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Quantum Discrete Sine-Gordon Model at Roots of 1: Integrable Quantum System on the Integrable Classical Background

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Dedicated to L.D. Faddeev on his 60th birthday

Abstract: The quantum discrete sine-Gordon model at roots of 1 is studied. It is shown that this model provides an example of an integrable quantum system in an integrable classical background. In particular, the spectrum of quantum integrals of motions in this model depends only on the values of integrals of motion of a background classical system.

1. Introduction

The sine-Gordon equation is a nonlinear differential equation for a scalar function ϕ of two variables:

$$-\partial_t^2\phi + \partial_x^2\phi = 4\sin\phi\,. \tag{1.1}$$

The Cauchy problem for this equation with initial data $\phi(x,t)|_{t=0} = \phi(x), \partial_t\phi(x,t)|_{t=0} = \pi(x)$ can be regarded as an infinite-dimensional Hamiltonian mechanical system. The functions $(\pi(x), \phi(x))$ are "natural canonical coordinate functions" on the phase space of this system with Poisson brackets:

$$\{\pi(x), \phi(y)\} = \delta(x-y)\,. \tag{1.2}$$

The Hamiltonian which generates evolution (1.1) on the phase space with the Poisson structure (1.2) is

$$\mathcal{H} = \int_{-\infty}^{+\infty} \left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}(\partial_x\phi(x))^2 + 4(1-\cos\phi(x))\right) dx\,, \tag{1.3}$$

where we assume the convergence of the integral.

The Hamiltonian system (1.2), (1.3) is integrable. See [FT] for more complete description of the sine-Gordon system.

The quantization of this model has been done in several steps. For a quasiclassical analysis, which includes the quantization of solitons in Eq. (1.1), see [FK]. A phenomenological scattering theory with a factorized S-matrix has been constructed in [ZZ]. The Bethe-ansatz solution has been found in [FST].

Since the sine-Gordon model has ultraviolet divergences, which are characteristic of quantum field-theoretical models, it is important to have a regularized counterpart of it. Ideally this should be a quantum field-theoretical model in discrete space-time.

The first integrable version of the sine-Gordon model with a discrete space variable was found in [IK]. In recent works [FV, BKP, FV1], the results of [IK] have been extended further and a local equation in discrete space-time, which approximates (1.1) together with its quantum discrete counterpart, has been found. The quantum sine-Gordon model depends on two parameters, $K \in \mathbb{R}$ and q, $|q| = 1$.

Here is a brief description of the quantum sine-Gordon model:

(i) The algebra of observables is a certain completion $\mathscr{A}_N(q)'$ of the algebra $\mathscr{A}_N(q), q \in \mathbb{C}^*$, generated by invertible elements $Q_n, n = 0, \dots, 2N-1$, with the determining relations

$$
\begin{aligned}
Q_{2n}\, Q_{2n-1} &= q^2 Q_{2n-1}\, Q_{2n}\,, \\
Q_{2n}\, Q_{2n+1} &= q^2 Q_{2n+1}\, Q_{2n}\,.
\end{aligned} \tag{1.4}
$$

Here we assume that subindices are taken modulo $2N$.

(ii) Let $r(K^2; U)$, be a solution of the difference equation

$$
r(K^2; Uq^2) = F(Uq^2)^{-1} r(K^2; U)\,, \tag{1.5}
$$

where

$$
F(Q) = \frac{K^2 + q^{-1}Q}{1 + K^2 q^{-1} Q}\,. \tag{1.6}
$$

Assume that the following products are defined in the completion of $\mathscr{A}_N(q)'$:

$$
\zeta(U) = \prod_{n=1}^{N} r(K^2; Q_{2n}^{-\varepsilon_n}), \quad U = \prod_{n=0}^{N-1} r(K^2; Q_{2n+1}^{\varepsilon_n})\,. \tag{1.7}
$$

Here $\varepsilon_n = 1$ if $n = 0 \pmod 2$, $\varepsilon_n = -1$, if $n = 1 \pmod 2$. Some examples of solutions $r(K^2, x)$ will be described in Sect. 4.

The evolution automorphism $\tau\colon \mathscr{A}_N(q)' \to \mathscr{A}_N(q)'$ is defined as

$$
\tau(a) = (\zeta(U)U)^{-1} a \zeta(U) U\,. \tag{1.8}
$$

For the details see Sect. 4.

(iii) It is easy to deduce from the definition of τ and $r_n(K^2; U)$ that elements $Q_{2n,\,2t} = \tau^t(Q_{2n}), Q_{2n+1,\,2t+1} = \tau^t(Q_{2n+1})$ satisfy the following nonlinear discrete-time evolution equations:

$$
Q_{n,\,t+2} Q_{n,\,t} = F(Q_{n+1,\,t+1}) F(Q_{n-1,\,t+1})\,. \tag{1.9}
$$

Notice again that here, the elements $Q_{n,\,t}$ are defined only for $n + t = 0 \pmod 2$.

(iv) The algebra (1.4) has two Casimir operators

$$
C_1 = \prod_{n=0}^{N-1} Q_{2n+1}^{\varepsilon_n}, \quad C_2 = \prod_{n=1}^{N} Q_{2n}^{\varepsilon_n}\,, \tag{1.10}
$$

where ε_n are the same as in (1.7). The evolution automorphism τ has $N-1$ independent nontrivial integrals $P_1, \dots, P_{N-1}$ with the generating function (5.19).

In this work we will investigate the structure of the quantum discrete model when q is a roof of 1. We will show that the model in this case combines both classical and quantum properties. The structure of the model in this case is similar to the structure of quantum groups at roots of 1 [DCK, DCP, R]. This last fact is not surprising since it is known that the sine-Gordon model is related to quantum $\widehat{sl_2}$.

This paper will follow a "quantum methodology." First we will describe a quantum model for generic q. Then a classical model and the models at $q^n = 1$ will appear as special limits of the quantum models for generic q.

In the second section we will clarify what we mean by quantum systems with discrete time, and the corresponding notion of integrability. The third section contains an exposition of the construction of discrete time Lax-integrable systems based upon inhomogeneous spin chains, a method dating back to the works [FR, FV-FV2]. In Sects. 4 and 5, we describe the quantum discrete sine-Gordon model. The main ideas and constructions of this section have their origin in [FV-FV2]. We found it necessary to include this exposition since in the work [FV], the equations of motion are given only in the light-cone form, not in terms of the algebra of observables $\mathscr{A}_N(q)$, but in terms of the bigger algebra $A_q^{\otimes 2N}$. Also, because the derivation of the discrete quantum sine-Gordon equation (1.9) given in [BKP] is rather brief. Section 6 contains an analysis of the classical limit. The behavior of the discrete sine-Gordon model at roots of 1 is studied in Sect. 7. In the conclusion we discuss possible applications of the model at roots of 1.

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2. Quantum Systems with Discrete Time

First let us make it clear what we mean by quantum discrete-time evolution equations. Let A be a complex algebra and I be a set. Consider an element

$$a_I = \{a_i\}_{i \in I} \in A^{\times I} \tag{2.1}$$

and a map

$$P : A^{\times I} \longrightarrow A^{\times I} . \tag{2.2}$$

Consider an automorphism $\phi : A \to A$. We say that the element a_I satisfies the (nonlinear) discrete-time evolution equation P with respect to the evolution ϕ if

$$\phi(a_i) = P_i(a_I) . \tag{2.3}$$

The sequence

$$(a_I)_t = \phi^t(a_I), \quad t = 0, 1, 2, \ldots \tag{2.4}$$

is called a trajectory of ϕ with the initial condition at $t = 0$.

Equation (2.3) implies that

$$(a_i)_{t+1} = P_i((a_I)_t) . \tag{2.5}$$

We will say that the evolution ϕ is Hamiltonian if there exists an invertible element $U \in A$ such that

$$\phi(a) = U^{-1} a U . \tag{2.6}$$

The element U is called the evolution operator. In many examples, U belongs to certain completions of A. In realistic physical models, the algebra A is given together with a $\mathbb{C}$-antilinear anti-involution $*$:

$$(\lambda a)^* = \bar{\lambda} a^*, \qquad (ab)^* = b^* a^* . \tag{2.7}$$

In this case it is usually assumed that $U^* = U^{-1}$, that there exists an involution $\sigma : I \to I$ with $a_i^* = a_{\sigma(i)}$ and that this condition is compatible with (2.3).

By the analogy with the classical case, we will say that the element $F \in A$ is an integral of the map ϕ if

$$\phi(F) = F . \tag{2.8}$$

Intuitively we can say that the quantum evolution is integrable if it has enough integrals. We will not attempt to give a precise notion of quantum integrability here.

