Solitons and Helices: The Search for a Math-Physics Bridge

Eric Zaslow★

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract: We present evidence for an undiscovered link between $N = 2$ supersymmetric quantum field theories and the mathematical theory of helices of coherent sheaves. We give a thorough review for nonspecialists of both the mathematics and physics involved, and invite the reader to take up the search for this elusive connection.

1. Introduction

Last year, Kontsevich noticed a similarity between the work of Cecotti and Vafa on classifying two-dimensional $N = 2$ supersymmetric field theories and the work of some algebraic geometers in Moscow [1]. In the independent and seemingly unrelated work of physicists and mathematicians, similar structures emerged. Both had found quasi-unipotent matrices satisfying certain Diophantine conditions, which supported the action of the braid group. Were they the same?

Behind this question lies a potential relationship between disparate fields and the opportunity for string theory and its offshoots to once again bring mathematicians and physicists together. Unfortunately, my search for this bridge was somewhat in vain. I cannot tout complete success; instead I offer an amalgam of evidence and observations supporting this conjecture, along with various approaches used in trying to find this elusive link. These diverse techniques span a breadth of physics and mathematics. This paper is intended to give a thorough treatment while remaining somewhat self-contained, perhaps at the expense of brevity.

The physics is the theory of classifying two-dimensional $N = 2$ supersymmetric field theories [2] and is closely related to topological-anti-topological (tt^*) fusion [3]. The idea for classification was to obtain information about the number of vacua and solitons between them in the infrared limit. Given a massive $N = 2$ theory (we will always consider two-dimensional theories), one can consider the whole renormalization group trajectory – its infrared and ultraviolet limits. In the conformal, or ultraviolet, limit, the (universality class of the) theory can be partially classified

by the structure of the chiral primary ring [4]. In particular one can compute the number of ring elements and their $U(1)$ charges. In the infrared limit, there is no superconformal symmetry; the excitations above the vacua become infinitely massive and we can investigate tunneling between vacua. These amplitudes will reveal the numbers of solitons connecting different vacua. These numbers, too, help classify the theory. In fact, the $U(1)$ charges of the theory at its conformal limit may be derived from this information. Our interest is in the topological sigma model associated to a Kähler target space. The physical picture is detailed in section two.

The mathematics involved regards a branch of study which has been developed in the past decade primarily at Moscow University [5]. The theory is that of collections of coherent sheaves called helices. The theory arose from the study of vector bundles over low dimensional projective spaces. Unfortunately, the math is almost as new as the physics, and few results are known rigorously. Helices are collections of sheaves over some complex base manifold obeying a sort of upper-triangularity condition (on the Euler characteristic between sheaves). These collections transform under mutations defining an action of the braid group on a finite collection of sheaves – the *foundation* – from which the helix is determined. A bilinear form over the foundation is defined; in several examples, we see that it is precisely the matrix derived from the physical theory on the same base manifold. Section three is composed of a detailed exploration of this mathematical subject.

Even without an explicit connection between the two disciplines, it may be possible to prove some sort of categorical equivalence between certain classes of $N = 2$ quantum field theories and foundations of helices. Such a description, while interesting, would not be as enlightening. For example, we have noticed that the matrices associated to topological sigma models over a given manifold correspond to the matrices associated with bundles over the same space. A categorical equivalence could not guarantee that the base space of the vector bundles and the sigma model should be in correspondence. We will discuss this further, along with several techniques which may be useful in finding a bridge, in section four.

A note on point of view: In the section labeled "The Physics," the first person "we" is taken to mean "we mathematicians." In the section of mathematics, it means "we physicists." This schizophrenic viewpoint reflects only the author's natural identification with those who feel inexpert.

2. The Physics

2.1. Overview. The physics we will discuss involves the realm of two-dimensional quantum field theories with two independent supersymmetry charges. These theories have many interesting properties which relate to various fields of mathematics including de Rham and Hodge cohomologies [6]; Morse theory [7]; singularities and Picard Lefschetz theory [2, 8]; variations of Hodge structure. The latter two are more closely related to our area of investigation, and are particular to the $N = 2$ case (these structures are absent in $N = 1$ supersymmetric models). Specifically, we will be working in the topological sector of such theories [9]. This is interpreted as follows. There exists a certain set of correlations functions in these theories which are independent of the positions of the fields on the two-dimensional surface. If we restrict the set of fields and correlation functions to those which have this property, we can use the $N = 2$ theory as a means of creating a "topological theory." Typically, the space of topological field theories is composed of finite-dimensional

components. This space can be thought of as the parameter space of the quantum field theory, where only the topologically relevant parameters – i.e. those which when perturbed change the topological correlation functions – are considered.

Thus, our spaces will be finite-dimensional spaces, each point of which will represent a topological field theory. Let us fix a component $\mathcal{M}$ of this space, and label its points by t. The fields ϕ_i in these theories create states $|i\rangle$ which comprise a Hilbert space, $\mathcal{H}_t$. The fields (and hence states) will typically transform amongst themselves under a perturbation. They thus define a vector bundle over moduli space, with a natural connection we will determine presently through field theoretic methods. The fields define a commutative associative algebra: $\phi_i \times \phi_j = C_{ij}{}^k \phi_k$. The correlation functions, being independent of position, can only depend on the types of fields and on the point t. We will be concerned with the variation of these parameters with the moduli.

As we will show, the Hilbert spaces can be thought of as spaces of *vacua*, i.e. states of zero energy. If we imagine a potential, the vacua lie at the minima, which we will take to be discrete and labeled x_n. If space is the real line, a field configuration $\phi(x)$ satisfying $\phi(\infty) = x_a$ and $\phi(-\infty) = x_b$ is said to be in the *ab* soliton sector. A minimal energy configuration is a soliton. The situation is decidedly more difficult to interpret in field theories without potentials. After our discussion of topological field theories, we will relate some of the quantities discussed above to the numbers of solitons in the theory. The sigma models are theories defined for a given Kähler manifold and Kähler form, $\mathbf{k}$. The moduli space of theories for a given manifold will be the Kähler cone. To each such manifold we will derive a matrix encoding the soliton numbers, which has several interesting properties. It transforms under the action of a braid group and is quasi-unipotent. We will liken this to a similar matrix derived through the theory of helices.

2.2. Topological Field Theory. Topological field theories are models in which correlation functions do not depend on the positions of the operators involved. They therefore depend only on the type of operators involved, and on topological properties of the space of field configurations. In the case of topological sigma models, in which the quantum fields are maps to some target manifold, the topology of the target manifold becomes crucial. Such a theory can be defined given any $N = 2$ quantum field theory. Topological theories constructed in this manner will be studied here.

Let us describe the twisting procedure which yields a topological theory from an $N = 2$ theory. To do so, let us first consider a theory defined on an infinite flat cylinder. A quantum theory with $N = 2$ supersymmetry is invariant under the $N = 2$ superalgebra. This algebra contains two fermionic generators Q^1, Q^2, as well as bosonic generators, which mix non-trivially. There is also an $SO(2)$ automorphism of this algebra rotating the Q's. We usually write $Q^\pm = \frac{1}{2}(Q^1 \mp iQ^2)$, where the sign denotes the charge under the $SO(2)$ generator, J. Further, since the supercharges are spinorial (they give spinors from bosons), their components have a chirality in two-dimensions. This gives us four charges: $Q_L^+, Q_R^+, Q_L^-, Q_R^-$. The algebra contains the two dimensional Lorentz group as well and reads:

$$\{Q_L^+, Q_L^-\} = 2H_L, \qquad \{Q_R^+, Q_R^-\} = 2H_R,$$
$$[L, Q_L^\pm] = \frac{1}{2}Q_L^\pm, \qquad [L, Q_R^\pm] = \frac{1}{2}Q_R^\pm,$$

$$[L, H_L] = H_L , \qquad [L, H_R] = -H_R ,$$

$$[J, Q_L^\pm] = \pm Q_L^\pm , \qquad [J, Q_R^\pm] = \pm Q_R^\pm , \qquad (2.1)$$

with all other (anti-)commutators vanishing.[1] Here L is the generator of the Euclidean rotation group $SO(2)$ and J is the generator of the $SO(2)$ rotation mixing the Q's. Also, $H_{L,R} = H \pm P$.

The topological theory is defined cohomologically by constructing a boundary operator from the Q's. Let us define, then,

$$Q_+ \equiv Q_L^+ + Q_R^+ .$$

Note that

$$(Q_+)^2 = 0 .$$

We use this operator to define cohomology classes, reducing the space of states to a finite-dimensional Hilbert space:

$$\mathscr{H} \equiv H^*(Q) = \frac{\{|\psi\rangle : Q_+|\psi\rangle = 0\}}{\{|\psi\rangle : |\psi\rangle = Q_+|\Lambda\rangle\}} .$$

Similarly, the fields are defined modulo commutators with Q_+. Topological invariance follows from the fact that derivatives with respect to z and $\bar{z}$ are represented by the action of H_L and H_R, respectively, on the fields. Since both H_L and H_R are exact ($H_{L,R} = \frac{1}{2}\{Q_+, Q_{L,R}^-\}$), all correlation functions between topological states will be invariant under infinitesimal variations of the positions.

We would like to extend this analysis to arbitrary Riemann surfaces. What prevents us from doing so now is that Q_+ is made up not of scalars but of pieces of spinors. When our surface was a flat torus with trivial spin connection, the separate components of Q_+ and Q_- were globally defined. These will not be globally defined on a general Riemann surface, and so what was cohomologically trivial in one set of coordinates may be nontrivial in another. To remedy this, we simply *declare* Q_+ to be a scalar. That is, we can redefine the spin of the fermions by adding a background gauge field proportional to the spin connection.

The topological fields form a ring, just as de Rham cohomology elements form a ring. The products are well-defined, since we can note $(\phi_1 + [Q_+, \Lambda])\phi_2 = \phi_1\phi_2 + [Q_+, \Lambda\phi_2] \equiv \phi_1\phi_2$, which follows from $[Q_+, \phi_2] = 0$. Let us choose a set of generators ϕ_i for the topological field space. The operator product can be captured through the structure constants $C_{ij}{}^k$ by writing

$$\phi_i \times \phi_j = C_{ij}{}^k \phi_k .$$

The field space is in one-to-one correspondence with the Hilbert space by the relation

$$\phi_i|0\rangle \equiv |i\rangle .$$

In the above we have used the unique vacuum $|0\rangle$ from the $N = 2$ quantum field theory.[2] The correlators are then all given by the ring coefficients $C_{ij}{}^k$ and the

---

[1] This algebra is modified in soliton sectors. There, it includes central terms which yield the Bogomolnyi bound.

[2] This vacuum is the unique vacuum of the Neveu–Schwartz sector.

two-point function
$$\eta_{ij} = \langle i | j \rangle \,. \tag{2.2}$$

Note that we don't take the adjoint state $\langle \bar{i} |$ in forming the topological metric. In (2.2), $\langle i |$ obeys
$$\langle i | Q_+ = 0 \,,$$

which insures topological invariance of the correlation functions. We note here that in the particular case where $|i\rangle$ is a ground state, and therefore annihilated by Q_+ and Q_-, we can take the regular adjoint and the correlation functions will still be topological. The discovery of such independence in correlators was made in particular models several years before topological field theory was systematically treated.

The analogy with de Rham cohomology can be extended to Hodge cohomology. We can interpret the Hamiltonian as the Laplace operator, with Q_+ and Q_- serving as the ∂ and $\bar{\partial}$ operators. Then, as with Hodge decomposition, we have the following statement. Every Q_+ cohomology class has a unique harmonic representative, i.e. a unique representative annihilated by Q_-. Noting that zero energy states are annihilated by Q_+ and Q_- we have the equivalence of several vector spaces:[3]

$$Q_+ \text{ cohomology} \leftrightarrow \text{Vacua} \leftrightarrow Q_- \text{ cohomology} \,. \tag{2.3}$$

The second equivalence in (2.3) is made simply by interchanging the rôles of Q_+ and Q_-.

The simple observation (2.3) will provide us with a rich source for exploration. Specifically, we will ask how the isomorphism between the Q_- and Q_+ cohomologies varies over the space of topological field theories.

To illustrate the structure of topological theories and provide us with our main object of study, we briefly discuss the structure of the chiral ring for the topological sigma models. By sigma model, we mean a quantum field theory in which the bosonic variables are maps (from a two-dimensional surface) to a target manifold. In the $N = 1$ and $N = 2$ supersymmetric theories, the fermionic structures mimic the forms of de Rham and Hodge cohomology. The action takes the form

$$S = 2t \int_{\Sigma} d^2 z \frac{1}{2} g_{IJ} \, \partial_z \phi^I \, \partial_{\bar{z}} \phi^J + i \psi_-^{\bar{i}} D_z \psi_-^i g_{i\bar{i}} + i \psi_+^{\bar{i}} D_{\bar{z}} \psi_+^i g_{i\bar{i}} + R_{i\bar{i}j\bar{j}} \psi_+^i \psi_+^{\bar{i}} \psi_-^j \psi_-^{\bar{j}} \,. \tag{2.4}$$

Here Σ represents the Riemann surface, which, for our purpose will always be of genus zero, g_{IJ} and $R_{i\bar{i}j\bar{j}}$ are respectively the metric and Riemann tensor of the target space. D is the pull-back onto Σ of the connection under the map, Φ. The $N = 2$ structure implies a holomorphic $U(1)$ current, by which we may twist the energy-momentum tensor. That is, we can redefine spins by adding a background gauge field equal to (one half) the spin connection. Mathematically, this is equivalent to redefining the bundles in which the fields live. As we have discussed, this will render the BRST charge Q_+ a scalar on the Riemann surface, so that the theory is defined for any genus. Specifically, we now take $\psi_+^i \in \Phi^*(T^{1,0})$ and $\psi_-^{\bar{i}} \in \Phi^*(T^{0,1})$. We put $\psi_+^{\bar{i}} \in \Omega^{1,0}(\Sigma; \Phi^*(T^{0,1}))$ and $\psi_-^i \in \Omega^{0,1}(\Sigma; \Phi^*(T^{1,0}))$; that is, they combine to form a one-form on Σ with values in the pull-back of the tangent space of K: call these components $\psi_z^{\bar{i}}$ and $\psi_{\bar{z}}^i$ respectively.

---

[3] We see $\langle \psi | H | \psi \rangle = 0 \Rightarrow \langle \psi | (Q_- Q_+ + Q_+ Q_-) | \psi \rangle = 0 \Rightarrow \| Q_+ | \psi \rangle \|^2 + \| Q_- | \psi \rangle \|^2 = 0$, since $(Q_-)^\dagger = Q_+$. Therefore both terms, being positive definite in a unitary theory, are zero separately.

The important aspect of these theories are that the energy-momentum tensor is Q_+-exact. This allows us to rescale the two-dimensional metric $\delta h_{\mu\nu} = \Lambda h_{\mu\nu}$ without affecting the correlators. As $\Lambda \to \infty$ – the topological limit – the only non-vanishing contribution to the path integral is from the instanton configurations, or classical minima. As the space of instantons, $\mathcal{M}$, is disconnected, the computation reduces to a sum over components of $\mathcal{M}$. Supersymmetry ensures the cancellation of the determinant from bosonic and fermionic oscillator modes. The zero mode integration yields just the number of instantons taking the insertion points to the Poincaré dual cycles representing the corresponding operators. This is how we derive the ring of observables.

