantization has been developed by Kostant [28], however the lack of a notion of polarization prevented him from completing the quantization. Here, the super-Kähler character of $(\mathscr{D}^{(1|2)}, \omega)$ singles out a natural invariant super-Kähler polarization which leads to the complete quantization. The coherent states associated to the atypical representations (atypical CS), their underlying supergeometry, and its quantization are described in Sect. 6. In particular the atypical coadjoint orbit is identified. Section 7 gathers additional results and discussions, while concluding remarks and possible extensions of the present work are displayed in Sect. 8. Our conventions and notations are presented in Appendix A, while useful constructions are relegated to Appendix B.

1.11. Before presenting the details, we would like to make clear some important points concerning our approach and strategy. Perelomov's construction of G-CS can be explicitly carried out starting simply from an infinitesimal version of a UIR of G [3]. This approach, used in [34, 35], is also adopted here. It might also seem strange that we start with a representation of the Lie superalgebra, evaluate its classical limit through the associated CS and then quantize the obtained supersymplectic supermanifold in order to construct a representation of the same Lie superalgebra! In fact, the representation we start with is, as we will see, an abstract representation (abstract algebraic module), while the second is an explicit one. The latter, is realized in a super-Hilbert space of superholomorphic sections of a line bundle sheaf over the considered coadjoint orbit. This representation is an important step towards constructing explicit super-UIR of Lie supergroups.

2. osp(2/2) Representations and Typical OSp(2/2)-CS

This section is devoted to two main purposes. We first describe the osp(2/2) Lie superalgebra and its lowest weight typical and atypical representations, then we construct the associated coherent states.

2.1. osp(2/2) Representations. The superalgebra we consider here is the real orthosymplectic Lie superalgebra osp$(2/2, \mathbb{R})$. Throughout, we will simply denote it osp(2/2). It is a real non-compact form of the basic classical simple Lie superalgebra osp$(2/2, \mathbb{C})$, which corresponds to $C(2)$ in Kac's classification [25]. Its $\mathbb{Z}_2$-grading, $\mathrm{osp}(2/2) = \mathrm{osp}(2/2)_{\bar{0}} \oplus \mathrm{osp}(2/2)_{\bar{1}}$, is such that the even component

$\mathrm{osp}(2/2)_{\bar{0}} = \mathrm{so}(2) \oplus \mathrm{sp}(2,\mathbb{R})$, and the odd component $\mathrm{osp}(2/2)_{\bar{1}}$ is an $\mathrm{osp}(2/2)_{\bar{0}}$-module. In what follows, using the isomorphism $\mathrm{sp}(2,\mathbb{R}) \cong \mathrm{su}(1,1)$, we will consider $\mathrm{su}(1,1)$ instead of $\mathrm{sp}(2,\mathbb{R})$.

As for Lie algebras, the construction of representations of osp(2/2) relies on its complexification $\mathrm{osp}(2/2,\mathbb{C})$. We now display a description of the latter superalgebra. Since $\mathrm{osp}(2/2,\mathbb{C})$ is of type I [25, 23], its odd component $\mathrm{osp}(2/2,\mathbb{C})_{\bar{1}}$ decomposes into two irreducible $\mathrm{osp}(2/2,\mathbb{C})_{\bar{0}}$-modules, namely, $\mathrm{osp}(2/2,\mathbb{C})_{\bar{1}} = \mathrm{osp}(2/2,\mathbb{C})_{-1} \oplus \mathrm{osp}(2/2,\mathbb{C})_{1}$. Moreover $\mathrm{osp}(2/2,\mathbb{C})$ admits the following $\mathbb{Z}_2$-compatible $\mathbb{Z}$-gradation:

$$\mathrm{osp}(2/2,\mathbb{C}) = \mathrm{osp}(2/2,\mathbb{C})_{-1} \oplus \mathrm{osp}(2/2,\mathbb{C})_{0} \oplus \mathrm{osp}(2/2,\mathbb{C})_{1} , \tag{2.1}$$

where $\mathrm{osp}(2/2,\mathbb{C})_0 = \mathrm{osp}(2/2,\mathbb{C})_{\bar{0}}$. Let $\{B, K_0, K_\pm; V_\pm, W_\pm\}$ be the Cartan–Weyl basis of $\mathrm{osp}(2/2,\mathbb{C})$. The above mentioned structures of $\mathrm{osp}(2/2,\mathbb{C})$ are explicitly displayed in the following defining commutation ($[\,,\,]$) and anticommutation ($[\,,\,]_+$) relations:

$$[K_0, K_\pm] = \pm K_\pm , \qquad [K_+, K_-] = -2K_0 , \tag{2.2a}$$

$$[B, K_\pm] = 0 , \qquad [B, K_0] = 0 , \tag{2.2b}$$

$$[K_0, V_\pm] = \pm\frac{1}{2} V_\pm , \qquad [K_0, W_\pm] = \pm\frac{1}{2} W_\pm , \tag{2.2c}$$

$$[K_\pm, V_\pm] = 0 , \qquad [K_\pm, W_\pm] = 0 , \tag{2.2d}$$

$$[K_\pm, V_\mp] = \mp V_\pm , \qquad [K_\pm, W_\mp] = \mp W_\pm , \tag{2.2e}$$

$$[B, V_\pm] = \frac{1}{2} V_\pm , \qquad [B, W_\pm] = -\frac{1}{2} W_\pm , \tag{2.2f}$$

$$[V_\pm, V_\pm]_+ = 0 , \qquad [W_\pm, W_\pm]_+ = 0 , \tag{2.2g}$$

$$[V_\pm, V_\mp]_+ = 0 , \qquad [W_\pm, W_\mp]_+ = 0 , \tag{2.2h}$$

$$[V_\pm, W_\pm]_+ = K_\pm , \qquad [V_\pm, W_\mp]_+ = K_0 \mp B . \tag{2.2i}$$

Clearly, the even component is spanned by $\{B, K_0, K_\pm\}$. The K-generators form an su(1,1) Lie subalgebra (see (2.2a)), and B spans a one dimensional center of $\mathrm{osp}(2/2,\mathbb{C})_{\bar{0}}$ (see (2.2b)). On the other hand, the odd component is the span of $V_\pm$ and $W_\pm$. More precisely, $\{V_\pm\}$ (resp. $\{W_\pm\}$) span $\mathrm{osp}(2/2,\mathbb{C})_1$ (resp. $\mathrm{osp}(2/2,\mathbb{C})_{-1}$). The fact that each of these two-dimensional vector spaces carries an irreducible representation of $\mathrm{osp}(2/2,\mathbb{C})_{\bar{0}}$ is transparent from Eqs. (2.2c)–(2.2f); $\mathrm{osp}(2/2,\mathbb{C})_1$ and $\mathrm{osp}(2/2,\mathbb{C})_{-1}$ are distinguished by the distinct eigenvalues of B in (2.2f).

In order to construct irreducible highest (or lowest) weight representations of osp(2/2) one needs to exhibit a Borel subsuperalgebra. In other words, we look for a decomposition of $\mathrm{osp}(2/2,\mathbb{C})$ of the following form:

$$\mathrm{osp}(2/2,\mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ , \tag{2.3}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathrm{osp}(2/2,\mathbb{C})_{\bar{0}}$, and $\mathfrak{n}^-$ and $\mathfrak{n}^+$ are subsuperalgebras of $\mathrm{osp}(2/2,\mathbb{C})$ such that $[\mathfrak{h}, \mathfrak{n}^\pm] \subseteq \mathfrak{n}^\pm$. The subsuperalgebras $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ are called respectively the positive and the negative Borel subsuperalgebras. In terms of the basis of $\mathrm{osp}(2/2,\mathbb{C})$ given above it clearly appears that $\mathfrak{h}$ is the complex span of $\{B, K_0\}$. Moreover, $\mathfrak{n}_{\bar{0}}^{+}$ (resp. $\mathfrak{n}_{\bar{0}}^{-}$) can be taken to be the span of $\{K_+\}$ (resp. $\{K_-\}$). Having fixed this, it can easily be shown that there exist three possible Borel subsuperalgebras of $\mathrm{osp}(2/2,\mathbb{C})$. Since $\mathfrak{b}^{\pm} = \mathfrak{h} \oplus \mathfrak{n}_{\bar{0}}^{\pm} \oplus \mathfrak{n}_{\bar{1}}^{\pm}$, we need only to exhibit the three possible $\mathfrak{n}_{\bar{1}}^{\pm}$. These are:

(i) $\mathfrak{n}_{\bar{1}}^{+} = \mathrm{span}\{V_+, V_-\}$ and $\mathfrak{n}_{\bar{1}}^{-} = \mathrm{span}\{W_+, W_-\}$,
(ii) $\mathfrak{n}_{\bar{1}}^{+} = \mathrm{span}\{W_+, W_-\}$ and $\mathfrak{n}_{\bar{1}}^{-} = \mathrm{span}\{V_+, V_-\}$,
(iii) $\mathfrak{n}_{\bar{1}}^{+} = \mathrm{span}\{V_+, W_+\}$ and $\mathfrak{n}_{\bar{1}}^{-} = \mathrm{span}\{V_-, W_-\}$.

A few remarks are now in order.

Remark 2.1. The situation in (**i**) is symmetric to that in (**ii**). Moreover, they both fit with the $\mathbb{Z}$-grading given in (2.1). Indeed, we have that $\mathrm{osp}(2/2,\mathbb{C})_0 = \mathfrak{n}_{\bar{0}}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{\bar{0}}^{+}$ and $\mathrm{osp}(2/2,\mathbb{C})_{\pm 1} = \mathfrak{n}_{\bar{1}}^{\pm}$.

Remark 2.2. The $\pm$ indices carried by the Vs and the Ws are misleading regarding the root space decomposition of $\mathrm{osp}(2/2,\mathbb{C})$ with respect to the $\mathfrak{h}$ given above. Indeed, one can see from (2.2i) that if V_+ (resp. W_+) is associated to some odd root α, then W_- (resp. V_-) is associated to $-\alpha$. This unconventional choice of notations is aimed at making the $\mathbb{Z}$-gradation of (2.1) explicit in the defining relations of $\mathrm{osp}(2/2,\mathbb{C})$ (see (2.2a)–(2.2i)).

Remark 2.3. Since the set of positive roots depends on the positive Borel subsuperalgebra considered, we have then here three possible sets of this type. While the set of positive roots arising from the case (**iii**) contains two odd simple roots, those arising from the cases (**i**) and (**ii**) contain one odd and one even simple roots (recall that rank $(\mathrm{osp}(2/2,\mathbb{C})) = 2$). The systems of simple roots of both (**i**) and (**ii**) can be connected to each other by an element of the Weyl group of $\mathrm{osp}(2/2,\mathbb{C})$ ($\mathscr{W}(\mathrm{osp}(2/2,\mathbb{C})) := \mathscr{W}(\mathrm{osp}(2/2,\mathbb{C})_{\bar{0}})$); they then give rise to the same Dynkin diagram. On the other hand, the system of simple roots of (**iii**) does not belong to the $\mathscr{W}(\mathrm{osp}(2/2,\mathbb{C}))$-orbit containing the two previous cases; a different Dynkin diagram arises then. This situation is a special feature of basic classical simple Lie superalgebras [40]. Indeed, usual complex simple Lie algebras admit a unique Dynkin diagram. The two Dynkin diagrams mentioned above exhaust all the possibilities for $\mathrm{osp}(2/2,\mathbb{C})$ [40]. Notice finally that Kac considered in his original classification [25] only those Borel subsuperalgebras that lead to the minimum number of odd simple roots. They were called *distinguished* Borel subsuperalgebras.

Remark 2.4. Contrary to the two other cases, the choice in (**iii**) leads to a very interesting $\mathbb{Z}$-grading of $\mathrm{osp}(2/2,\mathbb{C})$ [33]. Both different and finer than that of Eq. (2.1), it is given by:

$$\mathrm{osp}(2/2,\mathbb{C}) = \overset{(-2)}{\mathfrak{n}_{\bar{0}}^{-}} \oplus \overset{(-1)}{\mathfrak{n}_{\bar{1}}^{-}} \oplus \overset{(0)}{\mathfrak{h}} \oplus \overset{(1)}{\mathfrak{n}_{\bar{1}}^{+}} \oplus \overset{(2)}{\mathfrak{n}_{\bar{0}}^{+}}\ . \tag{2.4}$$

The abstract lowest-weight representations that will be described below are those obtained using the Borel subsuperalgebras of case (**iii**). This choice is justified simply by the fact that, up to some discrepancies that we correct here, the associated representations are those already used in [35]. A more mathematical description of these representations is given in [33].

An *abstract irreducible lowest-weight* $\mathrm{osp}(2/2)$*-module* is explicitly constructed starting from a lowest-weight state. According to our choice of Borel subsuperalgebras, namely the one made above in **(iii)**, the lowest-weight state is the state, temporarily denoted $|0\rangle$, which is simultaneously annihilated by K_-, V_- and W_-, and which moreover is a common eigenstate of both the Cartan subalgebra generators, B and K_0. Hence, $|0\rangle$ is such that,

$$K_0|0\rangle = \tau|0\rangle, \qquad B|0\rangle = b|0\rangle\ , \tag{2.5}$$

and

$$K_-|0\rangle = V_-|0\rangle = W_-|0\rangle = 0\ , \tag{2.6}$$

where $0 < \tau \in \mathbb{R}$ and $b \in \mathbb{R}$ completely specify a lowest-weight osp(2/2)-module, which is denoted $V(\tau, b)$ throughout. A basis of $V(\tau, b)$ is explicitly obtained by applying to $|0\rangle$ basis elements of the enveloping superalgebra of the Lie subsuperalgebra $\mathfrak{n}^+$ (as given in (2.3) and **(iii)** above). The results of this construction are now displayed; more details are given in Appendix B.

The following observation simplifies the construction. By restricting (2.5) and (2.6) to the su(1,1)-generators, K_0 and K_-, it clearly appears that $|0\rangle$ is also the lowest-weight vector of an irreducible su(1,1)-module $D(\tau) \subset V(\tau, b)$, namely,

$$D(\tau) \equiv \mathrm{span}\{|\tau, \tau + m\rangle, m \in \mathbb{N}\}\ . \tag{2.7}$$

This is the representation space of the well known *positive discrete series representations* of su(1,1). As a subspace of $V(\tau, b)$, $D(\tau)$ is an eigenspace of B with eigenvalue b. This is an immediate consequence of both (2.5) and the fact that B commutes with su(1,1) (see (2.2b)). Combined with (2.7), these facts suggest the following notation for the lowest-weight state of $V(\tau, b)$, namely $|0\rangle \equiv |b, \tau, \tau\rangle$.

As it is explicitly shown in Appendix B, $V(\tau, b)$ is built out of more than one irreducible su(1,1)-module. Two cases must however be distinguished, namely, either $|b| < \tau$ or $b = \pm\tau$.

Typical. When $|b| < \tau$, as a vector space, $V(\tau, b)$, turns out to be the direct sum of irreducible lowest-weight su(1,1)-modules (positive discrete series). More precisely,

$$V(\tau, b) \equiv D(\tau) \oplus 2 \cdot D\left(\tau + \frac{1}{2}\right) \oplus D(\tau + 1)\ . \tag{2.8}$$

Here $D(\tau + \frac{1}{2})$ appears with multiplicity 2. These two copies of $D(\tau + \frac{1}{2})$ are distinguished eigenspaces of B, with eigenvalues $b + \frac{1}{2}$ and $b - \frac{1}{2}$. The degeneracy is then raised when one considers the su(1,1)-modules appearing in (2.8) as (su(1,1) $\oplus$ so(2))-modules. An extra subscript has then to be added to our previous notation in order to take this fact into account. Observing that, as $D(\tau), D(\tau + 1)$ is a B-eigenspace with eigenvalue b, we can write the following finer decomposition of $V(\tau, b)$,

$$V(\tau, b) = D_b(\tau) \oplus D_{b+\frac{1}{2}}\left(\tau + \frac{1}{2}\right) \oplus D_{b-\frac{1}{2}}\left(\tau + \frac{1}{2}\right) \oplus D_b(\tau + 1)\ . \tag{2.9}$$

More precisely,

$$V(\tau,b) \equiv \operatorname{span}\left\{ |b,\tau,\tau+m\rangle, \left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle, \right.$$
$$\left. \left|b-\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle, |b,\tau+1,\tau+1+m\rangle; m\in\mathbb{N} \right\} . \quad (2.10)$$

Moreover, as a vector superspace, $V(\tau,b) = V_{\bar{0}}(\tau,b)\oplus V_{\bar{1}}(\tau,b)$, where $V_{\bar{0}}(\tau,b)\equiv D_b(\tau)\oplus D_b(\tau+1)$ and $V_{\bar{1}}(\tau,b)\equiv D_{b+\frac{1}{2}}(\tau+\frac{1}{2})\oplus D_{b-\frac{1}{2}}(\tau+\frac{1}{2})$. (The action of the generators of osp$(2/2,\mathbb{C})$ in $V(\tau,b)$ is explicitly given in Appendix B.) The irreducible osp(2/2)-modules, $V(\tau,b)$ for $|b|<\tau$, are usually called *typical representations* [23–25, 32].

Atypical. When $b=\tau$ (resp. $b=-\tau$), $V(\tau,\tau)$ (resp. $V(\tau,-\tau)$) is no longer irreducible. It contains an osp(2/2)-submodule, $V'(\tau,\tau)\equiv D_{\tau+\frac{1}{2}}(\tau+\frac{1}{2})\oplus D_\tau(\tau+1)$ (resp. $V'(\tau,-\tau)\equiv D_{-\tau-\frac{1}{2}}(\tau+\frac{1}{2})\oplus D_{-\tau}(\tau+1)$), generated by the *primitive vector* $V_+|0\rangle$ (resp. $W_+|0\rangle$). The quotient $V(\tau,\tau)/V'(\tau,\tau)$ (resp. $V(\tau,-\tau)/V'(\tau,-\tau)$) appears then as the appropriate irreducible osp(2/2)-module. More precisely,

$$U(\pm\tau)\equiv V(\tau,\pm\tau)/V'(\tau,\pm\tau)\equiv D_{\pm\tau}(\tau)\oplus D_{\pm(\tau-\frac{1}{2})}\left(\tau+\frac{1}{2}\right) . \quad (2.11)$$

These representations are known as the *atypical representations* [23–25, 32].

