Commun. Math. Phys. 175, 457–520 (1996)

Communications in
Mathematical
Physics

Geometry of Quantum Principal Bundles I

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Received: 29 August 1991/in revised form: 20 March 1995

Abstract: A theory of principal bundles possessing quantum structure groups and classical base manifolds is presented. Structural analysis of such quantum principal bundles is performed. A differential calculus is constructed, combining differential forms on the base manifold with an appropriate differential calculus on the structure quantum group. Relations between the calculus on the group and the calculus on the bundle are investigated. A concept of (pseudo)tensoriality is formulated. The formalism of connections is developed. In particular, operators of horizontal projection, covariant derivative and curvature are constructed and analyzed. Generalizations of the first Structure Equation and of the Bianchi identity are found. Illustrative examples are presented.

Contents

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1. Introduction

In diversity of mathematical concepts and theories a fundamental role is played by those giving a unified treatment of different and at a first sight mutually independent circles of problems.

As far as classical differential geometry is concerned, such a fundamental role is given to the theory of principal bundles [3]. Various basic concepts of theoretical physics are also naturally expressible in the language of principal bundles. Classical gauge theory is a paradigmic example.

In this work a quantum generalization of the theory of principal bundles will be presented. All constructions and considerations will be performed within a conceptual framework of noncommutative differential geometry [1, 2].

The generalization will be twofold. First of all, quantum groups will play the role of structure groups. Secondly, appropriate quantum spaces will play the role of base manifolds.

This paper is devoted to the study of quantum principal bundles over classical smooth manifolds.

The paper is organized as follows.

Section 2 begins with a definition of quantum principal bundles. For technical reasons, it will be assumed that a base manifold M is compact. Concerning a structure quantum group G, it will be a compact matrix quantum group (pseudogroup), in the sense of [8].

We shall prove that, as a consequence of an inherent geometrical inhomogeneity of quantum groups, there exists a natural correspondence between quantum principal bundles, and classical principal bundles over the same manifold M, with the structure group G_{cl} consisting of "classical points" of G. Informally speaking, if we start from a quantum principal bundle P then the corresponding classical principal bundle P_{cl} consists precisely of "classical points" of P. Conversely, starting from a G_{cl}-bundle P_{cl}, the bundle P can be recovered applying a variant of the classical procedure of extending structure groups.

Section 3 is devoted to the study of differential calculus on quantum principal bundles. At first, general properties for differential calculus on P will be formulated, including relations with differential structures over M and G. The main idea is that local trivializations of the bundle locally trivialize the calculus, too.

A differential calculus over M will be the standard one, specified by differential forms. A differential calculus on the structure quantum group G will be based on the *universal envelope* of an appropriate first-order differential calculus Γ. This universal envelope can be constructed by applying an extended bimodule technique [7, 9]. As we shall see, the mentioned local triviality property of the calculus on the bundle implies certain restrictions on the calculus Γ. Informally speaking, Γ should be compatible with all possible "transition functions" for P. Motivated by this observation, we shall introduce a notion of *admissibility* to distinguish first-order differential structures on G for which the mentioned compatibility holds.

The next theme of Sect. 3 is a construction of the calculus on P, starting from differential forms on M and a given admissible first-order calculus Γ over G. As a result we obtain a graded differential algebra $\Omega(P,\Gamma)$, representing the calculus on the bundle P. We shall prove the uniqueness of this algebra.

After this, various properties of $\Omega(P,\Gamma)$ will be studied (the existence of $*$-structures, the right covariance and the existence of the graded-differential extension of the dualized right action of G on P). These properties are closely related to similar properties of Γ. On the other hand, independently of the choice of Γ there exists a natural left coaction of G on $\Omega(P,\Gamma)$, becoming trivial in the classical case.

In Sect. 3 the structure of admissible calculi is studied, too. In particular, left-covariant admissible calculi are characterized in terms of the corresponding right ideals in the algebra $\mathcal{A}$ of "polynomial functions" on G. It turns out that there exists the "simplest" left-covariant admissible calculus (which is automatically bicovariant and *-covariant).

Finally, at the end of Sect. 3 we introduce and briefly analyze analogs of horizontal and verticalized differential forms on the bundle.

The study of connections on quantum principal bundles is the main topic of Sects. 4 and 5. Through these sections we shall assume that Γ is the simplest left-covariant admissible calculus.

In Sect. 4 we shall first generalize the classical concept of (pseudo) tensoriality. Together with certain considerations performed in Sect. 3 this will enable us to introduce connection forms, in analogy with classical geometry. We then pass to the study of local representations of connections, in terms of gauge potentials.

Further, we shall prove that each connection on P admits a decomposition into a "classical connection," interpretable as an ordinary connection on P_{cl}, and an appropriate "purely quantum" tensorial 1-form.

Each connection decomposes the algebra $\Omega(P,\Gamma)$ into a tensor product of spaces of horizontal forms and left-invariant forms on G. With the help of this decomposition we shall introduce the horizontal projection operator. This will enable us to define the analogs of covariant derivative and curvature operators, which will be studied in Sect. 5. In particular, we shall analyze local representations of covariant derivative and curvature, and find counterparts of the first Structure Equation and the Bianchi identity.

In Sect. 6 some concrete examples are worked out. Considerations are mainly confined to specific properties of the calculus on structure quantum group G, and to the presentation of "quantum phenomena" appearing at the level of connections. A particular care is devoted to the example with the quantum $SU(2)$ group. Finally, we shall briefly discuss a possible formulation of a "gauge theory" in the framework of quantum principal bundles.

The paper ends with three technical appendices. In Appendix A relevant properties of the set G_{cl} of classical points of G are collected. Some concrete examples are computed.

In the second appendix properties of universal envelopes of first-order differential structures are analyzed in detail. It is important to mention that, in the general case, the universal envelope of a bicovariant first-order calculus does not coincide with the exterior algebra constructed in [10], although in the case of ordinary Lie groups (and ordinary 1-forms on them) two structures coincide. We shall see that, quite generally, the universal envelope coincides with the graded-differential algebra constructed by applying the mentioned extended bimodule technique. A reason for our choice of higher-order calculus on G lies in the conceptual simplicity of the universal calculus, which is independent of the group structure on G (in contrast to the exterior algebra construction). Because of this, similar considerations can be applied to more general fiberings, for example of the type of associated bundles where fibers are diffeomorphic to an arbitrary quantum space. On the other hand, we are able to consider examples in which Γ is not bicovariant.

We shall also prove that $\Omega(P,\Gamma)$ can be understood as the universal envelope over its first-order part.

In Appendix C some properties of the already mentioned minimal admissible first-order calculi are collected.

Concerning the notation of quantum group entities, we shall follow [8]. A quantum group G will be represented as a pair $G = (A, u)$, where A is the C^*-algebra of "continuous functions" on the space G and $u \in M_n(A)$ is the matrix determining the group structure. The $*$-algebra representing "polynomial functions" on G will be denoted by $\mathcal{A}$. This $*$-algebra is generated by entries of u. The comultiplication, the counit and the antipode will be denoted by ϕ, ϵ and κ respectively.

We shall write symbolically

$$\phi(a) = a^{(1)} \otimes a^{(2)}$$

for each $a \in \mathcal{A}$. Similarly, the symbol $a^{(1)} \otimes \ldots a^{(n)}$ denotes the result of a $(n-1)$-fold comultiplication of $a \in \mathcal{A}$ (due to the coassociativity property of ϕ this is independent of the way in which comultiplications are performed).

We shall denote by $\mathrm{ad}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ the adjoint action of G on itself. Explicitly, this map is given by

$$\mathrm{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}.$$

If M is a smooth manifold we shall denote by $S(M)$ the $*$-algebra of complex smooth functions on M. Similarly, $S_c(M)$ will be the $*$-algebra consisting of smooth functions having a compact support.

2. Structure of Quantum Principal Bundles

Let us consider a compact matrix quantum group G. Let M be a compact smooth manifold.

Definition 2.1. A *(quantum) principal G-bundle* over M is a triplet of the form $P = (\mathcal{B}, i, F)$ where $\mathcal{B}$ is a (unital) $*$-algebra, $i : S(M) \to \mathcal{B}$ is a unital linear map and $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is a linear map such that for each $x \in M$ there exists an open set $U \subseteq M$ containing x and a $*$-homomorphism $\pi_U \colon \mathcal{B} \to S(U) \otimes \mathcal{A}$ such that the following properties hold:

(qpb1) We have

$$\pi_U i(f) = (f\restriction_U) \otimes 1$$

for each $f \in S(M)$.

(qpb2) If $q = i(\varphi)b$ where $\varphi \in S_c(U)$ then $\pi_U(q) = 0$ implies $q = 0$.

(qpb3) We have

$$(\mathrm{id} \otimes \phi)\pi_U = (\pi_U \otimes \mathrm{id})F \qquad \pi_U(\mathcal{B}) \supseteq S_c(U) \otimes \mathcal{A}.$$

A motivation for this definition comes from classical differential geometry. The map $i\colon S(M) \to \mathcal{B}$ is interpretable as the "dualized projection" of the bundle P on its base M. The map F plays the role of a dualized right action of G on P. Finally, maps π_U are dualized local trivializations of the bundle.

Let $P = (\mathcal{B}, i, F)$ be a principal G-bundle over M.

Definition 2.2. A *local trivialization* for P is a pair (U, π_U) consisting of a non-empty open set $U \subseteq M$ and a $*$-homomorphism $\pi_U\colon \mathcal{B} \to S(U) \otimes \mathcal{A}$ such that properties listed in the previous definition hold. A *trivialization system* for P is a family $\tau = (\pi_U)_{U \in \mathcal{U}}$, where $\mathcal{U}$ is a finite open cover of M and for each $U \in \mathcal{U}$ the pair (U, π_U) is a local trivialization for P.

Let $\tau = (\pi_U)_{U\in\mathscr{U}}$ be a trivialization system for P.

Lemma 2.1. *The family τ distinguishes elements of $\mathscr{B}$.*

Proof. Let us consider a partition of unity $\varpi = (\varphi_U)_{U\in\mathscr{U}}$ for $\mathscr{U}$. In other words $\varphi_U \in S_c(U)$ and

$$\sum_{U\in\mathscr{U}} \varphi_U = 1_M.$$

According to Definition 2.1 if b belongs to the intesection of kernels of maps π_U then $\pi_U(i(\varphi_U)b) = 0$, and hence $i(\varphi_U)b = 0$, for each $U \in \mathscr{U}$. Summing over $\mathscr{U}$ we conclude that $b = 0$. □

Lemma 2.2. (*i*) *The map $i\colon S(M) \to \mathscr{B}$ is a $*$-monomorphism.*

(*ii*) *The image $i\bigl(S(M)\bigr)$ is contained in the centre of $\mathscr{B}$.*

Proof. The following equalities hold

$$\begin{aligned}\pi_U\bigl(i(fg) - i(f)i(g)\bigr) &= (fg{\restriction_U}) \otimes 1 - (f{\restriction_U})(g{\restriction_U}) \otimes 1 = 0,\\ \pi_U\bigl(i(f^*)\bigr) - \pi_U\bigl(i(f)^*\bigr) &= (f^*{\restriction_U}) \otimes 1 - (f{\restriction_U})^* \otimes 1 = 0,\\ \pi_U\bigl(i(f)b - bi(f)\bigr) &= \bigl((f{\restriction_U}) \otimes 1\bigr)\pi_U(b) - \pi_U(b)\bigl((f{\restriction_U}) \otimes 1\bigr) = 0.\end{aligned}$$

Using Lemma 2.1 we conclude that i is a $*$-homomorphism and that (*ii*) holds. If $f \in \ker(i)$ then $f{\restriction_U} = 0$ for each $U \in \mathscr{U}$ and hence $f = 0$. □

Lemma 2.3. (*i*) *The map F is a unital *-homomorphism.*

(*ii*) *The following identities hold*

$$(F \otimes \mathrm{id})F = (\mathrm{id} \otimes \phi)F, \tag{2.1}$$

$$(\mathrm{id} \otimes \epsilon)F = \mathrm{id}. \tag{2.2}$$

(*iii*) *An element $b \in \mathscr{B}$ belongs to $i\bigl(S(M)\bigr)$ iff*

$$F(b) = b \otimes 1. \tag{2.3}$$

In other words F defines a right action of G on P. The corresponding "orbit space" coincides with the base manifold M.

Proof. According to Definition 2.1,

$$\begin{aligned}(\pi_U \otimes \mathrm{id})F(b^*) = (\mathrm{id} \otimes \phi)\pi_U(b^*) &= \bigl((\mathrm{id} \otimes \phi)\pi_U(b)\bigr)^*\\ &= \bigl((\pi_U \otimes \mathrm{id})F(b)\bigr)^* = (\pi_U \otimes \mathrm{id})\,(F(b))^*\end{aligned}$$

as well as

$$\begin{aligned}(\pi_U \otimes \mathrm{id})F(bq) &= (\mathrm{id} \otimes \phi)\pi_U(bq) = (\mathrm{id} \otimes \phi)\bigl(\pi_U(b)\pi_U(q)\bigr)\\ &= \bigl((\mathrm{id} \otimes \phi)\pi_U(b)\bigr)\bigl((\mathrm{id} \otimes \phi)\pi_U(q)\bigr)\\ &= \bigl((\pi_U \otimes \mathrm{id})F(b)\bigr)\bigl((\pi_U \otimes \mathrm{id})F(q)\bigr)\\ &= (\pi_U \otimes \mathrm{id})\bigl(F(b)F(q)\bigr)\end{aligned}$$

for each $U \in \mathscr{U}$. Hence, F is a *-homomorphism. Equations (2.1)–(2.2) as well as the identity

$$Fi(f) = i(f) \otimes 1$$

can be checked in a similar way.

Let us assume that $F(b) = b \otimes 1$. We have then

$$(\pi_U \otimes \mathrm{id})F\big(i(\varphi_U)b\big) = \pi_U\big(i(\varphi_U)b\big) \otimes 1 = (\mathrm{id} \otimes \phi)\pi_U(i(\varphi_U)b),$$

where $\big(\varphi_U\big)_{U\in\mathscr{U}}$ is a partition of unity for $\mathscr{U}$.

Acting by $\mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ on the second equality we obtain

$$\pi_U\big(i(\varphi_U)b\big) = \big[(\mathrm{id} \otimes \epsilon)\pi_U\big(i(\varphi_U)b\big)\big] \otimes 1.$$

It follows that

$$i(\varphi_U)b = i(\eta_U),$$

where $\eta_U = (\mathrm{id} \otimes \epsilon)\pi_U\big(i(\varphi_U)b\big)$. Summing over U's we obtain

$$b = i\bigg(\sum_{U\in\mathscr{U}} \eta_U\bigg).$$

Finally, the unitality of F directly follows from (iii) and from the unitality of i. □

We pass to the study of internal structure of quantum principal bundles, in terms of the corresponding "G-cocycles."

For a given open cover $\mathscr{U}$ of M, we shall denote by $N^k(\mathscr{U})$ the set of all k-tuples $(U_1, \dots, U_k)$, where $U_i \in \mathscr{U}$ are such that $U_1 \cap \dots \cap U_k \neq \emptyset$.

Definition 2.3. Let $\mathscr{U}$ be a finite open cover of M. A (*smooth, quantum*) G-*cocycle* over $(M, \mathscr{U})$ is a system $\mathscr{C} = \Big\{\psi_{UV} \mid (U, V) \in N^2(\mathscr{U})\Big\}$ of non-trivial $S(U \cap V)$-linear $*$-homomorphisms $\psi_{UV}\colon S(U \cap V) \otimes \mathscr{A} \to S(U \cap V) \otimes \mathscr{A}$ such that

(i) The diagram

$$\begin{array}{ccc} S(U\cap V)\otimes\mathscr{A} & \xrightarrow{\psi_{UV}} & S(U\cap V)\otimes\mathscr{A} \\ {\scriptstyle \mathrm{id}\otimes\phi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ S(U\cap V)\otimes\mathscr{A}\otimes\mathscr{A} & \xrightarrow[\psi_{UV}\otimes\mathrm{id}]{} & S(U\cap V)\otimes\mathscr{A}\otimes\mathscr{A} \end{array} \tag{2.4}$$

is commutative.

(ii) We have

$$\psi_{UV}\big[\psi_{VW}(\varphi)\big] = \psi_{UW}(\varphi), \tag{2.5}$$

for each $(U, V, W) \in N^3(\mathscr{U})$ and $\varphi \in S_c(U \cap V \cap W) \otimes \mathscr{A}$.

Let us observe that $S(U \cap V)$-linearity property of maps ψ_{UV} implies

$$\psi_{UV}\,\big[S_c(W) \otimes \mathscr{A}\big] \subseteq S_c(W) \otimes \mathscr{A}$$

for each (nonempty) open set $W \subseteq U \cap V$. Furthermore, maps ψ_{UV} are completely determined by their restrictions on $S_c(U \cap V)$.

The following proposition completely describes G-cocycles. Let G_{cl} be *the classical part* of G (Appendix A). This is a classical group (a "subgroup" of G) consisting of *points* of G (formally $*$-characters on $\mathscr{A}$).

Proposition 2.4. *For each G-cocycle* $\mathscr{C} = \left\{\psi_{UV} \mid (U,V) \in N^2(\mathscr{U})\right\}$ *there exists the unique collection of smooth maps* $g_{VU}\colon (U \cap V) \to G_{cl}$ *such that*

$$\psi_{UV}(\varphi \otimes a)\restriction_x = \varphi g_{VU}(x)(a^{(1)}) \otimes a^{(2)}. \tag{2.6}$$

Maps g_{UV} *form a classical* G_{cl}*-cocycle over* $(M, \mathscr{U})$.

Conversely, if g_{UV} *form a classical* G_{cl}*-cocycle then formula (2.6) determines a quantum G-cocycle over* $(M, \mathscr{U})$.

Proof. Let $\mathscr{C} = \left\{\psi_{UV} \mid (U,V) \in N^2(\mathscr{U})\right\}$ be a G-cocycle. For each $(U,V) \in N^2(\mathscr{U})$ let us define a map $\mu_{VU}\colon \mathscr{A} \to S(U \cap V)$ by

$$\mu_{VU}(a) = (\mathrm{id} \otimes \epsilon)\psi_{UV}(1 \otimes a). \tag{2.7}$$

Acting by $\mathrm{id} \otimes \epsilon \otimes \mathrm{id}$ on both wings of diagram (2.4) we obtain

$$\psi_{UV}(\varphi \otimes a) = \varphi\mu_{VU}(a^{(1)}) \otimes a^{(2)}. \tag{2.8}$$

Maps μ_{VU} are unital *-homomorphisms. Equivalently, they can be naturally understood as smooth maps $g_{VU}\colon (U \cap V) \to G_{cl}$, by exchanging the order of arguments:

$$\left[\mu_{VU}(a)\right](x) = \left[g_{VU}(x)\right](a).$$

We see that (2.6) holds. Now acting by $\mathrm{id} \otimes \epsilon$ on (2.5), using (2.6) and the definition of the product in G_{cl} we conclude that

$$g_{UV}g_{VW} = g_{UW} \tag{2.9}$$

for each $(U,V,W) \in N^3(\mathscr{U})$. In other words, maps g_{VU} form a classical G_{cl}-cocycle over $(M, \mathscr{U})$. The second part of the proposition easily follows from the coassociativity of ϕ and the definition of the product in G_{cl}. □

Property (2.6) implies that maps ψ_{UV} are bijective. Indeed, the inverse is explicitly given by

$$\psi_{UV}^{-1}(\varphi \otimes a)\restriction_x = \varphi g_{UV}(x)(a^{(1)}) \otimes a^{(2)}. \tag{2.10}$$

In particular, (2.5) implies

$$\psi_{UU}(f) = f \qquad \psi_{UV}^{-1} = \psi_{VU}.$$

We see that G-cocycles are in a natural correspondence with G_{cl}-cocycles. On the other hand, G_{cl}-cocycles are in a natural correspondence with classical principal G_{cl}-bundles over M (endowed with a trivialization system).

A similar correspondence holds between quantum G-cocycles and quantum principal G-bundles. Let $P = (\mathscr{B}, i, F)$ be a quantum principal G-bundle over M. For a given (nonempty) open set $V \subseteq M$ let us denote by I_V the lineal in $\mathscr{B}$ consisting of elements of the form $q = i(\varphi)b$, where $b \in \mathscr{B}$ and $\varphi \in S_c(V)$. Lemma 2.2 (*ii*) implies that I_V is a (two-sided) *-ideal in $\mathscr{B}$.

Let (U, π_U) be a local trivialization of P. The following lemma is a direct consequence of properties listed in Definition 2.1.

Lemma 2.5. *Let $V \subseteq U$ be a nonempty open set. Then*

$$\pi_U(I_V) \subseteq S_c(V) \otimes \mathscr{A}$$

*and the restriction $(\pi_U \restriction I_V): I_V \to S_c(V) \otimes \mathscr{A}$ is a *-isomorphism.* □

Let $\psi_U : S_c(U) \otimes \mathscr{A} \to \mathscr{B}$ be a *-monomorphism defined by

$$\psi_U = \left(\pi_U \restriction I_U\right)^{-1}. \tag{2.11}$$

Evidently, the diagram

$$\begin{array}{ccc} S_c(U) \otimes \mathscr{A} & \xrightarrow{\psi_U} & \mathscr{B} \\ {\scriptstyle \mathrm{id} \otimes \phi} \downarrow & & \downarrow {\scriptstyle F} \\ S_c(U) \otimes \mathscr{A} \otimes \mathscr{A} & \xrightarrow[\psi_U \otimes \mathrm{id}]{} & \mathscr{B} \otimes \mathscr{A} \end{array} \tag{2.12}$$

is commutative.

Let us consider a trivialization system $\tau = (\pi_U)_{U \in \mathscr{U}}$ for P.

Lemma 2.6. *There exists the unique G-cocycle $\mathscr{C}_\tau = \left\{\psi_{UV} \mid (U, V) \in N^2(\mathscr{U})\right\}$ satisfying*

$$\psi_{UV}(q) = \pi_U \psi_V(q) \tag{2.13}$$

for each $(U, V) \in N^2(\mathscr{U})$ and $q \in S_c(U \cap V) \otimes \mathscr{A}$.

Proof. The above formula defines maps ψ_{UV} on algebras $S_c(U \cap V) \otimes \mathscr{A}$. These maps are $S(U \cap V)$-linear. Because of this it is possible to extend them uniquely to *-homomorphisms $\psi_{UV} : S(U \cap V) \otimes \mathscr{A} \to S(U \cap V) \otimes \mathscr{A}$. Covariance property (2.4) follows from (2.12). Cocycle condition (2.5) is a direct consequence of the definition of maps ψ_{UV}. □

Let us consider an arbitrary G-cocycle $\mathscr{C} = \left\{\psi_{UV} \mid (U, V) \in N^2(\mathscr{U})\right\}$, and let us define a *-algebra $\mathscr{T}$ as a direct sum

$$\mathscr{T} = \sum_{U \in \mathscr{U}}^{\oplus} S(U) \otimes \mathscr{A}.$$

Let $\widetilde{\mathscr{B}}$ be a set consisting of elements $b \in \mathscr{T}$ satisfying

$$({}_U|_{U \cap V} \otimes \mathrm{id}) p_U(b) = \psi_{UV}({}_V|_{U \cap V} \otimes \mathrm{id}) p_V(b) \tag{2.14}$$

for each $(U, V) \in N^2(\mathscr{U})$, where p_U and ${}_U|_{U \cap V}$ are the corresponding coordinate projections and restriction maps.

All maps figuring in (2.14) are *-homomorphisms. Hence, $\widetilde{\mathscr{B}}$ is a *-subalgebra of $\mathscr{T}$. The formula

$$(p_U \otimes \mathrm{id}) F_{\mathscr{T}} = (\mathrm{id} \otimes \phi) p_U \tag{2.15}$$

determines a *-homomorphims $F_{\mathscr{T}}\colon \mathscr{T} \to \mathscr{T} \otimes \mathscr{A}$. Diagram (2.4) implies that $\widetilde{\mathscr{B}}$ is $F_{\mathscr{T}}$-invariant, in the sense that $F_{\mathscr{T}}(\widetilde{\mathscr{B}}) \subseteq \widetilde{\mathscr{B}} \otimes \mathscr{A}$. Let $\widetilde{F}\colon \widetilde{\mathscr{B}} \to \widetilde{\mathscr{B}} \otimes \mathscr{A}$ be the corresponding restriction map. The formula

$$p_U \tilde{\imath}(f) = (f\restriction_U) \otimes 1 \tag{2.16}$$

defines a *-homomorphism $\tilde{\imath}\colon S(M) \to \widetilde{\mathscr{B}}$. Let $\pi_U\colon \widetilde{\mathscr{B}} \to S(U) \otimes \mathscr{A}$ be the restrictions of coordinate projection maps.

Proposition 2.7. *The triplet $\widetilde{P} = (\widetilde{\mathscr{B}}, \tilde{\imath}, \widetilde{F})$ is a principal G-bundle over M. The family $\tau = (\pi_U)_{U \in \mathscr{U}}$ is a trivialization system for $\widetilde{P}$. The corresponding G-cocycle coincides with the initial one. In other words $\mathscr{C} = \mathscr{C}_\tau$.* □

The above proposition directly follows from the construction of $\widetilde{P}$. Let $P = (\mathscr{B}, i, F)$ be a principal G-bundle over M, with a trivialization system τ.

Lemma 2.8. *The following identities hold*

$$({}_U|_{U\cap V} \otimes \mathrm{id})\pi_U(b) = \psi_{UV}({}_V|_{U\cap V} \otimes \mathrm{id})\pi_V(b), \tag{2.17}$$

where ψ_{UV} are transition functions from $\mathscr{C}_\tau$.

Proof. It is sufficient to check that above equalities hold on elements of the form $q = i(\varphi)b$, where $\varphi \in S_c(U \cap V)$. However, this is equivalent to

$$\psi_{UV}\pi_V(q) = \pi_U(q)$$

which is the definition of ψ_{UV}. □

Proposition 2.9. *Let $\widetilde{P} = (\widetilde{\mathscr{B}}, \tilde{\imath}, \widetilde{F})$ be a principal G-bundle constructed from the G-cocycle $\mathscr{C}_\tau$. Then the *-homomorphism $j_\tau\colon \mathscr{B} \to \mathscr{T}$ defined by*

$$p_U j_\tau = \pi_U \tag{2.18}$$

isomorphically maps $\mathscr{B}$ onto $\widetilde{\mathscr{B}}$. Moreover, the following equalities hold

$$\widetilde{F} j_\tau = (j_\tau \otimes \mathrm{id})F, \tag{2.19}$$

$$j_\tau i = \tilde{\imath}. \tag{2.20}$$

Proof. According to Lemma 2.8 we have $j_\tau(\mathscr{B}) \subseteq \widetilde{\mathscr{B}}$. Further

$$p_U j_\tau i(\varphi) = (\varphi\restriction_U) \otimes 1 = p_U \tilde{\imath}(\varphi),$$

for each $\varphi \in S(M)$ and $U \in \mathscr{U}$. Thus (2.20) holds. Together with (2.18) this implies

$$j_\tau \psi_U = \widetilde{\psi}_U, \tag{2.21}$$

where $\widetilde{\psi}_U$ are the corresponding right inverses for $\widetilde{\mathscr{B}}$.

The map j_τ is surjective, because spaces $\widetilde{\psi}_U\left[S_c(U) \otimes \mathscr{A}\right]$ linearly span $\widetilde{\mathscr{B}}$. Injectivity of j_τ is a consequence of Lemma 2.1. Hence, $j_\tau\colon \mathscr{B} \leftrightarrow \widetilde{\mathscr{B}}$.