2.3. Topological-Anti-Topological Fusion and Classification of $N = 2$ Theories. We will be interested in the numbers of (Bogomolnyi saturating) solitons connecting ground states. These numbers were used by Cecotti and Vafa in their classification of $N = 2$ superconformal theories with massive deformations [2]. The idea is that in the infrared limit, the two-point functions of different vacua (choosing an appropriate basis) obey

$$\langle \bar{i} | j \rangle \sim \delta_{ij} + \text{ tunneling corrections}. \tag{2.5}$$

The tunneling corrections indicate the presence of solitons, and will depend on the size (Kahler class) of the manifold, or more generally the couplings of the theory. The tunneling corrections vanish in the infinite volume (conformal) limit, but the asymptotic behavior will indicate the number of solitons present (in a manner which will be made explicit). The dependence on the couplings is described by the tt^* equations of reference [3]. We review this technology below, then discuss the Diophantine constraints of classification.

In the previous section, we discussed how to make a topological field theory given any $N = 2$ theory by taking the Q_+ cohomology classes as states. Alternatively, we could have defined a theory with the Q_- cohomology. We can call this theory the "anti-topological" theory (it is still a topological field theory). In (2.3) we noted that the spaces of states were isomorphic. They can be thought of as different bases for a finite dimensional vector space. This means that each anti-topological state $|\bar{a}\rangle$ can be expressed in terms of the topological states: $|\bar{a}\rangle = \sum_b C_b |b\rangle$, for some coefficients C_b. More generally, we write

$$\langle \bar{a} | = \langle b | M^b{}_{\bar{a}},$$

with the sum over b understood. In this section we will describe how to compute this change of basis, and its variation on theory space. To do this, let us examine the relationship between the topological and anti-topological field theories.

The quantum field theory defines a metric on the topological Hilbert space,

$$g_{a\bar{b}} = \langle \bar{b} | a \rangle,$$

where we require the states to be ground states. This metric thus fuses the topological and anti-topological theory. In fact, by connecting two hemispherical regions along a common flat boundary, we can perform the topological twist on one half and the anti-topological twist on the other half. The long cylindrical middle projects states to their ground state representatives; flatness allows us to conjoin the different background metrics, used to make the topological twist, where they vanish. The resulting metric is the one described above, and is independent of the representatives of the topological states.

The topological theory defines a symmetric topological metric, given by intersections in an appropriate moduli space of classical minima:

$$\eta_{ab} = \langle a|b \rangle$$

($\eta_{\bar{a}b} = \eta_{ab}^*$). We note that $M = \eta^{-1}g$ and $(CPT)^2 = 1$ implies $MM^* = 1$ by relating topological and anti-topological states.

These structures are defined for any $N = 2$ theory, and become geometrical structures on the space of theories. We can coordinatize this space by coupling constants $\{t_i\}$. The key observation for us is that correlation functions of neighboring theories can be computed in terms of correlators of a given theory. So let us consider a theory described by an action $S(0)$ at some point $t = 0$ in parameter space. We can parametrize a neighborhood of 0 by perturbing by the local operators ϕ_i. Thus, we write

$$S(t) = S(0) - \sum_i \left[\int d^2\theta^- d^2z t_i \phi_i + \text{h.c.} \right] , \qquad (2.6)$$

where the perturbation is assumed to be small. The correlations functions are now t-dependent, but we can compute the variation given knowledge of the theory at $t = 0$. Indeed,

$$\partial_i \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle_{t=0} = \left\langle \int \phi_i(z) d^2z d^2\theta^- \, \phi_1(z_1) \ldots \phi_n(z_n) \right\rangle_{t=0} ,$$

where $\partial_i \equiv \partial/\partial t_i$ and the subscript indicates that the correlation functions are evaluated at $t = 0$. At each t, we have a chiral ring, isomorphic to the Ramond ground states of the theory. We thus have a vector bundle – the bundle of ground states – with the metric given above (now t-dependent). A ground state, characterized by its $U(1)$ charge, is then a section of this bundle; its wave function is therefore t-dependent, and we can thus consider the connection defined by

$$(A_i)_{a\bar{b}} = \langle \bar{b}|\partial_i|a \rangle .$$

Here we project out the change in $|a\rangle$ orthogonal to the ground states. The covariant derivative is then $D_i = \partial_i - A_i$. This connection is defined so that

$$D_i g_{a\bar{b}} = 0 ,$$

which follows simply.

The equations of topological-anti-topological fusion, the tt^* equations, describe the dependences of our geometrical constructions on the couplings t_i and $\bar{t}_{\bar{j}}$. The equations may be expressed covariantly, or in a particular choice of basis for the ground states (gauge). We will first show that the topological states (i.e. the Q_+ cohomology) constitute a "holomorphic basis," in which the anti-holomorphic part of the connection vanishes: $A_{\bar{i}a}{}^b = 0$. To prove this we note that in the path-integral formalism, the state $|a\rangle = |\phi_a\rangle$ is given by the path-integral over S_R, the right half of a sphere; so we have, from (2.6):

$$\partial_{\bar{i}}|a\rangle = \left| \int_{S_R} d^2\theta^+ d^2z \bar{\phi}_{\bar{i}}(z)\phi_a \right\rangle$$

$$= Q^+ \bar{Q}^+ \left| \int_{S_R} d^2z \bar{\phi}_{\bar{i}}(z)\phi_a \right\rangle$$

(we adopt the convention $Q^+ = D^+$). Then it is clear that the projection to states obeying $\langle c|Q^+ = 0$ kills $A_{\bar{\imath}}$:

$$(A_{\bar{\imath}})_a{}^b = \eta^{bc}\langle c|\partial_{\bar{\imath}}|a\rangle = 0\,.$$

Clearly, we could also choose the anti-topological basis, which would yield $A_{i\bar{a}}{}^{\bar{b}} = 0$. In the holomorphic basis, the covariant constancy of the metric determines the connection:

$$0 = D_i g_{a\bar{b}} = \partial_\iota g_{a\bar{b}} - A_{\iota a}{}^c g_{c\bar{b}} - g_{a\bar{c}} A_{i\bar{b}}{}^{\bar{c}} = \partial_\iota g_{a\bar{b}} - A_{\iota a}{}^c g_{c\bar{b}}$$

(the mixed index parts of the connection vanish by the Kähler condition). Thus,

$$A_\iota = \partial_\iota g \cdot g^{-1} = -g\partial_\iota g^{-1}\,. \tag{2.7}$$

The tt^* equations are derived by path integral manipulations like the ones used in finding the holomorphic basis. In fact, the existence of such a basis immediately tells us that the chiral ring matrices C_j (defined by $\phi_i\phi_j = (C_\iota)_j{}^k \phi_k$) obey $D_{\bar{\imath}} C_j = \partial_{\bar{\imath}} C_j = 0$, since the chiral ring has no $\bar{t}_{\bar{\imath}}$ dependence – the $\bar{t}$ terms in the action are Q^+-trivial. Similarly one shows that the topological metric, η, only has holomorphic dependence on the couplings.

We study succinctly express the tt^* equations by considering a family of connections indexed by a "spectral parameter," x:

$$\begin{aligned}\nabla_i &= D_\iota - xC_\iota\,, \\ \overline{\nabla}_{\bar{\imath}} &= \bar{D}_{\bar{\imath}} - x^{-1}\bar{C}_{\bar{\imath}}\,.\end{aligned} \tag{2.8}$$

The tt^* equations, conditions on the metric and the C_i, are then summarized by the statement that ∇ and $\overline{\nabla}$ are flat for all x. For example, from the term multiplying x in $[\nabla_j, \overline{\nabla}_{\bar{\imath}}] = 0$ we have the formula proved above:

$$[D_{\bar{\imath}}, C_j] = D_{\bar{\imath}} C_j = 0\,.$$

The terms independent of x in the same equation give

$$[D_j, D_{\bar{\imath}}] + [C_j, C_{\bar{\imath}}] = 0\,.$$

In the holomorphic basis, we know what the connection is from (2.7). Further, one finds the action of $\bar{C}_{\bar{\imath}}$ is by the matrix $gC_i^\dagger g^{-1}$. Thus, we arrive at

$$\bar{\partial}_{\bar{\imath}}(g\partial_j g^{-1}) = [C_j, gC_i^\dagger g^{-1}]\,.$$

In the foregoing, we have assumed that the two-dimensional space was an infinite cylinder with unit perimeter. A perimeter of β, which we will take as a scaling parameter, adds a factor of β^2 to the right-hand side. Now flow in the space of theories along the β direction – a change in scale, that is – is given by the renormalization group. We can study what the connection in the β direction looks like. One finds that the gauge field in the β direction corresponds to the Ramond charge matrix, q, in the conformal limit [10]. We define

$$Q_{ab} = -\frac{1}{2}(g\beta\partial_\beta g^{-1})_{ab} \tag{2.9}$$

(we have actually taken the direction defined by τ, where $\beta = e^{(\tau+\tau^*)/2}$). Then $q = Q$ as $\beta \to 0$. The reason for this relation is that under a scale transformation, a state – represented by a path integral with a circle boundary – changes by the trace of the energy momentum tensor plus the integral along the boundary of the topological twisting background gauge field coupled to the chiral fermion current:

$$\delta g = \varepsilon g \Rightarrow \delta|a\rangle = \left| \int \mathrm{Tr}\, T^{(\mathrm{top})} \phi_a \right\rangle = \left| \int T^\mu_\mu \varepsilon \phi_a \right\rangle + \left| \int \partial_\mu J^\mu \phi_a \right\rangle .$$

The matrix Q is the axial (left plus right) charge matrix, which is a conserved charge only at the conformal point (for example, any mass in the theory breaks this chiral $U(1)$). If we change our point of view, exchanging "space" and "time" in the path integral expression for Q, then the configurations being integrated are the solitons which run from vacuum a to vacuum b along the spatial line. The chiral fermion number then gets rewritten as the fermion number, since $j^5_\mu = i\varepsilon_{\mu\nu}j^\nu$ in two dimensions. The result is expressed as a limit as the spatial volume goes to infinity:

$$Q_{ab} = \lim_{L\to\infty} i\frac{\beta}{L}\mathrm{Tr}_{ab}(-1)^F F e^{-\beta H} . \tag{2.10}$$

The ab subscript indicates that the trace should be performed over the ab soliton sector of the Hilbert space. It is clear from the presentation (2.10) that the coefficient of the leading exponential for large β can be used to count the minimum energy solitons between a and b, weighted by $F(-1)^F$.

Let us now consider the following set of equations:

$$\nabla_i \Psi(x, w_a) = \bar{\nabla}_{\bar{\imath}} \Psi(x, w_a) = 0 . \tag{2.11}$$

In order to solve these equations simultaneously, we must require that ∇ and $\bar{\nabla}$ commute, i.e. they are *flat*; the consistency condition is thus tt^*. We can amend the connection to include the variable x so that the independence of the phase of β follows from requiring flatness. That this independence should hold follows from the freedom to redefine the phases of the fermions, eliminating an overall scale in the superpotential.[4] We write:

$$x\frac{\partial}{\partial x}\Psi = \left(\beta x C + Q - \beta x^{-1}\bar{C}\right)\Psi . \tag{2.12}$$

The matrix C is given by $C_i{}^j = \sum_k w_k C_{ki}{}^j$ (see the footnote for the definition of w_k) $\bar{C} = gC^\dagger g^{-1}$, and Q is as above. Here we have written that the scale of the superpotential is $\beta e^{i\theta}$ with $x = e^{i\theta}$, but the equations now make sense for x a complex variable. In general, there will be n solutions to (2.11), so we take Ψ to be an $n \times n$ matrix whose columns are solutions. The equations are singular at $x = 0, \infty$, which means the columns of Ψ will mix under monodromy $x \to e^{2\pi i}x : \Psi \to H \cdot \Psi$. In fact, the solutions can be expressed in terms of two regions of the x-plane. In the overlap of the regions, the solutions are matched by matrices which are related such that the total monodromy takes the form

$$H = S \cdot (S^{-1})^t .$$

---

[4] By "superpotential," we mean the values w_a which can be assigned to the different vacua such that the Bogomolnyi soliton masses (the central terms in the $N = 2$ algebra on the non-compact line) are given by the absolute values of differences of the w_a (these are the canonical coordinates of the theory). Note that $\beta = e^{-A}$, where A is the one-instanton action or area.

Furthermore, a flat connection guarantees that H (as well as S) is a *constant* matrix – independent of local variations the couplings of the theory. In particular, we can evaluate it in a convenient limit.

Consider $\beta \to 0$ with x small; this is the conformal limit, since the area goes to infinity. (We note that the scale of β, since $\ln \beta$ multiplies the actions, does not affect the configuration of the vacua.) In this limit, the equation in x takes the simple form

$$\frac{d}{d\theta} \Psi_i = q_{ij} \Psi_j \,,$$

where θ is the phases of x and $q = Q|_{\beta=0}$ is the Ramond charge matrix. The solution is $\Psi(\theta) = e^{2\pi i q\theta}$. It is then clear that

$$\text{Eigenvalues}[H] = \text{Eigenvalues}[e^{2\pi iq}] :$$

the phases of the eigenvalues of the monodromy around zero are precisely the Ramond charges. Since these charges must be real, the eigenvalues $\lambda_i = e^{2\pi i q_i}$ of the monodromy must satisfy $|\lambda_i| = 1$.

We can now go to the infrared limit (β large) to determine H, as it is in-dependent of β. Let us see how the (weighted) numbers of solitons enter in the calculation of H. As we discussed when deriving (2.10), these numbers appear as terms in the leading asymptotics of the matrix Q_{ab}. In the large β limit, the soliton states of minimal energy – the Bogomolnyi solitons – dominate the expression for Q, so that the leading behavior is

$$Q_{ij}|_{\beta \to \infty} = -\frac{i}{2\pi} A_{ij} m_{ij} \beta K_1(m_{ij}\beta) \,,$$

where m_{ij} is the soliton mass. Using the relation (2.9) between Q and $g^{-1}\partial g$, the asymptotic form of g is seen to be

$$g_{ij}^- = \delta_{ij} - \frac{i}{\pi} A_{ij} K_0(m_{ij}\beta) \,, \tag{2.13}$$

where the K_0 and K_1 are modified Bessel functions. Soliton numbers thus are directly related with solutions to the tt^* equations. The more rigorous analysis of Sect. 4 of [2] is needed to relate the solutions Ψ of (2.11) to the metric g_{ij}. The monodromy H is found to be related to the matrix A by the following expression:

$$H = S(S^{-1})^t \,,$$
$$S = 1 - A \,. \tag{2.14}$$

In a standard configuration of vacua, S is an upper-triangular matrix.

We have therefore seen that the soliton numbers counted with $F(-1)^F$ can be arranged in a monodromy matrix $H = (1 - A)(1 - A)^{-t}$ whose eigenvalues give the chiral charges of the vacua in the conformal limit (the integer part of the phases are defined by smoothly varying the identity matrix to A while counting the number of times the eigenvalue winds around the origin). We get constraints on H due to these facts. It must be integer valued. Its eigenvalues λ_i must obey $|\lambda_i| = 1$. Their phases must lie symmetrically around zero, due to fermion number conjugation. In addition, there is an action of the braid group, corresponding to changes in couplings which alter the configuration of the vacua in the W plane (defined abstractly for non-Landau–Ginzburg theories – see the previous footnote) and hence the number

of solitons connecting them. Specifically, the Diophantine constraints are that the characteristic polynomial of the $n \times n$ matrix H,

$$P(z) = \det(z - H)$$

must be quasi-unipotent (i.e. $(H^m \pm 1)^{k+1} = 0$ for some m, k) and obey

$$P(z) = \prod_{m \in N} (\Phi_m(z))^{v(m)},$$

where $v(m) \in \mathbf{N}$ (non-negative integers) are almost all zero, and $\Phi_m(z)$ is the cyclotomic polynomial of degree equal to $\phi(m)$, the number of numbers relatively prime to m. Further [2],

1. $\sum_m v(m)\phi(m) = n$,
2. $v(1) \equiv 1 \bmod 2$,
3. $n \in 2\mathbf{Z} \Rightarrow$ either $v(1) > 0$ or $\sum_{k \geq 1} v(p^k) \equiv 0 \bmod 2$, for all primes p.