The above typical and atypical osp(2/2)-modules can be turned into *super-unitary* irreducible representations [33] by equipping $V(\tau,b)$ with a *super-Hermitian* form $\langle\cdot|\cdot\rangle$. The latter notion was originally introduced in [41]. It has been used in the context of representation theory of Lie superalgebras in [24], and more recently in [33]. It is defined in the following way:

$$\langle\cdot|\cdot\rangle : V(\tau,b)\times V(\tau,b)\to\mathbb{C} \quad (2.12)$$

such that $\forall u,v$ two homogeneous elements of $V(\tau,b)$

$$\overline{\langle u|v\rangle} = (-1)^{\varepsilon(u)\varepsilon(v)}\langle v|u\rangle , \quad (2.13)$$

where the parity of a homogeneous element $w\in V(\tau,b)$ is $\varepsilon(w)\equiv 0(1)$ for $w\in V_{\bar{0}(\bar{1})}(\tau,b)$. The elements of $V_{\bar{0}(\bar{1})}$ are called even (odd) elements of $V(\tau,b)$. The super-Hermitian form is taken linear in the second argument. In what follows we will consider a *homogeneous* realization of (2.13), namely, $\forall u=u_0+u_1,\ v=v_0+v_1\in V(\tau,b)=V_{\bar{0}}(\tau,b)\oplus V_{\bar{1}}(\tau,b)$,

$$\langle u|v\rangle \equiv \langle u_0|v_0\rangle_0 + i\langle u_1|v_1\rangle_1 , \quad (2.14)$$

where $\langle\cdot|\cdot\rangle_0$ (resp. $\langle\cdot|\cdot\rangle_1$) is a Hermitian form, in the usual sense, on $V_{\bar{0}}(\tau,b)$ (resp. $V_{\bar{1}}(\tau,b)$). Since $\overline{\langle u_k|v_k\rangle_k}=\langle v_k|u_k\rangle_k$ for $k=0,1$, one clearly sees that (2.14) satisfies (2.13). The super-Hermitian form in (2.14) is even.

Besides being an osp(2/2)-module, $V(\tau,b)$ will be also considered as a left $\mathscr{B}$-module, where $\mathscr{B}=\mathscr{B}_0\oplus\mathscr{B}_1$ is a *complex Grassmann algebra* [22, 23]. The super Hermitian form (2.12)–(2.14) will then be extended to the *Grassmann envelope of* *the second kind* [22] $\tilde{V}(\tau,b)$ of $V(\tau,b)$,

$$\tilde{V}(\tau,b) \equiv (\mathscr{B} \otimes V(\tau,b))_0 \,, \tag{2.15}$$

such that

$$\langle \cdot | \cdot \rangle : \tilde{V}(\tau,b) \times \tilde{V}(\tau,b) \to \mathscr{B}_0 \,. \tag{2.16}$$

From the next section on $\mathscr{B}$ will assume a very specific form as the exterior algebra over $\mathbb{C}^4$, namely, $\mathscr{B} \equiv \bigwedge \mathbb{C}^4$ (see Appendix A for more details about (2.15) and (2.16)).

Remark 2.5. The super-Hermitian form in (2.12) and (2.14) is *positive definite* on $V(\tau,b)$ for $|b| < \tau$, in the sense that the Hermitian forms on both $V_{\bar{0}}(\tau,b)$ and $V_{\bar{1}}(\tau,b)$ are positive definite. In Appendix B, we show how this structure turns the typical module $V(\tau,b)$ into a super-unitary irreducible representation of osp(2/2).

Remark 2.6. The construction of atypical representations (2.11) looks very much like the construction of the so-called indecomposable representations which, for instance, usually intervene in the description of massless relativistic quantum elementary systems. This analogy is confirmed by the fact that the super-Hermitian form (2.14) is no longer positive definite on $V(\tau,\pm\tau)$. Indeed, as it is shown in Appendix B, $V'(\tau,\tau)$ (resp. $V'(\tau,-\tau)$) is an osp(2/2)-submodule of $V(\tau,\tau)$ (resp. $V(\tau,-\tau)$) made of *zero-norm* states. In fact, our atypical representations are indecomposable representations. Finally, notice that the atypical representations are, as the typical ones, simply expressed in terms of discrete series representations of su$(1,1)$ (see (2.11)).

Remark 2.7. It is important to note that the atypical modules $U(\pm\tau)$ in (2.11) are irreducible osp(1/2)-modules, where osp(1/2) stands here for the Lie subsuperalgebra of osp(2/2) whose Cartan–Weyl basis is $\{K_0, K_\pm, \frac{1}{\sqrt{2}}(V_\pm + W_\pm)\}$. This interesting observation will be discussed further on in Sect. 6.

We now briefly discuss Schur's lemma. Since osp$(2/2,\mathbb{C})$ is of rank 2, the center of its enveloping superalgebra is generated by two osp$(2/2,\mathbb{C})$-invariants which are, respectively, quadratic (the Casimir) and cubic in the generators of osp$(2/2,\mathbb{C})$. For simplicity, we exhibit below only the explicit expression of the former which we denote Q_2. Hence,

$$Q_2 = C_2 - B^2 + K_0 - W_+V_- - V_+W_- \,, \tag{2.17}$$

where C_2 is the Casimir invariant of the su$(1,1)$ subalgebra, namely,

$$C_2 = K_0^2 - K_0 - K_+K_- \,. \tag{2.18}$$

The irreducibility of $V(\tau,b)$ implies that on $V(\tau,b)$ both invariants are constant multiples of the identity. The exact value of the quadratic invariant on $V(\tau,b)$ is simply obtained by evaluating Q_2 on the lowest-weight state $|b,\tau,\tau\rangle$. Hence,

$$Q_2 \equiv (\tau^2 - b^2)\mathbb{I} \quad \text{on } V(\tau,b), \quad \text{since } C_2 \equiv \tau(\tau-1)\mathbb{I} \quad \text{on } D(\tau) \subset V(\tau,b) \,. \tag{2.19}$$

One clearly sees that Q_2 is identically zero on the atypical modules.

Finally, a lowest-weight irreducible osp(2/2)-module is said to be an *integrable module* whenever it is also a module over the Lie group Sp$(2,\mathbb{R}) \times$ SO(2) whose Lie algebra is osp$(2/2)_{\bar{0}}$. Since the maximal torus in OSp(2/2) (and in Sp$(2,\mathbb{R}) \times$ SO(2) too) is the $U(1) \times U(1)$ subgroup generated by K_0 and B, one easily sees from (2.5) that $V(\tau, b)$ is an integrable osp(2/2)-module if and only if both τ and b are half integers. Throughout we will only deal with integrable super-unitary irreducible osp(2/2)-modules, and we will no longer mention this fact. More precisely, we will restrict our attention to the typical integrable representations, knowing that all the constructions of the forthcoming sections carry over to the atypical case. However, some very interesting points concerning the latter are worth to be mentioned. They are gathered in Sect. 6.

2.2. OSp(2/2) Coherent States. In this section we construct the OSp(2/2)-coherent states associated to the typical representations described in the previous section. We should mention that up to some discrepancies that are cured here and a different choice of conventions (see Appendix A), this construction was originally carried out in [35].

As stressed in the introduction, the coherent states for a Lie group G are the quantum states belonging to a G-orbit through a fiducial vector in an irreducible $\mathfrak{g}$-module carrying a unitary representation of G. Here, $\mathfrak{g}$ stands for the Lie algebra of G. Recall also from the introduction that the minimal requirement for the construction of G-CS consists in an irreducible unitary $\mathfrak{g}$-module. For OSp(2/2), irreducible super-unitary osp(2/2)-modules are at our disposal.

Considering the lowest-weight state $|b,\tau,\tau\rangle \in V(\tau,b)$ as the fiducial state, taking into account Eqs. (2.5)–(2.6), and extending in a straightforward manner Perelomov's construction [3], the typical OSp(2/2)-coherent states are obtained as follows:

$$|a,\theta,\chi\rangle \equiv \mathcal{N}\exp(aK_+ + \theta V_+ + \chi W_+)|b,\tau,\tau\rangle\,. \tag{2.20}$$

They belong to $\tilde{V}(\tau,b)$ (see (2.10) and (2.15)), for $\mathscr{B}$ the Grassmann algebra generated by the complex anticommuting variables θ,χ, and their complex conjugates, $\bar{\theta}$ and $\bar{\chi}$. These are odd elements of $\mathscr{B}$, while a is an even element of $\mathscr{B}$. More details about $\mathscr{B}$ are given in Appendix A. In (2.20), $\mathcal{N}$ is a normalization factor which will be explicitly determined below.

Since $[K_+,V_+]=0$, $[K_+,W_+]=0$ and $\theta^2=\chi^2=0$, we can rewrite (2.20) as follows,

$$|a,\theta,\chi\rangle = \mathcal{N}[\exp(aK_+)]\left[1+\theta V_+ + \chi W_+ + \frac{1}{2}\chi\theta(V_+W_+ - W_+V_+)\right]|b,\tau,\tau\rangle\,. \tag{2.21}$$

A simple computation, based on (B.15)–(B.19), leads to:

$$\begin{aligned}|a,\theta,\chi\rangle = \mathcal{N}\left[\exp\left(\left(a+\frac{b}{2\tau}\chi\theta\right)K_+\right)\right]\Bigg(&|b,\tau,\tau\rangle + \theta\sqrt{\tau-b}\left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}\right\rangle\\ &+\chi\sqrt{\tau+b}\left|b-\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}\right\rangle\\ &+\chi\theta\sqrt{\frac{(\tau^2-b^2)(2\tau+1)}{2\tau}}|b,\tau+1,\tau+1\rangle\Bigg)\,.\end{aligned} \tag{2.22}$$

At this point, we introduce the new variable

$$z = a + \frac{b}{2\tau}\chi\theta \in \mathbb{C}\,. \tag{2.23}$$

Notice that for simplicity we are choosing here (and throughout) z to be a complex number. This choice is discussed in Sect. 7. Finally, using the known action of K_+ in $V(\tau,b)$ (see (B.18)), we get the explicit form of the typical coherent states (2.22):

$$\begin{aligned}|z,\theta,\chi\rangle = \mathcal{N}\Bigg[&\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m)}{m!\Gamma(2\tau)}}\,z^m|b,\tau,\tau+m\rangle\\
&+\theta\sqrt{\tau-b}\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m+1)}{m!\Gamma(2\tau+1)}}\,z^m\left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle\\
&+\chi\sqrt{\tau+b}\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m+1)}{m!\Gamma(2\tau+1)}}\,z^m\left|b-\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle\\
&+\chi\theta\sqrt{\frac{(\tau^2-b^2)(2\tau+1)}{2\tau}}\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m+2)}{m!\Gamma(2\tau+2)}}\,z^m|b,\tau+1,\tau+1+m\rangle\Bigg]\,.\end{aligned} \tag{2.24}$$

Now, using the super-Hermitian form (2.16), the fact that the basis of $V(\tau,b)$ in (2.10) is super-orthonormal with respect to (2.14) (see Appendix B), and the conventions of Appendix A, one easily evaluates $\mathcal{N}$. Hence,

$$\begin{aligned}\mathcal{N} = (1-|z|^2)^{\tau}\Bigg[&1+i\frac{\tau-b}{2}\frac{\bar{\theta}\theta}{1-|z|^2}+i\frac{\tau+b}{2}\frac{\bar{\chi}\chi}{1-|z|^2}\\
&-\frac{(\tau^2-b^2)(\tau-1)}{4\tau}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2}\Bigg]\,.\end{aligned} \tag{2.25}$$

Notice here that for obvious reasons we have considered $\mathcal{N}$ real. Moreover, the following identity,

$$\frac{1}{(1-x)^s}=\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}x^m,\quad \text{for } |x|<1\,, \tag{2.26}$$

has been used in order to write the result (2.25) in a compact form. It is worth mentioning at this point that when one sets to zero the odd variables θ and χ, all the previous formulae reduce exactly to those of the SU(1,1)-CS [3]. In that case, and clearly here too, z spans the unit disc $\mathcal{D}^{(1)}=\{z\in\mathbb{C};|z|<1\}$; $\mathcal{D}^{(1)}\equiv$ SU(1,1)/U(1).

For a later use we evaluate now the $|\bar{z},\bar{\theta},\bar{\chi}\rangle$ CS, where $\bar{z},\bar{\theta}$ and $\bar{\chi}$ are the complex conjugates of z,θ and χ. Straightforwardly,

$$\begin{aligned}|\bar{z},\bar{\theta},\bar{\chi}\rangle = \mathcal{N}'\Bigg[&\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m)}{m!\Gamma(2\tau)}}\,\bar{z}^m|b,\tau,\tau+m\rangle\\
&+\bar{\theta}\sqrt{\tau-b}\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m+1)}{m!\Gamma(2\tau+1)}}\,\bar{z}^m\left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle\end{aligned}$$

$$+ \bar{\chi}\sqrt{\tau+b} \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2\tau+m+1)}{m!\Gamma(2\tau+1)}} \bar{z}^{m} \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle$$

$$+ \bar{\chi}\bar{\theta}\sqrt{\frac{(\tau^2-b^2)(2\tau+1)}{2\tau}} \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2\tau+m+2)}{m!\Gamma(2\tau+2)}} \bar{z}^{m} |b, \tau+1, \tau+1+m\rangle \Bigg], \tag{2.27}$$

the normalizing constant being now

$$\mathcal{N}' = (1-|z|^2)^{\tau} \left[1 - i\frac{\tau-b}{2}\frac{\bar{\theta}\theta}{1-|z|^2} - i\frac{\tau+b}{2}\frac{\bar{\chi}\chi}{1-|z|^2} - \frac{(\tau^2-b^2)(\tau-1)}{4\tau}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2} \right]. \tag{2.28}$$

Having the explicit form of the typical OSp(2/2)-CS, we can now apply to them a super extension of Onofri's analysis [7] in order to reveal the (super)geometry underlying them, or equivalently, in order to evaluate the classical limit of the quantum theory described by the typical representation $V(\tau, b)$ of the previous subsection. This analysis is the main concern of the next two sections.

3. Supersymplectic Geometry and the OSp(2/2)-CS

In analogy with the non-super case [3], the results of the previous section suggest that the space parametrizing the typical OSp(2/2)-CS is the OSp(2/2)-homogeneous super space OSp(2/2)/$(U(1) \times U(1))$, realized in terms of the coordinates (z, θ, χ), where z parametrizes the unit disc $\mathscr{D}^{(1)} := \{z \in \mathbb{C}; |z| < 1\}$. This realization will be subsequently called the $N = 2$ super-unit disc, and it will be denoted $\mathscr{D}^{(1|2)}$. We recall that the $N = 1$ super-unit disc $\mathscr{D}^{(1|1)}$ was fully considered in [8, 9]. We will consider here $\mathscr{D}^{(1|2)}$ as a supermanifold, although its complete and precise geometric characterization will be only given in Sect. 4.

In Sect. 3.1, we carry out our super extension of Onofri's analysis [7] which only makes use of the explicit form of the CS. This analysis will provide us with a partial description of the geometric structure underlying the typical OSp(2/2)-CS. Some of the results obtained here will be revisited in Sect. 4 in light of the general theory of supermanifolds. Let us recall that in the case of a semi-simple Lie group, starting from the associated CS, Onofri's analysis allows one to equip the homogeneous space parametrizing these CS with an invariant symplectic form (i.e. a closed and non-degenerate 2-form) which is moreover Kähler. Analogously, here we equip $\mathscr{D}^{(1|2)}$ with an invariant *supersymplectic* form, in the sense of [22, 28, 38]. Moreover, as it will be discussed in Sect. 4, this form turns out to be *super-Kähler* in a sense to be defined (see also [8, 9]). In Sect. 3.2, extending Berezin's notion of covariant symbols, we evaluate the classical observables (superfunctions on $\mathscr{D}^{(1|2)}$) associated to the infinitesimal action of OSp(2/2) on $\mathscr{D}^{(1|2)}$. By doing this we exhibit a moment map that identifies $\mathscr{D}^{(1|2)}$ with an OSp(2/2)-coadjoint orbit. In the process, the Hamiltonian vector superfields associated to the classical observables are computed. Finally, in Sect. 3.3 we evaluate the *Liouville super-measure* on $\mathscr{D}^{(1|2)}$. This will be needed in Sect. 5.

3.1. The Supersymplectic Form. Onofri's analysis starts by evaluating a real function from the G-CS; it plays the role of a Kähler potential for a G-invariant Kähler form on the space parametrizing the G-CS. Here, emphasizing the symplectic output of this procedure, we extend it to our typical OSp(2/2)-CS. Hence, from Eq. (2.27), we evaluate the following superfunction on $\mathscr{D}^{(1|2)}$,

$$\begin{aligned} f(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi}) &:= \log|\langle 0|\bar{z},\bar{\theta},\bar{\chi}\rangle|^{-2} \\ &= 2\tau\left[-\log(1-|z|^2)+i\frac{\tau-b}{2\tau}\frac{\bar{\theta}\theta}{1-|z|^2}\right. \\ &\qquad \left.+i\frac{\tau+b}{2\tau}\frac{\bar{\chi}\chi}{1-|z|^2}-\frac{\tau^2-b^2}{4\tau^2}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2}\right]. \end{aligned} \tag{3.1}$$

Let now d be the exterior derivative on $\mathscr{D}^{(1|2)}$ given by $d=\delta+\bar{\delta}$, where

$$\delta = dz\frac{\partial}{\partial z}+d\theta\frac{\partial}{\partial\theta}+d\chi\frac{\partial}{\partial\chi}, \qquad \bar{\delta} = d\bar{z}\frac{\partial}{\partial\bar{z}}+d\bar{\theta}\frac{\partial}{\partial\bar{\theta}}+d\bar{\chi}\frac{\partial}{\partial\bar{\chi}}. \tag{3.2}$$

Notice that here $\varepsilon(d)=0$, i.e. d is an even quantity.

