A Contribution of the Trivial Connection to the Jones Polynomial and Witten's Invariant of 3d Manifolds, II

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Abstract: We extend the results of our previous paper [1] from knots to links by using a formula for the Jones polynomial of a link derived recently by N. Reshetikhin. We establish a relation between the parameters of this formula and the multivariable Alexander polynomial. This relation is illustrated by an example of a torus link. We check that our expression for the Alexander polynomial satisfies some of its basic properties. Finally we derive a link surgery formula for the loop corrections to the trivial connection contribution to Witten's invariant of rational homology spheres.

1. Introduction

This paper is an expansion of our previous work [1]. We will try to extend the results of that paper from knots to links. Our main tool will be the formula for the Jones polynomial of a link proposed recently by N. Reshetikhin[2] [2].

We start by briefly reviewing the notations of [1] (they will be used throughout this paper) as well as some of its results. Let $\mathscr{L}$ be an n-component link in a 3-dimensional manifold M. We assign an α_j-dimensional $SU(2)$ representation to each component $\mathscr{L}_j$ of $\mathscr{L}$. E. Witten introduced in [3] an invariant $Z_{\alpha_1,\dots,\alpha_n}(M,\mathscr{L};k)$ which is a path integral over the gauge equivalence classes of $SU(2)$ connection A_μ on M:

$$Z_{\alpha_1,\dots,\alpha_n}(M,\mathscr{L};k) = \int [\mathscr{D}A_\mu]\exp\left(\frac{i}{\hbar}S_{CS}\right) \prod_{j=1}^{n} \mathrm{Tr}_{\alpha_j}\mathrm{Pexp}\left(\oint_{\mathscr{L}_j} A_\mu dx^\mu\right),\qquad (1.1)$$

here S_{CS} is the Chern–Simons action

$$S_{CS} = \frac{1}{2}\mathrm{Tr}\,\varepsilon^{\mu\nu\rho}\int_M dx(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho),\qquad (1.2)$$

$\hbar$ is a "Planck's constant":

$$\hbar = \frac{2\pi}{k}, \quad k \in \mathbb{Z},\tag{1.3}$$

the trace Tr in Eq. (1.2) is taken in the fundamental (2-dimensional) representation and $\mathrm{Tr}_{\alpha_j}\mathrm{Pexp}\,(\oint_{\mathscr{L}_j} A_\mu dx^\mu)$ are the traces of holonomies along the link components $\mathscr{L}_j$ taken in the α_j-dimensional representations.

The path integral (1.1) can be calculated in the stationary phase approximation in the limit of large k. The stationary points of the Chern–Simons action (1.2) are flat connections and Witten's invariant is presented as a sum over connected pieces $\mathscr{M}_c$ of their moduli space $\mathscr{M}$:

$$Z_{\alpha_1,\dots,\alpha_n}(M,\mathscr{L};k) = \sum_{\mathscr{M}_c} Z^{(\mathscr{M}_c)}_{\alpha_1,\dots,\alpha_n}(M,\mathscr{L};k),$$

$$Z^{(\mathscr{M}_c)}_{\alpha_1,\dots,\alpha_n}(M,\mathscr{L};k) = \exp\frac{i}{\hbar}\left(S^{(c)}_{CS} + \sum_{n=1}^{\infty} S^{(c)}_n \hbar^n\right),\tag{1.4}$$

here S_{CS} is a Chern–Simons action of flat connections of $\mathscr{M}_c$ and $S^{(c)}_n$ are the quantum n-loop corrections to the contribution of $\mathscr{M}_c$. The 1-loop correction is a determinant of the quadratic form describing the small fluctuations of $S_{CS}(A_\mu)$ around a stationary phase point. Its major features were determined by Witten [3], Freed and Gompf [6], and Jeffrey [5] (some further details were added in [7]):

$$e^{iS^{(c)}_1} = \frac{(2\pi\hbar)^{\frac{\dim H^0_c - \dim H^1_c}{2}}}{\mathrm{Vol}(H_c)}\exp\left(\frac{i}{\pi}S_{CS} - \frac{i\pi}{4}N_{\mathrm{ph}}\right)$$

$$\times \int_{\mathscr{M}_c}\left[\sqrt{|\tau_R|}\prod_{j=1}^{n}\mathrm{Tr}_{\alpha_j}\mathrm{Pexp}\left(\oint_{\mathscr{L}_j} A_\mu dx^\mu\right)\right],\tag{1.5}$$

here H_c is an isotropy group of $\mathscr{M}_c$ (i.e. a subgroup of $SU(2)$ which commutes with the holonomies of connections $A^{(c)}_\mu$ of $\mathscr{M}_c$), N_{ph} is expressed [6] as

$$N_{\mathrm{ph}} = 2I_c + \dim H^0_c + \dim H^1_c + 3(1 + b^1_M),\tag{1.6}$$

I_c is a spectral flow of the operator $L_- = \star D + D\star$ acting on 1- and 3-forms, D being a covariant derivative, H^0_c and H^1_c are cohomologies of D, and b^1_M is the first Betti number of M. τ_R is a Reidemeister–Ray–Singer torsion. It was observed in [5] that $\sqrt{\tau_R}$ defines a ratio of volume forms on $\mathscr{M}_c$ and H_c.

In a particular case of a rational homology sphere (RHS), the 1-loop correction to the contribution of the trivial connection is

$$e^{iS^{(\mathrm{tr})}_1(M)} = \sqrt{2}\pi[K\,\mathrm{ord}\,H_1(M,\mathbb{Z})]^{-\frac{3}{2}}.\tag{1.7}$$

Based on our calculation of Witten's invariant of Seifert manifolds we conjectured in [7] that

$$S^{(\mathrm{tr})}_2(M) = 3\lambda_{CW}(M),\tag{1.8}$$

here λ_{CW} is the Casson–Walker invariant of M (it was calculated for Seifert manifolds by C. Lescop in [11]).

Witten has suggested in [3] a surgery formula for the invariant $Z(M;k)$. We need to introduce some notations in order to describe it. We pick two basic cycles

on the boundaries of the tubular neighborhoods $\text{Tub}(\mathcal{L}_j)$ of the link components $\mathcal{L}_j$. A cycle $C_1^{(j)}$ is a meridian of $\mathcal{L}_j$, it can be contracted through $\text{Tub}(\mathcal{L}_j)$. A cycle $C_2^{(j)}$ has a unit intersection number with $C_1^{(j)}$, it is defined only modulo $C_1^{(j)}$. We denote as l_{ij} the linking numbers of the link components. The self-linking number l_{jj} is a linking number between $\mathcal{L}_j$ and $C_2^{(j)}$.

A surgery on a link component $\mathcal{L}_j$ is determined by an $SL(2,\mathbb{Z})$ matrix $U^{(p_j,q_j)}$:

$$U^{(p_j,q_j)} = \begin{pmatrix} p_j & r_j \\ q_j & s_j \end{pmatrix} \in SL(2,\mathbb{Z}), \quad p_j s_j - q_j r_j = 1 . \tag{1.9}$$

The surgery means that we cut $\text{Tub}(\mathcal{L}_j)$ out and glue it back in such a way that the cycles $p_j C_1^{(j)} + q_j C_2^{(j)}$ and $r_j C_1^{(j)} + s_j C_2^{(j)}$ on the boundary of the complement $M\backslash\text{Tub}(\mathcal{L}_j)$ are glued to the cycles $C_1^{(j)}$ and $C_2^{(j)}$ on the boundary of $\text{Tub}(\mathcal{L}_j)$.

Let M' be a manifold constructed by n surgeries $U^{(p_j,q_j)}$ on the components of the link $\mathcal{L}$. Then, according to [3],

$$Z(M';k) = e^{i\phi_{\text{fr}}} \sum_{\alpha_1,\dots,\alpha_n=1}^{k+1} Z_{\alpha_1,\dots,\alpha_n}(M,\mathcal{L};k) \prod_{j=1}^{n} \tilde{U}_{\alpha_j 1}^{(p_j,q_j)} , \tag{1.10}$$

here $\tilde{U}_{\alpha\beta}^{(p,q)}$ is a representation of the group $SL(2,\mathbb{Z})$ in the $k+1$-dimensional space of affine $SU(2)$ characters:

$$\tilde{U}_{\alpha\beta}^{(p,q)} = i\frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(U^{(p,q)})} \sum_{\mu=\pm 1} \sum_{n=0}^{q-1} \mu$$

$$\times \exp\frac{i\pi}{2Kq}\left[p\alpha^2 - 2\alpha(2Kn+\mu\beta) + s(2Kn+\mu\beta)^2 \right] ,$$

$$1 \leqq \alpha,\beta \leqq K-1, \quad K = k+2 \tag{1.11}$$

(see e.g. [5] and references therein), $\Phi(U^{(p,q)})$ is the Rademacher function:

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q) , \tag{1.12}$$

$s(p,q)$ is a Dedekind sum:

$$s(p,q) = \frac{1}{4q}\sum_{j=1}^{q-1} \cot\left(\frac{\pi j}{q}\right) \cot\left(\frac{\pi p j}{q}\right) . \tag{1.13}$$

ϕ_{fr} is a framing correction (all Witten's invariants are reduced to the canonical framing, see e.g. [6]):

$$\phi_{\text{fr}} = \frac{\pi}{4}\frac{K-2}{K}\left[\sum_{j=1}^{n}\Phi(U^{(p_j,q_j)}) - 3\text{sign}\left(L^{(\text{tot})}\right)\right] , \tag{1.14}$$

here $L^{(\text{tot})}$ is an $n \times n$ matrix

$$L_{ij}^{(\text{tot})} = l_{ij} + \frac{p_j}{q_j}\delta_{ij} . \tag{1.15}$$

The mathematical proof of the invariance of Eq. (1.10) was given by N. Reshetikhin and V. Turaev [4]. They also formulated general conditions on the elements of that formula that would guarantee its invariance.