We are primarily interested in the sigma model case, for which the Ramond charges lie in the set $\{-d/2, -d/2 + 1, \ldots, d/2 - 1, d/2\}$, where d is the dimension of the Kähler manifold, M. Further, we restrict our attention to manifolds with diagonal Hodge numbers (or else the finite chiral ring would have nilpotent elements of non-zero fermion number, and no canonical basis – crucial to the derivations – would exist). In other words,

$$P(z) = (z - \varepsilon)^{\chi(M)},$$

where $\varepsilon = 1$ for d even and $\varepsilon = -1$ for d odd.

Let us now illustrate some solutions to these equations. The first example has an obvious physical interpretation; we will return to discuss the next example – affine Lie groups – in Sect. 2.4. The simply laced Lie groups are related to possible solutions for A as follows. Suppose the matrix $B = S + S^t$ is positive definite. Then $HBH^t = SS^{-t}(S + S^t)S^{-1}S^t = B$, which means that H is in the orthogonal group to the quadratic form, B, which tells us that H is simple and $|\lambda_i| = 1$. The simply laced Lie groups correspond to positive definite integral matrices through their Cartan matrices. B defines an inner product on $\mathbf{R}^n$, and if we take A to be upper triangular, with $A_{ij} = -B_{ij}/2$, $i < j$, then $H = (1 - A)(1 - A)^{-t}$ satisfies the Diophantine constraints. These matrices correspond to the $N = 2$ A-D-E minimal models, and have explicit realizations as Landau–Ginzburg theories. Weyl reflections of the lattice vectors produce different, though equivalent solutions to the Diophantine equations. These reflections correspond to perturbations of the superpotential, W, such that the vacua move through colinear configurations in the W plane. Such reconfigurations of the vacua produce a braid group action on the matrix H.

The affine Lie groups correspond to the case where $B = S + S^t$ has a single zero eigenvector, v, thus satisfying $S^t v = -Sv$. Then B defines a reduced matrix $\hat{B}$, on the orthogonal complement to $\mathbf{R}v$, which solves the Diophantine equations. We now note that

$$H^t v = S^{-1}S^t v = -S^{-1}Sv = -v,$$

so $|\lambda_v| = 1$ and we see that all the eigenvalues λ of H indeed have $|\lambda| = 1$. The remaining constraints on H are satisfied as well.

2.4. Mutations. We have already discussed a monodromy when one of the parameters of the theory is taken around the origin. In fact there are a number of discrete mutations or braidings of these theories. The treatment in this paper has been for general $N = 2$ theories, but this braiding is particularly intuitive in the case of Landau–Ginzburg models (all the results are valid in the general case). These models are described by a superpotential $W(X)$ and the vacua correspond to critical points X_i such that $\nabla W(X_i) = 0$. The locations of these points clearly depends on the parameters of W. Solitons, it is found, travel on straight lines in the W plane, and so a discrete shift in soliton number can occur when these vacua pass through a colinear configuration. This situation describes precisely the Picard–Lefschetz theory of vanishing cycles, (the inverse image of a point along the line in the W plane is a homology cycle, and solitons correspond to intersections of two homology cycles)[5] which undergo basis changes when crossed. Essentially, the change is

$$A_{ac} \rightarrow A_{ac} \pm A_{ab}A_{bc} \text{ (no sum)},$$

where the sign depends on the positive/negative orientation of the crossing. In a configuration in which the matrix S is upper triangular, the change of basis matrix implies

$$S \rightarrow PSP,$$

where $P = \begin{pmatrix} 0 & 1 \\ 1 & -S_{ij} \end{pmatrix}$ in the ij subsector. Note that P depends on S itself, and so the mutation is nonlinear.

We note here, too, that the canonical basis is only defined up to a sign. Further, reversing the orientation of the W plane – equivalently, taking the monodromy in the other direction – leads to $H \rightarrow H^{-1}$, i.e.

$$S \rightarrow S^t.$$

Thus, all matrices S obtained by any combination of the above transformations are related to the same (continuum class of theories associated to the) $N = 2$ quantum field theory.

Finally, we show how S and S^{-1} can be related. To the braid group of n objects, generated by $P_i, i = 1 \ldots n$, which denote braids of the ith object over the $(i + 1)$-th object, we define the element

$$v = P_1 P_2 P_1 P_3 P_2 P_1 \ldots P_{n-1} P_{n-2} \ldots P_2 P_1$$

consisting of $\binom{n}{2}$ transformations. This corresponds to reversing the orders of the elements. (Note that as a matrix v depends nonlinearly and nontrivially on S.) Then if $J = \delta_{i,n+1-i}$ is a reordering, we find

$$S^{-t} = JvSvJ.$$

In particular, S and S^{-1} are associated to the same $N = 2$ theory. We will use this point in our comparisons of results from math and physics.

---

[5] We note here that the mathematical theory to be discussed has been seen to parallel the Picard–Lefschetz theory as well, though a greater understanding of the relation is still unknown.

We conclude this subsection with a simple exercise. We take $S = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$.

Then P_1 is represented for this S by $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}$. One finds from $P_1 S P_1$ that the action is $P_1 : (a,b,c) \mapsto (-a,c,b-ac)$. Similarly, $P_2 : (a,b,c) \mapsto (b, a - bc, -c)$. The element $v = P_1 P_2 P_1$ then sends

$$(a,b,c) \mapsto (-a,c,b-ac) \mapsto (c, -a - c(b - ac), ac - b) \mapsto (-c, ac - b, -a).$$

Therefore, $v : S \to \tilde{S} = \begin{pmatrix} 1 & -c & ac - b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$, and it is simple to check that $J\tilde{S}J = S^{-t}$. Thus S and S^{-1} are related (taking the transpose, as discussed above).

Of course, for the general $N = 2$ theory there is no simple geometric interpretation of the placement of the vacua, though colinearity is still well-defined in terms of the $N = 2$ algebra. Still, the question of how perturbations of the theory affect the vacua is quite subtle. In addition, some of the perturbations may not make sense physically. For example, perturbations by nonrenormalizable terms are not allowed. We suspect that these phenomena are related to the question of constructability of helices, an issue we will return to from the mathematical viewpoint in Sect. 3.

2.5. Examples: *Projective Spaces, Grassmannians, and Orbifolds.* One of the few cases in which we can compute the soliton matrix by studying tt^* asymptotics directly is the simple case of the projective line $\mathbf{P}^1$. This theory has two chiral ring elements corresponding to its cohomology. Let us label them 1 and X. The quantum ring is $X^2 = \beta \equiv e^{-A}$, where A is the area of the $\mathbf{P}^1$, complexified so as to include the θ angle. Briefly, this comes about as follows. X has a non-vanishing one-point topological correlation (which we normalize to one), since there is a unique constant map taking the insertion point on the sphere to the chosen point on $\mathbf{P}^1$ representing X, the volume form. The only non-vanishing three-point correlator is $\langle XXX \rangle = \beta$, which has nonzero contribution at instanton number one arising from the unique holomorphic map from the (Riemann surface) sphere to the (target space) sphere taking the three insertion points to three specified points. This gives $X^2 = \beta$.

The metric $g_{a\bar{b}}$ is diagonal, due to a $\mathbf{Z}_2$ symmetry which is the leftover of the anomalous $U(1)$ symmetry (the chiral $U(1)$ is broken to $\mathbf{Z}_n$ on $\mathbf{P}^{n-1}$). This tells us that $\langle \bar{1} | X \rangle = \langle \bar{X} | 1 \rangle = 0$. Thus $g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, with a and b real, as g is hermitian. The metric η_{ab} is given by $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, since only X has a non-vanishing one-point correlation. The reality constraint, or CPT, tells us

$$\eta^{-1} g (\eta^{-1} g)^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \mathbf{1},$$

which gives $b = a^{-1}$.

We write the tt^* equations for variations with respect to β and $\bar{\beta}$. We thus need the matrix C_β, corresponding to the operator represented by varying β. Clearly A is the coefficient for X, and so $C_\beta = C_X \left(\frac{\partial A}{\partial \beta} \right) = -\frac{1}{\beta} \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}$; also $g C_\beta^\dagger g^{-1} = \begin{pmatrix} 0 & \bar{\beta} a^2 \\ a^{-2} & 0 \end{pmatrix}$. The tt^* equation is then

$$\partial_{\bar{\beta}} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \partial_\beta \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \frac{1}{|\beta|^2} \left[\begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bar{\beta} a^2 \\ a^{-2} & 0 \end{pmatrix} \right]. \qquad (2.15)$$

Both nontrivial components of (2.15) are equivalent. Further, a only depends on the absolute value of β, since the phase can be absorbed by a redefinition of the phase of the fermions [3]. Let $x = |\beta|$ and define $u = \ln(a^2 x)$. Then (2.15) gives

$$u'' + \frac{1}{x} u' = 4 \sinh u .$$

As often happens, requiring finiteness of the metric at $x = 0$ fixes the metric (i.e. one boundary condition is enough to impose). We need

$$u \to \ln x + c$$

as $x \to 0$. The asymptotic behavior at $x \to \infty$ has been analyzed in [11]. We compute the soliton number from the asymptotics of g in the canonical basis $O_\pm = (X \pm \sqrt{\beta})$. Extracting the lone soliton number A_{+-} from (2.13), we find

$$S = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}; \qquad H = SS^{-t} = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} .$$

Note that $\det(z - H) = (z + 1)^2 = \Psi_2(z)^2$ (Ψ_2 denotes the second cyclotomic polynomial), which gives the Ramond charges $N + \frac{1}{2}$. The integer part, N, can be determined by recording how many times the phases of the eigenvalues of $H(t)$ wrap around the origin as $S(t) = \begin{pmatrix} 1 & -2t \\ 0 & 1 \end{pmatrix}$ runs from the identity matrix to S while t spans the interval. One easily calculates that the two phases are

$$\theta_\pm = \pm \tan^{-1} \frac{2t\sqrt{1 - t^2}}{1 - 2t^2} ,$$

and so the charges are $\pm\frac{1}{2}$ as they should be. The braid group action is simple in this case. There is one mutation, P_1, which sends the matrix element $S_{12} \to -S_{12}$ (this can also be effected by a change of sign).

The classification program, in all its glory, has been illustrated by this simple example. Other spaces, such as the higher projective spaces and Grassmannians, are too unwieldy for a direct analysis. Too little is known about the solutions to the tt^* equations, which for $\mathbf{P}^n$ correspond to affine Toda equations. Perhaps the proposed relation between math and physics is best borne out by a rigorous analysis of these equations and their asymptotic properties.

Fortunately, the soliton numbers for $\mathbf{P}^{n-1}$ are computable by other methods, and the Grassmannians $G(k, N)$ (k-planes in $\mathbf{C}^N$) may be analyzed as well. First let us consider $\mathbf{P}^{n-1}$. These theories have an anomalous $U(1)$ charge (which is evident in the chiral ring, which has the simple form $X^n = \beta$). Instantons break this symmetry down to $\mathbf{Z}_n$. We can choose a basis for the vacua such that the $\mathbf{Z}_n$ symmetry cyclically rotates the n vacua. The soliton matrix μ_{ab} then depends

only on the difference $b - a$. Further, one can directly analyze the properties of the Stokes matrix by studying possible asymptotic solutions to (2.11) and (2.12) [2]. These considerations allow us to write the monodromy matrix H as $H = AB$, where A commutes with H. This condition allows us to conclude that the characteristic polynomial of B must also contain only products of cyclotomic polynomials. The real information, however, comes in the fact that B encodes the soliton numbers as well.

To see how this works, let us simply note that the Ramond charges of $\mathbf{P}^{n-1}$ have the form $q_k = k - (n - 1)/2$ and are thus half-integral or integral when n is even or odd, respectively. We therefore have

$$\det(z - H) = (z \pm 1)^n$$

for n even/odd. As a result, we conclude that, for n odd, say, $\det(z - B) = (z - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k z^k$, from which we may conclude

$$S_{ij} = (-1)^{j-i} \binom{n}{j - i} . \tag{2.16}$$

The minus signs may be removed by a redefinition of vacua $\mathrm{e}_a \to (-1)^a \mathrm{e}_a$, but we shall leave them in. Similarly, for n even we get the same result without the minus signs. We can reinsert them by performing the same change of basis.

These models have the special property of being integrable, for a special choice of the Kähler metric; i.e., they have an infinite number of conserved quantities such that the momenta of solitons are only permuted by interactions. Interactions of solitons in integrable models can be computed using the thermodynamic Bethe ansatz. Such an analysis shows the lightest solitons appear in fundamental multiplets of the original $SU(n)$ symmetry. These represent the n solitons of $\mu_{i,i+1}$. The interpretation of the other solitons of greater fermion number is that they are particles formed by anti-symmetric combinations of these n solitons [12]. Witten considered these solitons in Ref. [13], in which he formulated the $\mathbf{P}^{n-1}$ model by gauging a $U(1)$ action on fields $n^i \in \mathbf{C}^n$ constrained by Lagrange multipliers to satisfy $|n|^2 = 1$. The effective theory of the Lagrange multipliers and gauge field relates the topological charge of solitons to the $U(1)$ charge of the gauge field via the Gauss' law equation of motion. These n fields n^i are the fundamental solitons connecting neighboring vacua, and clearly transform under the fundamental representation of $SU(n)$. Other solitons are (anti-symmetric) composites of these fields.

The Grassmannian case is more subtle, and we must use a different technique, discussed in Sect. 8.2 and Appendix A of Ref. [2]. The Grassmannian $G(k, N)$ of k-planes in $\mathbf{C}^N$ is a complex manifold of dimension $k(N - k)$, and can be identified with the homogeneous space $U(N)/(U(k) \times U(N - k))$. Cecotti and Vafa have shown how to relate the observables to k copies of $\mathbf{P}^{N-1}$. The prescription is to take as vacua fully antisymmetric tensor products of k vacua for $\mathbf{P}^{N-1}$. There are $\binom{N}{k}$ such choices, equal in number to the Euler class of $G(k, N)$. The inner product of two vacua is then given in terms of the constituent $\mathbf{P}^{N-1}$ vacua. In this manner, the Grassmannian case can be reduced to the projective spaces. However, in the Grassmannian case, there is an ambiguity in asking what the soliton numbers are, as various vacua are aligned in the W plane. Such a configuration can lead to non-integer entries in the matrix S yielding monodromy matrices H which do not satisfy the classification equations. In the case of the Grassmannians, the matrix so-obtained does not satisfy the Diophantine equations, presumably for this very

reason. To fully resolve this difficulty, one would need to perturb the model so that the vacua were configured with no three of them colinear.

