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# Characterizing Invariants for Local Extensions of Current Algebras

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**Abstract:** Pairs $\mathscr{A} \subset \mathscr{B}$ of local quantum field theories are studied, where $\mathscr{A}$ is a chiral conformal quantum field theory and $\mathscr{B}$ is a local extension, either chiral or two-dimensional. The local correlation functions of fields from $\mathscr{B}$ have an expansion with respect to $\mathscr{A}$ into conformal blocks, which are non-local in general. Two methods of computing characteristic invariant ratios of structure constants in these expansions are compared: (a) by constructing the monodromy representation of the braid group in the space of solutions of the Knizhnik–Zamolodchikov differential equation, and (b) by an analysis of the local subfactors associated with the extension with methods from operator algebra (Jones theory) and algebraic quantum field theory. Both approaches apply also to the reverse problem: the characterization and (in principle) classification of local extensions of a given theory.

## 1. Introduction

The relevance of V. Jones' theory of (von Neumann) subfactors [1] for 2-dimensional (2D) models of critical behaviour was first recognized in the work of V. Pasquier on lattice models labelled by Dynkin diagrams [2]. A spectacular by-product of this realization was the ensuing *ADE* classification of $su(2)$ current algebra models and minimal conformal theories [3]. The above parallel was understood within the Haag–Kastler algebraic approach to local quantum field theory [4] in terms of the Doplicher–Haag–Roberts (DHR) theory of superselection sectors and particle statistics [5] applied to chiral algebras [6, 7], and provided an explanation for the Jones index as a measure for the violation of Haag duality (maximality of local observables) in a given representation, and relating it numerically to the statistical dimension [8].

In the cited work on subfactors in quantum field theory, the emphasis for the use of the theory of subfactors was its application to individual superselection sectors of a given local theory, and the derivation of invariant "charge quantum numbers" such as statistical dimensions and Markov traces. In contrast, here we shall consider a

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pair of local theories, one extending the other, as a subfactor (actually, a net of local subfactors) and apply the properly adapted Jones theory to describe the "position" of the subtheory in the ambient theory. This point of view opens the way to a detailed understanding of the behaviour of superselection sectors when one passes from one theory to the other by a generalized Mackey induction and restriction prescription [9].

In particular, given that the position of a subtheory in another theory is encoded and characterized by a subfactor, then subfactor theoretical methods can be applied to conformal models and their local extensions, and must give detailed answers comparable with the *ADE* classification and related results obtained by conventional methods of conformal quantum field theory.

The present article is a comparative study of conventional field theoretical methods on the one hand and the theory of subfactors on the other hand in application to the same problem: local extensions of local quantum field theories. A local extension is determined by the correlation functions of the extending fields. In chiral current algebra models of conformal field theory, the extending fields necessarily correspond to primary fields of the original theory with bosonic, i.e., integer conformal dimension $\Delta$. Their 4-point functions are linear (for chiral extensions) or bilinear (for 2D extensions) combinations of conformal block functions which are monodromy-free in spite of the non-trivial braid group transformation of the individual conformal blocks. Moreover, unlike the chiral vertex operators of the unextended theory whose fusion rules coincide with the intrinsic composition law of superselection charges provided by the DHR theory, the extending local fields must satisfy truncated fusion rules which involve only other bosonic fields, and which are therefore only majorized by the DHR fusion.