In our previous paper [1] we gave a "path-integral" proof of the following conjecture which P. Melvin and H. Morton [10] formulated for $M = S^3$:

Proposition 1.1. *The trivial connection contribution to the Jones polynomial of a knot $\mathcal{K}$ in a RHS M can be expressed as*

$$Z_\alpha^{(\mathrm{tr})}(M, \mathcal{K}; k) = Z^{(\mathrm{tr})}(M; k) \exp\left[\frac{i\pi}{2K} v(\alpha^2 - 1)\right] \alpha J(\alpha, K), \qquad (1.16)$$

here v is a self-linking number of $\mathcal{K}$ and $J(\alpha, K)$ is a function that has the following expansion in K^{-1} series:

$$J(\alpha, K) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} D_{m,n} \alpha^m K^{-n}. \qquad (1.17)$$

The dominant part of this expansion is related to the Alexander polynomial of $\mathcal{K}$:

$$\pi a \sum_{n=0}^{\infty} D_{n,n} a^n = [\mathrm{ord}\, H_1(M, \mathbb{Z})] \frac{\sin\left(\frac{\pi a}{m_2 d}\right)}{\Delta_A\left(M, \mathcal{K}; e^{2\pi i \frac{a}{m_2 d}}\right)}, \qquad (1.18)$$

the integer numbers m_2 and d are defined in [1], $m_2 = d = 1$ if $M = S^3$.

We combined the results of this proposition with the finite Poisson resummation formula in order to derive a knot surgery formula for the loop corrections to the trivial connection contribution to Witten's invariant of a RHS:

Proposition 1.2. *If M and M' are rational homology spheres and M' is constructed by a rational surgery $U^{(p,q)}$ on a knot $\mathcal{K}$ in M, which has a self-linking number v, then the trivial connection contribution to Witten's invariants of M and M' are related by the formula*

$$Z^{(\mathrm{tr})}(M'; k) = Z^{(\mathrm{tr})}(M; k) \frac{\mathrm{sign}\,(q)}{\sqrt{2K|q|}} e^{-i\frac{3}{4}\pi \mathrm{sign}\left(\frac{p}{q}+v\right)} \exp\left[\frac{i\pi}{2K}\left(12 s(p,q) - \left(\frac{p}{q}+v\right)\right.\right.$$

$$\left.\left. + 3\,\mathrm{sign}\left(\frac{p}{q}+v\right)\right)\right] \int\limits_{\substack{-\infty \\ [\alpha_* = 0]}}^{+\infty} d\alpha \sin\left(\frac{\pi\alpha}{Kq}\right) \alpha J(\alpha, K) \exp\left[\frac{i\pi}{2K}\left(\frac{p}{q}+v\right)\alpha^2\right],$$

$$(1.19)$$

here the function $J(\alpha, K)$ comes from Eq. (1.16), it is a Feynman diagram contribution of the trivial connection to the Jones polynomial of $\mathcal{K}$. The integral $\int_{\substack{-\infty \\ [\alpha_* = 0]}}^{+\infty}$ in Eq. (1.19) should be calculated in the following way: the preexponential factor $\sin(\frac{\pi\alpha}{Kq})\alpha J(\alpha, K)$ should be expanded in K^{-1} series with the help of Eq. (1.17), then each term should be integrated separately with the gaussian factor $\exp\left[\frac{i\pi}{2K}\left(\frac{p}{q}+v\right)\alpha^2\right]$.

Corollary 1.1. *Only a finite number of Vassiliev's invariants participate in a surgery formula for $Z^{(\mathrm{tr})}(M';k)$ at a given loop order.*

A 2-loop part of Eq. (1.19) coincides with Walker's surgery formula for the Casson–Walker invariant. This proves the conjectured relation (1.8).

A generalization of Eq. (1.16) for links was derived recently by N. Reshetikhin[3] [2]. He observed that if the dimensions α_i in Eq. (1.1) are big enough, then the representation spaces can be treated classically: the matrix elements of Lie algebra generators in the α_j-dimensional representation can be substituted by functions on the coadjoint orbit of radius α_j and a trace over the representation can be substituted by an integral over that orbit.

Proposition 1.3. *Let $\mathscr{L}$ be an n-component link in a RHS M. Then the trivial connection contribution to its Jones polynomial can be expressed as a multiple integral over the SU(2) coadjoint orbits:*

$$Z^{(\mathrm{tr})}_{\alpha_1,\ldots,\alpha_n}(M,\mathscr{L},k) = Z^{(\mathrm{tr})}(M;k) \int\limits_{|\vec{a}_j|=\frac{\alpha_j}{K}} \prod_{J=1}^{n} \left(\frac{K}{4\pi} \frac{d^2\vec{a}_J}{|\vec{a}_j|} \right)$$

$$\times \exp\left(\frac{i\pi K}{2} \sum_{m=2}^{\infty} L_m(\vec{a}_1,\ldots,\vec{a}_n) \right) \left[1 + \sum_{\substack{l,m=0 \\ l+m\neq 0}}^{\infty} K^{-m} P_{m,l}(\vec{a}_1,\ldots,\vec{a}_n) \right] \; ; (1.20)$$

here $\vec{a}_j$ are 3-dimensional vectors with fixed length

$$|\vec{a}_J| = \frac{\alpha_J}{K} \tag{1.21}$$

and $L_m(\vec{a}_1,\ldots,\vec{a}_n)$, $P_{l,m}(\vec{a}_1,\ldots,\vec{a}_n)$ are homogeneous invariant (under $SO(3)$ rotations) polynomials of degree m. In particular,

$$L_2(\vec{a}_1,\ldots,\vec{a}_n) = \sum_{i,J=1}^{n} l_{ij}\, \vec{a}_i \cdot \vec{a}_j \, , \tag{1.22}$$

l_{ij} is the linking number of the link components $\mathscr{L}_i$ and $\mathscr{L}_j$.

An example of this formula for a torus link is derived in Appendix 1.

In our paper [12] we proved this proposition by deriving a set of Feynman rules to calculate the coefficients of the polynomials L_m and P_{ml}. These rules allowed us to establish the following property of the polynomials L_m:

Proposition 1.4. *The polynomials $L_m(\vec{a}_1,\ldots,\vec{a}_n)$ are produced from invariant homogeneous polynomials $F_m(\vec{b}_1,\ldots,\vec{b}_m)$ of order m by substituting n vectors $\vec{a}_j$ in place of m vectors $\vec{b}_j$. The polynomials $F_m(\vec{b}_1,\ldots,\vec{b}_m)$, $m \geqq 3$ are equal to zero if at least $m-1$ of m vectors $\vec{b}_j$ are parallel.*

We also conjectured a relation between the coefficients of the polynomials L_m and Milnor's linking numbers:

Conjecture 1.1. *If $L_l(\vec{a}_1,\ldots,\vec{a}_n) = 0$ for all $l < m$, then the coefficients of the polynomial $L_m(\vec{a}_1,\ldots,\vec{a}_n)$ are proportional to the mth order Milnor's linking*

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[3] I am indebted to N. Reshetikhin for sharing the results of his unpublished research.

numbers $l^{(\mu)}_{i_1,\ldots,i_m}$ of the link $\mathscr{L}$:

$$L_m(\vec{a}_1,\ldots,\vec{a}_n) = \frac{(i\pi)^{m-2}}{m} \sum_{1 \leqq i_1,\ldots,i_m \leqq n} l^{(\mu)}_{i_1,\ldots,i_m} \mathrm{Tr}(\vec{\sigma}\cdot\vec{a}_{i_1})\cdots(\vec{\sigma}\cdot\vec{a}_{i_m}) , \qquad (1.23)$$

here $\vec{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is a 3-dimensional vector formed by Pauli matrices.

We will need especially the polynomials L_3, L_4 and $P_{0,2}$:

$$L_3(\vec{a}_1,\ldots,\vec{a}_n) = \sum_{i,j,k=1}^n l^{(3)}_{ijk} \vec{a}_i \cdot (\vec{a}_j \times \vec{a}_k) , \qquad (1.24)$$

$$L_4(\vec{a}_1,\ldots,\vec{a}_n) = \sum_{i,j,k,l=1}^n l^{(4)}_{ij,kl}(\vec{a}_i \times \vec{a}_j) \cdot (\vec{a}_k \times \vec{a}_l) , \qquad (1.25)$$

$$P_{0,2}(\vec{a}_1,\ldots,\vec{a}_n) = \sum_{i,j=1}^n p_{ij} \vec{a}_i \cdot \vec{a}_j . \qquad (1.26)$$

We demonstrated in [12] that the coefficients $l^{(3)}_{ijk}$ and $l^{(4)}_{ij,kl}$ are proportional to triple and quartic Milnor's linking numbers.