Finally, we have

$$(p_U j_\tau \otimes \mathrm{id})F = (\pi_U \otimes \mathrm{id})F = (\mathrm{id} \otimes \phi)\pi_U = (\mathrm{id} \otimes \phi)p_U j_\tau = (p_U \otimes \mathrm{id})\widetilde{F} j_\tau,$$

for each $U \in \mathscr{U}$. Consequently, (2.19) holds. □

In summary, the following natural correspondences hold:

$$\left\{\begin{matrix}\text{quantum principal}\\ G\text{-bundles}\end{matrix}\right\} \leftrightarrow \{G\text{-cocycles}\} \leftrightarrow \{G_{cl}\text{-cocycles}\} \leftrightarrow \left\{\begin{matrix}\text{classical principal}\\ G_{cl}\text{-bundles}\end{matrix}\right\}$$

In this sense, each quantum G-bundle P determines a classical G_{cl}-bundle P_{cl}, and vice versa.

The correspondence $P \leftrightarrow P_{cl}$ has a simple geometrical explanation. Each quantum group G is inherently inhomogeneous, because it always possesses a nontrivial classical part G_{cl} consisting of points of G (because of $\epsilon \in G_{cl}$) and (as far as $\mathcal{A}$ is not commutative) a nontrivial quantum part, imaginable as the "complement" to G_{cl} in G. It is clear that "transition functions" being diffeomorphisms at the level of spaces, preserve this intrinsic decomposition. As a result, because of the right covariance, transition functions are completely determined by their "restrictions" on G_{cl}.

In fact the correspondence $P \leftrightarrow P_{cl}$ can be formulated independently of trivialization systems τ. If $P = (\mathcal{B}, i, F)$ is given then the elements of P_{cl} are in a natural bijection with *-characters of $\mathcal{B}$. In other words, P_{cl} is consisting of *classical points* of P.

Conversely, if P_{cl} is given then P can be recovered by applying a variant of the classical construction of extending structure groups.

Let $r\colon g \mapsto r_g$ be the (left) action of G_{cl} on the algebra $S(P_{cl})$, induced by the right action of G_{cl} on P_{cl}. Let $\zeta^\star\colon g \mapsto \zeta^\star_g$ be the left action of G_{cl} on $\mathcal{A}$. Explicitly,

$$r_g(\varphi)(x) = \varphi(xg), \tag{2.22}$$

$$\zeta^\star_g = (g^{-1} \otimes \mathrm{id})\phi. \tag{2.23}$$

Operators $r_g \otimes \zeta^\star_g$ are automorphisms of a *-algebra $S(P_{cl}) \otimes \mathcal{A}$. Let $\mathcal{B}$ be the corresponding fixed-point subalgebra. It is easy to see that formulas

$$F(b) = (\mathrm{id} \otimes \phi)(b), \tag{2.24}$$

$$i(\varphi) = \varphi\pi_M \otimes 1, \tag{2.25}$$

where $\pi_M\colon P_{cl} \to M$ is the projection, define *-homomorphisms $i\colon S(M) \to \mathcal{B}$ and $F\colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ such that $P = (\mathcal{B}, i, F)$ is a principal G-bundle over M. The initial bundle P_{cl} is realized as the set of classical points of P.

3. Differential Calculus

Let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle over M. As the starting point for this section, we shall formulate three basic assumptions about a differential calculus over P. We shall assume that the calculus on P is based on a graded-differential algebra

$$\Omega_P = \sum_{k\geq 0}^{\oplus} \Omega_P^k$$

possessing the following properties:

(*diff1*) The algebra $\mathcal{B}$ is realized as the 0th order summand of Ω_P. In other words, $\Omega_P^0 = \mathcal{B}$.

(*diff2*) As a differential algebra, Ω_P is generated by $\mathcal{B}$.

The next (and the last) assumption expresses an idea of local triviality of the calculus. It relates the calculus over the bundle P with differential structures over the structure quantum group G and the base manifold M. The calculus over M will be the classical one, based on a graded-differential algebra $\Omega(M)$ consisting of differential forms. For each open set $U \subseteq M$ we shall denote by $\Omega(U)$ and $\Omega_c(U)$ algebras of differential forms on U (having compact supports).

Concerning the calculus over G, it will be based on the universal differential envelope $\Gamma^\wedge$ of a given first-order differential calculus Γ over G. Properties of such structures are collected in Appendix B. A symbol $\widehat{\otimes}$ will be used for the graded tensor product of graded (differential) algebras.

(*diff3*) Let (U, π_U) be a local trivialization for P and $\psi_U \colon S_c(U) \otimes \mathcal{A} \to \mathcal{B}$ the corresponding "right inverse." Then π_U and ψ_U are extendible to homomorphisms $\pi_U^\wedge \colon \Omega_P \to \Omega(U) \widehat{\otimes} \Gamma^\wedge$ and $\psi_U^\wedge \colon \Omega_c(U) \widehat{\otimes} \Gamma^\wedge \to \Omega_P$ of (graded-) differential algebras.

Property *diff2* as well as the fact that $\Omega_c(U) \widehat{\otimes} \Gamma^\wedge$ is generated, as a differential algebra, by $S_c(U) \otimes \mathcal{A}$, imply that homomorphisms $\pi_U^\wedge$ and $\psi_U^\wedge$ are uniquely determined. It is easy to see that

$$\pi_U^\wedge \psi_U^\wedge(w) = w \tag{3.1}$$

for each $w \in \Omega_c(U) \widehat{\otimes} \Gamma^\wedge$.

For a given open set $V \subseteq M$ let $I_V^\wedge \subseteq \Omega_P$ be the differential subalgebra generated by $I_V \subseteq \mathcal{B}$.

Lemma 3.1. *(i) Algebras $I_V^\wedge$ are ideals in Ω_P.*

(ii) If (U, π_U) is a local trivialization for P and if $V \subseteq U$ then

$$\psi_U^\wedge(\Omega_c(V) \widehat{\otimes} \Gamma^\wedge) = I_V^\wedge,$$
$$\pi_U^\wedge(I_V^\wedge) = \Omega_c(V) \widehat{\otimes} \Gamma^\wedge.$$

Proof. The second statement follows directly from Lemma 2.5 and definition (2.11). Concerning (i), let us prove it first in a special case described in (ii). It is sufficient to check that $b\psi_U^\wedge(f)$, $\psi_U^\wedge(f)b$, $db\psi_U^\wedge(f)$ and $\psi_U^\wedge(f)db$ belong to $I_V^\wedge = \psi_U^\wedge(\Omega_c(V) \widehat{\otimes} \Gamma^\wedge)$, for each $f \in \Omega_c(V) \widehat{\otimes} \Gamma^\wedge$ and $b \in \mathcal{B}$. Each $f \in \Omega_c(V) \widehat{\otimes} \Gamma^\wedge$ can be written as a sum of elements of the form $f_0 df_1 \dots df_k$, where $f_i \in S_c(V) \otimes \mathcal{A}$. We have $b\psi_U^\wedge(f_0 df_1 \dots df_k) = b\psi_U(f_0) d\psi_U(f_1) \dots d\psi_U(f_k) \in I_V^\wedge$ because $b\psi_U(f_0) \in \psi_U(S_c(V) \otimes \mathcal{A})$. Inclusions $\psi_U^\wedge(f)b \in I_V^\wedge$ follow in a similar manner. Further, $db\psi_U^\wedge(f) = d(b\psi_U^\wedge(f)) - b\psi_U^\wedge(df) \in I_V^\wedge$, and similarly $\psi_U^\wedge(f)db \in I_V^\wedge$.

Let $V \subseteq M$ be an arbitrary open set and $\tau = (\pi_U)_{U \in \mathcal{U}}$ an arbitrary trivialization system for P. It is then easy to see that $I_V^\wedge$ is linearly spanned by ideals $I_{V \cap U}^\wedge$, where $U \in \mathcal{U}$. Thus, $I_V^\wedge$ is an ideal in Ω_P. □

Lemma 3.2. *Let τ be a trivialization system for P. Then every map ψ_{UV} from the corresponding G-cocycle $\mathcal{C}_\tau$ is uniquely extendible to a graded-differential automorphism $\psi_{UV}^\wedge \colon \Omega(U \cap V) \widehat{\otimes} \Gamma^\wedge \to \Omega(U \cap V) \widehat{\otimes} \Gamma^\wedge$.*

Proof. It is sufficient to construct $\psi_{UV}^\wedge$ as automorphisms of $\Omega_c(U \cap V) \widehat{\otimes} \Gamma^\wedge$. For each $(U, V) \in N^2(\mathcal{U})$ let us define $\psi_{UV}^\wedge$ to be the composition of the isomorphisms $\psi_V^\wedge \colon \Omega_c(U \cap V) \widehat{\otimes} \Gamma^\wedge \to I_{U \cap V}^\wedge$ and $(\psi_U^\wedge)^{-1} \colon I_{U \cap V}^\wedge \to \Omega_c(U \cap V) \widehat{\otimes} \Gamma^\wedge$. By construction $\psi_{UV}^\wedge$ is a grade-preserving differential automorphism which extends the action of ψ_{UV}. Uniqueness follows from the fact that $S_c(U \cap V) \otimes \mathcal{A}$ generates the differential algebra $\Omega_c(U \cap V) \widehat{\otimes} \Gamma^\wedge$. □

Consequently, not all differential structures over G are relevant for our considerations. The calculus Γ must be compatible with transition functions ψ_{UV}. This is a motivation for the following

Definition 3.1. A first-order differential calculus Γ over G is called *admissible* iff for each G-cocycle $\mathscr{C}$ every transition map $\psi_{UV}\colon S(U\cap V)\otimes\mathcal{A}\to S(U\cap V)\otimes\mathcal{A}$ is extendible to a homomorphism $\psi_{UV}^{\wedge}\colon \Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^{\wedge}\to\Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^{\wedge}$ of differential algebras. Maps $\psi_{UV}^{\wedge}$ are grade preserving, bijective, $\Omega(U\cap V)$-linear and uniquely determined.

As we shall prove, each admissible calculus over G, together with requirements *diff1–3*, completely determines the corresponding calculus Ω_P over P. At first, the notion of admissibility will be analyzed in more detail.

As explained in Appendix A, the Lie algebra $\mathfrak{lie}(G_{cl})$ can be naturally understood as the space of (hermitian) functionals $X\colon\mathcal{A}\to\mathbb{C}$ satisfying

$$X(ab)=\epsilon(a)X(b)+\epsilon(b)X(a)$$

for each $a,b\in\mathcal{A}$. Hence, for each $X\in\mathfrak{lie}(G_{cl})$ the map

$$\ell_X=-(X\otimes\mathrm{id})\phi \tag{3.2}$$

is a derivation on $\mathcal{A}$. Further, $\ell\colon\mathfrak{lie}(G_{cl})\to\mathrm{Der}(\mathcal{A})$ is a monomorphism of Lie algebras. The image of ℓ consists precisely of right-invariant derivations on $\mathcal{A}$.

Let $\mathscr{C}=\left\{\psi_{UV}\mid(U,V)\in N^2(\mathscr{U})\right\}$ be a G-cocycle over $(M,\mathscr{U})$. For each $(U,V)\in N^2(\mathscr{U})$ we shall denote by $\partial^{UV}\colon\mathcal{A}\to\Omega(U\cap V)$ a linear map defined by

$$\partial^{UV}(a)=g_{VU}(a^{(1)})d\big(g_{UV}(a^{(2)})\big). \tag{3.3}$$

It is easy to see that

$$\partial^{UV}(ab)=\epsilon(a)\partial^{UV}(b)+\epsilon(b)\partial^{UV}(a) \tag{3.4}$$

for each $a,b\in\mathcal{A}$. Hence, ∂^{UV} can be understood in a natural manner as an element of the space $\Omega(U\cap V)\otimes\mathfrak{lie}(G_{cl})$.

Lemma 3.3. *A first-order calculus Γ over G is admissible iff the following implications hold*

$$\Big\{\sum_i a_i db_i=0\Big\}\Rightarrow\Big\{\sum_i \zeta_g^\star(a_i)d\zeta_g^\star(b_i)=0\Big\}, \tag{3.5}$$

$$\Big\{\sum_i a_i db_i=0\Big\}\Rightarrow\Big\{\sum_i a_i\ell_X(b_i)=0\Big\} \tag{3.6}$$

for each $g\in G_{cl}$ and $X\in\mathfrak{lie}(G_{cl})$.

Proof. Maps $\psi_{UV}^{\wedge}$ have the form

$$\psi_{UV}^{\wedge}(\alpha\otimes\vartheta)=\alpha\varphi_{UV}^{\wedge}(\vartheta), \tag{3.7}$$

where $\varphi_{UV}^{\wedge}\colon\ \to\Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^{\wedge}$ are (unique) graded-differential homomorphisms extending the maps

$$\varphi_{UV}(a) = g_{VU}(a^{(1)}) \otimes a^{(2)}. \tag{3.8}$$

If $\sum_i a_i db_i = 0$ then

$$\begin{aligned} 0 = \varphi_{UV}^{\wedge}\Bigl(\sum_i a_i db_i\Bigr) &= \sum_i \bigl(g_{VU}(a_i^{(1)}) \otimes a_i^{(2)}\bigr) d\bigl(g_{VU}(b_i^{(1)}) \otimes b_i^{(2)}\bigr) \\ &= \sum_i g_{VU}(a_i^{(1)}) d\bigl(g_{VU}(b_i^{(1)})\bigr) \otimes a_i^{(2)} b_i^{(2)} + \sum_i g_{VU}(a_i^{(1)}) g_{VU}(b_i^{(1)}) \otimes a_i^{(2)} db_i^{(2)} \\ &= \sum_i g_{VU}(a_i^{(1)} b_i^{(1)}) \partial^{VU}(b_i^{(2)}) \otimes a_i^{(2)} b_i^{(3)} + \sum_i g_{VU}(a_i^{(1)} b_i^{(1)}) \otimes a_i^{(2)} db_i^{(2)}, \end{aligned}$$

according to Definition 3.1. Comparing bidegrees we find

$$\sum_i g_{VU}(a_i^{(1)} b_i^{(1)}) \partial^{VU}(b_i^{(2)}) \otimes a_i^{(2)} b_i^{(3)} = 0,$$

$$\sum_i g_{VU}(a_i^{(1)} b_i^{(1)}) \otimes a_i^{(2)} db_i^{(2)} = 0.$$

Because of arbitrariness of the G-cocycle, the above equations imply (3.5)–(3.6). Conversely, if (3.5)–(3.6) hold then the formula

$$\sharp_{UV}(adb) = g_{VU}(a^{(1)} b^{(1)}) \otimes a^{(2)} db^{(2)} + g_{VU}(a^{(1)} b^{(1)}) \partial^{VU}(b^{(2)}) \otimes a^{(2)} b^{(3)}$$

consistently defines a linear map $\sharp_{UV} \colon \Gamma \to \Omega(U \cap V) \mathbin{\widehat{\otimes}} \Gamma^{\wedge}$. It is easy to check that

$$\sharp_{UV}(adb) = \varphi_{UV}(a) d\varphi_{UV}(b)$$

for each $a, b \in \mathcal{A}$. According to Proposition B.2 there exists the unique homomorphism $\varphi_{UV}^{\wedge} \colon \Gamma^{\wedge} \to \Omega(U \cap V) \mathbin{\widehat{\otimes}} \Gamma^{\wedge}$ of graded-differential algebras which extends both φ_{UV} and $\sharp_{UV}$. Let us define maps $\psi_{UV}^{\wedge}$ by (3.7). These maps are differential homomorphisms extending the cocycle maps ψ_{UV}. □

If implication (3.5) holds then the formula

$$\zeta_g^{\star}(adb) = \zeta_g^{\star}(a) d\zeta_g^{\star}(b) \tag{3.9}$$

consistently determines a left action of G_{cl} by automorphisms of Γ.

It is easy to see that if (3.5) holds then

$$\Bigl\{\sum_i a_i db_i = 0\Bigr\} \Rightarrow \Bigl\{\sum_i \ell_X(a_i) db_i + a_i d\ell_X(b_i) = 0\Bigr\} \tag{3.10}$$

for each $X \in \mathfrak{lie}(G_{cl})$. In other words, the formula

$$\ell_X(adb) = \ell_X(a) db + a d\ell_X(b) \tag{3.11}$$

consistently determines a linear map $\ell_X \colon \Gamma \to \Gamma$. Evidently, the following equalities hold

$$\ell_X(da) = d\ell_X(a) \qquad \begin{aligned} \ell_X(a\xi) &= \ell_X(a)\xi + a\ell_X(\xi) \\ \ell_X(\xi a) &= \ell_X(\xi)a + \xi\ell_X(a). \end{aligned}$$

Let us now suppose that (3.6) holds. In this case the formula

$$\iota_X(adb) = a\ell_X(b) \tag{3.12}$$

consistently determines a bimodule homomorphism $\iota_X\colon \Gamma \to \mathcal{A}$.

It is worth noticing that the mentioned left action of G_{cl} on Γ (and $\mathcal{A}$) is, according to Proposition B.2, uniquely extendible to the left action of G_{cl} by automorphisms of the graded-differential algebra $\Gamma^\wedge$. Moreover, operators ℓ_X and ι_X are uniquely extendible to a grade-preserving derivation $\ell_X\colon \Gamma^\wedge \to \Gamma^\wedge$ commuting with d, and an antiderivation $\iota_X\colon \Gamma^\wedge \to \Gamma^\wedge$ of order -1 respectively. Classical identities

$$\begin{aligned} \iota_X\iota_Y + \iota_Y\iota_X &= 0 \qquad & [\ell_X, \iota_Y] &= \iota_{[X,Y]}\\ \ell_X &= d\iota_X + \iota_X d \qquad & \ell_{[X,Y]} &= [\ell_X, \ell_Y] \end{aligned}$$

hold.

Lemma 3.4. *If G_{cl} is connected then the admissibility property is equivalent to implications (3.6) and (3.10).*

Proof. Let us suppose that $\sum_i a_i db_i = 0$. It is easy to see that

$$e^{t\ell_X}\Bigl(\sum_i a_i db_i\Bigr) = \sum_i \zeta^\star_{g^t}(a_i) d\zeta^\star_{g^t}(b_i) = 0 \tag{3.13}$$

for each $t \in \Re$ and $X \in \mathfrak{lie}(G_{cl})$, where $t \mapsto g^t$ is the 1-parameter subgroup of G_{cl} generated by X. Consequently, there exists an open set $N \ni \epsilon$ such that

$$\Bigl\{\sum_i a_i db_i = 0\Bigr\} \Rightarrow \Bigl\{\sum_i \zeta^\star_{g^N}(a_i) d\zeta^\star_{g^N}(b_i) = 0\Bigr\} \tag{3.14}$$

for each $g^N \in N$. If G_{cl} is connected then each $g \in G_{cl}$ is a product of some elements from N. Inductively applying (3.14) we find that (3.5) holds in the full generality. □

On the other hand, implications (3.6) and (3.10) are equivalent to the possibility of constructing the maps $\iota_X\colon \Gamma^\wedge \to \Gamma^\wedge$.

We pass to the construction of a calculus over P. Let us fix a trivialization system $\tau = (\pi_U)_{U\in\mathcal{U}}$ for P, and an admissible first-order calculus Γ over G.

For each $(U,V) \in N^2(\mathcal{U})$ the corresponding cocycle map ψ_{UV} admits a natural extension $\psi^\wedge_{UV}\colon \Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^\wedge \to \Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^\wedge$ characterized as the unique graded differential homomorphism extending ψ_{UV}. By definition, the maps $\psi^\wedge_{UV}$ are $\Omega(U\cap V)$-linear. In particular, subalgebras $\Omega_c(W)\mathbin{\widehat{\otimes}}\Gamma^\wedge$ are $\psi^\wedge_{UV}$-invariant for each open set $W \subseteq U\cap V$.

Lemma 3.5. (*i*) *The maps $\psi^\wedge_{UV}$ are bijective and*

$$(\psi^\wedge_{UV})^{-1} = \psi^\wedge_{VU}. \tag{3.15}$$

(*ii*) *We have*

$$\psi^\wedge_{UV}\psi^\wedge_{VW}(\varphi) = \psi^\wedge_{UW}(\varphi) \tag{3.16}$$

for each $(U,V,W) \in N^3(\mathcal{U})$ and $\varphi \in \Omega_c(U\cap V\cap W)\mathbin{\widehat{\otimes}}\Gamma^\wedge$.

Proof. Everything follows from similar properties of transition functions ψ_{UV}, and from the fact that $\psi^{\wedge}_{UV}$ are differential homomorphisms. □

Let us consider a graded-differential algebra

$$\mathscr{T}^{\wedge} = \sum_{U\in\mathscr{U}}^{\oplus} \Omega(U)\,\widehat{\otimes}\,\Gamma^{\wedge}$$

and let $\Omega(P,\tau,\Gamma) \subseteq \mathscr{T}^{\wedge}$ be a subset consisting of all $w \in \mathscr{T}^{\wedge}$ satisfying

$$\psi^{\wedge}_{UV}({}_V|_{U\cap V} \otimes \mathrm{id})p_V(w) = ({}_U|_{U\cap V} \otimes \mathrm{id})p_U(w) \tag{3.17}$$

for each $(U,V) \in N^2(\mathscr{U})$, where p_U are corresponding coordinate projections.

All maps figuring in (3.17) are graded-differential homomorphisms. This implies that $\Omega(P,\tau,\Gamma)$ is a graded-differential subalgebra of $\mathscr{T}^{\wedge}$.

The 0th part of $\Omega(P,\tau,\Gamma)$ can be, according to Proposition 2.9, identified with $\mathscr{B}$. By the use of the previous analysis, it can be shown easily that $\Omega(P,\tau,\Gamma) \subseteq \mathscr{T}^{\wedge}$ satisfies requirements *diff2* and *diff3* too.

We shall now prove that $\Omega(P,\tau,\Gamma)$ is, up to isomorphism, the unique graded-differential algebra satisfying conditions *diff1–3*.

Let $\mathscr{E}$ be an arbitrary algebra possessing this property.

Lemma 3.6. *We have*

$$\psi^{\wedge}_{UV}({}_V|_{U\cap V} \otimes \mathrm{id})\pi^{\wedge}_V(w) = ({}_U|_{U\cap V} \otimes \mathrm{id})\pi^{\wedge}_U(w) \tag{3.18}$$

for each $(U,V) \in N^2(\mathscr{U})$ and $w \in \mathscr{E}$.

Proof. Both sides of (3.18) are differential algebra homomorphisms coinciding on $\mathscr{B} = \mathscr{E}^0$, according to Lemma 2.8. Property *diff2* implies that they coincide on the whole $\mathscr{E}$. □

Lemma 3.7. *The system of maps $\tau^{\wedge} = (\pi^{\wedge}_U)_{U\in\mathscr{U}}$ distinguishes elements of $\mathscr{E}$.*

Proof. Let $(\varphi_U)_{U\in\mathscr{U}}$ be an arbitrary smooth partition of unity for $\mathscr{U}$, and let us assume that $w \in \ker(\pi^{\wedge}_U)$ for each $U \in \mathscr{U}$. Then $i(\varphi_U)w \in I^{\wedge}_U \cap \ker(\pi^{\wedge}_U)$ for each $U \in \mathscr{U}$. Hence, we have $i(\varphi_U)w = 0$. Summing over $\mathscr{U}$ we obtain $w = 0$. □

Proposition 3.8. (*i*) *There exists the unique homomorphism $j^{\wedge}_\tau \colon \mathscr{E} \to \Omega(P,\tau,\Gamma)$ of differential algebras, extending the map $j_\tau \colon \mathscr{B} \to \widetilde{\mathscr{B}}$.*

(*ii*) *The map $j^{\wedge}_\tau$ is bijective.*

Proof. Let us define a graded-differential homomorphism $j^{\wedge}_\tau \colon \mathscr{E} \to \mathscr{T}^{\wedge}$ by equalities

$$p_U j^{\wedge}_\tau = \pi^{\wedge}_U.$$

According to Lemma 3.6 we have

$$j^{\wedge}_\tau(\mathscr{E}) \subseteq \Omega(P,\tau,\Gamma).$$

The map $j^{\wedge}_\tau \colon \mathscr{E} \to \Omega(P,\tau,\Gamma)$ is injective, according to Lemma 3.7. The above equality implies

$$j^{\wedge}_\tau \psi^{\wedge}_U = \widetilde{\psi}^{\wedge}_U, \tag{3.19}$$

where $\widetilde{\psi}_U^\wedge\colon \Omega_c(U)\,\widehat{\otimes}\,\Gamma^\wedge \to \Omega(P,\tau,\Gamma)$ is the unique graded-differential extension of $\widetilde{\psi}_U\colon S_c(U)\otimes\mathcal{A} \to \widetilde{\mathcal{B}}$. Surjectivity of $j_\tau^\wedge$ now follows from the fact that $\Omega(P,\tau,\Gamma)$ is linearly generated by spaces $\mathrm{im}(\widetilde{\psi}_U)$. Uniqueness of $j_\tau^\wedge$ directly follows from property *diff2*. □

We see that $\Omega(P,\tau,\Gamma)$ is essentially independent of a trivialization system τ. For this reason we shall simplify the notation and write $\Omega(P,\Gamma)=\Omega(P,\tau,\Gamma)$. It is worth noticing that the algebra $\Omega(P,\Gamma)$ can be understood as the universal differential envelope of its first-order part (understood as a first-order calculus over $\mathcal{B}$).

In the rest of this section algebraic properties of $\Omega(P,\Gamma)$ will be analyzed in more detail. It will be assumed that a trivialization system τ is fixed.

Let us observe that the formula

$$\pi_U^\wedge i^\wedge(\alpha)=\alpha\restriction_U \tag{3.20}$$

determines (the unique) graded-differential homomorphism $i^\wedge\colon \Omega(M)\to\Omega(P,\Gamma)$ which extends the map i. The map $i^\wedge$ is injective and

$$i^\wedge(\alpha)w=(-1)^{\partial w\partial\alpha}wi^\wedge(\alpha) \tag{3.21}$$

for each $\alpha\in\Omega(M)$ and $w\in\Omega(P,\Gamma)$.

As we shall now see, it is possible to introduce a natural coaction of G on $\Omega(P,\Gamma)$, trivialized in classical geometry. Let $c\colon \Gamma^\wedge\otimes\mathcal{A}\to\Gamma^\wedge$ be a natural coaction map, defined in Appendix B.

Lemma 3.9. *The diagram*

$$\begin{array}{ccc}
\Big\{\Omega(U\cap V)\,\widehat{\otimes}\,\Gamma^\wedge\Big\}\otimes\mathcal{A} & \xrightarrow{\psi_{UV}^\wedge\otimes\mathrm{id}} & \Big\{\Omega(U\cap V)\,\widehat{\otimes}\,\Gamma^\wedge\Big\}\otimes\mathcal{A}\\
{\scriptstyle\mathrm{id}\otimes c}\downarrow & & \downarrow{\scriptstyle\mathrm{id}\otimes c}\\
\Omega(U\cap V)\,\widehat{\otimes}\,\Gamma^\wedge & \xrightarrow[\psi_{UV}^\wedge]{} & \Omega(U\cap V)\,\widehat{\otimes}\,\Gamma^\wedge
\end{array} \tag{3.22}$$

is commutative, for each $(U,V)\in N^2(\mathcal{U})$.