The truncated fusion rules and the ratios of structure constants (amplitudes of conformal block functions) in the said combinations are characteristic quantities for a pair of a chiral current algebra and its extension. They are computed by both methods. In the first part of the article (Sects. 2 and 3), we study the monodromy behaviour of the solutions of the Knizhnik–Zamolodchikov (KZ) equation and compute the braid invariant quadratic forms which determine the local 4-point functions of the two-dimensional extensions. Apart from the generic two-dimensional extension (corresponding to the $A$ series of the *ADE* classification), and the chiral $D$ series extensions which correspond to a global $\mathbb{Z}_2$ symmetry, we concentrate on the exceptional chiral $E_6$ and $E_8$ extensions of $su(2)$ current algebras. We compute explicitly the relative amplitudes of the $A$ and $E$ theories, which turn out to be rational numbers. In the second part (Sects. 4 and 5), we study the position of the operator algebra of the subtheory within the ambient theory in terms of the theory of subfactors. Remarkably, the relevant information already resides in a single pair of local von Neumann algebras. We analyze which quantities in the general theory of subfactors, when applied to a given local field extension, contain the desired information about the truncated fusion rules and the relevant ratios of structure constants. We describe how to compute these data in terms of the subtheory (interpreted as the physical observables) and its superselection structure.

While the first method will be easier to use in specific models and as long as one is interested only in 4-point functions, the second method is part of a general theory of local field extensions, confined neither to two dimensions nor to conformal quantum field theories. It covers also the standard situation of four-dimensional theories with a compact gauge group where one is interested in the subtheory of gauge invariant quantities. (In this latter case, the method essentially reduces to harmonic analysis and partial wave expansions based on the representation theory and Clebsch–Gordan coefficients of the gauge group.) It has the advantage to treat all $n$-point functions at one stroke. However, in practice it requires to solve in a first step a complicated non-linear system for the "generalized Clebsch–Gordan coefficients," which we have carried out only for the simplest model of a field extension which is not due to a gauge group.

The local extensions of chiral $su(2)$ current algebras studied in Sects. 2 and 3 are distinguished to have the same stress-energy tensor as the original theory, the stress-energy tensor implicitly entering the analysis through the KZ equations. If the extending fields are currents of dimension $\Delta = 1$, this condition means that the extension is a "conformal embedding" [12]. On the other hand, in Sects. 4 and 5, we assume the index of the inclusion to be finite. Indeed, for pairs of chiral current algebras, these two selection criteria are equivalent. Namely, both the finiteness of the index and the triviality of the coset stress-energy tensor are equivalent to the finiteness of the branching of the vacuum representation of the ambient theory upon restriction to the subtheory.

Let now $\mathscr{A} \subset \mathscr{B}$ be a conformal embedding [12] of two chiral quantum field theories like the current algebras $\mathscr{A}_{10}(A_1) \subset \mathscr{A}_1(B_2)$, where $A_1 = su(2)$ and $B_2 = spin(5) \simeq sp(4)$ refer to the Lie algebras underlying the current algebras, and the subscripts refer to the level 10 resp. 1 of the central extension. The embedding gives rise to a pair of braid-invariant quadratic forms $M$ and $\widetilde{M}$ in the space of 4-point conformal blocks of the subtheory $\mathscr{A}$ with four given external quantum numbers (superselection charges) such as isospins $I \leqq k/2$ for $\mathscr{A} = \mathscr{A}_k(A_1)$. The quadratic forms serve to express 2D correlation functions in terms of chiral conformal blocks, and turn out to completely characterize the model. The form $M$ corresponds to the "diagonal" WZNW theory [13] over $\mathscr{A}$, i.e., to the $A_{k+1}$ theory in the $ADE$ classification of $su(2)$ current algebra models at level $k$ [3]. The eigenvalues $D_\lambda^{(k,I)}$ of $M$, in the case of 4 equal external isospins $I$, are the squares of the structure constants

$$D_\lambda^{(k,I)} \equiv N_{II\lambda}^2 \quad (\lambda = 0, 1, \dots, \min(2I, k-2I) \equiv m_{kI}) \tag{1.1}$$

for the $s$-channel fusion of two of the isospin $I$ charges into isospin $\lambda$ intermediate states. We recall that for $4I > k$, the subspace of 4-point blocks with $\lambda > m_{kI}$ corresponds to "unphysical" correlations which violate positivity. Only the "physical" blocks contribute to $M$ and $\widetilde{M}$.