An even two-superform on $\mathscr{D}^{(1|2)}$ can now be obtained from the superfunction in (3.1) in the following way:

$$\omega \equiv -i\delta\bar{\delta}f = \omega_0+\omega_2+\omega_4\,, \tag{3.3}$$

where,

$$\omega_0 = \frac{-2i\tau}{(1-|z|^2)^2}\,dzd\bar{z}\,; \tag{3.4}$$

$$\begin{aligned} \omega_2 = &-(\tau-b)\left[\frac{1}{(1-|z|^2)}d\theta d\bar{\theta}+\frac{1}{(1-|z|^2)^2}[\theta\bar{z}\,dzd\bar{\theta}-\bar{\theta}z\,d\theta d\bar{z}]\right. \\ &\left.-\bar{\theta}\theta\frac{(1+|z|^2)}{(1-|z|^2)^3}dzd\bar{z}\right]-(\tau+b)\left[\frac{1}{(1-|z|^2)}d\chi d\bar{\chi}+\frac{1}{(1-|z|^2)^2}\right. \\ &\left.\times[\chi\bar{z}\,dzd\bar{\chi}-\bar{\chi}z\,d\chi d\bar{z}]-\bar{\chi}\chi\frac{(1+|z|^2)}{(1-|z|^2)^3}dzd\bar{z}\right]; \end{aligned} \tag{3.5}$$

$$\begin{aligned} \omega_4 = -i\frac{\tau^2-b^2}{2\tau(1-|z|^2)^2}&\left[-2\frac{(1+2|z|^2)}{(1-|z|^2)^2}\bar{\theta}\bar{\chi}\chi\theta\,dzd\bar{z}+\bar{\theta}\theta\,d\chi d\bar{\chi}+\bar{\chi}\chi\,d\theta d\bar{\theta}\right. \\ &-\bar{\theta}\chi\,d\theta d\bar{\chi}-\bar{\chi}\theta\,d\chi d\bar{\theta}+\frac{2\bar{\chi}\chi}{(1-|z|^2)}[\theta\bar{z}\,dzd\bar{\theta}-\bar{\theta}z\,d\theta d\bar{z}] \\ &\left.+\frac{2\,\bar{\theta}\theta}{(1-|z|^2)}[\chi\bar{z}\,dzd\bar{\chi}-\bar{\chi}z\,d\chi d\bar{z}]\right]. \end{aligned} \tag{3.6}$$

This two-superform belongs to the exterior algebra on $\mathscr{D}^{(1|2)}$. The latter is a bi-graded $\mathbb{Z} \times \mathbb{Z}_2$ algebra [28], where the $\mathbb{Z}$-gradation is the usual gradation of de Rham complexes, while the $\mathbb{Z}_2$-gradation is the natural gradation accompanying supersymmetry (i.e. the $\mathbb{Z}_2$-gradation of the Grassmann algebra $\mathscr{B}$). More precisely, for any two superforms β_1 and β_2 on $\mathscr{D}^{(1|2)}$, one has:

$$\beta_1\beta_2 = (-1)^{a_1a_2+b_1b_2}\beta_2\beta_1\,, \tag{3.7}$$

where a_i (resp. b_i) is the degree of the superform β_i with respect to the $\mathbb{Z}$ (resp. $\mathbb{Z}_2$) gradation. Hence, in (3.6), $dz\,d\bar{z} = -d\bar{z}\,dz$ (this is the usual wedge product), $dz\,d\bar{\theta} = -d\bar{\theta}\,dz$ and $d\theta\,d\bar{\theta} = d\bar{\theta}\,d\theta$.

Using these conventions one can check by explicit calculations that ω is closed, i.e. $d\omega = 0$. In fact, ω is closed by construction. This is a direct consequence of (3.3). Indeed, since $d = \delta + \bar{\delta}$, $d^2 = 0$ implies that $\delta^2 = \bar{\delta}^2 = \delta\bar{\delta} + \bar{\delta}\delta = 0$. Hence, $(\mathscr{D}^{(1|2)}, \omega)$ is a supersymplectic supermanifold. This particular point will be analyzed further on in Sect. 4.

3.2. The Classical Observables. In analogy with the non-super case, one can determine the classical observables associated to the generators of the supersymplectic action of OSp(2/2) on $\mathscr{D}^{(1|2)}$. This is achieved through the evaluation of the so-called Berezin covariant symbols [6]. As for the superfunction (3.1), these are obtained simply from the knowledge of the explicit form of the OSp(2/2)-CS and the representation $V(\tau, b)$ they belong to. Hence, the classical observable H^{cl} associated to an OSp(2/2)-generator $H \in \{B, K_0, K_\pm, V_\pm, W_\pm\}$ is given by the Berezin symbol:

$$H^{\text{cl}} \equiv H^{\text{cl}}(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi}) := \langle \bar{z},\bar{\theta},\bar{\chi}|H|\bar{z},\bar{\theta},\bar{\chi}\rangle\,. \tag{3.8}$$

After lengthy but straightforward computations based on (2.27) and results from Appendix B, one obtains:

$$B^{\text{cl}} = b + i\frac{\tau - b}{2}\frac{\bar{\theta}\theta}{1-|z|^2} - i\frac{\tau + b}{2}\frac{\bar{\chi}\chi}{1-|z|^2},$$

$$K_0^{\text{cl}} = \tau\frac{1+|z|^2}{1-|z|^2}\left[1 + i\frac{\tau - b}{2\tau}\frac{\bar{\theta}\theta}{(1-|z|^2)} + i\frac{\tau+b}{2\tau}\frac{\bar{\chi}\chi}{(1-|z|^2)} - \frac{\tau^2-b^2}{2\tau^2}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2}\right],$$

$$K_+^{\text{cl}} = \frac{2\tau z}{1-|z|^2}\left[1 + i\frac{\tau - b}{2\tau}\frac{\bar{\theta}\theta}{(1-|z|^2)} + i\frac{\tau+b}{2\tau}\frac{\bar{\chi}\chi}{(1-|z|^2)} - \frac{\tau^2-b^2}{2\tau^2}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2}\right],$$

$$K_-^{\text{cl}} = \frac{2\tau \bar{z}}{1-|z|^2}\left[1 + i\frac{\tau - b}{2\tau}\frac{\bar{\theta}\theta}{(1-|z|^2)} + i\frac{\tau+b}{2\tau}\frac{\bar{\chi}\chi}{(1-|z|^2)} - \frac{\tau^2-b^2}{2\tau^2}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2}\right],$$

$$V_+^{\text{cl}} = \frac{1}{1-|z|^2}\left[-i(\tau-b)\theta + (\tau+b)z\bar{\chi} - \frac{\tau^2-b^2}{2\tau}\frac{(iz\bar{\theta}+\chi)\bar{\chi}\theta}{(1-|z|^2)}\right],$$

$$W_-^{\text{cl}} = \frac{1}{1-|z|^2}\left[(\tau-b)\bar{\theta} - i(\tau+b)\bar{z}\chi - \frac{\tau^2-b^2}{2\tau}\frac{(\bar{z}\theta+i\bar{\chi})\bar{\theta}\chi}{(1-|z|^2)}\right],$$

$$V_-^{\mathrm{cl}} = \frac{1}{1-|z|^2}\left[-i(\tau-b)\bar{z}\theta + (\tau+b)\bar{\chi} - \frac{\tau^2-b^2}{2\tau}\frac{(\bar{z}\chi + i\bar{\theta})\bar{\chi}\theta}{(1-|z|^2)}\right],$$

$$W_+^{\mathrm{cl}} = \frac{1}{1-|z|^2}\left[(\tau-b)z\bar{\theta} - i(\tau+b)\chi - \frac{\tau^2-b^2}{2\tau}\frac{(iz\bar{\chi}+\theta)\bar{\theta}\chi}{(1-|z|^2)}\right]. \tag{3.9}$$

The obtained classical observables satisfy the following relations:

$$\overline{B^{\mathrm{cl}}} = B^{\mathrm{cl}}, \qquad \overline{K_0^{\mathrm{cl}}} = K_0^{\mathrm{cl}}, \qquad \overline{K_+^{\mathrm{cl}}} = K_-^{\mathrm{cl}}, \qquad \overline{V_+^{\mathrm{cl}}} = iW_-^{\mathrm{cl}}, \qquad \overline{V_-^{\mathrm{cl}}} = iW_+^{\mathrm{cl}}. \tag{3.10}$$

Remark 3.1. Notice that the Berezin symbols (3.8) are defined in terms of $|\bar{z}, \bar{\theta}, \bar{\chi}\rangle$ instead of $|z, \theta, \chi\rangle$. In order to justify this choice we need to anticipate future results. In fact, as it will be shown in Sect. 5, this choice leads to the classical theory whose quantization gives rise to a superholomorphic representation. The anti-superholomorphic one arises as the quantization of the classical theory obtained from the CS $|z, \theta, \chi\rangle$.

The Hamiltonian vector superfield X_H associated to a classical observable H^{cl} is the solution of the following defining equation [28],

$$X_H \lrcorner\, \omega = dH^{\mathrm{cl}}, \tag{3.11}$$

where "$\lrcorner$" stands for the interior product and ω is the supersymplectic form (3.3). Here we display the Hamiltonian vector fields associated to the above observables. A long computation leads to the following:

$$\begin{aligned}
X_B &= i\frac{\theta}{2}\frac{\partial}{\partial\theta} - i\frac{\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}} - i\frac{\chi}{2}\frac{\partial}{\partial\chi} + i\frac{\bar{\chi}}{2}\frac{\partial}{\partial\bar{\chi}},\\
X_{K_0} &= iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial\bar{z}} + i\frac{\theta}{2}\frac{\partial}{\partial\theta} - i\frac{\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}} + i\frac{\chi}{2}\frac{\partial}{\partial\chi} - i\frac{\bar{\chi}}{2}\frac{\partial}{\partial\bar{\chi}},\\
X_{K_+} &= iz^2\frac{\partial}{\partial z} - i\frac{\partial}{\partial\bar{z}} + iz\theta\frac{\partial}{\partial\theta} + iz\chi\frac{\partial}{\partial\chi},\\
X_{K_-} &= i\frac{\partial}{\partial z} - i\bar{z}^2\frac{\partial}{\partial\bar{z}} - i\bar{z}\bar{\theta}\frac{\partial}{\partial\bar{\theta}} - i\bar{z}\bar{\chi}\frac{\partial}{\partial\bar{\chi}},\\
X_{V_+} &= \left(\frac{\tau-b}{2\tau}\right)z\theta\frac{\partial}{\partial z} - i\left(\frac{\tau+b}{2\tau}\right)\bar{\chi}\frac{\partial}{\partial\bar{z}} + i\frac{\partial}{\partial\bar{\theta}} - \left(z + \left(\frac{\tau-b}{2\tau}\right)\chi\theta\right)\frac{\partial}{\partial\chi},\\
X_{W_-} &= \left(\frac{\tau+b}{2\tau}\right)\chi\frac{\partial}{\partial z} - i\left(\frac{\tau-b}{2\tau}\right)\bar{z}\bar{\theta}\frac{\partial}{\partial\bar{z}} - \frac{\partial}{\partial\theta} + i\left(\bar{z} - \left(\frac{\tau-b}{2\tau}\right)\bar{\theta}\bar{\chi}\right)\frac{\partial}{\partial\bar{\chi}},\\
X_{V_-} &= \left(\frac{\tau-b}{2\tau}\right)\theta\frac{\partial}{\partial z} - i\left(\frac{\tau+b}{2\tau}\right)\bar{z}\bar{\chi}\frac{\partial}{\partial\bar{z}} + i\left(\bar{z} + \left(\frac{\tau+b}{2\tau}\right)\bar{\theta}\bar{\chi}\right)\frac{\partial}{\partial\bar{\theta}} - \frac{\partial}{\partial\chi},\\
X_{W_+} &= \left(\frac{\tau+b}{2\tau}\right)z\chi\frac{\partial}{\partial z} - i\left(\frac{\tau-b}{2\tau}\right)\bar{\theta}\frac{\partial}{\partial\bar{z}} - \left(z - \left(\frac{\tau+b}{2\tau}\right)\chi\theta\right)\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\bar{\chi}}.
\end{aligned} \tag{3.12}$$

The above vector fields are such that:

$$\overline{X}_B = X_B, \quad \overline{X}_{K_0} = X_{K_0}, \quad \overline{X}_{K_+} = X_{K_-}, \quad \overline{X}_{V_+} = iX_{W_-}, \quad \overline{X}_{V_-} = iX_{W_+}. \tag{3.13}$$

The supersymplectic structure on $\mathscr{D}^{(1|2)}$ given in (3.3)–(3.6) defines a Poisson super-bracket structure on the space of smooth superfunctions on $\mathscr{D}^{(1|2)}$, turning it into a Poisson superalgebra. The Poisson super-bracket of any two smooth superfunctions g and h on $\mathscr{D}^{(1|2)}$ is defined by:

$$\{g,h\} \equiv -iX_g \,\lrcorner\, dh. \tag{3.14}$$

A simple computation shows that the classical observables in (3.9) form a Poisson subsuperalgebra isomorphic to osp(2/2). Hence, the classical observables provide one with a supersymplectic realization of osp(2/2). What precedes is equivalent to say that by evaluating the classical observables in (3.9) we have in fact exhibited an infinitesimally equivariant momentum map [18], $J : \mathscr{D}^{(1|2)} \to \mathrm{osp}(2/2)^* \equiv \mathrm{osp}(2/2)$, that identifies $\mathscr{D}^{(1|2)}$ with an OSp(2/2)-coadjoint orbit [28]. The latter is realized as a (2|4)-dimensional subsupermanifold of $\mathbb{R}^{(4|4)}$ defined by two constraints which correspond to the quadratic and the cubic osp(2/2)-invariants. Indeed, the latter are identically constant when evaluated in the obtained supersymplectic realization of osp(2/2). For instance, $Q_2^{\mathrm{cl}} \equiv \tau^2 - b^2$ (recall here that $|b| < \tau$).

This completes the first stage of our description of the geometry underlying the typical OSp(2/2)-CS. This turns out to be a supersymplectic geometry. The main purpose of the next and final stage (Sect. 4) is to situate the results of the present section within the already existing theory of supersymplectic supermanifolds. Moreover, in analogy with the non-super case [7], one is tempted to go one step further and consider $(\mathscr{D}^{(1|2)}, \omega)$ as an example of a *super-Kähler supermanifold*, a general notion which has not so far been seriously studied (see however [8, 9]). In Sect. 4, this notion will be given the legitimacy which will allow us to extend the full geometric quantization to the super-Kähler context (see Sect. 5).

Before carrying out this program, we display now a computation, the result of which will be needed later on in Sect. 5.

3.3. The Liouville Super-Measure. The notion of a Liouville measure on a symplectic manifold can be extended to the super context. Indeed, this can be done starting from the supersymplectic form and using Berezin's notion of a density [29]. For instance, up to a multiplicative constant, an OSp(2/2)-invariant measure on $\mathscr{D}^{(1|2)}$ is given by:

$$d\mu(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi}) = \frac{i}{\pi}\,\mathrm{sdet}\|\omega_{A\bar{B}}\|\,dz\,d\bar{z}\,d\theta\,d\bar{\theta}\,d\chi\,d\bar{\chi}, \tag{3.15}$$

where "sdet" stands for the superdeterminant (or Berezenian) [22], while $\|\omega_{A\bar{B}}\|$ stands for the supermatrix form of ω, namely,

$$\begin{aligned}\omega &\equiv dx^A \omega_{A\bar{B}}\, dx^{\bar{B}}\\ &= dz(a)d\bar{z} + dz(\alpha)d\bar{\theta} + dz(\beta)d\bar{\chi} + d\theta(\gamma)d\bar{z} + d\chi(\delta)d\bar{z}\\ &\quad + d\theta(y)d\bar{\theta} + d\theta(r)d\bar{\chi} + d\chi(s)d\bar{\theta} + d\chi(t)d\bar{\chi}.\end{aligned} \tag{3.16}$$

More precisely,

$$\|\omega_{A\bar{B}}\| = \begin{pmatrix} a & \alpha & \beta \\ \gamma & y & r \\ \delta & s & t \end{pmatrix} ; \tag{3.17}$$

the entries of $\|\omega_{A\bar{B}}\|$ are as follows:

$$a = -\frac{i}{x^2}\left[2\tau + i(\tau - b)\frac{1+|z|^2}{x}\,\bar{\theta}\theta + i(\tau + b)\frac{1+|z|^2}{x}\,\bar{\chi}\chi - \frac{\tau^2 - b^2}{\tau}\,\frac{1+2|z|^2}{x^2}\,\bar{\theta}\bar{\chi}\chi\theta\right];$$

$$\alpha = -\frac{1}{x^2}(\tau - b)\bar{z}\theta\left[1 + i\frac{\tau + b}{\tau}\,\frac{\bar{\chi}\chi}{x}\right]; \qquad \gamma = -\frac{i}{x^2}(\tau - b)z\bar{\theta}\left[1 + i\frac{\tau + b}{\tau}\,\frac{\bar{\chi}\chi}{x}\right];$$

$$\beta = -\frac{1}{x^2}(\tau + b)\bar{z}\chi\left[1 + i\frac{\tau - b}{\tau}\,\frac{\bar{\theta}\theta}{x}\right]; \qquad \delta = -\frac{i}{x^2}(\tau + b)z\bar{\chi}\left[1 + i\frac{\tau - b}{\tau}\,\frac{\bar{\theta}\theta}{x}\right];$$

$$y = -\frac{i}{x}(\tau - b)\left[1 + i\frac{\tau + b}{2\tau}\,\frac{\bar{\chi}\chi}{x}\right]; \qquad t = -\frac{i}{x}(\tau + b)\left[1 + i\frac{\tau - b}{2\tau}\,\frac{\bar{\theta}\theta}{x}\right];$$

$$r = -\frac{\tau^2 - b^2}{2\tau x^2}\,\bar{\theta}\chi; \qquad s = -\frac{\tau^2 - b^2}{2\tau x^2}\,\bar{\chi}\theta; \qquad x = 1 - |z|^2. \tag{3.18}$$

A simple computation leads then to:

$$d\mu(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi}) = \frac{-2\tau}{\pi(\tau^2 - b^2)}\,dz\,d\bar{z}\,d\theta\,d\bar{\theta}\,d\chi\,d\bar{\chi}. \tag{3.19}$$

Clearly, this measure is only valid in the typical case, $|b| < \tau$. (For the atypical case see Sect. 6.) Moreover, the choice of normalization in (3.19) is not innocent. Its usefulness will appear in Sect. 5 (see Remark 5.4). The OSp(2/2)-invariance of $d\mu$ is claimed without proof. In fact, we can show that this is true using the action of OSp(2/2) on $\mathscr{D}^{(1|2)}$. The latter can be derived by integrating the flows of the Hamiltonian vector fields (3.12) (see Sect. 7, 7.5).

At this point, it is worth mentioning that up to a slight variation in the conventions (see Appendix A), the super-measure (3.19) is exactly the one used in [35] in order to prove the resolution of the identity for the typical OSp(2/2)-CS. The same result holds here. In our notation, this means that the typical OSp(2/2)-CS (2.27) form an overcomplete basis of $V(\tau, b)$ (see Sect. 7, 7.1). From a computational point of view the way we evaluate here (3.19) is by far simpler than the one used in [35].

4. More About Supergeometry

The theory of supermanifolds was originally devised in order to provide physicists with a rigorous framework for studying supersymmetric field theories. Here we are interested in later developments of this theory that were oriented towards extending to the super context techniques of symplectic geometry and related methods such as geometric quantization. We start then this section by presenting a brief account on the key contributions in those directions (Sect. 4.1). The results of Sect. 3 will then be rediscussed in the light of the general theory (Sects. 4.2 and 4.3).