Proof. A direct computation gives

$$\begin{aligned}
\psi_{UV}^\wedge(\mathrm{id}\otimes c)(w\otimes a)&=\psi_{UV}^\wedge\big(1_{U\cap V}\otimes\kappa(a^{(1)})\big)w(1_{U\cap V}\otimes a^{(2)})\\
&=\big(g_{VU}\kappa(a^{(2)})\otimes\kappa(a^{(1)})\big)\psi_{UV}^\wedge(w)\big(g_{VU}(a^{(3)})\otimes a^{(4)}\big)\\
&=\big(g_{VU}(\epsilon(a^{(2)})1)\otimes\kappa(a^{(1)})\big)\psi_{UV}^\wedge(w)(1_{U\cap V}\otimes a^{(3)})\\
&=\big(1_{U\cap V}\otimes\kappa(a^{(1)})\big)\psi_{UV}^\wedge(w)(1_{U\cap V}\otimes a^{(2)})\\
&=(\mathrm{id}\otimes c)(\psi_{UV}^\wedge\otimes\mathrm{id})(w\otimes a).\square
\end{aligned}$$

Proposition 3.10. (*i*) *There exists the unique $\Delta\colon \Omega(P,\Gamma)\otimes\mathcal{A}\to\Omega(P,\Gamma)$ such that the diagram*

$$\begin{array}{ccc}
\Omega(P,\Gamma)\otimes\mathcal{A} & \xrightarrow{\pi_U^\wedge\otimes\mathrm{id}} & \Big\{\Omega(U)\,\widehat{\otimes}\,\Gamma^\wedge\Big\}\otimes\mathcal{A}\\
{\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\mathrm{id}\otimes c}\\
\Omega(P,\Gamma) & \xrightarrow[\pi_U^\wedge]{} & \Omega(U)\,\widehat{\otimes}\,\Gamma^\wedge
\end{array} \tag{3.23}$$

is commutative, for each $U \in \mathscr{U}$.

(*ii*) *The following identities hold*

$$\Delta(w \otimes ab) = \Delta\bigl(\Delta(w \otimes a) \otimes b\bigr), \tag{3.24}$$

$$\Delta(wu \otimes a) = \Delta(w \otimes a^{(1)})\Delta(u \otimes a^{(2)}), \tag{3.25}$$

$$\Delta(w \otimes 1) = w, \tag{3.26}$$

$$\Delta\bigl(i^{\wedge}(\alpha)w \otimes a\bigr) = i^{\wedge}(\alpha)\Delta(w \otimes a). \tag{3.27}$$

Proof. Uniqueness of Δ is a direct consequence of the fact that maps $\pi_U^{\wedge}$ distinguish elements of $\Omega(P,\Gamma)$. To prove the existence, consider a map $\tilde{\Delta}\colon \mathscr{T}^{\wedge} \otimes \mathscr{A} \to \mathscr{T}^{\wedge}$ defined by

$$p_U \tilde{\Delta}(w \otimes a) = (\mathrm{id} \otimes c)\bigl(p_U(w) \otimes a\bigr).$$

Lemma 3.9 implies that $\tilde{\Delta}\bigl(\Omega(P,\Gamma) \otimes \mathscr{A}\bigr) \subseteq \Omega(P,\Gamma)$. The restriction of $\tilde{\Delta}$ on $\Omega(P,\Gamma)$ gives the desired map $\Delta\colon \Omega(P,\Gamma) \otimes \mathscr{A} \to \Omega(P,\Gamma)$. Evidently, diagram (3.23) is commutative.

A direct computation gives

$$\begin{aligned}\pi_U^{\wedge}(\Delta(wu \otimes a)) &= \sum_{ij}(-1)^{\partial\vartheta_i\partial\beta_j}\alpha_i\beta_j \otimes c(\vartheta_i\eta_j \otimes a)\\ &= \sum_{ij}(-1)^{\partial\vartheta_i\partial\beta_j}\alpha_i\beta_j \otimes c(\vartheta_i \otimes a^{(1)})c(\eta_j \otimes a^{(2)})\\ &= \pi_U^{\wedge}(\Delta(w \otimes a^{(1)})\Delta(u \otimes a^{(2)})),\end{aligned}$$

Similarly

$$\begin{aligned}\pi_U^{\wedge}\bigl(\Delta(w \otimes ab)\bigr) = \sum_i \alpha_i \otimes c(\vartheta_i \otimes ab) &= \sum_i \alpha_i \otimes c(c(\alpha_i \otimes a) \otimes b)\\ &= \pi_U^{\wedge}\bigl(\Delta(\Delta(w \otimes a) \otimes b)\bigr),\end{aligned}$$

and finally

$$\pi_U^{\wedge}\Delta(i^{\wedge}(\alpha)w \otimes a) = \bigl((\alpha\restriction_U) \otimes 1\bigr)\sum_i \alpha_i \otimes c(\vartheta_i \otimes a) = \pi_U^{\wedge}\bigl(i^{\wedge}(\alpha)\Delta(w \otimes a)\bigr),$$

where $\pi_U^{\wedge}(w) = \sum_i \alpha_i \otimes \vartheta_i$ and $\pi_U^{\wedge}(u) = \sum_j \beta_j \otimes \eta_j$. Hence (3.24)–(3.27) hold. □

In the case when Γ admits the *-structure, or if it is right-covariant [9] the algebra $\Omega(P,\Gamma)$ possesses a similar property, too. To prove this we need a technical lemma.

Lemma 3.11. (*i*) *If Γ is a *-calculus then $\psi_{UV}^{\wedge}$ preserve the natural *-structure on $\Omega(U \cap V) \mathbin{\hat{\otimes}} \Gamma^{\wedge}$.*

(*ii*) *If Γ is right-covariant then the diagrams*

$$\begin{array}{ccc}\Omega(U \cap V) \mathbin{\hat{\otimes}} \Gamma^{\wedge} & \xrightarrow{\mathrm{id} \otimes \wp_{\Gamma}^{\wedge}} & \bigl\{\Omega(U \cap V) \mathbin{\hat{\otimes}} \Gamma^{\wedge}\bigr\} \otimes \mathscr{A}\\ \psi_{UV}^{\wedge}\Big\downarrow & & \Big\downarrow\psi_{UV}^{\wedge} \otimes \mathrm{id}\\ \Omega(U \cap V) \mathbin{\hat{\otimes}} \Gamma^{\wedge} & \xrightarrow[\mathrm{id} \otimes \wp_{\Gamma}^{\wedge}]{} & \bigl\{\Omega(U \cap V) \mathbin{\hat{\otimes}} \Gamma^{\wedge}\bigr\} \otimes \mathscr{A}\end{array} \tag{3.28}$$

are commutative. Here, $\wp_\Gamma^\wedge\colon \Gamma^\wedge \to \Gamma^\wedge \otimes \mathcal{A}$ *is a natural extension of the right action* $\wp_\Gamma\colon \Gamma \to \Gamma \otimes \mathcal{A}$.

Proof. Elements of the form $w = \alpha \otimes a_0 da_1 \dots da_n$, where $\alpha \in \Omega(U \cap V)$ and $a_0, a_1, \dots, a_n \in \mathcal{A}$, linearly span $\Omega(U \cap V) \mathbin{\hat{\otimes}} \Gamma^\wedge$. If Γ is *-covariant then

$$\begin{aligned}\psi_{UV}^\wedge(w^*) &= (-1)^{n(n-1)/2}\psi_{UV}^\wedge\bigl(\alpha^* \otimes d(a_n^*)\dots d(a_1^*)a_0^*\bigr)\\ &= (-1)^{n(n-1)/2}\alpha^* d[\varphi_{UV}(a_n^*)]\dots d[\varphi_{UV}(a_1^*)]\varphi_{UV}(a_0^*)\\ &= (-1)^{n(n-1)/2}\alpha^* d[\varphi_{UV}(a_n)]^*\dots d[\varphi_{UV}(a_1)]^*\varphi_{UV}(a_0)^*\\ &= \psi_{UV}^\wedge(w)^*,\end{aligned}$$

according to (3.8) and Proposition B.3. Similarly, if Γ is right-covariant then Proposition B.6 (*ii*) implies

$$\begin{aligned}(\mathrm{id} \otimes \wp_\Gamma^\wedge)\psi_{UV}^\wedge(w) &= (\mathrm{id} \otimes \wp_\Gamma^\wedge)\bigl(\alpha\varphi_{UV}(a_0)d[\varphi_{UV}(a_1)]\dots d[\varphi_{UV}(a_n)]\bigr)\\ &=\alpha[(\varphi_{UV} \otimes \mathrm{id})\phi(a_0)][(d\varphi_{UV} \otimes \mathrm{id})\phi(a_1)]\dots[(d\varphi_{UV} \otimes \mathrm{id})\phi(a_n)]\\ &=(\psi_{UV}^\wedge \otimes \mathrm{id})(\mathrm{id} \otimes \wp_\Gamma^\wedge)(w).\square\end{aligned}$$

Proposition 3.12. *If* Γ *is *-covariant then there exists the unique antilinear map* $*\colon \Omega(P,\Gamma) \to \Omega(P,\Gamma)$ *extending* $*\colon \mathcal{B} \to \mathcal{B}$, *satisfying* $(wu)^* = (-1)^{\partial w \partial u}u^*w^*$ *and commuting with* $d\colon \Omega(P,\Gamma) \to \Omega(P,\Gamma)$. *The following identities hold*

$$i^\wedge(\alpha^*) = i^\wedge(\alpha)^*, \tag{3.29}$$

$$(w^*)^* = w, \tag{3.30}$$

$$\Delta(w \otimes a)^* = \Delta(w^* \otimes \kappa(a)^*). \tag{3.31}$$

Proof. If Γ is a *-calculus then tensoring the natural *-structure on $\Omega(U)$ with the corresponding *-structure on $\Gamma^\wedge$ and taking the direct sum we obtain a *-structure on $\mathcal{T}^\wedge$. It is easy to see that

$$(wu)^* = (-1)^{\partial w \partial u}u^*w^* \qquad d(w^*) = d(w)^* \qquad i^\wedge(\alpha^*) = i^\wedge(\alpha)^*$$

for each $u, w \in \mathcal{T}^\wedge$ and $\alpha \in \Omega(M)$. According to Lemma 3.11 (*i*), the algebra $\Omega(P,\Gamma) \subseteq \mathcal{T}^\wedge$ is *-invariant. The restriction of the *-operation on $\Omega(P,\Gamma)$ gives the desired involution.

Applying the definition of Δ and elementary properties of c we obtain

$$\begin{aligned}\pi_U^\wedge\bigl[\Delta(w \otimes a)^*\bigr] = \sum_i \alpha_i^* \otimes c(\vartheta_i \otimes a)^* &= \sum_i \alpha_i^* \otimes c(\vartheta_i^* \otimes \kappa(a)^*)\\ &= \pi_U^\wedge\bigl[\Delta(w^* \otimes \kappa(a)^*)\bigr]\end{aligned}$$

for each $U \in \mathcal{U}$. Uniqueness of $*$ directly follows from property *diff2* for $\Omega(P,\Gamma)$. □

Proposition 3.13. (*i*) *If* Γ *is right-covariant then there exists the unique homomorphism* $F^\wedge\colon \Omega(P,\Gamma) \to \Omega(P,\Gamma) \otimes \mathcal{A}$ *which extends* F *and such that*

$$F^\wedge d = (d \otimes \mathrm{id})F^\wedge. \tag{3.32}$$

The following identities hold

$$F^\wedge i^\wedge(\alpha) = i^\wedge(\alpha) \otimes 1, \tag{3.33}$$

$$(\mathrm{id} \otimes \epsilon)F^\wedge = \mathrm{id}, \tag{3.34}$$

$$(\mathrm{id} \otimes \phi)F^\wedge = (F^\wedge \otimes \mathrm{id})F^\wedge, \tag{3.35}$$

$$F^\wedge \Delta(w \otimes a) = \sum_k \Delta(w_k \otimes a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)}, \tag{3.36}$$

where $F^\wedge(w) = \sum_k w_k \otimes c_k$.

(*ii*) *If* Γ *is also a* *-*calculus then* $F^\wedge$ *is hermitian, in a natural manner.*

Proof. If Γ is right-covariant then a map $F^\wedge_{\mathscr{T}} : \mathscr{T}^\wedge \to \mathscr{T}^\wedge \otimes \mathscr{A}$ defined by

$$(p_U \otimes \mathrm{id})F^\wedge_{\mathscr{T}} = (\mathrm{id} \otimes \wp^\wedge_\Gamma)p_U$$

is a homomorphism which, according to Proposition B.6 (*ii*), satisfies the following equations

$$F^\wedge_{\mathscr{T}} d = (d \otimes \mathrm{id})F^\wedge_{\mathscr{T}},$$
$$(\mathrm{id} \otimes \epsilon)F^\wedge_{\mathscr{T}} = \mathrm{id},$$
$$(\mathrm{id} \otimes \phi)F^\wedge_{\mathscr{T}} = (F^\wedge_{\mathscr{T}} \otimes \mathrm{id})F^\wedge_{\mathscr{T}},$$
$$F^\wedge_{\mathscr{T}}(\alpha) = \alpha \otimes \mathrm{id},$$

where $p_U(\alpha) \in \Omega(U) \otimes 1$ for each $U \in \mathscr{U}$. Now Lemma 3.11 (*ii*) implies that $\Omega(P, \Gamma) = \Omega(P, \tau, \Gamma)$ is a $F^\wedge_{\mathscr{T}}$-invariant subalgebra of $\mathscr{T}$, in other words we have the inclusion $F^\wedge_{\mathscr{T}}\bigl(\Omega(P,\Gamma)\bigr) \subseteq \Omega(P,\Gamma)\otimes\mathscr{A}$. The restriction of $F^\wedge_{\mathscr{T}}$ on $\Omega(P,\Gamma)$ gives the desired map $F^\wedge$.

According to Lemma B.7

$$\begin{aligned}(\pi^\wedge_U \otimes \mathrm{id})F^\wedge \Delta(w \otimes a) &= (\mathrm{id} \otimes \wp^\wedge_\Gamma)\Bigl[\sum_i \alpha_i \otimes c(\vartheta_i \otimes a)\Bigr] \\ &= \sum_{ij} \alpha_i \otimes c(\vartheta_{ij} \otimes a^{(2)}) \otimes \kappa(a^{(1)})c_{ij}a^{(3)} \\ &= (\pi^\wedge_U \otimes \mathrm{id})\Bigl[\sum_k \Delta(w_k \otimes a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)}\Bigr]\end{aligned}$$

for each $U \in \mathscr{U}$. Here, $\wp^\wedge_\Gamma(\vartheta_i) = \sum_j \vartheta_{ij} \otimes c_{ij}$. Uniqueness of $F^\wedge$ is a direct consequence of property *diff2*. If Γ is in addition *-covariant then $\Omega(P,\Gamma)$ is a *-subalgebra of $\mathscr{T}^\wedge$ and $F^\wedge_{\mathscr{T}}$ is hermitian, according to Proposition B.6. □

From this moment we shall assume that Γ is left-covariant. The space of left-invariant elements of Γ will be denoted by Γ_{inv}. Further, $\mathscr{R} \subseteq \ker(\epsilon)$ will be the right $\mathscr{A}$-ideal which canonically [9] determines this calculus.

Proposition 3.14. *A left-covariant calculus* Γ *is admissible iff*

$$(X \otimes \mathrm{id})\mathrm{ad}(\mathscr{R}) = \{0\} \tag{3.37}$$

for each $X \in \mathfrak{lie}(G_{cl})$.

Proof. If Γ is admissible (and left-covariant) then the following equality holds

$$\varphi^{\wedge}_{UV}\pi(a) = \partial^{VU}(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)} + 1_{U\cap V} \otimes \pi(a). \tag{3.38}$$

Indeed,

$$\begin{aligned}\varphi^{\wedge}_{UV}\pi(a) &= \varphi^{\wedge}_{UV}\left(\kappa(a^{(1)})da^{(2)}\right)\\ &= g_{VU}\kappa(a^{(2)})dg_{VU}(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\ &\quad + g_{VU}\left(\kappa(a^{(2)})a^{(3)}\right) \otimes \kappa(a^{(1)})da^{(4)}\\ &= g_{UV}(a^{(2)})dg_{VU}(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\ &\quad + g_{VU}\left(\epsilon(a^{(2)})1\right) \otimes \kappa(a^{(1)})da^{(3)}\\ &= \partial^{VU}(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)} + 1_{U\cap V} \otimes \pi(a)\end{aligned}$$

according to (3.3) and (B.29).

If $a \in \mathcal{R}$ then

$$\partial^{VU}(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)} = 0. \tag{3.39}$$

It is easy to see that, because of arbitrariness of τ, Eqs. (3.39) are equivalent to Eqs. (3.37).

Conversely, let us assume that (3.37) holds for each $X \in \mathfrak{lie}(G_{cl})$. To prove admissibility of Γ it is sufficient to check implication (3.6), because (3.5) is satisfied automatically for left-covariant differential structures. As a consequence of (3.37), the formula

$$\rho_X\left(\pi(a)\right) = X(a^{(2)})\kappa(a^{(1)})a^{(3)} \tag{3.40}$$

consistently defines a linear map $\rho_X : \Gamma_{inv} \to \mathcal{A}$, for each $X \in \mathfrak{lie}(G_{cl})$.

Now if $\sum_i a_i db_i = 0$ then

$$\begin{aligned}0 = \sum_i a_i b_i^{(1)} \rho_X\left(\pi(b_i^{(2)})\right) &= \sum_i a_i b_i^{(1)} X(b_i^{(3)})\kappa(b_i^{(2)})b_i^{(4)}\\ &= \sum_i a_i X(b_i^{(2)})\epsilon(b_i^{(1)})b_i^{(3)} = -\sum_i a_i \ell_X(b_i)\end{aligned}$$

because of (B.31) and the fact that Γ is free over Γ_{inv} as a left/right $\mathcal{A}$-module. □

There exists "the simplest" left-covariant admissible calculus. It is based on the right $\mathcal{A}$-ideal $\widehat{\mathcal{R}}$ consisting of all elements $a \in \ker(\epsilon)$ annihilated by operators $(X \otimes \mathrm{id})\mathrm{ad}$. This calculus is also bicovariant and *-covariant. It is analyzed in more details in Appendix C.

Now we are going to construct the total "pull back" for the right action of G on P. We shall assume that Γ is bicovariant. As shown in Proposition B.11, the comultiplication map admits a natural extension $\widehat{\phi}: \Gamma^{\wedge} \to \Gamma^{\wedge} \mathbin{\widehat{\otimes}} \Gamma^{\wedge}$, which is a graded differential algebra homomorphism.

Lemma 3.15. *The diagram*

$$\begin{array}{ccc}
\Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^\wedge & \xrightarrow{\psi_{UV}^\wedge} & \Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^\wedge \\
{\scriptstyle \mathrm{id}\otimes\widehat{\phi}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\widehat{\phi}} \\
\Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^\wedge\mathbin{\widehat{\otimes}}\Gamma^\wedge & \xrightarrow[\psi_{UV}^\wedge\otimes\mathrm{id}]{} & \Omega(U\cap V)\mathbin{\widehat{\otimes}}\Gamma^\wedge\mathbin{\widehat{\otimes}}\Gamma^\wedge
\end{array} \tag{3.41}$$

is commutative.

Proof. All maps figuring in this diagram are homomorphisms of graded differential algebras, and $\Omega(U\cap V)$-linear in a natural manner. Hence, it is sufficent to check the commutativity in the 0th order level. However, this is just the covariance condition for the cocycle maps. □

Proposition 3.16. (*i*) *There exists the unique homomorphism*

$$\widehat{F}\colon \Omega(P,\Gamma)\rightarrow\Omega(P,\Gamma)\mathbin{\widehat{\otimes}}\Gamma^\wedge$$

of graded-differential algebras which extends the map F.

(*ii*) *The diagram*

$$\begin{array}{ccc}
\Omega(P,\Gamma) & \xrightarrow{\widehat{F}} & \Omega(P,\Gamma)\mathbin{\widehat{\otimes}}\Gamma^\wedge \\
{\scriptstyle \widehat{F}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\widehat{\phi}} \\
\Omega(P,\Gamma)\mathbin{\widehat{\otimes}}\Gamma^\wedge & \xrightarrow[\widehat{F}\otimes\mathrm{id}]{} & \Omega(P,\Gamma)\mathbin{\widehat{\otimes}}\Gamma^\wedge\mathbin{\widehat{\otimes}}\Gamma^\wedge
\end{array} \tag{3.42}$$

is commutative and the following identities hold

$$F^\wedge=(\mathrm{id}\otimes p_0)\widehat{F}, \tag{3.43}$$

$$(\mathrm{id}\otimes\epsilon^\wedge)\widehat{F}=\mathrm{id}, \tag{3.44}$$

$$\widehat{F}i^\wedge(\alpha)=i^\wedge(\alpha)\otimes 1. \tag{3.45}$$

(*iii*) *If* Γ *is in addition *-covariant then* $\widehat{F}$ *preserves canonical *-structures.*

Proof. Let us consider a linear map $\widehat{F}_{\mathcal{T}}\colon\mathcal{T}^\wedge\rightarrow\mathcal{T}^\wedge\mathbin{\widehat{\otimes}}\Gamma^\wedge$ given by

$$(p_U\otimes\mathrm{id})\widehat{F}_{\mathcal{T}}=(\mathrm{id}\otimes\widehat{\phi})p_U.$$

This map is a homomorphism of graded-differential algebras and $\widehat{F}_{\mathcal{T}}(\alpha)=\alpha\otimes 1$ for each $\alpha\in\mathcal{T}_M$, where

$$\mathcal{T}_M=\sum_{U\in\mathcal{U}}^{\oplus}\Omega(U).$$

According to Lemma 3.15 the algebra $\Omega(P,\Gamma)=\Omega(P,\tau,\Gamma)$ is $\widehat{F}_{\mathcal{T}}$-invariant, in the sense that $\widehat{F}_{\mathcal{T}}\bigl(\Omega(P,\Gamma)\bigr)\subseteq\Omega(P,\Gamma)\mathbin{\widehat{\otimes}}\Gamma^\wedge$.

Let $\widehat{F}\colon \Omega(P,\Gamma) \to \Omega(P,\Gamma) \mathbin{\widehat{\otimes}} \Gamma^\wedge$ be the corresponding restriction. The diagram (3.42) and Eq. (3.44) directly follow from (B.38) and (B.39).

Let us consider a map $(\mathrm{id} \otimes p_0)\widehat{F}\colon \Omega(P,\Gamma) \to \Omega(P,\Gamma) \otimes \mathcal{A}$. Evidently, this is a homomorphism which extends F. Moreover,

$$(\mathrm{id} \otimes p_0)\widehat{F} d(w) = (\mathrm{id} \otimes p_0)(d \otimes \mathrm{id} + (-1)^{\partial *}\mathrm{id} \otimes d)\widehat{F}(w) = (d \otimes p_0)\widehat{F}(w)$$

for each $w \in \Omega(P,\Gamma)$. Proposition 3.13 implies that $(\mathrm{id} \otimes p_0)\widehat{F} = F^\wedge$. Uniqueness of $\widehat{F}$ follows from property *diff2*.

Finally, if Γ is $*$-covariant then $\widehat{\phi}$ is hermitian. This implies that $\widehat{F}_{\mathcal{T}}$ is hermitian, too. Hermicity of $\widehat{F}$ also directly follows from hermicity of F, and hermicity of all differentials appearing in the game. □

Let us define the graded $*$-algebra of *horizontal forms* to be the tensor product

$$\mathfrak{hor}(P) = \Omega(M) \otimes_M \mathcal{B}. \tag{3.46}$$

This algebra can be understood as a subalgebra of $\Omega(P,\Gamma)$ consisting of all w satisfying

$$\pi_U^\wedge(w) \in \Omega(U) \otimes \mathcal{A} \tag{3.47}$$

for each $U \in \mathcal{U}$. By construction, $\mathfrak{hor}(P)$ is independent of a choice of Γ.

Let us now define a graded algebra of "verticalized" differential forms to be, as a graded vector space

$$\mathfrak{ver}(P,\Gamma) = \mathcal{B} \otimes \Gamma_{inv}^\wedge \tag{3.48}$$

while the product is specified by

$$(q \otimes \eta)(b \otimes \vartheta) = \sum_k q b_k \otimes (\eta \circ c_k)\vartheta, \tag{3.49}$$

where $\sum_k b_k \otimes c_k = F(b)$. Here, $\circ$ is the left-invariant restriction of the coaction map c. Associativity of this product easily follows from the main properties of F and $\circ$. We see that $\mathcal{B}$ and $\Gamma_{inv}^\wedge$ are subalgebras of $\mathfrak{ver}(P,\Gamma)$, in a natural manner. For each $U \in \mathcal{U}$ the map

$$\pi_U \otimes \mathrm{id}\colon \mathfrak{ver}(P,\Gamma) \to S(U) \otimes \mathcal{A} \otimes \Gamma_{inv}^\wedge \cong S(U) \otimes \Gamma^\wedge$$

becomes a homomorphism of graded algebras. Actually this property characterizes the product in $\mathfrak{ver}(P,\Gamma)$, because the maps $\pi_U \otimes \mathrm{id}$ distinguish elements of this algebra.

The algebra $\mathfrak{ver}(P,\Gamma)$ can be equipped with a natural differential, defined by

$$d_v(b \otimes \vartheta) = \sum_k b_k \otimes \pi(c_k)\vartheta + b \otimes d\vartheta. \tag{3.50}$$

We have

$$(\pi_U \otimes \mathrm{id})d_v(b \otimes \vartheta) = \sum_i \left[\alpha_i \otimes a_i^{(1)} \otimes \pi(a_i^{(2)})\vartheta + \alpha_i \otimes a_i \otimes d\vartheta\right], \tag{3.51}$$

where $\pi_U(b) = \sum_i \alpha_i \otimes a_i$. We see that locally

$$d_v \leftrightarrow (\mathrm{id} \otimes d)\colon S(U) \otimes \Gamma^\wedge \to S(U) \otimes \Gamma^\wedge.$$

Furthermore, right actions of G on $\mathcal{B}$ and $\Gamma_{inv}^\wedge$ naturally induce the right action F_v of G on $\mathfrak{ver}(P,\Gamma)$. More precisely,

$$F_v(b \otimes \vartheta) = \sum_{kl} b_k \otimes \vartheta_l \otimes c_k d_l, \tag{3.52}$$

where $\varpi^\wedge(\vartheta) = \sum_l \vartheta_l \otimes d_l$. This action can be also characterized by relations

$$(\pi_U \otimes \mathrm{id}^2)F_v = (\mathrm{id} \otimes \varrho_\Gamma^\wedge)(\pi_U \otimes \mathrm{id}). \tag{3.53}$$

The differential d_v is F_v-covariant, in the sense that

$$F_v d_v = (d_v \otimes \mathrm{id})F_v. \tag{3.54}$$

Indeed, we have

$$\begin{aligned} F_v d_v(b \otimes \vartheta) &= \sum_{kl} \Bigl(b_k \otimes \pi(c_k^{(3)})\vartheta_l \otimes c_k^{(1)}\kappa(c_k^{(2)})c_k^{(4)}d_l + b_k \otimes d\vartheta_l \otimes c_k d_l\Bigr) \\ &= \sum_{kl} \Bigl(b_k \otimes \pi(c_k^{(1)})\vartheta_l \otimes c_k^{(2)}d_l + b_k \otimes d\vartheta_l \otimes c_k d_l\Bigr) = (d_v \otimes \mathrm{id})F_v(b \otimes \vartheta). \end{aligned}$$

Graded-differential algebra $\mathfrak{ver}(P,\Gamma)$ can be also obtained from $\Omega(P,\Gamma)$ by factoring through horizontal forms. More precisely, let H be the ideal in $\Omega(P,\Gamma)$ generated $di\bigl(S(M)\bigr)$. Then $\mathfrak{ver}(P,\Gamma)$ is naturally isomorphic to the factor-algebra $\Omega(P,\Gamma)/H$. Moreover, H is a right-invariant ideal and, according to (3.51) and (3.53) the factorized $F^\wedge$ and d coincide with F_v and d_v respectively. We shall denote by π_v the factor projection map.

The homomorphism $\pi_v\colon \Omega(P,\Gamma) \to \mathfrak{ver}(P,\Gamma)$ possesses the following properties

$$(\pi_v \otimes \mathrm{id})F^\wedge = F_v \pi_v, \tag{3.55}$$

$$\pi_v d = d_v \pi_v, \tag{3.56}$$

$$\pi_v(b) = b \otimes 1. \tag{3.57}$$

The last two properties uniquely characterize π_v.