An obvious condition

$$Z^{(\mathrm{tr})}_{1,\ldots,1}(M,\mathscr{L};k) = Z^{(\mathrm{tr})}(M;k) \qquad (1.27)$$

imposes a relation between the polynomials L_m and $P_{m,l}$. It allows us to express the numbers $P_{m,0}$ through the coefficients of other polynomials. For example,

$$P_{1,0} = -\frac{i\pi}{2}\sum_{j=1}^n l_{jj} . \qquad (1.28)$$

In this paper we will extend Propositions 1.1 and 1.2 to links by using Reshetikhin's formula (1.20) for the Jones polynomial of a link as a generalization of Eq. (1.16). In Sect. 2 we derive a formula for the multivariable Alexander polynomial of a link in terms of the components of Reshetikhin's formula (1.20) (Proposition 2.1). In Sect. 3 we calculate the first terms in the Taylor series expansion of the multivariable Alexander polynomial. In Sect. 4 we check that the Alexander polynomial as given by Eq. (2.15) does satisfy some of its basic properties (Propositions 4.1 and 4.3). In Sect. 5 we use Reshetikhin's formula in order to derive the link surgery formula for loop corrections to the trivial connection contribution to Witten's invariant of a RHS (Proposition 5.1). In Appendix 1 we derive Reshetikhin's presentation for the Jones polynomial of a torus link. In Appendix 2 we give a brief description of the structure of the moduli space of flat connections in a link complement in the vicinity of the trivial connection. We demonstrate that those connections are in one-to-one correspondence with the stationary points of the phase in Reshetikhin's formula (Proposition A2.1) at least in the linear approximation around the trivial connection.

Path integral arguments are used for the derivation of propositions of Sects. 2 and 5. Therefore these propositions are not mathematically rigorous and should be considered as "physical." The arguments of Sects. 3 and 4 do not involve path integrals. The propositions of these sections are rigorously derived from "physical"

propositions of Sect. 1 and 2. The calculations in the appendices do not rely on physical methods. They are completely clean.

2. The Multivariable Alexander Polynomial

We will follow the method of Sect. 2 of [1] in order to relate Eq. (1.20) to the multivariable Alexander polynomial which we define here as the inverse of the Reidemeister–Ray–Singer torsion of the link complement:

$$\Delta_A(M, \mathscr{L}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}) = \tau_R^{-1}(M \backslash \mathrm{Tub}(\mathscr{L}); e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}), \qquad (2.1)$$

here $e^{2\pi i a_i}$ are the holonomies of the $U(1)$ flat connection in $M \backslash \mathrm{Tub}(\mathscr{L})$ around the meridians $C_1^{(j)}$ of the link components $\mathscr{L}_j$.

We take the limit $K \to \infty$ of the integral in Eq. (1.20) while keeping the ratios α_j/K fixed. Then according to Eq. (1.4) the partition function can be presented as a sum over flat connections in the link complement which satisfy (up to a conjugation) the boundary condition for each meridian $C_1^{(j)}$:

$$\mathrm{Pexp}\left(\oint_{C_1^{(j)}} A_\mu dx^\mu\right) = \exp\left(\frac{2\pi i}{K}\alpha_j\right), \qquad 1 \leqq j \leqq n. \qquad (2.2)$$

In contrast to the knot complement considered in Sect. 2 of [1], there may be irreducible flat connections in $M \backslash \mathrm{Tub}(\mathscr{L})$ even if the phases α_j/K are arbitrarily small (see, e.g. Appendix 2). Besides, there is not just one but 2^{n-1} reducible flat connections due to the fact that a diagonal $SU(2)$ holonomy fixed up to a conjugation by Eq. (2.2) corresponds to two $U(1)$ holonomies related by a Weyl reflection, i.e. differing by the sign of the exponent (the overall change of signs however does not change the gauge equivalence class of the $SU(2)$ connection).

We calculate the integral of Eq. (1.20) by the stationary phase approximation method. Let us first assume that all $|\vec{a}_j| \ll 1$. Then we should look for the extrema of the quadratic form (1.22) constrained by conditions (1.21). These extrema satisfy equations

$$\left(\sum_{j=1}^n l_{ij}\, \vec{a}_j\right) \times \vec{a}_i = 0, \qquad 1 \leqq i \leqq n. \qquad (2.3)$$

The solutions to these equations do indeed correspond (up to an overall $SO(3)$ rotation) to flat connections in the link complement for small phases $|\vec{a}_j|$ (see Appendix 1, for more details on flat connections in the link complement see [12]). Equations (2.3) are obviously satisfied when all the vectors $\vec{a}_j$ are parallel:

$$\vec{a}_j^{(0)} = a_j \vec{n}, \qquad (2.4)$$

here $\vec{n}$ is a unit vector and

$$|a_j| = \frac{\alpha_j}{K}. \qquad (2.5)$$

There are 2^{n-1} such inequivalent configurations depending on the choice of signs for a_j in Eqs. (2.5). They correspond to 2^{n-1} reducible flat connections. If the phases $|\vec{a}_j|$ are not small, then we should also account for the higher order polynomials

L_m, $m \geqq 3$. However Proposition 1.4 guarantees that the parallel configurations (2.4) still remain the stationary phase points of the full phase in Eq. (1.20).

The arguments of Sect. 2 of [1] suggest that the 1-loop (that is, leading in the K^{-1} expansion) approximation to the contribution of "reducible" stationary phase point (2.4) to the integral (1.20) is proportional to the Reidemeister–Ray–Singer torsion of the link complement and inversely proportional to the multivariable Alexander polynomial as defined by Eq. (2.1). To obtain this approximation we introduce the local coordinates $\vec{x}_j$ in the vicinity of the stationary phase point (2.4):

$$\vec{a}_j = \vec{a}_j^{(0)} + a_j \vec{x}_j + \frac{1}{2} \vec{a}_j^{(0)} \vec{x}_j^2 + \mathcal{O}(x^3), \quad \vec{n} \cdot \vec{x}_j = 0 . \tag{2.6}$$

We may retain only a quadratic part of the exponent in Eq. (1.20):

$$\frac{i\pi K}{2} \sum_{i,j=1}^{n} \sum_{\mu,\nu=1}^{2} M_{ij,\mu\nu}(\vec{a}_1, \ldots, \vec{a}_n) x_\mu^{(i)} x_\nu^{(j)} , \tag{2.7}$$

here $x_\mu^{(j)}$ are coordinates of the vectors $\vec{x}_j$. A quadratic form $M_{ij,\mu\nu}$ may receive contributions from all the polynomials L_m:

$$\sum_{\mu,\nu=1}^{2} M_{ij,\mu\nu}(\vec{a}_1, \ldots, \vec{a}_n) x_\mu^{(i)} x_\nu^{(j)}$$

$$= L_{ij} \vec{x}_i \cdot \vec{x}_j + \sum_{m=3}^{\infty} L_m(a_1 \vec{n}, \ldots, a_i \vec{x}_i, \ldots, a_j \vec{x}_j, \ldots, a_n \vec{n}) . \tag{2.8}$$

The matrix L_{ij} comes from L_2:

$$L_{ij} = l_{ij} a_i a_j - \delta_{ij} \sum_{k=1}^{n} l_{ik} a_i a_k . \tag{2.9}$$

In our approximation the integration measure for $\vec{x}_i$ is reduced to

$$\prod_{j=1}^{n} \frac{K}{4\pi} |a_j| d^2 \vec{x}_j . \tag{2.10}$$

Also we should retain only the following part of the preexponential factor in Eq. (1.20):

$$1 + \sum_{l=2}^{\infty} P_{0,l}(a_1 \vec{n}, \ldots, a_n \vec{n}) \tag{2.11}$$

(the polynomials $P_{0,l}(a_1 \vec{n}, \ldots, a_n \vec{n})$ do not depend on the orientation of $\vec{n}$). What remains is a gaussian integral over $\vec{x}_j$, which would produce a square root of the determinant of the $2n \times 2n$ matrix $M_{ij,\mu\nu}$. However there is a small problem: this matrix has two zero modes:

$$x_\mu^{(j)} = \delta_{\mu 1}, \; 1 \leqq j \leqq n \quad \text{and} \quad x_\mu^{(j)} = \delta_{\mu 2}, \; 1 \leqq j \leqq n , \tag{2.12}$$

which originate from $SO(3)$ rotations of $\vec{n}$. Zero modes appear quite often in calculations of the Alexander polynomial. They should be removed from the determinant and the integration over the direction of $\vec{n}$ should be performed with an appropriate measure. The removal of the zero modes is achieved either by taking a second

derivative of the characteristic polynomial of $M_{ij,\mu\nu}$ at zero, or by taking any of the non-zero second rank minors of two diagonal elements:

$$\det{}' M_{ij,\mu\nu} = \frac{1}{2}\partial_x^2 \det(M_{ij,\mu\nu} + x\delta_{ij}\delta_{\mu\nu})\big|_{x=0} \equiv n^2 \det M'', \tag{2.13}$$

here M'' is a $(2n-2) \times (2n-2)$ matrix obtained from $M_{ij,\mu\nu}$ by crossing out the columns and rows to which the two diagonal elements $M_{ii,11}$ and $M_{jj,22}$ belong (det M'' does not depend on the choice of i and j). Finally after using Eq. (1.7) as the 1-loop formula for $Z^{(\mathrm{tr})}(M;k)$, we get the following formula for the contribution to the Jones polynomial coming from the reducible flat connection related to the configuration (2.4):

$$Z^{(\mathrm{red})}_{(a_1,\ldots,a_n)}(M,\mathscr{L};k) = \exp\left(\frac{i\pi K}{2}\sum_{i,j=1}^n l_{ij}a_i a_j\right)\frac{1}{\sqrt{2K}}e^{\frac{i\pi}{4}\operatorname{sign}(M_{ij,\mu\nu})}[\operatorname{ord}H_1(M,\mathbb{Z})]^{-\frac{3}{2}}$$

$$\times (2\pi)^{2-n}\left(\prod_{i=1}^n |a_i|\right)|\det M''|^{-\frac{1}{2}}\left(1 + \sum_{l=2}^\infty P_{0,l}(a_1\vec{n},\ldots,a_n\vec{n})\right), \tag{2.14}$$

here sign $(M_{ij,\mu\nu})$ is the difference between the numbers of positive and negative eigenvalues of $M_{ij,\mu\nu}$. It is easy to relate the factors of this expression to those of Eq. (1.5): $\pi^2 \sum_{i,j=1}^n l_{ij}a_i a_j$ is the classical Chern–Simons action, $1/\sqrt{2K}$ is the factor $\frac{\sqrt{2\pi\hbar}}{\operatorname{Vol}U(1)}$ and $[\operatorname{ord}H_1(M,\mathbb{Z})]^{-1/2}$ is the contribution of the diagonal part of $SU(2)$ to the square root of the Reidemeister–Ray–Singer torsion. What remains is (up to a phase) the $U(1)$ torsion. According to Eq. (2.1) its inverse is the multivariable Alexander polynomial.