The form $\widetilde{M}$ corresponds to the diagonal theory over the ambient chiral theory $\mathscr{B}$. Since the local fields of the latter are in general non-diagonal with respect to $\mathscr{A}$, the form $\widetilde{M}$ is a non-diagonal matrix in the $s$-channel basis of conformal block functions which diagonalizes $M$. The ratios of the diagonal elements of the form $\widetilde{M}$ to the corresponding eigenvalues (1.1) of $M$ are invariant under rescaling of the 4-point blocks and thus provide a basis-independent characteristics of the non-diagonal theory associated with the form $\widetilde{M}$. Such ratios were already considered in the above-mentioned pioneer work by Pasquier [2], and have later been computed for specific conformal embeddings [14]. We shall provide in Sect. 3 below an independent computation using previous work on monodromy representations of the braid group [15, 16].

Let us turn to the subfactor point of view. As we shall see, one can characterize a local field extension $\mathscr{B}$ of a given theory $\mathscr{A}$ in terms of a triple $(\varrho, W, X)$. Here $\varrho$ is a localized endomorphism of $\mathscr{A}$ equivalent to a reducible representation $\pi$ of $\mathscr{A}$ (the restriction of the vacuum representation of $\mathscr{B}$), $W$ is an isometric observable (i.e., $W^*W = \mathbb{1}$) such that $E = WW^*$ projects onto the vacuum representation $\pi_0$ of $\mathscr{A}$ contained in $\pi$, and $X$ is a second isometric observable satisfying a system of identities with $W$, involving $\varrho$, which guarantees the possibility to recover the local extension from these data. The field net $\mathscr{B}$ is generated by the observable net $\mathscr{A}$ and one isometric "charged" operator on the Hilbert space of $\pi$. It contains fields which create the states in the non-trivial subsectors of $\pi$ from the vacuum. The operator $X$ may be considered as a generating functional for all the relevant "generalized Clebsch–Gordan coefficients" associated with the inclusion. The mathematical concept behind this notion is a "harmonic analysis" for subfactors, which generalizes the ordinary harmonic analysis in the case of a compact gauge symmetry. The coefficients determine both the truncated operator product expansions and the amplitudes in the "partial wave" decomposition of correlation functions of local charged fields. The partial waves due to the subfactor harmonic analysis of charged fields will be identified with the conformal blocks in chiral current algebra models, and the Clebsch–Gordan coefficients coincide with the structure constants entering the quadratic forms as discussed before.

It is important to note that also in this general context, there is always a "standard" extension (corresponding to the generic braid-invariant quadratic form $M$ in the case of chiral current algebras) which can be used to fix the normalizations, i.e., to absorb the uncontrolled kinematical model characteristics, by computing invariant double ratios of amplitudes.

Our article is organized as follows. We review in Sect. 2 the monodromy representation of the mapping class group $\mathfrak{B}_4$ of the 2-sphere with 4 punctures in the space of solutions of the KZ equation, and write down the generic braid invariant form corresponding to the $A$ series in the $ADE$ classification. In Sect. 3, the explicit computations are done for two models of special interest, the $E_{\text{even}}$ series conformal embeddings labelled $E_6$ and $E_8$.

In Sect. 4, we turn to the theory of subfactors (of finite index) and introduce some of the basic concepts which are of particular relevance for the application to (local) field extensions. In Sect. 5, the connection with chiral vertex operators is established, and the general method to compute relative structure constants in terms of subfactors is presented. The method is then applied to the $E_6$ inclusion and reproduces the results obtained in Sect. 3.

The two parts consisting of Sects. 2, 3 and Sects. 4, 5, respectively, are to a large extent independent of each other. The reader may start with either part according to personal preference. Our point is the comparison of the conceptually different guises under which the same objects and quantities arise in the two approaches.