Finally if Γ is *-covariant then H is *-invariant and there exists the unique *-structure on $\mathfrak{ver}(P,\Gamma)$ such that π_v is hermitian. Explicitly, this *-structure is given by

$$(b \otimes \vartheta)^* = \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*). \tag{3.58}$$

4. Connections and Pseudotensorial Forms

This section is devoted to the study of counterparts of (pseudo)tensorial forms. In particular, we shall develop the formalism of connections.

At first, the classical concept of pseudotensoriality will be translated into the noncommutative context. Let us assume for a moment that the bundle is classical. Let us consider a representation $\widetilde{\rho}\colon G \to \mathrm{lin}(V)$ in a vector space V. Then a V-valued k-form $\widetilde{w}$ on P is called *pseudotensorial* of $(\widetilde{\rho}, V)$-type [3] iff

$$g^*(\widetilde{w}) = \rho(g^{-1})\widetilde{w}$$

for each $g \in G$, where g^* is the pull back of the corresponding right action. The form $\widetilde{w}$ is called *tensorial,* if it vanishes whenever at least one argument is vertical.

The pseudotensoriality property can be equivalently formulated in terms of the map $w\colon V^* \to \Omega(P)$, where $w(\vartheta) = \vartheta\widetilde{w}$, via the following diagram

$$\begin{array}{ccc} V^* & \xrightarrow{\;w\;} & \Omega(P) \\ {\scriptstyle\rho(g)}\big\downarrow & & \big\downarrow{\scriptstyle g^*} \\ V^* & \xrightarrow[\;w\;]{} & \Omega(P) \end{array} \tag{4.1}$$

where ρ is the contragradient representation of $\widetilde{\rho}$. Moreover, $\widetilde{w}$ is tensorial iff $w(\vartheta)$ is horizontal for each $\vartheta \in V^*$.

Let us turn back to the noncommutative context. Let $P = (\mathcal{B}, i, F)$ be a quantum principal G-bundle over M and $\rho\colon L \to L \otimes \mathcal{A}$ a (nonsingular) representation [8] of G in a complex vector space L. Let Γ be an admissible right-covariant calculus over G. The above diagram naturally suggests to define pseudotensorial forms as linear maps $w\colon L \to \Omega(P, \Gamma)$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\;w\;} & \Omega(P,\Gamma) \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle F^\wedge} \\ L\otimes\mathcal{A} & \xrightarrow[\;w\otimes\mathrm{id}\;]{} & \Omega(P,\Gamma)\otimes\mathcal{A} \end{array} \tag{4.2}$$

is commutative.

Let us denote by $\psi(P, \rho, \Gamma)$ the space of corresponding pseudotensorial forms. This space is naturally graded

$$\psi(P, \rho, \Gamma) = \sum_{i\geq 0}^{\oplus} \psi^i(P, \rho, \Gamma), \tag{4.3}$$

where the grading is induced from $\Omega(P, \Gamma)$. Strictly speaking the above decomposition holds if L is finite-dimensional. The space $\psi(P, \rho, \Gamma)$ is a bimodule over $\Omega(M)$, in a natural manner. According to (3.32), the space of pseudotensorial forms is invariant under compositions with $d\colon \Omega(P, \Gamma) \to \Omega(P, \Gamma)$.

We shall denote by $\tau(P, \rho)$ the subspace consisting of *tensorial forms* w, characterized by

$$w(L) \subseteq \mathfrak{hor}(P). \tag{4.4}$$

Actually $\tau(P,\rho)$ is a graded $\Omega(M)$-submodule of $\psi(P,\rho,\Gamma)$. Let us observe that $\tau(P,\rho)$ is independent of a specification of Γ.

If L is endowed with an antilinear involution $*\colon L\rightarrow L$ such that ρ is hermitian, in a natural manner, and if Γ is a *-calculus then the formula

$$w^*(\vartheta)=(w(\vartheta^*))^*$$

defines a *-structure on $\psi(P,\rho,\Gamma)$. The space $\tau(P,\rho)$ is *-invariant.

Tensorial forms possess a simple local representation.

Proposition 4.1. *(i) For each $w\in\tau(P,\rho)$ and $U\in\mathscr{U}$ there exists the unique linear map $\varphi_U\colon L\rightarrow\Omega(U)$ such that*

$$\pi_U^\wedge w=(\varphi_U\otimes\mathrm{id})\rho. \tag{4.5}$$

We have

$$\bigl(\varphi_V(\vartheta)\restriction_{U\cap V}\bigr)=\sum_k\bigl(\varphi_U(\vartheta_k)\restriction_{U\cap V}\bigr)g_{UV}(c_k) \tag{4.6}$$

for each $\vartheta\in L$ and $(U,V)\in N^2(\mathscr{U})$, where $\sum_k\vartheta_k\otimes c_k=\rho(\vartheta)$.

(ii) Conversely, if maps φ_U satisfy equalities (4.6) then there exists the unique $w\in\tau(P,\rho)$ such that (4.5) holds.

Proof. We have

$$\pi_U^\wedge w(L)\subseteq\Omega(U)\otimes\mathscr{A}$$

for each $w\in\tau(P,\rho)$ and $U\in\mathscr{U}$. On the other hand (4.2) is equivalent to the following equations

$$(\mathrm{id}\otimes\phi)\bigl[\pi_U^\wedge w(\vartheta)\bigr]=\bigl(\pi_U^\wedge w\otimes\mathrm{id}\bigr)\rho(\vartheta). \tag{4.7}$$

Acting by $\mathrm{id}\otimes\epsilon\otimes\mathrm{id}$ on both sides of this equation we obtain (4.5) with $\varphi_U=(\mathrm{id}\otimes\epsilon)\pi_U^\wedge w$. Conversely, a direct verification shows that (4.7) follows from (4.5).

Let us now analyze how φ_U and φ_V are related on the overlapping of regions U and V.

For an arbitrary system of linear maps $\varphi_U\colon L\rightarrow\Omega(U)$, the formula (4.5) determines a linear map $w\colon L\rightarrow\mathscr{T}^\wedge$. According to (3.17) a necessary and sufficient condition for the inclusion $w(L)\subseteq\Omega(P,\tau,\Gamma)$ can be written in the form

$$({}_U|_{U\cap V}\otimes\mathrm{id})(\varphi_U\otimes\mathrm{id})\rho(\vartheta)=\sum_k {}_V|_{U\cap V}\bigl(\varphi_V(\vartheta_k)\bigr)g_{VU}(c_k^{(1)})\otimes c_k^{(2)}, \tag{4.8}$$

which is equivalent to (4.6). □

From this moment it will be assumed that Γ is the simplest left-covariant admissible calculus. Explicitly, Γ is a first-order calculus based on the right ideal $\widehat{\mathscr{R}}$ consisting of all $a\in\ker(\epsilon)$ such that $(X\otimes\mathrm{id})\mathrm{ad}(a)=0$ for each $X\in\mathfrak{lie}(G_{cl})$. As explained in Appendix C, this is a bicovariant *-calculus.

Furthermore, we shall restrict the consideration to the case

$$L=\Gamma_{inv}\qquad\rho=\varpi.$$

In this case we shall simplify the notation and write $\Omega(P)$, $\mathfrak{ver}(P)$, $\tau(P)$ and $\psi(P)$ for the corresponding algebras and modules.

Finally, we shall fix a section $\eta\colon \mathscr{L}^* \to \Gamma_{inv}$ of $\nu\colon \Gamma_{inv} \to \mathscr{L}^*$ (Appendix C) which intertwines *-structures and adjoint actions of G_{cl}. Hence we can write

$$\Gamma_{inv} = \mathscr{L}^* \oplus \ker(\nu), \tag{4.9}$$

with $\eta\nu$ playing the role of the projection on the first factor.

If φ_U are local representatives of $w \in \tau(P)$ then maps

$$\varphi_U^{cl} = \varphi\eta\nu \qquad \varphi_U^{\perp} = \varphi_U(1-\eta\nu)$$

satisfy (4.6), too. This, together with Proposition 4.1, enables us to introduce the "classical" and the "quantum" component of w, by

$$\pi_U^{\wedge} w_{cl} = (\varphi_U^{cl} \otimes \mathrm{id})\varpi \qquad \pi_U^{\wedge} w_{\perp} = (\varphi_U^{\perp} \otimes \mathrm{id})\varpi.$$

By construction,

$$w = w_{cl} + w_{\perp}.$$

We shall denote by $\tau_{cl}(P)$ and $\tau_{\perp}(P)$ corresponding mutually complementary graded *-$\Omega(M)$-submodules of $\tau(P)$. Elements of $\tau_{cl}(P)$ will be called *classical* tensorial forms.

Proposition 4.2. *A tensorial form w is classical iff the diagram*

$$\begin{array}{ccc} \Gamma_{inv} \otimes \mathcal{A} & \xrightarrow{w \otimes \mathrm{id}} & \Omega(P) \otimes \mathcal{A} \\ \circ\big\downarrow & & \big\downarrow\Delta \\ \Gamma_{inv} & \xrightarrow[w]{} & \Omega(P) \end{array} \tag{4.10}$$

is commutative.

Proof. Let us suppose that w is classical. In local trivialization terms, this means

$$\varphi_U(\vartheta \circ a) = \epsilon(a)\varphi_U(\vartheta), \tag{4.11}$$

for each $\vartheta \in \Gamma_{inv}$, $U \in \mathscr{U}$ and $a \in \mathcal{A}$. On the other hand, according to (3.23) and (B.20), commutativity of (4.10) is equivalent to equalities

$$\begin{aligned} \pi_U^{\wedge}\bigl[w(\vartheta \circ a)\bigr] &= (\varphi_U \otimes \mathrm{id})\varpi(\vartheta \circ a) = \sum_k \varphi_U(\vartheta_k \circ a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)} \\ &= \sum_k \varphi_U(\vartheta_k) \otimes \kappa(a^{(1)})c_k a^{(2)} = \pi_U^{\wedge}\Delta\bigl[w(\vartheta) \otimes a\bigr], \end{aligned} \tag{4.12}$$

where $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$.

If (4.11) holds then, evidently, (4.12) holds. Conversely, if (4.12) holds then acting by $\mathrm{id} \otimes \epsilon$ on both sides of the third equality we obtain (4.11). □

We pass to the study of connection forms.

Definition 4.1. A *connection* on P is every pseudotensorial 1-form ω satisfying

$$\omega(\vartheta^*) = \omega(\vartheta)^*, \tag{4.13}$$

$$\pi_v \omega(\vartheta) = 1 \otimes \vartheta \tag{4.14}$$

for each $\vartheta \in \Gamma_{inv}$.

Condition (4.14) plays the role of the classical requirement that connections map fundamental vector fields into their generators. Connections naturally form an infinite-dimensional affine space (as far as Γ_{inv} is non-trivial).

Lemma 4.3. (*i*) *Each quantum principal bundle P admits a connection.*

(*ii*) *For an arbitrary connection ω on P, and a linear map $\alpha\colon \Gamma_{inv} \to \Omega(P)$, the map $\alpha + \omega$ is a connection iff α is a hermitian 1-order tensorial form.*

Proof. Let us consider an arbitrary smooth partition of unity $(\rho_U)_{U\in\mathcal{U}}$ for $\mathcal{U}$, and define a map $\omega\colon \Gamma_{inv} \to \Omega(P)$ by

$$\omega(\vartheta) = \sum_{U\in\mathcal{U}} \psi_U^\wedge(\rho_U \otimes \vartheta). \tag{4.15}$$

This map is a connection on P. The second statement easily follows from Definition 4.1. □

Let $\mathrm{con}(P)$ be the affine space of all connections on P. The following proposition describes connections in terms of gauge potentials.

Proposition 4.4. (*i*) *For each $\omega \in \mathrm{con}(P)$ there exist the unique system of linear maps $A_U\colon \Gamma_{inv} \to \Omega(U)$ such that*

$$\pi_U^\wedge \omega(\vartheta) = \sum_k A_U(\vartheta_k) \otimes c_k + 1_U \otimes \vartheta \tag{4.16}$$

for each $U \in \mathcal{U}$, where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$. These maps are hermitian and

$$\bigl(A_V(\vartheta){\restriction}_{U\cap V}\bigr) = \sum_k \bigl(A_U(\vartheta_k){\restriction}_{U\cap V}\bigr) g_{UV}(c_k) + \partial_{UV}(\vartheta) \tag{4.17}$$

for each $(U, V) \in N^2(\mathcal{U})$, where $\partial_{UV}\pi = \partial^{UV}$.

(*ii*) *Conversely, if hermitian maps $A_U\colon \Gamma_{inv} \to \Omega(U)$ are given such that (4.17) holds, then the formula (4.16) determines a connection on P.*

Proof. The proof is essentially the same as for Proposition 4.1. □

Definition 4.2. A connection ω is called *classical* iff the diagram

$$\begin{array}{ccc} \Gamma_{inv} \otimes \mathcal{A} & \xrightarrow{\ \circ\ } & \Gamma_{inv} \\ {\scriptstyle \omega \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \omega} \\ \Omega(P) \otimes \mathcal{A} & \xrightarrow[\Delta]{} & \Omega(P) \end{array} \tag{4.18}$$

is commutative.

Proposition 4.5. *A connection ω is classical iff*

$$A_U\eta\nu = A_U \iff A_U(\vartheta\circ a) = \epsilon(a)A_U(\vartheta),$$

for each $U \in \mathcal{U}$.

Proof. A similar reasoning as in the proof of Proposition 4.2. □

Every connection can be written as a sum of a classical connection, and a "purely quantum" part.

Proposition 4.6. *For each $\omega \in \mathfrak{con}(P)$ there exist the unique classical connection ω_{cl} and hermitian tensorial 1-form $\omega_\perp \in \tau_\perp(P)$ such that*

$$\omega = \omega_{cl} + \omega_\perp. \tag{4.19}$$

Proof. Let us start from the corresponding gauge potentials A_U and define

$$A_U^{cl} = A_U\eta\nu \qquad A_U^\perp = A_U - A_U^{cl}.$$

From (4.17) it follows that

$$\big(A_V^{cl}(\vartheta){\restriction}_{U\cap V}\big) = \sum_k \big(A_U^{cl}(\vartheta_k){\restriction}_{U\cap V}\big)g_{UV}(c_k) + \partial_{UV}(\vartheta),$$
$$\big(A_V^{\perp}(\vartheta){\restriction}_{U\cap V}\big) = \sum_k \big(A_U^{\perp}(\vartheta_k){\restriction}_{U\cap V}\big)g_{UV}(c_k).$$

It is easy to see that A_U^{cl} and $A_U^\perp$ are hermitian. Hence, there exist a classical connection ω_{cl} and a hermitian element $\omega_\perp \in \tau_\perp^1(P)$ such that

$$\pi_U^\wedge\omega_{cl}(\vartheta) = (A_U^{cl}\otimes\mathrm{id})\varpi(\vartheta) + 1_U\otimes\vartheta,$$
$$\pi_U^\wedge\omega_\perp(\vartheta) = (A_U^{\perp}\otimes\mathrm{id})\varpi(\vartheta)$$

for each $\vartheta \in \Gamma_{inv}$. Evidently, (4.19) holds. This decomposition is unique, because of mutual complementarity between $\tau_{cl}(P)$ and $\tau_\perp(P)$. □

From this moment it will be assumed that the subalgebra $\Gamma_{inv}^\wedge$ of left-invariant elements is realized as a complement to the space $S_{inv}^\wedge \subseteq \Gamma_{inv}^\otimes$, with the help of a linear section $\iota\colon \Gamma_{inv}^\wedge \to \Gamma_{inv}^\otimes$ of the factorization map, which intertwines *-structures and adjoint actions of G. Here $S_{inv}^\wedge$ is the left-invariant part of the ideal $S^\wedge \subseteq \Gamma^\otimes$ and $\Gamma_{inv}^\otimes$ is the tensor algebra over Γ_{inv} (Appendix B).

It is easy to see (for example, applying a quantum analog of the method of group projectors) that ι always exists. If Γ_{inv} is finite-dimensional then ι can be constructed by identifying $\Gamma_{inv}^\wedge$ with the orthocomplement of $S_{inv}^\wedge$, with respect to an appropriate scalar product.

However, it is important to mention that in various interesting situations (for example, if $G = S_\mu U(2)$ and $\mu \in (-1,1)\setminus\{0\}$) the space Γ_{inv} will be infinite-dimensional.

For each connection ω, let us denote by $\omega^\otimes\colon \Gamma_{inv}^\otimes \to \Omega(P)$ the corresponding unital multiplicative extension. Let $\omega^\wedge\colon \Gamma_{inv}^\wedge \to \Omega(P)$ be the composition of maps ι and $\omega^\otimes$.

Proposition 4.7. *(i) The diagram*

$$\begin{array}{ccc} \Gamma_{inv}^{\wedge} & \xrightarrow{\omega^{\wedge}} & \Omega(P) \\ \varpi^{\wedge}\big\downarrow & & \big\downarrow F^{\wedge} \\ \Gamma_{inv}^{\wedge} \otimes \mathcal{A} & \xrightarrow[\omega^{\wedge} \otimes \mathrm{id}]{} & \Omega(P) \otimes \mathcal{A} \end{array} \tag{4.20}$$

is commutative.

(ii) We have

$$\pi_v \omega^{\wedge}(\vartheta) = 1 \otimes \vartheta \tag{4.21}$$

for each $\vartheta \in \Gamma_{inv}^{\wedge}$.

(iii) The map $\omega^{\wedge}$ *is *-preserving.*

(iv) If ω *is classical then* $\omega^{\wedge}$ *is multiplicative and the diagram*

$$\begin{array}{ccc} \Gamma_{inv}^{\wedge} \otimes \mathcal{A} & \xrightarrow{\omega^{\wedge} \otimes \mathrm{id}} & \Omega(P) \otimes \mathcal{A} \\ \circ\big\downarrow & & \big\downarrow \Delta \\ \Gamma_{inv}^{\wedge} & \xrightarrow[\omega^{\wedge}]{} & \Omega(P) \end{array} \tag{4.22}$$

is commutative.

Proof. Property (i) is a simple consequence of the pseudotensoriality of ω and of the $\varpi^{\otimes}$-invariance of $\iota(\Gamma_{inv}^{\wedge})$. Property (ii) follows from (4.14), and the multiplicativity of π_v.

To prove (iii), it is sufficient to observe that $\omega^{\otimes}$ intertwines *-structures on $\Gamma_{inv}^{\otimes}$ and $\Omega(P)$.

Let us assume that ω is classical. We shall prove that $\omega^{\otimes}$ vanishes on the ideal $S_{inv}^{\wedge} \subseteq \Gamma_{inv}^{\otimes}$. In accordance with considerations performed in Appendix B, it is sufficient to check that

$$\omega^{\otimes}\left[\pi(a^{(1)}) \otimes \pi(a^{(2)})\right] = 0 \tag{4.23}$$

for each $a \in \widehat{\mathcal{R}}$. In the local trivialization system, this is equivalent to the following equalities

$$\begin{aligned} \left[(A_U \otimes \mathrm{id})\varpi\pi(a^{(1)})\right]\pi(a^{(2)}) + \pi(a^{(1)})\left[(A_U \otimes \mathrm{id})\varpi\pi(a^{(2)})\right] \\ + \left[(A_U \otimes \mathrm{id})\varpi\pi(a^{(1)})\right]\left[(A_U \otimes \mathrm{id})\varpi\pi(a^{(2)})\right] = 0. \end{aligned}$$

A direct calculation shows that the last term, as well as the sum of the first two, vanishes. Consequently $\omega^{\wedge}$ is multiplicative.

Commutativity of (4.22) is a direct consequence of (3.25), (B.27) and (4.10), and the multiplicativity of $\omega^{\wedge}$. □

With the help of $\omega^\wedge$ the space $\Omega(P)$ can be naturally decomposed into a tensor product of $\mathfrak{hor}(P)$ and $\Gamma_{inv}^\wedge$.

Let us suppose that $\mathfrak{vh}(P) = \mathfrak{hor}(P) \otimes \Gamma_{inv}^\wedge$ is endowed with a graded *-algebra structure, via the natural identification

$$\mathfrak{hor}(P) \otimes \Gamma_{inv}^\wedge \leftrightarrow \Omega(M) \otimes_M \mathfrak{ver}(P). \tag{4.24}$$

The algebra $\mathfrak{vh}(P)$ represents "vertically-horizontally" decomposed forms on the bundle. We shall denote by F_{vh} the natural right action of G on $\mathfrak{vh}(P)$.

For each $\omega \in \mathfrak{con}(P)$ the formula

$$m_\omega(\varphi \otimes \vartheta) = \varphi\omega^\wedge(\vartheta) \tag{4.25}$$

defines a linear grade-preserving map $m_\omega \colon \mathfrak{vh}(P) \to \Omega(P)$.

Proposition 4.8. *(i) The map m_ω is bijective.*

(ii) The diagram

$$\begin{array}{ccc} \mathfrak{vh}(P) & \xrightarrow{\ m_\omega\ } & \Omega(P) \\ {\scriptstyle F_{vh}}\big\downarrow & & \big\downarrow{\scriptstyle F^\wedge} \\ \mathfrak{vh}(P)\otimes\mathcal{A} & \xrightarrow[m_\omega\otimes\mathrm{id}]{} & \Omega(P)\otimes\mathcal{A} \end{array} \tag{4.26}$$

is commutative.

*(iii) If ω is classical then m_ω is an isomorphism of graded *-algebras.*

Proof. At first we prove that m_ω is injective. Each $\alpha \in \mathfrak{vh}(P)\setminus\{0\}$ can be written in the form $\alpha = \sum_i w_i \otimes \vartheta_i + \psi$, where $\vartheta_i \in \Gamma_{inv}^{\wedge k}$ are homogeneous linearly independent elements and $w_i \neq 0$, while ψ is the element having the second degrees less than k. If $m_\omega(\alpha) = 0$ then

$$\sum_i \pi_U(w_i)\vartheta_i = 0$$

for each $U \in \mathcal{U}$. This implies $\sum_i w_i \otimes \vartheta_i = 0$, which is a contradiction.

In order to prove that m_ω is surjective, it is sufficient to check that

$$\psi_U^\wedge\bigl(\Omega_c(U) \otimes \Gamma^{\wedge k}\bigr) \subseteq m_\omega\bigl(\mathfrak{vh}(P)\bigr)$$

for each $U \in \mathcal{U}$ and $k \geq 0$.

For $k = 0$ the statement is obvious. Let us suppose that the above inclusion holds for degrees up to some fixed k. Equation (4.16) together with the definition of $\omega^\wedge$ gives

$$\pi_U^\wedge\bigl[m_\omega(w \otimes \vartheta)\bigr] = \sum_i \alpha_i \otimes a_i\vartheta + \beta, \tag{4.27}$$

where $\vartheta \in (\Gamma_{inv}^\wedge)^{k+1}$ and $w = \psi_U^\wedge\Bigl(\sum_i \alpha_i \otimes a_i\Bigr)$, while $\beta \in \Omega_c(U) \otimes \Gamma^\wedge$, with the second degrees less than $k+1$.

Acting by $\psi_U^\wedge$ on both sides of (4.27) we get

$$\psi_U^\wedge\Bigl(\sum_i \alpha_i \otimes a_i\vartheta\Bigr) = m_\omega(w \otimes \vartheta) - \psi_U^\wedge(\beta).$$

By the inductive assumption, the right-hand side of the above equality belongs to $\mathrm{im}(m_\omega)$. Hence m_ω is bijective.

The commutativity of (4.26) is a direct consequence of (4.25), and Proposition 4.7 (i).

Finally, let us suppose that ω is classical. According to Proposition 4.7 (iv) and definition (3.49) of the product in $\mathfrak{ver}(P)$, we have

$$(u \otimes \vartheta)(w \otimes \eta) = (-1)^{\partial\vartheta\partial w}\sum_k uw_k \otimes (\vartheta \circ c_k)\eta, \tag{4.28}$$

and hence

$$\begin{aligned} m_\omega\bigl[(u \otimes \vartheta)(w \otimes \eta)\bigr] &= (-1)^{\partial w\partial\vartheta}\sum_k uw_k\omega^\wedge(\vartheta \circ c_k)\omega^\wedge(\eta) \\ &= (-1)^{\partial w\partial\vartheta}\sum_k uw_k\Delta\bigl(\omega^\wedge(\vartheta) \otimes c_k\bigr)\omega^\wedge(\eta) \\ &= u\omega^\wedge(\vartheta)w\omega^\wedge(\eta) = m_\omega(u \otimes \vartheta)m_\omega(w \otimes \eta). \end{aligned}$$

Here $F^\wedge(w) = \sum_k w_k \otimes c_k$ and we have used the identity

$$\sum_k w_k\Delta(\alpha \otimes c_k) = (-1)^{\partial\alpha\partial w}\alpha w, \tag{4.29}$$

where α is arbitrary (and w is horizontal). Similarly, the *-structure on $\mathfrak{vh}(P)$ is given by

$$(w \otimes \vartheta)^* = \sum_k w_k^* \otimes (\vartheta^* \circ c_k^*), \tag{4.30}$$

and hence

$$\begin{aligned} m_\omega\bigl[(w \otimes \vartheta)^*\bigr] &= \sum_k w_k^*\omega^\wedge(\vartheta^* \circ c_k^*) = \sum_k w_k^*\Delta\bigl(\omega^\wedge(\vartheta)^* \otimes c_k^*\bigr) \\ &= (-1)^{\partial w\partial\vartheta}\omega^\wedge(\vartheta)^*w^* = \bigl[m_\omega(w \otimes \vartheta)\bigr]^*. \square \end{aligned}$$

It is of some interest to analyze in more detail the question of the multiplicativity of $\omega^\wedge$.

Definition 4.3. A connection ω is called *multiplicative* iff

$$\omega^\otimes(S_{inv}^\wedge) = \{0\}.$$

Equivalently, ω is multiplicative iff $\omega^\wedge$ is a multiplicative map. In this case $\omega^\wedge$ is independent of the embedding ι, and coincides with $\omega^\otimes/S_{inv}^\wedge$. As already mentioned, the multiplicativity of $\omega^\wedge$ is equivalent to (4.23). This gives a quadratic constraint in $\mathfrak{con}(P)$. In the general case the left-hand side of (4.23) determines a linear map $r_\omega\colon \widehat{\mathscr{R}} \to \Omega(P)$. This map "measures" a lack of multiplicativity of ω.

Proposition 4.9. *We have*

$$r_\omega = m_\Omega(w_\perp\pi \otimes \omega_\perp\pi)\phi, \tag{4.31}$$

where m_Ω *is the product map in* $\Omega(P)$*. In local terms*

$$\pi_U^\wedge r_\omega = (r_\omega^U \otimes \mathrm{id})\big(\mathrm{ad}\restriction\widehat{\mathscr{R}}\big), \tag{4.32}$$

where $r_\omega^U(a) = A_U^\perp\pi(a^{(1)})A_U^\perp\pi(a^{(2)})$*. In particular* r_ω *is a horizontally-valued map.*

Proof. Using local expressions for ω_{cl} and $\omega_\perp$, Eqs. (4.23) and (B.30), and Proposition 4.5 we obtain

$$\begin{aligned}
\pi_U^\wedge r_\omega(a) - \pi_U^\wedge m_\Omega(\omega_\perp\pi \otimes \omega_\perp\pi)\phi(a) &= \pi_U^\wedge m_\Omega(\omega_\perp\pi \otimes \omega_{cl}\pi)\phi(a)\\
&\quad + \pi_U^\wedge m_\Omega(\omega_{cl}\pi \otimes \omega_\perp\pi)\phi(a)\\
&= A_U^\perp\pi(a^{(2)})A_U^{cl}\pi(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\
&\quad + A_U^{cl}\pi(a^{(2)})A_U^\perp\pi(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\
&\quad + A_U^\perp\pi(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)}\pi(a^{(4)})\\
&\quad - A_U^\perp\pi(a^{(3)}) \otimes \pi(a^{(1)})\kappa(a^{(2)})a^{(4)}\\
&= A_U^\perp\pi(a^{(3)})A_U^{cl}\pi\big(\kappa(a^{(2)})a^{(4)}\big) \otimes \kappa(a^{(1)})a^{(5)}\\
&\quad + A_U^\perp\pi(a^{(3)}) \otimes \kappa(a^{(2)})a^{(4)}\pi\big(\kappa(a^{(1)})a^{(5)}\big),
\end{aligned}$$

for each $a \in \widehat{\mathscr{R}}$. Remembering that $\widehat{\mathscr{R}}$ is ad-invariant we conclude that the above terms vanish. Hence (4.31) holds. Property (4.32) simply follows from (4.31). □

5. Horizontal Projection, Covariant Derivative and Curvature

For each $\omega \in \mathfrak{con}(P)$ let $h_\omega\colon \Omega(P) \to \Omega(P)$ be a linear map given by

$$h_\omega = m_\omega(\mathrm{id} \otimes p_{inv}^0)m_\omega^{-1}. \tag{5.1}$$

Let $D_\omega\colon \Omega(P) \to \Omega(P)$ be a linear map defined as a composition

$$D_\omega = h_\omega d. \tag{5.2}$$

Evidently, both maps are $\mathfrak{hor}(P)$-valued.

