

A Contribution of the Trivial Connection to the Jones Polynomial and Witten’s Invariant of 3d Manifolds, I

L. Rozansky¹

Physics Department, University of Miami, P. O. Box 248046, Coral Gables, FL 33124, U.S.A.

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Abstract: We use a path integral formulation of the Chern–Simons quantum field theory in order to give a simple “semi-rigorous” proof of a recently conjectured limitation on the $1/K$ expansion of the Jones polynomial of a knot and its relation to the Alexander polynomial. A combination of this limitation with the finite version of the Poisson resummation allows us to derive a surgery formula for the contribution of the trivial connection to Witten’s invariant of rational homology spheres. The 2-loop part of this formula coincides with Walker’s surgery formula for the Casson–Walker invariant. This proves a conjecture that the Casson–Walker invariant is proportional to the 2-loop correction to the trivial connection contribution. A contribution of the trivial connection to Witten’s invariant of a manifold with nontrivial rational homology is calculated for the case of Seifert manifolds.

1. Introduction

In his paper [1], Witten defined a topological invariant of a 3d manifold M with an n -component link \mathcal{L} inside it as a partition function of a quantum Chern–Simons theory. Let us attach representations $V_{\alpha_i}, 1 \leq i \leq n$ of a simple Lie group G to the components of \mathcal{L} (in our notations α_i are the highest weights shifted by $\rho = \frac{1}{2} \sum_{\lambda_i \in A_+} \lambda_i, A_+$ is a set of positive roots of G). Then Witten’s invariant is equal to the path integral over all gauge equivalence classes of G connection on M :

$$Z_{z_1, \dots, z_n}(M, \mathcal{L}; k) = \int [\mathcal{D}A_\mu] \exp\left(\frac{i}{\hbar} S_{CS}\right) \prod_{i=1}^n \text{Tr}_{\alpha_i} \text{Pexp}\left(\oint_{L_i} A_\mu dx^\mu\right), \quad (1.1)$$

here A_μ is a connection, S_{CS} is its Chern–Simons action,

$$S_{CS} = \frac{1}{2} \text{Tr} \int_M e^{\mu\nu\rho} dx \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right), \quad (1.2)$$

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Tr is a trace in the fundamental representation (so that $\text{Tr } \lambda_i^2 = 2$ for long roots of G), \hbar is a Planck's constant:

$$\hbar = \frac{2\pi}{k}, \quad k \in \mathbb{Z}. \tag{1.3}$$

$\text{Tr}_{\alpha_i} \text{Pexp}(\oint_{L_i} A_\mu dx^\mu)$ are the traces of holonomies along the link components L_i taken in the representations V_{α_i} . Witten showed that for a link in S^3 his invariant was proportional to the Jones polynomial of that link. In what follows we will refer to Eq. (1.1) as the definition of the Jones polynomial and its normalization.

Witten derived a surgery algorithm for an exact calculation of the path integral (1.1). We review it briefly in order to set our notations. Consider a manifold M with a knot \mathcal{K} inside it. Let us choose a basis of cycles on the boundary of its tubular neighborhood $\text{Tub}(\mathcal{K})$. C_1 is a cycle contractible through the tubular neighborhood (i.e. C_1 is the meridian of \mathcal{K}). C_2 is a cycle which has a unit intersection number with C_1 (C_2 is defined only modulo C_1). Cut out the tubular neighborhood $\text{Tub}(\mathcal{K})$ and glue it back in such a way that the cycles $pC_1 + qC_2$ and $rC_1 + sC_2$ on the boundary of the complement of \mathcal{K} are identified with the cycles C_1 and C_2 on the boundary of $\text{Tub}(\mathcal{K})$. As a result of this surgery, a new manifold M' is constructed.

The integer numbers p, q, r, s form a unimodular matrix

$$U^{(p,q)} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \mathbb{Z}), \quad ps - qr = 1. \tag{1.4}$$

The group $SL(2, \mathbb{Z})$ has a unitary representation in the space of affine characters of G which is in fact a Hilbert space of the Chern–Simons theory corresponding to $T^2 = \partial \text{Tub}(\mathcal{K})$. The basis vectors of this space $|\alpha, 1\rangle$ ($\alpha \in \Delta_G = A^w / (W \times K A^R) \setminus \text{walls}$, $K = k + c_V$, c_V is a dual Coxeter number of G , $c_V = N$ for $SU(N)$) are the eigenstates of the holonomy operator along the cycle C_1 :

$$\text{Pexp} \left(\oint_{C_1} \hat{A}^\mu dx^\mu \right) |\alpha, 1\rangle = \exp \left(\frac{2\pi i}{K} \alpha \right) |\alpha, 1\rangle, \tag{1.5}$$

here \hat{A}^μ is an operator corresponding to the classical field A^μ . The matrix elements of $U^{(p,q)}$ represented in this basis are (for a simply laced group)

$$\begin{aligned} \tilde{U}_{\alpha\beta}^{(p,q)} &= \frac{[i \text{sign}(q)]^{|\Delta_+|}}{(K|q|)^{\text{rank } G/2}} \exp \left[-\frac{i\pi}{12} \dim G \Phi(U^{(p,q)}) \right] \left(\frac{\text{Vol } A^w}{\text{Vol } A^R} \right)^{\frac{1}{2}} \\ &\times \sum_{n \in A^R/qA^R} \sum_{w \in W} (-1)^{|w|} \exp \frac{i\pi}{Kq} \\ &\times [p\alpha^2 - 2\alpha \cdot (Kn + w(\beta)) + s(Kn + w(\beta))^2], \end{aligned} \tag{1.6}$$

here $|\Delta_+|$ is a number of positive roots in G , W is the Weyl group and $\Phi(U^{(p,q)})$ is the Rademacher function defined as follows:

$$\Phi \begin{bmatrix} p & r \\ q & s \end{bmatrix} = \frac{p+s}{q} - 12s(s, q), \tag{1.7}$$

$s(s, q)$ is a Dedekind sum:

$$s(m, n) = \frac{1}{4n} \sum_{j=1}^{n-1} \cot\left(\frac{\pi j}{n}\right) \cot\left(\frac{\pi m j}{n}\right). \tag{1.8}$$

The formula (1.6) was derived by L. Jeffrey [3] for $G = SU(2)$:

$$\begin{aligned} \tilde{U}_{\alpha\beta}^{(p, q)} &= i \frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4} \Phi(U^{(p, q)})} \sum_{\mu=\pm 1} \sum_{n=0}^{q-1} \mu \exp \frac{i\pi}{2Kq} \\ &\times [p\alpha^2 - 2\alpha(2Kn + \mu\beta) + s(2Kn + \mu\beta)^2], \quad A_{SU(2)} : 1 \leq \alpha, \beta \leq K - 1. \end{aligned} \tag{1.9}$$

According to Witten [1], the invariant of the manifold M' constructed by a $U^{(p, q)}$ surgery on a knot \mathcal{K} in a manifold M can be expressed through the Jones polynomial of that knot and the representation (1.8) of the surgery matrix:

$$Z(M'; k) = e^{i\phi_{\text{fr}}} \sum_{\alpha \in \Delta_G} Z_{\alpha}(M, \mathcal{K}; k) \tilde{U}_{\alpha\rho}^{(p, q)} \tag{1.10}$$

(recall that ρ is a shifted highest weight of the trivial representation). The phase ϕ_{fr} is a framing correction. If both invariants are reduced to canonical framing, then

$$\phi_{\text{fr}} = \frac{\pi}{12} \frac{K - c_V}{K} \dim G \left[\Phi(U^{(p, q)}) - 3 \text{sign} \left(\frac{p}{q} + v \right) \right], \tag{1.11}$$

here v is a self-linking number of \mathcal{K} defined as a linking number between C_2 and \mathcal{K} .

For a more general case when a surgery is performed on a link \mathcal{L} in M Witten concluded that

$$Z(M'; k) = e^{i\phi_{\text{fr}}} \sum_{\alpha_1, \dots, \alpha_n \in \Delta_G} Z_{\alpha_1, \dots, \alpha_n}(M, \mathcal{L}; k) \tilde{U}_{\alpha_1\rho}^{(p_1, q_1)} \dots \tilde{U}_{\alpha_n\rho}^{(p_n, q_n)}. \tag{1.12}$$

Reshetikhin and Turaev showed in [2] that Eq. (1.12) is invariant under Kirby moves. Therefore they proved that $Z(M; k)$ is a topological invariant of the manifold without invoking the path integral representation (1.1) which still lacks mathematical rigor. They also established a general set of conditions on the components of the r.h.s. of Eq. (1.12) which guarantee its topological invariance.