## 2. Braid Invariant Positive Forms in the Space of Solutions of the KZ Equation

We study the $2I+1$-dimensional space of solutions to the KZ differential equation [18] for the Möbius invariant 4-point functions

$$w(z_1,z_2,z_3,z_4) = \left(\frac{z_{13}z_{24}}{z_{12}z_{34}z_{14}z_{23}}\right)^{2\Delta} \cdot f(\eta)\,, \tag{2.1}$$

where $z_{ij}$ are coordinate differences, $\eta$ is the invariant cross ratio

$$z_{ij} = z_i - z_j, \qquad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}\,, \tag{2.2}$$

and $\Delta$ is the conformal dimension associated with the isospin $I$:

$$\Delta = \Delta_I = \frac{I(I+1)}{k+2} \quad (2I = 0, 1, \ldots, k) ; \tag{2.3}$$

the functions $w$ (and $f$) are $SU(2)$-invariant tensors in the tensor product of four representations $\mathscr{V}_i$ of $SU(2)$ with isospin $I(i = 1, \ldots, 4)$. The KZ equation reads in terms of the reduced functions $f$

$$\left((k+2)\frac{d}{d\eta} - \frac{C_{12}}{\eta} + \frac{C_{23}}{1-\eta}\right) f(\eta) = 0 , \tag{2.4}$$

where $C_{ij}$ are the $SU(2)$ Casimir invariants in $\mathscr{V}_i \otimes \mathscr{V}_j$. The solutions to the KZ equations are referred to as 4-point (conformal) blocks.

The quantum field theoretical background for this set-up can be summarized as follows. One starts with the algebra of chiral observables $\mathscr{A}_k = \mathscr{A}_k(A_1)$ generated by the $su(2)$ current algebra with the central extension of level $k$. This algebra contains the chiral affine-Sugawara stress-energy tensor. The primary chiral vertex operators $V_I$ [17] which intertwine the vacuum sector with the superselection sector of charge $I$ (= positive-energy representation of $\mathscr{A}_k$ with lowest energy eigenstates of isospin $I$) are assumed to have homogeneous local commutation relations with the currents ("local gauge covariance") and with the stress-energy tensor ("reparametrization covariance"). These assumptions imply that the 4-point correlation functions of chiral vertex operators satisfy the KZ equation [18], as well as the relation (2.3) between isospin and conformal (scaling) dimension. The 4-point correlation functions of 2D local conformal fields are then given as braid-invariant bilinear combinations of chiral conformal blocks, to be studied in Sect. 2B.

*2A. The Mapping Class Group and its Monodromy Representations.* The $2I+1$- dimensional space of all 4-point solutions of the KZ equation (2.4) carries a (projective) representation of the mapping class group $\mathfrak{B}_4$ of the 2-sphere with 4 punctures. We first construct this representation, into which the level $k$ enters only via the complex phase

$$q = \exp\left(\frac{i\pi}{k+2}\right) . \tag{2.5}$$

Unless $k$ is a positive integer, this space of solutions violates the positivity of correlation functions, and the representation of $\mathfrak{B}_4$ is not unitarizable. Yet, it is computationally advantageous to deal with generic $q$ in a first step. At a given level $k \in \mathbb{N}$, positivity is still violated for $4I > k$, and one has therefore, in a second step, to restrict to the $(m_{kI}+1)$-dimensional invariant "physical" subspace spanned by the $s$-channel blocks $s_\lambda^{(I)}$ with $\lambda$ in the range of (1.1).