Definition 5.1. Operators h_ω and D_ω are called *the horizontal projection* and *the covariant derivative* associated to ω.

The following statement easily follows from the analysis of the previous section.

Proposition 5.1. (*i*) *The map* h_ω *is* $\Omega(M)$*-linear and projects* $\Omega(P)$ *onto* $\mathfrak{hor}(P)$*.*

(*ii*) *We have*

$$(D_\omega - d)\big(\Omega(M)\big) = \{0\} \qquad D_\omega(w\varphi) = (dw)h_\omega(\varphi) + (-1)^{\partial w}wD_\omega(\varphi) \tag{5.3}$$

for each $w \in \Omega(M)$ *and* $\varphi \in \Omega(P)$*.*

(*iii*) *Maps* h_ω *and* D_ω *are invariant under the action of G. In other words, the diagrams*

$$\begin{array}{ccc} \Omega(P) & \xrightarrow{F^\wedge} & \Omega(P)\otimes\mathcal{A} \\ {\scriptstyle h_\omega}\downarrow & & \downarrow{\scriptstyle h_\omega\otimes\mathrm{id}} \\ \Omega(P) & \xrightarrow[F^\wedge]{} & \Omega(P)\otimes\mathcal{A} \end{array} \qquad \begin{array}{ccc} \Omega(P) & \xrightarrow{F^\wedge} & \Omega(P)\otimes\mathcal{A} \\ {\scriptstyle D_\omega}\downarrow & & \downarrow{\scriptstyle D_\omega\otimes\mathrm{id}} \\ \Omega(P) & \xrightarrow[F^\wedge]{} & \Omega(P)\otimes\mathcal{A} \end{array} \tag{5.4}$$

are commutative.

(*iv*) *If ω is classical then h_ω is a *-homomorphism. Furthermore*

$$D_\omega(\psi\varphi) = D_\omega(\psi)h_\omega(\varphi) + (-1)^{\partial\psi}h_\omega(\psi)D_\omega(\varphi) \tag{5.5}$$

for each $\psi,\varphi\in\Omega(P)$. □

By construction, the space $\mathfrak{hor}(P)$ is D_ω-invariant. The corresponding restriction is described by the following

Proposition 5.2. *If $\varphi\in\mathfrak{hor}(P)$ then*

$$D_\omega(\varphi) = d(\varphi) - (-1)^{\partial\varphi}m_\Omega(\mathrm{id}\otimes\omega\pi)F^\wedge(\varphi). \tag{5.6}$$

In local terms,

$$\pi_U^\wedge D_\omega(\varphi) = \sum_i\Bigl\{d(\alpha_i)\otimes a_i - (-1)^{\partial\alpha}\alpha_i A_U\pi(a_i^{(1)})\otimes a_i^{(2)}\Bigr\}, \tag{5.7}$$

where $\sum_i \alpha_i\otimes a_i = \pi_U^\wedge(\varphi)$.

Proof. We have

$$\pi_U^\wedge d(\varphi) = \sum_i d(\alpha_i)\otimes a_i + (-1)^{\partial\alpha}\alpha_i\otimes a_i^{(1)}\pi(a_i^{(2)}),$$

and hence

$$\begin{aligned}\pi_U^\wedge D_\omega(\varphi) &= \sum_i d(\alpha_i)\otimes a_i - (-1)^{\partial\alpha}\sum_i \alpha_i A_U\pi(a_i^{(3)})\otimes a_i^{(1)}\kappa(a_i^{(2)})a_i^{(4)} \\ &= \sum_i\Bigl\{d(\alpha_i)\otimes a_i - (-1)^{\partial\alpha}\alpha_i A_U\pi(a_i^{(1)})\otimes a_i^{(2)}\Bigr\}\end{aligned}$$

according to Definition 5.1. This proves (5.7). Let us compute the right-hand side of (5.6). We have

$$\begin{aligned}\pi_U^\wedge\bigl[d(\varphi) - (-1)^{\partial\varphi}m_\Omega(\mathrm{id}\otimes\omega\pi)F^\wedge(\varphi)\bigr] &= \sum_i d(\alpha_i)\otimes a_i \\ &+ (-1)^{\partial\alpha}\sum_i \alpha_i\otimes a_i^{(1)}\pi(a_i^{(2)}) \\ &- (-1)^{\partial\alpha}\sum_i \alpha_i A_U\pi(a_i^{(1)})\otimes a_i^{(2)} \\ &- (-1)^{\partial\alpha}\sum_i \alpha_i\otimes a_i^{(1)}\pi(a_i^{(2)}) \\ = \sum_i\Bigl\{d(\alpha_i)\otimes a_i - (-1)^{\partial\alpha}\alpha_i A_U\pi(a_i^{(1)})\otimes a_i^{(2)}\Bigr\} &= \pi_U^\wedge D_\omega(\varphi).\end{aligned}$$ □

For given linear maps $\alpha, \beta\colon \Gamma_{inv} \to \Omega(P)$ we shall denote by $[\alpha, \beta]$ and $\langle \alpha, \beta \rangle$ linear maps defined by

$$[\alpha, \beta] = m_\Omega(\alpha \otimes \beta)c^\top, \tag{5.8}$$

$$\langle \alpha, \beta \rangle = m_\Omega(\alpha \otimes \beta)\delta, \tag{5.9}$$

where $c^\top\colon \Gamma_{inv} \to \Gamma_{inv} \otimes \Gamma_{inv}$ is the "transposed commutator" map [9] explicitly given by (C.11) and $\delta\colon \Gamma_{inv} \to \Gamma_{inv} \otimes \Gamma_{inv}$ is the "embedded differential" defined by

$$\delta(\vartheta) = \iota d(\vartheta). \tag{5.10}$$

If $\alpha, \beta \in \psi(P)$ then $\langle \alpha, \beta \rangle, [\alpha, \beta] \in \psi(P)$, according to Lemma C.4. In particular these brackets map $\tau(P) \times \tau(P)$ into $\tau(P)$. Similar brackets can be introduced for maps valued in an arbitrary algebra.

According to Proposition 5.1 the space $\psi(P)$ is mapped, via compositions with h_ω and D_ω, into $\tau(P)$. In particular $\tau(P)$ is D_ω-invariant.

Proposition 5.3. (*i*) *We have*

$$(\pi_U^\wedge D_\omega \varphi)(\vartheta) = (\left\{d\varphi_U - (-1)^{\partial\varphi}[\varphi_U, A_U]\right\} \otimes \mathrm{id})\varpi(\vartheta), \tag{5.11}$$

where φ_U are local representatives of $\varphi \in \tau(P)$.

(*ii*) *The following identity describes the action of D_ω on tensorial forms*

$$D_\omega \varphi = d\varphi - (-1)^{\partial\varphi}[\varphi, \omega]. \tag{5.12}$$

Proof. We have

$$\pi_U^\wedge d\varphi(\vartheta) = \sum_k d\varphi_U(\vartheta_k) \otimes c_k + (-1)^{\partial\varphi}\varphi_U(\vartheta_k) \otimes c_k^{(1)}\pi(c_k^{(2)}),$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$. Taking the horizontal projection we obtain

$$\begin{aligned}
(\pi_U^\wedge D_\omega \varphi)(\vartheta) &= \sum_k \left\{d\varphi_U(\vartheta_k) \otimes c_k - (-1)^{\partial\varphi}\varphi_U(\vartheta_k)A_U\pi(c_k^{(3)}) \otimes c_k^{(1)}\kappa(c_k^{(2)})c_k^{(4)}\right\}\\
&= \sum_k d\varphi_U(\vartheta_k) \otimes c_k - (-1)^{\partial\varphi}\varphi_U(\vartheta_k)A_U\pi(c_k^{(1)}) \otimes c_k^{(2)}\\
&= \sum_k \big(d\varphi_U - (-1)^{\partial\varphi}[\varphi_U, A_U]\big)(\vartheta_k) \otimes c_k.
\end{aligned}$$

A computation of the right-hand side of (5.12) gives

$$\begin{aligned}
\pi_U^\wedge\left\{d\varphi - (-1)^{\partial\varphi}[\varphi, \omega]\right\} &= \sum_k d\varphi_U(\vartheta_k) \otimes c_k - (-1)^{\partial\varphi}\varphi_U(\vartheta_k)A_U\pi(c_k^{(1)}) \otimes c_k^{(2)}\\
&= (\pi_U^\wedge D_\omega \varphi)(\vartheta). \square
\end{aligned}$$

Let $q_\omega\colon \psi(P) \to \psi(P)$ be a linear map defined by

$$q_\omega(\varphi) = \langle \omega, \varphi \rangle - (-1)^{\partial\varphi}\langle \varphi, \omega \rangle - (-1)^{\partial\varphi}[\varphi, \omega]. \tag{5.13}$$

By definition, this map is $\Omega(M)$-linear from the right.

Proposition 5.4. *The space $\tau(P)$ is q_ω-invariant.*

Proof. For a given $\vartheta \in \Gamma_{inv}$ let us choose $a \in \ker(\epsilon)$ satisfying conditions listed in Lemma C.5 (i). We have then

$$\begin{aligned}-(-1)^{\partial\varphi}\bigl(\pi_U^\wedge q_\omega(\varphi)\bigr)(\vartheta) = \sum_k \Bigl\{\varphi_U(\vartheta_k)A_U\pi(c_k^{(1)}) \otimes c_k^{(2)} + \varphi_U(\vartheta_k) \otimes c_k^{(1)}\pi(c_k^{(2)})\Bigr\}\\ - \varphi_U\pi(a^{(2)})A_U\pi(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\ - \varphi_U\pi(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)}\pi(a^{(4)})\\ + (-1)^{\partial\varphi}A_U\pi(a^{(2)})\varphi_U\pi(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\ + \varphi_U\pi(a^{(3)}) \otimes \pi(a^{(1)})\kappa(a^{(2)})a^{(4)},\end{aligned}$$

for each $\varphi \in \tau(P)$.

On the other hand, applying (B.30) and (B.25) we find

$$\varphi_U\pi(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)}\pi(a^{(4)}) - \varphi_U\pi(a^{(3)}) \otimes \pi(a^{(1)})\kappa(a^{(2)})a^{(4)} = \sum_k \varphi_U(\vartheta_k) \otimes c_k^{(1)}\pi(c_k^{(2)}).$$

Combining the above equalities we obtain finally

$$\bigl(\pi_U^\wedge q_\omega(\varphi)\bigr)(\vartheta) = \bigl(q_\omega^U(\varphi) \otimes \mathrm{id}\bigr)\varpi(\vartheta), \tag{5.14}$$

where

$$q_\omega^U(\varphi) = \langle A_U, \varphi_U\rangle - (-1)^{\partial\varphi}\langle\varphi_U, A_U\rangle - (-1)^{\partial\varphi}[\varphi_U, A_U]. \tag{5.15}$$

We see that $q_\omega(\varphi)$ is tensorial. □

If ω is classical then the operator q_ω vanishes on tensorial forms. Indeed, in this case

$$A_U\pi(ab) = \epsilon(a)A_U\pi(b) + \epsilon(b)A_U\pi(a)$$

which, together with (5.8)–(5.9), implies

$$\begin{aligned}[\varphi_U, A_U](\vartheta) &= \varphi_U\pi(a^{(2)})A_U\pi\bigl(\kappa(a^{(1)})a^{(3)}\bigr)\\ &= \varphi_U\pi(a^{(1)})A_U\pi(a^{(2)}) - (-1)^{\partial\varphi}A_U\pi(a^{(1)})\varphi_U\pi(a^{(2)})\\ &= -\bigl(\langle\varphi_U, A_U\rangle - (-1)^{\partial\varphi}\langle A_U, \varphi_U\rangle\bigr)(\vartheta).\end{aligned}$$

Consequently, in the general case the operator $q_\omega\restriction\tau(P)$ depends only on the quantum part $\omega_\perp$ of ω, and can be written in an explicitly tensorial form

$$\begin{aligned}q_\omega(\varphi) &= \langle\omega_\perp, \varphi\rangle - (-1)^{\partial\varphi}\langle\varphi, \omega_\perp\rangle - (-1)^{\partial\varphi}[\varphi, \omega_\perp],\\ q_\omega^U(\varphi) &= \langle A_U^\perp, \varphi_U\rangle - (-1)^{\partial\varphi}\langle\varphi_U, A_U^\perp\rangle - (-1)^{\partial\varphi}[\varphi_U, A_U^\perp].\end{aligned} \tag{5.16}$$

The rest of the section is devoted to the introduction and the analysis of the curvature form.

Definition 5.2. A tensorial 2-form

$$R_\omega = D_\omega\omega \tag{5.17}$$

is called *the curvature* of ω.

This definition directly follows classical differential geometry. However, in contrast to the classical case, the curvature is generally δ-dependent.

Proposition 5.5. *We have*

$$\pi_U^\wedge R_\omega(\vartheta) = (F_U \otimes \mathrm{id})\varpi(\vartheta), \tag{5.18}$$

where

$$F_U = dA_U - \langle A_U, A_U\rangle. \tag{5.19}$$

Proof. A direct calculation gives

$$\begin{aligned}
-\big(\pi_U^\wedge\omega^\wedge\big)(d\vartheta) &= 1_U \otimes \pi(a^{(1)})\pi(a^{(2)}) + A_U\pi(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)}\pi(a^{(4)})\\
&\quad - A_U\pi(a^{(3)}) \otimes \pi(a^{(1)})\kappa(a^{(2)})a^{(4)}\\
&\quad + A_U\pi(a^{(2)})A_U\pi(a^{(3)}) \otimes \kappa(a^{(1)})a^{(4)}\\
&= 1_U \otimes \pi(a^{(1)})\pi(a^{(2)}) + \sum_k\Big\{A_U(\vartheta_k) \otimes c_k^{(1)}\pi(c_k^{(2)}) - \langle A_U, A_U\rangle(\vartheta_k) \otimes c_k\Big\}.
\end{aligned}$$

On the other hand

$$\big(\pi_U^\wedge d\omega\big)(\vartheta) = -1_U \otimes \pi(a^{(1)})\pi(a^{(2)}) + \sum_k\Big\{dA_U(\vartheta_k) \otimes c_k - A_U(\vartheta_k) \otimes c_k^{(1)}\pi(c_k^{(2)})\Big\}.$$

Here $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$ and $a \in \ker(\epsilon)$ is chosen as explained in Lemma C.5.

Combining the above expressions we find

$$\pi_U\big(d\omega(\vartheta) - \omega^\wedge(d\vartheta)\big) = \sum_k\Big\{dA_U(\vartheta_k) \otimes c_k - \langle A_U, A_U\rangle(\vartheta_k) \otimes c_k\Big\}. \tag{5.20}$$

To complete the proof it is sufficient to observe that two summands in the right-hand side of the above equation are horizontal while the left second summand is completely "vertical." □

Now, the analogs of classical Structure Equation and Bianchi identity will be derived.

Proposition 5.6. *The following identities hold*

$$R_\omega = d\omega - \langle\omega, \omega\rangle, \tag{5.21}$$
$$D_\omega R_\omega - q_\omega(R_\omega) = \langle\omega_\perp, \langle\omega_\perp, \omega_\perp\rangle\rangle - \langle\langle\omega_\perp, \omega_\perp,\rangle, \omega_\perp\rangle. \tag{5.22}$$

Proof. The previous proposition and Eqs. (5.20) imply

$$(\pi_U^\wedge d\omega)(\vartheta) = \big(\pi_U^\wedge\omega^\wedge\big)(d\vartheta) + \big(\pi_U^\wedge R_\omega\big)(\vartheta) = \{\pi_U^\wedge(R_\omega + \langle\omega, \omega\rangle)\}(\vartheta),$$

for each $\vartheta \in \Gamma_{inv}$ and $U \in \mathscr{U}$. Hence (5.21) holds.

Equation (5.15) and Proposition 5.3 imply

$$\big[\pi_U^\wedge\big(D_\omega R_\omega - q_\omega(R_\omega)\big)\big](\vartheta) = \sum_k ddA_U(\vartheta_k)\otimes c_k - \sum_k \langle dA_U, A_U\rangle(\vartheta_k)\otimes c_k$$

$$+\sum_k \langle A_U, dA_U\rangle(\vartheta_k)\otimes c_k + \sum_k \Big\{\langle F_U, A_U\rangle(\vartheta_k)\otimes c_k - \langle A_U, F_U\rangle(\vartheta_k)\otimes c_k\Big\}$$

$$= \sum_k \big(\langle A_U, \langle A_U, A_U\rangle\rangle - \langle\langle A_U, A_U\rangle, A_U\rangle\big)(\vartheta_k)\otimes c_k.$$

On the other hand, using Lemma C.6 we conclude that

$$\langle A_U, \langle A_U, A_U\rangle\rangle - \langle\langle A_U, A_U\rangle, A_U\rangle = \langle A_U^\perp, \langle A_U^\perp, A_U^\perp\rangle\rangle - \langle\langle A_U^\perp, A_U^\perp\rangle, A_U^\perp\rangle.$$

This is the local expression for the right-hand side of (5.22). □

If ω is classical then (5.21)–(5.22) are equivalent to the classical Structure Equation and Bianchi identity for ω, if ω is understood as a (standard) connection on P_{cl}.

More generally, if ω is multiplicative then the right-hand side of (5.22) vanishes. Indeed in this case we have

$$\langle \omega_\perp, \omega_\perp\rangle\pi(a) = -\omega_\perp\pi(a^{(1)})\omega_\perp\pi(a^{(2)})$$

for each $a\in\mathcal{A}$.

It is important to mention that the proofs of identities contained in Propositions 5.4–5.6, the choice of an embedding ι figures only via its restriction on $d(\Gamma_{inv})$, which determines the embedded differential map δ.

Generally, a map δ can be constructed by fixing a $*\kappa$-invariant ad-invariant complement $\mathcal{L}\subseteq\ker(\epsilon)$ of $\widehat{\mathcal{R}}$, and defining

$$-\delta = (\pi\otimes\pi)\phi(\pi{\restriction}\mathcal{L})^{-1}. \tag{5.23}$$

If, in addition, $\phi(\mathcal{L})\subseteq 1\otimes\mathcal{L}+\mathcal{L}\otimes 1+\mathcal{L}\otimes\mathcal{L}$ then the above δ satisfies

$$(\delta\otimes\mathrm{id})\delta = (\mathrm{id}\otimes\delta)\delta$$

and right-hand side of (5.22) vanishes identically.

Our restriction to the minimal admissible left-covariant calculus Γ is not essential. All considerations can be performed using an arbitrary admissible bicovariant *-calculus. Moreover, if the bundle is trivial we can abandon the assumption of admissibility, and work in a fixed global trivialization.

For example if we take $\mathcal{R}=\{0\}$ then Γ becomes the "maximal" calculus. In this case $\Gamma_{inv}=\ker(\epsilon)$ and $\Gamma^\wedge=\Gamma^\otimes$ is the universal differential envelope of $\mathcal{A}$ (modulo the relation $d1=0$). Because of $S^\wedge=\{0\}$, every connection is multiplicative and δ is uniquely determined.

6. Examples

In this section we consider some illustrative examples related to the presented theory. We shall discuss "nonclassical" phenomena appearing in the formalism of connections, as well as interesting properties of appropriate differential caluli over the structure group G.

Two types of G will be considered. The case of a classical Lie group G, and the quantum case $G=S_\mu U(2)$.

As a possible application in theoretical physics, we shall briefly describe a "gauge theory" based on quantum principal bundles.

Classical Structure Groups

Let us assume that G is a classical compact Lie group ($\mathcal{A}$ is commutative and $G_{cl} = G$). The corresponding principal bundles are objects of classical differential geometry.

The minimal admissible calculus over G coincides with the classical one, based on standard 1-forms. The corresponding universal differential envelope gives the classical higher-order calculus on G, based on standard differential forms.

The classical calculus on G, together with the classical calculus on the base manifold M, induces the classical differential calculus on corresponding principal bundles. The whole theory presented in this paper is equivalent to the classical theory.

However, if we start from a *nonstandard* differential calculus on G then, generally, "quantum phenomena" will enter the game.

Let Γ be an arbitrary admissible bicovariant *-calculus over G, and let $\mathcal{R} \subseteq \ker(\epsilon)$ be the corresponding $\mathcal{A}$-ideal. We have

$$\mathcal{R} \subseteq \ker(\epsilon)^2$$

because of the admissibility of Γ.

For example, if $\mathcal{R} = \ker(\epsilon)^k$ with $k \geq 2$, then Γ_{inv} is naturally isomorphic to the space of $(k-1)$-jets in the neutral element $\epsilon \in G$.

Let P be a principal G-bundle over M and $\omega \in \mathfrak{con}(P)$. After choosing a splitting (4.9) the "classical-quantum" decomposition of ω can be performed. Components of the field $\omega_\perp$ are "labeled" by elements of the space $\ker(\nu)$. The field $\omega_\perp$ figures in "quantum terms" introduced in the previous two sections. Generally these terms do not vanish. Moreover they already figure in the case of a *finite* group G.

The Minimal Admissible Calculus For Quantum SU(2)

This subsection is devoted to the analysis of the minimal admissible left-covariant calculus Γ over the group $G = S_\mu U(2)$. We shall also briefly discuss certain features of corresponding principal bundles.

As first, let us assume that $\mu \in (-1,1) \setminus \{0\}$. As explained in Appendix A, $G_{cl} = U(1)$ in a natural manner. The (complex) Lie algebra of G_{cl} is spanned by a single element $X : \mathcal{A} \to \mathbb{C}$ determined by

$$X(\alpha) = -X(\alpha^*) = \frac{1}{2} \qquad X(\gamma) = X(\gamma^*) = 0. \tag{6.1}$$

The correspondence $X \leftrightarrow 1$ enables us to identify $\mathfrak{lie}(G_{cl}) = \mathbb{C}$. In particular, the space Γ_{inv} can be viewed (via the map ρ) as a certain subspace of $\mathcal{A}$.

Proposition 6.1. *The map $\rho \colon \Gamma_{inv} \to \mathcal{A}$ is a bijection onto the subalgebra $\mathcal{Q} \subseteq \mathcal{A}$ consisting of left $U(1)$-invariant elements. A natural basis in $\mathcal{Q}$ is given by elements $\xi_{n,k}$, where $n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$ and*

$$\xi_{n,k} = \begin{cases} (-\mu)^n (\gamma\gamma^*)^k \gamma^n \alpha^n & \text{if } n \geq 0 \\ (\alpha^*)^{-n} (\gamma^*)^{-n} (\gamma^*\gamma)^k & \text{if } n \leq 0. \end{cases} \tag{6.2}$$

Proof. According to [7] the elements $\alpha^n\gamma^k\gamma^{*r}$ form a basis in $\mathcal{A}$ (by definition $\alpha^{-n} = \alpha^{*n}$). It is easy to see that $g \in U(1)$ acts on the left by multiplying these elements by z^{n-k+r}, where $z = g(\alpha)$. Hence, $\mathcal{Q}$ is spanned by basis elements satisfying $n - k + r = 0$. Equivalently, elements (6.2) form a basis in $\mathcal{Q}$.

We have to verify that $\mathcal{Q} = \rho\big(\Gamma_{inv}\big)$. According to Lemma C.7 (i) the image of ρ is contained in $\mathcal{Q}$. It is easy to see that

$$\begin{aligned} \rho\pi(\alpha) &= \frac{1}{2} - \gamma\gamma^* \qquad & \rho\pi(\gamma^*) &= \alpha^*\gamma^* \\ \rho\pi(\alpha^*) &= \mu^2\gamma\gamma^* - \frac{1}{2} \qquad & \rho\pi(\gamma) &= -\alpha\gamma. \end{aligned} \tag{6.3}$$

Furthermore, a straightforward calculation gives

$$\xi_{n,k} \circ \alpha = \mu^{-2k-|n|}\xi_{n,k} + (\mu^{|n|} - \mu^{-2k-|n|})\xi_{n,k+1}, \tag{6.4}$$

$$\xi_{n,k} \circ \alpha^* = \mu^{2k+|n|}\xi_{n,k} + \mu^2(\mu^{|n|} - \mu^{2k+3|n|})\xi_{n,k+1}, \tag{6.5}$$

$$\xi_{n,k} \circ \gamma = (1 - \mu^{2(k+n)})\xi_{n+1,k}, \quad n \geq 0, \tag{6.6}$$

$$\xi_{n,k} \circ \gamma^* = (1 - \mu^{2(k-n)})\xi_{n-1,k}, \quad n \leq 0, \tag{6.7}$$

$$\xi_{n,k} \circ \gamma = (1 - \mu^{-2k})\xi_{n+1,k+1} + \mu^{-2k}(1 - \mu^{2(k-n)})\xi_{n+1,k+2} \quad n < 0, \tag{6.8}$$

$$\xi_{n,k} \circ \gamma^* = (1 - \mu^{-2k})\xi_{n-1,k+1} + \mu^{-2k}(1 - \mu^{2(k+n)})\xi_{n-1,k+2} \quad n > 0. \tag{6.9}$$

The $\circ$ operation is given by $\xi \circ a = \kappa(a^{(1)})\xi a^{(2)}$. We see that $\mathcal{Q}$ is invariant under $\circ$. Above formulas imply that $\mathcal{Q}$ is generated, as a right $\mathcal{A}$-module, by elements (6.3). Having in mind that $\rho(\Gamma_{inv})$ is a right $\mathcal{A}$-submodule of $\mathcal{Q}$ (as follows from (C.7)) we conclude that ρ is surjective. □

The following proposition describes the right $\mathcal{A}$-ideal $\widehat{\mathcal{R}}$ corresponding to the calculus Γ.

Proposition 6.2. *We have*

$$\widehat{\mathcal{R}} = \big(\mu^2\alpha + \alpha^* - (1 + \mu^2)1\big)\ker(\epsilon). \tag{6.10}$$

Proof. Let $\mathcal{R}$ be the right-hand side of (6.10). According to Lemma C.7 (ii) the space $\mathcal{R}$ is contained in $\widehat{\mathcal{R}}$.