Orbifolds of $\mathbf{P}^1 \cong S^2$ by discrete subgroups of $SO(3)$ are interesting cases in which the tt^* equations may be explicitly used to compute soliton numbers. This procedure was employed in [14] and [15]. What makes these theories workable is that the orbifold theory possesses a symmetry which simplifies the metric, $g_{a\bar{b}}$. The only new subtlety is in the computation of the quantum rings, which are obtained by an analysis of equivariant holomorphic maps from covering surfaces, branched over the insertion points of twisted observables. This theory was developed in [16] and [17].[6]

The case of the dihedral groups yields the following results, easily generalizable though not yet proved for the tetrahedral, dodecahedral, and icosahedral groups [17]. A discrete subgroup G of $SO(3)$ lifts under the $\mathbf{Z}_2$ covering $SU(2) \to SO(3)$ to a subgroup $\tilde{G}$ of $SU(2)$, which is associated to a Dynkin diagram in the following way, due to McKay [18]. The fundamental representation of $SU(2)$ defines a two-dimensional representation R of any subgroup. If we label the irreducible representations of $\tilde{G}$ by V_i, we can define a matrix A_{ij} by the tensor decomposition

$$V_i \otimes R \cong \bigoplus_j N_{ij} V_j \, .$$

McKay's theorem states that the matrix N is the adjacency matrix of a Dynkin diagram of an affine Lie algebra. As N is symmetric with zeroes on the diagonal, we can write $N = A + A^t$, where A is upper triangular. For example, to compute the matrix for the $\mathbf{Z}_N$ orbifold of a sphere, we compute the matrix N for the double cover $\mathbf{Z}_{2N}$. The Dynkin diagram corresponds to the Lie group $\tilde{A}_{2N-1}$, and looks like a circular chain of $2N$ dots. For the dihedral groups D_N, the matrix A obtained from this procedure yields the Dynkin diagram for the affine Lie group $\tilde{D}_{N+2}$ (we should not be disconcerted by the mismatch of numbers; there need be no relation). This analysis is verified physically by computing asymptotics of $g_{a\bar{b}}$ from the tt^* equations.

Of course, we have already proven that these affine Dynkin diagrams yield solutions to the classification constraints. The discussion here allows us to identify these solutions as orbifolds.

3. The Math

3.1. Overview. We have stressed that no link has been found between the physics we just discussed and the mathematics we will introduce, so our motivation is indirect. Nevertheless, the evidence that some link exists is compelling. Certainly, none will be found without a thorough understanding of the structures at hand.

The evidence is the following. In the previous section we constructed, given a Kähler manifold with positive first Chern class (to guarantee asymptotic freedom, so that the quantum field theory makes sense) and diagonal Hodge numbers (so that a canonical basis exists), a quasi-unipotent matrix satisfying certain Diophantine

---

[6] The orbifold of a quantum field theory – a model with target space M/G, where G is a discrete group acting on M, includes states in the Hilbert space which correspond to strings running between points on M related by an element of G. The holomorphic maps, or instantons, to this singular space (as G may have fixed points) are analyzed by studying holomorphic maps between G-covers, which are equivariant with respect to the G action.

equations regarding its eigenvalues. This matrix – the matrix of (properly counted) soliton numbers between vacua – had an action of the braid group on it. The new matrices so constructed also satisfied the equations in question. Here we will start with a topological space and consider sheaves over that space (for the time being we can think of the sheaves as vector bundles). We will construct a set of "basis" sheaves, i.e. a set such that all other sheaves are "equivalent" to (i.e. have a resolution in terms of) a direct sum of sheaves from the basis set. Now from this set we simply consider the bilinear form which is the Euler character between two sheaves, i.e. the alternating sum of dimensions of cohomology classes. This matrix will be quasi-unipotent. The choice of basis set will not be unique, and the different choices will yield different matrices which satisfy the same properties. Further, we will be able to show for the projective spaces that the matrices are *exactly the same* as in the physics case. We will also explore other examples which have not yet been solved physically.

To make this clear, let us learn about coherent sheaves, helices, and braiding. Finally, we will look at some examples – projective spaces and Grassmannians – in detail.

3.2. Coherent Sheaves. We begin with a brief discussion of sheaves, following a brief summary of the algebraic geometry we will need. The treatment here borrows liberally from Ref. [9]. Heuristically, we can think of a sheaf as the generalization of a vector bundle when we replace a vector space by an abelian group. Thus, given a topological space, X, a sheaf $\mathscr{F}$ on X gives a set of sections $\mathscr{F}(U)$ which are abelian groups with the following properties. Given open sets $U \subset V$ we have a restriction map $r_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ satisfying for all $U \subset V \subset W$, $\sigma \in \mathscr{F}(A)$, $\tau \in \mathscr{F}(B)$,

- $r_{V,U} \circ r_{W,V} = r_{W,U}$ (thus we can write $\sigma|_U$ for $r_{V,U}\sigma$)

- $\sigma|_{A \cap B} = \tau|_{A \cap B} \Rightarrow \exists \rho \in \mathscr{F}(A \cup B)$ s.t. $\rho|_A = \sigma$, $\rho|_B = \tau$.

Roughly speaking, this says that sections are determined by their values on open sets.

The most important example of a sheaf for us will be $\mathscr{O}_n$, the sheaf of holomorphic functions in n variables (usually, we will drop the subscript n). Thus, $\mathscr{O}(U)$ consists of the ring of holomorphic functions on $U \subset \mathbf{C}^n$. Clearly $\mathscr{O}(X)$, when X is a compact complex manifold, is equal to the globally holomorphic functions, or constants.

The algebra of sheaves will be important to us, so we briefly review the pertinent aspects. We should have little difficulty with the constructions, as they closely parallel manipulations of abelian groups or vector bundles. A map between sheaves

$$f : \mathscr{F} \to \mathscr{G}$$

over X is a collection of group homomorphisms[7] $f_U : \mathscr{F}(U) \to \mathscr{G}(U)$. As with abelian groups, given such a sheaf map we can define its kernel and cokernel. The sections of the kernel are simply the kernels of the section maps f_U. Thus, $\mathrm{Ker}(\mathscr{F})(U) = \mathrm{Ker}(f_U)$. The cokernel is slightly subtle. We cannot take the naïve

---

[7] Recall a homomorphism f between abelian groups A, B is a map commuting with the group addition: $f(a + b) = f(a) + f(b)$

definition of Cok, which would read $\text{Cok}(\mathscr{F})(U)\,``="\,\text{Cok}(f_U)$, as it does not satisfy the properties required of sheaves. For example, consider the sheaves $\mathscr{O}$ and $\mathscr{O}^*$ of holomorphic and non-vanishing holomorphic maps on $\mathbf{C} - 0$. The abelian groups are addition and multiplication, respectively. Then the exponential map $\exp: h(z) \mapsto e^{2\pi i h(z)}$ is a homomorphism, with kernel $\mathbf{Z}$, the sheaf with integer-valued sections. Generalizing the notions of algebra, we would like to have the sequence

$$0 \to \mathbf{Z} \xrightarrow{i} \mathscr{O} \xrightarrow{\exp} \mathscr{O}^* \to 0 \tag{3.1}$$

be exact, i.e. have $\mathscr{O}^* = \mathscr{O}/\mathbf{Z} = \text{Cok}(i)$. However, with our simple definition, the cokernel is not even a sheaf. Consider the section $h(z) = z \in \mathscr{O}^*(\mathbf{C} - 0)$ which is not in the image $\exp(\mathscr{O}(\mathbf{C} - 0))$. Clearly, its restrictions to the contractible sets $U_1 \equiv \{-\varepsilon < \text{Arg}\,z < \pi + \varepsilon\}$ and $U_2 \equiv \{\pi - \varepsilon < \text{Arg}\,z < \pi\}$ are in the image of $\exp: \mathscr{O}(U_i) \to \mathscr{O}^*(U_i)$, $i = 1, 2$. Thus the second condition above is not satisfied. To remedy this, we define the sections of the cokernel of f over U to be given by a set of sections σ_α of a cover $\{U_\alpha\}$ of U satisfying

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in f(\mathscr{F}(U_\alpha \cap U_\beta)) \,.$$

We then identify two sections (i.e. collections $(U_\alpha, \sigma_\alpha)$ and (V_β, ρ_β)) if at all points p in U and open sets U_α, V_β containing p, there exists a neighborhood $N \subset U_\alpha \cap V_\beta$ such that the restrictions to this neighborhood of σ_α and ρ_β differ by the image of a section on N. This procedure allows the cokernel to be defined on open sets which would otherwise be "too big," by allowing the equivalence to be true up to refinements. It can be checked that the sequence (3.1) is now exact for any complex manifold, i.e. $\mathscr{O}^* = \text{Cok}(i)$.

Generally, a sequence of sheaves

$$\ldots \mathscr{F}_{n-1} \xrightarrow{f_{n-1}} \mathscr{F}_n \xrightarrow{f_n} \mathscr{F}_{n+1} \to \ldots$$

is said to be exact if $f_n \circ f_{n-1} = 0$ and

$$0 \to \text{Ker}(f_{n-1}) \to \mathscr{F}_{n-1} \to \text{Ker}(f_n) \to 0$$

is exact for all n. We will henceforth assume that a fine enough cover (U_α) exists such that the subtleties discussed are erased, i.e. such that each induced sequence of sections over U_α is exact.

We are interested in topological properties of the sheaves we will consider. Sheaf cohomology measures global properties of sheaves by comparisons on intersections of a cover. Let (U_α) be a cover of a manifold, M, and $\mathscr{F}$ a sheaf over M. Then we define the sheaf cohomology by taking the cohomology of the following complex. Define C^m to be the disjoint union of the sections of all $(m+1)$-fold intersections of the U_α:

$$C^m(U, \mathscr{F}) = \prod_{\alpha_i \neq \alpha_j} \mathscr{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_m}) \,.$$

Then the coboundary $\delta: C^m(U, \mathscr{F}) \to C^{m+1}(U, \mathscr{F})$ is defined by

$$(\delta \sigma)_{\alpha_0, \ldots, \alpha_{m+1}} = \sum_{j=0}^{m+1} (-1)^j \sigma_{\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_{m+1}} |U_{\alpha_0} \cap \cdots \cap U_{\alpha_{m+1}} \,.$$

The sheaf cohomology $H^*(M, \mathscr{F})$ is just the cohomology $\mathrm{Ker}(\delta)/\mathrm{Im}(\delta)$ of this complex, provided we choose an appropriately fine cover.[8] As an example, we observe that $H^0(M, \mathscr{F}) = \mathrm{Ker}(\delta) \subset C^0$, which is the set of collections $\{\sigma_\alpha\}$ obeying $(\delta\sigma)_{\alpha\beta} = \sigma_\beta - \sigma_\alpha = 0$ (with the restriction to $U_\alpha \cap U_\beta$ understood). This is precisely the data which determines a global section. Thus, $H^0(M, \mathscr{F}) = \mathscr{F}(M)$. We note that this property is independent of the covering.

Sheaves differ from vector bundles mainly due to the fact that the abelian groups involved, i.e. $\mathscr{F}(U)$, need not be free. In the cases we will study, all our sheaves will be $\mathcal{O}$-modules. Thus, the sections admit multiplication by locally holomorphic functions. In this case, and if its stalks are finitely generated (the condition is actually slightly different from this, as we will see), then we can the treat the sheaves as we would ordinary modules. As the sheaves are generally not made up of free abelian groups (and are thus not vector bundles), defining notions similar to the Euler characteristic will be quite subtle. To this end, we will briefly review homological properties of commutative algebra before discussing how to generalize these concepts to sheaves.

In the following, we will describe point-wise and then global constructions. So to begin, instead of considering sheaves as $\mathcal{O}$-modules, we will consider modules of the ring

$$O_n \equiv \lim_{\{0\} \in U} \mathcal{O}(U), \quad U \subset \mathbf{C}^n .$$

Thus, O_n is the ring of convergent power series in $\vec{z}$. Some facts about this ring. It has a unique maximal ideal equal to the (power series of) functions vanishing at the origin (they clearly are an ideal since $f(0) = 0 \Rightarrow f \cdot g(0) = 0$). The ring O_n is also Noetherian, meaning all ideals are finitely generated.

The sheaves we will consider are global versions of O_n-modules, which for us will always be finitely generated (as O_n-modules; they may be infinite dimensional vector spaces). Any O_n-module M defines a module of relations, R as follows. If $\{m_1, \ldots, m_k\}$ is a set of generators, then

$$R = \{(\lambda_1, \ldots, \lambda_k) : \lambda_1 m_1 + \cdots + \lambda_k m_k = 0\} .$$

R, it can be shown, is also finitely generated. We then have that the sequence of O_n-modules,

$$0 \to R \to O_n^{(k)} \to M \to 0 ,$$

is exact, where $O_n^{(k)} \equiv O_n \oplus O_n \oplus \cdots \oplus O_n$ (k times). The global analogue of finite-dimensionality is the notion of a coherent sheaf. A coherent sheaf is one which has a local presentation

$$\mathcal{O}^{(p)} \to \mathcal{O}^{(q)} \to \mathscr{F} \to 0 .$$

By "local presentation," we mean that for each point p there exists a neighborhood $U \ni p$ such that the above sequence is an exact sequence of modules when restricted to U. The $\mathcal{O}^{(q)}$ means that $\mathscr{F}|_U$ is finitely generated (not just the stalk $\mathscr{F}|_p$) and the $\mathcal{O}^{(p)}$ means that there are a finite number of relations among these generators.

---

[8] The actual sheaf cohomology is defined as a limit of the stated cohomologies under refinements of the cover. If the cover is *acyclic*, meaning the cohomology of any multiple intersection is trivial (for example if they are contractible, for sheaves of holomorphic p-forms), then the sequence yields the proper sheaf cohomology.

356

E. Zaslow

The gist of this definition is that it allows us to carry properties of finite-dimensional modules over to sheaves.

Of course we have the usual properties of modules. Given two O_n-modules M and N, we can construct the O_n-modules $M \oplus N$, $M \otimes_{O_n} N$, and $\text{Hom}_{O_n}(M,N)$. Note that tensoring and Hom do not necessarily preserve exactness. We have instead the following. Given the exact sequence

$$0 \to P \to Q \to R \to 0$$

of O_n-modules, and an O_n-module M, we have the following exact sequences:

$$P \otimes_{O_n} M \to Q \otimes_{O_n} M \to R \otimes_{O_n} M \to 0 \,,$$

$$0 \to \text{Hom}_{O_n}(M,P) \to \text{Hom}_{O_n}(M,Q) \to \text{Hom}_{O_n}(M,R) \,. \tag{3.2}$$

The maps are the obvious ones: e.g., if $\varphi : P \to Q$ then $\tilde{\varphi} : \text{Hom}_{O_n}(M,P) \to \text{Hom}_{O_n}(M,Q)$ sends f to $\varphi \circ f$. The operations (functors) on complexes of $\otimes_{O_n} M$ and $\text{Hom}_{O_n}(M,*)$ are said to be right exact and left exact, respectively. Naïvely, we would expect the sequences in (3.2) to extend to a short exact sequence, and indeed this is the case if the module M is projective ($\Leftrightarrow$ free; we discuss such modules shortly). The same functors apply in the category of sheaves, as well.