The disadvantage of Eqs. (1.10) and (1.12) is that they do not make the relation between Witten’s invariant and classical topological invariants of 3d manifolds quite transparent (The Alexander polynomial was the only quantum invariant which had a clear topological nature since it was originally constructed from the fundamental group of the knot complement). A possible way to deal with this problem is to consider a large k asymptotics of the path integral (1.1) by applying a stationary phase approximation. The stationary phase points are flat connections. Therefore the invariant is presented as a sum over connected pieces \mathcal{M}_c of the moduli space \mathcal{M} of flat connections on M :

$$\begin{aligned} Z_{\alpha_1, \dots, \alpha_n}(M, \mathcal{L}; k) &= \sum_{\mathcal{M}_c} Z_{\alpha_1, \dots, \alpha_n}^{(\mathcal{M}_c)}(M, \mathcal{L}; k), \\ Z_{\alpha_1, \dots, \alpha_n}^{(\mathcal{M}_c)}(M, \mathcal{L}; k) &= \exp \frac{i}{\hbar} \left(S_{CS}^{(c)} + \sum_{n=1}^{\infty} S_n^{(c)} \hbar^n \right), \end{aligned} \tag{1.13}$$

here S_{CS} is a Chern–Simons action of flat connections of \mathcal{M}_c and $S_n^{(c)}$ are the quantum n -loop corrections to the contribution of \mathcal{M}_c . The 1-loop correction is a determinant of the quadratic form describing the small fluctuations of $S_{CS}(A_\mu)$ around a stationary phase point. Its major features were determined by Witten [1], Freed and Gompf [4], and Jeffrey [3] (some further details were added in [5]):

$$e^{iS_1^{(c)}} = \frac{(2\pi\hbar)^{\frac{\dim H_c^0 - \dim H_c^1}{2}}}{\text{Vol}(H_c)} \exp\left(\frac{i}{2\pi} c_V S_{CS} - \frac{i\pi}{4} N_{\text{ph}}\right) \times \int_{\mathcal{M}_c} \left[\sqrt{|\tau_R|} \prod_{i=1}^n \text{Tr}_{x_i} \text{Pexp} \left(\oint_{L_i} A_\mu dx^\mu \right) \right], \tag{1.14}$$

here H_c is an isotropy group of \mathcal{M}_c (i.e. a subgroup of G which commutes with the holonomies of connections $A_\mu^{(c)}$ of \mathcal{M}_c), N_{ph} is expressed [4] as

$$N_{\text{ph}} = 2I_c + \dim H_c^0 + \dim H_c^1 + (1 + b_M^1) \dim G, \tag{1.15}$$

I_c is a spectral flow of the operator $L_- = \star D + D \star$ acting on 1- and 3-forms, D being a covariant derivative, H_c^0 and H_c^1 are cohomologies of D , and b_M^1 is the first Betti number of M . τ_R is a Reidemeister–Ray–Singer torsion. It was observed in [3] that $\sqrt{|\tau_R|}$ defines a ratio of volume forms on \mathcal{M}_c and H_c .

The higher loop corrections $S_n^{(c)}$ are calculated by Feynman rules. They are expressed as multiple integrals of the products of propagators taken over the manifolds M and the link \mathcal{L} . Such representation might make the nature of invariants $S_n^{(c)}$ more transparent. Bar-Natan [6] and Kontsevich [7] studied the Feynman diagrams related to the link. These diagrams produce Vassiliev invariants. In particular, Bar-Natan observed that the 2-loop correction to the $SU(2)$ invariant of the knot in S^3 is proportional to the second derivative of its Alexander polynomial.

In their recent paper [8] Melvin and Morton conjectured² a rather strict limitation on the possible powers of α in the K^{-1} expansion of the $SU(2)$ Jones polynomial $Z_\alpha(S^3, \mathcal{K}; k)$ as well as a relation between the dominant part of this expansion and the Alexander polynomial which generalizes the result of [6].

The properties of Feynman diagrams related to the manifold were studied in early papers [10, 11] and then by Axelrod and Singer [12] and Kontsevich [13]. A convergence of those diagrams was proven, however no multiloop diagrams were explicitly calculated. An “experimental” approach to their study was initiated in [4] and [3]. Freed and Gompf checked the 1-loop formula (1.14) by comparing it numerically to the surgery formula (1.12) applied to some lens spaces and Seifert homology spheres. L. Jeffrey transformed the surgery formula for lens spaces and some mapping tori into the asymptotic form (1.13) thus obtaining all the loop corrections for those manifolds. This program was further extended to Seifert manifolds in [5]. It was observed there among other things that the 2-loop correction to the contribution of the trivial connection was proportional to the Casson–Walker invariant as calculated by C. Lescop [14].

In this paper we study the trivial connection contribution to Witten’s invariant of a knot, a link and a manifold. In Sect. 2 we prove the relation between the Jones and Alexander polynomials of a knot (Proposition 2.1) conjectured in [8] by relating the

² This conjecture was proven recently by D. Bar-Natan and S. Garoufalidis [9] at the level of weight systems.

former to the Reidemeister–Ray–Singer torsion of the knot complement. We also generalize this result to the case of an arbitrary rational homology sphere (RHS). In Sect. 3 we derive a knot surgery formula for the trivial connection contribution to Witten’s invariant of a RHS (Proposition 3.1). We show that at the 2-loop level this formula coincides with Walker’s formula [15] for Casson–Walker invariant. This proves the relation between the 2-loop correction to the contribution of the trivial connection and the Casson–Walker invariant (Proposition 3.2) conjectured in [5]. In Sect. 4 we try to go beyond RHS by considering a Seifert manifold with nontrivial rational homology. We derive a formula for the trivial connection contribution to its Witten’s invariant (Proposition 4.3) and compare its properties to the partition function of a 2d gauge theory studied by Witten [16]. The results of Sect. 2 are illustrated in the Appendix, where a large k asymptotics of the Jones polynomial of a torus knot is calculated. The contributions of reducible and irreducible connections in the knot complement are identified. Similarly to the results of [5], the contribution of the irreducible connections appears to be 2-loop exact.

It should be emphasized that the derivation of the results of Sects. 2 and 3 involves the use of path integral. Therefore these results lack mathematical rigor. The propositions which state them should be understood as “physical propositions.” At the same time, the calculations of Sect. 4 and the Appendix are perfectly rigorous.

2. The Jones Polynomial and the Reidemeister–Ray–Singer Torsion

We are going to study the Jones polynomial of a knot \mathcal{K} in a rational homology sphere M (i.e. $b_M^1 = 0$). We start with the case of $M = S^3$. Then the $SU(2)$ Jones polynomial (in Witten’s normalization (1.1)) can be expanded in K^{-1} :

$$Z_\alpha(S^3, \mathcal{K}; k) = \sum_{m, n \geq 0} C_{m, n} \alpha^m K^{-n}. \tag{2.1}$$

Melvin and Morton [8] suggested³ the following.

Proposition 2.1. *If the knot \mathcal{K} is canonically framed (i.e. the linking number ν between the cycle C_2 which determines the framing and \mathcal{K} is zero), then*

$$C_{m, n} = 0 \quad \text{if } m > n. \tag{2.2}$$

Moreover,

$$\sum_{n \geq 0} C_{n, n} a^n = \sqrt{\frac{2}{K}} \frac{\sin(\pi a)}{\Delta_A(S^3, \mathcal{K}; \exp(2\pi i a))}, \quad 0 \leq a \leq 1, \tag{2.3}$$

here $\Delta_A(S^3, \mathcal{K}; \exp(2\pi i a))$ is the Alexander polynomial of \mathcal{K} normalized in such a way that $\Delta(S^3, \text{unknot}; \exp(2\pi i a)) = 1$, $\Delta_A(M, \mathcal{K}; \exp(2\pi i a))$ is real.

It was established by Milnor [17] and Turaev [18] that in this normalization Δ_A is related to the Reidemeister torsion of the knot complement:

$$\Delta_A(M, \mathcal{K}; \exp(2\pi i a)) = \frac{2 \sin(\pi a)}{\tau_R(M \setminus \text{Tub}(\mathcal{K}); \exp(2\pi i a))}. \tag{2.4}$$

³ I am thankful to D. Bar-Natan and S. Garoufalidis for drawing my attention to the paper [8].

Some simple quantum field theory arguments were used in [19] to show that the Alexander polynomial was related by Eq. (2.4) to the Ray–Singer torsion of the knot complement. Here we will apply the same arguments to the Jones polynomial $Z_\alpha(S^3, \mathcal{K}; k)$.

Consider the values of α of order K . We introduce a new variable

$$a = \frac{\alpha}{K}, \quad 0 \leq a \leq 1. \tag{2.5}$$

Let us split the path integral (1.1) for a knot \mathcal{K} into an integral over the connection A_μ inside the tubular neighborhood $\text{Tub}(\mathcal{K})$ and inside its complement $S^3 \setminus \text{Tub}(\mathcal{K})$ with certain boundary conditions on the boundary $T^2 = \partial\text{Tub}(\mathcal{K})$, as well as an integral over these boundary conditions. According to [20], one possible set of boundary conditions requires that the gauge fields $A_{1,2}$ on T^2 should belong to the Cartan subalgebra, the curvature F_{12} should be zero and the integral $I_1 = \oint_{C_1} A_\mu dx^\mu$ should be fixed. In fact, it was established in [20] that in accordance with Eq. (1.5), the path integral over connections on $\text{Tub}(\mathcal{K})$ is proportional to $\delta(I_1 - 2\pi ia)$. Therefore the Jones polynomial $Z_\alpha(S^3, \mathcal{K}; k)$ is equal to the path integral over connections on $S^3 \setminus \text{Tub}(\mathcal{K})$

$$Z_\alpha(S^3, \mathcal{K}; k) = \int_{[S^3 \setminus \text{Tub}(\mathcal{K})]} [\mathcal{D}A_\mu] \exp\left(\frac{i}{\hbar} S'_{CS}\right) \tag{2.6}$$

taken with the boundary condition

$$P \exp\left(\oint_{C_1} A_\mu dx^\mu\right) = \exp(2\pi ia). \tag{2.7}$$

The Chern–Simons action is modified [20] by the boundary term

$$S'_{CS} = S_{CS} + \frac{1}{2} \text{Tr} \int_{T^2} A_1 A_2 d^2x, \tag{2.8}$$

which is necessary for the choice (2.7) of boundary conditions.