3. Quantum Integrable Systems and the Yang–Baxter Equation

In this section we will describe a method for generating integrable discrete-time models from solutions of the Yang–Baxter equations. This construction goes back to works [FR, FV]. It is part of a more general scheme [R1], which provides a quantization of the approach to integrable classical systems based on Lie–Poisson groups. First let us specify the following data:

i) A vector space V and the function $R(z)$ on $\mathbb{C}$ with values in $V \otimes V$.

ii) We assume that $R(z)$ is a solution to the Yang–Baxter equation

$$R_{12}(y) R_{13}(y+z) R_{23}(z) = R_{23}(z) R_{13}(y+z) R_{12}(y) , \tag{3.1}$$

which is invertible for all $z \in \mathbb{C}$, except for a finite number of points.

iii) An algebra A and the element $L(z) \in \mathrm{End}(V) \otimes A$, $z \in \mathbb{C}$, such that

$$R_{12}(y) L_1(y+z) L_2(z) = L_2(z) L_1(y+z) R_{12}(y) . \tag{3.2}$$

iv) An element $\overline{R}(z) \in A \otimes A$, $z \in \mathbb{C}$ (or of an appropriate completion of $A \otimes A$) such that

$$\overline{R}(y)(L(y+z) \dot{\otimes} L(z)) = (L(z) \dot{\otimes} L(y+z)) \overline{R}(y) . \tag{3.3}$$

Here $\dot{\otimes}$ denotes the multiplication $(\mathrm{End}(V) \otimes A)^{\otimes 2} \to \mathrm{End}(V) \otimes A \otimes A$ which is the identity operation on $A \otimes A$ and the usual multiplication in $\mathrm{End}(V)$ (see [FRT, TTF]).

Note that we do not require the Yang–Baxter equation for $\overline{R}(x)$.

Consider the algebra $A_N = A^{\otimes 2N}$ for some integer $N > 0$ and the following elements of $\mathrm{End}(V) \otimes A_N$:

$$L_{2n-1}^{(+)}(z) = (\mathrm{id} \otimes \phi_{2n-1})(L(z+\kappa)) , \tag{3.4}$$

$$L_{2n}^{(-)}(z) = (\mathrm{id} \otimes \phi_{2n})(L(z-\kappa)) . \tag{3.5}$$

Here $\kappa \in \mathbb{C}$ and

$$\phi_n(a) = 1 \otimes \cdots \otimes \underset{\hat{n}}{a} \otimes \cdots \otimes 1 . \tag{3.6}$$

Consider also an element

$$U = \prod_{n=0}^{N-1} \overline{R}_{2n+1,\,2n}(2\kappa) , \tag{3.7}$$

where

$$\overline{R}_{2n+1,\,2n}(2\kappa) = (\phi_{2n+1} \otimes \phi_{2n})(\overline{R}(2\kappa)) \tag{3.8}$$

and assume that $\overline{R}(2\kappa)$ is invertible.

Define the automorphism of the cyclic shift $\zeta : A_N \to A_N$:

$$\zeta(a_n) = a_{n+1} , \tag{3.9}$$

where n is taken modulo $2N$ and a_n is given by (3.6).

Define the following automorphisms:

$$\tau_{\pm}(a) = U^{-1}\zeta^{\pm 1}(a)U . \tag{3.10}$$

Proposition 3.1. *The following relations hold:*

$$\tau_+\tau_- = \tau_-\tau_+ , \tag{3.11}$$

$$\tau_+\tau_-^{-1} = \zeta^2 . \tag{3.12}$$

These relations follow immediately from the definition of $\tau_{\pm}$.

Proposition 3.2. *The elements* $L_{n,\,t}^{(\pm)}(z) \in A_N, t \geqq 0$, *are determined, using the following recursive procedure:*

$$L_{n+1,t+1}^{(\pm)}(z) = \tau_+(L_{n,\,t}^{(\pm)}(z)) , \tag{3.13}$$

$$L_{n-1,t+1}^{(\pm)}(z) = \tau_-(L_{n,\,t}^{(\pm)}(z)) \tag{3.14}$$

with the initial condition $L_{n,\,0}^{(\pm)}(z) \equiv L_n^{(\pm)}(z)$.

Important Remark 1. We will regard the variable t as discrete time. At the point $t = 0$ the elements $L_n^{(-)}$ are defined only for even n and the elements $L_n^{(+)}$ are defined only for odd n. Therefore:

$$L_{n,\,t}^{(-)} \text{ is defined when } n + t = 0 \bmod (2) ,$$

$$L_{n,\,t}^{(+)} \text{ is defined when } n + t = 1 \bmod (2) . \tag{3.15}$$

Important Remark 2. For the action of $\tau_{\pm}^t$, $t \in \mathbb{Z}_+$, we have:

$$\tau_{\pm}^t(a) = U_t^{(\pm)-1}\zeta^{\pm t}(a)U_t^{(\pm)} , \tag{3.16}$$

where

$$U_t^{(\pm)} = \zeta^{\pm t \mp 1}(U) \cdots \zeta^{\pm 1}(U)U . \tag{3.17}$$

This gives the following explicit expressions for $L^{\pm}_{n,t}(z)$ in terms of

$$L_n(z) = (\mathrm{id} \otimes \phi_n)(L(z))\,. \tag{3.18}$$

For even t we have:

$$L^{(-)}_{2n,t}(z) = U_t^{(+)^{-1}} L_{2n}(z-\kappa)^{-1} U_t^{(+)}\,, \tag{3.19}$$

$$L^{(+)}_{2n-1,t}(z) = U_t^{(+)^{-1}} L_{2n-1}(z+\kappa) U_t^{(+)}\,. \tag{3.20}$$

For odd t we have:

$$L^{(-)}_{2n-1,t}(z) = U_t^{(+)^{-1}} L_{2n-1}(z-\kappa)^{-1} U_t^{(+)}\,, \tag{3.21}$$

$$L^{(+)}_{2n,t}(z) = U_t^{(+)^{-1}} L_{2n}(z+\kappa) U_t^{(+)}\,. \tag{3.22}$$

Theorem 3.2. *The elements $L^{(\pm)}_{n,t}$ satisfy the following quadratic equations*:

$$L^{(-)}_{n,t+1}(z) L^{(+)}_{n,t}(z) = L^{(+)}_{n-1,t+1}(z) L^{(-)}_{n-1,t}(z)\,. \tag{3.23}$$

Proof. We must to prove the relation

$$\tau_+(L^{(-)}_{2n,t}(z)) L^{(+)}_{2n+1,t}(z) = \tau_-(L^{(+)}_{2n+1,t}(z)) L^{(-)}_{2n,t}(z)\,. \tag{3.24}$$

From the definition of $L^{(\pm)}_{n,t}$, it is enough to check it at $t=0$. In this case, the identity (3.24) is equivalent to

$$U^{-1} L_{2n+1}(z-\kappa)^{-1} U L_{2n+1}(z+\kappa) = U^{-1} L_{2n}(z+\kappa) U \cdot L_{2n}(z-\kappa)^{-1}\,, \tag{3.25}$$

or to the identity:

$$L_{2n+1}(z-\kappa)^{-1} \bar{R}_{2n+1,2n}(2\kappa) L_{2n+1}(z+\kappa) = L_{2n}(z+\kappa) \bar{R}_{2n+1,2n}(2\kappa) L_{2n}(z-\kappa)^{-1}\,, \tag{3.26}$$

which is the same as

$$\bar{R}_{2n+1,2n}(2\kappa) L_{2n+1}(z+\kappa) L_{2n}(z-\kappa) = L_{2n+1}(z-\kappa) L_{2n}(z+\kappa) \bar{R}_{2n+1,2n}(2\kappa)\,. \tag{3.27}$$

This identity is part of our data. □

The automorphisms $\tau_\pm$ and relations (3.23) determine the evolution

$$\tau = \tau_+ \tau_- \tag{3.28}$$

of the algebra A_N and the nonlinear evolution equations for matrix elements of $L^{\pm}_n(z)$. Notice that this evolution is Hamiltonian with the evolution operator $\zeta(U)U$:

$$\tau(a) = (\zeta(U) \cdot U)^{-1} \cdot a \cdot (\zeta(U) \cdot U)\,. \tag{3.29}$$

This evolution is integrable in the sense of the definition in Sect. 2. The quantum Lax operator can be easily constructed from $L^{(\pm)}_{n,t}(z)$.