It is worth stressing that the most important point of this section is the extension of Rothstein's characterization of supersymplectic supermanifolds to the Kähler context (Sect. 4.3). This extension was already discussed in [8,9] where a definition and a non-trivial example of the notion of a super-Kähler supermanifold were exhibited. Here, that definition is made more precise, explicit general formulae are given, and another non-trivial example is discussed, namely $(\mathscr{D}^{(1|2)}, \omega)$.

4.1. Supersymplectic Supermanifolds. It is known that the geometry of a manifold can be recovered from its structure sheaf, i.e. the algebra of functions on that manifold. A supermanifold is defined by extending such an algebra to a supercommutative superalgebra. The supergeometry of this supermanifold is then extracted from the superstructure sheaf thus obtained through known techniques [28]. As for usual manifolds, three types of supermanifolds emerge, namely, the C^∞, the real-analytic and the complex-analytic (or holomorphic) supermanifolds. From the results of the previous section it clearly appears that we are dealing here with the last type of supermanifolds.

Definition 4.1. [42,43] *A $(p|q)$-dimensional holomorphic supermanifold is a pair $(M, \mathscr{A}_M)$, where M is a p-dimensional complex manifold with holomorphic structure sheaf $\mathscr{O}_M$ and $\mathscr{A}_M$ is a sheaf of supercommutative superalgebras on M, such that:*

(a) *$\mathscr{A}_M/\mathscr{N}$ is isomorphic to $\mathscr{O}_M$, $\mathscr{N}$ being the subsheaf of nilpotent elements of $\mathscr{A}_M$, and*

(b) *$\mathscr{A}_M$ is locally isomorphic to the exterior sheaf $\wedge\mathscr{E}$, where $\mathscr{E} \equiv \mathscr{N}/\mathscr{N}^2$ is a locally free sheaf over $\mathscr{O}_M$; equivalently, for $\{U_\alpha\}$ an open cover of M, $\mathscr{A}_M(U_\alpha)$ is locally isomorphic to $\mathscr{O}_M(U_\alpha) \otimes \wedge\mathbb{C}^q \equiv \wedge\mathscr{E}(U_\alpha)$.*

Here $\wedge\mathbb{C}^q$ stands for the exterior algebra on $\mathbb{C}^q$, and p (resp. q) is the even (resp. odd) complex dimension of $(M, \mathscr{A}_M)$. Moreover, local supercoordinates on $(M, \mathscr{A}_M)$ are given by a set $(z^1, \ldots, z^p; \theta^1, \ldots, \theta^q)$, where $(z^1, \ldots, z^p)$ are local coordinates on M and $(\theta^1, \ldots, \theta^q)$ form a basis of $\mathscr{E}$ over $\mathscr{O}_M$. Notice that up to obvious modifications, Definition 4.1 applies equally well to C^∞ and real-analytic supermanifolds.

It is well known that a holomorphic vector bundle over M is completely specified once its sheaf of holomorphic sections is given. This sheaf is a locally free sheaf over $\mathscr{O}_M$. Hence, $\mathscr{E}$ in Definition 4.1 represents the sheaf of sections of a rank-q holomorphic vector bundle $\mathbb{F}$ over M. To any holomorphic supermanifold $(M, \mathscr{A}_M)$ one can then canonically associate the holomorphic supermanifold $(M, \mathscr{O}_M(\wedge\mathbb{F}))$, where $\mathscr{O}_M(\wedge\mathbb{F})$ is the sheaf of sections of the exterior bundle $\wedge\mathbb{F} \to M$. Condition **(b)** in Definition 4.1 above implies that $(M, \mathscr{A}_M)$ and $(M, \mathscr{O}_M(\wedge\mathbb{F}))$ are locally isomorphic. The same holds true for C^∞ and real-analytic supermanifolds. In these two last instances, this local isomorphism always extends to a global but non-canonical one [44]. However, this is not always true in the holomorphic case [42,43].

As for usual manifolds, the tangent sheaf is defined as the sheaf of superderivations of $\mathscr{A}_M$, and the cotangent sheaf $\Omega^1(\mathscr{A}_M)$ as its dual. Then a super de Rham complex can be constructed by introducing a coboundary operator d. From these ingredients one defines the notion of a supersymplectic supermanifold.

Definition 4.2. [28] *A supersymplectic supermanifold is a triple $(M, \mathscr{A}_M, \omega)$, where ω is a closed and non-degenerate even 2-superform on $(M, \mathscr{A}_M)$.*

Rothstein's characterization of C^∞ supersymplectic supermanifolds in terms of the geometry of vector bundles over the usual symplectic manifolds constitutes a major contribution to this topic [38]. Indeed, for ω *at most quadratic in the odd coordinates*, Rothstein's theorem [38] allows one to completely identify ω in terms of a symplectic structure on M and extra structures in the vector bundle sector. More precisely, using the global isomorphism mentioned above, it states that to any C^∞ supersymplectic supermanifold $(M, \mathscr{A}_M, \omega)$ there corresponds a set $(M, \omega_0, \mathbb{E}, g, \nabla^g)$, where (M, ω_0) is a symplectic manifold, $\mathbb{E}$ is a vector bundle over M with metric g and g-compatible connection ∇^g, such that $\mathscr{E}$ is the sheaf of linear functionals on $\mathbb{E}$ and ω is completely determined in terms of (ω_0, g, ∇^g) as follows:

$$\omega = \omega_0 - d\alpha_2, \quad \text{where } \alpha_2 = -g_{ab}\theta^a D\theta^b . \tag{4.1}$$

Here D is an operator defined on $\wedge\mathscr{E}$ with values in $\Omega^1(\mathscr{A}_M)$, such that:

$$D\theta^a \equiv d\theta^a - A^a_{ib}\theta^b dx^i, \quad a, b \in \{1, \dots, q\},\ i \in \{1, \dots, p = 2n\} , \tag{4.2}$$

where $(x^i; \theta^a)$ are now real supercoordinates on $(M, \mathscr{A}_M, \omega)$, and A^a_{ib} are the components of ∇^g in the basis of the generators θ^a of $\mathscr{E}$. The explicit form of (4.1) is given by:

$$\omega = \omega_0 + \frac{1}{2} g_{ab} R^b_{ijc}\theta^c\theta^a dx^i dx^j + g_{ab} D\theta^a D\theta^b , \tag{4.3}$$

where R^b_{ijc} are the components of $\operatorname{curv}\nabla^g$.

The correspondence mentioned above is one to one only in the C^∞ case. In the complex-analytic case either one considers supersymplectic holomorphic supermanifolds in the form $(M, \mathscr{O}_M(\wedge\mathbb{F}), \omega)$, for $\mathbb{F}$ a holomorphic vector bundle over M, or uses only the one way correspondence of Rothstein's theorem [38]. In both situations, equations similar to (4.1)–(4.3) hold; they are explicitly derived in Sect. 4.3. In what follows, *Rothstein's data* will refer to the set $(M, \omega_0, \mathbb{E}, g, \nabla^g)$ associated to a supersymplectic supermanifold $(M, \mathscr{A}_M, \omega)$.

Few precisions are now in order. To be able to decompose without any ambiguity an even two-superform as a sum of homogeneous components in the anticommuting variables, one needs to restrict bundle automorphisms of $\wedge\mathscr{E}$ to those automorphisms induced from bundle automorphisms of $\mathscr{E}$ [38]. Moreover, when ω contains higher order terms in the odd coordinates (more than quadratic), the identification of Rothstein's data requires the second part of Rothstein's theorem [38] which states that there exists a superdiffeomorphism ρ of $(M, \mathscr{A}_M)$ such that ρ is the identity modulo $\wedge^2\mathscr{E}$, and $\rho^*(\omega)$ is at most quadratic in the odd coordinates. Hence, the first part of Rothstein's theorem can be applied to $\rho^*(\omega)$. In other words, for $\omega = \omega_0 + \omega_2 + \omega_4 + \cdots$, where the subscripts refer to the degree of homogeneity in the odd coordinates, one needs first to find the superdiffeomorphism ρ (which depends only on $\omega_4 + \cdots$), then one uses it in order to transform ω into a 2-superform at most quadratic in the odd coordinates. Finally, one can identify Rothstein's data for ω. On the other hand, given those data one cannot reconstruct the original 2-superform. Indeed, only the transformed one is at reach, since ρ cannot be deduced from the above data. The supersymplectic form considered here (see (3.3)–(3.6)) is obviously of the preceding form, i.e. it is quartic in the odd coordinates. We will nevertheless show in Sect. 4.3 that ω_4 can be obtained explicitly from Rothstein's data of $\omega_0 + \omega_2$ without having recourse to any superdiffeomorphism ρ (see (4.18)). In other words, the whole supersymplectic form can be obtained from the simple knowledge of $(M, \omega_0, \mathbb{E}, g, \nabla^g)$. This seems to be a common feature of super coadjoint orbits of the type considered here.

Our task is now twofold. First, identify $\mathscr{D}^{(1|2)}$ as a holomorphic supermanifold, then identify Rothstein's data for the supersymplectic $(\mathscr{D}^{(1|2)}, \omega)$. The first part is straightforward. For the second one, we will make use of a very useful lemma proved in [8] in the case of $\mathscr{D}^{(1|1)} \equiv \mathrm{OSp}(1/2)/\mathrm{U}(1)$, and which applies to more general situations, in particular to the one in hand.

4.2. $\mathscr{D}^{(1|2)}$ as a Holomorphic Supermanifold. As a holomorphic supermanifold, the OSp(2/2)-homogeneous superspace $\mathscr{D}^{(1|2)}$ obtained in Sect. 3 corresponds to the pair $(\mathscr{D}^{(1)}, \mathscr{A}_{\mathscr{D}^{(1)}})$, where $\mathscr{D}^{(1)}$ is the unit disc, and $\mathscr{A}_{\mathscr{D}^{(1)}} \equiv \mathscr{A}^{(1|2)} = \mathscr{O}_{\mathscr{D}^{(1)}} \otimes \wedge \mathbb{C}^2$ is the defining superstructure sheaf. A general section h of $\mathscr{A}^{(1|2)}$ is a superholomorphic function,

$$h(z, \theta, \chi) = h_1(z) + \theta h_2(z) + \chi h_3(z) + \chi\theta h_4(z)\,, \tag{4.4}$$

where the h_i's are holomorphic functions on $\mathscr{D}^{(1)}$. Moreover, the vector bundle $\mathbb{F}$ is nothing but a trivial rank 2 holomorphic vector bundle over $\mathscr{D}^{(1)}$.

Finally, notice here that the super observables in (3.9) are sections of the complexified superstructure sheaf $\mathscr{A}^{(1|2)}_{\mathbb{C}} \equiv C^{\infty}_{\mathbb{C}} \otimes \wedge \mathbb{C}^4$; and the Hamiltonian vector superfields in (3.12) are derivations of $\mathscr{A}^{(1|2)}_{\mathbb{C}}$. The vector bundle canonically associated to $(\mathscr{D}^{(1)}, \mathscr{A}^{(1|2)}_{\mathbb{C}})$ is $\mathbb{F} \oplus \overline{\mathbb{F}}$, where $\overline{\mathbb{F}}$ is the complex conjugate bundle of $\mathbb{F}$.

4.3. $\mathscr{D}^{(1|2)}$ as a Super-Kähler Supermanifold. Let us now identify the data $(M, \omega_0, \mathbb{E}, g, \nabla^g)$ for $(\mathscr{D}^{(1|2)}, \omega)$. First of all, it is straightforward from (3.6) that the symplectic manifold (M, ω_0) is here the symplectic unit disc $(\mathscr{D}^{(1)}, \omega_0)$, where $\omega_0 \equiv -2i\tau(1-|z|^2)^{-2} dz d\bar{z}$. Indeed, observe that $\mathrm{SU}(1,1) \times \mathrm{U}(1)$ is the body of OSp(2/2), hence the body of $\mathscr{D}^{(1|2)} \equiv \mathrm{OSp}(2/2)/(\mathrm{U}(1) \times \mathrm{U}(1))$ is $\mathrm{SU}(1,1)/\mathrm{U}(1) \equiv \mathscr{D}^{(1)}$. On the other hand, when one sets the odd variables to zero, the OSp(2/2)-CS become the SU(1,1)-CS, the underlying geometry of which is known to be given by $(\mathscr{D}^{(1)}, \omega_0)$. Furthermore, according to Rothstein's theorem the vector bundle $\mathbb{E}$ is just $\mathbb{F}^*$, the dual of $\mathbb{F}$.

The identification of the remaining data is highly simplified if one makes use of the results of [8]. Indeed, in [8] it has been shown that Rothstein's data of the OSp(1/2) coadjoint orbit studied there can be directly read off from the superfunction f generating the supersymplectic form. More precisely, if one writes f as $f_0 + f_2 + f_4 + \cdots$, where f_{2n} designates that component of f which is homogeneous of degree $2n$ in the odd coordinates, then it appears that f_2 assumes the following form:

$$f_2 = -i g_{a\bar{b}} \theta^a \bar{\theta}^b\,, \tag{4.5}$$

where g is the sought for Rothstein's metric on $\mathbb{E}$. The θ^a's are the odd supercoordinates of the considered holomorphic supermanifold. They can also be viewed as a (local) frame field of $\mathbb{F}$ over $\mathscr{D}^{(1)}$. The notations in (4.5) are those commonly used in complex geometry, see for example [45].

At this point it is worth anticipating by mentioning that the above considerations are valid only in the *particular complex-analytic* case of a *super-Kähler* supermanifold, a notion defined in [8] and rediscussed below. Both $\mathscr{D}^{(1|1)}$ [8] and $\mathscr{D}^{(1|2)}$ are non-trivial examples of such a notion.

We now write the explicit form of g for $\mathscr{D}^{(1|2)}$. If the odd coordinates θ and χ are now denoted, respectively, by θ^1 and θ^2, then f_2 in (3.1) is given by:

$$f_2 = i(\tau - b)\frac{\theta^1\bar{\theta}^1}{(1-|z|^2)} + i(\tau + b)\frac{\theta^2\bar{\theta}^2}{(1-|z|^2)} \,. \tag{4.6}$$

A simple comparison of (4.5) and (4.6) leads to the following matrix form $\|g\|$ of g in the frame field of $\mathbb{E}$ over $\mathscr{D}^{(1)}$ which is dual to that of $\mathbb{F}$ given above:

$$\|g\| = \begin{pmatrix} \dfrac{\tau - b}{(1-|z|^2)} & 0 \\ 0 & \dfrac{\tau + b}{(1-|z|^2)} \end{pmatrix} . \tag{4.7}$$

This is a diagonal metric which is clearly Hermitian [45] since we are considering $|b| < \tau$ (typical CS).

It remains now to identify the g-compatible connection ∇^g. For this purpose we rederive Rothstein's results (4.1)–(4.2) in our *particular complex-analytic* setting. Once again this task is highly simplified, thanks to those observations in [8] that led to formula (4.5). We recall that Rothstein's formulae (4.1)–(4.3) were derived in the real C^∞ case [38].

The complex-analytic counterpart of the even 1-superform α_2 appearing in (4.1) can be obtained from (4.5) as follows:

$$\alpha_2 \equiv -\frac{i}{2}(\delta - \bar{\delta})f_2 \,. \tag{4.8}$$

A direct computation based on (4.5), with g a Hermitian metric on a holomorphic vector bundle $\mathbb{E} \to M$, leads to:

$$\alpha_2 = \frac{1}{2} g_{a\bar{b}}(\theta^a D\bar{\theta}^b + \bar{\theta}^b D\theta^a) \,, \tag{4.9}$$

where now

$$D\theta^a = d\theta^a + \Gamma^a_{ib}\theta^b dz^i, \qquad D\bar{\theta}^a = d\bar{\theta}^a + \Gamma^{\bar{a}}_{\bar{i}\bar{b}}\bar{\theta}^b d\bar{z}^i \,, \tag{4.10}$$

and

$$\Gamma^a_{ib} = g^{a\bar{c}}\frac{\partial g_{b\bar{c}}}{\partial z^i} \,. \tag{4.11}$$

One easily recognizes here the Γ^a_{ib}'s as the components of the (canonical) *Hermitian connection* associated to the Hermitian metric g on $\mathbb{E}$ [45]. Notice that these connection components are expressed in the frame field of $\mathbb{E}$, while in the C^∞ case of (4.1)–(4.3) the A^a_{ib}'s are expressed in the frame field of $\mathbb{F}$, dual to that of $\mathbb{E}$. As an endomorphism of a fibre, one is minus the transpose of the other [45]. This explains the sign difference and the change in the notation.

Before stating the final result concerning $(\mathscr{D}^{(1|2)}, \omega)$, let us carry on and evaluate the complex-analytic counterpart of (4.3). From (4.9)–(4.11), a straightforward computation gives

$$d\alpha_2 = g_{a\bar{b}} D\theta^a D\bar{\theta}^b - g_{a\bar{b}} R^a_{\bar{i}\bar{j}c}\theta^c\bar{\theta}^b dz^i d\bar{z}^j \,, \tag{4.12}$$

where

$$R^a_{\bar{j}ic} \equiv \frac{\partial \Gamma^a_{ic}}{\partial \bar{z}^j} \,. \tag{4.13}$$

From what precedes, we have the following immediate result:

$$\omega_2 \equiv -d\alpha_2 \,, \tag{4.14}$$

where ω_2 is given in (3.5) and $d\alpha_2$ is (4.12) for g the Hermitian metric in (4.7). Hence, in summary, we have proven the following:

Theorem 4.3. *Rothstein's data for* $(\mathscr{D}^{(1|2)}, \omega)$ *are*: $(\mathscr{D}^{(1)}, \omega_0, \mathbb{E}, g, \nabla^g)$, *where* $\mathscr{D}^{(1)}$ *in the unit disc,* ω_0 *is the* SU(1,1)*-invariant two-form on* $\mathscr{D}^{(1)}$ (3.4), $\mathbb{E} = \mathbb{F}^*$ *is a rank* 2 *trivial holomorphic vector bundle over* $\mathscr{D}^{(1)}, g$ *is the Hermitian metric* (4.7) *on* $\mathbb{E}$ *and* ∇^g *is the corresponding* (*canonical*) *Hermitian connection* (*the components of which can be explicitly evaluated using* (4.11)).