It is instructive to study how these functors can fail to be exact. This situation can arise when the module M – or sheaf in the global case – is not locally free. Consider the ideal $I \subset O_1$ generated by z^m. Then we have the exact sequence of O_1-modules

$$0 \to I \xrightarrow{i} O_1 \xrightarrow{\pi} O_1/I \to 0 \,. \tag{3.3}$$

Clearly O_1/I is generated by $\{1, z, \ldots, z^{m-1}\}$ and is an O_1 module in the obvious way. Let us now apply $\otimes_{O_1} O_1/I$ to this sequence to get

$$I \otimes_{O_1} O_1/I \xrightarrow{\tilde{i}} O_1 \otimes_{O_1} O_1/I \xrightarrow{\tilde{\pi}} O_1/I \otimes_{O_1} O_1/I \to 0 \,.$$

If this is not exact, then $\tilde{i}$ has a kernel.[9] Let us enumerate the generators. $\{z^m \otimes z^j, j < m\}$ generate $I \otimes_{O_1} O_1/I$, since $z^{m+a} \otimes z^b \sim z^m \otimes z^{a+b} = 0$ for $a + b \geqq m$. Note that we cannot move the z^m across the tensor product since z^m cannot be written as something in I times something in O_1 other than a scalar.[10] The generators for $O_1 \otimes_{O_1} O_1/I$ are $\{1 \otimes z^j, j < m\}$, since now any zs on the left side can be moved over by the tensor equivalence. Under the map $\tilde{i}$, $z^m \otimes z^j$ is mapped to $z^m \otimes z^j \sim z^m \cdot 1 \otimes z^j \sim 1 \otimes z^{m+j} = 0$. Hence the map $\tilde{i}$ is trivial, and $\text{Ker}(\tilde{i}) = I \otimes_{O_1} O_1/I$. The nontriviality of the kernel clearly had to do with the torsion of O_1/I, or the existence of zero divisors. It is also instructive to check exactness at the middle of this sequence, and to find the non-surjectivity of the right-hand side of (3.2) when we apply $\text{Hom}(O_1/I, *)$ to the original sequence (3.3). In what follows, we will define modules which measure the non-exactness of the functors $\otimes M$ and $\text{Hom}(M, *)$.

---

[9] The surjectivity of the map $\tilde{\pi}$ follows from surjectivity of π. For $a \otimes b \in O_1/I \otimes_{O_1} O_1/I$, we have $a = \pi(p)$ for some p and thus $a \otimes b = \tilde{\pi}(p \otimes b)$. The property is clearly true in general.

[10] Recall the definition of the tensor product of R-modules: $A \otimes_R B \equiv (A \times B)/I$, where I is the ideal generated by elements $(ra, b) - (a, rb)$.

Important to us will be the notions of projective modules and projective resolutions. A projective module is one for which the following diagram holds:
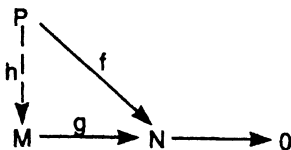
That is, given surjective maps f and g, there is a map h such that $g \circ h = f$. Another way to put this is that $M \to N \to 0$ is exact implies $\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0$ is also exact. Thus for projective modules P, $\mathrm{Hom}(P, *)$ is a (right and left) exact functor. (The sequence could have been extended to a short exact sequence by adding $\mathrm{Ker}(g)$ on the left.) The same is true of $\otimes P$. It can be proven that projectivity of a module coincides exactly with its being free (no relations among the generators, hence isomorphic to a vector space). In the sheaf language, projectivity is defined similarly, and it coincides with a sheaf's being *locally* free, i.e. isomorphic to $\mathcal{O}^{(d)}$.

We will use projective modules to "resolve" general modules. In this way, the complexity of a module will be borne out in the cohomology of the resolution. A (left) projective resolution of a module M is an exact sequence

$$0 \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that each P_i is projective and $d_j \circ d_{j+1} = 0$. Every finite-dimensional module has a projective resolution. One builds it iteratively, beginning with the sequence of relations (3.3) and continues to use (3.3) on the kernel of the left-most term in the resolution.

When discussing sheaves, we call a resolution by locally free sheaves a syzygy. Coherent sheaves have syzygies for the same reasons as above. Given a syzygy of some sheaf E, we can analyze the complexity of the sheaf, loosely speaking, by measuring the extent to which $\mathrm{Hom}(E, *)$ fails to be exact, for example. To do this we define Ext as follows, first for modules. Give O-modules M, N, we can first construct a resolution $\{P_i\}$ of M then create the complex $C^{\cdot}$:

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \mathrm{Hom}(P_2, N) \to \cdots \qquad (3.4)$$

(not exact). We then define

$$\mathrm{Ext}^i(M, N) = H^i(C^{\cdot}) \, .$$

To define the vector spaces $\mathrm{Ext}^i(\mathscr{F}, \mathscr{G})$ (which can be thought of as trivial sheaves) we first resolve the sheaf $\mathscr{F}$ with a syzygy $\{\mathscr{P}_i\}$. We note that the stalk of the sheaf, $\mathscr{F}_p$, at a point p is nothing other than an O module; likewise for $\mathscr{G}$. The sheaf $\underline{\mathrm{Ext}}^i(\mathscr{F}, \mathscr{G})$ has the natural property that $\underline{\mathrm{Ext}}^i(\mathscr{F}, \mathscr{G})_p = \mathrm{Ext}^i(\mathscr{F}_p, \mathscr{G}_p)$ and is built by applying Ext to the local syzygy. The global version, $\mathrm{Ext}^i(\mathscr{F}, \mathscr{G})$, must be defined more carefully. We have a complex of sheaves $\{\mathrm{Hom}(\mathscr{P}_i, \mathscr{G})\}$, analogous to (3.4), which gives rise to a double complex. The horizontal differential is just the natural one from the complex, while the vertical direction is defined by the Čech

cohomology of the sheaves, reviewed in this section. $\text{Ext}^i(\mathcal{F}, \mathcal{G})$ is the ith term of the cohomology of the total complex of this double complex.[11] We note without proof that if $\underline{\text{Ext}}^q(\mathcal{F}, \mathcal{G}) = 0$ for $q < k$ then $\text{Ext}^k(\mathcal{F}, \mathcal{G})$ is equal to the global sections of $\underline{\text{Ext}}^k(\mathcal{F}, \mathcal{G})$.

3.3. Exceptional Collections and Helices.

A coherent sheaf E over a variety M is called *exceptional* if

$$\text{Ext}^i(E, E) = 0, \quad i \geq 1,$$

$$\text{Ext}^0(E, E) \cong \mathbf{C}, \quad i = 0.$$

In the case of sheaves over projective spaces, these conditions imply that E is locally free, i.e. we can think of E as the sheaf of sections of some vector bundle. We sometimes refer to E itself as a vector bundle.

An ordered collection of exceptional sheaves $\varepsilon = (E_1, \ldots, E_k)$ is called an exceptional collection if for all $1 \leq m < n \leq k$ we have

$$\text{Ext}^i(E_n, E_m) = 0, \quad i \geq 0.$$

The most important property that exceptional collections enjoy is that they can be transformed to get new exceptional collections. We will define right and left transformations or mutations. Together, they will represent an action of the braid group on the set of possible such collections.

Mutations of collections by the action of the braid group are performed by making replacements of neighboring pairs. Take a neighboring pair (E_i, E_{i+1}) of sheaves in an exceptional collection, $\varepsilon = (E_1, \ldots, E_n)$. Suppose the following condition is true: $\text{Hom}(E_i, E_{i+1}) \neq 0$ and

$$\text{Hom}(E_i, E_{i+1}) \otimes E_i \xrightarrow{\text{ev}} E_{i+1} \to 0$$

is exact (i.e., ev is surjective), where the map ev is the canonical one. Then we can define a new sheaf $L_{E_i} E_{i+1}$ to be the kernel of this map:

$$0 \to L_{E_i} E_{i+1} \to \text{Hom}(E_i, E_{i+1}) \otimes E_i \to E_{i+1} \to 0.$$

For brevity, we usually write this new sheaf as LE_{i+1}. Thus we write $L^2 E_{i+1}$ for $L_{E_{i-1}}(L_{E_i} E_{i+1})$, etc. If we then replace the pair (E_i, E_{i+1}) by (LE_{i+1}, E_i) in the exceptional collection, we have the following [20].

Theorem. *The new collection* $\varepsilon' = (E_1, \ldots, E_{i-1}, LE_{i+1}, E_i, E_{i+2}, \ldots, E_n)$ *is exceptional.*

Sometimes the canonical map $\text{Hom}(E_i, E_{i+1}) \otimes E_i \to E_{i+1} \to 0$ is not surjective. If, however, it is injective, we can define LE_{i+1} to be the cokernel of this map instead of the kernel, and the theorem is still true. Finally, if $\text{Hom}(E_i, E_{i+1}) = 0$ but $\text{Ext}^1(E_i, E_{i+1}) \neq 0$, we can define the left mutation $L_{E_i} E_{i+1}$ to be the universal extension, defined by its property of making the sequence

$$0 \to E_{i+1} \to L_{E_i} E_{i+1} \to \text{Ext}^1(E_i, E_{i+1}) \to 0 \tag{3.5}$$

exact.

---

[11] Given a double complex C^{ij}, $i, j > 0$, with horizontal differential, d and vertical differential δ obeying $d\delta + \delta d = 0$ (if d and δ commute, we can redefine the signs of odd d's so the two anti-commute), we define the total complex $C^n = \oplus C^{n-i, i}$ with differential $D = d + \delta$.

To demonstrate how ev : $\text{Hom}(A,B) \otimes A \to B$ may not be surjective, consider the simple example of modules (not sheaves) on the complex plane, $\mathbf{C}$. Define $A = \mathcal{O}/I_2$, $B = \mathcal{O}/I_4$, where I_m is the ideal generated by z^m. Then $\{1,z\}$ generate A and $\{1,z,z^2,z^3\}$ generate B. Now $f \in \text{Hom}_{\mathcal{O}}(A,B)$ must satisfy $f(z^n a) = z^n f(a)$, and since $z^2 = 0$ in A, we have $z^2 f(a) = 0$ in B. Thus, $f(1) = c_1 z^2 + c_2 z^3, c_i \in \mathbf{C}$. Clearly, the image of ev is $\mathbf{C}z^2 \oplus \mathbf{C}z^3$, which is not all of B.

We can perform right mutations as well, under a different set of conditions. Since $\text{Hom}(E_i, E_{i+1})$ as a vector space has the indentity morphism in

$$\text{Hom}((\text{Hom}(E_i, E_{i+1}), \text{Hom}(E_i, E_{i+1})) \cong \text{Hom}(E_i, E_{i+1})^* \otimes \text{Hom}(E_i, E_{i+1}),$$

we get a canonical map from E_i to $\text{Hom}(E_i, E_{i+1})^* \otimes E_{i+1}$. To understand this, it may help to consider the finite-dimensional vector space example. Choose bases a_i, b_j for E_i, E_{i+1} and dual bases $\tilde{a}^i, \tilde{b}^j$ for E_i^*, E_{i+1}^*. Then the map sends $a \mapsto \sum_j (a \otimes \tilde{b}^j) \otimes b_j$. If this map is injective, then the right mutation is then defined by the cokernel as shown:

$$0 \to E_i \to \text{Hom}(E_i, E_{i+1})^* \otimes E_{i+1} \to R_{E_{i+1}} E_i \to 0 . \tag{3.6}$$

Similarly to the left mutation, we can define the right mutation if this map is surjective, or if $\text{Ext}^1(E_i, E_{i+1}) \neq 0$.

We note here – though it should not be seen as obvious – that if the exceptional pair (A,B) admits a left mutation, then the pair $L(A,B) = (LB,A)$ is exceptional (by the theorem above) and its right mutation is such that $R(LB,A) = (A,B)$. Thus, if the left operation is seen as a braiding (we have yet to show this), then the right mutation is an unbraiding. The right operation is dual in the following sense. If $\varepsilon = (E_1,\ldots,E_n)$ is exceptional then so is $\varepsilon^* \equiv (E_n^*,\ldots,E_1^*)$, and if ε admits a left transformation abbreviated $L\varepsilon$, then ε^* admits a right transformation. We then have $(L\varepsilon)^* = R\varepsilon^*$. If $R\varepsilon$ exists then so does $L\varepsilon^*$, and in that case $(R\varepsilon)^* = L\varepsilon^*$.

These mutations amount to an action of the braid group of n-objects on helices, as they obey the Yang–Baxter relations. Specifically, if L_i represents mutating (E_i, E_{i+1}) to $(L_{E_i}E_{i+1}, E_i)$ – i.e. a shift at the ith entry of the foundation – then

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1} \tag{3.7}$$

is satisfied. The same is true of the right shifts.

A *helix* is an infinite collection of coherent sheaves $\{E_i\}_{i \in \mathbf{Z}}$ such that

1) for any $i \in \mathbf{Z}$, $(E_{i+1},\ldots,E_{i+n})$ is an exceptional collection,

2) $R^{n-1} E_i = E_{i+n}$.

Thus when you move n steps to the right, you come back to where you were, up to a translation. This explains the terminology. It is clear that a helix is uniquely determined by any of its foundations, and conversely any exceptional collection determines a helix.

We are finally in a position to define the bilinear form which corresponds to the matrix S of soliton numbers. Let us assume we have a helix ε with foundation $(E_1,\ldots,E_n)$. In the case of bundles, this form is the relative Euler characteristic, and is defined generally to be

$$\chi(E_i, E_j) \equiv \sum_{k=0}^{n} (-1)^k \dim \text{Ext}^k(E_i, E_j) . \tag{3.8}$$

We took care to define this form even for sheaves which do not correspond to bundles. For vector bundles, though, computations are simplified by the following derivation. Since these sheaves E_ι are locally free (they are sections of a vector bundle), the syzygy or projective resolution is trivial: it has the form $0 \to P_0 \to E_i \to 0$, with $P_0 = E_i$. Since the complex $\{P\}$ just contains a single element, the double complex used to compute Ext reduces to a single complex, and $\mathrm{Ext}(E_i, E_j)$ becomes the ordinary sheaf cohomology of $\mathrm{Hom}(E_\iota, E_j)$. Thus

$$\dim \mathrm{Ext}^k(E_i, E_j) = \dim H^k(\mathrm{Hom}(E_\iota, E_j)),$$

and so

$$\chi(E_\iota, E_j) = \sum_k (-1)^k \dim H^k(\mathrm{Hom}(E_i, E_j))$$

$$= \sum_k (-1)^k \dim H^k(E_i^* \otimes E_j)$$

$$= \chi(E_\iota^* \otimes E_j)$$

$$= \int_M (\mathrm{ch}(E_j)/\mathrm{ch}(E_i)) \mathrm{td}(TM),$$

where in the second line we considered the sheaves as vector bundles, using the equivalence of sheaf and vector-valued cohomology (the de Rham theorem). The last line expresses the Euler characteristic in terms of characteristic classes of bundles in de Rham cohomology, and is equivalent to the Riemann–Roch theorem.

3.4. Examples: *Projective Spaces, Grassmannians, Orbifolds, and Blow-Ups.* The simplest spaces for us to consider are the projective spaces. We recall the connection between divisors and line bundles. A hyperplane is described by the zero locus of a linear polynomial in homogeneous coordinates (which is itself a section of a line bundle). Thus, up to isomorphism, all hyperplanes have the form $X_0 = 0$. To find the corresponding line bundle, look at the set $U_1 = \{X_1 \neq 0\}$. Then the hyperplane H is described by $z_0 \equiv X_0/X_1 = 0$. In $U_2 = \{X_2 \neq 0\}$, it is given by $w_0 \equiv X_0/X_2 = 0$, and on the intersection the two functions are related by a nonzero function: $z_0 = (X_2/X_1)w_0$, which defines a transition function between open sets. The set of all such transition functions determines a line bundle. More generally, a divisor defined by a function f_α in U_α and f_β in U_β defines holomorphic transition functions $s_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ by

$$f_\alpha = s_{\alpha\beta} f_\beta.$$

Different defining functions for equivalent divisors yield isomorphic line bundles. We find that the isomorphism classes of line bundles are given by the degrees of the defining polynomials (with negative degrees associated to divisors along poles), and so all line bundles can be written as powers of the hyperplane bundle described above. We denote the dth power of the hyperplane bundle by $\mathcal{O}(d)$.