The system (3.23) can be regarded as the compatibility condition for the following recursive system in A_N:

$$\psi_{n+1}^{t+1}(z) = L_{n,t}^{(+)}(z)\psi_n^t(z), \tag{3.30}$$

$$\psi_{n-1}^{t+1}(z) = L_{n-1,t}^{(-)}(z)\psi_n^t(z). \tag{3.31}$$

If we iterate this system twice in the n-direction we obtain a difference equation:

$$\begin{aligned}\psi_{n+2}^t(z) &= (L_{n+1,t}^{(-)}(z))^{-1}L_{n,t}^{(+)}(z)\psi_n^t(z)\\ &= L_{n+1,t-1}^{(+)}(z)(L_{n,t-1}^{(-)}(z)^{-1})\psi_n^t(z).\end{aligned} \tag{3.32}$$

Iterating the system (3.30), (3.31) twice in the t-direction we get

$$\begin{aligned}\psi_n^{t+2} &= L_{n-1,t+1}^{(+)}\,L_{n-1,t}^{(-)}\psi_n^t\\ &= L_{n,t+1}^{(-)}\,L_{n,t}^{(+)}\psi_n^t.\end{aligned} \tag{3.33}$$

For even t and n, introduce the matrices

$$\begin{aligned}\mathscr{L}_n^t &= (L_{n+1,t}^{(-)})^{-1}\,L_{n,t}^{(+)} = L_{n+1,t-1}^{(+)}\,(L_{n,t-1}^{(-)})^{-1},\\ \mathscr{M}_n^t &= L_{n,t+1}^{(-)}\,L_{n,t}^{(+)} = L_{n-1,t+1}^{(+)}\,L_{n-1,t}^{(-)}.\end{aligned} \tag{3.34}$$

Proposition 3.3. *The matrices* (3.34) *satisfy the following relation*:

$$\mathscr{L}_n^{t+2} = \mathscr{M}_{n+2}^t\,\mathscr{L}_n^t(\mathscr{M}_n^t)^{-1}. \tag{3.35}$$

The proof is an elementary consequence of relations (3.23). The relation (3.35) is called quantum Lax equation for the evolution (3.29).

Consider the monodromy matrix of (3.32) at $t = 0$ on the interval $1 \leqq n \leqq 2N$:

$$T(z) = L_{2N}(z-\kappa)L_{2N-1}(z+\kappa)L_{2N-2}(z-\kappa)\cdots L_1(z+\kappa). \tag{3.39}$$

The trace of this matrix over the space V is called the transfer matrix:

$$t_V(z) = (tr_V \otimes \mathrm{id})(T(z)). \tag{3.40}$$

It is an element of A_N. We will keep the subindex V and use it in case we have transfer matrices corresponding to different auxiliary vector spaces V.

The relations (3.2) generates the commutativity of the family (3.34)

$$[t_V(z), t_V(z')] = 0 \tag{3.41}$$

for any z, $z' \in \mathbb{C}$. The relations (3.3) imply

$$\tau(t_V(z)) = t_V(z). \tag{3.42}$$

Therefore, $t_V(z)$ is a generating function for commuting quantum integrals of motions for the evolution (3.29).

4. Quantum Complex Sine-Gordon Equation

Now consider an example of the procedure described in the previous section which provides a quantum version of the discrete complex sine-Gordon model.

Let A_q be an algebra generated by invertible elements u and v:

$$uv = qvu \,, \tag{4.1}$$

where $q \in \mathbb{C}^*$.

Define the elements $R(z) \in \mathrm{End}(\mathbb{C}^{2\otimes 2})$, $L(z) \in \mathrm{End}(\mathbb{C}^2) \otimes A$ as

$$\begin{aligned} R(z) = {} & (xq - x^{-1}q^{-1})(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) \\ & + (x - x^{-1})(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) \\ & + (q - q^{-1})(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) \,, \end{aligned} \tag{4.2}$$

$$L(z) = \begin{pmatrix} u & -x^{-1}v^{-1} \\ x^{-1}v & u^{-1} \end{pmatrix} . \tag{4.3}$$

Here $x = e^z$. From this point, until the end of this section, we will assume that $|q| < 1$.

These elements satisfy the relations (3.1), (3.2) [IK]. This fact has a simple meaning in terms of the quantum universal enveloping algebra $U_q(\widehat{b_+})$, where $\widehat{b_+}$ is a Borel subalgebra in $\widehat{sl_2}$. The matrices (4.2), (4.3) determine a "minimal" representation of $U_q(\widehat{sl_2})$ (when $q^n = 1$ it is closely related to a minimal cylic representation of $U_q(\widehat{sl_2})$ [DJMM]).

Consider the following matrices with coefficients in $\widehat{A_q^{\otimes 2}}$:

$$L_i(x) = \begin{pmatrix} u_i & -x^{-1}v_i^{-1} \\ x^{-1}v_i & u_i \end{pmatrix} , \tag{4.4}$$

where $u_i = \phi_i(u), v_i = \phi_i(v), i = 1, 2$ and ϕ_i are as in (3.6). Let $r(x; U)$ be a function and $\widehat{A_q^{\otimes 2}}$ be a completion of $A_q^{\otimes 2}$ where the element $r(x; v_1^{-1}v_2u_1^{-1}u_2^{-1})$ is defined.

Proposition 4.1. [*V*] *The element* $r(x, v_1^{-1}v_2u_1^{-1}u_2^{-1})$ *satisfies the following relation*:

$$r(x, v_1^{-1}v_2u_1^{-1}u_2^{-1})L_1(xy)L_2(y) = L_1(y)L_2(xy)r(x, v_1^{-1}v_2u_1^{-1}u_2^{-1}) \,, \tag{4.5}$$

if and only if the function $r(x, U)$ *satisfies Eq.* (1.5).

Proof. After some simple algebra, one can reduce (4.5) to the following three relations:

$$r(x, U)u_1u_2 = u_1u_2r(x, U) \,, \tag{4.6}$$

$$r(x, U)v_1^{-1}v_2 = v_1^{-1}v_2r(x, U) \,, \tag{4.7}$$

$$r(x, U)(xV + qV \cdot U) = (V + qxV \cdot U)r(x, U) \,. \tag{4.8}$$

Here we denote:

$$V = u_1 v_2^{-1}, \quad U = v_1^{-1} v_2 u_1^{-1} u_2^{-1} .$$

Clearly relations (4.6, 4.7) hold for these elements. The equality (4.8) is equivalent to the difference equation

$$r(x, q^2 U) = \left(\frac{1 + qxU}{x + qU} \right) r(x, U) . \tag{4.9}$$

Let us describe completions $\widehat{A_q^{\otimes 2}}$ which will be used in Sect. 6 and 7. Denote by q_0 a primitive root of 1 of odd degree l. Assume $q = q_0 e^h$, where h is a formal variable. Denote by $\widehat{A_q^{\otimes 2}}$ the algebra generated by $U^{\pm 1}$, $u_1^{\pm 1}$, $u_2^{\pm 1}$, $v_1^{\pm 1}$, $v_2^{\pm 1}$ with the same determining relations between $u_i^{\pm 1}, v_i^{\pm 1}$ as in $A^{\otimes 2}$ and one extra:

$$v_1^{-1} v_2 u_1^{-1} u_2^{-1} = U . \tag{4.10}$$

As a linear space $\widehat{A_q}$ consists of Laurent polynomials in u_i, v_i and power series in h and $U^l - 1$.

Consider the element

$$r(x; U) = \exp\left(-\frac{P^2}{4h} \right) (-xU^{-1}q; q^2)_\infty^{-1} (-xUq; q^2)_\infty^{-1} , \tag{4.11}$$

in the space $\widehat{A_q^{\otimes 2}}[[h^{-1}]]$. Here $U^l = \exp(lP)$ and x is a nonzero complex number and

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n) .$$

The function (4.11) provides a solution to (4.9).

Let $\widehat{A_q^{\otimes 3}}$ be the algebra generated by $U^{\pm 1}$, $V^{\pm 1}$, $u_i^{\pm 1}$, $v_i^{\pm 1}$, $i = 1, 2, 3$, with the extra relations:

$$\begin{aligned} U &= v_2^{-1} v_1 u_2^{-1} u_1^{-1} , \\ V &= v_3^{-1} v_2 u_3^{-1} u_2^{-1} , \\ UV &= q^2 VU . \end{aligned} \tag{4.12}$$

As the vector space $\widehat{A_q^{\otimes 3}}$ consists of power series over h, $U^l - 1$ and $V^l - 1$, and Laurent polynomials in $u_i^{\pm 1}$, $v_i^{\pm 1}$.