Proposition 2.1. *The formula for the multivariable Alexander polynomial of the link $\mathscr{L}$ is*

$$\Delta_A(M,\mathscr{L};e^{2\pi i a_1},\ldots,e^{2\pi i a_n}) = -ie^{-\frac{i\pi}{4}\operatorname{sign}(M_{ij,\mu\nu})}[\operatorname{ord}H_1(M,\mathbb{Z})](2\pi)^{n-2}\frac{|\det M''|^{\frac{1}{2}}}{\prod_{j=1}^n a_j}$$

$$\times \left[1 + \sum_{l=2}^\infty P_{0,l}(a_1\vec{n},\ldots,a_n\vec{n})\right]^{-1}, \tag{2.15}$$

here $\det M''$ may be expressed through the characteristic polynomial of the matrix $M_{ij,\mu\nu}$ according to Eq. (2.13), while the matrix $M_{ij,\mu\nu}$ itself is expressed by Eq. (2.8).

Let us denote as $S_{\Delta_A=0}$ the set of zeros of the Alexander polynomial Δ_A in the space of the phases $a_1,\ldots,a_n$. Obviously, $S_{\Delta_A=0}$ is an invariant of $\mathscr{L}$. Let us assume that there exists a finite neighborhood $\mathscr{V}$ of the origin of the a-space, in which the series $\sum_{l=2}^\infty P_{0,l}(a_1\vec{n},\ldots,a_n\vec{n})$ is at least asymptotically convergent. Then according to Eq. (2.15) in this neighborhood $S_{\Delta_A=0}$ is also a set of zeros of the function

$$\Delta_A^{(\mu)}(a_1,\ldots,a_n) = \frac{|\det M''|^{\frac{1}{2}}}{\prod_{j=1}^n a_j}. \tag{2.16}$$

Neither the function $\Delta_A^{(\mu)}$ nor the factor $1 + \sum_{l=2}^{\infty} P_{0,l}(a_1\vec{n}, \ldots, a_n\vec{n})$ are the invariants of the link $\mathscr{L}$ by themselves because they depend on the choice of the zero-points on the link components and the choice of the propagators (i.e., the choice of the gauge fixing) as described in [12]. Only their ratio in Eq. (2.15) is an invariant. However the set of zeros of $\Delta_A^{(\mu)}$ coincides with $S_{\Delta_A=0}$ in the vicinity $\mathscr{V}$ and is the invariant of $\mathscr{L}$.

Let us now extend Conjecture 1.1 beyond the well-defined lowest level Milnor's numbers:

Conjecture 2.1. *For any choice of parallels and meridians of the link components $\mathscr{L}_j$ which determines the values of all Milnor's linking numbers through the Magnus expansion, there exists a choice of propagators in the quantum Chern–Simons theory and a choice of the zero-points on $\mathscr{L}_j$ for the Feynman diagrams of* [12] *such that Eq.* (1.23) *holds for all values of m.*

A combination of Eqs. (1.23) and (2.8) expresses the coefficients of the matrix $M_{ij,\mu\nu}$ entering Eq. (2.16) in terms of Milnor's linking numbers $l_{i_1,\ldots,i_m}^{(\mu)}$. Therefore we conclude that the set $S_{\Delta_A=0}$ of zeros of the Alexander polynomial in the vicinity of the point $a_1 = \cdots = a_n = 0$ is determined by Milnor's linking numbers. In other words, the set of zeroes of $\Delta_A^{(\mu)}$ is an invariant of the higher order Milnor's linking numbers. Note that this set is generally not empty, since if the link has at least 3 components, then $\Delta_A|_{a_1=\cdots=a_n=0} = 0$.

Usually it is said that the higher order Milnor's linking numbers are the invariants of the link only modulo the greatest common factor of the lower order numbers. It seems that the set of zeroes of $\Delta_A^{(\mu)}$ is a sharper invariant. For example, if all ordinary linking numbers of a 3-component link are equal to 1 then the higher linking numbers modulo the lower ones are obviously equal to zero. If there exists a choice of meridians and parallels which actually makes them all equal to zero, then according to Eq. (3.4) the set of zeroes of $\Delta_A^{(\mu)}$ is a flat plane

$$a_1 + a_2 + a_3 = 0 \,. \tag{2.17}$$

However this is not necessarily the case for a general 3-component link with unit linking numbers.

3. Taylor Series

The bilinear form $\sum_{i,j=1}^{n} \sum_{\mu\nu=1}^{2} M_{ij,\mu\nu} x_\mu^{(i)} x_\nu^{(j)}$ includes only two basic bilinear structures coming from the r.h.s. of Eq. (2.8): $\vec{x}_i \cdot \vec{x}_j$ and $\vec{n} \cdot (\vec{x}_i \times \vec{x}_j)$. Therefore the matrix $M_{ij,\mu\nu}$ has the following block structure:

$$M = \left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array} \right) \tag{3.1}$$

here A and B are a symmetric and an antisymmetric $n \times n$ matrices. As a result, the characteristic polynomial of $M_{ij,\mu\nu}$ is a square of another polynomial of x and matrix elements $M_{ij,\mu\nu}$. Also a matrix element $M_{ij,\mu\nu}$ is proportional to a_i and a_j. This together with the particular form of the zero modes (2.12) guarantees that det M'' is proportional to $(\prod_{j=1}^{n} a_j)^2$. Thus we conclude that the r.h.s. of Eq. (2.15) can be expanded in Taylor series in phases a_i. The first two terms of this expansion are

used in C. Lescop's surgery formula for the Casson–Walker invariant, so we are going to find their expression.

To get the first term in the Taylor series we retain only the terms $L_{i,j} \vec{x}_i \cdot \vec{x}_j$ in the r.h.s. of Eq. (2.8). Then the matrix $M_{ij,\mu\nu}$ splits into a direct sum of two equal matrices L_{ij}:

$$M_{ij,\mu\nu} = L_{ij}\delta_{\mu\nu} + \mathcal{O}(a^3). \tag{3.2}$$

Therefore

$$ie^{-\frac{i\pi}{4}\,\mathrm{sign}\,(M_{ij,\mu\nu})}\big|\det M''\big|^{\frac{1}{2}} = \frac{i^{n-2}}{n}\det{}'L = \frac{i^{n-2}}{n}\partial_x \det(L_{ij} + x\delta_{ij})\big|_{x=0}$$

$$= i^{n-2}\det L', \tag{3.3}$$

here L' is any of the minors of diagonal elements of L_{ij} (they are all equal). Thus the first term in the Taylor expansion of the Alexander polynomial is a polynomial in a_i of degree $n - 2$:

$$\Delta_A^{(n-2)}(M, \mathscr{L}; a_1, \ldots, a_n) = -(-2\pi i)^{n-2}[\mathrm{ord}\,H_1(M, \mathbb{Z})]\frac{\det L'(a_1, \ldots, a_n)}{\prod_{j=1}^{n} a_j}. \tag{3.4}$$

This expression coincides[4] with the formula of [13].

Obviously, Eq. (3.4) provides the leading term in the Taylor series expansion of the multivariable Alexander polynomial if $\det L'(a_1, \ldots, a_n)$ is non-zero. If the linking numbers l_{ij} are zero then, in view of Conjecture 1.1, the dominant term will be expressed through higher Milnor's invariants.

To get the second term in the Taylor expansion we have to account for the polynomials L_3 and L_4 in the r.h.s. of Eq. (2.8) as well as for the polynomial $P_{0,2}$ in the preexponential factor of Eq. (2.14). We can expand the exponential of Eq. (1.20) in L_3 and L_4. Only the second power of L_3 and the first power of L_4 contribute to the leading power in K. We make a simple rearrangement

$$(\vec{a}_{i_1} \times \vec{a}_{i_2}) \cdot (\vec{a}_{j_1} \times \vec{a}_{j_2}) = \det \begin{pmatrix} (\vec{a}_{i_1} \cdot \vec{a}_{j_1}) & (\vec{a}_{i_1} \cdot \vec{a}_{j_2}) \\ (\vec{a}_{i_2} \cdot \vec{a}_{j_1}) & (\vec{a}_{i_2} \cdot \vec{a}_{j_2}) \end{pmatrix}, \tag{3.5}$$

$$[\vec{a}_{i_1} \cdot (\vec{a}_{i_2} \times \vec{a}_{i_3})][\vec{a}_{j_1} \cdot (\vec{a}_{j_2} \times \vec{a}_{j_3})]$$

$$= \det \begin{pmatrix} (\vec{a}_{i_1} \cdot \vec{a}_{j_1}) & (\vec{a}_{i_1} \cdot \vec{a}_{j_2}) & (\vec{a}_{i_1} \cdot \vec{a}_{j_3}) \\ (\vec{a}_{i_2} \cdot \vec{a}_{j_1}) & (\vec{a}_{i_2} \cdot \vec{a}_{j_2}) & (\vec{a}_{i_2} \cdot \vec{a}_{j_3}) \\ (\vec{a}_{i_3} \cdot \vec{a}_{j_1}) & (\vec{a}_{i_3} \cdot \vec{a}_{j_2}) & (\vec{a}_{i_3} \cdot \vec{a}_{j_3}) \end{pmatrix}. \tag{3.6}$$