The (projectively represented) mapping class group $\mathfrak{B}_4$ can be identified as the braid group of 4 strands on the sphere with generators $B_i$, $i = 1, 2, 3$, such that

$$B_1B_3 = B_3B_1, \qquad B_iB_{i+1}B_i = B_{i+1}B_iB_{i+1} \quad (i = 1, 2) , \tag{2.6}$$

$$B_1B_2B_3^2B_2B_1 = B_3B_2B_1^2B_2B_3 = q^{-4I(I+1)} , \tag{2.7}$$

satisfying the additional relation

$$(B_1B_2B_3)^4 = q^{-8I(I+1)} . \tag{2.8}$$

(In the standard definition of $\mathfrak{B}_4$, the relations (2.7) and (2.8) are assumed to hold with $q = 1$; here we are dealing with a projective representation, or equivalently, with a central extension of the mapping class group.) It can be proven, using only the above relations, that the monodromy operators $B_1^2$ and $B_3^2$ are equal. It then follows from (2.7) that the "fusion" matrix $F$ has square 1:

$$B_1 B_2 B_1 \equiv B_2 B_1 B_2 =: (-1)^{2I} q^{-2I(I+1)} F, \quad F^2 = \mathbb{1} \,. \tag{2.9}$$

$F$ plays the role of a $6j$ symbol (in general, for 4-point blocks of different isospins $I_i$, its matrix elements require 6 isospin labels $F_{\lambda\mu} = F_{\lambda\mu}^{I_1 I_2 I_3 I_4}$).

An analysis of the solutions of the KZ equation shows that (in the case at hand with four equal isospins $I$), actually the generators $B_1$ and $B_3$ coincide:

$$B_1 = B_3 \,. \tag{2.10}$$

Moreover, there exists a basis of solutions [15] for which the fusion matrix has only non-zero elements on the second diagonal,

$$F_{\lambda\mu} = \delta_{\lambda+\mu, 2I} \quad (\lambda, \mu = 0, 1, \ldots, 2I) \,, \tag{2.11}$$

while $B_1$ is upper triangular:

$$(B_1)_{\lambda\mu} = (-1)^{2I-\mu} q^{\mu(\lambda+1)-2I(I+1)} \begin{bmatrix} 2I - \lambda \\ \mu - \lambda \end{bmatrix} . \tag{2.12}$$

Here, $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ are the (real) $q$-binomial coefficients vanishing for $n < m$ and otherwise given by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}, \quad [n]! = [n][n-1]!, \quad [0]! = 1 \,, \tag{2.13}$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sin \frac{n\pi}{k+2}}{\sin \frac{\pi}{k+2}} \,. \tag{2.14}$$

We are using a non-unitary basis (even for $4I \leqq k$ when $B_1$ is unitarizable) which has the following advantages:

(i) it exhibits no singularities for $4I \geqq k + 2$ ($2I \leqq k$, $q$ given by (2.5));

(ii) the entries of the braid matrices and of the invariant forms are elements of the cyclotomic field $\mathbb{Q}(q^{1/2})$ (or $\mathbb{Q}(q)$ for integer $I$; $q^{k+2} = -1$).

We anticipate here, that the ratios of structure constants we are finally interested in (Eqs. (3.8), (3.9), and (3.15) below) turn out rational and are therefore invariant under Galois automorphisms $q \mapsto q^n$ ($n$ and $2k + 4$ coprime) of this field.

The second generator, $B_2$, of $\mathfrak{B}_4$ is a conjugate to $B_1$ by $F$:

$$B_2 := F B_1 F \quad (B_1 = F B_2 F) \,, \tag{2.15}$$

and appears as a lower triangular matrix.

It is noteworthy that this monodromy representation of $\mathfrak{B}_4$ can in fact be derived without a detailed study of the solutions of the KZ equations. Indeed, the eigenvalues of $B_1$ are already read off the 3-point block functions, which are just powers of the coordinate differences. In a basis in which the fusion matrix $F$ has the form (2.11) and $B_1$ is upper triangular, the non-diagonal entries of $B_1$ and the matrix $B_2$ are determined by (2.9) up to a rescaling of the basis. As it was already noted, the ratios of interest will turn out to be invariant under such a rescaling, too.