Define the element $r(x, V)$ similarly to (4.11). Then, in $\widehat{A_q^{\otimes 3}}$ we have the following identity:

$$r(x; U) r(xy; V) r(y; U) = r(y; V) r(xy; U) r(x; V) . \tag{4.13}$$

For the proof see [BR](similar statements can be found in [K, V]). We will not use the relation (4.13) in this paper but, in [BR] we show that it provides the star-triangular relation for the chiral Potts model.

The remarkable fact about the expression (4.11) is that though it does not lie in $\widehat{A_q^{\otimes 3}}$ the similarity transformation by $r(x, U)$ and by $r(x, V)$ provide automorphisms of this algebra.

Now $K = \exp\kappa$ and choose

$$\bar{R}(z) = r(e^z, v^{-1}u^{-1} \otimes vu^{-1}), \tag{4.14}$$

where $r(x, U)$ is a solution to (1.5) and the right side is considered in an appropriate completion of $A_q^{\otimes 2}$. Now let us construct the dynamics of the discrete sine-Gordon model on the algebra $A_q^{\otimes 2N}$. Define the evolution operator (3.7) using the R-matrix (4.14):

$$U = \prod_{n=0}^{N-1} r(K^2, v_{2n+1}^{-1}u_{2n+1}^{-1}v_{2n}u_{2n}^{-1}), \tag{4.15}$$

where $K = \exp\kappa$. Then formulas (3.10), (3.29) provide automorphisms $\tau, \tau_\pm$ of appropriate completion of $A_q^{\otimes 2N}$ (defined similar to the completions described above).

Define the trajectories of $u_n^{\pm 1}$, $v_n^{\pm 1}$ inductively as follows:

$$a_{n\pm 1, t+1} = \tau_\pm(a_{n,t}), \tag{4.16}$$

where $a := u^{\pm 1}, v^{\pm 1}$.

Proposition 4.2. *The elements $u_{n,t}^{\pm 1}$, $v_{n,t}^{\pm 1}$ satisfy the following relations*:
t-even:

$$u_{2n-1,t}\, u_{2n-2,t} = u_{2n-1,t+1}\, u_{2n-2,t+1}\,,$$

$$v_{2n-1,t}^{-1}\, v_{2n-2,t} = v_{2n-1,t+1}^{-1}\, v_{2n-2,t+1}\,,$$

$$Ku_{2n-1,t}\, v_{2n-2,t}^{-1} + K^{-1}v_{2n-1,t}\, u_{2n-2,t}^{-1}$$

$$= K^{-1}u_{2n-1,t+1}\, v_{2n-2,t+1}^{-1} + Kv_{2n-1,t+1}^{-1}\, u_{2n-2,t+1}^{-1}\,, \tag{4.17}$$

t-odd:

$$u_{2n,t}\, u_{2n-1,t} = u_{2n,t+1}\, u_{2n-1,t+1}\,,$$

$$v_{2n,t}^{-1}\, v_{2n-1,t} = v_{2n,t+1}^{-1}\, v_{2n-1,t+1}\,,$$

$$Ku_{2n,t}\, v_{2n-1,t}^{-1} + K^{-1}v_{2n,t}^{-1}\, u_{2n-1,t}^{-1} = K^{-1}u_{2n,t+1}\, v_{2n-1,t+1}^{-1} + Kv_{2n,t+1}^{-1}\, u_{2n-1,t+1}^{-1}\,. \tag{4.18}$$

Proof. These relations follow immediately from Eq. (3.22) if we identify $L_{n,t}^{\mp}(x)$ as follows:

$$L_{n,t}^{+}(x) = \begin{pmatrix} u_{n,t}, & -K^{-1}x^{-1}v_{n,t}^{-1} \\ K^{-1}x^{-1}v_{n,t}, & u_{n,t}^{-1} \end{pmatrix}, \tag{4.19}$$

$$L_{n,t}^{-}(x) = \begin{pmatrix} u_{n,t}, & -Kx^{-1}v_{n,t}^{-1} \\ Kx^{-1}v_{n,t}, & u_{n,t}^{-1} \end{pmatrix}^{-1}, \tag{4.20}$$

where $x = e^z$.

The physically interesting case corresponds to $|q| = 1$ and $K \in \mathbb{R}$. The algebra A_q in this case is regarded as the algebra with the antilinear antiinvolution $*$ acting as:

$$u_i^* = u_i^{-1}, \qquad v_i^* = v_i^{-1}. \tag{4.21}$$

It is easy to check that in this case the coefficients of the generating function (3.40) are $*$-invariant integrals of motion. We will consider in detail such cases when q is a root of 1. For the case when $|q| = 1$ but when it is not a root of 1, see [F].

5. Equations of Motion and Integrals of Motion for the Quantum Sine-Gordon Equation

In this section we will introduce the subalgebra of "sine-Gordon variables" in the algebra $A_q^{\otimes 2N}$, and will show that the evolution τ, with U given by (4.15), determines an integrable quantum system for this subalgebra.

For even t, define the following elements:

$$\begin{aligned} U_{2n,\,t} &= v_{2n,\,t}^{-1}\, v_{2n-1,\,t}\, u_{2n,\,t}^{-1}\, u_{2n-1,\,t}^{-1} \\ &= v_{2n,\,t-1}^{-1}\, v_{2n-1,\,t-1}\, u_{2n,\,t-1}^{-1}\, u_{2n-1,\,t-1}^{-1} = \tau^t(U_{2n}), \\ U_{2n-1,\,t} &= v_{2n-1,\,t}^{-1}\, v_{2n-2,\,t}\, u_{2n-1,\,t}^{-1}\, u_{2n-2,\,t}^{-1} \\ &= v_{2n-1,\,t+1}^{-1}\, v_{2n-2,\,t+1}\, u_{2n-1,\,t+1}^{-1}\, u_{2n-2,\,t+1}^{-1} = \tau^t(U_{2n-1}). \end{aligned} \tag{5.1}$$

Here the second lines in the equalities follow from equations of motion (4.17), (4.18).

Elements $U_n = U_{n,0}$ obey the following relations:

$$\begin{aligned} U_{2n} U_{2n-1} &= q^2 U_{2n-1} U_{2n}, \\ U_{2n} U_{2n+1} &= q^{-2} U_{2n+1} U_{2n}, \end{aligned} \tag{5.2}$$

and generate the subalgebra of $A_q^{\otimes 2N}$ which we denote $\mathscr{A}_N(q)$. This algebra has a center generated by the elements

$$C_1 = \prod_{n=0}^{N-1} U_{2n+1}, \quad C_2 = \prod_{n=1}^{N} U_{2n}. \tag{5.3}$$

Clearly the sequence of elements $U_{n,\,t}$, $t = 0, 1, 2, \ldots$ is the trajectory of U_n with respect to the evolution τ:

$$U_{n,\,t} = \tau^t(U_n). \tag{5.4}$$

Theorem 5.1. *Elements $U_{n,\,t}$ satisfy the following nonlinear evolution equations:*

$$U_{2n,\,t+2} = \frac{K^2 + q^{-1} U_{2n+1,\,t}}{1 + K^2 q^{-1} U_{2n+1,\,t}} \cdot \frac{1 + K^2 q \cdot U_{2n-1,\,t}}{K^2 + q \cdot U_{2n-1,\,t}}\, U_{2n,\,t}, \tag{5.5}$$

$$U_{2n+1,\,t-2} = U_{2n+1,\,t} \frac{1 + K^2 q U_{2n+2,\,t}}{K^2 + q U_{2n+2,\,t}} \cdot \frac{K^2 + q^{-1} U_{2n,\,t}}{1 + K^2 q^{-1} U_{2n,\,t}}. \tag{5.6}$$

Theorem 5.2. *The trace* $t_1(x)$ *of the quantum monodromy matrix* (3.39), *with* $L^{(\pm)}(z)$ *given by* (4.19) *and* (4.20), *depends only on the elements* $U_n^{\pm 1}$, $n = 1, \dots, N$, *and therefore belongs to the subalgebra* $\mathscr{A}_N(q)$ *in* $A_q^{\otimes 2N}$. *The element* $t_1(x)$ *is the polynomial of degree 2N. Its coefficients determine N non-trivial integrals of the evolution* τ.

Let us first prove Theorem 5.1. One can do this in two natural ways. The first, "the Hamiltonian" approach, is to use the definition of the evolution τ and compute the equations of motion directly from this definition. The second, the "Lagrangian" approach, is to use the definition of $U_{n,t}$ in terms of $u_{n,t}^{\pm 1}$, $v_{n,t}^{\pm 1}$. Here we present both approaches.