Multiplying the preexponential factor of the integral of Eq. (1.20) by an extra scalar product $(\vec{a}_i \cdot \vec{a}_j)$ is equivalent to taking a derivative $\partial_{l_{ij}}$. By applying this

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[4] I am thankful to C. Lescop for checking this.

trick to the factors (3.5) and (3.6) we arrive at the formula

$$\Delta_A^{(n)}(M, \mathcal{L}; a_1, \ldots, a_n)$$

$$= -(-2\pi i)^{n-2}[\operatorname{ord} H_1(M, \mathbb{Z})]\left[-\sum_{i,j=1}^n p_{i,j} a_i a_j + 4 \sum_{i_1,i_2,j_1,j_2=1}^n l_{i_1 i_2, j_1 j_2}^{(4)} a_{i_1} a_{j_1} \partial_{l_{i_2 j_2}} \right.$$

$$\left. -\frac{9}{2} \sum_{i_1,i_2,i_3,j_1,j_2,j_3=1}^n l_{i_1 i_2 i_3}^{(3)} l_{j_1 j_2 j_3}^{(3)} a_{i_1} a_{j_1} \partial_{l_{i_2 j_2}} \partial_{l_{i_3 j_3}} \right] \frac{\det L'}{\prod_{j=1}^n a_j} . \tag{3.7}$$

We consider l_{ij} and l_{ji} as independent variables when we take derivatives in this formula. The coefficients p_{ij}, $l_{i_1 i_2 i_3}^{(3)}$ and $l_{i_1 i_2, j_1 j_2}^{(4)}$ come from Eqs. (1.26), (1.24) and (1.25).

Proposition 3.1. *The expression* (2.15) *for the multivariable Alexander polynomial can be expanded in the Taylor series in phases* a_j:

$$\Delta_A(M, \mathcal{L}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}) = \sum_{j=0}^\infty \Delta_A^{(n-2+2j)}(M, \mathcal{L}; a_1, \ldots, a_n) . \tag{3.8}$$

Each term $\Delta_A^{(n-2+2j)}(M, \mathcal{L}; a_1, \ldots, a_n)$ *is a polynomial of degree* $n-2+2j$. *The first two terms in this expansion are given by Eqs.* (3.4) *and* (3.7).

4. Basic Properties of the Alexander Polynomial

We are going to check whether the r.h.s. of Eq. (1.20) satisfies some basic properties of the multivariable Alexander polynomial. Let us find the value of $\Delta_A(M, \mathcal{L}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_n})$ when $a_n = 0$. Consider the matrix $M_{ij,\mu\nu}''$. Suppose for simplicity that the diagonal elements $M_{nn,11}$ and $M_{nn,22}$ do not belong to the two columns and rows that were removed from $M_{ij,\mu\nu}$. Then it is not hard to see that the part of $\det M''$ which is proportional only to the second power of a_n, must include both these elements. As a result,

$$\Delta_A(M, \mathcal{L}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_{n-1}}, 1) = -ie^{-\frac{i\pi}{4}\operatorname{sign}(M_{[n]})}(2\pi)^{n-2}[\operatorname{ord} H_1(M, \mathbb{Z})]$$

$$\times \left(1 + \sum_{l=2}^\infty P_{0,l}(a_1 \vec{n}, \ldots, a_n \vec{n}) \right)^{-1} \frac{|\det M_{[n]}''(a_1, \ldots, a_{n-1})|^{\frac{1}{2}}}{\prod_{j=1}^{n-1} a_j} \left(i \sum_{j=1}^{n-1} l_{jn} a_j \right), \tag{4.1}$$

here $M_{[n]}''$ is a $(n-4) \times (n-4)$ matrix obtained by "reducing" the $(n-2) \times (n-2)$ matrix M'': two rows and two columns containing the elements $M_{nn,11}$ and $M_{nn,22}$ are removed and $a_n = 0$ is substituted in all other matrix elements.

Suppose now that we remove the nth component $\mathcal{L}_n$ of the link $\mathcal{L}$. We denote the remaining link as $\mathcal{L}_{[n]}$. To calculate its Jones polynomial we have to substitute $\alpha_n = 1$ in Eq. (1.20). Then $|\vec{a}_n| = 1/K$ and the contribution of the configuration (2.4)

for $1 \leq j \leq n-1$ to the integral (1.20) in the leading order in K is equal to

$$Z^{(\mathrm{red})}_{(a_1,\ldots,a_{n-1})}(M, \mathscr{L}_{[n]}; k)$$

$$= \exp\left(\frac{i\pi K}{2} \sum_{i,j=1}^{n-1} l_{ij} a_i a_j\right) \frac{1}{\sqrt{2K}} e^{\frac{i\pi}{4} \mathrm{sign}(M_{[n]})} [\mathrm{ord}\, H_1(M, \mathbb{Z})]^{-\frac{3}{2}} (2\pi)^{3-n} \frac{\prod_{j=1}^{n-1} |a_j|}{|\det M''_{[n]}|^{\frac{1}{2}}}$$

$$\times \left(1 + \sum_{l=2}^{\infty} P_{0,l}(a_1 \vec{n}, \ldots, a_{n-1}\vec{n}, 0)\right) \int_{|\vec{\rho}|=1} \frac{d^2\vec{\rho}}{4\pi} \exp\left[i\pi\left(\sum_{j=1}^{n-1} l_{jn} a_j\right) \vec{\rho} \cdot \vec{n}\right]$$

$$= \exp\left(\frac{i\pi K}{2} \sum_{i,j=1}^{n-1} l_{ij} a_i a_j\right) \frac{1}{\sqrt{2K}} e^{\frac{i\pi}{4} \mathrm{sign}(M_{[n]})} [\mathrm{ord}\, H_1(M, \mathbb{Z})]^{-\frac{3}{2}} (2\pi)^{3-n} \frac{\prod_{j=1}^{n-1} |a_j|}{|\det M''_{[n]}|^{\frac{1}{2}}}$$

$$\times \left(1 + \sum_{l=2}^{\infty} P_{0,l}(a_1 \vec{n}, \ldots, a_{n-1}\vec{n}, 0)\right) \frac{\sin\left(\pi \sum_{j=1}^{n-1} l_{jn} a_j\right)}{\pi \sum_{j=1}^{n-1} l_{jn} a_j}. \tag{4.2}$$

After extracting the $U(1)$ Reidemeister-Ray-Singer torsion from this expression we find that

$$\Delta_A(M, \mathscr{L}_{[n]}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_{n-1}})$$

$$= -i e^{-\frac{i\pi}{4} \mathrm{sign}(M_{[n]})} (2\pi)^{n-3} [\mathrm{ord}\, H_1(M, \mathbb{Z})] \left(1 + \sum_{l=2}^{\infty} P_{0,l}(a_1 \vec{n}, \ldots, a_{n-1}\vec{n}, 0)\right)^{-1}$$

$$\times \frac{|\det M''_{[n]}|^{\frac{1}{2}}}{\prod_{j=1}^{n-1} a_j} \frac{\pi \sum_{j=1}^{n-1} l_{jn} a_j}{\sin\left(\pi \sum_{j=1}^{n-1} l_{jn} a_j\right)}. \tag{4.3}$$

Comparing Eqs. (4.1) and (4.3) we conclude that

Proposition 4.1. *The multicolored Alexander polynomial as defined by Eq.* (2.15) *satisfies the following property*:

$$\Delta_A(M, \mathscr{L}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_{n-1}}, 1)$$

$$= 2i \sin\left(\pi \sum_{j=1}^{n-1} l_{jn} a_j\right) \Delta_A(M, \mathscr{L}_{[n]}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_{n-1}}), \tag{4.4}$$

here $\mathscr{L}_{[n]}$ *is the link* $\mathscr{L}$ *with* n^{th} *component removed*.

Now let us see what happens if we perform a $U^{(p,q)}$ surgery on the nth component of $\mathscr{L}$ thus constructing a new RHS M' with the link $\mathscr{L}_{[n]}$ inside it. According

to the surgery formula (1.10) and the results of Sect. 3 of [1],

$$Z^{(\mathrm{tr})}_{(\alpha_1,\ldots,\alpha_{n-1})}(M',\mathscr{L}_{[n]};k)$$

$$= Z^{(\mathrm{tr})}(M;k)e^{-\frac{3}{4}i\pi\,\mathrm{sign}\,(\frac{p}{q}+l_{nn})}\frac{2\,\mathrm{sign}\,(q)}{\sqrt{2K|q|}}\exp\frac{i\pi}{2K}\left[12s(p,q)-\frac{p}{q}+3\,\mathrm{sign}\,\left(\frac{p}{q}+l_{nn}\right)\right]$$

$$\times \int_0^\infty K\,da_n \int_{\substack{|\vec{a}_j|=\alpha_j/K\\|\vec{a}_n|=a_n}}\prod_{j=1}^n\left(\frac{K}{4\pi}\frac{d^2\vec{a}_j}{|\vec{a}_j|}\right)\exp\left[\frac{i\pi K}{2}\left(\sum_{m=2}^\infty L_m(\vec{a}_1,\ldots,\vec{a}_n)+\frac{p}{q}\vec{a}_n^2\right)\right]$$

$$\times \sin\left(\pi\frac{a_n}{q}\right)\left[1+\sum_{\substack{l,m=0\\l+m\neq0}}^\infty K^{-m}P_{m,l}(\vec{a}_1,\ldots,\vec{a}_n)\right],\tag{4.5}$$

or, equivalently,

$$Z^{(\mathrm{tr})}_{(\alpha_1,\ldots,\alpha_{n-1})}(M',\mathscr{L}_{[n]};k)$$

$$= Z^{(\mathrm{tr})}(M;k)e^{-\frac{3}{4}i\pi\,\mathrm{sign}\,(\frac{p}{q}+l_{nn})}\frac{2\,\mathrm{sign}\,(q)}{\sqrt{2K|q|}}$$

$$\times \exp\frac{i\pi}{2K}\left[12s(p,q)-\frac{p}{q}+3\,\mathrm{sign}\,\left(\frac{p}{q}+l_{nn}\right)\right]$$

$$\times \int\frac{K^2}{4\pi}d^3\vec{a}_n\int_{|\vec{a}_j|=\alpha_j/K}\prod_{j=1}^{n-1}\left(\frac{K}{4\pi}\frac{d^2\vec{a}_j}{|\vec{a}_j|}\right)\exp\left[\frac{i\pi K}{2}\left(\sum_{m=2}^\infty L_m(\vec{a}_1,\ldots,\vec{a}_n)+\frac{p}{q}\vec{a}_n^2\right)\right]$$

$$\times \frac{1}{|\vec{a}_n|}\sin\left(\pi\frac{|\vec{a}_n|}{q}\right)\left[1+\sum_{\substack{l,m=0\\l+m\neq0}}^\infty K^{-m}P_{m,l}(\vec{a}_1,\ldots,\vec{a}_n)\right].\tag{4.6}$$