On the other hand, the space of ad-invariant elements of $\mathcal{A}$ consists precisely of polynomials of $\mu^2\alpha + \alpha^*$ and we have

$$\mathrm{ad}(ba) = b\,\mathrm{ad}(a)$$

for each $a \in \mathcal{A}$ and an ad-invariant element $b \in \mathcal{A}$. In particular, corresponding multiple irreducible subspaces are closed under the left multiplication by ad-invariant elements. Furthermore, primitive elements for nonsinglet multiple irreducible subspaces of ad are of the form $p(\mu^2\alpha + \alpha^*)\gamma^k$ and $p(\mu^2\alpha + \alpha^*)\gamma^{*k}$, corresponding to spin k highest and lowest weights respectively. Hence, in the decomposition of the factorized adjoint action on $\ker(\epsilon)/\mathcal{R}$ each irreducible multiplet appears no more than once. On the other hand, elements $\rho\pi(\gamma^n)$, $\rho\pi(\gamma^{*n})$ and $\rho\pi(\mu^2\alpha+\alpha^*)$ are all non-zero (as follows from (6.3), (6.6)–(6.7) and (C.7)). Therefore, for each spin value, the representation ad contains at least one irreducible multiplet. Consequently $\mathcal{R} = \widehat{\mathcal{R}}$. □

We pass to the detailed analysis of the adjoint action ϖ. In terms of the identification $\Gamma_{inv} = \mathcal{Q}$ we have

$$\varpi = (\phi \restriction \mathcal{Q}).$$

Let us assume that Γ_{inv} is endowed with a natural ϖ-invariant scalar product, induced by the Haar measure (as explained in Appendix C). We are going to decompose ϖ into irreducible multiplets. Let us consider operators

$$K_{\pm} = (\mathrm{id} \otimes X_{\pm})\varpi \qquad K_3 = (\mathrm{id} \otimes X)\varpi, \tag{6.11}$$

which are counterparts for the "creation" and "anihilation," as well as the "third spin component" operator. Here $X_{\pm} : \mathcal{A} \to \mathbb{C}$ are linear functionals satisfying

$$X_{\pm}(ab) = X_{\pm}(a)\chi(b) + \epsilon(a)X_{\pm}(b), \tag{6.12}$$

where $\chi : \mathcal{A} \to \mathbb{C}$ is a multiplicative functional determined by

$$\chi(\alpha) = \frac{1}{\mu} \qquad \chi(\alpha^*) = \mu \quad \chi(\gamma) = \chi(\gamma^*) = 0.$$

We shall adopt the following normalization

$$X_{\pm}(\alpha) = X_{\pm}(\alpha^*) = X_+(\gamma) = X_-(\gamma^*) = 0 \quad -\mu X_+(\gamma^*) = X_-(\gamma) = 1.$$

It turns out that the following identities hold

$$K_+K_- - \mu^2 K_-K_+ = \frac{1-\mu^{-4K_3}}{1-\mu^{-2}}, \tag{6.13}$$

$$K_3K_+ - K_+K_3 = K_+ \qquad K_3K_- - K_-K_3 = -K_-, \tag{6.14}$$

$$K_3(\vartheta\eta) = K_3(\vartheta)\eta + \vartheta K_3(\eta), \tag{6.15}$$

$$K_{\pm}(\vartheta\eta) = K_{\pm}(\vartheta)\chi_{\varpi}(\eta) + \vartheta K_{\pm}(\eta), \tag{6.16}$$

where $\chi_{\varpi} = (\mathrm{id} \otimes \chi)\varpi$. Furthermore, we have

$$\chi_{\varpi}(\xi_{n,k}) = \mu^{-2n}\xi_{n,k} \qquad K_3(\xi_{n,k}) = n\xi_{n,k}, \tag{6.17}$$

$$\begin{aligned} K_+(\xi_{n,k}) &= \frac{1-\mu^{2k}}{\mu^{n+2}(1-\mu^2)}\xi_{n+1,k-1} \qquad & n \geq 0, \\ K_-(\xi_{n,k}) &= \frac{\mu^{1-n}(1-\mu^{2k})}{1-\mu^2}\xi_{n-1,k-1} \qquad & n \leq 0. \end{aligned} \tag{6.18}$$

Now (6.17)–(6.18) imply that

$$\mathcal{Q} = \sum_{k\geq 0}^{\oplus} \mathcal{Q}_k,$$

where $\mathcal{Q}_k$ are irreducible subspaces for the k-spin representation. In particular

$$\mathcal{Q}_k = \sum_{|m|\leq k}^{\oplus} \mathcal{Q}_{k,m}, \tag{6.19}$$

where $\mathcal{Q}_{k,m} = \ker(mI - K_3) \cap \mathcal{Q}_k$. The spaces $\mathcal{Q}_{k,m}$ are 1-dimensional. Hence it is possible to construct an orthonormal basis in $\mathcal{Q}$ by choosing unit vectors $\zeta_{k,m} \in$

$\mathcal{Q}_{k,m}$. A priori, there exists an ambiguity for this choice, one phase factor for each $\zeta_{k,m}$. However, requiring that non-vanishing matrix elements of $K_\pm$ are positive, the ambiguity is reduced to one phase factor for each multiplet. According to [7], we have

$$K_+\zeta_{k,m} = v_{k,m+1}\zeta_{k,m+1} \qquad K_-\zeta_{k,m} = v_{k,m}\zeta_{k,m-1}, \tag{6.20}$$

where

$$v_{k,m} = \mu^{1-m-k}\big((k+m)_\mu(k-m+1)_\mu\big)^{1/2} \qquad n_\mu = \frac{1-\mu^{2n}}{1-\mu^2}.$$

Let $\mathscr{P}$ be the space of one-variable polynomials. It is easy to see that

$$\zeta_{0,k} = p_k(\gamma^*\gamma), \tag{6.21}$$

where $p_k \in \mathscr{P}$ are kth order polynomials orthonormal with respect to a scalar product given by

$$(p,q) = \int p^* q. \tag{6.22}$$

Here $\int : \mathscr{P} \to \mathbb{C}$ is a linear functional given by

$$\int x^n = (n+1)_\mu^{-1}. \tag{6.23}$$

We shall assume that leading coefficients of polynomials p_k are positive. This completely fixes vectors $\zeta_{k,m}$.

Proposition 6.3. (*i*) *Polynomials* p_k *are given by*

$$p_k(x) = (-1)^k c_k \partial^k\Big[x^k \prod_{j=1}^{k}(1-\mu^{2-2j}x)\Big], \tag{6.24}$$

where $c_k > 0$ *are normalization constants and* $\partial\colon \mathscr{P} \to \mathscr{P}$ *is a linear map specified by*

$$\partial(x^n) = n_\mu x^{n-1}. \tag{6.25}$$

(*ii*) *The following identities hold*

$$\begin{aligned} \zeta_{k,m} &= (-1)^m \mu^{km-m}\left(\frac{(k-m)_\mu!}{(k+m)_\mu!}\right)^{1/2}(\partial^m p_k)(\gamma\gamma^*)\gamma^m\alpha^m, \\ \zeta_{k,-m} &= \mu^{km}\alpha^{*m}\gamma^{*m}\left(\frac{(k-m)_\mu!}{(k+m)_\mu!}\right)^{1/2}(\partial^m p_k)(\gamma\gamma^*), \end{aligned} \tag{6.26}$$

where $m \in \{0,\dots,k\}$ *and* $n_\mu! = \prod_{j=1}^{n} j_\mu$.

Proof. The map ∂ satisfies the following "Leibniz rule"

$$\partial(pq)(x) = (\partial p)(x)q(x) + p(\mu^2 x)(\partial q)(x), \tag{6.27}$$

as it directly follows from (6.25). More generally

$$\partial^n(pq)(x) = \sum_{k=0}^{n} \binom{n}{k}_\mu (\partial^{n-k} p)(\mu^{2k} x)(\partial^k q)(x) \tag{6.28}$$

for each $n \in \mathbb{N}$. In the above formula

$$\binom{n}{k}_\mu = \frac{n_\mu!}{k_\mu!(n-k)_\mu!}.$$

It is easy to see that

$$\int \partial(p) = p(1) - p(0) \tag{6.29}$$

for each $p \in \mathscr{P}$. Inductively using (6.27) and (6.29) we obtain the following "partial integration" rule

$$\int q\partial^n(p) = \sum_{k=1}^{n} (-1)^{k-1} \mu^{-k(k-1)} \Big\{ (\partial^{n-k} p)(\mu^{2k-2} x)(\partial^{k-1} q)(x) \Big|_0^1 \Big\} + (-1)^n \mu^{-n(n-1)} \int (\partial^n q) p(\mu^{2n} x).$$

It is now easy to prove that polynomials p_k given by (6.24) are mutually orthogonal. Furthermore, leading coefficients of these polynomials are positive. Having in mind that p_k are normed we conclude that (6.21) holds.

To prove (*ii*) it is sufficient to act by $K_\pm^m$ on both sides of (6.21), and to apply (6.18) and (6.20). □

It is worth noticing that $\mathscr{Q}$ is *-invariant. The map $*\colon \mathscr{Q} \to \mathscr{Q}$ corresponds to the canonical *-structure on Γ_{inv}. We have

$$\zeta_{k,-m}^* = (-\mu)^m \zeta_{k,m}.$$

In the classical limit the algebra $\mathscr{A}$ consists of polynomial functions on the group $SU(2)$. The subalgebra $\mathscr{Q}$ then consists of polynomial functions invariant under left translations by diagonal matrices from $U(1)$. Equivalently, $\mathscr{Q}$ can be described as the algebra of polynomial functions on the 2-sphere S^2, because the above mentioned action defines the Hopf fibering $S^3 \to S^2$. In this picture $\zeta_{k,m}$ become spherical harmonics, and K_3, $K_\pm$ correspond to standard angular momentum operators.

Of course, for $\mu = 1$ the minimal admissible calculus is just the classical 3-dimensional one. As we shall see later, a similar situation holds for $\mu = -1$.

In the general case the algebra $\mathscr{Q}$ represents polynomial functions on a "quantum 2-sphere" [5]. At the level of spaces, the inclusion $\mathscr{Q} \hookrightarrow \mathscr{A}$ is interpretable as the "quantum Hopf fibering."

Proposition 6.4. *The space $S_{inv}^{\wedge 2}$ consists precisely of elements of the form*

$$q = 1 \otimes \left[\pi(b) + \frac{2\mu^2}{1-\mu^2}(\gamma^*\gamma) \circ b\right] + \left[\pi(b) + \frac{2\mu^2}{1-\mu^2}(\gamma^*\gamma) \circ b\right] \otimes 1$$
$$- \frac{2\mu^2}{1-\mu^2}\left[(1+\mu^2)\gamma\gamma^* \otimes \gamma\gamma^* + \mu\alpha^*\gamma^* \otimes \alpha\gamma + \frac{1}{\mu}\alpha\gamma \otimes \alpha^*\gamma^*\right](\circ \otimes \circ)\phi(b),$$

where $b \in \ker(\epsilon)$.

Proof. The statement follows from Lemma B.10, Proposition 6.2, and properties (B.30) and (6.3). □

Let us now consider a quantum principal G-bundle P over a compact manifold M. According to the results of Sect. 2, the structure of P is completely determined by its classical part P_{cl}, which is a classical $U(1)$-bundle over M. Let us consider a connection ω, and describe its components ω_{cl} and $\omega_\perp$. At first, we have to specify a splitting (4.9). Modulo the identification $\Gamma_{inv} = \mathscr{Q}$ we have $\nu = (\epsilon\restriction\mathscr{Q})$. With the help of ν, let us identify $\mathscr{L}^*$ with the 1-dimensional subspace in Γ_{inv} generated by 1. The elements of the subspace $\mathscr{L}^*$ are characterized by $\xi \circ a = \epsilon(a)\xi$.

Therefore, the classical component ω_{cl} is locally determined by 1-form $A_U(1)$. From the point of view of classical geometry, this 1-form is a gauge potential of ω_{cl}, understood as a connection on P_{cl}. On the other hand, the quantum component $\omega_\perp$ is locally determined by a collection of 1-forms $A_U(\xi_{n,k})$, where $(n,k) \neq (0,0)$. Globally, we have a collection of tensorial 1-forms on P_{cl}.

It is important to mention that such a classical reinterpretation of connections destroys the information about the irreducible multiplet structure of corersponding gauge potentials. Because of mutual *incompatibility* of decompositions (4.9) and (6.19).

Let us now describe a construction of the embedded differential map δ. In the context of this example, δ can be naturally introduced with the help of a splitting $\ker(\epsilon) = \widehat{\mathscr{R}} \oplus \mathscr{L}$, where $\mathscr{L} \subseteq \ker(\epsilon)$ is the minimal ad-invariant lineal which contains $\mu^2\alpha + \alpha^* - (1+\mu^2)1$ and γ^k, for each $k \in \mathbb{N}$. Explicitly, this lineal can be constructed by extracting irreducible multiplets from $\mathrm{ad}(\gamma^k)$. The map δ is given by (5.23).

According to (5.19) the local expression for the curvature is given by

$$F_U\pi(a) = A_U\pi(a) + A_U\pi(a^{(1)})A_U\pi(a^{(2)}),$$

where $a \in \mathscr{L}$.

Let us consider the case $\mu = -1$. As explained in Appendix A, the classical part of G is isomorphic to a semidirect product of groups $U(1)$ and $\mathbb{Z}_2 = \{-1, 1\}$. The corresponding Lie algebra is generated by a single element X, as in the previous example. Let Γ be the minimal admissible left-covariant calculus. Equations (6.3) reduce to

$$\rho\pi(\gamma^*) = \alpha^*\gamma^* \quad \rho\pi(\gamma) = \gamma\alpha, \qquad (6.30)$$
$$\rho\pi(\alpha) = -\rho\pi(\alpha^*) = \frac{1}{2} - \gamma\gamma^*.$$

The $\circ$-structure is given by

$$\pi(\gamma) \circ \{\alpha, \alpha^*\} = -\pi(\gamma) \quad \pi(\gamma^*) \circ \{\alpha, \alpha^*\} = -\pi(\gamma^*), \qquad (6.31)$$
$$\pi\{\gamma, \gamma^*\} \circ \{\gamma, \gamma^*\} = \{0\} \quad \pi(\alpha) \circ a = \epsilon(a)\pi(\alpha).$$

Consequently, elements

$$\eta_+ = \pi(\gamma) \quad \eta_3 = \pi(\alpha - \alpha^*) \quad \eta_- = \pi(\gamma^*)$$

form a basis in Γ_{inv}.

From (6.31) and Lemma B.13 it follows that the flip-over operator σ is just the standard transposition. Furthermore, the space $S^{\wedge}_{inv}$ is consisting precisely of symmetric elements of $\Gamma^{\otimes 2}_{inv}$.

It is worth noticing that the map δ is uniquely determined, because $\Gamma^{\otimes 2}_{inv}$ contains only one irreducible triplet. Explicitly,

$$\begin{aligned}\delta(\eta_+) &= (\eta_3 \otimes \eta_+ - \eta_+ \otimes \eta_3)/2 \\ \delta(\eta_-) &= (\eta_- \otimes \eta_3 - \eta_3 \otimes \eta_-)/2\end{aligned} \qquad \delta(\eta_3) = \eta_+ \otimes \eta_- - \eta_- \otimes \eta_+, \tag{6.32}$$

and hence

$$\delta = -\frac{1}{2}c^{\top} \tag{6.33}$$

in accordance with Lemma C.5 (*ii*). Furthermore, we have

$$\widehat{\mathscr{R}} = \ker(\epsilon)^2 \tag{6.34}$$

as in the classical case.

The formalism of connections, based on this calculus Γ, becomes essentially the same as in the classical $SU(2)$ case. In particular, because of the symmetricity of $S^{\wedge 2}_{inv}$, every connection is multiplicative. Hence, the right-hand side of the Bianchi identity vanishes. Further, the "perturbation" q_ω also vanishes, as follows directly from (6.32)–(6.33) and (5.16). The presence of the decomposition $\omega = \omega_{cl} + \omega_\perp$ is the only nonclassical phenomena appearing at the level of connections.

Trivial Bundles and Non-Admissible Structures

According to the previous example, compatibility conditions between a left-covariant differential calculus Γ over $G = S_\mu U(2)$, and "transition functions" of an appropriate principal bundle can be fulfilled only in the infinite-dimensional case. This automatically rules out various interesting finite-dimensional differential structures.

Such obstructions can be avoided if we restrict the formalism on *trivial principal bundles*. In this case $\mathscr{B} = S(M) \otimes \mathscr{A}$, and a differential calculus on P can be constructed by taking the product $\Omega(M) \widehat{\otimes} \Gamma^{\wedge} = \Omega(P)$.

Of course, such a calculus over P does not satisfy the property *diff3*. On the other hand, if Γ is an arbitrary bicovariant *-calculus then essentially all considerations of Sects. 4 and 5 can be repeated in this "trivial" framework. The only exception is that there exist no analogs for classical connections. Because it is no longer possible to construct the restriction map $\nu\colon \Gamma_{inv} \to \mathscr{L}^*$.

Each connection ω possesses a global gauge potential $A^\omega\colon \Gamma_{inv} \to \Omega(M)$, given by

$$\omega(\vartheta) = (A^\omega \otimes \mathrm{id})\varpi(\vartheta) + 1_M \otimes \vartheta. \tag{6.35}$$

The curvature is of the form

$$R_\omega = (F^\omega \otimes \mathrm{id})\varpi \qquad F^\omega = dA^\omega - \langle A^\omega, A^\omega\rangle. \tag{6.36}$$

As a concrete illustration, let us consider the case $G = S_\mu U(2)$ where $\mu \in (-1,1)\setminus\{0\}$, and let Γ be a 4-dimensional calculus described in [9]. By definition, the corresponding right $\mathcal{A}$-ideal $\mathcal{R}$ is generated by multiplets

$$1 = \Big\{\, a\big(\mu^2\alpha + \alpha^* - (1+\mu^2)1\big) \,\Big\}, \qquad 3 = \Big\{\, a\gamma,\ a(\alpha - \alpha^*),\ a\gamma^* \,\Big\},$$
$$5 = \Big\{\, \gamma^2,\ \gamma(\alpha-\alpha^*),\ \mu^2\alpha^{*2} - (1+\mu^2)(\alpha\alpha^* - \gamma\gamma^*) + \alpha^2,\ \gamma^*(\alpha-\alpha^*),\ \gamma^{*2} \,\Big\},$$

where $a = \mu^2\alpha + \alpha^* - (\mu^3 + 1/\mu)1$. It turns out that the elements

$$\tau = \pi(\mu^2\alpha + \alpha^*) \quad \eta_+ = \pi(\gamma) \quad \eta_3 = \pi(\alpha - \alpha^*) \quad \eta_- = \pi(\gamma^*) \tag{6.37}$$

form a basis in Γ_{inv}. The canonical right $\mathcal{A}$-module structure on Γ_{inv} is given by

$$\begin{gathered}
\tau\circ\gamma = \frac{(1-\mu)(1-\mu^3)}{\mu}\eta_+ \qquad \tau\circ\alpha^* = \frac{1+\mu^4}{\mu(1+\mu^2)}\tau - \frac{\mu(1-\mu)(1-\mu^3)}{1+\mu^2}\eta_3\\
\tau\circ\gamma^* = \frac{(1-\mu)(1-\mu^3)}{\mu}\eta_- \qquad \tau\circ\alpha = \frac{1+\mu^4}{\mu(1+\mu^2)}\tau + \frac{(1-\mu)(1-\mu^3)}{\mu(1+\mu^2)}\eta_3\\
\eta_+\circ\gamma^* = \eta_-\circ\gamma = -\frac{(1+\mu)(1-\mu^2)}{\mu(1+\mu^2)(1-\mu^3)}\tau - \frac{1-\mu^2}{\mu(1+\mu^2)}\eta_3\\
\eta_3\circ\gamma = -\frac{1-\mu^2}{\mu}\eta_+ \quad \eta_+\circ\gamma = \eta_-\circ\gamma^* = 0 \quad \eta_3\circ\gamma^* = -\frac{1-\mu^2}{\mu}\eta_-\\
-\eta_3\circ\alpha^* = \frac{(1+\mu)(1-\mu^2)}{\mu(1+\mu^2)(1-\mu^3)}\tau - \frac{2\mu}{1+\mu^2}\eta_3 \qquad \eta_+\circ\alpha = \eta_+ = \eta_+\circ\alpha^*\\
\eta_3\circ\alpha = \frac{\mu(1+\mu)(1-\mu^2)}{(1+\mu^2)(1-\mu^3)}\tau + \frac{2\mu}{1+\mu^2}\eta_3 \qquad \eta_-\circ\alpha = \eta_- = \eta_-\circ\alpha^*.
\end{gathered}\tag{6.38}$$

The ideal $\mathcal{R}$ is ad, $*\kappa$-invariant. This means [9] that Γ is a bicovariant *-calculus. By the use of (B.33) and (B.37) it is easy to determine the *-involution and the adjoint action ϖ. We have

$$\begin{gathered}
\eta_+^* = \mu\eta_- \qquad \eta_3^* = -\eta_3 \qquad \mu\eta_-^* = \eta_+\\
\tau^* = -\tau \qquad \varpi(\tau) = \tau\otimes 1\\
\varpi(\eta_+) = \eta_+\otimes\alpha^2 - \eta_3\otimes\alpha\gamma + \mu^2\eta_-\otimes\gamma^2\\
\varpi(\eta_3) = (1+\mu^2)\eta_+\otimes\gamma^*\alpha + \eta_3\otimes(\alpha\alpha^* - \gamma\gamma^*) - (1+\mu^2)\eta_-\otimes\gamma\alpha^*\\
\varpi(\eta_-) = \eta_+\otimes\gamma^{*2} + \eta_3\otimes\alpha^*\gamma^* + \eta_-\otimes\alpha^{*2}.
\end{gathered}\tag{6.39}$$

We see that τ form a singlet, while $\{\eta_+, \eta_3, \eta_-\}$ form a triplet, relative to ϖ.

We are now going to compute the space $S_{inv}^{\wedge 2} \subseteq \Gamma_{inv}^{\otimes 2}$. Acting by $(\pi\otimes\pi)\phi$ on the generating elements of $\mathcal{R}$, using (6.38) and (B.30), and taking linear combinations we obtain a lineal spanned by

$$5 = \left\{ \begin{matrix} \eta_+ \otimes \eta_+ & \mu\eta_3 \otimes \eta_3 + \mu^4\eta_+ \otimes \eta_- + \eta_- \otimes \eta_+ & \eta_- \otimes \eta_- \\ \mu^2\eta_+ \otimes \eta_3 + \eta_3 \otimes \eta_+ & \eta_- \otimes \eta_3 + \mu^2\eta_3 \otimes \eta_- & \end{matrix} \right\},$$

$$3 = \left\{ \frac{1+\mu^4}{1-\mu^3}(\tau \otimes \eta_j + \eta_j \otimes \tau) + (1-\mu)\varkappa_j \middle| j \in \{+,-,3\} \right\}, \qquad (6.40)$$

$$1 = \left\{ \frac{(1+\mu)(1+\mu^3)}{\mu(1-\mu)(1-\mu^3)}\tau \otimes \tau + \mu\eta_3 \otimes \eta_3 - (1+\mu^2)(\eta_+ \otimes \eta_- + \mu^2\eta_- \otimes \eta_+) \right\},$$

where we have used the following abbreviations

$$\varkappa_+ = \eta_+ \otimes \eta_3 - \mu^2\eta_3 \otimes \eta_+ \qquad \varkappa_- = \eta_3 \otimes \eta_- - \mu^2\eta_- \otimes \eta_3,$$
$$\varkappa_3 = (1-\mu^2)\eta_3 \otimes \eta_3 + \mu(1+\mu^2)(\eta_+ \otimes \eta_- - \eta_- \otimes \eta_+).$$

Lemma 6.5. *It turns out that $S^{\wedge 2}_{inv}$ coincides with the lineal generated by the above elements.*

Proof. According to equation (B.42) elements of $S^{\wedge 2}_{inv}$ are σ-invariant, where σ is the canonical flip-over operator. On the other hand, the space $\ker(I-\sigma)$ is 10-dimensional, spanned by the above elements and $\tau \otimes \tau$. Consequently, in order to determine $S^{\wedge 2}_{inv}$, it is sufficient to analyze elements of the form $(\pi \otimes \pi)\phi(a)$, where $a \in \mathscr{R}$ is ad-invariant. This follows from the fact that $(\pi \otimes \pi)\phi$ intertwines ad and $\varpi^{\otimes 2}$. However, ad-invariant elements of $\mathscr{R}$ are just linear combinations of terms of the form

$$r_n = \left(\mu^2\alpha + \alpha^* - (\mu^3 + 1/\mu)1\right)\left(\mu^2\alpha + \alpha^* - (1+\mu^2)1\right)(\mu^2\alpha + \alpha^*)^n.$$

Inductively using (B.30) and (6.38) we find

$$(\pi \otimes \pi)\phi(r_n) = \mu^{-2n}(1+\mu^6)^n(\pi \otimes \pi)\phi(r_0).$$

On the other hand, the last (singlet) term in (6.40) coincides with the element $\left(\mu(1+\mu^2)/(1-\mu)(1-\mu^5)\right)(\pi \otimes \pi)\phi(r_0)$. Hence, elements (6.40) generate $S^{\wedge 2}_{inv}$. □

Let us compute the differential $d\colon \Gamma^\wedge \to \Gamma^\wedge$. As first, let us observe that

$$\pi(a) = \frac{\mu}{(1-\mu)(1-\mu^3)}(\tau \circ a - \epsilon(a)\tau), \qquad (6.41)$$

for each $a \in \mathscr{A}$. Indeed, it is evident that (6.41) holds for $a = 1$, and from (6.38) we conclude that it holds for $a \in \{\alpha, \alpha^*, \gamma, \gamma^*\}$. Remembering that $\{\alpha, \alpha^*, \gamma, \gamma^*\}$ generate $\mathscr{A}$ and using (B.30) and linearity of both sides of (6.41) we conclude that the above equality holds for all $a \in \mathscr{A}$.

As a consequence of identities (6.41) and (B.31)–(B.32) we find

$$d\vartheta = \frac{\mu}{(1-\mu)(1-\mu^3)}(\tau\vartheta - (-1)^{\partial\vartheta}\vartheta\tau) \qquad (6.42)$$

for each $\vartheta \in \Gamma^\wedge$.