1. "*Hamiltonian*" *proof of Theorem 5.1.* Let us use the definition of the automorphisms $\tau_\pm$ and the difference equation for $r(x; U)$. We obtain the following expressions for the action of $\tau_\pm$ on U_n:

$$\begin{aligned}
\tau_+(U_{2n}) &= U_{2n+1}\,, \\
\tau_-(U_{2n}) &= U_{2n-1}\,, \\
\tau_+(U_{2n-1}) &= F(U_{2n+1})U_{2n}F(U_{2n-1})^{-1}\,, \\
\tau_-(U_{2n+1}) &= \tau_+(U_{2n-1})\,,
\end{aligned} \tag{5.7}$$

where $F(U)$ is the same as in (1.6). From this we have for $\tau = \tau_+\tau_- = \tau_-\tau_+$:

$$\begin{aligned}
\tau(U_{2n}) &= F(U_{2n+1})U_{2n}F(U_{2n-1})^{-1}\,, \\
\tau(U_{2n+1}) &= F(\tau(U_{2n+2}))U_{2n+1}F(\tau(U_{2n}))^{-1}\,,
\end{aligned} \tag{5.8}$$

which implies (5.5), (5.6).

2. "*Lagrangian*" *proof of Theorem 5.1.* Let us factorize the L-operator (4.3) as follows:

$$L = \begin{pmatrix} u & -x^{-1}v^{-1} \\ x^{-1}v & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^{-1}v \end{pmatrix} u\mathscr{M}_1 \begin{pmatrix} 1 & 0 \\ 0 & u^{-1}v^{-1} \end{pmatrix}, \tag{5.9}$$

where

$$\mathscr{M}_\alpha = \begin{pmatrix} 1 & -\alpha x^{-1} \\ \alpha x^{-1}q & 1 \end{pmatrix}. \tag{5.10}$$

Then, for the product of two L-operators we have:

$$\begin{aligned}
L_{2n-1,t}(xK)L_{2n,t}(xK^{-1}) = &\begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v)_{2n-1,t} \end{pmatrix} u_{2n-1,t}\mathscr{M}_{1/K} \\
&\begin{pmatrix} 1 & 0 \\ 0 & U_{2n-1,t} \end{pmatrix} u_{2n-2,t}\mathscr{M}_K \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v^{-1})_{2n-2,t} \end{pmatrix}.
\end{aligned} \tag{5.11}$$

The Lax equation (3.23) implies the compatibility condition

$$\mathscr{L}^+_{2n+1,t}\mathscr{L}^-_{2n,t}\mathscr{L}^+_{2n-1,t}\mathscr{L}^-_{2n-2,t} = \mathscr{L}^-_{2n+1,t+1}\mathscr{L}^+_{2n,t+1}\mathscr{L}^-_{2n-1,t+1}\mathscr{L}^+_{2n-2,t+1}\,. \tag{5.12}$$

Now, if we use the factorization (5.9) and the definition of $U_{n,t}$, this relation can be rewritten in terms of $L^{\pm}_{n,t}(z)$ defined in (4.19), (4.20), as

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v)_{2n+1,t} \end{pmatrix} u_{2n+1,t} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & U_{2n+1,t} \end{pmatrix} u_{2n,t} \\
&\mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & U_{2n,t} \end{pmatrix} u_{2n-1,t} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & U_{2n-1,t} \end{pmatrix} u_{2n-2,t} \\
&\mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v^{-1})_{2n-2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v)_{2n+1,t+1} \end{pmatrix} u_{2n+1,t+1} \\
&\mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & U_{2n+1,t} \end{pmatrix} u_{2n,t+1} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & U_{2n,t+2} \end{pmatrix} u_{2n-1,t+1} \\
&\mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & U_{2n-1,t} \end{pmatrix} u_{2n-2,t+1} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v^{-1})_{2n-2,t+1} \end{pmatrix} .
\end{aligned} \tag{5.13}$$

Moving all $u_{n,t}, u_{n,t+1}$ to the left and cancelling by the factor

$$(u_{2n+1}\, u_{2n}\, u_{2n-1}\, u_{2n-2})_{t+1} = (u_{2n+1}\, u_{2n}\, u_{2n-1}\, u_{2n-2})_t \,,$$

one gets

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v)_{2n+1,t} \end{pmatrix} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2n+1,t} \end{pmatrix} \mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2n,t} \end{pmatrix} \\
&\mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2n-1,t} \end{pmatrix} \mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v^{-1})_{2n-2,t} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v)_{2n+1,t+1} \end{pmatrix} \mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2n+1,t} \end{pmatrix} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2n,t+2} \end{pmatrix} \\
&\mathcal{M}_K \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2n-1,t} \end{pmatrix} \mathcal{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & (u^{-1}v^{-1})_{2n-2,t+1} \end{pmatrix} .
\end{aligned} \tag{5.14}$$

The element 11 of this equality depends on only U's and yields

$$\begin{aligned}
&(K^2 + q^{-1}U_{2n+1,t})U_{2n,t}(1 + K^2q^{-1}U_{2n-1,t}) \\
&\quad = (1 + K^2q^{-1}U_{2n+1,t})U_{2n,t+2}(K^2 + q^{-1}U_{2n-1,t}) .
\end{aligned} \tag{5.15}$$

In a similar way as for even t in (5.12) we can obtain the following equation for odd t:

$$\begin{aligned}
&(K^2 + q^{-1}U_{2n+2,t})U_{2n+1,t-2}(1 + K^2q^{-1}U_{2n,t}) \\
&\quad = (1 + K^2q^{-1}U_{2n+2,t})U_{2n+1,t}(K^2 + q^{-1}U_{2n,t}) .
\end{aligned} \tag{5.15'}$$

Now let us prove Theorem 5.2. Direct calculation, similar to the one used in the proof of the previous proposition, yields

$$T(x) = u_{2N}\cdots u_1 \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}(u^{-1}v)_{2N} \end{pmatrix} F(x; U_2, \ldots, U_{2N}) \begin{pmatrix} 1 & 0 \\ 0 & q(v^{-1}u^{-1})_1 \end{pmatrix}, \quad (5.16)$$

where

$$F(x; U_2, \ldots, U_{2N}) = \mathscr{M}_K \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2N} \end{pmatrix} \mathscr{M}_{1/K} \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_{2N-1} \end{pmatrix} \cdots$$
$$\ldots \mathscr{M}_K \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}U_2 \end{pmatrix} \mathscr{M}_{1/K}\,. \quad (5.17)$$

Denote the elements of the matrix $F(x; U_2, \ldots, U_{2N-1})$ by

$$F(x;U) = \begin{pmatrix} \bar{A}(x;U) & \bar{B}(x;U) \\ \bar{C}(x;U) & \bar{D}(x;U) \end{pmatrix}. \quad (5.18)$$

One gets for the trace of the matrix (over $\mathbb{C}^2$)

$$\begin{aligned} t_1(x) &= u_{2N-1}\ldots u_0\{\bar{A}(x;U_2,\ldots,U_{2N}) \\ &\quad +(u^{-1}v)_{2N}\bar{D}(x;U_2,\ldots,U_{2N})(v^{-1}u^{-1})_1\} \\ &= u_{2N-1}\ldots u_0\{\bar{A}(x;U_2,\ldots,U_{2N}) + \bar{D}(x;U_2,\ldots,U_{2N-1},q^2U_{2N})U_1\}\,. \quad (5.19) \end{aligned}$$

The product $u_{2N},\ldots,u_1$ can also be expressed in terms of $U_1,\ldots,U_{2N}$ (in an appropriate extension of $A_{qN}(q)$ by $\sqrt{C_1}$ and $\sqrt{C_2}$)

$$u_{2N}\cdots u_1 = \sqrt{U_{2N}^{-1}U_{2N-1}^{-1}\ldots U_1^{-1}}\, q^{1-N}, \quad (5.20)$$

which completes the proof.

It is easy to see from the structure of $L(x)$ that $t_1(x)$ is a polynomial of degree $2N$ in x^{-2}. It is also a simple computation which shows that when $x \to \infty$,

$$\begin{aligned} t_1(x) &= u_{2N}\cdots u_1 + u_{2N}^{-1}\cdots u_1^{-1} + O(x^{-2}) \\ &= (C_1C_2^{-1})^{\frac{1}{2}}q^{1-N} + (C_2C_1^{-1})^{\frac{1}{2}}q^{N-1} + O(x^{-2})\,. \end{aligned}$$

Here, C_1 and C_2 are defined in (1.10). All other coefficients do not lie in the center of $A_N(q)$ and therefore provide nontrivial integrals of the evolution τ. Although we do not prove it here, it is quite reasonable to assume that they are algebraically independent.