Theorem. [20]. *The collection* $\{\mathcal{O}(m) : m \in \mathbf{Z}\}$ *of sheaves on* $\mathbf{P}^n$ *is exceptional, and* $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ *is a foundation of the helix. We define* $E_\iota \equiv \mathcal{O}(i)$.

To show that this collection is a helix, we have to study the mutations. Here we will just consider the first right mutation. Consider the following exact sequence of

sheaves:

$$0 \to \mathcal{O} \to V \otimes \mathcal{O}(1) \to T \to 0 \ .$$

Here V is the $n + 1$ dimensional vector space of which $\mathbf{P}^n$ is the projectivization (the notation V above thus signifies the trivial rank $n + 1$ vector bundle) and T is the holomorphic tangent space. This sequence is called the Euler sequence. Locally, for some choice of basis for V, sections of $V \otimes \mathcal{O}(1)$ look like $(f_0, \ldots, f_n)$ and are mapped to the vector field $\sum_i f_i \partial_i$ (easily checked to be well-defined when the f_i are in $\mathcal{O}(1)$), where we have the relation $\sum_i X_i \partial_i = 0$. The one-dimensional kernel is the image of the left map. We need to show that the maps are the canonical ones of (3.6) and that $V \cong \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1))^*$. We show only the latter. In fact, this is readily checked. The notation Hom stands for the global sections, here, and since $V \cong V^*$ as vector spaces, we need to show $\dim H^0 \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = n + 1$. Since Hom is taken over $\mathcal{O}$, we can identify $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) \cong \mathcal{O}(1)$. The global sections of $\mathcal{O}(1)$ are linear functionals on V, and there are $n + 1$ of them – the $n + 1$ coordinate functions, for example. Thus, the above sequence gives the right mutation and $R\mathcal{O} = T$.

Further right mutations (by "exterior products" of this sequence) show that as we move $\mathcal{O}$ right by mutations we find $R^k \mathcal{O} = \Lambda^k T$, and indeed $R^n \mathcal{O} = \Lambda^n T = \mathcal{O}(n+1)$, as it must [21].

Now that we have an exceptional collection of vector bundles, we may compute the bilinear form

$$\chi(E_i, E_j) = \chi(E_i^* \otimes E_j) \ .$$

The Todd class and the Chern characters of vector bundles are computed in terms of the Chern roots – two-forms computed from the splitting principle, or diagonalizing the curvature. For the tangent bundle $\mathbf{TP}^n$, each Chern root x_i is equal to x, the Kähler two-form. The Todd class is given by

$$\mathrm{td}(E) \equiv \prod_{i=1}^{r} \frac{x_i}{1 - \mathrm{e}^{-x_i}} = \frac{x^{n+1}}{(1 - \mathrm{e}^{-x})^{n+1}} \quad \text{for } T\mathbf{P}^n \ .$$

The Chern character is defined to be $\mathrm{ch}(E) = \sum \mathrm{e}^{x_i}$. We have

$$\mathrm{ch}(\mathcal{O}_{\mathbf{P}^n}(m)) = \mathrm{e}^{mx} \ .$$

Therefore,

$$\chi(E_i, E_j) = \chi(E_i^* \otimes E_j) = \chi(\mathcal{O}(i)^* \otimes \mathcal{O}(j))$$

$$= \chi(\mathcal{O}(j - i))$$

$$= \int_{\mathbf{P}^n} \mathrm{ch}(\mathcal{O}(j - i))\mathrm{td}(T\mathbf{P}^n)$$

$$= \int_{\mathbf{P}^n} \mathrm{e}^{(j-i)x} \frac{x^{n+1}}{(1 - \mathrm{e}^{-x})^{n+1}} \ .$$

The integrand is understood as a polynomial in the two-form x. For $\mathbf{P}^n$, we have $\int_{\mathbf{P}^n} x^n = 1$, so we simply need to extract the coefficient of x^n. To do this, we multiply

by x^{-n-1} and take the residue:

$$\chi(E_i, E_j) = \oint \frac{e^{(j-i)x}}{(1 - e^{-x})^{n+1}} dx$$

$$= \oint (1 - y)^{-(j-i)-1} y^{-n-1} dy, \quad y = 1 - e^{-x}$$

$$= \frac{1}{n!} \left(\frac{d}{dy}\right)^n (1 - y)^{-(j-i)-1}|_{y=0}$$

$$= \frac{1}{n!} ((j-i) + 1)((j-i) + 2) \ldots ((j-i) + n)$$

$$= \binom{n + (j-i)}{(j-i)} .$$

Indeed, this is related to the result we obtained for the (weighted) soliton numbers of the topological sigma model on $\mathbf{P}^n$. Specifically, we have found the inverse of the matrix (2.16). As we discussed in Sect. 2.4, a matrix and its inverse are equivalent under braiding. Therefore, the conjectured math-physics link has been demonstrated. What is more, the fundamental solitons of the physical theory were shown to be given by the coordinate functions of the $\mathbf{C}^{n+1}$ which fibers over $\mathbf{P}^n$. These are none other than the global sections of the bundle $\mathcal{O}(1)$. Note that

$$\dim H^0(\mathcal{O}(1)) = \dim \mathrm{Ext}^0(\mathcal{O}(k), \mathcal{O}(k+1)) = \chi(\mathcal{O}(k), \mathcal{O}(k+1)) .$$

Here, then, the physical and mathematical calculations are counting the same things! Unfortunately, the correspondence does not seem so direct for other examples.

Exceptional collections for Grassmannians and other flag manifolds were considered by M. M. Kapranov in [22] and in the sixth article in [5]. These authors were able to construct exceptional collections by relating vector bundles over homogeneous spaces to representations of the coset group in the standard way. Namely, on G/H we can define a vector bundle given any representation of H by the associated bundle of the principal H bundle $G \to G/H$.[12] The Grassmannian $G(k, N)$ of k-planes in $\mathbf{C}^N$ is equated with $U(N)/(U(k) \times U(N-k))$ (by considering that $U(N)$ acts transitively on the set of planes, with $U(k) \times U(N-k)$ fixing a given plane), and is thus a homogeneous space. We can take representations of $U(k)$ alone to define vector bundles. These representations are described by Young diagrams, where now we allow negative indices, as the totally antisymmetric representation acts by the determinant (which is trivial for $SU(k)$ but not for $U(k)$), which can be raised to any (positive or negative) power.

Kapranov has shown that an exceptional collection is defined by the Young tableaux with the property that all entries are non-negative and no row has more than $(N-k)$ elements. We note that there are $\binom{n}{k}$ such diagrams, equal in number to the Euler character and the dimension of the Grothendieck group (the Hodge diamond is diagonal).

In order to compute the bilinear form, we note that the tensor products of these line bundles are nothing other than the bundles associated to the tensor products

---

[12] Given a principal H bundle, G, with transition functions $h_{\alpha\beta}$ and a representation $\rho : H \to$ Aut(V) of H, we construct the associated rank $n = \dim V$ vector bundle by considering the transition functions $s_{\alpha\beta} = \rho(h_{\alpha\beta})$. Equivalently, we take the space $G \times_\rho V$ defined to be $G \times V$ modulo the relation $(gh, v) \sim (g, hv)$.

of the representations. Further, the dual bundle is described by the dual representation, defined by reversing the sign and order of the Young tableau indices (thus $(\alpha_1, \ldots, \alpha_k)^* = (-\alpha_k, \ldots, -\alpha_1)$). Therefore, in order to compute the Euler character $\chi(E, F)$ we just decompose the tensor product $E^* \otimes F$ and take the Euler character of each component separately. Kapranov [23] has shown that this quantity is nonzero only when all of the Young indices are non-negative, and equal to the dimension of the representation (as a representation of $U(N)$) in this case. Therefore, we have reduced the problem of computing this bilinear form to a question of representation theory.

Note, too, that there is a partial ordering on the representations in terms of inclusion of Young diagrams. As $E^* \otimes F$ contains positive parts only when E appears as a subdiagram of F, the upper-triangularity of the bilinear form follows immediately. These results are readily extended to the flag manifolds $U(N)/(U(n_1) \times \cdots \times U(n_r))$, $\sum n_j = N$. See Ref. [22].

Let us compute the first nontrivial example: the Grassmannian $G(2,4) \cong U(4)/ (U(2) \times U(2))$. The basis $\{e_1, \ldots, e_6\}$ for the bundles is given by the diagrams

$$(0,0), (1,0), (2,0), (1,1), (2,1), (2,2),$$

where the jth entry denotes the length of the jth row of the tableau. Note that the totally anti-symmetric $(1,1)$ diagram, for example, is trivial as a representation of $SU(2)$ but is in fact the one-dimensional representation in which a matrix in $U(2)$ acts by its determinant. We will do a sample computation of a matrix element in the bilinear form. Consider $\chi_{2,5} = \chi(e_2, e_5)$. We need to decompose

$$e_2^* \otimes e_5 = (0,-1) \otimes (2,1) = ((-1,-1) \otimes (1,0)) \otimes ((1,1) \otimes (1,0)) = (1,0) \otimes (1,0),$$

where in the last steps we have factored out the $(\det)^{\pm 1}$ representations (which cancel when tensored). Using the usual rules of decomposition, we find $(1,0) \otimes (1,0) \cong (2,0) \oplus (1,1)$, where the second summand is not trivial in $U(2)$. We find $\chi_{2,5} = \dim_{U(4)}(1,1) + \dim_{U(4)}(2,0)$, where the subscript indicates that we take the dimension of the representation as a representation of $U(4)$. Hence $\chi_{2,5} = 10 + 6 = 16$. Proceeding straightforwardly, we find

$$\chi = \begin{pmatrix} 1 & 4 & 10 & 6 & 20 & 20 \\ 0 & 1 & 4 & 4 & 16 & 20 \\ 0 & 0 & 1 & 0 & 4 & 10 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As it must, the matrix χ satisfies $\det(z - \chi \chi^{-t}) = (z-1)^6$. Other Grassmannian spaces – including projective spaces – can be computed similarly. We note that $G(N-1, N)$, equivalent to $G(1, N) = \mathbf{P}^{N-1}$, gives a basis for which the bilinear form is the same as in the physics case (no matrix inversion is necessary).

The case of orbifolds, discussed from the physical viewpoint in Sect. 2.5, has not been studied from the mathematical viewpoint in the context of helices. However, Bondal and Kapranov [24] have discussed the derived category of complexes of $\mathbf{Z}_3$-equivariant sheaves over $\mathbf{P}^1$, and have found collections of sheaves leading to results similar to those derived in Ref. [15]. The sheaves are no longer locally free, but include those associated to the fixed points of the $\mathbf{Z}_3$ action. Similar results

hold for any finite group, leading us to conjecture that the bilinear forms will be associated to Dynkin diagrams, as was previously discussed.

It is interesting to consider blow-ups of $\mathbf{P}^2$ at n points. We will encounter an example of such a space (when $n = 1$) in some detail in Sect. 4.4. For now, we just note that these spaces, which we shall denote $\widetilde{\mathbf{P}^2_n}$, have isomorphisms

$$\widetilde{\mathbf{P}^2_n} - \bigcup_{i=1}^{n} E_i \cong \mathbf{P}^2 - \bigcup_{i=1}^{n} \{p_i\} .$$

That is, each point p_i, $i = 1 \ldots n$ is replaced by an "exceptional divisor" E_i, isomorphic to $\mathbf{P}^1$. The isomorphism arises from a map $\pi : \tilde{\mathbf{P}}^2_n \to \mathbf{P}^2$, with $\pi^{-1}(p_i) = E_i$.

An exceptional collection can be defined for these spaces, though whether they generate helices or not is now known (for most examples; for $\tilde{\mathbf{P}}^2_1$, see Sect. 4.4). The collection is given by $\mathcal{O}(i)$, $i = 0 \ldots 2$ – sections of the ith power of the hyperplane bundle, pulled back under π – and the sheaves $\mathcal{O}_j$ $j = 1 \ldots n$, which have support only on E_j. That is $\mathcal{O}_j(U) = \mathcal{O}(U)/I(U)$, where I is the ideal of holomorphic functions vanishing on E_j. Clearly, $\mathcal{O}_j(U) = 0$ for $U \cap E_j = \emptyset$. One finds for the bilinear form (say, for $n = 3$) [25]:

$$\chi = \begin{pmatrix} 1 & 3 & 6 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

We note that $H = \chi \chi^{-t}$ is unipotent, yielding the correct charges (Hodge numbers) for the topological sigma model, including the integer parts (see $\mathbf{P}^1$ example in Sect. 2.5). Namely, the eigenvalues of H, all equal to unity, wrap around the origin an integral number of times when χ is given a t dependence and is taken from the identity to χ smoothly. The integer value is precisely the form degree. The Hodge numbers of the space are determined as follows: each exceptional divisor, isomorphic to $\mathbf{P}^1$, contributes one $(1,1)$ form, in addition to the $\mathbf{P}^2$ cohomology which pulls back from π; thus $h^{ij} = \text{diag}(1, 4, 1)$, which is what is found.

We note that if $n > 8$, then these spaces have negative first Chern classes, and therefore do not have a simple geometric interpretation as asymptotically free quantum field theories. Nevertheless, the mathematical constraints are satisfied. The difference may arise from the difference between an exceptional collection and a helix: admissibility of mutations and the "periodicity" requirement of the shifts R^n and L^n.

4. Links

4.1. Parallel Structures, Categorical Equivalence. One way of formalizing the parallel structures shared by topological field theories and exceptional collections is by describing a categorical equivalence between the two. In fact, one approach to the theory of helices is through their categorical definition. Many of the structures we have discussed are structures which can be defined given an abelian category and its derived category.

For those of us unfamiliar with categorical constructions, we recall only the very basics. One constructs a category of objects (e.g., sets, topological spaces, sheaves, groups, vector spaces, complexes) and composable morphisms (e.g., functions, continuous functions, maps of sheaves, homomorphisms, linear maps, morphisms of complexes). Categories may have additional structures such as addition of objects (e.g., direct sums of vector spaces or complexes). Functors are maps between categories which map objects to objects and morphisms to morphisms, respecting composition. For example, the fundamental group π_1 is a functor from topological spaces to groups. Continuous functions are mapped to group homomorphisms.

Equivalence of two categories, $\mathscr{A}$ and $\mathscr{B}$, is provided by constructing a bijective functor – an invertible recipe for getting $\mathscr{B}$ objects and maps from such $\mathscr{A}$ structures. First, let us consider the example of the category of coherent sheaves. As every coherent sheaf $\mathscr{F}$ has a syzygy $\mathscr{P}_n \to \cdots \mathscr{P}_0$ with homology $H_0(\mathscr{P}^{\cdot}) = \mathscr{F}$, we may focus our attention on the category of complexes of sheaves, defined up to (co)homology, instead of just the category of sheaves. This is called the derived category of coherent sheaves.

We now pass from sheaves to algebras using the following construction [26]. Consider a sheaf, $\mathscr{F}$, an object of the category $\mathscr{A}$ of coherent sheaves. Then we have an algebra $A = \mathrm{Hom}(\mathscr{F},\mathscr{F})$, and we can use $\mathscr{F}$ to construct the map (functor)

$$F_{\mathscr{F}}(\mathscr{G}) = \mathrm{Ext}^{\cdot}(\mathscr{F},\mathscr{G})$$

from the category of sheaves to (bounded) complexes of representations of A, or $D^b (\mathrm{mod}-A)$. Here the differential map on the complex of Ext's is the zero map. Of course the interesting question is when is this functor a category equivalence. A.I. Bondal has shown that when $\mathscr{F}$ includes sufficiently many summands – usually, $\mathscr{F}$ will be a direct sum of generators for $\mathscr{A}$ – then this is so [26].