Notice now that the symplectic manifold $(\mathscr{D}^{(1)}, \omega_0)$ is moreover a Kähler manifold since $\omega_0 = -i\partial\bar{\partial} f_0$, where f_0, the odd-coordinates-independent part of f in (3.1), is a Kähler potential for ω_0, and $\partial = dz\frac{\partial}{\partial z}$. Hence, one clearly sees that $(\mathscr{D}^{(1|2)}, \omega)$ is a non-trivial example of the following definition of a super-Kähler supermanifold:

Definition 4.4. *A super-Kähler supermanifold* $(M, \mathscr{A}_M, \omega)$ *is a holomorphic super-symplectic supermanifold, whose Rothstein's data,* $(M, \omega_0, \mathbb{E}, g, \nabla^g)$, *are such that* (M, ω_0) *is a Kähler manifold,* $(\mathbb{E}, g)$ *is a holomorphic Hermitian vector bundle over* M *and* ∇^g *is the canonical Hermitian connection.*

As already mentioned the present situation allows us to go beyond Rothstein's theorem [38]. Indeed, ω_4 in (3.3)–(3.6) can also be rewritten in terms of Rothstein's data of $(\mathscr{D}^{(1|2)}, \omega)$ obtained above. This is achieved by simply noticing that in (3.1),

$$f_4 = \frac{1}{4\tau}(f_2)^2 \,. \tag{4.15}$$

Then

$$\alpha_4 \equiv -\frac{i}{2}(\delta - \bar{\delta}) f_4 = \frac{1}{2\tau} f_2 \alpha_2 \,, \tag{4.16}$$

such that

$$\omega_4 \equiv -d\alpha_4 = \frac{1}{2\tau}[f_2\omega_2 - df_2\alpha_2] \,. \tag{4.17}$$

A simple computation based on (4.5),(4.9) and (4.12), leads to the following:

$$\omega_4 = \frac{i}{2\tau} g_{a\bar{b}} g_{c\bar{d}} [\theta^a \bar{\theta}^d D\theta^c D\bar{\theta}^b + \theta^a \bar{\theta}^b (D\theta^c D\bar{\theta}^d - R^c_{i\bar{j}e} \theta^e \bar{\theta}^d dz^i d\bar{z}^j)] \,. \tag{4.18}$$

When g is the Hermitian metric in (4.7), ω_4 above is exactly the ω_4 of (3.6). The present extension of Rothstein's theorem is mainly based on the observation made in (4.15). This is in fact an intrinsic property of super-Kähler coadjoint orbits of some simple Lie supergroups. For such orbits, Rothstein's data determine the complete super-Kähler form. A detailed description of a general framework will be given elsewhere.

In analogy with the non-super case, and in view of (3.3) it is worth calling the real superfunction f in (3.1) a super-Kähler potential for the super-Kähler form ω given in (3.3)–(3.6). Such a potential is defined up to the addition of a superholomorphic or/and an anti-superholomorphic function on $\mathscr{D}^{(1|2)}$. Clearly, Rothstein's data for a super-Kähler supermanifold are encoded in its super-Kähler potential $f = f_0 + f_2 + \cdots$. Indeed, the body f_0 of f is a Kähler potential for the Kähler manifold (M, ω_0), while f_2 provides the Hermitian structure on the holomorphic bundle $\mathbb{E}$ (see 4.5).

The description of super-Kähler geometry given above is sufficient for our purpose. However, it is worth mentioning that Definition 4.4 can be made much more precise. Indeed, a deeper analysis of Rothstein's data for $(\mathscr{D}^{(1|2)}, \omega)$ shows that they determine an Einstein–Hermitian vector bundle [45] (see Sect. 7, 7.3). This very interesting observation deserves further investigations. More details will be given in a forthcoming publication.

The next section addresses the geometric quantization of $(\mathscr{D}^{(1|2)}, \omega)$. The super-Kähler character of the latter leads naturally to the existence of a super-Kähler polarization which makes the complete quantization program successful.

5. Geometric Quantization

By *geometric quantization* we refer here to the celebrated method, independently devised by Kostant [17] and Souriau [18], which associates to a given classical mechanics a quantum counterpart. Kostant–Souriau quantization procedure meets Kirillov's *method of orbits* [16] when the classical mechanics is described by a coadjoint orbit of a Lie group G. The quantum output comes then in the form of a Unitary Irreducible Representation (UIR) of the considered group G. This is the case we are interested in extending to the super context. From now on we will focus our attention on this kind of situation.

In practice, geometric quantization proceeds in two steps. The first one, called *prequantization*, consists in exhibiting a complex line bundle over $(M \equiv G/H, \omega)$, with Hermitian structure and compatible connection ∇, such that $\operatorname{curv} \nabla = \omega$. Such a line bundle exists whenever $[\omega]$ is an integral cohomology class (integrality condition). When lifted to this line bundle, the transitive (and symplectic) action of G on M gives rise to a unitary but reducible representation of G. The second step consists in using a *polarization* in order to select an irreducible subrepresentation of the prequantum representation. More precisely, the group action is restricted to the subspace of those L^2 sections of the prequantum line bundle which are covariantly constant along the vector fields generating the polarization. More details concerning the general procedure of geometric quantization can be found in [46, 47].

Super-prequantization was partly developed by Kostant [28]. Tuynman [48] completed that construction by equipping Kostant's super-prequantum bundle with a super-Hermitian structure compatible with the connection. We will follow here that construction, assuming that the reader is at least familiar with Kostant's work. On the other hand, because of the lack of a notion of polarization, the second part of the program was not considered in [28]. Here, as a super-Kähler supermanifold $(\mathscr{D}^{(1|2)}, \omega)$ is naturally equipped with a *super-Kähler polarization* that allows us to carry out the whole quantization program.

Before giving the details of the construction, it is worth mentioning that the notion of a polarization in the super context and in connection with geometric quantization already appeared in the literature. For instance, in the real case, a general definition of a polarization was given in [48]; it was then used to quantize the BRST charge. On the other hand, a super-Kähler polarization was introduced in [49] in order to quantize a field theory with fermionic degrees of freedom, i.e. an infinite dimensional flat phase space. Here we use a general notion in the complex-analytic case which allows quantizing non-trivial super phase spaces such as coadjoint orbits.

Finally, let us mention that recently another quantization method, namely the deformation quantization, has been extended to the super context in [50]; this method was applied in particular to the super-unit disc $\mathscr{D}^{(1|1)}$. Geometric quantization of the latter is considered in [9].

5.1. Super-Prequantization . Following Kostant's general scheme [28], in order to prequantize $(\mathscr{D}^{(1|2)}, \omega)$, one needs to exhibit a complex *line bundle sheaf* $\mathscr{L}^{(1|2)}$ with connection ∇ over $\mathscr{D}^{(1|2)}$, such that $\operatorname{curv} \nabla = \omega$. Such a line bundle sheaf exists if and only if there exists a complex line bundle $\mathbb{L}$ with connection ∇_0 over $\mathscr{D}^{(1)}$, such that $\operatorname{curv} \nabla_0 = \omega_0$ (i.e. *iff* $(\mathscr{D}^{(1)}, \omega_0)$ is prequantizable) [28]. Since our ω_0 is exact, it is known that such an $\mathbb{L}$ always exists. It is then not hard to see that,

$$\mathscr{L}^{(1|2)} \equiv \mathscr{L}^{(1)} \otimes \mathscr{A}_{\mathbb{C}}^{(1|2)} \cong \mathscr{A}_{\mathbb{C}}^{(1|2)} , \tag{5.1}$$

where $\mathscr{L}^{(1)}$ is the sheaf of C^∞ sections of $\mathbb{L}$.

To the classical observables in (3.9) one can associate the so-called prequantum operators which act in the space of C^∞ sections of $\mathscr{L}^{(1|2)}$. These operators are obtained using the following formula [28],

$$\widehat{H} \equiv -i\nabla_{X_H} + H, \quad \text{where } \nabla_{X_H} = X_H - iX_H \lrcorner \vartheta , \tag{5.2}$$

and

$$\begin{aligned}\vartheta \equiv -i\delta f = -2i\tau \frac{\bar{z}dz}{1-|z|^2} &\left[1 + i\frac{\tau - b}{2\tau} \frac{\bar{\theta}\theta}{1-|z|^2} + i\frac{\tau + b}{2\tau} \frac{\bar{\chi}\chi}{1-|z|^2} \right. \\ &\left. - \frac{\tau^2 - b^2}{2\tau^2} \frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2} \right] + (\tau - b)\frac{d\theta\,\bar{\theta}}{1-|z|^2} \left[1 + i\frac{\tau + b}{2\tau} \frac{\bar{\chi}\chi}{1-|z|^2} \right] \\ &+ (\tau + b)\frac{d\chi\,\bar{\chi}}{1-|z|^2} \left[1 + i\frac{\tau - b}{2\tau} \frac{\bar{\theta}\theta}{1-|z|^2} \right]\end{aligned} \tag{5.3}$$

is a 1-superform potential for the connection ∇ equipping the prequantum bundle $\mathscr{L}^{(1|2)}$; it is such that $\omega = -d\vartheta$. A straightforward computation gives,

$$\widehat{B} = \frac{\theta}{2}\frac{\partial}{\partial\theta} - \frac{\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}} - \frac{\chi}{2}\frac{\partial}{\partial\chi} + \frac{\bar{\chi}}{2}\frac{\partial}{\partial\bar{\chi}} + b ,$$

$$\widehat{K}_0 = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}} + \frac{\theta}{2}\frac{\partial}{\partial\theta} - \frac{\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}} + \frac{\chi}{2}\frac{\partial}{\partial\chi} - \frac{\bar{\chi}}{2}\frac{\partial}{\partial\bar{\chi}} + \tau ,$$

$$\widehat{K}_+ = z^2\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}} + z\theta\frac{\partial}{\partial\theta} + z\chi\frac{\partial}{\partial\chi} + 2\tau z ,$$

$$\widehat{K}_- = \frac{\partial}{\partial z} - \bar{z}^2\frac{\partial}{\partial\bar{z}} - \bar{z}\bar{\theta}\frac{\partial}{\partial\bar{\theta}} - \bar{z}\bar{\chi}\frac{\partial}{\partial\bar{\chi}} ,$$

$$\widehat{V}_+ = -i\left(\frac{\tau-b}{2\tau}\right)z\theta\frac{\partial}{\partial z} - \left(\frac{\tau+b}{2\tau}\right)\bar{\chi}\frac{\partial}{\partial\bar{z}} + \frac{\partial}{\partial\bar{\theta}}$$
$$+ i\left(z + \left(\frac{\tau-b}{2\tau}\right)\chi\theta\right)\frac{\partial}{\partial\chi} - i(\tau-b)\theta\,,$$
$$\widehat{W}_- = -i\left(\frac{\tau+b}{2\tau}\right)\chi\frac{\partial}{\partial z} - \left(\frac{\tau-b}{2\tau}\right)\bar{z}\bar{\theta}\frac{\partial}{\partial\bar{z}} + i\frac{\partial}{\partial\theta} + \left(\bar{z} - \left(\frac{\tau-b}{2\tau}\right)\bar{\theta}\bar{\chi}\right)\frac{\partial}{\partial\bar{\chi}}\,,$$
$$\widehat{V}_- = -i\left(\frac{\tau-b}{2\tau}\right)\theta\frac{\partial}{\partial z} - \left(\frac{\tau+b}{2\tau}\right)\bar{z}\bar{\chi}\frac{\partial}{\partial\bar{z}} + \left(\bar{z} + \left(\frac{\tau+b}{2\tau}\right)\bar{\theta}\bar{\chi}\right)\frac{\partial}{\partial\bar{\theta}} + i\frac{\partial}{\partial\chi}\,,$$
$$\widehat{W}_+ = -i\left(\frac{\tau+b}{2\tau}\right)z\chi\frac{\partial}{\partial z} - \left(\frac{\tau-b}{2\tau}\right)\bar{\theta}\frac{\partial}{\partial\bar{z}}$$
$$+ i\left(z - \left(\frac{\tau+b}{2\tau}\right)\chi\theta\right)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\bar{\chi}} - i(\tau+b)\chi\,. \tag{5.4}$$

One easily verifies that these operators close to the osp(2/2) Lie superalgebra. They provide thus a representation of osp(2/2) in the space of C^∞ sections of $\mathscr{L}^{(1|2)}$.

In the next subsection, using a natural *invariant super-Kähler polarization* on $\mathscr{D}^{(1|2)}$ we will select a subsheaf $\mathscr{L}_{\mathrm{p}}^{(1|2)}$ of $\mathscr{L}^{(1|2)}$; $\mathscr{L}_{\mathrm{p}}^{(1|2)}$ will then be equipped with a ∇-compatible *super-Hermitian structure*. When restricted to $\mathscr{L}_{\mathrm{p}}^{(1|2)}$ the above prequantum representation will reduce to a *super-unitary irreducible representation* of osp(2/2).

5.2. Super-Polarization. As for the Kähler unit disc $\mathscr{D}^{(1)}$ a natural super-polarization, called here *super-Kähler* and denoted $\mathscr{P}$, exists on $\mathscr{D}^{(1|2)}$. It is spanned by the vector superfields $\frac{\partial}{\partial z}, \frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\chi}$. One easily verifies that this is actually a good candidate for a super-polarization. Indeed, $\mathscr{P}$ fulfills all the required conditions, namely,

(i) $\mathscr{P}$ is involutive;

(ii) $\mathscr{P}$ is maximal isotropic, i.e. $\omega(Z,Y)=0, \forall\, Z,Y\in\mathscr{P}$ and $\dim\mathscr{P}=(1|2)=\dim\mathscr{D}^{(1|2)}$ as a holomorphic supermanifold.

Moreover, one can easily verify that,

(iii) $\mathscr{P}\cap\overline{\mathscr{P}}=\{0\}$;

(iv) $\mathscr{P}$ is invariant, namely, for X_H one of the Hamiltonian vector fields in (3.12), $[X_H,Z]\in\mathscr{P}, \forall\, Z\in\mathscr{P}$ (brackets denote here a commutator or an anticommutator).

Property **(iii)** confirms, in agreement with the results of Sect. 4, that $\mathscr{P}$ deserves to be called a *super Kähler* polarization for $\mathscr{D}^{(1|2)}$. On the other hand, property **(iv)** means that the Hamiltonian flows of the classical observables (3.9) preserve $\mathscr{P}$; the importance of this property will be stressed soon. Moreover, as it will be shown at the end of this section, $\mathscr{P}$ is *positive*. This property ensures that the final representation space is non-trivial. In summary, $\mathscr{P}$ is a positive invariant super-Kähler polarization.

This polarization is now used in order to select a subsheaf $\mathscr{L}_{\mathrm{p}}^{(1|2)}$ of $\mathscr{L}^{(1|2)}$. The latter consists of the sections $\psi(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi})$ of $\mathscr{L}^{(1|2)}(\mathscr{D}^{(1)})$ which are covariantly constant along $\overline{\mathscr{P}}$, i.e. those sections ψ which are such that:

$$\nabla_{\frac{\partial}{\partial \bar{z}}} \psi = 0, \quad \nabla_{\frac{\partial}{\partial \bar{\theta}}} \psi = 0, \quad \text{and} \quad \nabla_{\frac{\partial}{\partial \bar{\chi}}} \psi = 0\,, \tag{5.5}$$

where from now on the covariant superderivation is taken as in (5.2) but with ϑ, the superpolarization-adapted 1-superform potential for ∇, replaced by the *real* 1-superform potential α given by,

$$\begin{aligned}
\alpha &\equiv -\frac{i}{2}(\delta - \bar{\delta})f \\
&= i\tau \frac{z\,d\bar{z} - \bar{z}\,dz}{1-|z|^2}\left[1 + i\frac{\tau - b}{2\tau}\frac{\bar{\theta}\theta}{1-|z|^2} + i\frac{\tau + b}{2\tau}\frac{\bar{\chi}\chi}{1-|z|^2} - \frac{\tau^2 - b^2}{2\tau^2}\frac{\bar{\theta}\bar{\chi}\chi\theta}{(1-|z|^2)^2}\right] \\
&\quad + \frac{\tau - b}{2}\frac{d\theta\,\bar{\theta} + d\bar{\theta}\,\theta}{1-|z|^2}\left[1 + i\frac{\tau + b}{2\tau}\frac{\bar{\chi}\chi}{1-|z|^2}\right] \\
&\quad + \frac{\tau + b}{2}\frac{d\chi\,\bar{\chi} + d\bar{\chi}\,\chi}{1-|z|^2}\left[1 + i\frac{\tau - b}{2\tau}\frac{\bar{\theta}\theta}{1-|z|^2}\right]\,.
\end{aligned} \tag{5.6}$$

In (5.6) $f = f(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi})$ is the super-Kähler potential (3.1). Solutions to (5.5) are of the form

$$\psi(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi}) = \exp(-f/2)\phi(z,\theta,\chi)\,, \tag{5.7}$$

where ϕ is a superholomorphic section of $\mathscr{L}^{(1|2)}$, and $\exp(-f/2) \equiv \mathscr{N}'$ (see (2.28) and (3.1)). Notice from (5.7) that $\mathscr{L}_{\mathrm{p}}^{(1|2)}$ is isomorphic to $\mathscr{A}^{(1|2)}$, the superstructure sheaf of $\mathscr{D}^{(1|2)}$ viewed as a holomorphic supermanifold. Moreover, property (**iv**) above ensures that the action of the prequantum operators (5.4) in $\mathscr{L}_{\mathrm{p}}^{(1|2)}$ leaves the latter invariant.

Let us now equip $\mathscr{L}_{\mathrm{p}}^{(1|2)}(\mathscr{D}^{(1)})$ with a super-Hermitian structure $(\,\cdot\,,\,\cdot\,)$. For $\psi = \psi_{\bar{0}} + \psi_{\bar{1}}$ and $\psi' = \psi'_{\bar{0}} + \psi'_{\bar{1}} \in \mathscr{L}_{\mathrm{p}}^{(1|2)}(\mathscr{D}^{(1)}) = \mathscr{L}_{\mathrm{p}}^{(1|2)}(\mathscr{D}^{(1)})_{\bar{0}} \oplus \mathscr{L}_{\mathrm{p}}^{(1|2)}(\mathscr{D}^{(1)})_{\bar{1}}$, this is given by:

$$(\psi',\psi) \equiv \overline{\psi'}\psi = \overline{\psi'_{\bar{0}}}\psi_{\bar{0}} + \overline{\psi'_{\bar{1}}}\psi_{\bar{1}} + \overline{\psi'_{\bar{0}}}\psi_{\bar{1}} + \overline{\psi'_{\bar{1}}}\psi_{\bar{0}}\,. \tag{5.8}$$

The latter clearly satisfies (2.13). Notice however that $(\,\cdot\,,\,\cdot\,)$ is *not* homogeneous, i.e. it is not of the form (2.14). Moreover, it is not hard to verify that the superconnection ∇ on $\mathscr{L}^{(1|2)}$ is compatible with (5.8), i.e.

$$X(\psi',\psi) = (\nabla_X\psi',\psi) + (-1)^{\varepsilon(\psi')\varepsilon(X)}(\psi',\nabla_X\psi)\,, \tag{5.9}$$

where ψ' is now a homogeneous section of $\mathscr{L}_{\mathrm{p}}^{(1|2)}(\mathscr{D}^{(1)})$ of parity $\varepsilon(\psi')$, X is a real vector superfield on $\mathscr{D}^{(1|2)}$ homogeneous of parity $\varepsilon(X)$, and $\nabla_X = X - iX \,\lrcorner\, \alpha$, for α given in (5.6).