The arguments of Sect. 3 show that the discrete quantum sine-Gordon equation (4.38) describes an integrable quantum system with a generating function $t_1(x)$ for commuting quantum integrals.

Denote by $T^{(1)}(z)$ the quantum monodromy matrix (3.39) for the quantum sine-Gordon model described in Sect. 4. It is known that $T^{(1)}(z)$ corresponds to the irreducible 2-dimensional $U_q(sl_2)$-module. Using the fusion procedure for the elements $L(z)$ [KSR], one can obtain quantum monodromy matrices corresponding to any finite-dimensional $U_q(sl_2)$-modules. Denote by $T^{(l)}(z)$, $(l = 1, 2, \ldots)$ the corresponding $(l+1)$-dimensional representation. For an explicit description of the fusion procedure which gives $T^{(l)}(z)$ via a certain algebraic operation on $T^{(1)}(z)$, see [KR]. The element $T^{(l)}(z) \in \mathrm{End}(\mathbb{C}^{l+1}) \otimes A_N$ has the following form:

$$T^{(l)}(z) = L_{2N}^{(l)}(zK^{-1})L_{2N-1}^{(l)}(zK) \cdots L_2^{(l)}(zK^{-1})L_1^{(l)}(zK)\,, \tag{5.21}$$

where $L_n^{(l)}(z)$ are obtained by an appropriate fusion of several copies of $L(z)$ [KR].

Introduce the elements

$$t_l(z) = \mathrm{tr}(T^{(l)}(z))\,, \tag{5.22}$$

where the trace is taken over $\mathbb{C}^{l+1}$, so that $t_l(z) \in A_N$. Represent the element $T^{(1)}(z)$ as a 2×2 matrix with elements from A_N:

$$T^{(1)}(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}\,. \tag{5.22$'$}$$

Proposition 5.3. [KR] *Elements $t_l(z)$ satisfy the following relations*:

$$[t_l(z), t_{l'}(w)] = 0\,, \tag{5.23}$$

for any $l, l' = 1, 2, \ldots$, $z, w \in \mathbb{C}$ and the recursive relations

$$t_1(z)t_l(zq) = t_{l+1}(z) + d(z) \cdot t_{l-1}(zq^2)\,, \tag{5.24}$$

$$t_l(z)t_1(zq^l) = t_{l+1}(z) + d(zq^{l+1})t_{l-1}(z)\,, \tag{5.24$'$}$$

where $d(z) = A(zq)D(z) - B(zq)C(z)$ is a quantum determinant of $T^{(1)}(z)$.

For the proof of this proposition see [KR].

Remark 5.4. Let $\mathscr{A}$ be the algebra generated by the elements $A(z), B(z), C(z), D(z)$. Topologically $\mathscr{A}$ can be regarded in several ways: elements in the complex plane parameterize the generators of $\mathscr{A}$, or A, B, C, D are generating functions like $f(z) = \sum_{n \geqq 0} f_n z^n$ (or in some other appropriate sense) with relations

$$R(z)T_1(zw)T_2(w) = T_2(w)T_1(zw)R(z)\,. \tag{5.25}$$

Proposition 5.3 holds in this, more general, situation.

Remark 5.5. For the monodromy matrix of the quantum sine-Gordon equation, the quantum determinant can be computed explicitly

$$d(z) = (1 + z^{-1}q^{-1}K^2)^N(1 + z^{-1}q^{-1}K^{-2})^N\,. \tag{5.26}$$

The solution to the recursion (5.24) can be written explicitly as:

$$t_l(z) = \det \begin{bmatrix} t_1(z) & 1 & 0 & \ldots & 0 & 0 \\ d(z) & t_1(zq) & 1 & \ldots & 0 & 0 \\ 0 & d(zq) & t_1(zq^2) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & t_1(zq^{l-2}) & \\ 0 & 0 & 0 & \ldots & d(zq^{l-2}) & t_1(zq^{l-1}) \end{bmatrix}\,, \tag{5.27}$$

or, in terms of generating functions,

$$(1 - Xt_1(z) + X^2 d(z))^{-1} = \sum_{l \geqq 0} X^l t_l(z)\,, \tag{5.28}$$

where

$$t_1(z)X = Xt_1(zq)\,,$$
$$d(z)X = Xd(zq)\,. \tag{5.29}$$

The coefficients of the polynomial $t_1(z)$ are integrals of motion of the evolution (3.28). The coefficients of the polynomials $t_l(z)$, $l > 1$, do not generate new integrals because of (5.27).

Now define the discrete sine-Gordon variables $Q_{n,t}$, which are defined only for $n + t = 0(\mathrm{mod}\ 2)$:

$$Q_{n,t} = \begin{cases} U_{n,t}\,, & n+t = 0\,(\mathrm{mod}\ 4), \quad t\text{-even} \\ U_{n,t}^{-1}\,, & n+t = 2\,(\mathrm{mod}\ 4), \quad t\text{-even} \\ U_{n,t-1}\,, & n+t = 2\,(\mathrm{mod}\ 4), \quad t\text{-odd} \\ U_{n,t-1}^{-1}\,, & n+t = 0\,(\mathrm{mod}\ 4), \quad t\text{-odd} \end{cases}. \tag{5.30}$$

We also assume that $Q_{2n} := Q_{2n,0}$, $Q_{2n+1} := Q_{2n+1,1}$.

There are several reasons to call these elements sine-Gordon variables. In the continuum limit they approach $\exp(i\phi(x))$, where $\phi(x)$ is a continuous Sine-Gordon variable (see the next section). Another reason is that in the classical limit $Q_{n,t}$ has a natural interpretation in terms of angles between edges of discrete K-surfaces (see the conclusion and [BP] for details).

If we associate the elements $Q_{n,t}$ with the unshaded faces of a two-dimensional square lattice (see Fig. 1) the evolution equations for $Q_{n,t}$ can be written as

$$Q_u = F(Q_l)F(Q_r)Q_d^{-1}\,, \tag{5.31}$$

where u, l, r, d are the up, left, right and down unshaded faces surrounding any shaded face.

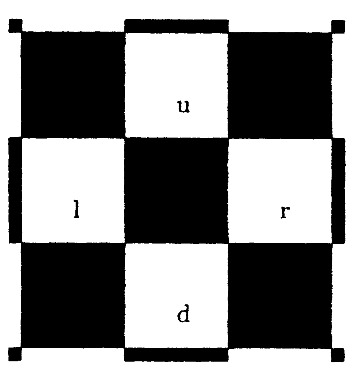

Fig. 1.

6. The Classical Discrete Sine–Gordon System

In the previous section we described a quantum complex sine-Gordon system as a quantum mechanical system with a finitely generated algebra of observables. From the point of view of quantum field theory it is a finite quantum mechanical system.

The quasiclassical limit of the system (5.13), (5.14) corresponds to the limit $q \to 1$. The classical system which we recover in this limit has the phase space $M_{2N} \simeq \mathbb{T}^{\times 2N}$, where $\mathbb{T}$ is the (complex) torus. In natural coordinates in M_{2N}, the Poisson bracket between coordinate functions will have the following form:

$$\{Q_{2n}, Q_{2n\pm 1}\} = 2Q_{2n}Q_{2n\pm 1} . \tag{6.1}$$

All other brackets $\{Q_n, Q_m\}$ vanish. The equations of motion (5.13) will become

$$Q_{n,t+2}\, Q_{n,t} = F_{cl}(Q_{n+1,t+1})F_{cl}(Q_{n-1,t+1}) , \tag{6.2}$$

$$F_{cl}(x) = \frac{K^2 + x}{1 + K^2 x} . \tag{6.3}$$

They can be regarded as a symplectomorphism $\phi : M_{2N} \to M_{2N}$ given by the formula

$$\tau_{cl}(Q_n) = F_{cl}(Q_{n+1})F_{cl}(Q_{n-1})Q_n^{-1} \tag{6.4}$$

in natural coordinates in M_{2N}.