*2B. The Generic* $\mathfrak{B}_4$ *Invariant Symmetric Form.* The local 4-point function of the two-dimensional theory is defined by a hermitian braid invariant form $M$ in the space of 4-point blocks:

$$\langle \Phi_I \Phi_I \Phi_I \Phi_I \rangle \propto G_4 = \sum_{\lambda\mu} \bar{f}_\lambda M_{\lambda\mu} f_\mu \quad \text{with } M^+ = M = B^+ M B \quad (B \in \mathfrak{B}_4)\,, \tag{2.16}$$

where an appropriate power of the coordinate differences has been split off as in (2.1), and $f$ resp. $\bar{f}$ depend only on the conformally invariant cross ratios (2.2) of coordinate differences on the left- resp. right-moving light-cone. (For further details on the choice of basis $f_\lambda$ see [15].)

The above non-unitary realization of $B_i$ has the advantage that the inverse generators are just given by the complex conjugate matrices

$$B_i^{-1} = \overline{B}_i \quad \text{since} \quad \bar{q} = q^{-1}\,. \tag{2.17}$$

The same is trivially true for $F$.

We are thus looking for a real symmetric form $M = (M_{\lambda\mu}) = {}^tM$ satisfying the braid invariance condition

$${}^tB_i M = M B_i \quad (i = 1,2)\,. \tag{2.18}$$

**Proposition 2.1** [16]. *For every $q \neq 0$ there exists a diagonalizable $\mathfrak{B}_4$ invariant symmetric form in the space of 4-point solutions of the KZ equation with four isospins $I$,*

$$M = {}^tSDS, \quad \text{where } D_{\lambda\mu} = D_\lambda \delta_{\lambda\mu}\,. \tag{2.19}$$

*At the values $q = e^{\frac{i\pi}{k+2}}$ ($k \in \mathbb{N}$), the diagonal matrix $D$ has $m_{kI} + 1$ non-zero elements (with $m_{kI}$ given by (1.1)):*

$$D_\lambda \equiv D_\lambda^{(k,I)} = \left\{ \frac{[\lambda]![2I+1+\lambda]!}{[2I+1]![2\lambda]!} \right\}^2 \frac{1}{[2\lambda+1]}\,, \quad (\lambda = 0, \ldots, m_{kI})\,. \tag{2.20}$$

*If $4I > k$, then $D_\lambda$ vanish for $m_{kI} < \lambda \leqq 2I$. The transformation matrix $S$ is a real upper triangular matrix with elements*

$$S_{\lambda\mu} = (-1)^{\mu-\lambda} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[2I-\lambda]![2\lambda+1]!}{[2I-\mu]![\lambda+\mu+1]!} \quad \text{for } 0 \leqq \lambda \leqq m_{kI} \tag{2.21}$$

*and $S_{\lambda\mu} = \delta_{\lambda\mu}$ for $\lambda > m_{kI}$.*

*Sketch of a Proof.* We consider the similarity transformation

$$B \mapsto B^{(s)} := SBS^{-1}\,. \tag{2.22}$$

The specific block form $S = \begin{pmatrix} \Sigma & \Sigma' \\ 0 & \mathbb{1} \end{pmatrix}$ – where $\Sigma$ is given by $S$ in (2.21) and the rectangular block $\Sigma'$ is only present when $4I > k$ – implies the block form of the inverse matrix $S^{-1} = \begin{pmatrix} \Sigma^{-1} & -\Sigma^{-1}\Sigma' \\ 0 & \mathbb{1} \end{pmatrix}$ with

$$S^{-1}_{\lambda\mu} = \Sigma^{-1}_{\lambda\mu} = \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[2I-\lambda]![\lambda+\mu]!}{[2I-\mu]![2\mu]!} \quad \text{for } 0 \leqq \lambda \leqq \mu \leqq m_{kI} . \tag{2.23}$$

The transformation (2.22) brings $B_1$ in a reduced form for $4I > k$ and diagonalizes it for $4I \leqq k$; in both cases

$$(B_1^{(s)})_{\lambda\mu} = \delta_{\lambda\mu}(-1)^{2I-\lambda} q^{\lambda(\lambda+1)-2I(I+1)} \quad \text{for } \lambda, \mu \leqq m_{kI} . \tag{2.24}$$

In particular, the basis $s_\lambda = S_{\lambda\mu} f_\mu$ of conformal blocks has definite $B_1$ monodromy on the physical subspace $0 \leqq \lambda \leqq m_{kI}$. (For this reason we call $s_\lambda$ the $s$-channel basis.)