As for the usual geometric quantization in the presence of a Kähler polarization, using the super-Hermitian structure (5.8), the superspace of sections ψ of $\mathscr{L}_{\mathrm{p}}^{(1|2)}(\mathscr{D}^{(1)})$ can be equipped with a super-inner product given by:

$$\langle\langle\psi',\psi\rangle\rangle_{\tau,b} \equiv \int_{\mathscr{D}^{(1|2)}} (\psi',\psi)\,d\mu\,, \tag{5.10}$$

where $d\mu = d\mu(z,\bar{z},\theta,\bar{\theta},\chi,\bar{\chi})$ is the OSp(2/2)-invariant super-measure on $\mathscr{D}^{(1|2)}$ obtained from ω in (3.19). Because of the isomorphism mentioned after (5.7), this super-inner product can be understood as an inner product on the space of sections ϕ of $\mathscr{A}^{(1|2)}$ (i.e. the space of superholomorphic sections of $\mathscr{L}^{(1|2)}$). Hence, using (5.7), we can write

$$\langle\langle \phi', \phi \rangle\rangle_{\tau,b} = \int_{\mathscr{D}^{(1|2)}} e^{-f} (\phi', \phi)\, d\mu\,. \tag{5.11}$$

Let us now investigate the status of this super-inner product from the Hilbert space point of view.

Since $|b| < \tau$, one can always write a superholomorphic function $\phi(z,\theta,\chi)$ on $\mathscr{D}^{(1|2)}$ as follows:

$$\begin{aligned}\phi(z,\theta,\chi) &= \phi_1(z) + \theta\sqrt{\tau - b}\,\phi_2(z)\\ &\quad + \chi\sqrt{\tau+b}\,\phi_3(z) + \chi\theta\sqrt{\frac{(\tau^2-b^2)(2\tau+1)}{2\tau}}\,\phi_4(z)\,,\end{aligned} \tag{5.12}$$

where ϕ_i, $i = 1,\ldots,4$, are holomorphic functions on $\mathscr{D}^{(1)}$.

The super integration in (5.11) can be partially carried out. Indeed, replacing (3.1), (3.19), (5.8) and (5.12) in (5.11), and using Berezin integration over the odd Grassmann variables [22], namely, the only non-zero integral being $\int d\theta\, d\bar{\theta}\, d\chi\, d\bar{\chi}\, (\bar{\chi}\chi\bar{\theta}\theta) = 1$, one obtains:

$$\begin{aligned}\langle\langle \phi', \phi \rangle\rangle_{\tau,b} &= \langle \phi'_1, \phi_1\rangle_{k=\tau} + i\langle \phi'_2, \phi_2\rangle_{k=\tau+\frac{1}{2}}\\ &\quad + i\langle \phi'_3, \phi_3\rangle_{k=\tau+\frac{1}{2}} + \langle \phi'_4, \phi_4\rangle_{k=\tau+1} \in \mathbb{C}\,,\end{aligned} \tag{5.13}$$

where

$$\langle \phi', \phi\rangle_k \equiv \frac{2k-1}{\pi} \int_{\mathscr{D}^{(1)}} \overline{\phi'(z)}\,\phi(z) \frac{dz\, d\bar{z}}{(1-|z|^2)^{2-2k}}, \quad \text{for } k > \frac{1}{2}\,, \tag{5.14}$$

is the usual inner product on the representation space of the holomorphic (positive) discrete series $D(k)$ of SU(1, 1) which arise through geometric quantization of the unit disc $(\mathscr{D}^{(1)}, \omega_0)$ [12, 51, 52].

We can now define a natural notion of square integrability of superholomorphic sections of a prequantum bundle sheaf, and thus that of a *super-Hilbert space.* First observe that $\langle\langle \cdot\,, \cdot \rangle\rangle_{\tau,b}$ is an *even super-Hermitian form* on the space of superholomorphic sections of $\mathscr{L}^{(1|2)}$. Indeed,

$$\overline{\langle\langle \phi', \phi \rangle\rangle}_{\tau,b} = (-1)^{\varepsilon(\phi')\varepsilon(\phi)} \langle\langle \phi, \phi' \rangle\rangle_{\tau,b}\,, \tag{5.15}$$

and moreover,

$$\langle\langle \phi', \phi \rangle\rangle_{\tau,b} = \langle\langle \phi'_{\bar{0}}, \phi_{\bar{0}} \rangle\rangle_{\tau,b,\bar{0}} + i \langle\langle \phi'_{\bar{1}}, \phi_{\bar{1}} \rangle\rangle_{\tau,b,\bar{1}}\,. \tag{5.16}$$

The new quantities and notation in the above equation can be easily identified in terms of the old ones appearing in (5.7), (5.13) and (5.14). A simple comparison of (5.15) and (5.16) with respectively (2.13) and (2.14) confirms our claim. This suggests the following natural definition:

Definition 5.1. *A superholomorphic section $\phi = \phi_{\bar{0}} + \phi_{\bar{1}}$ of $\mathscr{L}^{(1|2)}$ is said to be square integrable if both $\phi_{\bar{0}}$ and $\phi_{\bar{1}}$ are respectively square integrable with respect to the Hermitian forms $\langle\langle \cdot\,, \cdot \rangle\rangle_{\tau,b,\bar{0}}$ and $\langle\langle \cdot\,, \cdot \rangle\rangle_{\tau,b,\bar{1}}$.*

A definition of a *super-Hilbert space*, different from those proposed in [27, 53, 54], immediately follows.

Definition 5.2. *A super-Hilbert space is a pair* $(\mathscr{H}, \langle\langle \cdot , \cdot \rangle\rangle)$, *where* $\mathscr{H} \equiv \mathscr{H}_{\bar{0}} \oplus \mathscr{H}_{\bar{1}}$ *is a complex superspace equipped with a super-Hermitian form,* $\langle\langle \cdot , \cdot \rangle\rangle = \langle\langle \cdot , \cdot \rangle\rangle_{\bar{0}} + i\langle\langle \cdot , \cdot \rangle\rangle_{\bar{1}}$, *such that* $(\mathscr{H}_{\bar{0}}, \langle\langle \cdot , \cdot \rangle\rangle_{\bar{0}})$ *and* $(\mathscr{H}_{\bar{1}}, \langle\langle \cdot , \cdot \rangle\rangle_{\bar{1}})$ *are both Hilbert spaces.*

According to these definitions, the space of superholomorphic sections of $\mathscr{L}^{(1|2)}$ equipped with the super-Hermitian form (5.13) can be turned into a super-Hilbert space simply by taking ϕ_i, $i = 1, \ldots, 4$, to be L^2 functions on the unit disc with respect to the corresponding Hermitian forms, namely, that $\|\phi_1\|^2_{k=\tau} < \infty$, $\|\phi_2\|^2_{k=\tau+\frac{1}{2}} < \infty$, $\|\phi_3\|^2_{k=\tau+\frac{1}{2}} < \infty$ and $\|\phi_4\|^2_{k=\tau+1} < \infty$. Accordingly, $\phi(z, \theta, \chi)$ in (5.12) can be called an L^2 superholomorphic section of $\mathscr{L}^{(1|2)}$ with respect to the super-Hermitian form (5.11). The obtained super-Hilbert space, denoted subsequently $\mathscr{H}_{\tau,b}$, constitutes then the representation superspace carrying an *explicit* realization of the typical irreducible representation $V(\tau, b)$ of osp(2/2).

Remark 5.3.We could have equipped $\mathscr{L}^{(1|2)}(\mathscr{D}^{(1)})$ with an *even* super-Hermitian structure $((\cdot , \cdot))$ instead of the one of indefinite parity introduced in (5.8). The former would assume the following form $((\psi', \psi)) \equiv \overline{\psi'_{\bar{0}}}\psi_{\bar{0}} + \overline{\psi'_{\bar{1}}}\psi_{\bar{1}}$. In this case (5.9) will be true only modulo *odd* quantities. However, $\langle\langle \cdot , \cdot \rangle\rangle_{\tau,b} \equiv \int_{\mathscr{D}^{(1|2)}} (\cdot , \cdot)\, d\mu = \int_{\mathscr{D}^{(1|2)}} ((\cdot , \cdot))\, d\mu$, since $((\cdot , \cdot))$ and $(\cdot , \cdot)$ differ only by *odd* quantities which disappear when one integrates over anti-commuting variables using Berezin integration. For the same reason, all the results following (5.10) will still hold true.

Remark 5.4.The important result in (5.13)–(5.14) follows from the special form adopted for ϕ in (5.12), and the choice of normalization made for $d\mu$ in Sect. 3.3, namely, $\int_{\mathscr{D}^{(1|2)}} \exp(-f)\, d\mu = 1$.

The generators of osp(2/2) are represented in $\mathscr{H}_{\tau,b}$ by the super holomorphic restrictions of the prequantum operators (5.4). More precisely,

$$\widehat{B} = \frac{\theta}{2}\frac{\partial}{\partial\theta} - \frac{\chi}{2}\frac{\partial}{\partial\chi} + b\,, \qquad \widehat{K}_0 = z\frac{\partial}{\partial z} + \frac{\theta}{2}\frac{\partial}{\partial\theta} + \frac{\chi}{2}\frac{\partial}{\partial\chi} + \tau\,,$$

$$\widehat{K}_+ = z^2\frac{\partial}{\partial z} + z\theta\frac{\partial}{\partial\theta} + z\chi\frac{\partial}{\partial\chi} + 2\tau z\,, \qquad \widehat{K}_- = \frac{\partial}{\partial z}\,,$$

$$\widehat{V}_+ = -i\left(\frac{\tau - b}{2\tau}\right) z\theta\frac{\partial}{\partial z} + i\left(z + \left(\frac{\tau - b}{2\tau}\right)\chi\theta\right)\frac{\partial}{\partial\chi} - i(\tau - b)\theta\,,$$

$$\widehat{W}_- = -i\left(\frac{\tau + b}{2\tau}\right)\chi\frac{\partial}{\partial z} + i\frac{\partial}{\partial\theta}\,,$$

$$\widehat{V}_- = -i\left(\frac{\tau - b}{2\tau}\right)\theta\frac{\partial}{\partial z} + i\frac{\partial}{\partial\chi}\,,$$

$$\widehat{W}_+ = -i\left(\frac{\tau + b}{2\tau}\right) z\chi\frac{\partial}{\partial z} + i\left(z - \left(\frac{\tau + b}{2\tau}\right)\chi\theta\right)\frac{\partial}{\partial\theta} - i(\tau + b)\chi\,. \quad (5.17)$$

The obtained representation is *super-unitary*. Indeed, in agreement with (5.9), the *superadjoint* (or *super Hermitian conjugate*) $\widehat{X}^\dagger$ of an operator $\widehat{X}$ acting in $\mathscr{H}_{\tau,b}$ is defined as follows:

$$\langle\langle \widehat{X}^\dagger \phi', \phi\rangle\rangle_{\tau,b} = (-1)^{\varepsilon(\phi')\varepsilon(X)}\langle\langle \phi', \widehat{X}\phi\rangle\rangle_{\tau,b}\,, \tag{5.18}$$

for ϕ' a homogeneous superholomorphic section of $\mathscr{L}^{(1|2)}$ and $\widehat{X}$ a homogeneous operator. Hence, the quantum counterpart $\widehat{H}$ of a *real* classical observable $H^{\rm cl}$ is a *self-superadjoint* operator acting in $\mathscr{H}_{\tau,b}$, i.e. $\widehat{H}^\dagger = \widehat{H}$. Specifically, since both $B^{\rm cl}$ and $K_0^{\rm cl}$ are real (see (3.10)), the associated quantum operators in (5.17) are self-superadjoint:

$$\widehat{B}^\dagger = \widehat{B} \quad \text{and} \quad \widehat{K}_0^\dagger = \widehat{K}_0\,. \tag{5.19}$$

On the other hand, the reality of $(K_+^{\rm cl} + K_-^{\rm cl})$, $(V_+^{\rm cl} + iW_-^{\rm cl})$ and of similar combinations that can be obtained from (3.10) leads to the following,

$$(\widehat{K}_+)^\dagger = \widehat{K}_-, \quad (\widehat{V}_+)^\dagger = i\widehat{W}_- \quad \text{and} \quad (\widehat{V}_-)^\dagger = i\widehat{W}_+\,. \tag{5.20}$$

As it should be, these results are in perfect agreement with the relations we started with at the level of the abstract representation theory (compare (5.18)–(5.20), with (B.10), (B.13) and (B.14)).

Geometric quantization of $(\mathscr{D}^{(1|2)}, \omega)$ is hence completed. It remains now to integrate the obtained super-unitary irreducible representation of osp(2/2) to a representation of OSp(2/2) (see Sect. 7, 7.5). The integrability condition, which we assumed from the beginning, ensures that such a procedure leads actually to a non-trivial representation of OSp(2/2). Indeed, recall that we started with an *integrable* typical osp(2/2)-module $V(\tau, b)$, i.e. both b and $\tau \in \frac{1}{2}\mathbb{N}$ (with $\tau > \frac{1}{2}$, in agreement with (5.14)).

Finally, we briefly discuss the positivity of our super-Kähler polarization $\mathscr{P}$. In the non-super context, the positivity of a Kähler polarization ensures that the unitary irreducible representation obtained is non-trivial, i.e. the corresponding Hilbert space does not reduce to the zero function [46]. Because $\mathscr{H}_{\tau,b}$ is clearly non-trivial, our polarization $\mathscr{P}$ is then positive with respect to the putative extension to the super context of the positivity condition. An analysis of the present situation and of the one considered in [9] leads to a natural supergeometric definition of the positivity of a super-Kähler polarization. This will be given elsewhere.

6. Atypical OSp(2/2)-CS and Associated Orbits

We now briefly discuss the main results concerning the OSp(2/2) atypical representations: the atypical coherent states and their underlying supergeometry. Recall that the atypical representations occur when $|b| = \tau$. It is then not hard to see that by taking $b \to \tau$ or $b \to -\tau$, almost all the results obtained in the previous sections reduce to atypical analogs. Here we emphasize the main features of the case $b = -\tau$ (similar considerations applied to the case $b = \tau$ lead to the same results).

If $b = -\tau$, the lowest-weight vector is $|-\tau, \tau, \tau\rangle \equiv |\tau, \tau\rangle$. As it is shown in Appendix B, $W_+|\tau,\tau\rangle$ is a primitive vector generating an osp(2/2)-submodule of zero-norm states. Hence, $|\tau,\tau\rangle$ satisfies not only (2.5)–(2.6), but it is also such that

$$W_+|\tau,\tau\rangle = 0\,. \tag{6.1}$$

The appropriate irreducible osp(2/2)-module is no longer $V(\tau, b)$, given in (2.9), but the one given in (2.11). Instead of having four families of states, as in (2.10) or (B.1), we have only two:

$$K_+^m|\tau,\tau\rangle \quad \text{and} \quad K_+^m V_+^m|\tau,\tau\rangle, \; m \geqq 0 . \tag{6.2}$$

In other words,

$$U(\tau) \equiv V(\tau,-\tau)/V'(\tau,-\tau) \equiv \mathrm{span}\left\{|\tau,\tau+m\rangle, \left|\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle, \; m \in \mathbb{N}\right\} . \tag{6.3}$$

The same techniques as in Sect. 2 lead to the following atypical CS in $\tilde{U}(\tau)$. Starting with $|\tau,\tau\rangle$, these CS appear to be parametrized only by two variables. Indeed, as a consequence of (6.1),

$$\begin{aligned} |a,\theta,\chi\rangle &\equiv \mathcal{M} \exp(aK_+ + \theta V_+ + \chi W_+)|\tau,\tau\rangle , \\ &\equiv \mathcal{M} \exp(zK_+ + \theta V_+)|\tau,\tau\rangle \equiv |z,\theta\rangle , \end{aligned} \tag{6.4}$$

where $a \in \mathscr{B}_0$, and $\theta, \chi \in \mathscr{B}_1$ such that $z = a - \frac{1}{2\tau}\chi\theta \in \mathbb{C}$. A simple comparison of (6.4) with the corresponding equation for the typical CS in Sect. 2 shows that the atypical CS are simply obtained from the typical ones by taking $b = -\tau$; they clearly do not depend on χ. More precisely,

$$\begin{aligned} |z,\theta\rangle = \mathcal{M} \Bigg[&\sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2\tau+m)}{m!\Gamma(2\tau)}}\, z^m|\tau,\tau+m\rangle + \theta\sqrt{2\tau}\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(2\tau+m+1)}{m!\Gamma(2\tau+1)}} \\ &\times z^m \left|\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle \Bigg] , \end{aligned} \tag{6.5}$$

where the real normalization constant is

$$\mathcal{M} = (1-|z|^2)^\tau \left[1 + i\tau\frac{\bar{\theta}\theta}{1-|z|^2}\right] . \tag{6.6}$$

At this point, it is worth noticing in connection with Remark 2.7 that the atypical CS are nothing but OSp(1/2)-CS [34, 8], where here OSp(1/2) is the Lie subsupergroup of OSp(2/2) whose Lie superalgebra has $\{K_0, K_\pm, F_\pm \equiv (V_\pm + W_\pm)/\sqrt{2}\}$ as a Cartan–Weyl basis. More precisely,

$$|z,\theta\rangle = \mathcal{M} \exp(zK_+ + \sqrt{2}\theta F_+)|\tau,\tau\rangle ; \tag{6.7}$$

once again this is a direct consequence of (6.1). Hence, up to the above rescaling of θ, the analysis of the supergeometry underlying the OSp(2/2) atypical CS and its geometric quantization are *almost* the same as those already considered in [8, 9]. The main differences are discussed below.