The integral over $\vec{a}_n$ should be calculated in the following way. We first separate the part of the sum $\sum_{m=2}^\infty L_m(\vec{a}_1,\ldots,\vec{a}_n)$ which is linear in $\vec{a}_n$:

$$\vec{a}_n\cdot\sum_{m=2}^\infty\vec{L}_m^{(n)}(\vec{a}_1,\ldots,\vec{a}_{n-1}),\tag{4.7}$$

and introduce a new variable $\vec{x}$ instead of $\vec{a}_n$:

$$\vec{a}_n=\vec{x}-\frac{1}{2}\frac{q}{p+ql_{nn}}\sum_{m=2}^\infty\vec{L}_m^{(n)}(\vec{a}_1,\ldots,\vec{a}_{n-1}).\tag{4.8}$$

After substituting Eq. (4.8) into the integral (4.6) we separate the terms of the exponent that do not depend on $\vec{x}$. These terms form the exponent of the representation Eq. (1.20) for $Z^{(\mathrm{tr})}_{\alpha_1,\ldots,\alpha_{n-1}}(M',\mathscr{L}_{[n]};k)$. We leave the term $\frac{p+ql_{nn}}{q}\vec{x}^2$ in the exponent of Eq. (4.6) and expand that exponent in all other terms which are at least quadratic in $\vec{x}$ (there are no linear terms thanks to the substitution (4.8)). This expansion mixes up with expansions in powers of $\vec{x}$ of two preexponential factors $1+\sum_{\substack{l,m=0\\l+m\neq0}}^\infty K^{-m}P_{m,l}(\vec{a}_1,\ldots,\vec{a}_n)$ and $\sin(\pi\frac{|\vec{a}_n|}{q})|\vec{a}_n|^{-1}$ (the latter factor is in fact

analytic in $\vec{a}_n$ since its expansion contains only even powers of $|\vec{a}_n|$). Thus, similar to the integral (1.19), what remains is a bunch of gaussian integrals over $\vec{x}$. The limit on the powers of K versus powers of $\vec{x}$ in the expansion of the preexponential factor is weaker than that of Eq. (1.19) (e.g. we now have positive powers of K). However it is easy to see that the main property still holds (cf. Corollary 1.1):

Proposition 4.2. *Only a finite number of the polynomials L_m and $P_{m,l}$ of Eq. (4.6) are needed to express a given polynomial $L_m, P_{m,l}$ or a given term in the $1/K$ expansion of $Z^{(\mathrm{tr})}(M';k)$ participating in the expression* (1.20) *for $Z^{(\mathrm{tr})}_{\alpha_1,\dots,\alpha_{n-1}}(M',\mathscr{L}_{[n]};k)$.*

To determine what happens to the multivariable Alexander polynomial under the surgery $U^{(p,q)}$ on the link component $\mathscr{L}_n$ we have to find the contribution of the configuration (2.4) to the integral (4.5) to the leading order in K. In view of Proposition 1.4, the integral over a_n is dominated by the stationary phase point

$$a_n^{(\mathrm{st})} = -\frac{q}{p+ql_{nn}}\sum_{j=1}^{n-1}l_{jn}a_j\,. \tag{4.9}$$

We need only the 1-loop approximation to this integral:

$$Z^{(\mathrm{red})}_{(a_1,\dots,a_{n-1})}(M',\mathscr{L}_n;k)$$

$$= \sqrt{\frac{2}{K}}\pi[\mathrm{ord}\,H_1(M,\mathbb{Z})]^{-\frac{3}{2}}|p+ql_{nn}|^{-\frac{1}{2}}2i\sin\left(\frac{\pi\sum_{j=1}^{n-1}l_{jn}a_j}{p+ql_{nn}}\right)\mathrm{sign}\,(p+ql_{nn})$$

$$\times\int\limits_{\substack{|\vec{a}_j|=|a_j|\\|\vec{a}_n|=|a_n^{(\mathrm{st})}|}}\prod_{j=1}^{n}\left(\frac{K}{4\pi}\frac{d^2\vec{a}_j}{|\vec{a}_j|}\right)\exp\left(\frac{i\pi K}{2}\sum_{m=2}^{\infty}L_m(\vec{a}_1,\dots,\vec{a}_n)\right)$$

$$\times\left[1+\sum_{\substack{l,m=0\\l+m\neq 0}}^{\infty}K^{-m}P_{m,l}(\vec{a}_1,\dots,\vec{a}_n)\right]+\mathcal{O}(K^{-\frac{3}{2}})\,, \tag{4.10}$$

and we have to take only the contribution of the configuration (2.4) with $a_n = a_n^{(\mathrm{st})}$ to this integral. Comparing this expression with Eqs. (1.5) and the surgery formula for the 1-loop correction

$$e^{iS_1^{(\mathrm{tr})}(M')} = |p+vq|^{-\frac{3}{2}}e^{iS_1^{(\mathrm{tl})}(M)}\,, \tag{4.11}$$

we conclude that

Proposition 4.3. *If a $U^{(p,q)}$ surgery on the nth component of a link L in RHS M produces another RHS M', then for the remaining link $\mathscr{L}_{[n]}$,*

$$\Delta_A(M',\mathscr{L}_{[n]};e^{2\pi ia_1},\dots,e^{2\pi ia_{n-1}}) = 2i\sin\left(\frac{\pi\sum_{j=1}^{n-1}l_{jn}a_j}{p+ql_{nn}}\right)\mathrm{sign}\,(p+ql_{nn})$$

$$\times\,\Delta_A\left(M,\mathscr{L};e^{2\pi ia_1},\dots,e^{2\pi ia_{n-1}},\exp\left(-\frac{2\pi iq}{p+ql_{nn}}\sum_{j=1}^{n-1}l_{jn}a_j\right)\right)\,. \tag{4.12}$$

5. The Link Surgery Formula

Now we turn to the subject of our main concern: the surgery formula for the contribution of the trivial connection to Witten's invariant. Suppose that a RHS M contains an N-component link $\mathscr{L}$ and we perform $U^{(p_j, q_j)}$ surgeries on its components in order to obtain a new RHS M'. Applying the arguments of Sect. 3 of [1] to Eq. (1.20) instead of Eq. (1.16) we conclude that

Proposition 5.1. *The trivial connection contribution to Witten's invariants of RHSM and M' connected by $U^{(p_j, q_j)}$ surgeries on components of a link $\mathscr{L}$ in M, are related by the following equation:*

$$
\begin{aligned}
&Z^{(\mathrm{tr})}(M'; k) \\
&= Z^{(\mathrm{tr})}(M; k) \exp\left(-\frac{3}{4} i\pi \operatorname{sign}(L^{(\mathrm{tot})})\right) \\
&\times \exp\frac{i\pi}{2K}\left[3\operatorname{sign}(L^{(\mathrm{tot})}) + \sum_{j=1}^{n}\left(12 s(p_i, q_i) - \frac{p_i}{q_i}\right)\right] \\
&\times \int_{[\vec{a}_j = 0]} \prod_{j=1}^{n}\left(\frac{K^2}{4\pi} d^3\vec{a}_j \frac{2\operatorname{sign}(q_j)}{\sqrt{2K|q_j|}} \frac{\sin\left(\pi\frac{|\vec{a}_j|}{q_j}\right)}{|\vec{a}_j|}\right)\left[1 + \sum_{\substack{l,m=0 \\ l+m \neq 0}}^{\infty} K^{-m} P_{m,l}(\vec{a}_1, \ldots, \vec{a}_n)\right] \\
&\times \exp\left(\frac{i\pi K}{2}\sum_{i,j=1}^{n} L_{ij}^{(\mathrm{tot})} \vec{a}_i \cdot \vec{a}_j\right).
\end{aligned}
\tag{5.1}
$$

The symbol $\int_{[\vec{a}_j = 0]}$ means that we take only the contribution of the stationary phase point $\vec{a}_j = 0$, $1 \leq j \leq n$, which should be calculated in the following way: all the factors except for the last exponential should be expanded in powers of $\vec{a}_j$ and then the gaussian integrals with polynomial prefactors should be calculated one by one.

Although in contrast to Eq. (1.19) there will be positive powers of K in the preexponential series, still.