Let us calculate the path integral (2.6) by the stationary phase approximation method (1.13). First of all, we look for stationary phase points, i.e. flat connections satisfying the boundary condition (2.7). There is only one such connection for $a < a_0$ ($a_0 > 0$ being a critical value depending on \mathcal{K}). This connection is reducible: all the holonomies belong to the maximal torus $U(1) \subset SU(2)$. For this connection $S_{CS} = 0$. Since the linking number ν of C_2 and \mathcal{K} is zero, the homology class of C_2 in $S^3 \setminus \text{Tub}(\mathcal{K})$ is trivial. Therefore $A_2 = 0$ and the boundary term in Eq. (2.8) is also zero. Thus the whole classical Chern–Simons action S'_{CS} is zero.

We will estimate the 1-loop correction (1.14) up to a phase factor $\exp(-\frac{i\pi}{4} N_{\text{ph}})$. The flat $U(1)$ connection on $S^3 \setminus \text{Tub}(\mathcal{K})$ satisfying Eq. (2.7) has no moduli, so $\dim H_c^1 = 0$. The isotropy group is $H_c = U(1)$, so $\text{Vol} H_c = 2\sqrt{2}\pi$ (recall that the radius of $U(1)$ is $\sqrt{2}$), while $\dim H_c^0 = 1$. The determinants in the $SU(2)$ Ray–Singer torsion τ_R split into three factors for three Lie algebra components of A_μ which have the definite $U(1)$ charge. The chargeless Cartan subalgebra (i.e. diagonal) component of A_μ contributes 1, while each of the two off-diagonal components contribute the square root of the $U(1)$ torsion $\tau_R(S^3 \setminus \text{Tub}(\mathcal{K}); \exp(2\pi ia))$. As a result of all this and Eq. (2.4) we conclude that

Proposition 2.2. *The loop formula (1.13) for the Jones polynomial of a knot in S^3 can be presented in the form*

$$Z_z(S^3, \mathcal{K}; k) = \sqrt{\frac{2}{K}} \frac{\sin(\pi a)}{\Delta_A(S^3, \mathcal{K}; \exp(2\pi i a))} \exp \left[i \sum_{n=1}^{\infty} \left(\frac{2\pi}{K} \right)^n S_{n+1}(a) \right], \quad (2.9)$$

here $S_n(a)$ are the higher loop corrections for the path integral (2.6), they depend on the boundary holonomy $\exp(2\pi i a)$.

We will show later that $e^{-iN_{\text{ph}}} = 1$.

The substitution (2.5) turns the r.h.s. of this equation into the expansion (2.1) with limitations (2.2) and property (2.3). We also learn that the sum of the terms

$$\sum_{m \geq 0} C_{m, m+n} \alpha^m K^{-m-n} \quad (2.10)$$

comes from the n -loop Feynman diagrams (including disconnected ones) in the knot complement $S^3 \setminus \text{Tub}(\mathcal{K})$.

Consider now a general RHS M with a knot \mathcal{K} inside it. This time there may be many flat connections (both reducible and irreducible) with a given holonomy (2.7) even if a is very small. Each of them will contribute to the stationary phase approximation of the path integral (2.6) turning it into the sum (1.13). We will concentrate on the reducible $U(1)$ connections because their 1-loop contributions can again be related to the Alexander polynomial of \mathcal{K} .

Some changes have to be made to Eq. (2.9). Let b define the holonomy along C_2 for a reducible flat connection on $M \setminus \text{Tub}(\mathcal{K})$:

$$\text{Pexp} \left(\oint_{C_2} A_\mu dx^\mu \right) = \exp(2\pi i b). \quad (2.11)$$

The holonomies (2.7) and (2.11) are related by the fact that the homomorphism

$$H_1(\partial \text{Tub}(\mathcal{K}), \mathbb{Z}) \rightarrow H_1(M \setminus \text{Tub}(\mathcal{K}), \mathbb{Z}) \quad (2.12)$$

has a kernel. Let the cycle

$$C_0 = d(m_1 C_1 + m_2 C_2), \quad d, m_1, m_2 \in \mathbb{Z}, \quad m_1, m_2 - \text{coprime} \quad (2.13)$$

be its generator. Then

$$\text{Pexp} \left(\oint_{C_0} A_\mu dx^\mu \right) \equiv \exp[2\pi i d(m_1 a + m_2 b)] = 1, \quad (2.14)$$

so that

$$b = -\frac{1}{m_2} \left(m_1 a + \frac{n}{d} \right), \quad n \in \mathbb{Z}, \quad 0 \leq n < d. \quad (2.15)$$

If we smoothly reduce a to zero, then the flat connection on $M \setminus \text{Tub}(\mathcal{K})$ will transform into a flat connection on M . Let $S_{CS,0}$ be its Chern–Simons invariant. Then according to [21] and Eq. (2.8), the Chern–Simons action of the original connection is

$$S'_{CS} = -\pi^2 \left(\frac{m_1}{m_2} a^2 + 2 \frac{na}{m_2 b} \right) + S_{CS,0}. \quad (2.16)$$

In particular, if the flat connection on M at $a = 0$ is trivial, then $S_{CS} = 0$ and $n = 0$, so that

$$S'_{CS} = -\pi^2 \frac{m_1}{m_2} a^2. \tag{2.17}$$

The Reidemeister–Ray–Singer torsion for the Cartan subalgebra part of A_μ is known to be equal to $\text{ord}H_1(M, \mathbb{Z})$. As for the off-diagonal Lie algebra components of A_μ , we can use again Eq. (2.4). However the argument of the torsion is related to the holonomy along the generator of the \mathbb{Z} part of $H_1(M \setminus \text{Tub}(\mathcal{K}), \mathbb{Z})$. The holonomy along the cycle C'_0 which has the unit intersection number with C_0 , is

$$\text{Pexp} \left(\oint_{C'_0} A_\mu dx^\mu \right) = \exp \left(2\pi i \frac{a}{m_2} \right). \tag{2.18}$$

This cycle generates the $d\mathbb{Z}$ subgroup of \mathbb{Z} , so the holonomy along the generator of \mathbb{Z} is $\exp \left(2\pi i \frac{a}{m_2 d} \right)$. Combining all pieces together we get the following

Proposition 2.3. *If M is a RHS and \mathcal{K} is a knot inside it, then the contribution of the trivial connection to the Jones polynomial of \mathcal{K} is given by the formula*

$$\begin{aligned} Z_\alpha^{(\text{tr})}(M, \mathcal{K}; k) &= \sqrt{\frac{2}{K}} [\text{ord}H_1(M, \mathbb{Z})]^{-\frac{1}{2}} \frac{\sin \left(\frac{\pi}{K} \frac{\alpha}{m_2 d} \right)}{A_A \left(M, \mathcal{K}, \exp \left(\frac{2\pi i}{K} \frac{\alpha}{m_2 d} \right) \right)} \\ &\times \exp \left(-\frac{i\pi}{2K} \frac{m_1}{m_2} \alpha^2 \right) \exp \left[\sum_{n=1}^{\infty} \left(\frac{2\pi}{K} \right)^n S_{n+1} \left(\frac{\alpha}{K} \right) \right]. \end{aligned} \tag{2.19}$$

We dropped the factor $e^{\frac{i\pi}{4} N_{\text{ph}}}$, we will show later that it is equal to 1.

Assuming that $\alpha \ll K$ (that is, $a \ll 1$), we can present $Z_\alpha^{(\text{tr})}(M, \mathcal{K}; k)$ in a slightly different form by applying the stationary phase approximation directly to the path integral (1.1) taken over connections on the whole manifold M :

$$Z_\alpha^{(\text{tr})}(M, \mathcal{K}; k) = Z^{(\text{tr})}(M; k) \exp \left[\frac{i\pi}{2K} v(\alpha^2 - 1) \right] \alpha J(\alpha, K). \tag{2.20}$$

In this formula $Z^{(\text{tr})}(M; k)$ is a contribution of the trivial connection to Witten’s invariant of M itself; it contains Feynman diagrams which are not connected to the knot \mathcal{K} . The function $J(\alpha, K)$ is a contribution of Feynman diagrams attached to the knot, except for two factors that we separated out explicitly: the framing factor $\exp \left[\frac{i\pi}{2K} v(\alpha^2 - 1) \right]$ and the dimension of the representation $\dim V_\alpha = \alpha$, which appears when the trace of the holonomy is taken in Eq. (1.1). Note the relation between the self-linking number of the knot v and the numbers m_1, m_2 :

$$v = -\frac{m_1}{m_2}. \tag{2.21}$$

The function $J(\alpha, K)$ can be expanded in K^{-1} ,

$$J(\alpha, K) = \sum_{m, n \geq 0} D_{m, n} \alpha^m K^{-n}. \tag{2.22}$$

The numbers $D_{m,n}$ are type n Vassiliev invariants. By comparing Eqs. (2.19) and (2.20) we conclude that

$$D_{m,n} = 0 \quad \text{for } m > n. \tag{2.23}$$

Moreover, according to [6] $D_{0,0} = 1$, $D_{0,1} = D_{1,1} = D_{1,2} = 0$, $D_{2,2} = -D_{2,0}$, so that

$$J(\alpha, K) = 1 + D_{2,2} \frac{\alpha^2 - 1}{K^2} + \mathcal{O}(K^3). \tag{2.24}$$

The value of $D_{2,2}$ can be deduced by comparing Eqs. (2.19) and (2.20):

$$D_{2,2} = \frac{2\pi^2}{m_2^2 d^2} \left(\Delta_A'' - \frac{1}{12} \right). \tag{2.25}$$

This relation was first obtained by Bar-Natan in [6].