The variables z and θ above parametrize the $N = 1$ super-unit disc OSp(1/2)/U(1) $\equiv \mathscr{D}^{(1|1)}$ [8]. The super-Kähler potential on $\mathscr{D}^{(1|1)}$ is given by,

$$f(z,\bar{z},\theta,\bar{\theta}) = \log|\langle\tau,\tau|\bar{z},\bar{\theta}\rangle|^{-2} = -2\tau\left[\log(1-|z|^2) - i\frac{\bar{\theta}\theta}{1-|z|^2}\right] , \tag{6.8}$$

and the super-Kähler superform is simply,

$$\omega = \frac{-2i\tau}{(1-|z|^2)^2}\left[1 + i\bar{\theta}\theta\frac{1+|z|^2}{(1-|z|^2)}\right] dz\, d\bar{z} - \frac{2\tau}{1-|z|^2} d\theta\, d\bar{\theta}$$

$$- \frac{2\tau}{(1-|z|^2)^2}[\theta\bar{z}\, dz\, d\bar{\theta} - \bar{\theta} z\, d\theta\, d\bar{z}]\,. \tag{6.9}$$

The osp(2/2) classical observables are obtained from (3.9) by taking $b = -\tau$, or equivalently by formally setting $\chi = 0$. When evaluated as in the typical case, the Hamiltonian vector superfields assume now the following form,

$$X_B = i\frac{\theta}{2}\frac{\partial}{\partial\theta} - i\frac{\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}}, \qquad X_{K_0} = iz\frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial\bar{z}} + i\frac{\theta}{2}\frac{\partial}{\partial\theta} - i\frac{\bar{\theta}}{2}\frac{\partial}{\partial\bar{\theta}}\,, \tag{6.10}$$

$$X_{K_+} = iz^2\frac{\partial}{\partial z} - i\frac{\partial}{\partial\bar{z}} + iz\theta\frac{\partial}{\partial\theta}, \qquad X_{K_-} = i\frac{\partial}{\partial z} - i\bar{z}^2\frac{\partial}{\partial\bar{z}} - i\bar{z}\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\,, \tag{6.11}$$

$$X_{V_+} = z\theta\frac{\partial}{\partial z} + i\frac{\partial}{\partial\bar{\theta}}, \qquad X_{W_-} = -i\bar{z}\bar{\theta}\frac{\partial}{\partial\bar{z}} - \frac{\partial}{\partial\theta}\,, \tag{6.12}$$

$$X_{V_-} = \theta\frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial\bar{\theta}}, \qquad X_{W_+} = -i\bar{\theta}\frac{\partial}{\partial\bar{z}} - z\frac{\partial}{\partial\theta}\,. \tag{6.13}$$

This corresponds to simultaneously taking $b = -\tau$ and eliminating χ in (3.12).

From these explicit expressions one immediately identifies the superalgebra of the isotropy subsupergroup G_0 at the origin ($z = 0,\ \theta = 0$) of the atypical phase space. Clearly, X_B, X_{K_0}, X_{V_-} and X_{W_+} act trivially at the origin. Hence, the isotropy subsupergroup is generated by $\{B, K_0, V_-, W_+\}$. From the commutation relations (2.2) it appears that $G_0 \equiv \mathrm{U}(1/1)$, and thus the atypical OSp(2/2)-CS are parametrized by the super-Kähler homogeneous space OSp(2/2)/U(1/1) $\equiv (\mathscr{D}^{(1|1)}, \omega)$. This could have been already deduced from (2.5)–(2.6) and (6.1).

Rothstein's data for $(\mathscr{D}^{(1|1)}, \omega)$ are given in [8]. They can be rederived by simply using results of Sect. 4. Indeed, the rank 2 holomorphic vector bundles $\mathbb{E}$ and $\mathbb{F}$ reduce both to a holomorphic line bundle. Moreover, from (4.7) one easily sees that when $b = -\tau$, we are left with the Hermitian metric $\|g\| = g_{1\bar{1}} = 2\tau/(1-|z|^2)$. This is twice the metric obtained in [8].

On the other hand geometric quantization leads to the super-Hilbert space obtained in [9]. This result corresponds to taking $b = -\tau$ in Sect. 5. Generators of osp(2/2) are however represented by the following operators,

$$\widehat{B} = \frac{\theta}{2}\frac{\partial}{\partial\theta} - \tau, \qquad \widehat{K}_0 = z\frac{\partial}{\partial z} + \frac{\theta}{2}\frac{\partial}{\partial\theta} + \tau\,, \tag{6.14}$$

$$\widehat{K}_+ = z^2\frac{\partial}{\partial z} + z\theta\frac{\partial}{\partial\theta} + 2\tau z, \qquad \widehat{K}_- = \frac{\partial}{\partial z}\,, \tag{6.15}$$

$$\widehat{V}_+ = -iz\theta\frac{\partial}{\partial z} - 2i\tau\theta, \qquad \widehat{W}_- = i\frac{\partial}{\partial\theta}, \tag{6.16}$$

$$\widehat{V}_- = -i\theta\frac{\partial}{\partial z}, \qquad \widehat{W}_+ = iz\frac{\partial}{\partial\theta}\,. \tag{6.17}$$

Finally, the super-measure on $\mathscr{D}^{(1|1)}$ can not be obtained simply as the limit $b \longrightarrow -\tau$ of (3.19). It has to be evaluated starting from (6.9) using the same technique as in Sect. 3.3. This leads to,

$$d\mu(z,\bar{z},\theta,\bar{\theta}) = \frac{i}{\pi(1-|z|^2)}\left[1 + i\frac{\bar{\theta}\theta}{1-|z|^2}\right] dz\, d\bar{z}\, d\theta\, d\bar{\theta}\,. \tag{6.18}$$

7. Miscellaneous Results and Discussions

Here are gathered a few consequences of the main results of the paper. A few other important points are discussed further.

7.1. Square integrability. When speaking about coherent states the first of their properties that comes to mind is the so-called resolution of the identity. In the non-super case this property reflects the square integrability of the unitary irreducible representation these special states belong to. Does this notion extend to the super case? The answer is yes. A simple computation based on (3.19), (2.27) and the Berezin integration leads to the following:

$$\int_{\mathscr{D}^{(1|2)}} |\bar{z},\bar{\theta},\bar{\chi}\rangle\langle\bar{z},\bar{\theta},\bar{\chi}|\, d\mu = \mathbb{I}\,. \tag{7.1}$$

Here $\mathbb{I} \equiv \mathbb{I}_{V(\tau,b)}$. A similar identity holds for the atypical CS. Hence, this allows a straightforward super extension of the definition of a square integrable representation. The above identity (7.1) provides a new argument that can be added to those already listed in [33] in order to justify calling the super-unitary irreducible representations of osp(2/2) considered here *discrete series representations.* As for usual CS, another immediate consequence of (7.1) is that $\mathscr{H}_{\tau,b}$ is a reproducing super-Hilbert space. This applies to both typical and atypical super-Hilbert spaces.

7.2. Status of z. For simplicity, the variable z was considered from the beginning as a usual complex variable (see (2.23)). The main reason behind this choice is to make the connection between the results of Sects. 2 and 3, and Rothstein's approach to supersymplectic supergeometry as described in Sect. 4, free of any change of coordinates. The same argument applies to the integrals over the unit disc $\mathscr{D}^{(1)}$ that appear in Sect. 5 (see (5.13)–(5.14)). Indeed, if the soul of z was different from zero, then those integrals and the complex geometry of Sect. 4 would be meaningless, unless a change of coordinates transforming z into a "soulless" variable is performed. Hence, our initial choice prevents us from making any change of coordinates.

7.3. Whitney sum and Einstein–Hermitian vector bundles. The rank 2 holomorphic vector bundle $\mathbb{E}$ intervening in Rothstein's data for the typical orbits (see Theorem 4.3) is the Whitney sum of two holomorphic line bundles over the unit disc $\mathscr{D}^{(1)}$, i.e. $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$. Independently, each component of $\mathbb{E}$ provides Rothstein's data for a super-Kähler subsupermanifold of $(\mathscr{D}^{(1|2)},\omega)$. More precisely, $(\mathscr{D}^{(1)},\omega_0,\mathbb{E}_i,g_i,\nabla^{g_i})$ for $i=1$ or 2, are such data for a $(1|1)$-dimensional super-Kähler supermanifold, denoted $\mathscr{D}^{(1|1)}_{\tau\pm b}$, where g_1 (resp. g_2) is the Hermitian metric on $\mathbb{E}_1$ (resp. $\mathbb{E}_2$) given in (4.7) as the first (resp. second) diagonal entry, and ∇^{g_i} are the associated Hermitian connections. These supermanifolds are $N=1$ super-Kähler extensions of $(\mathscr{D}^{(1)}, \omega_0)$. Their 2-superforms are obtained from (3.3)–(3.6) by setting $\chi = 0$ in the first instance and $\theta = 0$ in the second. These two super-unit discs are in fact OSp(2/2)-homogeneous spaces superdiffeomorphic to $\mathscr{D}^{(1|1)}$, the $N=1$ super-unit disc which is a supersymplectic homogeneous space for both OSp(2/2) and OSp(1/2) (see Sect. 6). Since both $\mathbb{E}_1$ and $\mathbb{E}_2$ are trivial line bundles over $\mathscr{D}^{(1)}$, Rothstein's data for $\mathscr{D}^{(1|1)}_{\tau+b}$ and $\mathscr{D}^{(1|1)}_{\tau-b}$ differ only by the constant factor, $\tau \pm b$, in front of the Hermitian structure on these bundles (see (4.7)).

The above observations lead to the following interesting picture. Let us denote by $\mathbf{R}_\lambda$ the data $(\mathscr{D}^{(1)}, \omega_0, \mathbb{E}_\lambda, g_\lambda, \nabla^{g_\lambda})$, where ω_0 is given in (3.4), $\mathbb{E}_\lambda$ is a trivial holomorphic line bundle over $\mathscr{D}^{(1)}$, $\|g_\lambda\| = \lambda(1-|z|^2)^{-1}$ (with $\lambda \geqq 0$) is a Hermitian structure on $\mathbb{E}_\lambda$, and ∇^{g_λ} is the corresponding Hermitian connection. Then we have: (**i**) Rothstein's data for the atypical OSp(2/2)-coadjoint orbit realized as $\mathscr{D}^{(1|1)}$ are given by $\mathbf{R}_{2\tau}$; (**ii**) Rothstein's data for the typical OSp(2/2)-coadjoint orbit realized as $\mathscr{D}^{(1|2)}$ are given by $\mathbf{R}_{\lambda_1} \oplus \mathbf{R}_{\lambda_2}$ such that $\lambda_1 + \lambda_2 = 2\tau$. Here the symbol "$\oplus$" indicates that we have to take the Whitney sum of $(\mathbb{E}_{\lambda_1}, g_{\lambda_2})$ and $(\mathbb{E}_{\lambda_2}, g_{\lambda_2})$. We recall at this point that the information in $\mathbf{R}_{\tau-b} \oplus \mathbf{R}_{\tau+b}$ is sufficient to entirely reconstruct the 2-superform (3.3)–(3.6) (see Sect. 4). Hence, this suggests that $\mathbf{R}_\lambda$ is the basic building block for describing super-Kähler coadjoint orbits of simple Lie supergroups super extending SU(1,1).

A deeper analysis of $\mathbf{R}_\lambda = (\mathscr{D}^{(1)}, \omega_0, \mathbb{E}_\lambda, g_\lambda, \nabla^{g_\lambda})$ shows that $(\mathbb{E}_\lambda, g_\lambda, \nabla^{g_\lambda})$ is an *Einstein–Hermitian vector bundle* [45] over the Kähler SU(1,1)-homogeneous space $(\mathscr{D}^{(1)}, \omega_0)$. Similarly, one can show that $\mathbb{E}_1 \oplus \mathbb{E}_2$ is also an Einstein–Hermitian vector bundle over $(\mathscr{D}^{(1)}, \omega_0)$. These very interesting and important observations not only improve the characterization of a super-Kähler coadjoint orbit, but they suggest a way of extending to the super context the known classification of irreducible bounded symmetric Hermitian domains. This direction is now under investigation.

7.4. Realizations of the typical and atypical representations. In agreement with the descriptions of Sect. 2 and Appendix B, results of Sect. 5 show that the typical super-Hilbert space $H_{\tau,b}$ of L^2 superholomorphic sections of $\mathscr{L}^{(1|2)}$ is the direct sum of Hilbert spaces of four holomorphic discrete series representations of SU(1,1). More precisely, as a vector superspace

$$\mathscr{H}_{\tau,b} = \mathscr{H}_{k=\tau} \oplus 2 \cdot \mathscr{H}_{k=\tau+\frac{1}{2}} \oplus \mathscr{H}_{k=\tau+1}\,, \tag{7.2}$$

where $\mathscr{H}_k$ is the Hilbert space carrying the holomorphic discrete series representation $D(k)$ of SU(1,1) (see (5.14) and (2.8)–(2.10)). This suggests that the osp(2/2) operators obtained in (5.17) and which act in the left-hand side of (7.2) can be replaced by matrix valued and thus anticommutating-variables free operators acting in the right-hand side of (7.2) [55]. The former realization is much more convenient than the latter. Indeed, for example for osp(N/2) the matrices can be of dimension $2^N \times 2^N$. We insist here on the fact that our main goal in Sect. 5 was to show that geometric quantization extends to the super context, at least when applied to coadjoint orbits admitting a super-Kähler polarization. We not only succeeded in achieving this, but the above observation confirms that our output constitutes an intrinsically supersymmetric alternative to the matrix realization. The same discussion applies to the atypical representations.

7.5. OSp(2/2) representations. Throughout, we have been considering only representations of the Lie superalgebra. Explicit representations of OSp(2/2) can be in fact obtained from those of osp(2/2) exhibited in Sects. 5 and 6. The procedure does not present any difficulties. The first step towards this construction consists in finding the explicit action of OSp(2/2) on $\mathscr{D}^{(1|2)}$ and $\mathscr{D}^{(1|1)}$ by integrating the Hamiltonian vector fields in (3.12) and (6.10)–(6.13). This amounts to solving super-Riccati differential equations which have already been considered in [56]. The full construction will be given elsewhere.

8. Conclusions and Outlook

Although our present contribution treats a specific example, the obtained results pave the way to harmonic superanalysis. It must be regarded as the first important step of a program aimed at classifying Lie supergroups' coadjoint orbits and the associated irreducible representations.

In this work several closely related questions have been addressed, and several new notions have been introduced. The consistency of our conventions is manifest throughout the paper, from abstract to explicit representation theory via super-Kähler geometry. The main results are now summarized:

(**a**) Starting with a comprehensive description of the abstract typical and atypical representations of osp(2/2), the associated OSp(2/2) coherent states are constructed.

(**b**) Their underlying geometries are exhibited, and are shown to be those of OSp(2/2) coadjoint orbits. The latter are OSp(2/2)-supersymplectic homogeneous spaces: $\mathscr{D}^{(1|2)} \equiv \mathrm{OSp}(2/2)/(\mathrm{U}(1) \times \mathrm{U}(1))$ for the typical CS, and $\mathscr{D}^{(1|1)} \equiv \mathrm{OSp}(2/2)/\mathrm{U}(1/1)$ for the atypical CS.

(**c**) The identification of Rothstein's data for $\mathscr{D}^{(1|2)}$ and $\mathscr{D}^{(1|1)}$ draws us to generalizing Rothstein's theorem to the complex-analytic setting. This leads to a natural definition of a super-Kähler supermanifold, $\mathscr{D}^{(1|2)}$ and $\mathscr{D}^{(1|1)}$ being nontrivial examples of such a notion. We moreover show that in this context, Rothstein's theorem can be refined. More precisely, the *complete* supersymplectic structure of a super-Kähler coadjoint orbit can be encoded in an elementary building block of the type mentioned in point 7.3 of the previous section.

(**d**) Finally, geometric quantization is successfully extended to the super-Kähler context examplified by the typical and atypical coadjoint orbits of OSp(2/2). A super-Kähler polarization is exhibited in each case. This leads to an explicit super-unitary irreducible typical (atypical) representation of osp(2/2) in a super-Hilbert space of square integrable superholomorphic sections of a complex line bundle sheaf over $\mathscr{D}^{(1|2)}$ ($\mathscr{D}^{(1|1)}$).

Possible generalizations of our results are numerous and worth considering. At both the representation theoretic and the geometric levels, the present work relies essentially on known results from the non-super context. For instance, the representation theory of osp(2/2) is based on that of $\mathrm{su}(1,1) \subset \mathrm{osp}(2/2)$, while the supergeometry of the $N = 2$ (resp. $N = 1$) super-unit disc $\mathscr{D}^{(1|2)}$ (resp. $\mathscr{D}^{(1|1)}$) is based on that of the unit disc $\mathscr{D}^{(1)}$. Our results show precisely how the super extension occurs (see Sect. 7, 7.3). One can now seriously consider other Lie supergroups, and look for a classification of their super-Kähler homogeneous spaces along the known classification of the Kähler homogeneous spaces of their body Lie groups. Geometric quantization will then provide a classification of their associated representations.

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A. Conventions and Notation

One of the main features of the present work is the complete consistency of the conventions used throughout. The latter are displayed here. They concern the superalgebra $\mathscr{B}$ introduced in Sect. 2, and its interactions with both the Lie superalgebra and modules over it.

A complex superalgebra is a complex vector superspace (i.e. a $\mathbb{Z}_2$-graded linear space) $\mathscr{B} = \mathscr{B}_0 \oplus \mathscr{B}_1$ equipped with a $\mathbb{Z}_2$-compatible product, namely $\mathscr{B}_k \cdot \mathscr{B}_l \subset \mathscr{B}_{k+l}$; $\mathscr{B}$ is considered associative and unital. Note that, $\mathscr{B}_0$ (resp. $\mathscr{B}_1$) is called the *even* (resp. *odd*) part of $\mathscr{B}$. Accordingly, elements of $\mathscr{B}_0$ (resp. $\mathscr{B}_1$) are called *even* (resp. *odd*) elements of $\mathscr{B}$. A homogeneous element of $\mathscr{B}$ is either even or odd. The *parity* of such an element $u \in \mathscr{B}_k$, denoted $\varepsilon(u)$, is defined by $\varepsilon(u) = k$. The superalgebra $\mathscr{B}$ is supercommutative if

$$uv = (-1)^{\varepsilon(u)\varepsilon(v)} vu \,, \tag{A.1}$$

for u and v two homogeneous elements of $\mathscr{B}$.