Now we shall compute the braid operator $\sigma\colon \Gamma^{\otimes 2}_{inv} \to \Gamma^{\otimes 2}_{inv}$. Using Lemma B.13, and properties (6.38)–(6.39) we obtain the following expressions

$$\sigma(\eta_+\otimes\eta_+)=\eta_+\otimes\eta_+ \qquad \sigma(\eta_-\otimes\eta_-)=\eta_-\otimes\eta_- \qquad \sigma(\vartheta\otimes\tau)=\tau\otimes\vartheta$$

$$\sigma(\tau\otimes\eta_-)=\frac{1+\mu^6}{\mu^2(1+\mu^2)}\eta_-\otimes\tau+\frac{(1-\mu)(1-\mu^3)}{\mu^2}\varkappa_-$$

$$\sigma(\tau\otimes\eta_3)=\frac{1+\mu^6}{\mu^2(1+\mu^2)}\eta_3\otimes\tau+\frac{(1-\mu)(1-\mu^3)}{\mu^2}\varkappa_3$$

$$\sigma(\tau\otimes\eta_+)=\frac{1+\mu^6}{\mu^2(1+\mu^2)}\eta_+\otimes\tau+\frac{(1-\mu)(1-\mu^3)}{\mu^2}\varkappa_+$$

as well as

$$\begin{aligned}\sigma(\eta_3\otimes\eta_3)&=(3-\mu^2-\frac{1}{\mu^2})\eta_3\otimes\eta_3\\&\quad+\frac{1-\mu^4}{\mu}(\eta_-\otimes\eta_+-\eta_+\otimes\eta_-)-\frac{(1+\mu)(1-\mu^2)^2}{\mu^2(1-\mu^3)}\eta_3\otimes\tau\end{aligned}$$

$$\sigma(\eta_+\otimes\eta_3)=\eta_3\otimes\eta_+-\frac{(1+\mu)(1-\mu^2)}{\mu^2(1-\mu^3)}\eta_+\otimes\tau+(1-\frac{1}{\mu^2})\eta_+\otimes\eta_3$$

$$\sigma(\eta_-\otimes\eta_3)=\eta_3\otimes\eta_-+\frac{(1+\mu)(1-\mu^2)}{1-\mu^3}\eta_-\otimes\tau+(1-\mu^2)\eta_-\otimes\eta_3$$

$$\sigma(\eta_3\otimes\eta_+)=\eta_+\otimes\eta_3+\frac{(1+\mu)(1-\mu^2)}{1-\mu^3}\eta_+\otimes\tau+(1-\mu^2)\eta_3\otimes\eta_+$$

$$\sigma(\eta_3\otimes\eta_-)=\eta_-\otimes\eta_3-\frac{(1+\mu)(1-\mu^2)}{\mu^2(1-\mu^3)}\eta_-\otimes\tau+(1-\frac{1}{\mu^2})\eta_3\otimes\eta_-$$

$$\sigma(\eta_+\otimes\eta_-)=\eta_-\otimes\eta_+-\frac{1-\mu^2}{\mu(1+\mu^2)}\eta_3\otimes\eta_3-\frac{(1+\mu)(1-\mu^2)}{\mu(1+\mu^2)(1-\mu^3)}\eta_3\otimes\tau$$

$$\sigma(\eta_-\otimes\eta_+)=\eta_+\otimes\eta_-+\frac{1-\mu^2}{\mu(1+\mu^2)}\eta_3\otimes\eta_3+\frac{(1+\mu)(1-\mu^2)}{\mu(1+\mu^2)(1-\mu^3)}\eta_3\otimes\tau.$$

Furthermore, $\mathrm{sp}(\sigma)=\{1,-\mu^2,-1/\mu^2\}$. The operator σ is diagonalized in the basis consisting of vectors (6.40), $\tau\otimes\tau$, and the following two $\varpi^{\otimes 2}$-triplets

$$\tau\otimes\eta_+-\mu^2\eta_+\otimes\tau+\frac{1-\mu^3}{1+\mu}\varkappa_+ \qquad \mu^2\tau\otimes\eta_+-\eta_+\otimes\tau-\frac{1-\mu^3}{1+\mu}\varkappa_+$$

$$\tau\otimes\eta_3-\mu^2\eta_3\otimes\tau+\frac{1-\mu^3}{1+\mu}\varkappa_3 \qquad \mu^2\tau\otimes\eta_3-\eta_3\otimes\tau-\frac{1-\mu^3}{1+\mu}\varkappa_3$$

$$\tau\otimes\eta_--\mu^2\eta_-\otimes\tau+\frac{1-\mu^3}{1+\mu}\varkappa_- \qquad \mu^2\tau\otimes\eta_--\eta_-\otimes\tau-\frac{1-\mu^3}{1+\mu}\varkappa_-$$

corresponding to values $-\mu^2$ and $-1/\mu^2$ respectively.

It is interesting to observe that there exists an indefinite ϖ-invariant scalar product on Γ_{inv}, such that σ is unitary, relative to the induced product in $\Gamma_{inv}^{\otimes 2}$. Such a product is given by

$$\begin{aligned}(\tau,\tau)&=-\frac{(1-\mu^3)^2(1+\mu^2)}{(1+\mu)^2}\\(\eta_+,\eta_+)=\mu^2 \qquad (\eta_3,\eta_3)&=1+\mu^2 \qquad (\eta_-,\eta_-)=1/\mu^2,\end{aligned}\tag{6.43}$$

while $\eta_\pm, \eta_3, \tau$ are assumed to be mutually orthogonal. The unitarity of σ easily follows from the ϖ-invariance of the introduced scalar product, and the identity

$$(\vartheta \circ a, \eta) = (\vartheta, \eta \circ \kappa^2(a)^*). \tag{6.44}$$

The product is uniquely determined by the above conditions, up to a scalar multiple. There exists a natural splitting $\ker(\epsilon) = \mathscr{R} \oplus \mathscr{L}$, where $\mathscr{L}$ is the lineal spanned by elements $\{\gamma, \gamma^*, \alpha - 1, \alpha^* - 1\}$. This splitting enables us to introduce the embedded differential map δ. A direct calculation gives

$$-(1+\mu^2)\delta(\tau\ ) = \tau \otimes \tau + \mu^2\eta_3 \otimes \eta_3 - \mu(1+\mu^2)(\eta_+ \otimes \eta_- + \mu^2\eta_- \otimes \eta_+)$$
$$-(1+\mu^2)\delta(\eta_\zeta) = \tau \otimes \eta_\zeta + \eta_\zeta \otimes \tau + \varkappa_\zeta, \quad \zeta \in \{+, 3, -\}.$$

The map δ is coassociative, by construction.

According to (6.35)–(6.36) the curvature has the form

$$F^\omega(\tau) = dA^\omega(\tau) + \mu(1-\mu^2)A^\omega(\eta_-)A^\omega(\eta_+)$$
$$F^\omega(\eta_3) = dA^\omega(\eta_3) + 2\mu A^\omega(\eta_+)A^\omega(\eta_-)$$
$$F^\omega(\eta_-) = dA^\omega(\eta_-) + A^\omega(\eta_3)A^\omega(\eta_-) \qquad F^\omega(\eta_+) = dA^\omega(\eta_+) + A^\omega(\eta_+)A^\omega(\eta_3).$$

It is worth noticing that essentially the same expressions for *singlet* and *triplet* components of δ and F^ω can be obtained in the framework of the previous example.

Gauge Theories

Classical principal bundles provide a natural mathematical framework for the study of gauge theories [4]. It is therefore interesting to see what will be the counterparts of these theories, in the context of quantum principal bundles [6].

In analogy with the classical case, the simplest possibility is to consider lagrangians of the form

$$L(\omega) = \sum_\vartheta \bigl(F^\omega(\vartheta), F^\omega(\vartheta)\bigr)_M, \tag{6.45}$$

where elements ϑ form an orthonormal system in Γ_{inv} with respect to an ad-invariant scalar product, and $(\,)_M$ is the scalar product in $\Omega(M)$, induced by a metric on M (here M plays the role of space-time).

Properties of such "quantum gauge" theories essentially depend, besides on the "symmetry group" G, on the following two prespecifications.

First, it is necessary to fix a bicovariant *-calculus Γ. This determines kinematical degrees of freedom, as well as "infinitezimal gauge transformations."

Secondly, we have to choose a map δ. This influences dynamical properties of the theory, because δ implicitly figures in the self-interacting part of (6.45). In the classical case the curvature is δ-independent.

For instance, in the context of the previous example, we find a four-component gauge field consisting of mutually interacting singlet and triplet fields. However if we change δ and define

$$\delta(\vartheta) = \frac{\mu}{(1-\mu)(1-\mu^3)}(\tau \otimes \vartheta + \vartheta \otimes \tau),$$

then (6.45) will describe non-interacting fields. On the other hand, in the context of the second example, we find a self-interacting infinite-component gauge field with all integer spin multiplets in the game.

Closely related with this line of thinking is a question of "gauge transformations." The most direct way of introducing gauge trannsformations as vertical automorphisms of P gives nothing new. Every such automorphism of P preserves the classical part P_{cl}, and moreover it is completely determined by the corresponding "restriction," which is a classical gauge transformation of P_{cl}. In such a way we obtain an isomorphism between gauge groups for P and P_{cl}. However, a proper quantum generalization of gauge transformations can be introduced via the concepts of quantum (infinitesimal) gauge bundles. These are the bundles associated to P, relative to the adjoint actions $\{\mathrm{ad}, \varpi\}$ respectively. It turns out that operators h_ω, D_ω and R_ω are covariant with respect to natural actions of these bundles on P. Moreover, the lagrangian (6.45) is gauge-invariant, in the appropriate sense.

A. Classical Points

Let G be a compact matrix quantum group. We have denoted by G_{cl} the set of $*$-characters of $\mathcal{A}$. The elements of G_{cl} are interpretable as classical points of G.

The quantum group structure on G induces a classical group structure on G_{cl}, in a natural manner. The product and the inverse are given by

$$gf = (g \otimes f)\phi, \tag{A.1}$$

$$g^{-1} = g\kappa. \tag{A.2}$$

The counit $\epsilon\colon \mathcal{A} \to \mathbb{C}$ is the neutral element of G_{cl}.

Lemma A.1. (*i*) *The formula*

$$\iota_u(g)_{ij} = g(u_{ij})$$

defines a monomorphism $\iota_u\colon G_{cl} \to GL(n)$.

(*ii*) *The image* $\iota_u(G_{cl})$ *is compact.*

Proof. Without a lack of generality we can assume [8] that u is a unitary matrix. In this case matrices $\iota_u(g)$ belong to $U(n)$. We have

$$\begin{aligned}\iota_u(gf)_{ij} &= (gf)(u_{ij}) = (g \otimes f)\phi(u_{ij})\\ &= \sum_{k=1}^{n} g(u_{ik})f(u_{kj}) = \sum_{k=1}^{n} \iota_u(g)_{ik}\iota_u(f)_{kj}.\end{aligned}$$

Hence ι_u is a group homomorphism. This map is injective, because $\mathcal{A}$ is generated, as a $*$-algebra, by the matrix elements u_{ij}.

Because of the compactness of $U(n)$, it is sufficient to prove that the image of ι_u is closed. Let us suppose that a sequence of matrices $\iota_u(g_k)$ converges to $T \in U(n)$. This means that the sequence of numbers $g_k(u_{ij})$ converges to T_{ij} for each $i, j \in \{1, \dots, n\}$. It follows that a sequence $g_k(a)$ is convergent, for each $a \in \mathcal{A}$. Now the formula

$$g(a) = \lim_k g_k(a) \tag{A.3}$$

consistently defines a $*$-character $g\colon \mathcal{A} \to \mathbb{C}$ with the property $\iota_u(g) = T$. □

The monomorphism ι_u enables us to interpret G_{cl} as a compact group of matrices. In particular, G_{cl} is a Lie group in a natural manner. Furthermore the space G_{cl} is an algebraic submanifold of $U(n)$. The Hopf *-algebra $\mathcal{A}_{cl}$ of polynomial functions on G_{cl} is generated by elements $u^{cl}_{ij}(g) = g(u_{ij})$. Let $\upharpoonright_{cl}\colon \mathcal{A} \to \mathcal{A}_{cl}$ be the restriction homomorphism. Let $\mathfrak{lie}(G_{cl})$ be the (complex) Lie algebra of G_{cl}, understood as the tangent space to G_{cl}, in the point ϵ.

The formula

$$X(a) = d\big(\upharpoonright_{cl}(a)\big)_{\epsilon}(X) \tag{A.4}$$

enables us to interpret elements $X \in \mathfrak{lie}(G_{cl})$ as certain linear functionals on $\mathcal{A}$.

Lemma A.2. *(i) We have*

$$X(ab) = \epsilon(a)X(b) + \epsilon(b)X(a) \tag{A.5}$$

for each $a, b \in \mathcal{A}$. Conversely, if $X\colon \mathcal{A} \to \mathbb{C}$ is a hermitian linear functional such that (A.5) holds then X is interpretable via (A.4) as a real element of $\mathfrak{lie}(G_{cl})$.

(ii) In terms of the above identification, the Lie brackets are given by

$$[X, Y](a) = X(a^{(1)})Y(a^{(2)}) - Y(a^{(1)})X(a^{(2)}). \tag{A.6}$$

Proof. It is clear that functionals X given by (A.4) satisfy (A.5). If X is a hermitian functional satisfying (A.5) then the formula

$$g^t(a) = \epsilon\Big[\sum_{k=0}^{\infty} \frac{1}{k!}\big((\mathrm{id}\otimes X)\phi\big)^k(a)t^k\Big] \tag{A.7}$$

determines a 1-parameter subgroup of G_{cl}. The corresponding generator coincides with X, in the sense of (A.4). Finally, (A.6) directly follows from (A.4), and the definition of Lie brackets. □

In terms of the identification (A.4) the conjugation in $\mathfrak{lie}(G_{cl})$ is given by

$$X^*(a) = X(a^*)^*.$$

Let $F \in M_n(\mathbb{C})$ be the canonical intertwiner [8] between u and its second contragradient u^{cc}. Then

Lemma A.3. *We have*

$$\iota_u(g)F = F\iota_u(g),$$

for each $g \in G_{cl}$.

Proof. According to definitions of F and u^{cc}, we have

$$FuF^{-1} = u^{cc} = (\mathrm{id}\otimes\kappa^2)u.$$

Acting by $g \in G_{cl}$ on this equality, and remembering that $g\kappa^2 = g$, we conclude that F and $\iota_u(g)$ commute. □

In a generic case when all eigenvalues of F are mutually different, the group G_{cl} will be very small, because every element $U \in \iota_u(G_{cl})$ is a function of F. In particular G_{cl} will be Abelian.

Furthermore, a rough information about the minimal size of G_{cl} is cointained in F. According to the results of [8] we have $F^{it} \in \iota_u(G_{cl})$, for each $t \in \Re$. Hence, the closure of this 1-parameter subgroup is contained in $\iota_u(G_{cl})$. This closure is isomorphic to a torus the dimension of which is equal to the number of rationally linearly independent elements of the spectrum of $\log(F)$.

In the rest of this appendix classical parts of some concrete quantum groups will be computed.

The Classical Case

Let us assume that $\mathcal{A}$ is commutative. Then so is A and according to [8], G is an ordinary compact matrix group consisting of characters of A. Since every compact matrix group is an algebraic manifold in the corresponding matrix space, the restriction map $g \mapsto g\restriction\mathcal{A}$ is an isomorphism between G and G_{cl}.

Quantum $SU(2)$ groups

By definition [7], the C^*-algebra representing continuous functions on the group $G = S_\mu U(2)$ is generated by elements α and γ, and relations

$$\begin{gathered} \alpha\alpha^* + \mu^2\gamma\gamma^* = 1 \qquad \alpha^*\alpha + \gamma^*\gamma = 1 \\ \alpha\gamma = \mu\gamma\alpha \qquad \alpha\gamma^* = \mu\gamma^*\alpha \qquad \gamma\gamma^* = \gamma^*\gamma \end{gathered} \tag{A.8}$$

while

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

Let us consider the case $\mu \in (-1, 1) \setminus \{0\}$. Relations (A.8) imply that every $g \in G_{cl}$ satisfies

$$|g(\alpha)| = 1 \quad g(\gamma) = g(\gamma^*) = 0.$$

Consequently g is completely determined by the number $g(\alpha) \in U(1)$. Moreover, the correspondence $G_{cl} \ni g \mapsto g(\alpha) \in U(1)$ is a group isomorphism.

If $\mu = -1$ relations (A.8) give the following constraints

$$\begin{aligned} |g(\alpha)| = 1 \quad g(\gamma) = g(\gamma^*) = 0, \text{ or} \\ |g(\gamma)| = 1 \quad g(\alpha) = g(\alpha^*) = 0. \end{aligned}$$

In this case

$$G_{cl} = U(1) \wedge \mathbb{Z}_2$$

in a natural manner.

Quantum $SU(n)$ groups

Let us assume that $\mu \in (-1,1)\backslash\{0\}$. By definition [10] the C^*-algebra A representing continuous functions on $G = S_\mu U(n)$ groups is generated by elements u_{ij}, where $i,j \in \{1,\dots,n\}$, and relations

$$\sum_{j=1}^{n} u_{ij}u_{kj}^* = \delta_{ik}I \qquad \sum_{j=1}^{n} u_{ji}^*u_{jk} = \delta_{ik}I, \tag{A.9}$$
$$\sum\nolimits_* u_{i_1j_1}\dots u_{i_nj_n}E_{j_1\dots j_n} = E_{i_1\dots i_n}I.$$

The last summation is performed over indexes j, and

$$E_{i_1\dots i_n} = (-\mu)^{I(i)},$$

where $I(i)$ is the number of inversions in the sequence $i = (i_1,\dots,i_n)$, if the sequence is a permutation. Other components of E vanish, by definition.

The fundamental representation of G is irreducible. Let us compute the canonical intertwiner F. The conjugate representation u^c can be naturally realized as a subrepresentation of the $(n-1)$th tensor power of u. The carrier space H is spanned by vectors

$$x_k = \sum\nolimits_* E_{kj_1\dots j_{n-1}}e_{j_1}\otimes\dots\otimes e_{j_{n-1}}.$$

Here e_i are absolute basis vectors in $\mathbb{C}^n$, and the summation is performed over indexes j. We have

$$F = cj^\dagger j,$$

where $c > 0$ and $j\colon \mathbb{C} \to H$ is the canonical antilinear map defined by $j(e_k) = x_k$. Now, a direct computation gives

$$Fe_k = \mu^{2k-n-1}e_k \tag{A.10}$$

for each $k \in \{1,\dots,n\}$.

According to Lemma A.3, matrices $\iota_u(g)$ are diagonal. Relations (A.9) imply that corresponding diagonal elements $\iota_{ii}(g)$ are complex units, and that

$$\prod_i \iota_{ii}(g) = 1.$$

The same relations imply that conversely for any sequence of numbers $z_1,\dots z_n \in U(1)$ satisfying $\prod_i z_i = 1$ there exists the unique $g \in G_{cl}$ such that $\iota_{ii}(g) = z_i$. In summary, G_{cl} is isomorphic to the $(n-1)$-dimensional torus.

Abelian Quantum Groups

If G is Abelian then every subgroup of G is Abelian, too. In particular G_{cl} is an Abelian compact matrix group, and as such it is isomorphic to a product of a torus with a finite Abelian group.

According to [8] there exist a discrete finitely generated group Γ, Hilbert space H and a unitary representation $U\colon \Gamma \to U(H)$ (the square of which is contained in its multiple) such that $\mathscr{A}$ is isomorphic to the *-algebra generated by the image of U. Furthermore

$$\phi\bigl(U(\gamma)\bigr) = U(\gamma) \otimes U(\gamma) \quad \epsilon\bigl(U(\gamma)\bigr) = 1 \quad \kappa\bigl(U(\gamma)\bigr) = U(\gamma)^{-1}$$

for each $\gamma \in \Gamma$. Since operators $U(\gamma)$ are mutually linearly independent [8], every character $g \in G_{cl}$ can be viewed as a character on Γ, via

$$g(\gamma) = g\bigl(U(\gamma)\bigr),$$

and vice versa. In other words G_{cl} is isomorphic to the group of characters of Γ.

Universal Unitary Quantum Matrix Groups

Let us consider a positive matrix $F \in M_n(\mathbb{C})$ such that

$$\operatorname{tr}(F) = \operatorname{tr}(F^{-1}).$$

Let A_F be a C^*-algebra generated by elements u_{ij}, where $i,j \in \{1,\dots,n\}$, and relations

$$\begin{aligned} &\sum_{j=1}^n u_{ij}u_{kj}^* = \delta_{ik}I && \sum_{j=1}^n u_{ji}^*u_{jk} = \delta_{ik}I \\ &\sum_{j=1}^n u_{ij}^*u_{kj}^F = \delta_{ik}I && \sum_{j=1}^n u_{ji}^F u_{jk}^* = \delta_{ik}I, \end{aligned} \tag{A.11}$$

where $u^F = FuF^{-1}$.

The pair $G_F = (A_F, u)$ is a compact matrix quantum group. We are going to describe the category of unitary representations of G_F. Let $\mathscr{T}$ be a concrete monoidal W^*-category [10] generated by elements u and u^c, with carrier Hilbert spaces $H_u = \mathbb{C}^n$ and $H_{u^c} = H_u^*$. It will be assumed that H_u is endowed with the standard scalar product, while the product in H_u^* is specified by $(x,y) = (x, Fy)$. The objects of $\mathscr{T}$ are just the words of u and u^c (including the unit object). By definition, morphisms between objects of $\mathscr{T}$ are generated by "elementary morphisms" $t\colon \mathbb{C} \to H_u \otimes H_u^*$ and $\bar{t}\colon H_u^* \otimes H_u \to \mathbb{C}$, which are given by

$$t(1) = \sum_{i=1}^n e_i \otimes j(e_i) \qquad \bar{t}(x \otimes y) = (j^{-1}x, y),$$

where $j\colon H_u \to H_u^*$ is the complex conjugation map. By construction u and u^c are mutually conjugate objects.

Then $G_F = (A_F, u)$ is the universal $\mathscr{T}$-admissible pair (u is a distinguished object). In other words G_F is a compact matrix quantum group corresponding to $\mathscr{T}$, in the framework of the Tannaka-Krein duality [10]. The antipode acts as follows

$$\kappa(u_{ij}) = u_{ji}^* \qquad \kappa(u_{ij}^*) = u_{ji}^F.$$

The map $F = j^\dagger j$ is just the canonical intertwiner between u and u^{cc}. According to Lemma A.3 and relations (A.11), the elements of $\iota_u\bigl(G_F^{cl}\bigr)$ are precisely unitary matrices commuting with F. Hence,

$$G_F^{cl} = U(n_1) \times \cdots \times U(n_k)$$

where n_i are multiplicities of eigenvalues of F.

B. Universal Differential Envelopes

Let $\mathcal{A}$ be a complex unital associative algebra and Γ a first-order calculus [9] over $\mathcal{A}$. Let $\Gamma^{\otimes}$ be the corresponding "tensor bundle" algebra, and let $S^{\wedge}$ be the ideal in $\Gamma^{\otimes}$ generated by elements of the form

$$Q = \sum_i da_i \otimes_{\mathcal{A}} db_i, \quad \text{where} \quad \sum_i a_i db_i = 0. \tag{B.1}$$

By definition, $S^{\wedge}$ is a graded ideal in $\Gamma^{\otimes}$ and its first (generally) nontrivial component coincides with the set of elements Q of the form (B.1).

Let $\Gamma^{\wedge} = \Gamma^{\otimes}/S^{\wedge}$ be the corresponding factor-algebra.

Proposition B.1. *There exists the unique linear map $d\colon \Gamma^{\wedge} \to \Gamma^{\wedge}$ extending the derivation $d\colon \mathcal{A} \to \Gamma$ such that*

$$d^2 = 0$$
$$d(\vartheta\eta) = d(\vartheta)\eta + (-1)^{\partial\vartheta}\vartheta d(\eta)$$

for each $\vartheta, \eta \in \Gamma^{\wedge}$.

Proof. The formula

$$d\Bigl(\sum_i a_i db_i\Bigr) = \sum_i da_i db_i \tag{B.2}$$

consistently defines a linear map $d\colon \Gamma \to \Gamma^{\wedge}$. We have

$$dd(a) = 0, \tag{B.3}$$
$$d(a\vartheta) = (da)\vartheta + ad(\vartheta), \tag{B.4}$$
$$d(\vartheta a) = d(\vartheta)a - \vartheta(da) \tag{B.5}$$

for each $a \in \mathcal{A}$ and $\vartheta \in \Gamma$. Equalities (B.4)–(B.5) imply that maps d admit the unique extension $d\colon \Gamma^{\otimes} \to \Gamma^{\wedge}$ satisfying

$$d(w \otimes_{\mathcal{A}} u) = d(w)\Pi(u) + (-1)^{\partial w}\Pi(w)d(u), \tag{B.6}$$

where $\Pi\colon \Gamma^{\otimes} \to \Gamma^{\wedge}$ is the projection map. Equations (B.3) and (B.6) imply that $S^{\wedge} \subseteq \ker(d)$. Consequently, there exists the unique map $d\colon \Gamma^{\wedge} \to \Gamma^{\wedge}$ defined as a factorization of the previous d through Π. This map possesses all desired properties. □

The differential algebra $\Gamma^{\wedge}$ possesses the following *universality property*.

Proposition B.2. *Let Ω be a differential algebra with a differential $d_\Omega\colon \Omega \to \Omega$.*

(i) Let $\varphi\colon \mathcal{A} \to \Omega$ be a homomorphism admitting the extension $\sharp_\varphi\colon \Gamma \to \Omega$, given by

$$\sharp_\varphi\bigl(ad(b)\bigr) = \varphi(a)d_\Omega\varphi(b).$$

Then there exists the unique differential algebra homomorphism $\varphi^{\wedge}\colon \Gamma^{\wedge} \to \Omega$ extending both φ and $\sharp_\varphi$.

(ii) Similarly, if $\varphi\colon \mathcal{A} \to \Omega$ is an antimultiplicative linear map and if there exists $\sharp_\varphi\colon \Gamma \to \Omega$ satisfying

$$\sharp_{\varphi}\big(ad(b)\big) = d_{\Omega}\varphi(b)\varphi(a),$$

then φ *and* $\sharp_{\varphi}$ *admit the unique extension* $\varphi^{\wedge}\colon \Gamma^{\wedge} \to \Omega$ *satisfying*

$$\begin{aligned} \varphi^{\wedge} d &= d_{\Omega}\varphi^{\wedge}, \\ \varphi^{\wedge}(\vartheta\eta) &= (-1)^{\partial\vartheta\partial\eta}\varphi^{\wedge}(\eta)\varphi^{\wedge}(\vartheta) \end{aligned}$$

for each $\vartheta, \eta \in \Gamma^{\wedge}$.

Proof. We shall check the statement (i). The maps φ and $\sharp_{\varphi}$ admit the unique common multiplicative extension $\varphi^{\otimes}\colon \Gamma^{\otimes} \to \Omega$. It is easy to see that $\varphi^{\otimes}(Q) = 0$, for each Q given by (B.1). In other words, $S^{\wedge} \subseteq \ker\big(\varphi^{\otimes}\big)$ and hence $\varphi^{\wedge}$ can be factorized through Π. In such a way we obtain the desired map $\varphi^{\wedge}$. The uniqueness follows from the fact that $\Gamma^{\wedge}$ is generated by $\mathcal{A}$, as a differential algebra. □

A similar statement can be formulated for antilinear maps φ. As a simple corollary we obtain

Proposition B.3. *Let us assume that* $\mathcal{A}$ *is a *-algebra and that* Γ *is a *-calculus. There exists the unique antilinear involution* $*\colon \Gamma^{\wedge} \to \Gamma^{\wedge}$ *extending *-involutions on* $\mathcal{A}$ *and* Γ *and satisfying*

$$\begin{aligned} d(\vartheta^*) &= d(\vartheta)^*, \\ (\vartheta\eta)^* &= (-1)^{\partial\eta\partial\vartheta}\eta^*\vartheta^* \end{aligned} \tag{B.7}$$

for each $\vartheta, \eta \in \Gamma^{\wedge}$.□

Let us consider some examples of universal envelopes, interesting from the point of view of quantum principal bundles.

Proposition B.4. (i) *Let* M *be a compact manifold. Then*

$$\Omega(M) = [\Omega^1(M)]^{\wedge}.$$

(ii) *If* P *is a quantum principal bundle over* M *and* Γ *an arbitrary admissible calculus over* G *then*

$$\Omega(P, \Gamma) = [\Omega^1(P, \Gamma)]^{\wedge}.$$

In other words $\Omega(M)$ *and* $\Omega(P, \Gamma)$ *are understandable as universal envelopes.*

Proof. We shall prove the statement (i). The proof of (ii) is based on (i) and the universality of $\Gamma^{\wedge}$.

The space $\Omega^1(M)\otimes_M \Omega^1(M)$ is naturally isomorphic to a $S(M)$-module of covariant 2-tensors. To prove (i) it is sufficient to check that $S^{\wedge 2}$ coincides with the space Σ of symmetric 2-tensors. According to universality of $\Omega^1(M)^{\wedge}$ we have $S^{\wedge 2} \subseteq \Sigma$. Conversely, elements of the form $q = df \otimes_M df$, where $f \in S(M)$, generate the module Σ. Every such element belongs to $S^{\wedge 2}$, because of the identity $fdf - d(f^2)/2 = 0$. Hence, $\Sigma \subseteq S^{\wedge 2}$. □

The algebra $\Gamma^\wedge$ can be alternatively constructed by applying a method of extended bimodules [1, 7, 9].