The trivial connection contribution $Z^{(\text{tr})}(M; k)$ can be expanded in asymptotic series in K^{-1} . The leading 1-loop term is given by Eq. (1.14):

$$Z_{1\text{-loop}}^{(\text{tr})}(M; k) = \frac{1}{\text{Vol } SU(2)} \left(\frac{2\pi\hbar}{\text{ord } H_1(M, \mathbb{Z})} \right)^{\frac{3}{2}} \equiv \sqrt{2\pi} (K \text{ ord } H_1(M, \mathbb{Z}))^{-\frac{3}{2}}. \tag{2.26}$$

Comparing Eqs. (2.26), (2.20) and (2.19) we see that $N_{\text{ph}} = 0$ and the term $e^{-\frac{i\pi}{4} N_{\text{ph}}}$ can indeed be dropped from Eqs. (2.19) and (2.9).

All the formulas of this section can be easily generalized to the case of a general simple Lie group G . Equation (2.19) transforms into

$$\begin{aligned} Z_x^{(\text{tr})}(M, \mathcal{K}; k) &= [2K \text{ ord } H_1(M, \mathbb{Z})]^{-\frac{\text{rank } G}{2}} \exp\left(-\frac{i\pi}{K} \frac{m_1}{m_2} \alpha^2\right) \\ &\times \left[\prod_{\lambda_i \in \mathcal{A}_+} \frac{2 \sin\left(\frac{\pi}{K} \frac{\alpha \cdot \lambda_i}{m_2 d}\right)}{\Delta_A\left(M, \mathcal{K}; \exp\left(\frac{2\pi i}{K} \frac{\alpha \cdot \lambda_i}{m_2 d}\right)\right)} \right] \exp\left[i \sum_{n=1}^{\infty} \left(\frac{2\pi}{K}\right)^n S_{n+1}\left(\frac{\alpha}{K}\right) \right]. \end{aligned} \tag{2.27}$$

The generalization of Eqs. (2.20) and (2.24) is

$$Z_x^{(\text{tr})}(M, \mathcal{K}; k) = Z^{(\text{tr})}(M; k) \exp\left(\frac{i\pi}{K} v(\alpha^2 - \rho^2)\right) \left[\prod_{\lambda_i \in \mathcal{A}_+} \frac{(\alpha \cdot \lambda_i)}{(\rho \cdot \lambda_i)} \right] J(\alpha, K), \tag{2.28}$$

$$J(\alpha, K) = 1 + D_{2,2} \frac{12\rho^2}{\dim G} \frac{\alpha^2 - \rho^2}{K^2} + \mathcal{O}(K^{-3}). \tag{2.29}$$

3. A Trivial Connection Contribution to Witten’s Invariant

Suppose that a manifold M' is constructed by a $U^{(p,q)}$ surgery on a knot \mathcal{K} in a manifold M . Then Witten’s invariant of M' can be calculated by the surgery formula (1.10). The large k limit of the r.h.s of this formula contains implicitly the contributions of all flat connections on M' . We will try to separate the contribution of the trivial connection in the case when both M and M' are rational homology spheres.

We start with the case of $G = SU(2)$. Our main tool is the finite version of the Poisson resummation formula. The Poisson resummation formula states that for any function $f(\alpha)$.

$$\sum_{\alpha \in \mathbb{Z}} f(\alpha) = \sum_{l \in \mathbb{Z}} \int d\alpha \exp(2\pi i l \alpha) f(\alpha). \tag{3.1}$$

Therefore we would like to extend the sum in Eq. (1.10) from $1 \leq \alpha \leq K - 1$ to \mathbb{Z} . First of all, since $\tilde{U}_{\alpha 1}^{(p,q)}$ is equal to zero at $\alpha = 0, K$, we can add these points to the range of summation. We can also double this range:

$$\sum_{\alpha=0}^K \rightarrow \frac{1}{2} \sum_{\alpha=-K+1}^K \tag{3.2}$$

because the summand is even. Finally, we use a ‘‘regularization’’ formula,

$$\sum_{\alpha=-K+1}^K f(\alpha) = 2K \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{2}} \sum_{\alpha \in \mathbb{Z}} e^{-\pi \varepsilon \alpha^2} f(\alpha), \quad \text{if } f(\alpha + 2K) = f(\alpha). \tag{3.3}$$

Thus we obtain a sum $\sum_{\alpha \in \mathbb{Z}}$ to which we apply Eq. (3.1):

$$Z(M'; k) = e^{i\phi_{\text{fr}}} K \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{2}} \sum_{l \in \mathbb{Z}} \int_{-\infty}^{+\infty} d\alpha e^{-\pi \varepsilon \alpha^2} \exp(2\pi i l \alpha) Z_{\alpha}(M, \mathcal{K}; k) \tilde{U}_{\alpha 1}^{(p,q)}. \tag{3.4}$$

At this point we make the following assumption: the large k limit of the integral over α in Eq. (3.4) is equal to the sum of the contributions of the special points α_* of the integrand, e.g. stationary phase points, breaks, poles, *etc.* Their contribution depends on the local properties of the integrand. Apart from the regularization factor $e^{-\pi \varepsilon \alpha^2}$, the set of critical points for all $l \in \mathbb{Z}$ and their respective contributions exhibit the same symmetries as the original summand of Eq. (1.10), i.e. they are even and have a period of $2K$. The role of the factor $e^{-\pi \varepsilon \alpha^2}$ to the leading power in ε is to suppress the contribution of each critical point α_* by a factor $e^{-\varepsilon \pi \alpha_*^2}$. Therefore we can play Eq. (3.3) backwards: we drop $K \lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{2}}$ and $e^{-\pi \varepsilon \alpha^2}$ while limiting ourselves to the contribution of only those critical points which belong to the fundamental domain

$$0 \leq \alpha_* \leq K. \tag{3.5}$$

In other words,

$$Z(M'; k) = e^{i\phi_{\text{fr}}} \sum_{l \in \mathbb{Z}} \int_{\substack{-\infty \\ [0 \leq \alpha_* \leq K]}}^{+\infty} d\alpha e^{2\pi i l \alpha} Z_{\alpha}(M, \mathcal{K}; k) \tilde{U}_{\alpha 1}^{(p,q)}, \tag{3.6}$$

here the symbol $\int_{\substack{-\infty \\ [0 \leq \alpha_* \leq K]}}^{+\infty}$ means that we take only the contributions of the special points (3.5). If $\alpha_* = 0, K$, then its contribution to the integral of Eq. (3.6) carries an extra boundary factor $\frac{1}{2}$.

We assume that each of the special points in the domain (3.5) corresponds to one or several connected pieces \mathcal{M}'_c of the moduli space \mathcal{M}' of flat connections on M' . Consider a cycle in M' which corresponds to the cycle C_2 on the boundary of a tubular neighborhood of the knot \mathcal{K} in M . We will also call it C_2 . According to Eq. (1.5), the holonomy of a flat connection related to a special point α_* along C_2

is equal to $\exp(\frac{2\pi i}{K}\alpha_*)$, so the contribution of the trivial connection on M' should come from the point $\alpha_* = 0$.

We now concentrate on the point $\alpha_* = 0$, so we are interested only in the values $\alpha \ll K$. Therefore we can use a 1-loop approximation given by the formulas (1.13) and (1.14) for the partition function $Z_x(M, \mathcal{H}; k)$. Some of the terms $Z_x^{(\mathcal{M}_c)}(M, \mathcal{H}; k)$ may have a special point $\alpha_* = 0$. We have to determine which of them do contribute to $Z^{(tr)}(M'; k)$.

We are going to present some arguments why the contribution to $Z^{(tr)}(M'; k)$ may actually come only from $Z_x^{(tr)}(M, \mathcal{H}; k)$. Suppose that a contribution of the special point $\alpha_* = 0$ of a particular term $Z_x^{(\mathcal{M}_c)}(M, \mathcal{H}; k)$ to the r.h.s. of Eq. (3.6) corresponds to the contribution $Z^{(tr)}(M'; k)$ of the trivial connection to the r.h.s of that equation. Let us multiply the integrand of the path integral for $Z(M'; k)$ by an “observable” factor

$$\mathcal{O}(C, \beta) = \text{Tr}_\beta \text{Pexp} \left(\oint_C A_\mu dx^\mu \right) \tag{3.7}$$

for $\beta \ll K$, thus turning it into $Z_\beta(M', C; k)$. According to Eq. (1.14) in the 1-loop approximation,

$$Z_\beta^{(tr)}(M', C; k) = \beta Z^{(tr)}(M'; k). \tag{3.8}$$

A surgery formula (3.6) can also work for $Z_\beta(M', C; k)$ if we add the factor (3.7) to $Z_x(M, \mathcal{H}, k)$ transforming it into the $Z_{x\beta}(M, (\mathcal{H}, C); k)$. According to Eq. (1.14), the effect of the factor $\mathcal{O}(C, \beta)$ on $Z_x^{(\mathcal{M}_c)}(M, \mathcal{H}; k)$ is (if we forget for a moment about an integral over \mathcal{M}_c) to multiply it by a factor $\text{Tr}_\beta \text{Pexp}(\oint_C A_\mu^{(c)} dx^\mu)$. This factor turns into β if \mathcal{M}_c is a trivial connection. This is not only a sufficient but also a necessary condition if \mathcal{M}_c is a point.