This statement is profound. It allows us to shift our focus to finite-dimensional algebras. What is more, we can analyze the algebraic properties associated to exceptional collections and helices. What are the special algebras and representations which result from this construction? The algebras turn out to be those associated to *quivers*, which we will briefly review here.

[Before discussing quivers, it should be noted that helices can be defined not only for collections of sheaves, but for an arbitrary triangulated category (a triangulated category is an abelian category $\mathscr{C}$, with an automorphism functor $T : \mathscr{C} \to \mathscr{C}$ which satisfies certain axioms; the paradigm is the category of complexes of sheaves, with automorphism a shift in Čech degree). We will not discuss these matters in detail, except to say that the constructions are rather general. We feel the geometric link to physics is of primary importance, and have therefore concentrated our attention on this application of helix theory. For more on the categorical approach to helices, see the first and seventh articles in Ref. [5].]

A quiver is a set of labeled points with some number of labeled directed arrows between them. An ordered quiver is one in which the points are ordered and arrows connect points labeled with a lesser number to those with a greater label. For example, the following diagram defines an ordered quiver:

$$i,j = 1...3 \qquad (4.1)$$

Path composition defines an algebra, A, associated to a quiver, with vertices p_i corresponding to projections in the algebra: $p_i \cdot p_i = p_i$, and products equaling zero if paths don't line up tip to tail. We may impose relations in the algebra. In the example above, we may put

$$f_i \cdot g_j = f_j \cdot g_i . \tag{4.2}$$

In this case, we say we have a quiver with relations.

The projections p_i decompose representations V of A (or left A-modules) into (non-invariant) subspaces V_i, via

$$V = \bigoplus_i p_i V = \bigoplus_i V_i ,$$

and arrows determine morphisms $V_i \to V_j$. Now suppose W is a *right* A-module. Then we can also decompose W into subspaces $G_i W \equiv W p_i$ via

$$W = \bigoplus_i W p_i = \bigoplus_i G_i W .$$

We can then consider A as a right module over itself and define submodules $P_k = p_k A$ (closed under A). Now the algebra of a quiver with $n+1$ vertices looks like $A = \bigoplus_0^n P_i$, which is naturally identified with

$$A = \mathrm{Hom}_A(A,A) = \mathrm{Hom}_A \left(\bigoplus_{i=0}^n P_i, \bigoplus_{j=0}^n P_j \right) = \bigoplus_{i,j} \mathrm{Hom}(P_i, P_j) .$$

Therefore the algebra of a quiver is the algebra of morphisms between modules. Also note that $\mathrm{Hom}(P_i, P_j) = 0$ for $i > j$, if the quiver is ordered.

Suppose we have a strong exceptional collection, by which we mean an exceptional collection satisfying the additional requirement that $\mathrm{Ext}^k(E_i, E_j) = 0$ for all i and j when $k \neq 0$. (As stated, we will work here with sheaves over X.) Then if the derived category is generated by the collection, we can write $E = \oplus E_i$ and define $A = \mathrm{Hom}(E, E)$. Then A is the algebra of a quiver with relations, and Bondal has proven that this mapping from sheaves to right A-modules is an equivalence of categories:

$$D^b(\mathrm{Sheaf}(X) \cong D^b(\mathrm{mod} - A) .$$

An example of a strong exceptional collection is $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n))$ over $\mathbf{P}^n$. Consider $n = 2$. Then

$$A = \mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) ,$$

which is the algebra of the quiver (4.1) with the relations (4.2). The vertices p_i correspond to the one-dimensional spaces $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i))$, and the arrows correspond to the three independent generators of $\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$ (see calculation for $\mathbf{P}^n$ in Sect. 3.4), which compose according to the quiver relations. Further, mutations act similarly on exceptional collections and modules.[13]

We remark here that not all exceptional collections have all mutations admissible, nor is it known whether a strong exceptional collection can always be found.

---

[13] We define the (right) mutation of representation spaces $R(V_i, V_{i+1}) = (V_i', V_{i+1}')$ by $V_i' = V_{i+1}$ and $V_{i+1}' = {}^\perp V_{i+1} \cap V_i \oplus V_{i+1}$.

Further, the helix generated by an exceptional collection might not yield a quasi-unipotent bilinear form (if $R^n E$ were not represented by tensoring by a line bundle, for example).

Another notion which these remarks cannot address is the question of constructivity. Can any exceptional collection be generated by mutations of a given one? For $\mathbf{P}^1$ and $\mathbf{P}^2$, for example, it has been proven by Drezet and Rudakov that this is so. For $\mathbf{P}^2$, the conditions of being exceptional yield the Markov equation, $x^2 + y^2 + z^2 - 3xyz = 0$, for the ranks of the bundles involved [27]. For higher dimensional spaces, little more is known. Note that the Markov equation is precisely the equation of classification for topological sigma models with three vacua (see Sect. 6.2 of Ref. [2]). The constructivity of the helix then tells us that this is the only such model with three vacua (up to continuous deformations, as always). It would be interesting to see these conditions translated into the simple – or more intuitive – structure of quivers.

4.2. Localization. The difficulty of the classification program is that finding the soliton numbers of a physical theory is a daunting task, especially in the sigma model case. One must first find the quantum ring of the topological sigma model, which itself demands a detailed knowledge of rational curves on the manifold. Then, one must construct and solve the tt^* equations. Very few solutions to these equations are known. At best, some asymptotics have been calculated in a limited number of cases. This is what is needed in order to extract the soliton numbers. These numbers may be calculable numerically, but that program, too, is formidable. A quick way of deriving soliton numbers would be a godsend. This is why a link would be so interesting.

In this section, we shall outline a possible approach to this problem. While difficulties remain, we hope that these obstacles can be overcome. The idea is to localize the vacua by adding a potential to the sigma model. Normally, this would destroy the $N = 2$ invariance, as the potential would have to be a holomorphic function of the superfield coordinates (and the only holomorphic functions on a compact manifold are the constants, which don't affect the Lagrangian due to the integration over superspace). However, this difficulty can be circumvented if the manifold admits a holomorphic Killing vector for its Kähler metric. We recall that any manifold admits a supersymmetric sigma model – a natural extension to superfields of the ordinary nonlinear sigma model. If the manifold is Kähler with metric $g_{i\bar{j}}$, then the separation into holomorphic and anti-holomorphic tangent spaces (preserved by the connection) leads to a second independent supersymmetry. A potential is added to this model by inserting a more general form for the Lagrangian, accompanied by new transformation laws. Conditions on the potential terms arise from requiring this Lagrangian to be invariant under supersymmetry.

We report here the results of this procedure, calculated first in [28]. Let us call the Lagrangian of the usual topological sigma model $S_0(\Phi)$ (here Φ denotes all of the fields). Then the sigma model with potential has the form

$$S_0(\Phi) + m^2 g_{\mu\nu} V^\mu V^\nu + m \bar{\psi}^\mu D_\mu V_\nu \psi^\nu . \tag{4.3}$$

Here V is the holomorphic Killing vector, which by definition obeys

$$D_\mu V_\nu + D_\nu V_\mu = 0 ,$$

$$\partial_i V_{\bar{j}} + \partial_{\bar{j}} V_i = 0 .$$

Since V is holomorphic, we have $V^{\bar{i}} = (V^i)^*$, and the second condition above tells us that we may write

$$V_a = i\partial_a U ,$$

where U is a real (not analytic) function (the metric relates our holomorphic Killing vector to a closed one-form, which is therefore the exterior derivative of a function, as the diagonality of the Hodge numbers implies $H^1 = 0$). We may proceed with the topological twisting in the usual fashion, redefining the bundles of which the fermions are sections. To obtain the topological theory, we need the action of Q_+. We find

$$[Q_+, \phi^i] = i\chi^i ,$$

$$\{Q_+, \chi^i\} = -iV^i ,$$

$$\{Q_+, \rho_{\bar{z}}^i\} = \partial_{\bar{z}}\chi^i - i\Gamma^i_{jk}\chi^j\rho_{\bar{z}}^k .$$

Note that if we now try to do the usual game of relating local observables to differential forms by $\chi^i \leftrightarrow dz^i$ etc., we find that $Q = d + mi_V \equiv d_m$, where i_V is contraction by V and m is a parameter which scales with V (we have defined $Q \equiv Q_+$). It is clear then that

$$Q^2 = m(di_V + i_V d) = m\mathscr{L}_V ,$$

where $\mathscr{L}_V$ is the Lie derivative in the V direction (the second equality above is true for differential forms). Thus, the notion of Q-cohomology doesn't make sense unless we are talking about V-invariant forms.

The simplification of this procedure is that the bosonic action now has a potential term, so the vacua are localized at the minima of the potential. Since the potential is essentially $|V|^2$, the minima are at the zeros of V. To simplify the discussion, we will assume that we can choose a holomorphic vector field with isolated zeros. In fact, by a mathematical theorem of Carrell and Lieberman, this property requires the Hodge numbers to be diagonal [29]. These are just the manifolds in which we are interested from the point of view of classification, as they have finite chiral rings. Furthermore, such manifolds have the property that the d_m cohomology is isomorphic (as a vector space) to the ordinary de Rham cohomology. Therefore, no number of observables is lost in the addition of the potential to our theory.

The physical classification of these theories rests on the calculation of soliton numbers. In the ordinary sigma model, the space of minimum bosonic configurations is the entire target manifold (constant maps) and the solitons are derived from a quantum-mechanical analysis. Here things are much simpler: the vacua are points, as in Landau–Ginzburg theories. As usual, we consider an infinite cylinder with compactified time. Let us label the vacua (zeros of V) x_a. The solitons in the ab sector correspond to time-independent field configurations with $\phi(-\infty) = x_a$ and $\phi(+\infty) = x_b$. The solitons which saturate the Bogomolnyi bound minimize the energy functional. We have, for (the bosonic part of) time-independent configurations,

$$E = \int dx[g_{i\bar{j}}\partial_x\phi^i\partial_x\phi^{\bar{j}} + m^2\partial_i U\partial_{\bar{j}}Ug^{i\bar{j}}]$$

$$= \int dx[\partial_x\phi^i \pm m\partial^i U][\partial_x\phi^{\bar{j}} \pm m\partial^{\bar{j}}U]g_{i\bar{j}} \mp m\int dx(\partial_x U) ,$$

from which we derive the Bogomolnyi bound

$$E \geqq m|U(\infty) - U(-\infty)|$$

(m can be incorporated into U as well; henceforth we will put $m = 1$). This bound is saturated for trajectories for which

$$\partial_x \Phi = J \cdot V , \qquad (4.4)$$

where J is the action of the complex structure. Thus the solitons move along paths defined by the vector JV, which is V rotated by the complex structure tensor, i.e. $\Phi' = JV$ (here we are speaking of the real vector field $V + \bar{V}$). We note here that any trajectory obeying (4.4), transformed by the flow defined by V, will still obey (4.4). This follows because U and the metric are invariant with respect to V.

The subtleties of these theories are two-fold. First, we need to know how to compute the chiral ring and ensure that the tt^* structure is left intact. Secondly, we have a continuous set of classical soliton trajectories which needs to be quantized by the method of collective coordinates. When we perform the quantization,[14] the collective coordinate becomes a quantum-mechanical particle. This technique is standard. The first subtlety amounts to asking whether there is continuity at $m = 0$, i.e. do the soliton numbers change discontinuously when we turn on this vector field. As we have discussed previously, the cohomologies are isomorphic, but the theories may be different. We may suddenly be describing a massive $N = 2$ theory with a different classification – after all, the configuration space now only includes V-invariant forms. Without fully resolving these issues, we will walk through a simple example, highlighting the general features.

Our example is just the sphere, $\mathbf{P}^1$, endowed with its usual (Fubini-Study) metric,

$$g_{z\bar{z}} = \partial\bar{\partial} \ln\left(1 + |z|^2\right) .$$

The holomorphic vector field is just a rotation of the sphere, say about the polar axis ($\varphi \to \varphi + c$). The vector field V has the form

$$V = iz\partial_z - i\bar{z}\partial_{\bar{z}}$$

and the function U is

$$U = -\frac{|z|^2}{(1 + |z|^2)}$$

(we sometimes use the notations V for just the holomorphic piece, too), and so $iV^z = z\partial_z$, which generates dilatations or lines of longitude emanating from the north pole. Specifically, the solitons are

$$z(t) = \rho e^{i\varphi_0} e^{mx} ,$$

where ρ and φ_0 are arbitrary real parameters associated to translation invariance and the $U(1)$ invariance generated by V. The translational symmetry means that solitons can appear with arbitrary momentum. The minimal energy will have zero momentum. The collective coordinate φ_0 is simply the spatially constant part of the azimuthal angle φ and becomes a free particle upon quantization. Explicitly,

---

[14] See, for example, Ref. [30]. One extracts the parameter describing the different solutions from the path integral. Restoring the time dependence amounts to treating it as a quantum-mechanical particle, so the space of solitons includes a one (or more) particle Hilbert space, from which we will choose the state of lowest energy.

we define $z = \rho e^{i\varphi}$, then single out the zero mode $\varphi = \tilde{\varphi}(x,t) + \varphi_0(t)$ and write the bosonic action as

$$S_{\text{bos}} = \int dx\, dt \frac{1}{(1+\rho^2)^2}[\partial_\mu\rho\partial^\mu\rho + \rho^2\partial_\mu\tilde{\varphi}\partial^\mu\tilde{\varphi} - m^2\rho^2(1+\rho^2)]$$

$$+ \int dt\, \dot{\varphi}_0^2 A + \dot{\varphi}_0 B\ .$$

Here A and B are defined in terms of the other fields:

$$A = \int dx \frac{\rho^2}{(1+\rho^2)^2};\qquad B = \int dx \frac{\rho^2\partial_x\tilde{\varphi}}{(1+\rho^2)^2}\ .$$

Expanding around the classical soliton solution and performing the integration, we have

$$A = \frac{1}{2m};\qquad B = 0\ .$$

The action for φ_0 is thus a standard free single particle quantum-mechanical action.

We know the full Hilbert space of a free particle in one (bound) dimension, and its energy is minimized by the $n = 0$ ground state. This analysis would thus tell us that there is just one Bogolomonyi soliton. However, if we could eliminate this state, the ground state(s) would appropriately be a doublet ($n = \pm 1$). This is indeed the correct representation under $U(1)$ induced by the doublet of $SU(2)$. That is, the theory without potential – what we are interested in, after all – has an $SU(2)$ symmetry and the solitons have been shown (by independent analysis) to lie in the doublet of $SU(2)$. The potential breaks the $SU(2)$ to $U(1)$, and the resulting solitons should thus have charges $\pm\frac{1}{2}$ under this $U(1)$. Unfortunately, to eliminate the ground state we had to impose the known solution. A possible resolution of this conundrum may come from the form of the $N = 2$ algebra in the models with potentials from holomorphic Killing vectors. In Ref. [28] it was found that the algebra contains central terms proportional to $\mathscr{L}_V$. The difference from the usual $N = 2$ algebra suggests that we are studying the equivariant quantum cohomology ring in this example (which may indeed be equivalent in this case when the vector field is set to zero). The soliton structure of these theories may not be continuous as $V \rightarrow 0$.

4.3. $\tilde{\mathbf{P}}^2$: A Good Test Case.