Corollary 5.1. *Only a finite number of the polynomials L_m and $P_{m,l}$ are required to express $Z^{(\mathrm{tr})}(M'; k)$ at a given order in $1/K$ expansion.*

We can use Eq. (5.1) in order to derive explicit surgery formulas for the first two loop corrections to $Z^{(\mathrm{tr})}(M; k)$:

$$
e^{iS_1^{(\mathrm{tr})}(M')} = \left|\det L^{(\mathrm{tot})} \prod_{j=1}^{n} q_n\right|^{-\frac{3}{2}} e^{iS_1^{(\mathrm{tr})}(M)},
\tag{5.2}
$$

which is consistent with Eq. (1.7) and

$$
S_2^{(\mathrm{tr})}(M') = S_2^{(\mathrm{tr})}(M) + 3\Delta_{\mathrm{CW}},
\tag{5.3}
$$

here

$$\Delta_{\mathrm{CW}} = \frac{1}{4}\mathrm{sign}\left(L^{(\mathrm{tot})}\right) + \sum_{j=1}^{n}\left((s(p_j,q_j)) - \frac{1}{12}\left(\frac{p_j}{q_j} + l_{jj}\right)\right)$$

$$- \frac{1}{\pi^2\det L^{(\mathrm{tot})}}\left[-\frac{1}{2}\sum_{i,j=1}^{n}\left(p_{ij} - \frac{1}{6}\frac{\pi^2}{q_i^2}\delta_{ij}\right)\partial_{l_{ij}} + \frac{3}{2}\sum_{i_1,i_2,j_1,j_2=1}^{n}l^{(4)}_{i_1i_2j_1j_2}\partial_{l_{i_1j_1}}\partial_{l_{i_2j_2}}\right.$$

$$\left. + \frac{3}{4}\sum_{i_1,i_2,i_3,j_1,j_2,j_3=1}^{n}l^{(3)}_{i_1i_2i_3}l^{(3)}_{j_1j_2j_3}\partial_{l_{i_1j_1}}\partial_{l_{i_2j_2}}\partial_{l_{i_3j_3}}\right]\det L^{(\mathrm{tot})}. \qquad (5.4)$$

In view of Eq. (1.8) we assume that

$$\lambda_{\mathrm{CW}}(M') = \lambda_{\mathrm{CW}}(M) + \Delta_{\mathrm{CW}}. \qquad (5.5)$$

We did not compare Eqs. (5.5), (5.4), (3.4) and (3.7) directly to the surgery formula of [13]. However the latter formula was derived from Walker's surgery formula by using the properties (4.4) and (4.12) of the multivariable Alexander polynomial. Since our formula also satisfies these properties, we assume that it is consistent with the results of [13].

6. Discussion

The results of this paper are based on Reshetikhin's formula (1.20) which separates the exponent of order K from the preexponential factor of order at most K^0. This separation allowed us to extract the large k asymptotics of the Jones polynomial of a link and of the link surgery formula (1.10).

Assuming that Conjecture 1.1 is correct, we see the relation between the leading part of the multivariable Alexander polynomial when its arguments are close to 1, and Milnor's linking numbers of the knot. Slightly generalizing the results of [14] and [15] we may say that the Alexander polynomial and Milnor's linking numbers are the algebraic tools for the study of irreducible deformations of reducible flat connections in the knot complement: the zeros of the Alexander polynomial indicate the points where the deformation can be carried out and Milnor's numbers determine the possible directions of the deformation.

The surgery formula for the loop corrections $S_n^{(\mathrm{tr})}$ to the trivial connection contribution to Witten's invariant of a rational homology sphere as defined by Eq. (1.4) was derived in [1] at the "physical" level of rigor. The extension of this formula to links provided by Proposition 5.1 gives it a better chance to acquire a rigorous mathematical proof. In other words, the invariance of the r.h.s. of Eq. (5.1) under Kirby moves has to be established.

Equation (5.1) is a surgery formula for the perturbative invariants $S_n^{(\mathrm{tr})}$ defined in canonical framing. The same invariants can be calculated through Feynman diagrams which require a certain regularization [17]. The relation between this regularization and the choice of framing still remains to be understood.

Another open question is a calculation of the contributions of nontrivial connections as well as the extension of this discussion beyond the rational homology spheres. Some experimental results on Witten's invariant for these cases are provided in [6, 5, 7 and 1], while a study of Casson's invariant of the manifolds with

nontrivial rational homology was carried out in [13]. However, all these results seem to require a more detailed analysis.

Appendix 1

We are going to derive Reshetikhin's formula (1.20) for the type (n, mn) torus link $\mathscr{L}_{(n,mn)}$. This is a very simple link which consists of n parallel components which are twisted m times.

Its Jones polynomial is easy to calculate:

$$Z_{\alpha_1,\ldots,\alpha_n}(S^3, \mathscr{L}_{(n,mn)}; k)$$

$$= -\frac{i^{\frac{1}{2}}}{K\sqrt{m}} \exp\left[-\frac{i\pi}{K}\frac{m}{2}\left(1 + \sum_{j=1}^{n}(\alpha_i^2 - 1)\right)\right]$$

$$\times \sum_{\beta=1}^{K-1} \frac{\prod_{j=1}^{n}\sin\left(\frac{\pi}{K}\alpha_i\beta\right)}{\sin^{n-1}\left(\frac{\pi}{K}\beta\right)} \sum_{l=0}^{m-1}\sum_{\mu=\pm 1} \mu \exp\left[-\frac{i\pi}{2Km}(\beta + 2Kl + \mu)^2\right] . \quad (A1.1)$$

The sum over l is a nuisance because it is nowhere present in the r.h.s. of Eq. (1.20). However by applying the methods of Sect. 4 of [1] we can show that the contribution of the terms with $l \neq 0$ is related only to irreducible connections which appear only when the values of the phases α_i/K are large enough (the $n + 1$ numbers α_i/K and $2l/m$ should satisfy "polygon inequality" conditions). Ultimately for small values of α_i/K we can use the expression

$$Z_{\alpha_1,\ldots,\alpha_n}(S^3, \mathscr{L}_{(n,mn)}; k) = \frac{2i^{\frac{3}{2}}}{K\sqrt{m}} \exp\left[-\frac{i\pi}{K}\frac{m}{2}\left(1 - n + \frac{1}{m^2} + \sum_{j=1}^{n}\alpha_j\right)\right]$$

$$\times \int_{-\infty}^{+\infty} \frac{\prod_{j=1}^{n}\sin\left(\frac{\pi}{K}\alpha_i\beta\right)}{\sin^{n-1}\left(\frac{\pi}{K}\beta\right)} \sin\left(\frac{\pi}{K}\frac{\beta}{m}\right) \exp\left(-\frac{i\pi}{2K}\frac{\beta^2}{m}\right) . \qquad (A1.2)$$
$$\scriptstyle [0 \leq \beta \leq K]$$

A simple formula

$$\int_{|\vec{a}|=\frac{a}{K}} \frac{K}{4\pi} \frac{d^2\vec{a}}{|\vec{a}|} \exp\left(i\pi K \vec{a} \cdot \vec{b}\right) = \frac{\sin(\pi\alpha|\vec{b}|)}{\pi|\vec{b}|} \qquad (A1.3)$$

allows us to rewrite Eq. (A1.2) as

$$
Z_{\gamma_1,\ldots,\gamma_n}(S^3, \mathscr{L}_{(n,mn)}; k) = Z(S^3; k) \left(\frac{1}{2}\frac{K}{m}\right)^{\frac{3}{2}} \exp\left[\frac{i\pi}{2K}m\left((n-1)-m^{-2}\right)\right]
$$

$$
\times \int\limits_{|\vec{a}_j|=\frac{\gamma_j}{K}} \prod_{j=1}^{n}\left(\frac{K}{4\pi}\frac{d^2\vec{a}_j}{|\vec{a}_j|}\right) \int d^3\vec{b} \left(\frac{\pi|\vec{b}|}{\sin(\pi|\vec{b}|)}\right)^{n-1} \frac{\sin\left(\pi\frac{|\vec{b}|}{m}\right)}{\pi\frac{|\vec{b}|}{m}}
$$

$$
\times \exp\left[-\frac{i\pi K}{2}\left(m\sum_{j=1}^{n}\vec{a}_j^2 - 2\vec{b}\cdot\sum_{j=1}^{n}\vec{a}_j + \frac{\vec{b}^2}{m}\right)\right]. \tag{A1.4}
$$

After changing the integration variable from $\vec{b}$ to

$$
\vec{x} = \vec{b} - m\sum_{j=1}^{n}\vec{a}_j, \tag{A1.5}
$$

expanding the preexponential factor in powers of $\vec{x}^2$ and calculating gaussian integrals over $\vec{x}$, we obtain the formula

$$
Z_{\gamma_1,\ldots,\gamma_n}(S^3, \mathscr{L}_{(n,mn)}; k)
$$

$$
= Z(S^3; k) \int\limits_{|\vec{a}_j|=\frac{\gamma_j}{K}} \prod_{j=1}^{n}\left(\frac{K}{4\pi}\frac{d^2\vec{a}_j}{|\vec{a}_j|}\right) \exp\left[\frac{i\pi K}{2}m\sum_{\substack{i,j=1\\i\neq j}}^{n}\vec{a}_i\cdot\vec{a}_j\right]
$$

$$
\times \exp\left[\frac{i\pi}{2K}m(n-1-m^{-2})\right] \frac{\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\right)} \sum_{l=0}^{\infty}\left(-\frac{im}{2\pi K}\right)^l \frac{(2l+1)!}{l!(2l)!}\partial_\gamma^{(2l)}
$$

$$
\times \left[\left(\frac{\pi y}{\sin(\pi y)}\right)^{n-1}\frac{\sin\left(\pi\frac{y}{m}\right)}{\left(\pi\frac{y}{m}\right)}\right]\Bigg|_{y=m\left|\sum_{j=1}^{n}\vec{a}_j\right|}. \tag{A1.6}
$$