If \mathcal{M}_c has a nonzero dimension and there is a nontrivial integral in Eq. (1.14), then we may use the following reasoning. Characters form a basis in the space of functions on the maximal torus of the Lie group (in our case it is $U(1) \subset SU(2)$). Therefore we can take a linear combination of observables with different values of β so that they form a smooth slowly varying function on the space of conjugation classes of holonomies $\text{Pexp}(\oint_C A_\mu dx^\mu)$, which is equal to 1 at identity. This new observable is equal to 1 on $Z^{(tr)}(M'; k)$ (that is, if we multiply the integrand of the path integral for $Z^{(tr)}(M'; k)$ by that observable, then the value of the path integral does not change at 1-loop). However different choices of the smooth function will affect the value of the integral over \mathcal{M}_c in Eq. (1.14). Therefore we conclude that the contribution of the trivial connection on M' to $Z(M'; k)$ comes only from the contribution $Z_x^{(tr)}(M, \mathcal{H}; k)$ of the trivial connection on M to $Z_x(M, \mathcal{H}; k)$.

According to Eq. (2.20), if M is a RHS, then $\alpha = 0$ is not a singular point of $Z_x^{(tr)}(M, \mathcal{H}; k)$. Therefore its only chance to contribute to the integral (3.6) is to be a stationary phase point. According to Eqs. (1.9), (2.19) and (2.20), the relevant part of the phase is

$$\frac{i\pi}{2K} \left(\frac{p}{q} + v \right) \alpha^2 + 2\pi i \left(l + \frac{n}{q} \right) \alpha. \tag{3.9}$$

We see that $\alpha = 0$ can indeed be a stationary phase point if we put $n = 0$ in Eq. (1.9) and $l = 0$ in Eq. (3.6). Now it remains to substitute Eqs. (1.9) and (2.20) into the r.h.s. of Eq. (3.6) and add an extra boundary factor $\frac{1}{2}$. Then we come to the following

Proposition 3.1. *If M and M' are rational homology spheres and M' is constructed by a rational surgery $U^{(p,q)}$ on a knot \mathcal{K} in M , which has a self-linking number v , then the trivial connection contribution to Witten's invariants of M and M' are related by the formula*

$$\begin{aligned}
 Z^{(\text{tr})}(M'; k) &= Z^{(\text{tr})}(M; k) \frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-i\frac{3}{4}\pi\text{sign}\left(\frac{p}{q} + v\right)} \\
 &\times \exp \left[\frac{i\pi}{2K} \left(12s(p, q) - \left(\frac{p}{q} + v \right) + 3\text{sign} \left(\frac{p}{q} + v \right) \right) \right] \\
 &\times \int_{\substack{+\infty \\ -\infty \\ [x_* = 0]}} d\alpha \sin \left(\frac{\pi\alpha}{Kq} \right) \alpha J(\alpha, K) \exp \left[\frac{i\pi}{2K} \left(\frac{p}{q} + v \right) \alpha^2 \right]; \quad (3.10)
 \end{aligned}$$

here the function $J(\alpha, K)$ comes from Eq. (2.20), it is a Feynman diagram contribution of the trivial connection to the Jones polynomial of \mathcal{K} and satisfies the properties (2.22)–(2.25).

The integral $\int_{[x_* = 0]}^{+\infty}$ in Eq. (3.5) should be calculated in the following way: the preexponential factor $\sin\left(\frac{\pi\alpha}{Kq}\right)\alpha J(\alpha, K)$ should be expanded in K^{-1} series with the help of Eq. (2.22), then each term should be integrated separately with the gaussian factor $\exp\left[\frac{i\pi}{2K}\left(\frac{p}{q} + v\right)\alpha^2\right]$.

According to this prescription a term $D_{m,n}\alpha^m K^{-n}$ in the expansion (2.22) contributes up to the $(n - \frac{m}{2})^{\text{th}}$ order in the loop expansion of $Z^{(\text{tr})}(M'; k)$. Therefore the limitation (2.23) leads to the following

Corollary 3.1. *Only a finite number of Vassiliev's invariants participate in a surgery formula for $Z^{(\text{tr})}(M'; k)$ at a given loop order.*

In particular, we present explicit surgery formulas for the first two loop corrections. In the notations of Eq. (1.13),

$$e^{iS_1^{(\text{tr})}(M')} = |p + vq|^{-\frac{3}{2}} e^{iS_1^{(\text{tr})}(M)}, \quad (3.11)$$

$$\begin{aligned}
 S_2^{(\text{tr})}(M') &= S_2^{(\text{tr})}(M) + 3 \left[s(p, q) - \frac{1}{12} \left(\frac{p}{q} + v \right) + \frac{1}{4} \text{sign} \left(\frac{p}{q} + v \right) \right. \\
 &\left. + \frac{1}{\left(\frac{p}{q} + v \right)} \left(\frac{D_{2,2}}{2\pi^2} - \frac{1}{12} \frac{1}{q^2} \right) \right]. \quad (3.12)
 \end{aligned}$$

The first formula is consistent with Eq. (1.14) which predicts that for a RHS,

$$e^{iS_1^{(\text{tr})}(M)} = \sqrt{2}\pi [K \text{ ord } H_1(M, \mathbb{Z})]^{-\frac{3}{2}}. \quad (3.13)$$

As for Eq. (3.12), it transforms into Walker's surgery formula [15] for Casson's invariant of a RHS⁴ if we substitute

$$S_2^{(\text{tr})}(M) = 3\lambda_{CW}(M) \quad (3.14)$$

⁴ I am thankful to K. Walker for checking this.

and recall the relation (2.25) between $D_{2,2}$ and the second derivative of the Alexander polynomial. Thus we conclude that the conjecture of [5] can indeed be extended to all RHS:

Proposition 3.2. *If M is a rational homology sphere, then the 2-loop correction to the contribution of the trivial connection to its Witten’s invariant (defined by Eq. (1.13)) is related to the Casson–Walker invariant according to Eq. (3.14).*

In the case of a general simply laced Lie group G the surgery formula (3.10) takes the form

$$\begin{aligned}
 Z^{(\text{tr})}(M'; k) &= Z^{(\text{tr})}(M; k) \frac{[2\text{sign}(q)]^{|D_+|}}{(K|q|)^{\frac{\text{rank } G}{2}}} \frac{1}{|W|} e^{-\frac{i\pi}{4} \dim G \text{sign}\left(\frac{p}{q} + v\right)} \\
 &\times \exp \left[\frac{i\pi}{K} \rho^2 \left(12s(p, q) - \left(\frac{p}{q} + v\right) + 3\text{sign}\left(\frac{p}{q} + v\right) \right) \right] \\
 &\times \int_{[\alpha_* = 0]} d\alpha \left[\prod_{\lambda_i \in D_+} \frac{(\alpha \cdot \lambda_i)}{(\rho \cdot \lambda_i)} \sin \left(\frac{\pi}{Kq} (\alpha \cdot \lambda_i) \right) \right] \\
 &\times J(\alpha, K) \exp \left[\frac{i\pi}{K} \left(\frac{p}{q} + v \right) \alpha^2 \right], \tag{3.15}
 \end{aligned}$$

here $|W|$ is the number of elements in the Weyl group and the integral goes over the Cartan subalgebra. The first two loop corrections are

$$e^{iS_1^{(\text{tr})}(M)} = \frac{1}{\text{Vol } G} \left[\frac{2\pi}{K \text{ord } H_1(M, \mathbb{Z})} \right]^{\frac{\dim G}{2}}, \tag{3.16}$$

$$S_2^{(\text{tr})}(M) = 6\rho^2 \lambda_{CW}(M). \tag{3.17}$$

A simple formula

$$\begin{aligned}
 &\int d\alpha \exp \left[\frac{i\pi}{K} \left(\frac{p}{q} + v \right) \alpha^2 \right] \prod_{\lambda_i \in D_+} \frac{(\alpha \cdot \lambda_i)}{(\rho \cdot \lambda_i)} \sin \left[\frac{\pi}{Kq} (\alpha \cdot \lambda_i) \right] \\
 &= e^{\frac{i\pi}{4} \dim G \text{sign}\left(\frac{p}{q} + v\right)} |W| \left[\frac{\text{sign}(q)}{2|q|} \right]^{|D_+|} \\
 &\times K^{\frac{\text{rank } G}{2}} \left| \frac{p}{q} + v \right|^{\frac{-\dim G}{2}} \exp \left(-\frac{i\pi}{K} \frac{\rho^2}{q\left(\frac{p}{q} + v\right)} \right) \tag{3.18}
 \end{aligned}$$

allows an easy check of an obvious generalization of relations (3.11) and (3.12).

4. Beyond the Rational Homology Spheres

If a manifold $M(M')$ is not a RHS then the trivial connection is a point on a connected piece $\mathcal{M}_0(\mathcal{M}'_0)$ of the moduli space of flat connections. Equations

(2.20), (2.22) and (2.24) are no longer valid, since the 1-loop contribution of \mathcal{M}_0 to the partition function $Z_\alpha(M, \mathcal{K}; k)$ includes an integral

$$\int_{\mathcal{M}_0} \sqrt{|\tau_R|} \text{Tr}_\alpha \text{Pexp} \left(\oint_{\mathcal{K}} A_\mu dx^\mu \right), \tag{4.1}$$

which may have singularities (e.g. poles or breaks) at $\alpha = 0$.

We will determine the contribution of \mathcal{M}'_0 to Witten's invariant $Z(M'; k)$ when M' is a Seifert manifold $X_g \left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} \right)$. X_g can be produced by n surgeries $U^{(p_i, q_i)}$ on fibers of $S^1 \times \Sigma_g$; Σ_g is a g -handle Riemann surface. We will sketch the calculation leaving the details for [22].