Let $\Gamma^\wedge\{X\}$ be the graded differential algebra generated by $\Gamma^\wedge$, a first-order element X, and the following relations

$$\begin{gathered} X^2 = 0 \qquad d(X) = 0 \\ X\vartheta - (-1)^{\partial\vartheta}\vartheta X = d(\vartheta). \end{gathered} \tag{B.8}$$

On the other hand, let $\widetilde{\Gamma}$ be the extended bimodule

$$\widetilde{\Gamma} = \mathcal{A}\widetilde{X} \oplus \Gamma$$

with a right $\mathcal{A}$-module structure specified by

$$\widetilde{X}a = a\widetilde{X} + d(a). \tag{B.9}$$

Proposition B.5. *There exists the unique homomorphism $\Pi^\star\colon \widetilde{\Gamma}^\otimes \to \Gamma^\wedge\{X\}$ satisfying $\Pi^\star(\widetilde{X}) = X$ and extending the factorization map Π. The kernel of $\Pi^\star$ coincides with the ideal in $\widetilde{\Gamma}^\otimes$ generated by $\widetilde{X} \otimes_{\mathcal{A}} \widetilde{X}$.* □

In other words, $\Gamma^\wedge$ can be viewed as a differential subalgebra of $\widetilde{\Gamma}^\otimes/\ker(\Pi^\star)$ generated by $\mathcal{A}$.

Let us turn to the quantum group context, and assume that $\mathcal{A}$ represents polynomial functions on a compact matrix quantum group G. The following statement is a direct corollary of Proposition B.2.

Proposition B.6. *(i) Let Γ be a left-covariant calculus over G, with the corresponding left action $\ell_\Gamma\colon \Gamma \to \mathcal{A} \otimes \Gamma$. Then there exists the unique map $\ell_\Gamma^\wedge\colon \Gamma^\wedge \to \mathcal{A} \otimes \Gamma^\wedge$ which is multiplicative, extends ϕ and such that*

$$\ell_\Gamma^\wedge d = (\mathrm{id} \otimes d)\ell_\Gamma^\wedge. \tag{B.10}$$

This map also extends ℓ_Γ and satisfies

$$(\epsilon \otimes \mathrm{id})\ell_\Gamma^\wedge = \mathrm{id}, \tag{B.11}$$

$$(\phi \otimes \mathrm{id})\ell_\Gamma^\wedge = (\mathrm{id} \otimes \ell_\Gamma^\wedge)\ell_\Gamma^\wedge. \tag{B.12}$$

*If Γ is also a *-calculus then $\ell_\Gamma^\wedge$ is hermitian, in a natural manner.*

(ii) Similarly, if Γ is right-covariant then there exists the unique homomorphism $\wp_\Gamma^\wedge\colon \Gamma^\wedge \to \Gamma^\wedge \otimes \mathcal{A}$ extending ϕ and satisfying

$$\wp_\Gamma^\wedge d = (d \otimes \mathrm{id})\wp_\Gamma^\wedge. \tag{B.13}$$

This homomorphism also extends the right action map $\wp_\Gamma\colon \Gamma \to \Gamma \otimes \mathcal{A}$ and satisfies

$$(\mathrm{id} \otimes \epsilon)\wp_\Gamma^\wedge = \mathrm{id}, \tag{B.14}$$

$$(\wp_\Gamma^\wedge \otimes \mathrm{id})\wp_\Gamma^\wedge = (\mathrm{id} \otimes \phi)\wp_\Gamma^\wedge. \tag{B.15}$$

*If, in addition, the calculus Γ is *-covariant then $\wp_\Gamma^\wedge$ preserves corresponding *-structures.*

(iii) If Γ is bicovariant then so is $\Gamma^\wedge$, that is

$$(\mathrm{id} \otimes \wp_\Gamma^\wedge)\ell_\Gamma^\wedge = (\ell_\Gamma^\wedge \otimes \mathrm{id})\wp_\Gamma^\wedge. \square \tag{B.16}$$

There exists a natural grade-preserving coaction map $c\colon \Gamma^\wedge \otimes \mathcal{A} \to \Gamma^\wedge$, given by

$$c(\vartheta \otimes a) = \kappa(a^{(1)})\vartheta a^{(2)}. \tag{B.17}$$

The same formula determines the coaction of G on $\Gamma^\otimes$. We have

$$c(\vartheta \otimes 1) = \vartheta \qquad c\big(c(\vartheta \otimes a) \otimes b\big) = c\big(\vartheta \otimes (ab)\big). \tag{B.18}$$

If Γ is $*$-covariant then

$$c(\vartheta \otimes a)^* = c\big(\vartheta^* \otimes \kappa(a)^*\big) \tag{B.19}$$

for each $\vartheta \in \Gamma^\wedge$ and $a \in \mathcal{A}$.

Lemma B.7. *Let us assume that Γ is right-covariant. Then the following identity holds*

$$\wp_\Gamma^\wedge c(\vartheta \otimes a) = \sum_k c(\vartheta_k \otimes a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)}, \tag{B.20}$$

where $\sum_k \vartheta_k \otimes c_k = \wp_\Gamma^\wedge(\vartheta)$.

Proof. We compute

$$\begin{aligned}\wp_\Gamma^\wedge c(\vartheta \otimes a) = \wp_\Gamma^\wedge\left(\kappa(a^{(1)})\vartheta a^{(2)}\right) &= \sum_k \kappa(a^{(2)})\vartheta_k a^{(3)} \otimes \kappa(a^{(1)})c_k a^{(4)}\\ &= \sum_k c(\vartheta_k \otimes a^{(2)}) \otimes \kappa(a^{(1)})c_k a^{(3)}.\square\end{aligned}$$

Definition B.1. A first-order calculus Γ over G is called *κ-covariant* iff there exists a linear map $\sharp_\kappa\colon \Gamma \to \Gamma$ such that

$$d\kappa(a) = \sharp_\kappa d(a), \tag{B.21}$$

$$\sharp_\kappa(a\vartheta) = \sharp_\kappa(\vartheta)\kappa(a) \tag{B.22}$$

for each $a \in \mathcal{A}$ and $\vartheta \in \Gamma$.

The map $\sharp_\kappa$ is uniquely determined by the above conditions. Furthermore it is bijective and

$$\sharp_\kappa(\vartheta a) = \kappa(a)\sharp_\kappa(\vartheta). \tag{B.23}$$

According to Proposition B.2 the map $\sharp_\kappa$ can be extended to a d-preserving graded-antiautomorphism $\kappa^\wedge\colon \Gamma^\wedge \to \Gamma^\wedge$. If Γ is $*$-covariant then

$$\kappa^\wedge(\kappa^\wedge(\vartheta^*)^*) = \vartheta \tag{B.24}$$

for each $\vartheta \in \Gamma^\wedge$.

Proposition B.8. *If the calculus Γ is left-covariant then κ-covariance is equivalent to bicovariance.* $\square$

From this moment we assume that Γ is left-covariant. Let us denote by $\Gamma_{inv}^{\star}$ the space of left-invariant elements of $\Gamma^{\star}$, for $\star \in \{\otimes, \wedge\}$. The space $\Gamma_{inv}^{\otimes}$ is naturally identificable with the tensor algebra over Γ_{inv}. Proposition B.6 (i) implies that $\Gamma_{inv}^{\wedge}$ is a graded-differential subalgebra of $\Gamma^{\wedge}$. This algebra is generated by Γ_{inv}.

Let $\ell_{\Gamma}^{\otimes}\colon \Gamma^{\otimes} \to \mathcal{A} \otimes \Gamma^{\otimes}$ be the left action of G on $\Gamma^{\otimes}$. The ideal $S^{\wedge}$ is $\ell_{\Gamma}^{\otimes}$-invariant and $\ell_{\Gamma}^{\wedge}$ coincides with the factorized $\ell_{\Gamma}^{\otimes}$ through Π. The ideal $S^{\wedge}$ is decomposable as

$$S^{\wedge} \leftrightarrow \mathcal{A} \otimes S_{inv}^{\wedge}.$$

It is easy to see that $\Pi(\Gamma_{inv}^{\otimes}) = \Gamma_{inv}^{\wedge}$. In other words

$$\Gamma_{inv}^{\otimes}/S_{inv}^{\wedge} = \Gamma_{inv}^{\wedge}.$$

The spaces $\Gamma_{inv}^{\star}$ are c-invariant, and hence the formula

$$\vartheta \circ a = c(\vartheta \otimes a) \tag{B.25}$$

determines a right $\mathcal{A}$-module structure on them. The following identities hold

$$1 \circ a = \epsilon(a)1, \tag{B.26}$$

$$(\vartheta\eta) \circ a = (\vartheta \circ a^{(1)})(\eta \circ a^{(2)}). \tag{B.27}$$

If Γ is $*$-covariant then the spaces $\Gamma_{inv}^{\star}$ are $*$-invariant and we can write

$$(\vartheta \circ a)^* = \vartheta^* \circ \kappa(a)^*. \tag{B.28}$$

Let $\pi\colon \mathcal{A} \to \Gamma_{inv}$ be a linear map given by

$$\pi(a) = \kappa(a^{(1)})d(a^{(2)}). \tag{B.29}$$

The map π is surjective, and $\pi(1) = 0$.

Lemma B.9. *The following identities hold*

$$\pi(a) \circ b = \pi\bigl(ab - \epsilon(a)b\bigr), \tag{B.30}$$

$$d(a) = a^{(1)}\pi(a^{(2)}), \tag{B.31}$$

$$d\pi(a) = -\pi(a^{(1)})\pi(a^{(2)}). \tag{B.32}$$

Proof. All these equalities follow by straightforward transformations, applying the definition of π. □

We can write

$$\Gamma_{inv} = \ker(\epsilon)/\mathcal{R},$$

where $\mathcal{R} = \ker(\epsilon) \cap \ker(\pi)$ is the right $\mathcal{A}$-ideal which, in the sense of [9], canonically determines the structure of Γ. According to [9], the calculus is $*$-covariant iff $\kappa(\mathcal{R})^* = \mathcal{R}$. In this case

$$\pi(a)^* = -\pi\bigl(\kappa(a)^*\bigr) \tag{B.33}$$

for each $a \in \mathcal{A}$.

Lemma B.10. *The space $S^{\wedge 2}_{inv} \subseteq \Gamma_{inv} \otimes \Gamma_{inv}$ consists precisely of elements of the form*

$$q = \pi(a^{(1)}) \otimes \pi(a^{(2)}), \tag{B.34}$$

where $a \in \mathscr{R}$.

Proof. The space $S^{\wedge 2}_{inv}$ consists of left-invariant projections of elements Q given by (B.1). In terms of the identification $\Gamma^{\otimes} \leftrightarrow \mathcal{A} \otimes \Gamma^{\otimes}_{inv}$ we have

$$Q = \sum_i a_i^{(1)} b_i^{(1)} \otimes \left\{ \left(\pi(a_i^{(2)}) \circ b_i^{(2)}\right) \otimes \pi(b_i^{(3)}) \right\}$$

and hence

$$(\epsilon \otimes \mathrm{id})(Q) = \sum_i \pi(a_i b_i^{(1)}) \otimes \pi(b_i^{(2)}) - \sum_i \epsilon(a_i) \pi(b_i^{(1)}) \otimes \pi(b_i^{(2)}).$$

The first summand on the right-hand side of the above equality vanishes, because of $\sum_i a_i db_i = \sum_i a_i b_i^{(1)} \otimes \pi(b_i^{(2)}) = 0$. On the other hand, the elements of the form $r = \sum_i \epsilon(a_i) b_i$ cover the whole space $\ker(\pi) = \mathbb{C}1 \oplus \mathscr{R}$. □

Actually the space $S^{\wedge 2}_{inv}$ generates the whole ideal $S^{\wedge}_{inv}$ in $\Gamma^{\otimes}_{inv}$. In other words, $\Gamma^{\wedge}_{inv}$ is a quadratic algebra.

Proposition B.11. *The following conditions are equivalent*

(*i*) *The calculus Γ is bicovariant.*

(*ii*) *The coproduct map ϕ is (necessarily uniquely) extendible to the homomorphism $\widehat{\phi}\colon \Gamma^{\wedge} \to \Gamma^{\wedge} \widehat{\otimes} \Gamma^{\wedge}$ of differential algebras.*

Proof. Let us suppose that (*i*) holds. Let $\widehat{\phi}\colon \Gamma \to \Gamma^{\wedge} \widehat{\otimes} \Gamma^{\wedge}$ be a map given by

$$\widehat{\phi}(\vartheta) = \ell_{\Gamma}(\vartheta) \oplus \wp_{\Gamma}(\vartheta). \tag{B.35}$$

Proposition B.3 implies that this map, together with ϕ, can be further extended to a differential homomorphism $\widehat{\phi}\colon \Gamma^{\wedge} \to \Gamma^{\wedge} \widehat{\otimes} \Gamma^{\wedge}$. Conversely, if (*ii*) holds then formula (B.35) defines the left and the right actions of G on Γ. In other words the calculus is bicovariant. □

Let us assume that Γ is bicovariant. This is equivalent [9] to $\mathrm{ad}(\mathscr{R}) \subseteq \mathscr{R} \otimes \mathcal{A}$. The spaces $\Gamma^{\star}_{inv}$ are invariant under the right action of G.

Let $\varpi^{\star}\colon \Gamma^{\star}_{inv} \to \Gamma^{\star}_{inv} \otimes \mathcal{A}$ be the corresponding restriction maps. The following identity holds

$$\varpi^{\star}(\vartheta \circ a) = \sum_k \vartheta_k \circ a^{(2)} \otimes \kappa(a^{(1)}) c_k a^{(3)}, \tag{B.36}$$

where $\sum_k \vartheta_k \otimes c_k = \varpi^{\star}(\vartheta)$.

Explicitly, the map $\varpi\colon \Gamma_{inv} \to \Gamma_{inv} \otimes \mathcal{A}$ is given by

$$\varpi\pi = (\pi \otimes \mathrm{id})\mathrm{ad}. \tag{B.37}$$

The map $\widehat{\phi}$ possesses the property

$$(\mathrm{id} \otimes \widehat{\phi})\widehat{\phi} = (\widehat{\phi} \otimes \mathrm{id})\widehat{\phi} \tag{B.38}$$

as follows from the coassociativity of ϕ. Let $\epsilon^\wedge \colon \Gamma^\wedge \to \mathbb{C}$ be a homomorphism acting as ϵ on $\mathcal{A}$, and vanishing on higher-order components. Then

$$(\epsilon^\wedge \otimes \mathrm{id})\widehat{\phi} = (\mathrm{id} \otimes \epsilon^\wedge)\widehat{\phi} = \mathrm{id}. \tag{B.39}$$

If in addition Γ admits a *-structure then $\widehat{\phi}$ is a hermitian map. Let us denote by $m^\wedge$ the multiplication map in $\Gamma^\wedge$.

Proposition B.12. *The following identity holds*

$$m^\wedge(\kappa^\wedge \otimes \mathrm{id})\widehat{\phi} = m^\wedge(\mathrm{id} \otimes \kappa^\wedge)\widehat{\phi} = 1\epsilon^\wedge. \tag{B.40}$$

Proof. It follows from the definition of $\kappa^\wedge$, $\epsilon^\wedge$ and $\widehat{\phi}$. □

Let $\sigma \colon \Gamma \otimes_{\mathcal{A}} \Gamma \to \Gamma \otimes_{\mathcal{A}} \Gamma$ be the canonical braid operator [9]. This map intertwines the corresponding left and right actions. In particular it is reduced in the space $\Gamma_{inv}^{\otimes 2}$. Its left-invariant restriction is explicitly given by

Lemma B.13. *We have*

$$\sigma(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k) \tag{B.41}$$

for each $\vartheta, \eta \in \Gamma_{inv}$, *where* $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$.

Proof. Using the definition [9] of σ and performing direct transformations we obtain

$$\begin{aligned}\sigma(\eta \otimes \vartheta) &= \sum_k \sigma(\eta \otimes_{\mathcal{A}} (\vartheta_k \kappa(c_k^{(1)}))c_k^{(2)}) = \sum_k \vartheta_k \kappa(c_k^{(1)}) \otimes_{\mathcal{A}} \eta c_k^{(2)} \\ &= \sum_k \big(\vartheta_k \kappa(c_k^{(1)}) c_k^{(2)}\big) \otimes_{\mathcal{A}} (\eta \circ c_k^{(3)}) = \sum_k \vartheta_k \otimes (\eta \circ c_k).\square\end{aligned}$$

Let $\Gamma^\vee$ be the braided exterior algebra [9] built over Γ. In view of the universality of $\Gamma^\wedge$ there exists the unique homomorphism $\between \colon \Gamma^\wedge \to \Gamma^\vee$ of graded differential algebras reducing to the identity on Γ and $\mathcal{A}$. In particular

$$S^{\wedge 2} \subseteq \ker(I - \sigma). \tag{B.42}$$

This also follows from (B.30), (B.41) and Lemma B.10. The map $\between$ is surjective, but generally not injective. Moreover, the algebra $\Gamma^\vee$ is generally not quadratic.

C. The Minimal Admissible Calculus

Let $\widehat{\mathcal{R}}$ be the set of elements $a \in \ker(\epsilon)$ satisfying

$$(X \otimes \mathrm{id})\mathrm{ad}(a) = 0 \tag{C.1}$$

for each $X \in \mathfrak{lie}(G_{cl})$.

Lemma C.1. *The space* $\widehat{\mathcal{R}}$ *is a right* $\mathcal{A}$*-ideal and*

$$\mathrm{ad}(\widehat{\mathcal{R}}) \subseteq \widehat{\mathcal{R}} \otimes \mathcal{A}, \tag{C.2}$$

$$\kappa(\widehat{\mathcal{R}})^* = \widehat{\mathcal{R}}. \tag{C.3}$$

Proof. Let us assume that $a \in \widehat{\mathcal{R}}$ and $b \in \ker(\epsilon)$. A direct computation gives

$$\begin{aligned}(X \otimes \mathrm{id})\mathrm{ad}(ab) &= X(a^{(2)}b^{(2)})\kappa(a^{(1)}b^{(1)})a^{(3)}b^{(3)}\\ &= X(a^{(2)})\kappa(b^{(1)})\kappa(a^{(1)})a^{(3)}b^{(2)} + \epsilon(a)X(b^{(2)})\kappa(b^{(1)})b^{(3)} = 0.\end{aligned}$$

Hence $\widehat{\mathcal{R}}$ is a right ideal in $\mathcal{A}$. Properties (C.2)–(C.3) follow from the definition of $\widehat{\mathcal{R}}$, applying elementary properties of maps figuring in the game. □

Let Γ be the left-covariant calculus which canonically, in the sense of [9], corresponds to $\widehat{\mathcal{R}}$. Then property (C.2) implies that Γ is bicovariant, while (C.3) shows that Γ admits a *-structure. According to Proposition 3.14 the calculus Γ is admissible. By construction, it is the *minimal admissible* left-covariant calculus.

Let $\mathcal{L}^*$ be the dual space of $\mathfrak{lie}(G_{cl})$. It turns out that Γ_{inv} can be naturally embedded in $\mathcal{L}^* \otimes \mathcal{A}$. First, let us observe that the formula

$$\big(\nu\pi(a)\big)(X) = \nu_X\pi(a) = X(a) \tag{C.4}$$

consistently defines a surjective linear map $\nu\colon \Gamma_{inv} \to \mathcal{L}^*$. Now, according to the definition of $\widehat{\mathcal{R}}$, a linear map $\rho\colon \Gamma_{inv} \to \mathcal{L}^* \otimes \mathcal{A}$ given by

$$\rho = (\nu \otimes \mathrm{id})\varpi \tag{C.5}$$

is injective.

Lemma C.2. *The following identities hold*

$$(\mathrm{id} \otimes \phi)\rho = (\rho \otimes \mathrm{id})\varpi, \tag{C.6}$$

$$\rho(\vartheta \circ a) = \sum_k \varphi_k \otimes \kappa(a^{(1)})c_k a^{(2)}, \tag{C.7}$$

where $\sum_k \varphi_k \otimes c_k = \rho(\vartheta)$.

Proof. Property (C.6) is a direct consequence of the definition of ρ, and the comodule structure property of ϖ. Equality (C.7) follows from Lemma B.7 and the following equation

$$\nu(\vartheta \circ a) = \epsilon(a)\nu(\vartheta), \tag{C.8}$$

which easily follows from (A.5), (B.30) and (C.4). □

In the following, $\mathcal{L}^*$ will be endowed with the natural *-structure, induced from $\mathfrak{lie}(G_{cl})$. Then maps ν and ρ are hermitian.

Let $(\,)_{cl}$ be a scalar product in $\mathcal{L}^*$, with respect to which the *-operation is antiunitary. Let $h\colon \mathcal{A} \to \mathbb{C}$ be the Haar measure [8] of G. The formula

$$<\varphi \otimes a, \psi \otimes b> = (\varphi, \psi)_{cl} h(a^* b) \tag{C.9}$$

defines a scalar product in $\mathcal{L}^* \otimes \mathcal{A}$. This enables us to introduce a scalar product $<>$ in Γ_{inv}, by requiring that ρ is isometrical.

Lemma C.3. *The introduced scalar product is ϖ-invariant.* □

The above statement follows from the invariance of h. Let $\varkappa\colon \Gamma_{inv} \to \Gamma_{inv}$ be a linear map defined by

$$\varkappa\pi(a) = \pi\bigl(\kappa^2(a)\bigr). \tag{C.10}$$

Consistency of this formula is a consequence of the bicovariance of Γ. The following identities hold

$$\nu\varkappa(\vartheta) = \nu(\vartheta) \qquad \varkappa(\vartheta)^* = \varkappa^{-1}(\vartheta^*) \qquad \varpi\varkappa = (\varkappa \otimes \kappa^2)\varpi$$
$$(\vartheta, \varkappa(\eta)) = (\varkappa(\vartheta), \eta) \qquad (\vartheta^*, \eta^*) = (\varkappa^{-1}(\eta), \vartheta).$$

The scalar product on Γ_{inv} can be naturally extended to a scalar product on $\Gamma_{inv}^{\otimes}$, by tensoring and taking the direct sum. Let us assume that the maps $\varkappa$ and $*$ are extended from Γ_{inv} to $\Gamma_{inv}^{\otimes}$ by requiring multiplicativity and graded-antimultiplicativity respectively. Such extended maps, together with the adjoint action $\varpi^{\otimes}$ satisfy the same relations as initial maps.

Let us assume that the ideal $S_{inv}^{\wedge}$ can be orthocomplemented in $\Gamma_{inv}^{\otimes}$, relative to the constructed scalar product. Then the space $\Gamma_{inv}^{\wedge}$ is naturally realizable as the orthocomplement of $S_{inv}^{\wedge}$. In particular, we can introduce an embedded differential map $\delta\colon \Gamma_{inv} \to \Gamma_{inv} \otimes \Gamma_{inv}$. The space $\Gamma_{inv}^{\wedge} = S_{inv}^{\perp}$ is invariant under ϖ, $*$ and $\varkappa$.

Let $c^{\top}\colon \Gamma_{inv} \to \Gamma_{inv} \otimes \Gamma_{inv}$ be the "transposed Lie commutator" map [9]. This map can be defined by

$$c^{\top} = (\mathrm{id} \otimes \pi)\varpi. \tag{C.11}$$

Maps δ and $c^{\top}$ are both covariant with respect to the adjoint action of G. In other words

Lemma C.4. *The following identities hold*

$$(\delta \otimes \mathrm{id})\varpi = \varpi^{\otimes 2}\delta \qquad (c^{\top} \otimes \mathrm{id})\varpi = \varpi^{\otimes 2}c^{\top}. \tag{C.12}$$

Proof. Applying (C.11) and (B.37) we obtain

$$\begin{aligned}\varpi^{\otimes 2}c^{\top}(\vartheta) = \varpi^{\otimes 2}\Bigl(\sum_k \vartheta_k \otimes \pi(c_k)\Bigr) &= \sum_k \vartheta_k \otimes \pi(c_k^{(3)}) \otimes c_k^{(1)}\kappa(c_k^{(2)})c_k^{(4)}\\ &= \sum_k \vartheta_k \otimes \pi(c_k^{(1)}) \otimes c_k^{(2)} = (c^{\top} \otimes \mathrm{id})\varpi(\vartheta),\end{aligned}$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$. The second equality follows from the covariance of the differential $d\colon \Gamma_{inv}^{\wedge} \to \Gamma_{inv}^{\wedge}$. □

Lemma C.5. (*i*) *For each $\vartheta \in \Gamma_{inv}$ there exists $a \in \ker(\epsilon)$ such that*

$$\begin{aligned}\vartheta &= \pi(a),\\ \delta(\vartheta) &= -\pi(a^{(1)}) \otimes \pi(a^{(2)}).\end{aligned} \tag{C.13}$$

(*ii*) *The following identity holds*

$$c^{\top} = \sigma\delta - \delta. \tag{C.14}$$

Proof. Let us choose $c \in \ker(\epsilon)$ such that $\pi(c) = \vartheta$. According to Lemma B.9 we have $d\vartheta = -\pi(c^{(1)})\pi(c^{(2)})$. According to Lemma B.10 there exists $b \in \widehat{\mathscr{R}}$ such that

$$\delta(\vartheta) = -\pi(c^{(1)}) \otimes \pi(c^{(2)}) - \pi(b^{(1)}) \otimes \pi(b^{(2)}).$$

Now $a = b + c$ satisfies (C.13).

To prove (C.14) let us choose, for a given $\vartheta \in \Gamma_{inv}$, an element $a \in \ker(\epsilon)$ as above. Applying (B.37), (B.30) and (C.11) we obtain

$$\begin{aligned}
-\sigma\delta(\vartheta) &= \sigma\big(\pi(a^{(1)}) \otimes \pi(a^{(2)})\big) \\
&= \pi(a^{(3)}) \otimes \pi(a^{(1)}) \circ \big(\kappa(a^{(2)})a^{(4)}\big) \\
&= \pi(a^{(3)}) \otimes \pi\big[\big(a^{(1)} - \epsilon(a^{(1)})1\big)\kappa(a^{(2)})a^{(4)}\big] \\
&= \pi(a^{(1)}) \otimes \pi(a^{(2)}) - \pi(a^{(2)}) \otimes \pi\big(\kappa(a^{(1)})a^{(3)}\big) = -c^{\top}(\vartheta) - \delta(\vartheta). \square
\end{aligned}$$

Lemma C.6. *We have*

$$(\nu_X \otimes \mathrm{id})\delta(\vartheta) - (\mathrm{id} \otimes \nu_X)\delta(\vartheta) = (\mathrm{id} \otimes X)\varpi(\vartheta) \tag{C.15}$$

for each $\vartheta \in \Gamma_{inv}$ *and* $X \in \mathfrak{lie}(G_{cl})$. □

The following lemma gives a rough information about the "size" of the space Γ_{inv}. For each $g \in G_{cl}$ let $\varpi^g \colon \mathscr{L}^* \to \mathscr{L}^*$ be the induced adjoint action, given by

$$\varpi^g \nu = (\nu \otimes g)\varpi.$$

Lemma C.7. (i) *We have*

$$(\varpi^g \otimes \zeta_g^{\star})\rho = \rho \tag{C.16}$$

for each $g \in G_{cl}$.

(ii) *Let* $a \in \ker(\epsilon)$ *be an arbitrary* ad-*invariant element. Then*

$$a\big(\ker(\epsilon)\big) \subseteq \widehat{\mathscr{R}}. \tag{C.17}$$

Proof. The statement (i) directly follows from the definition of ρ. Let us prove (ii). For arbitrary $b \in \ker(\epsilon)$ and ad-invariant $a \in \ker(\epsilon)$ we have

$$(X \otimes \mathrm{id})\mathrm{ad}(ab) = X(ab^{(2)})\kappa(b^{(1)})b^{(3)} = X(a)\epsilon(b)1 + \epsilon(a)(X \otimes \mathrm{id})\mathrm{ad}(b) = 0.$$

This shows that $ab \in \widehat{\mathscr{R}}$, and hence (C.17) holds. □

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Communicated by A. Connes

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