In this section we will present materials necessary for studying the manifold $\tilde{\mathbf{P}}^2$, by which we mean the blow-up of the projective space $\mathbf{P}^2$ at a point (in Sect. 3.4 this was denoted $\tilde{\mathbf{P}}_1^2$). This manifold is particularly interesting for several reasons. It is a diagonal Fano variety with $c_1 > 0$, so it defines a good quantum field theory. Further, it is not a coset space. Coset spaces may prove to satisfy the proposed link due to simplifications from a representation-theoretic description. However, this space has no simple treatment, so the equivalence here would show that the link was more robust. Another reason this space is interesting is that though it is not simple, we do have some tools available to help our treatment. $\tilde{\mathbf{P}}^2$ is the blow-up of a projective space. Due to this fact, we will be able to give an exceptional collection and compute the bilinear form from the mathematical point of view. Happily, this space is also a toric variety. These spaces were studied by Batyrev, who showed that their quantum cohomology rings have a very simple description. This knowledge is necessary for computing the soliton numbers through

the tt^* equations in order to compare with the mathematical results. What is more, this space has holomorphic vector fields, since it is a toric variety. Therefore, it may be treatable by the method of localization described in Sect. 4.2. Finally, the space has only four cohomology classes, so the calculations are not too messy.

Let us first describe the blow-up procedure. We begin by recalling the blow-up of $\mathbf{C}^2$ at the origin, denoted $\tilde{\mathbf{C}}^2$. It is the subset of $\mathbf{C}^2 \times \mathbf{P}^1$ defined by

$$\mathbf{C}^2 = \{(z_1, z_2; \lambda_1, \lambda_2) \in \mathbf{C}^2 \times \mathbf{P}^1 : z_1 \lambda_2 - z_2 \lambda_1 = 0\} .$$

Thus the λ part is determined to be the unique line through (z_1, z_2) whenever $(z_1, z_2) \neq (0, 0)$. At the origin, we have the entire $\mathbf{P}^1$ (all lines through 0). We call this $\mathbf{P}^1$ the exceptional divisor, E. Note that $\tilde{\mathbf{C}}^2 - E \cong \mathbf{C}^2 - 0$. To define the blow-up of $\mathbf{P}^2$, we choose a point $(0, 0, 1)$ and replace a neighborhood isomorphic to $\mathbf{C}^2$ by its blow-up. Thus,

$$\tilde{\mathbf{P}}^2 = \{(\mu_1, \mu_2, \mu_3; \lambda_1, \lambda_2) \in \mathbf{P}^2 \times \mathbf{P}^1 : \mu_1 \lambda_2 - \mu_2 \lambda_2 = 0\} .$$

This is the zero locus of a homogeneous function of bi-degree $(1, 1)$ in $\mathbf{P}^2 \times \mathbf{P}^1$.

We can also understand this space as a toric variety, being one of the rational, ruled, or Hirzebruch, surfaces. The study of toric manifolds is quite a broad subject. A readable hands-on introduction is given by Batyrev in his paper on quantum rings of toric varieties, the results of which we shall use presently [31].

In the language of toric varieties, the space $\tilde{\mathbf{P}}^2$ is described by the diagram (drawn on a plane)
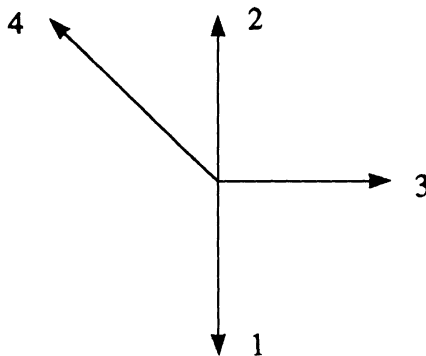
which is interpreted as follows. We define a complex variable for each arrow, beginning with the space $\mathbf{C}^4 - V$. Here V is an open set containing points which must be removed so that the group action by which we mod out has no fixed points.

From the diagram we get that[15]

$$V = \{z_1 = z_2 = 0\} \cup \{z_3 = z_4 = 0\} .$$

We then mod out by the action of $(\mathbf{C}^*)^2$, which is derived from the independent relations among the arrows: $\vec{v}_1 + \vec{v}_2 = 0$, $\vec{v}_1 + \vec{v}_3 + \vec{v}_4 = 0$. Thus we act without fixed point by $(\lambda, \rho) \in (\mathbf{C}^*)^2$ on $\mathbf{C}^2 - V$ by sending

$$(z_1, z_2, z_3, z_4) \mapsto (\rho \lambda z_1, \lambda z_2, \rho z_3, p z_4) . \tag{4.5}$$

We can see here that $(z_1, z_2, z_3, z_4) \to (z_3, z_4) \in \mathbf{P}^1$ represents a fibering over $\mathbf{P}^1$ equal to the projectivization of the bundle $\mathcal{O} \oplus \mathcal{O}(1)$, as the λ action represents the projective equivalence on the fiber and the ρ action on z_1 denotes that it is a coordinate of an $\mathcal{O}(1)$ bundle. The nth Hirzebruch surface, H_n, gives $\vec{v}_4$ a height of n (instead of one) and is equal to the total space of the bundle $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(n))$.

In order to compute the tt^* equations for this manifold, we need the chiral ring coefficients $C_{ij}{}^k$. In fact these can be computed in the purely topological theory, since $C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle$, and the indices are raised with the topological metric η_{ij}. The topological correlation functions are obtained by passing to the topological limit, in which the path integral becomes an integral over the moduli space of instantons (not to be confused with parameter space). The instantons are holomorphic maps. The topological observables, as we have discussed, correspond to cohomology classes of forms, and can be chosen to have support on their Poincaré dual cycles, L_i. The correlators $\langle \phi_i(p_i) \phi_j(p_j) \phi_k(p_k) \rangle$ just count the number of holomorphic maps taking p_m to L_m (when that number is finite[16]), weighted by $\exp(-dA)$, where d is the degree of the instanton and A is the area of the image (which depends on the Kähler class of the target space).

Batyrev has calculated the ring coefficients for toric varieties. We quote his results without proof for the nth Hirzebruch surface, H_n, obtained by taking $\vec{v}_4 = (-1, n)$ (our example is $n = 1$) and generalizing the action of $(\mathbf{C}^*)^2$ in (4.5) by changing the $\rho \lambda z_1$ to $\rho^n \lambda z_1$ on the right-hand side. For this space, Batyrev's prescription gives a ring with two generators, z_1 and z_2, and the following relations:

$$z_1^2 = e^{-\beta} z_2^n ,$$

$$z_2^2 = e^{-\alpha} - n z_1 z_2 .$$

In the above, the α and β are parameters describing the Kähler class. As they represent the areas of the two homology cycles, they should both be positive. We note here that in the large radius limit, where the Ricci curvature goes to zero and these areas go to infinity, we recover the ordinary cohomology ring or intersection of H_n.

The ring gives us the values $C_{ij}{}^k$. To compute all the correlators, we need C_{ijk} which we can get by lowering indices with $\eta_{ij} = C_{ij0} = C_{ij}{}^m \eta_{m0}$ (where the subscript 0 represents the identity element of the ring). But η_{m0} represents the one-point

---

[15] This set is defined by taking the union of all sets obtained by setting to zero all coordinates corresponding to vectors of a primitive collection. A primitive collection consists of vectors not generating a single cone (there are four two-dimensionl and four one-dimensional cones in the diagram above), though any subset of those vectors generates a single cone of the diagram.

[16] If the formal dimension of such maps is zero but there is a continuous family, then the "number" of maps is replaced by the Euler characteristic of a vector bundle over this family.

function, which is determined by the anomalous charge conservation. That is, the chiral charge (i.e. the form degree) is violated by $2d$ (the dimension of the space) units, so only the top form ($z_1 z_2$ here) can have a nonzero one-point function. We can normalize this to be unity. Thus we have all the information necessary to write down the full tt^* equations.

The tt^* equations will be nonlinear differential equations for the metric $g_{i\bar{j}}$. Since this is a matrix, they are coupled differential equations representing the different entries. It is almost certain that they cannot be solved analytically by today's methods. One might hope to obtain the soliton numbers through a numerical analysis. A similar analysis was performed in [32] in a computation involving Landau–Ginzburg theories. In that paper, the author started from the conformal (homogeneous) point and iterated out to the infrared limit. To do so, one needs the values, and first derivatives of the metric at the conformal point. In the Landau–Ginzburg case, explicit formulas for these values provide necessary ingredients. It was found that convergence of the solution also determines the boundary conditions. This is consistent with the cases which have been solved analytically. For the topological sigma model, certain asymptotic expressions for the metric are known (see Sect. 5 of Ref. [2]). One expects that these data and regularity of the solutions would once again determine the boundary conditions needed to proceed with a numerical computation of the soliton numbers from the physical viewpoint.

We should mention here that toric varieties all have holomorphic actions by vector fields (and one can write metrics invariant with respect to such actions, so that they are holomorphic Killing vectors). A toric variety is constructed from some open set in $\mathbf{C}^n$ by modding out by $(\mathbf{C}^*)^r$. Roughly speaking, this leaves us with at least $(\mathbf{C}^*)^{n-r}$ independent $\mathbf{C}^*$ actions of the form $z_j \mapsto \lambda z_j$. In our example, two remaining actions are

$$(z_1, z_2, z_3, z_4) \mapsto (s z_1, z_2, t z_3, z_4), \quad s, t \in \mathbf{C}^* \,.$$

Let us consider the S^1 action defined by setting $s = t^2 \in U(1)$. The vector field which generates this action has four isolated fixed points, in one-to-one correspondence with the number of vacua or cohomology elements for this space. They are $(0, 1, 0, 1)$, $(0, 1, 1, 0)$, $(1, 0, 0, 1)$, and $(1, 0, 1, 0,)$. This space is clearly able to be analyzed by the method of localization. However, it will certainly be necessary to clean up the $\mathbf{P}^1$ case before proceeding in this direction.

From the mathematical point of view, exceptional collections over $\tilde{\mathbf{P}}^2$, as well as the other rational, ruled surfaces, were studied by Kvichansky and Nogin in [5], and by Nogin in [33]. They found exceptional collections, as well as those which generate helices, in which shifting a sheaf right n times corresponds to tensoring by the canonical bundle of $\tilde{\mathbf{P}}^2$. In fact, these authors found foundations consisting of line bundles. We describe line bundles using the equivalence of divisors and line bundles discussed for $\mathbf{P}^n$ in 3.2.[17] The two homology cycles correspond to one on $\mathbf{P}^2$ and the exceptional divisor. Denote these by F and C, respectively; products of their corresponding line bundles L_F and L_C will be denoted $(L_F)^a \otimes (L_C)^b = aF + bC = (a, b)$. The authors showed that the collections $\{\mathcal{O}, \mathcal{O}(1, k), \mathcal{O}(1, k + 2), \mathcal{O}(2, 2)\}$ are helices for all k. It would be very interesting to use these data to compute the

---

[17] For other varieties the hyperplane bundle represents the pull-back bundle under an imbedding into projective space.

"mathematical" soliton numbers $\chi(E_i, E_j)$ and compare with results derived from physics.

5. Prolegomena to Any Future Math-Physics

We have detailed an interesting open problem in mathematics and physics, along with several proposals for establishing a link between classification of $N = 2$ theories and helices of exceptional sheaves. Currently, there is no definite connection, though the two areas have been shown to be related through examples. Further, a categorical link to the algebras of quivers has been discussed.

Clearly, there are many approaches to solving this problem and much work needs to be done in all directions. One would like to amass more "experimental" evidence through a detailed exploration of a wide range of examples. Physicists would like the mathematical theory to be more mature, in order to develop better intuition for why solitons could have such an abstruse origin. The situation is much like the status of supersymmetry and de Rham theory and Morse theory, before Witten's famous papers relating the two. Many roads can be taken; is one likely to lead to such a fruitful discovery? In the absence of hard facts, we have amassed circumstantial evidence that some bridge between the theories should exist. The author trusts the reader to be well-skilled to investigate this problem, and invites him/her to establish this elusive connection.

References

 1. Kontsevich, M.: Private communication
 2. Cecotti, S., Vafa, C.: Commun. Math. Phys. **158**, 569 (1993)
 3. Cecotti, S., Vafa, C.: Nucl. Phys. **B367**, 3543 (1991)
 4. Lerche, W., Vafa, C., Warner, N.: Nucl. Phys. **B324**, 427 (1989)
 5. Rudakov, A.N. et al.: Helices and Vector Bundles. Seminaire Rudakov, London Mathematical Society Lecture Note Series **148**, Cambridge: Cambridge University Press, 1990
 6. Witten, E.: Mirror Manifolds and Topological Field Theory. In: Essays on Mirror Manifolds Hong Kong: International Press, 1992
 7. Witten, E.: J. Diff. Geom. **17**, 661 (1982)
 8. Cecotti, S., Girardello, L., Pasquinucci, A.: Intl. J. Mod. Phys. **A6**, 2427 (1991)
 9. Witten, E.: Commun. Math. Phys. **118**, 411 (1988); Cecotti, S.: Intl. J. Mod. Phys. **A6**, 1749 (1991)
10. Cecotti, S., Fendley, P., Intriligator, K., Vafa, C.: Nucl. Phys. **B386**, 405 (1992)
11. Its, A.R., Novokshenov, V.Yu.: The Isomonodromic Deformation Method in the Theory of Painlevé Equations. Lecture Notes in Mathematics **1191**, New York: Springer-Verlag, 1986; McCoy, B.M., Tracy, C.A., Wu, T.T.: J. Math Phys. **18**, 1058 (1977)
12. Intriligator, K.: Private communication
13. Witten, E.: Nucl. Phys. **B149**, 285 (1979)
14. Cecotti, S., Vafa, C.: Mod. Phys. Lett. **A7**, 1715 (1992)
15. Zaslow, E.: Nucl. Phys. **B415**, 155 (1994)
16. Hamidi, S., Vafa, C.: Nucl. Phys. **B279**, 465 (1989)

17. Zaslow, E.: Commun. Math. Phys. **156**, 301 (1993)
18. McKay, J.: Cartan Matrices,.Finite Groups of Quaternions, and Kleinian Singularities. Proc. Am. Math. Soc. (1981) 153; Slodowy, P.: Simple Singularities and Simple Algebraic Groups. Lecture Notes in Mathematics **815**, New York: Springer-Verlag, 1980
19. Griffths P., Harris, J.: Principles of Algbraic Geometry. New York: Wiley-Interscience, 1978
20. Gorodentsev, A.L., Rudakov, A.N.: Duke Mathematical Journal **54**, 115 (1987)
21. Gorodentsev, A.L.: Math. USSR Izv. **33**, 67 (1989)
22. Kapranov, M.M.: Invent. Math. **92**, 479 (1988); Kapranov, M.M.: Math. USSR Izv. **24**, 183 (1985)
23. Kapranov, M.M.: Private communication
24. Bondal, A.I., Kapranov, M.M.: Math. USSR Izv. **35**, 519 (1990)
25. Polischuk, A.E.: Private communication
26. Bondal, A.I.: Math. USSR Izv. **34**, 23 (1990)
27. Rudakov, A.N.: Math. USSR Izv. **32**, 99 (1989)
28. Alvarez-Gaumé, L., Freedman, D.Z.: Commun. Math. Phys. **91**, 87 (1983)
29. Carrell, J.B., Lieberman, D.I.: Inv. Math. **21**, 303 (1973)
30. Rajaraman, R.: Solitons and Instantons. Amsterdam: Elsevier Science Publishers, 1987
31. Batyrev, V.V.: Quantum Cohomology Rings of Toric Manifolds. MSRI preprint, 1993
32. Leaf-Herrmann, W.A.: Nucl. Phys. **B389**, 611 (1993), and Harvard preprint HUTP-92/A060
33. Nogin, D.Yu.: Algebraic Geometry. Lecture Notes in Mathematics **1479**, Berlin, Heidelberg, New York: Springer, 1991

Communicated by S.-T. Yau