The Seifert manifold $X_g \left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n} \right)$ can be constructed by an S surgery⁵ on a special knot \mathcal{K} belonging to the manifold M which is a connected sum of n lens spaces L_{-p_i, q_i} and $2g$ manifolds $S^1 \times T^2$. The Jones polynomial of the knot \mathcal{K} (in Witten's normalization) is

$$Z_\alpha(M, \mathcal{K}; k) = \prod_{i=1}^n \tilde{U}_{\alpha 1}^{(-q_i, p_i)} \tilde{S}_{\alpha 1}^{1-2g-n}. \tag{4.2}$$

We put $n = 0$ in Eq. (1.9) in order to extract the contribution of \mathcal{M}_0 to $Z_\alpha(M, \mathcal{K}; k)$:

$$Z_{\alpha}^{(\mathcal{M}_0)}(M, \mathcal{K}; k) = \left(\frac{2}{K} \right)^{\frac{1}{2}-g} \exp \left(-\frac{i\pi}{2K} \frac{H}{P} \alpha^2 \right) \times \left[\prod_{j=1}^n \frac{\text{sign}(p_j)}{\sqrt{|p_j|}} e^{-\frac{i\pi}{4} \Phi(U^{(-q_j, p_j)})} \right] \frac{\prod_{j=1}^n \sin \left(\frac{\pi}{K} \frac{\alpha}{p_j} \right)}{\sin^{n+2g-1} \left(\frac{\pi}{K} \alpha \right)}, \tag{4.3}$$

here

$$P = \prod_{j=1}^n p_j, \quad \frac{H}{P} = \sum_{j=1}^n \frac{q_j}{p_j}, \quad H = \text{ord } H_1(X_g, \mathbb{Z}). \tag{4.4}$$

The expression in the r.h.s. of Eq. (4.3) has a pole at $\alpha = 0$ of order $2g - 1$.

After substituting Eq. (4.3) into the surgery formula (1.10) and taking into account that $U^{(p, q)} = S$ and

$$\phi_{\text{fr}} = \frac{\pi}{4} \frac{K-2}{K} \left[\sum_{j=1}^n \Phi(U^{(-q_j, p_j)}) + 3 \text{sign} \left(\frac{H}{P} \right) \right], \tag{4.5}$$

we get the following expression:

$$\left(\frac{2}{K} \right)^{1-g} \sum_{\alpha=1}^{K-1} \exp \left(-\frac{i\pi}{2K} \frac{H}{P} \alpha^2 \right) \frac{\text{sign}(P)}{\sqrt{|P|}} \frac{e^{i\pi \frac{3}{4} \frac{K-2}{K} \text{sign} \left(\frac{H}{P} \right)}}{\sin^{n+2g-2} \left(\frac{\pi}{K} \alpha \right)} \times \prod_{j=1}^n e^{-\frac{i\pi}{2K} \Phi(U^{(-q_j, p_j)})} \sin \left(\frac{\pi}{K} \frac{\alpha}{p_j} \right). \tag{4.6}$$

⁵ $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z})$.

We should be careful in converting it to the integral (3.6), because the points $\alpha = 0, K$ are singular. We shift the argument of the singular factor:

$$\sin^{-n-2g+2} \left(\frac{\pi}{K} \alpha \right) \rightarrow \lim_{\varepsilon \rightarrow 0} \sin^{-n-2g+2} \left[\frac{\pi}{K} (\alpha - i\varepsilon) \right]. \tag{4.7}$$

Now we can add the points $\alpha = 0, K$ to the sum (4.6). The factor $\prod_{j=1}^n \sin(\frac{\pi}{K} \frac{\alpha}{p_j})$ makes the contribution to $\alpha = 0$ equal to zero, while the contribution of $\alpha = K$ does not affect the local behavior at $\alpha = 0$. Thus the contribution of the piece \mathcal{M}'_0 containing the trivial connection to Witten’s invariant $Z(X_g; k)$ is equal to the contribution of the special point $\alpha_* = 0$ to the integral in the following expression:

$$\begin{aligned} Z^{(\mathcal{M}'_0)}(X_g; k) &= \frac{1}{2} \left(\frac{2}{K} \right)^{1-g} \frac{\text{sign}(P)}{\sqrt{|P|}} e^{i\pi \frac{3}{4} \text{sign}(\frac{H}{P})} \\ &\times \exp \frac{i\pi}{2K} \left(\frac{H}{P} - 3\text{sign} \left(\frac{H}{P} \right) - 12 \sum_{j=1}^n s(q_j, p_j) \right) \\ &\times \lim_{\varepsilon \rightarrow 0} \sum_{l \in \mathbb{Z}} \int_{\substack{+\infty \\ \alpha_* = 0 \\ -\infty}} d\alpha \frac{\prod_{j=1}^n \sin \left(\frac{\pi}{K} \frac{\alpha}{p_j} \right)}{\sin^{n+2g-2} \left(\frac{\pi}{K} (\alpha - i\varepsilon) \right)} e^{2\pi i \alpha l} \\ &\times \exp \left(-\frac{i\pi}{2K} \frac{H}{P} \alpha^2 \right). \end{aligned} \tag{4.8}$$

For $l = 0, \alpha = 0$ is a stationary phase point. Similar to the previous section we conclude that

Proposition 4.1. *The contribution of the $l = 0$ term to the expression (4.8) for $Z^{(\mathcal{M}'_0)}(X_g; K)$ is equal to*

$$\begin{aligned} Z_{l=0}^{(\mathcal{M}'_0)} &= \frac{1}{2} \left(\frac{2}{K} \right)^{1-g} \frac{\text{sign}(P)}{\sqrt{|P|}} e^{i\pi \frac{3}{4} \text{sign}(\frac{H}{P})} \exp \frac{i\pi}{2K} \left(\frac{H}{P} - 3\text{sign} \left(\frac{H}{P} \right) - 12 \sum_{j=1}^n s(q_j, p_j) \right) \\ &\times \int_{\substack{+\infty \\ \alpha_* = 0 \\ -\infty}} d\alpha \sin^{-n-2g+2} \left(\frac{\pi}{K} \alpha \right) \left[\prod_{j=1}^n \sin \left(\frac{\pi}{K} \frac{\alpha}{p_j} \right) \right] \exp \left(-\frac{i\pi}{2K} \frac{H}{P} \alpha^2 \right). \end{aligned} \tag{4.9}$$

To calculate the integral we have to expand the preexponential factor in Laurent series in K^{-1} :

$$\begin{aligned} \frac{\prod_{j=1}^n \sin \left(\frac{\pi}{K} \frac{\alpha}{p_j} \right)}{\sin^{n+2g-2} \left(\frac{\pi}{K} (\alpha - i\varepsilon) \right)} &= \frac{1}{P} \left(\frac{\pi}{K} \alpha \right)^{2g-2} - \frac{1}{6P} \left(\frac{\pi}{K} \alpha \right)^{4-2g} \\ &\times \left(2 - 2g - n + \sum_{j=1}^n \frac{1}{p_j^2} \right) + \dots, \end{aligned} \tag{4.10}$$

and then integrate each term separately with the help of the formula

$$\int_{-\infty}^{+\infty} d\alpha \alpha^{2m} \exp\left(-\frac{i\pi}{2K} \frac{H}{P} \alpha^2\right) = \left(\frac{2K}{\pi} \left|\frac{P}{H}\right|\right)^{m+\frac{1}{2}} e^{-\frac{i\pi}{2}(m+\frac{1}{2})\text{sign}(\frac{H}{P})} \Gamma\left(m + \frac{1}{2}\right), \tag{4.11}$$

which works for both positive and negative m .

Thus the contribution of $l = 0$ is expressed as an asymptotic series in K^{-1} :

$$\begin{aligned} Z_{l=0}^{(\mathcal{M}'_0)} &= 2^{\frac{3}{2}-2g} \pi^{\frac{1}{2}-g} K^{2g-\frac{3}{2}} |H|^{-\frac{3}{2}} \left(\frac{H}{P}\right)^g \Gamma\left(\frac{3}{2} - g\right) \\ &\times \left\{ 1 + \frac{i\pi}{2K} \left[\frac{P}{H} \left(1 - \frac{2}{3}g\right) \left(2 - 2g - n + \sum_{j=1}^n \frac{1}{p_j^2}\right) - 3\text{sign}\left(\frac{H}{P}\right) \right. \right. \\ &\left. \left. + \frac{H}{P} - 12\sum_{j=1}^n s(q_j, p_j)\right] + \mathcal{O}(K^{-2}) \right\}. \end{aligned} \tag{4.12}$$

The stationary phase points of the integral in Eq. (4.8) for $l \neq 0$ are

$$\alpha_l^{(\text{st})} = 2K \frac{H}{P} l. \tag{4.13}$$

The integration contour of steepest descent for these points is

$$\text{Im } \alpha = \text{sign}\left(\frac{H}{P}\right) (\alpha_l^{(\text{st})} - \text{Re } \alpha). \tag{4.14}$$