Taking the limit $q \to 1$ of the generating function for quantum evolution we obtain the generating function for the symplectomorphism τ_{cl}:

$$\tau_{cl}(a) = \exp(H) \circ a = \sum_{n \geqq 0} \frac{1}{n!}\{\mathscr{H}\{\ldots\{\mathscr{H}, a\}\}\} , \tag{6.5}$$

with

$$\mathscr{H} = H(\mathscr{H}_+, \mathscr{H}_-) . \tag{6.6}$$

Here $H(a, b)$ is a Campbell–Hausdorff function for the multiplication in $A_N(e^h)$:

$$a * b = a \cdot b + \frac{h}{2}\{a, b\} + O(h^2) , \tag{6.7}$$

$$H(a, b) = h \log\left(\exp\left(\frac{a}{h}\right) * \exp\left(\frac{b}{h}\right)\right) \bmod h , \tag{6.8}$$

and

$$\mathscr{H}_- = \frac{1}{2}\sum_{n=1}^{N}(Li_2(-Q_{2n}K^2) + Li_2(-Q_{2n}^{-1}K^2) + \frac{1}{2}\log^2(Q_{2n})) , \tag{6.9}$$

$$\mathscr{H}_+ = \frac{1}{2}\sum_{n=1}^{N}(Li_2(-Q_{2n-1}K^2) + Li_2(-Q_{2n-1}^{-1}K^2) + \frac{1}{2}\log^2(Q_{2n-1})) . \tag{6.10}$$

Here $Li_2(z)$ is the Euler dilogarithm

$$Li_2(z) = -\int_0^z \frac{\log(1-x)}{x}\,dx\,.$$

Here we assume that the limit $h \to 0$ is taken in the completion of $A_N(e^h)$ by power series in K^2 and $(Q_n - 1)$. In this completion, the evolution operator (1.7) is given by the product of the elements (4.11).

As for the quantum case, transfer matrix $t_1(z)$ generates classical integrals of motion.

Let us demonstrate how one can recover the continuum sine-Gordon system from the system (6.2), (6.3) in the limit $K \to 0$.

Assume that $K \to 0$, $N \to \infty$ as $N = \frac{L}{K}$ for some fixed value of L, and

$$Q_n = Q(nK)\,, \tag{6.11}$$

where $Q(x)$ is some smooth periodic function on the interval $[0, L]$. Then if $t = \frac{\sigma}{K}$ there exists a limit

$$Q_{n,t} \to Q(x,\sigma), \quad K \to 0\,, \tag{6.12}$$

and the limiting function is the solution of the classical sine-Gordon equation

$$\partial_\sigma((\partial_\sigma Q)Q^{-1}) - \partial_x((\partial_x Q)Q^{-1}) = 2(Q^{-1} - Q)\,. \tag{6.13}$$

Indeed we can rewrite the system (6.2), (6.3) as

$$Q_{n,t+2}\,Q_{n,t}\,Q^{-1}_{n-1,t+1}\,Q^{-1}_{n+1,t+1} = \frac{1 + K^2 Q^{-1}_{n-1,t+1}}{1 + K^2 Q_{n-1,t+1}} \cdot \frac{1 + K^2 Q^{-1}_{n+1,t+1}}{1 + K^2 Q_{n+1,t+1}}\,. \tag{6.14}$$

Taking the limit $K \to 0$ with the assumption (6.11) we get (6.13).

In the coordinates $Q = e^{i\phi}$, Eq. (6.13) looks more familiar:

$$\partial_\sigma^2 \phi - \partial_x^2 \phi = -4 \sin\phi\,. \tag{6.15}$$

7. Quantum Discrete Sine-Gordon System at Roots of 1

Consider the discrete sine-Gordon system in the limit $q \to q_0$, where $q_0^l = 1$. We assume also that l is odd and q_0 is a primitive root of degree l.

It is easy to establish the following properties of the algebra $\mathscr{A}_N(q_0)$:

- The center $Z(\mathscr{A}_N(q_0))$ of the algebra $\mathscr{A}_N(q_0)$ is generated by the elements Q_n^l.
- The following brackets determine Poisson algebra structure on $Z(\mathscr{A}_N(q_0))$ and the Poisson action of the center by derivations on $\mathscr{A}_N(q_0)$:

$$\{a, b\} = \lim_{\tau \to 0} \frac{a * b - b * a}{\tau}\,. \tag{7.1}$$

Here $*$ is the multiplication in $\mathscr{A}_N(e^\tau q_0)$ and at least one of a or b belongs to $Z(\mathscr{A}_N(e^\tau q_0))$. We also assume an identification of the vector spaces $\mathscr{A}_N(q)$ for all q (for example by choosing some normal ordering).

It is easy to compute the Poisson brackets (7.1) between the generators Q_n^l of $Z(\mathscr{A}_N(q_0))$ and their action on Q_n:

$$\{Q_n^l, Q_m^l\} = \begin{cases} 4l^2 Q_n^l Q_m^l, & |n-m| = 1 \\ 0, & \text{otherwise} \end{cases}, \tag{7.2}$$

$$\{Q_n^l, Q_m\} = \begin{cases} 2l Q_n^l Q_m, & |n-m| = 1 \\ 0, & \text{otherwise} \end{cases}. \tag{7.3}$$

Now let us consider the equations of motion for $Q_{n,t}^l$.

Theorem 7.1. *The evolution equations* (5.13), (5.13′) *provide the following equations for* $Q_{n,t}^l$:

$$Q_{n,t+2}^l Q_{n,t}^l = \frac{K^{2l} + Q_{n+1,t+1}^l}{1 + K^{2l} Q_{n+1,t+1}^l} \cdot \frac{K^{2l} + Q_{n-1,t+1}^l}{1 + K^{2l} Q_{n-1,t+1}^l}. \tag{7.4}$$

Proof. From the commutation relations (5.2) we conclude that

$$F(U_{n-1,t}) G(U_{n+1,t}) U_{n,t}^{-1} = U_{n,t}^{-1} F(q^{\pm 2} U_{n-1,t}) G(q^{\pm 2} U_{n+1,t}) \tag{7.5}$$

for any rational functions $F(x)$ and $G(x)$. Here we have pluses or minuses in the r.h.s. depending on the parity of n.

Write Eq. (5.13) as

$$Q_{n,t+2} = F(Q_{n-1,t}) F(Q_{n+1,t}) Q_{n,t}^{-1},$$

where

$$F(x) = \frac{K^2 + q^{-1}x}{q + K^2 q^{-1} x}. \tag{7.6}$$

Then, if we will use the definition of Q-elements and the relation (7.5) we will have:

$$Q_{n,t+2}^l = F_l(Q_{n-1,t}) F_l(Q_{n+1,t}) Q_{n,t}^{-l}, \tag{7.7}$$

where

$$F_l(x) = F(x) F(xq^{-2}) \cdots F(xq^{-2(l-1)}).$$

If q_0 is a root of 1 of degree l with odd l we have

$$F_l(x) = \frac{K^{2l} + x^l}{1 + K^{2l} x^l}. \tag{7.8}$$

Therefore, if we introduce new variables $\{Q_{n,t}^l\}$ they will satisfy the classical equations of motion for the discrete sine-Gordon model. Thus, the dynamics of the quantum sine-Gordon model at roots of 1 has the following structure:

- It determines the classical dynamics on $Z(A_N(q_0))$ with the equations of motion (7.4).
- The algebra $A_N(q_0)$ is finite-dimensional over its center and therefore we can regard the quantum dynamics (5.13), as "finite-dimensional quantum fluctuations" over the classical trajectory of Q_n^l.

Let us consider the evolution automorphism for quantum dynamics at roots of 1. First, let us determine the asymptotics of $r(x,z)$ from (4.5) when $q \to q_0$.