The *complex supercommutative superalgebra with unit* $\mathscr{B}$ considered in the present work is the complex Grassmann algebra [22, 23] generated by (θ, χ) and their complex conjugates $(\bar{\theta}, \bar{\chi})$. These are anticommuting variables. In other words $\mathscr{B}$ is the complex exterior algebra over $\mathbb{C}^4 = \mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$. Its even (resp. odd) part is spanned by the products of an even (resp. odd) number of generators, and the dimension of $\mathscr{B}$ is 16. The decomposition of any element $\Theta \in \mathscr{B}$ in a given basis of $\mathscr{B}$ assumes the following form:

$$\Theta = \tilde{\Theta} \cdot \mathrm{I} + \Theta_{\mathrm{nil}} \,, \tag{A.2}$$

where the purely nilpotent component Θ_{nil} is called the *soul* of Θ while the component $\tilde{\Theta}$ along the identity of $\mathscr{B}$ is called the *body* of Θ.

The complex conjugation maps $\mathbb{C}^2 \ni (\theta, \chi) \mapsto (\bar{\theta}, \bar{\chi}) \in \overline{\mathbb{C}}^2$. Its extension to $\mathscr{B}$ is completely defined by the following rule:

$$\overline{\Theta_1 \, \Theta_2} = \bar{\Theta}_1 \, \bar{\Theta}_2, \quad \forall \Theta_1, \Theta_2 \in \mathscr{B} \,. \tag{A.3}$$

The other properties are:

$$\bar{\bar{\Theta}} = \Theta \quad \text{and} \quad \overline{\mathrm{w}\,\Theta} = \bar{\mathrm{w}}\,\bar{\Theta}, \qquad \forall \mathrm{w} \in \mathbb{C} \quad \text{and} \quad \Theta \in \mathscr{B} \,. \tag{A.4}$$

An element $\Theta \in \mathscr{B}$ is *real* if $\bar{\Theta} = \Theta$. Using (A.3) one easily sees that $\Theta\bar{\Theta}$ is real for $\Theta \in \mathscr{B}_0$ and imaginary for $\Theta \in \mathscr{B}_1$.

It is important to note that our convention in (A.3) is different from the one introduced by Berezin [21, 22] and commonly used in the literature (see [53] and references therein). In that case the complex conjugate of a product is such that

$$\overline{\Theta_1 \, \Theta_2} = \bar{\Theta}_2 \, \bar{\Theta}_1, \quad \forall \Theta_1, \Theta_2 \in \mathscr{B} \,. \tag{A.5}$$

Hence, for Θ a homogeneous element of $\mathscr{B}, \Theta\bar{\Theta}$ is real independently of the parity of Θ. Using these conventions one faces serious inconsistencies. The most obvious one was encountered in [8] (see also [50]), where the author followed Berezin's conventions already used in [34]; the super-Kähler 2-form obtained there was neither real nor imaginary! As a consequence, the classical observables and their associated Hamiltonian vector fields were not satisfying any property of the type of (3.10) and (3.13) which are crucial in identifying a real observable and then the associated self-superadjoint operator. Combined with the notion of a super-Hermitian structure introduced in [41], our convention (A.3) cures this discrepancy. More precisely, the arguments invoked in [53] in order to justify the choice in (A.5) apply to our choice (A.3) too, provided one considers the notion of a super-Hermitian structure (2.12)–(2.16) and its consequences. (The problem in [8] mentioned above is cured in [9].)

Finally, all vector superspaces appearing in this work are considered as left $\mathscr{B}$-modules. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be such a $\mathscr{B}$-module. Then, for v and Θ homogeneous elements in V and $\mathscr{B}$ respectively, we have

$$\Theta v = (-1)^{\varepsilon(\Theta)\varepsilon(v)} v\Theta \,. \tag{A.6}$$

This applies equally well to $V = \mathrm{osp}(2/2, \mathbb{C})$, $V = V(\tau, b)$ or $V = U(\pm\tau)$. When V is equipped with an additional structure, such as a super-Hermitian form $\langle \cdot | \cdot \rangle$ (see (2.12)–(2.14)) or a super-bracket $[\,,]$, then that structure can be extended to the *Grassmann envelope of second type* $\tilde{V}$ of V. The latter is defined as follows [22],

$$\tilde{V} \equiv (\mathscr{B} \otimes V)_0 = (\mathscr{B}_0 \otimes V_0) \oplus (\mathscr{B}_1 \otimes V_1) \,, \tag{A.7}$$

and the above structures are extended in the following way:

$$\langle \Theta_1 v | \Theta_2 u \rangle = (-1)^{\varepsilon(v)\varepsilon(\Theta_2)} \bar{\Theta}_1 \Theta_2 \langle v | u \rangle$$

and

$$[\Theta_1 X, \Theta_2 Y] = (-1)^{\varepsilon(X)\varepsilon(\Theta_2)} \Theta_1 \Theta_2 [X, Y] \,, \tag{A.8}$$

where Θ_2, v and X are homogeneous elements in respectively $\mathscr{B}$, $V(\tau, b)$ and $\mathrm{osp}(2/2, \mathbb{C})$. We end this appendix by giving a formula which is useful for some of the computations of Sect. 2 and Appendix B. Let Θ and v be homogeneous elements of respectively $V(\tau, b)$ and $\mathscr{B}$, the super Hermitian conjugate of $|\Theta v\rangle \in \tilde{V}(\tau, b)$ with respect to the super Hermitian form (2.13)–(2.16) is obtained as follows:

$$(\Theta|v\rangle)^\dagger = \bar{\Theta}(|v\rangle)^\dagger = (i)^{\varepsilon(v)} \bar{\Theta} \langle v| \,. \tag{A.9}$$

B. osp(2/2) Representations: More Details

Finite and infinite dimensional irreducible representations of osp(2/2) have already been studied in [32, 33, 35]. The description of the infinite dimensional ones given in [35] is the most convenient for our purpose, but since it suffers from some discrepancies we consider important to reexpose the construction. This appendix must be viewed as a complement to Sect. 2.

The equations defining the lowest-weight vector $|0\rangle$ (2.5)–(2.6), together with the observation that $|0\rangle$ is the lowest-weight vector of a discrete series representation $D(\tau)$ of su(1, 1) (2.7), are our starting points.

The osp(2/2)-module $V(\tau,b)$ of Sect. 2 is generated by applying arbitrary polynomials in the generators of $\mathfrak{n}^+ = \mathrm{span}\{K_+, V_+, W_+\}$ (2.3) to $|0\rangle$. Using the commutation relations and the results mentioned above, one can see that $V(\tau,b)$ is spanned by the following vectors:

$$K_+^m|0\rangle,\quad K_+^m V_+|0\rangle,\quad K_+^m W_+|0\rangle,\quad K_+^m V_+ W_+|0\rangle,\quad m\in\mathbb{N}\,. \tag{B.1}$$

The latter are eigenstates of B:

$$\begin{aligned}
B(K_+^m|0\rangle) &= b(K_+^m|0\rangle)\,,\\
B(K_+^m V_+|0\rangle) &= \left(b+\frac{1}{2}\right)(K_+^m V_+|0\rangle)\,,\\
B(K_+^m W_+|0\rangle) &= \left(b-\frac{1}{2}\right)(K_+^m W_+|0\rangle)\,,\\
B(K_+^m V_+ W_+|0\rangle) &= b(K_+^m V_+ W_+|0\rangle)\,,
\end{aligned} \tag{B.2}$$

and also of K_0:

$$\begin{aligned}
K_0(K_+^m|0\rangle) &= (\tau+m)(K_+^m|0\rangle)\,,\\
K_0(K_+^m V_+|0\rangle) &= \left(\tau+\frac{1}{2}+m\right)(K_+^m V_+|0\rangle)\,,\\
K_0(K_+^m W_+|0\rangle) &= \left(\tau+\frac{1}{2}+m\right)(K_+^m W_+|0\rangle)\,,\\
K_0(K_+^m V_+ W_+|0\rangle) &= (\tau+1+m)(K_+^m V_+ W_+|0\rangle)\,,
\end{aligned} \tag{B.3}$$

but not all of them are eigenstates of the su$(1,1)$ Casimir C_2 given in (2.18):

$$\begin{aligned}
C_2(K_+^m|0\rangle) &= \tau(\tau-1)(K_+^m|0\rangle)\,,\\
C_2(K_+^m V_+|0\rangle) &= \left(\tau+\frac{1}{2}\right)\left(\tau-\frac{1}{2}\right)(K_+^m V_+|0\rangle)\,,\\
C_2(K_+^m W_+|0\rangle) &= \left(\tau+\frac{1}{2}\right)\left(\tau-\frac{1}{2}\right)(K_+^m W_+|0\rangle)\,,\\
C_2(K_+^m V_+ W_+|0\rangle) &= (\tau+1)\tau(K_+^m V_+ W_+|0\rangle) - (\tau+b)(K_+^{m+1}|0\rangle)\,.
\end{aligned} \tag{B.4}$$

The last equation suggests to use another family of states instead of $K_+^m V_+ W_+|0\rangle$. Indeed, notice that the vectors

$$(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle,\quad (\text{or } (-2\tau K_+^m W_+ V_+ + (\tau-b)K_+^{m+1})|0\rangle)\,, \tag{B.5}$$

are such that

$$\begin{aligned}
B(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle &= b(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle\,,\\
C_2(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle &= (\tau+1)\tau(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle\,,\\
K_0(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle &= (\tau+1+m)(2\tau K_+^m V_+ W_+ - (\tau+b)K_+^{m+1})|0\rangle\,.
\end{aligned} \tag{B.6}$$

The previous results suggest to use the following notation:

$$\begin{aligned}
&|b,\tau,\tau\rangle \equiv |0\rangle, \qquad \left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}\right\rangle \propto V_+|0\rangle\,,\\
&|b,\tau+1,\tau+1\rangle \propto (2\tau V_+ W_+ - (\tau+b)K_+)|0\rangle, \qquad \left|b-\frac{1}{2},\ \tau+\frac{1}{2},\tau+\frac{1}{2}\right\rangle \propto W_+|0\rangle\,.
\end{aligned} \tag{B.7}$$

The osp(2/2)-module $V(\tau,b)$ obtained in this way is irreducible only when $b \neq \pm\tau$. For $b=\pm\tau$, $V(\tau,b)$ contains a *primitive vector*. Indeed, using (2.2a)–(2.2i) and (2.5)–(2.6), one easily sees that:

$$\text{when}\quad b=\tau, \qquad K_-(V_+|0\rangle) = V_-(V_+|0\rangle) = W_-(V_+|0\rangle) = 0\,, \tag{B.8}$$

and

$$\text{when}\quad b=-\tau, \qquad K_-(W_+|0\rangle) = V_-(W_+|0\rangle) = W_-(W_+|0\rangle) = 0\,. \tag{B.9}$$

These two situations being very similar, we focus here only on the second one. Hence, in that instance, $W_+|0\rangle$ generates an osp(2/2)-submodule of $V(\tau,b)$, denoted $V'(\tau,-\tau)$. An irreducible osp(2/2)-module emerges then as the quotient $V(\tau,-\tau)/V'(\tau,-\tau) \equiv U(-\tau)$. Notice that $V'(\tau,-\tau)$ is spanned by the last two series in (B.1), while $U(-\tau)$ is spanned by the two first ones modulo the two last ones.

In order to obtain the proportionality constants in (B.7) it is necessary to equip $V(\tau,b)$ with a super-Hermitian form [41, 24] of the type described in (2.12)–(2.14). The *super-adjoint* $A^\dagger$ (denoted $\tilde{A}$ and called differently in [24]) of a homogeneous operator A acting in $V(\tau,b)$ is defined as follows:

$$\langle A^\dagger u|v\rangle = (-1)^{\varepsilon(u)\varepsilon(A)}\langle u|Av\rangle, \quad \forall u,v\in V(\tau,b) \quad \text{with } u \text{ homogeneous}\,. \tag{B.10}$$

Hence, A is self-superadjoint if $A^\dagger = A$. Moreover, one can check from (B.10) that

$$(A^\dagger)^\dagger = A \quad\text{and}\quad (AB)^\dagger = (-1)^{\varepsilon(A)\varepsilon(B)}B^\dagger A^\dagger\,. \tag{B.11}$$

Consequently,

$$[A,B]^\dagger = -[A^\dagger, B^\dagger]\,. \tag{B.12}$$

Using this equation, the relations in (2.2a)–(2.2i) and the root space decomposition of osp(2/2, $\mathbb{C}$) given in Sect. 2, one easily shows that

$$B^\dagger = B, \qquad K_0^\dagger = K_0, \qquad (K_\pm)^\dagger = K_\mp\,, \tag{B.13}$$

$$(V_\pm)^\dagger = iW_\mp \quad\text{and}\quad (W_\pm)^\dagger = iV_\mp\,. \tag{B.14}$$

Using the above results, Eq. (A.9), and assuming that the vectors on the left-hand side of the equations in (B.7) are normalized to one with respect to $\langle\cdot|\cdot\rangle_{0(1)}$ (according to their parity) one obtains:

$$V_+|b,\tau,\tau\rangle = \sqrt{\tau-b}\left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}\right\rangle , \tag{B.15}$$

$$W_+|b,\tau,\tau\rangle = \sqrt{\tau+b}\left|b-\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}\right\rangle , \tag{B.16}$$

$$(2\tau V_+W_+ - (\tau+b)K_+)|b,\tau,\tau\rangle = \sqrt{(\tau^2-b^2)(2\tau+1)2\tau}\,|b,\tau+1,\tau+1\rangle . \tag{B.17}$$

Clearly, these results are valid only if $|b| \leqq \tau$. When $b \neq \pm\tau$, this means that the super-Hermitian structure introduced above, turns the irreducible osp(2/2)-modules $V(\tau,b)$ into super-unitary representations only if $|b| < \tau$. On the other hand, when $b = \pm\tau$ one sees from (B.15)–(B.17) that the primitive vectors generating the submodules $V'(\tau,\pm\tau)$ are zero-norm states, and thus the entire submodules are made of zero-norm states. Moding out the latter from $V(\tau,\pm\tau)$ turns $U(\pm\tau)$ into super-unitary irreducible modules. Moreover, (2.9) and (2.11) are direct consequences of (B.15)–(B.17) and (2.2a)–(2.2i). Indeed, the four states $|b,\tau,\tau\rangle$, $V_+|b,\tau,\tau\rangle$, $W_+|b,\tau,\tau\rangle$ and $(2\tau V_+W_+ - (\tau+b)K_+)|b,\tau,\tau\rangle$ are orthogonal with respect to the super Hermitian structure, and each of them is a lowest state of an su(1, 1) irreducible module. The latter are generated from the former su(1, 1) lowest-weight states through the action of powers of K_+:

$$K_+^m|\cdot,k,k\rangle = \sqrt{\frac{m!\Gamma(2k+m)}{\Gamma(2k)}}\,|\cdot,k,k+m\rangle; \quad k=\tau,\tau\pm\frac{1}{2},\text{or } \tau+1 . \tag{B.18}$$

We end this appendix by displaying the action of the osp(2/2) generators on the different vectors. The following formulae are straightforward consequences of (2.2a)–(2.2i) and the results described above;

$$K_+|b,\tau,\tau+m\rangle = \sqrt{(2\tau+m)(m+1)}\,|b,\tau,\tau+m+1\rangle ,$$

$$K_-|b,\tau,\tau+m\rangle = \sqrt{(2\tau+m-1)m}\,|b,\tau,\tau+m-1\rangle ,$$

$$V_+|b,\tau,\tau+m\rangle = \sqrt{\frac{(\tau-b)(2\tau+m)}{2\tau}}\left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle ,$$

$$V_-|b,\tau,\tau+m\rangle = \sqrt{\frac{(\tau-b)m}{2\tau}}\left|b+\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m-1\right\rangle ,$$

$$W_+|b,\tau,\tau+m\rangle = \sqrt{\frac{(\tau+b)(2\tau+m)}{2\tau}}\left|b-\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m\right\rangle ,$$

$$W_-|b,\tau,\tau+m\rangle = \sqrt{\frac{(\tau+b)m}{2\tau}}\left|b-\frac{1}{2},\tau+\frac{1}{2},\tau+\frac{1}{2}+m-1\right\rangle ,$$

$$V_+ \left| b+\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = 0\,, \qquad V_- \left| b+\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = 0\,,$$

$$W_+ \left| b+\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = \sqrt{\frac{(\tau-b)(m+1)}{2\tau}}\, |b, \tau, \tau+m+1\rangle$$
$$- \sqrt{\frac{(\tau+b)(2\tau+m+1)}{2\tau}}\, |b, \tau+1, \tau+1+m\rangle\,,$$

$$W_- \left| b+\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = \sqrt{\frac{(\tau-b)(2\tau+m)}{2\tau}}\, |b, \tau, \tau+m\rangle$$
$$- \sqrt{\frac{(\tau+b)m}{2\tau}}\, |b, \tau+1, \tau+1+m-1\rangle\,,$$

$$V_+ \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = \sqrt{\frac{(\tau+b)(m+1)}{2\tau}}\, |b, \tau, \tau+m+1\rangle$$
$$+ \sqrt{\frac{(\tau-b)(2\tau+m+1)}{2\tau}}\, |b, \tau+1, \tau+1+m\rangle\,,$$

$$V_- \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = \sqrt{\frac{(\tau+b)(2\tau+m)}{2\tau}}\, |b, \tau, \tau+m\rangle$$
$$+ \sqrt{\frac{(\tau-b)m}{2\tau}}\, |b, \tau+1, \tau+1+m-1\rangle\,,$$

$$W_+ \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = 0\,, \qquad W_- \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle = 0\,,$$

$$V_+ |b, \tau+1, \tau+1+m\rangle = -\sqrt{\frac{(\tau+b)(m+1)}{2\tau}} \left| b+\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m+1 \right\rangle\,,$$

$$V_- |b, \tau+1, \tau+1+m\rangle = -\sqrt{\frac{(\tau+b)(2\tau+m+1)}{2\tau}} \left| b+\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle\,,$$

$$W_+ |b, \tau+1, \tau+1+m\rangle = \sqrt{\frac{(\tau-b)(m+1)}{2\tau}} \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m+1 \right\rangle\,,$$

$$W_- |b, \tau+1, \tau+1+m\rangle = \sqrt{\frac{(\tau-b)(2\tau+m+1)}{2\tau}} \left| b-\frac{1}{2}, \tau+\frac{1}{2}, \tau+\frac{1}{2}+m \right\rangle\,. \tag{B.19}$$

The discrepancies mentioned at the beginning of this appendix can be easily seen by comparing (B.19) with its analog in [35].

Note added in proof. The definition of self-superadjointness given in Sect. 5 is purely formal. It does not take into account the necessary domain considerations. The latter are indeed necessary in the present situation since the first order differential superoperators (quantum operators) obtained in Sects. 5 and 6 are unbounded. These considerations are described and analyzed in detail in [55].

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