The original integration contour $\text{Im } \alpha = 0$ should be rotated into the contour (4.14). During this rotation it crosses the pole $\alpha = i\varepsilon$ of the integrand of (4.8) if $\text{sign}(\frac{H}{P}) \times \alpha_l^{(\text{st})} > 0$. Therefore the pole contributes its residue to the integral (4.8) if

$$l \geq 1. \tag{4.15}$$

The total contribution of $l \neq 0$ to $Z^{(\mathcal{M}'_0)}(X_g; k)$ is equal to

$$\begin{aligned} Z_{l \neq 0}^{(\mathcal{M}'_0)}(X_g; k) &= \frac{1}{2} \left(\frac{2}{K}\right)^{1-g} \frac{\text{sign}(P)}{\sqrt{|P|}} e^{i\pi\frac{3}{4}\text{sign}(\frac{H}{P})} \exp \frac{i\pi}{2K} \\ &\times \left(\frac{H}{P} - 3\text{sign}\left(\frac{H}{P}\right) - 12\sum_{j=1}^n s(q_j, p_j)\right) \\ &\times 2\pi i \lim_{\varepsilon \rightarrow 0} \text{Res}_{\alpha=i\varepsilon} \left[\frac{\exp\left(-\frac{i\pi}{2K} \frac{H}{P} \alpha^2\right)}{e^{-i\pi\alpha} - 1} \frac{\prod_{j=1}^n \sin\left(\frac{\pi}{K} \frac{\alpha}{p_j}\right)}{\sin^{n+2g-2}\left(\frac{\pi}{K}(\alpha - i\varepsilon)\right)} \right]. \end{aligned} \tag{4.16}$$

The $\varepsilon \rightarrow 0$ limit in this expression is nonsingular, so after changing the variable to $x = \frac{\pi}{K}\alpha$, we conclude that

Proposition 4.2. *The contribution of $l \neq 0$ terms to the expression (4.8) for $Z^{(\mathcal{M}'_0)}(X_g; K)$ is equal to*

$$\begin{aligned}
 Z_{l \neq 0}^{(\mathcal{M}'_0)}(X_g; k) &= -i\pi \left(\frac{2}{K}\right)^{1-g} \frac{\text{sign}(P)}{\sqrt{|P|}} e^{i\pi \frac{3}{4} \text{sign}\left(\frac{H}{P}\right)} \\
 &\quad \times \exp \frac{i\pi}{2K} \left(\frac{H}{P} - 3\text{sign}\left(\frac{H}{P}\right) - 12 \sum_{j=1}^n s(q_j, p_j) \right) \\
 &\quad \times \text{Res}_{x=0} \frac{K \exp\left(-\frac{iK}{2\pi} \frac{H}{P} x^2\right) \prod_{j=1}^n \sin\left(\frac{x}{p_j}\right)}{\pi \frac{e^{iKx} - 1}{\sin^{n+2g-2}(x)}}. \tag{4.17}
 \end{aligned}$$

The residue is a polynomial $P_{2g-2}(K)$ in K of order $2g - 2$:

$$\begin{aligned}
 Z_{l \neq 0}^{(\mathcal{M}'_0)}(X_g; k) &= -\frac{i\pi}{2^{g-1}} \frac{\text{sign}(P)}{\sqrt{|P|}} e^{i\pi \frac{3}{4} \text{sign}\left(\frac{H}{P}\right)} \\
 &\quad \times \exp \frac{i\pi}{2K} \left(\frac{H}{P} - 3\text{sign}\left(\frac{H}{P}\right) - 12 \sum_{j=1}^n s(q_j, p_j) \right) K^{g-1} P_{2g-2}(K), \tag{4.18}
 \end{aligned}$$

here

$$P_{2g-2}(K) = \frac{i^{2g-3} K^{2g-2}}{\pi P} \frac{B_{2g-2}}{(2g-2)!} - \frac{i^{2g-4} K^{2g-3} H}{2\pi^2 P^2} \frac{B_{2g-4}}{(2g-4)!} + \dots, \tag{4.19}$$

B_n are Bernoulli numbers.

The following statement summarizes our calculations:

Proposition 4.3. *The contribution of the connected piece \mathcal{M}'_0 of the moduli space of flat connections on a Seifert manifold $X_g\left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}\right)$, which contains the trivial connection, is given by a sum of two terms,*

$$Z^{(\mathcal{M}'_0)}(X_g; k) = Z_{l \neq 0}^{(\mathcal{M}'_0)}(X_g; k) + Z_{l=0}^{(\mathcal{M}'_0)}(X_g; k), \tag{4.20}$$

which are expressed by Eqs. (4.17) and (4.9). The first term is a finite polynomial in K , while the second one is an asymptotic series in K^{-1} .

There is an obvious similarity between these results and the calculations of [16]. Technically it comes from the similarity between the sum (4.6) and the formula for a partition function of the 2d gauge theory. The asymptotic expansion of both expressions can be calculated with the help of the same technical tricks. The connected piece \mathcal{M}'_0 of the moduli space of flat connections on $X_g\left(\frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}\right)$ which contains the trivial connection is isomorphic to the moduli space of trivial connections on Σ_g . Therefore the polynomial $P_{2g-2}(K)$ may also be related to some intersection numbers on \mathcal{M}'_0 . Note however that the degree of this polynomial is bigger than that of its counterpart in [16].

Similar to [16], the $l = 0$ contribution (4.10) should be related to the singularity of \mathcal{M}'_0 at the trivial connection. It carries the fractional power of K , but it is an asymptotic series rather than one term as in [16].

It remains to be determined if either of the second terms in Eqs. (4.12) or (4.14) might be related to Casson–Walker invariant of X_g . According to C. Lescop [23] this invariant should be zero for $g \geq 2$. Note that when $g \geq 2$, $Z_{l \neq 0}^{(\mathcal{M}'_0)}(X_g; k)$ starts dominating $Z_{l=0}^{(\mathcal{M}'_0)}(X_g; k)$ in the large k limit.

5. Discussion

Even apart from invoking the path integral representation of Witten’s invariant, some arguments of Sects. 2 and 3 were not quite rigorous. A more careful study of boundary conditions for Witten’s invariant of manifolds with boundary and their relation to the boundary conditions used in [24] to define the analytic torsion, is needed. The formula (3.5) might also require a more rigorous proof. We take however some encouragement from the fact that it contains Walker’s surgery formula [15] for Casson–Walker invariant, whose correctness has been checked.

The further study of the loop expansion for Witten’s invariant of manifolds with boundaries may be a useful tool in understanding Vassiliev invariants. According to formulas of Sect. 2, Vassiliev invariants are rearranged and related to Feynman diagrams on a knot complement. In this approach the emphasis would be on cubic vertices rather than on chords of the knot diagrams.

The appearance of the Casson–Walker invariant as a 2-loop correction to the contribution of the trivial connection looks strange. After all, the Casson–Walker invariant is rather a “number” of flat $SU(2)$ connections than a local property of Chern–Simons action near the trivial connection. However this fact has its precedent. The 1-loop correction to the contribution of the trivial connection (as well as other connections) to the $U(1)$ Witten’s invariant of a RHS M is proportional to the square root of the Reidemeister–Ray–Singer torsion of M , which is known to be equal to $\text{ord } H_1(M, \mathbb{Z})$. On the other hand, the order of homology is equal to the number of flat $U(1)$ connections on M and may be called a $U(1)$ Casson–Walker invariant. Going further along this way we can expect to find the surgery formulas for Casson–Walker invariants of other groups among the higher loop pieces of Eq. (3.15).

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Appendix

We illustrate some calculations of Sect. 2 by considering an example: a large k asymptotics of the Jones polynomial of the type (m, n) torus knot $\mathcal{K}_{m,n}$ in S^3 . The $1/K$ expansion of this polynomial has been worked out by H. Morton in [25]. Here we take a limit in which the ratio $a = \alpha/K$ is kept constant as $k \rightarrow \infty$. This allows us to identify the contributions of various flat connections in the knot complement. According to Proposition 2.2, the contribution of the reducible flat connection will provide the usual $1/K$ expansion of the Jones polynomial.

The Jones polynomial $Z_\alpha(S^3, \mathcal{K}_{m,n}; k)$ is expressed as a sum

$$Z_\alpha(S^3, \mathcal{K}_{m,n}; k) = \frac{1}{2} \sum_{\mu=\pm 1} \sum_{\substack{\alpha-1 \\ \beta=-\alpha+1 \\ \beta-\alpha \text{ odd}}} \sqrt{\frac{2}{K}} \sin \left[\frac{\pi}{K} (m\beta + \mu) \right] \times \exp \frac{i\pi}{2K} \left[\frac{n}{m} (m\beta + \mu)^2 - \frac{n}{m} - nm(\alpha^2 - 1) \right]. \quad (A.1)$$

The sum over β can be converted into an integral with the help of the Poisson resummation formula. In this particular case

$$\sum_{\substack{\alpha-1 \\ \beta=-\alpha+1 \\ \beta-\alpha \text{ odd}}} = \frac{1}{2} \sum_{l \in \mathbb{Z}} \int_{-\alpha}^{\alpha} d\beta \exp[i\pi l(\alpha + \beta + 1)], \quad (A.2)$$

so that

$$Z_\alpha(S^3, \mathcal{K}_{m,n}; k) = \frac{1}{8i} \sqrt{\frac{2}{K}} \sum_{\mu_{1,2}=\pm 1} \mu_1 \mu_2 \sum_{l \in \mathbb{Z}} \int_{-\alpha}^{\alpha} d\beta \exp \frac{i\pi}{K} \left[\frac{mn}{2} \beta^2 + (n\mu_1 + m\mu_2 + Kl)\beta + \mu_1 \mu_2 - \frac{mn}{2} (\alpha^2 - 1) + Kl(\alpha + 1) \right]. \quad (A.3)$$