The solution $r(x,z)$ to Eq. (1.5) which is regular at $x=0$ has the following asymptotics when $q = q_0 e^{\tau}$ and $\tau \to 0$:

$$r(x,z) \simeq \text{const} \cdot \exp\left(-\frac{1}{2l^2\tau}L(x^l,z^l)\right)\left(\frac{1+z^{-l}x^l}{1+z^lx^l}\right)^{\frac{1-l}{2l}} \times \prod_{n=0}^{l-1}\left(\frac{1+xz^{-1}q_0^{-2n-1}}{1+xzq_0^{2n+1}}\right)^{-\frac{n}{l}}(1+O(\tau)), \tag{7.9}$$

where

$$L(x^l,z^l) = Li_2(-z^lx^l) + Li_2(-z^{-l}x^l) + \frac{1}{2}\log^2(z^l). \tag{7.10}$$

Taking the limit $q \to q_0$ in the evolution automorphism τ, with $\bar{R}(2\kappa)$ determined by (4.14), we obtain the following expression for the action of τ in the completion of $\mathscr{A}_N(q_0)$ by the power series in K^2 and $Q_n - 1$:

$$\tau(a) = \bar{U}^{-1}\tau_{\text{cl}}(a)\bar{U}, \tag{7.11}$$

where

$$\bar{U} = \tau_{\text{cl}}(\bar{U}_+)\bar{U}_-, \tag{7.12}$$

$$\tau_{\text{cl}}(a) = \exp\left(\frac{\mathscr{H}_+^{\text{cl}}}{l^2}\right) \circ \exp\left(\frac{\mathscr{H}_-^{\text{cl}}}{l^2}\right) \circ a, \tag{7.13}$$

$$\bar{U}_- = \prod_{n=1}^{N}\prod_{m=0}^{l-1}\left(\frac{1+K^2Q_{2n}^{-1}q_0^{-2m-1}}{1+K^2Q_{2n}q_0^{2m+1}}\right)^{-\frac{m}{l}}, \tag{7.15}$$

$$\bar{U}_+ = \prod_{n=1}^{N}\prod_{m=0}^{l-1}\left(\frac{1+K^2Q_{2n-1}^{-1}q_0^{-2m-1}}{1+K^2Q_{2n-1}q_0^{2m+1}}\right)^{-\frac{m}{l}}. \tag{7.16}$$

Here, $\exp(a) \circ b$ is the Poisson action (6.6) of $\exp(a)$ on b. Functions $\mathscr{H}_\pm^{\text{cl}}$ are determined by Eqs. (6.10), (6.11):

$$\mathscr{H}_\pm^{\text{cl}}(Q) = \mathscr{H}_\pm(Q^l).$$

Notice that in the center of $\mathscr{A}_N(q_0)$ the map τ_{cl} acts as a classical evolution automorphism

$$\tau_{\text{cl}}(a) = \sum_{n\geq 0}\frac{\left(\frac{1}{l^2}\right)^n}{n!}\{\mathscr{H}^{\text{cl}}\cdots\{\mathscr{H}^{\text{cl}},a\}\cdots\}, \tag{7.17}$$

where $\mathscr{H}^{\text{cl}} = H(\mathscr{H}_+^{\text{cl}},\mathscr{H}_-^{\text{cl}})$, and $H(a,b)$ is the Campbell–Hausdorff function (6.8).

Now consider integrals of motion for the evolution (7.11).

In Sect. 5 we described transfer matrices corresponding to $(l+1)$-dimensional representations of $U_q(sl_2)$ for the quantum discrete model (5.6) with $l = 0,1,2,\ldots$. It is known (see for example [L]) that the structure of $U_q(sl_2)$-modules is completely different when q is a root of 1. This is why in this case the transfer matrices corresponding to higher irreducible representations of $U_q(sl_2)$ have a different "fusion algebra" from (5.23). Let $t_m(z)$ be the transfer matrix (see Sect. 5) corresponding to $(m+1)$-dimensional irreducible representation of $U_{q_0}(sl_2)$ with q_0 as above.

Let $t_1^{\mathrm{cl}}(z;K;U_n)$ be the classical limit ($q \to 1$) of the quantum transfer matrix (5.19). Consider the elements

$$A^{(l)}(z) = A(z) \cdots A(zq_0^{l-1}), \tag{7.18}$$

$$D^{(l)}(z) = D(z) \cdots D(zq_0^{l-1}), \tag{7.19}$$

where $A(z)$, $D(z)$ are elements of the quantum monodromy matrix (5.22′).

Theorem 7.2. 1. *The transfer matrices $t_m(z)$ form a commutative family* ($n, m \leqq l-1$)

$$[t_m(z), t_n(z')] = 0 . \tag{7.20}$$

2. *If* $1 \leqq n \leqq l-1$, *the transfer matrix $t_m(z)$ can be computed via* (5.26).
3. *The elements $A^{(l)}(z)$, $D^{(l)}(z)$ are related to $t_1^{\mathrm{cl}}(z)$ as*:

$$A^{(l)}(z) + D^{(l)}(z) = t_1^{\mathrm{cl}}(z^l;K^l;U_n^l) \equiv t^{(l)}(z) . \tag{7.21}$$

4. *The following identity holds*:

$$1 + t_1^{\mathrm{cl}}(z) + d^{\mathrm{cl}}(z) = \det \begin{bmatrix} t_1(z) & 1 & 0 & \dots & 0 & 0 \\ d(z) & t_1(zq_0) & 1 & \dots & 0 & 0 \\ 0 & d(zq_0) & t_1(zq_0^2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_1(zq_0^{l-2}) & 1 \\ 0 & 0 & 0 & \dots & d(zq_0^{l-2}) & t_1(zq_0^{l-1}) \end{bmatrix}, \tag{7.22}$$

where $d^{(l)}(z) = \prod_{j=0}^{l-1} d(zq_0^j) = (1 + z^{-l}K^{2l})^N(1 + z^{-l}K^{-2l})^N$.

The proof of this theorem essentially follows from work [BS]. It is very technical and we will omit it here.

8. Conclusion

In this paper we have studied the discrete quantum sine-Gordon model at roots of 1. From the physical point of view the main feature of this model is that it may be regarded as an integrable quantum field theoretic model (as $N \to \infty$) with classical background.

The discrete sine-Gordon model has an interesting geometrical application which we briefly present here, for details of which we refer the reader to [BP]. The sine-Gordon equation

$$\phi_{\xi\eta} = \sin \phi$$

was first derived in the 19th century in the context of differential geometry to describe surfaces with a constant negative curvature (K-surfaces). These surfaces allow a Chebyshev net parameterization $F(\eta, \xi)$,

$$F \colon \mathbb{R}^2 \to \mathbb{R}^3 ,$$

which is an asymptotic line parameterization with $\|F_\xi\| = \|F_\eta\| = \text{constant}$. The function $\phi(\xi,\eta)$ describes the angles between the asymptotic lines.

This geometric picture can be discretized in a natural and elegant way. A discrete K-surface is a map $F : \mathbb{Z}^2 \to \mathbb{R}^3$ with the properties

(1) any four edges emanating from a common vertex $F_{n,m}$ are coplanar,

(2) all the edges have a constant length.

In fact, this discretization has already been suggested in the 1950's [W]. Starting with this geometric definition one can derive an equation for the angles between the edges of neighboring vertices (see [BP]), i.e. an equation with the same geometric meaning as the one of the sine-Gordon equations in the smooth case. So it is therefore not surprising that this equation turned out to be the discrete sine-Gordon equation (6.2), (6.3), where $Q_{n,m} = \exp(i\phi_{n,m})$ and $\phi_{n,m}$ is the angle between the edges at vertex $F_{n,m}$.

It would be interesting to find a natural geometric interpretation of the quantum sine-Gordon system, i.e. to define quantum discrete K-surfaces.

The system can also be regarded as an example of a dynamical system in the vector bundle. Indeed, the algebra $\mathscr{A}_N(q_0)$ can be regarded as a (nontrivial) vector bundle over an N-dimensional complex torus with an l^2-dimensional fiber. It is known as a quantum torus [C]. The evolution operator (7.11) is an endomorphism of this bundle which projects to the classical symplectomorphism (7.13) on the base. It would be interesting to investigate this dynamical system in more detail, for example, to compute its non-commutative entropy [C].

Another interesting physical application of our results is that the evolution operator (7.12) can be regarded as the transfer matrix for the chiral Potts model [BR]. The chiral Potts model, whose discovery was originated in [VGR, HKDN] and finalized in [B], has already been found in the center of various interesting results [BS, BB, DJMM]. First, the Boltzmann weights of the model require high-genus algebraic functions for their parameterization [B]. The second (related) feature of the model is there is no a "difference property," and therefore one cannot directly apply various methods used for other two-dimensional solvable models. In addition, the chiral Potts model has a three dimensional interpretation and can be considered as a particular two-layer case of the three-dimensional solvable lattice model of [BB]. All these suggest that the integrability of the chiral Potts model of statistical mechanics has a nature different from other known models. We hope that a new interpretation of this model [BR] as a quantum discrete sine-Gordon model in a constant classical background would lead to the further progress in this area. In particular this interpretation opens the possibility of studying even more interesting models like the chiral–Potts-type system in other classical backgrounds (e.g. in a periodic background which might not in general be rational relative to the original lattice).

We did not attempt here to describe the spectrum of the transfer matrix $t_1(x)$ at $q = q_0$, although Eq. (7.21), regarded as a functional equation for the eigenvalues of $t_1(z)$, describes possible eigenvalues as points on the corresponding algebraic curve. We hope that it is possible to describe the spectrum even more explicitly, particularly in the limit $N \to \infty$. This problem is especially important because of its relation to Chiral Potts and Hoffshtater [WiZ, FKa, Ku] models.

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