This is Reshetikhin's formula (1.20). In particular,

$$
l_{ij} = m \quad \text{for} \quad i\neq j, \quad l_{ij} = 0 \tag{A1.7}
$$

and

$$
L_m(\vec{a}_1,\ldots,\vec{a}_n) = 0 \quad \text{for } m \geq 3, \tag{A1.8}
$$

(this seems to be a general property of torus links), while

$$
1 + \sum_{\substack{l,m=0\\l+m\neq 0}}^{\infty} K^{-m}P_{m,l}(\vec{a}_1,\ldots,\vec{a}_n)
$$

$$
= \exp\left[\frac{i\pi}{2K}m(n-1-m^{-2})\right] \frac{\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\right)} \sum_{l=0}^{\infty}\left(-\frac{im}{2\pi K}\right)^l \frac{(2l+1)!}{l!(2l)!}\partial_y^{(2l)}
$$

$$
\times \left[\left(\frac{\pi y}{\sin(\pi y)}\right)^{n-1}\frac{\sin\left(\pi\frac{y}{m}\right)}{\left(\pi\frac{y}{m}\right)}\right]\Bigg|_{y=m\left|\sum_{j=1}^{\infty}\vec{a}_j\right|}. \tag{A1.9}
$$

It is easy to check the relation (2.15) between Eq. (A1.6) and the Alexander polynomial if we recall that

$$\Delta_{\mathscr{A}}(S^3, \mathscr{L}_{(n,mn)}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}) = (2i)^{n-2} \frac{\sin^{n-1}\left(\pi m \sum_{j=1}^n a_j\right)}{\sin\left(\pi \sum_{j=1}^n a_j\right)}, \qquad \text{(A1.10)}$$

and that in our case

$$-i e^{-\frac{i\pi}{4} \operatorname{sign}(M_{IJ,\mu\nu})} |\det M''|^{\frac{1}{2}} = i^{n-2} m^{n-1} \left(\sum_{j=1}^n a_j\right)^{n-2} \prod_{j=1}^n a_j. \qquad \text{(A1.11)}$$

The torus link $\mathscr{L}_{(n,mn)}$ provides an example of existence of irreducible flat connections in the link complement even for arbitrarily small phases $|\vec{a}_j|$. Equation (2.3), which in view of Eq. (A1.8) is exactly valid for small phases, is reduced to a condition

$$\left(\sum_{i=1}^n \vec{a}_i\right) \times \vec{a}_j = 0, \quad 1 \leqq j \leqq n, \qquad \text{(A1.12)}$$

which is obviously satisfied if

$$\sum_{j=1}^n \vec{a}_j = 0. \qquad \text{(A1.13)}$$

The necessary and sufficient condition for the existence of this configuration is that the phases $|\vec{a}_j|$ satisfy "polygon inequalities":

$$\sum_{\substack{i=1 \\ i \neq j}}^n |\vec{a}_i| \leqq |\vec{a}_j|, \quad 1 \leqq j \leqq n. \qquad \text{(A1.14)}$$

These inequalities can indeed be satisfied even for arbitrarily small phases. Note that the extremal cases of these inequalities, i.e.

$$|\vec{a}_j| = \sum_{\substack{i=1 \\ i \neq j}}^n |\vec{a}_i|, \qquad \text{(A1.15)}$$

are parallel configurations and also zeros of the multivariable Alexander polynomial (A1.10).

It is possible to combine the calculations of Appendix in [1] and Eq. (A1.6) into Reshetikhin's representation of the Jones polynomial of a general p-component torus link $\mathscr{L}_{(mp,np)}$ (m and n are coprime). We present here the result without derivation (it is similar to the one for Eq. (A1.6)):

$$Z_{\alpha_1, \ldots, \alpha_p}(S^3, \mathscr{L}_{(mp,np);k})$$

$$= Z(S^3; k) \int_{|\vec{a}_j| = \frac{\alpha_j}{k}} \prod_{j=1}^p \left(\frac{K}{4\pi} \frac{d^2 \vec{a}_j}{|\vec{a}_j|}\right) \exp\left[\frac{i\pi K}{2} mn \sum_{\substack{i,j=1 \\ i \neq j}}^p \vec{a}_i \cdot \vec{a}_j\right]$$

$$\times \exp\left[\frac{i\pi}{2K} \frac{m^2 n^2 p - m^2 - n^2}{mn}\right] \frac{\left(\frac{\pi}{K}\right)}{\sin\left(\frac{\pi}{K}\right)} \sum_{l=0}^\infty (2\pi i K)^{-l} \frac{(2l+1)!}{l!(2l)!} \partial_y^{(2l)}$$

$$\times \left[\left(\frac{\pi mn y}{\sin(\pi mn y)}\right)^p \frac{\sin(\pi m y)}{\pi m y} \frac{\sin(\pi n y)}{\pi n y}\right]\Bigg|_{y=\left|\sum_{j=1}^p \vec{a}_j\right|}. \qquad \text{(A1.16)}$$

Equations (A1.4) of [1] and (A1.6) are particular cases of this equation (set $p = 1$ or set $n = 1$ and put n instead of p).

The Alexander polynomial of $\mathscr{L}_{(mp,\,np)}$ is

$$
\Delta_{\mathscr{A}}(S^3, \mathscr{L}_{(mp,\,np)}; e^{2\pi i a_1}, \ldots, e^{2\pi i a_p}) = (2i)^{p-2} \frac{\sin^p\left(\pi mn \sum_{j=1}^{p} a_j\right)}{\sin\left(\pi m \sum_{j=1}^{p} a_j\right) \sin\left(\pi n \sum_{j=1}^{p} a_j\right)},
$$

$$\tag{A1.17}$$

its relation to Eq. (A1.16) is easy to observe.

Appendix 2

Here we will briefly review the structure of flat connections in a link complement and show that Eq. (2.3) describes them approximately in close vicinity of the trivial connection. For more details on the structure of flat connections in the link complement and their relation to Milnor's linking numbers, see for example [12].

Consider an n-component link $\mathscr{L}$ in S^3. We use Wirtinger's presentation for the group $\pi_1(S^3 \backslash \mathrm{Tub}(\mathscr{L}))$. We project the link $\mathscr{L}$ onto a 2-dimensional plane and denote as $L_{i,j}$ the pieces into which a link component L_i is split when it is overcrossed. With each such piece we associate an element $g_{i,j} \in \pi_1(S^3 \backslash \mathrm{Tub}(\mathscr{L}))$. These elements generate the whole group $\pi_1(S^3 \backslash \mathrm{Tub}(\mathscr{L}))$ modulo certain relations. Let $P_{i,j}^{k,l}$ be a crossing point where a piece $L_{k,l}$ overcrosses a junction of two pieces $L_{i,j}$ and $L_{i,j+1}$. Let sign $\left(P_{i,j}^{k,l}\right)$ be a signature of this crossing. In other words, sign $\left(P_{i,j}^{k,l}\right)$ is either $+1$ or -1 depending on mutual orientation of L_i and L_k at the point of crossing. The linking number of two link components can be expressed in terms of the signatures of crossings:

$$
l_{ik} = \sum_{j,l} \mathrm{sign}\left(P_{i,j}^{k,l}\right). \tag{A2.1}
$$

The relation between the group elements corresponding to the crossing point $P_{i,j}^{k,l}$ is

$$
g_{i,j+1} = g_{k,l}^{\mathrm{sign}\left(P_{i,j}^{k,l}\right)} g_{i,j} g_{k,l}^{-\mathrm{sign}\left(P_{i,j}^{k,l}\right)}. \tag{A2.2}
$$

The relations (A2.2) describe the structure of $\pi_1(S^3 \backslash \mathrm{Tub}(\mathscr{L}))$.

Suppose that we have a one-parametric family of homomorphisms

$$
\pi_1(S^3 \backslash \mathrm{Tub}(\mathscr{L})) \to G,
$$

$$
g_{i,j} \mapsto \exp\left(\sum_{m=1}^{\infty} \lambda_{i,j}^{(m)} t^m\right), \quad t \geqq 0, \tag{A2.3}
$$

here G is a Lie group and $\lambda_{i,j}^{(m)}$ are elements of its Lie algebra. The homomorphisms (A2.3) describe (up to a conjugation) a family of flat connections in $S^3 \backslash \mathrm{Tub}(\mathscr{L})$ which includes the trivial connection at $t = 0$.

We substitute the images of the homomorphism (A2.3) into the relations (A2.2) and expand them in powers of t. At order t^1 we observe that the elements $\lambda_{i,j}^{(1)}$ do not depend on j, so we denote them simply as

$$\lambda_i = \lambda_{i,j}^{(1)} \; . \tag{A2.4}$$

At order t^2 we get a relation

$$\lambda_{i,j+1}^{(2)} - \lambda_{i,j}^{(2)} = \text{sign}\left(P_{i,j}^{k,l}\right)[\lambda_k, \lambda_l] \; . \tag{A2.5}$$

If we go around a link component $\mathscr{L}_i$ and add together all the relations (A2.5), then we arrive at the equation

$$\sum_{k,l,j} \text{sign}\left(P_{i,j}^{k,l}\right)[\lambda_k, \lambda_i] = 0 \; , \tag{A2.6}$$

which in view of Eq. (A2.1) is equivalent to Eq. (2.3) for the case of $G = SU(2)$.

Proposition A2.1. *The stationary points of the phase in Reshetikhin's formula (1.20) are in one-to-one correspondence with the flat connections on the link complement in the linear approximation around the trivial connection.*

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