To find the large k asymptotics of this expression we apply a stationary phase approximation to the integral over β in the way similar to [5]. We start with the contribution of the boundary points $\beta = \pm\alpha$. They contribute the same amount due to the symmetry of the integrand, so we simply double the contribution of the point $\beta = -\alpha$. After shifting the integration variable $\beta \rightarrow \beta - \alpha$ and some additional transformations we get the following formula:

$$\begin{aligned} Z_\alpha^{(\pm\alpha)}(S^3, \mathcal{K}_{m,n}; k) &= \frac{1}{4i} \sqrt{\frac{2}{K}} \sum_{\mu_{1,2}=\pm 1} \mu_1 \mu_2 \exp \frac{i\pi}{K} \left[-\alpha(\mu_1 n + \mu_2 m) + \frac{m^2 n^2 - m^2 - n^2}{2mn} \right] \\ &\times \sum_{l \in \mathbb{Z}} e^{i\pi l} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{Kmn}{2\pi i} \right)^j \partial_\varepsilon^{(2j)} \left\{ \exp \left[\frac{i\pi}{K} \varepsilon \left(\frac{\mu_1}{m} + \frac{\mu_2}{n} \right) \right] \right. \\ &\times \left. \int_0^\infty d\beta \exp \left[\frac{i\pi}{K} \beta (\varepsilon - mn\alpha + Kl) \right] \right\} \Big|_{\varepsilon=0} \\ &= \frac{1}{4i} \sqrt{\frac{2}{K}} \sum_{\mu_{1,2}=\pm 1} \mu_1 \mu_2 \exp \frac{i\pi}{K} \left[-\alpha(\mu_1 n + \mu_2 m) + \frac{m^2 n^2 - m^2 - n^2}{2mn} \right] \\ &\times \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{Kmn}{2\pi i} \right)^j \partial_\varepsilon^{(2j)} \left\{ \exp \left[\frac{i\pi}{K} \varepsilon \left(\frac{\mu_1}{m} + \frac{\mu_2}{n} \right) \right] \left[\frac{iK}{\pi} \sum_{l \in \mathbb{Z}} \frac{e^{i\pi l}}{\varepsilon - mn\alpha - Kl} \right] \right\} \Big|_{\varepsilon=0} \\ &= \sqrt{\frac{2}{K}} \sum_{\mu_{1,2}=\pm 1} \mu_1 \mu_2 \exp \frac{i\pi}{K} \left[\frac{m^2 n^2 - m^2 - n^2}{2mn} \right] \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{2\pi i}{K} mn \right)^{-j} \\ &\times \partial_{\tilde{\alpha}}^{(2j)} \left[\frac{\sin \left(\frac{\pi}{K} n \tilde{\alpha} \right) \sin \left(\frac{\pi}{K} m \tilde{\alpha} \right)}{\sin \left(\frac{\pi}{K} mn \tilde{\alpha} \right)} \right] \Big|_{\tilde{\alpha}=\alpha}. \end{aligned} \quad (A.4)$$

If we substitute Eq. (2.5) then we find that in the 1-loop approximation,

$$Z_x^{(\pm\alpha)}(S^3, \mathcal{K}_{m,n}; k) = \sqrt{\frac{2}{K}} \frac{\sin(\pi na)\sin(\pi ma)}{\sin(\pi mna)} + \mathcal{O}(K^{-\frac{3}{2}}). \tag{A.5}$$

This is a particular case of relation (2.9) if we recall the formula for the Alexander polynomial of the torus knot,

$$\Delta_A(S^3, \mathcal{K}_{m,n}; \exp(2\pi ia)) = \frac{\sin(\pi mna)\sin(\pi a)}{\sin(\pi ma)\sin(\pi na)}. \tag{A.6}$$

Thus we see that the contribution of the boundary points $\beta = \pm\alpha$ is in fact the contribution of the reducible flat connection in the knot complement satisfying the boundary condition (2.7). The final expression of Eq. (A.4) is well defined not only for α of order K (where it was derived) but also for α of order 1, where it becomes a formula for $Z_x(S^3, \mathcal{K}_{m,n}; k)$.

The stationary phase points of the integral (A.3),

$$\beta_l^{(st)} = -\frac{Kl}{mn}, \tag{A.7}$$

also contribute to the Jones polynomial $Z_x(S^3, \mathcal{K}_{m,n}; k)$. The contributions of the points $\beta_l^{(st)}$ and $\beta_{-l}^{(st)}$ are equal and should be combined into

$$\begin{aligned} Z_x^{(l)}(S^3, \mathcal{K}_{m,n}; k) &= \frac{1}{4i} \sqrt{\frac{2}{K}} \sum_{\mu_1, \mu_2 = \pm 1} \mu_1 \mu_2 \int_{-\infty}^{+\infty} d\beta \\ &\times \exp \frac{i\pi}{K} \left[\frac{mn}{2} \beta^2 + (\mu_1 n + \mu_2 m + Kl) + \mu_1 \mu_2 - \frac{mn}{2} (\alpha^2 - 1) + Kl(\alpha + 1) \right] \\ &= \frac{i}{2} e^{i\frac{\pi}{4} \text{sign}(mn) + i\pi l} \exp \left[-i\pi K \frac{mn}{2} \left(a - \frac{l}{mn} \right)^2 \right] \frac{4 \sin \left(\pi \frac{l}{m} \right) \sin \left(\pi \frac{l}{n} \right)}{\sqrt{|mn|}} \\ &\times \exp \left[\frac{i\pi}{2K} \left(\frac{m^2 n^2 - m^2 - n^2}{mn} \right) \right]. \end{aligned} \tag{A.8}$$

The contributions $Z_x^{(l)}(S^3, \mathcal{K}_{m,n}; k)$ come from the irreducible flat connections in the knot complement satisfying Eq. (2.7). The classical exponent as well as the ingredients of the 1-loop formula (1.14) can be easily identified in the expression (A.8). Note that the whole expression is 2-loop exact similar to the results of [5].

A particular value of l may contribute to the integral (A.3) only if the point $\beta_l^{(st)}$ lies within the integration range, that is

$$0 < l < mna. \tag{A.9}$$

This condition cannot be satisfied for sufficiently small values of a . We see that the first irreducible connection appears only for $a > \frac{1}{mn}$. New irreducible connections emerge at critical values

$$a_l^{(cr)} = \frac{l}{mn}, \quad 0 < l < mn. \tag{A.10}$$

These are also the zeros of the Alexander polynomial (A.6). This is not surprising (see e.g. [26, 27]), since the zeros signal the presence of a zero mode in one of the determinants comprising the Ray–Singer analytic torsion. This mode is responsible for the “off-diagonal” deformation of the flat reducible connection.

The formula (A.4) becomes singular near the critical points (A.10). However in this case it only means that the calculation of the contribution of the boundary points $\beta = \pm\alpha$ has to be modified when one of the stationary phase points (A.7) is close to the boundary. More specifically, the integral over β in Eq. (A.4) has to be recalculated for $l = l_0$ if

$$\alpha = K \frac{l_0}{mn} + \gamma, \quad \gamma \ll \sqrt{K}, \quad 0 < l_0 < mn. \tag{A.11}$$

As a result, the combined contribution of the stationary phase points $\beta_{\pm l_0}^{(st)}$ and boundary points $\pm\alpha$ to $Z_\alpha(S^3, \mathcal{K}_{m,n}; k)$ is given by the formula

$$\begin{aligned} Z_{K \frac{l_0}{mn} + \gamma}^{(l_0, \pm\alpha)}(S^3, \mathcal{K}_{m,n}; k) &= i e^{i \frac{\pi}{4} \text{sign}(mn) + i\pi l_0} \frac{\sin\left(\pi \frac{l_0}{m}\right) \sin\left(\pi \frac{l_0}{n}\right)}{\sqrt{|mn|}} \\ &\times \exp\left[\frac{i\pi}{2K} \left(-mn\gamma^2 + \frac{m^2 n^2 - m^2 - n^2}{mn}\right)\right] \\ &+ \sqrt{\frac{2}{k}} e^{i\pi l_0} \exp\left[\frac{i\pi}{K} \left(\frac{m^2 n^2 - m^2 - n^2}{mn}\right)\right] \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{2\pi i}{K} mn\right)^{-j} \\ &\times \partial_{\tilde{\gamma}}^2 \left[\frac{\sin \pi \left(\frac{l_0}{m} + \frac{n}{K} \tilde{\gamma}\right) \sin \pi \left(\frac{l_0}{n} + \frac{m}{K} \tilde{\gamma}\right)}{\sin\left(\frac{\pi}{K} mn \tilde{\gamma}\right)} \right. \\ &\left. - \frac{K}{\pi} \frac{\sin\left(\pi \frac{l_0}{m}\right) \sin\left(\pi \frac{l_0}{n}\right)}{mn \tilde{\gamma}} \exp\left(-\frac{i\pi}{K} mn \tilde{\gamma} \gamma\right) \right] \Bigg|_{\tilde{\gamma}=\gamma} \tag{A.12} \end{aligned}$$

which demonstrates the smooth behavior of $Z_\alpha(S^3, \mathcal{K}_{m,n}; k)$ in the vicinity of critical points (A.10).

The following proposition summarizes our calculations:

Proposition A.1. *A large k asymptotics of the Jones polynomial of a type (m, n) torus knot in the limit when $a = \frac{\alpha}{K}$ is kept fixed, contains the contribution of the reducible connection (A.4) as well as the contributions of irreducible connections (A.8) for the values of l satisfying the condition (A.9).*

The formula (A.4) works also in the limit when α is fixed. It becomes a contribution of the trivial connection and provides a usual $1/K$ expansion of the Jones polynomial $Z_\alpha(S^3, \mathcal{K}_{m,n}; k)$.

The expression (A.4) has a singular behavior at zeros of the Alexander polynomial (A.6), however the whole Jones polynomial is smooth (see Eq. (A.12)).

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