It follows that $B_1^{(s)}$ commutes with $D$ and hence (2.18) holds for $i = 1$. Verification of invariance of $M$ with respect to $B_2$ or $F$ requires more work. One could either use the explicit form of $M$:

$$M_{\lambda\mu} = \frac{(-1)^{\lambda+\mu}[\lambda]![\mu]!}{[2I-\lambda]![2I-\mu]![2I+1]!^2} \sum_{\nu=0}^{m_{kI}} \frac{[2I+\nu+1]!^2[2I-\nu]!^2[2\nu+1]}{[\lambda+\nu+1]![\mu+\nu+1]![\lambda-\nu]![\mu-\nu]!} \tag{2.25}$$

or transform $F$ to the $s$-channel basis ($F \mapsto F^{(s)} = SFS^{-1}$) – see below.

*Remarks.* ▷ An expression of the type (2.19), (2.25) for the invariant form was first derived in [16, Sect. 6] using quantum group techniques. The present formulae differ slightly because of a different normalization of the basis. They are related by $[2I+1]^2 M_{\lambda\mu} = \begin{bmatrix} 2I \\ \lambda \end{bmatrix} \begin{bmatrix} 2I \\ \mu \end{bmatrix} Z_{\lambda\mu}$. Such a change of basis does not affect the ratios of structure constants to be computed below.

▷ The proposition explicitly provides the transition matrix to the $s$-channel basis, from which, together with the spectrum (2.24) of the braid matrix, all the basis-independent quantities of interest in the sequel will be obtained by direct computations.

The braid invariant 2D 4-point function now assumes a diagonal form in the physical $s$-channel basis of conformal blocks $s_\lambda$ with $\lambda \leqq m_{kI}$,

$$G_4 = \sum_{\lambda=0}^{m_{kI}} D_\lambda^{(k,I)} \bar{s}_\lambda s_\lambda . \tag{2.26}$$

Summing up we see that, at the quantized values (2.5) of $q$, and more generally for any $q$ such that $q^{k+2} = -1$, the $(2I+1)$-dimensional representation $\mathfrak{B}_4$ of the mapping class group is reducible when $4I > k$. It is also non-unitarizable, the generators $B_i$ being not diagonalizable (for $4I \geqq k+2$). It is the kernel of the form $M$ that carries a non-unitary factor representation. The $(m_{kI}+1)$-dimensional subrepresentation $\mathfrak{B}_4^{(k,I)}$ preserves a non-degenerate positive form (2.26) and is hence unitarizable. The resulting $(m_{kI}+1)$-dimensional representation may, in general, still be reducible. As we shall see in Sect. 3A, this fact is responsible for the possible existence of non-diagonal local extensions.

The $s$-channel reflection matrix $F^{(s)}$ (which is related to the exchange of the factors 1 and 3 in (2.16) and which, for four generic isospins, encodes the entire fusion information of the model) is, not surprisingly, considerably more complicated than the original expression (2.11). We have computed it from

$$F^{(s)} = SFS^{-1} = SU^{-1} = US^{-1}$$

in terms of the above $s$-channel transition matrix $S$ which diagonalizes $B_1$, and the $u$-channel transition matrix $U = SF$ which diagonalizes $B_2$:

$$U_{\lambda\mu} = S_{\lambda,2I-\mu} = (-1)^{2I-\lambda-\mu} \begin{bmatrix} 2I-\mu \\ \lambda \end{bmatrix} \frac{[2\lambda+1]![2I-\lambda]!}{[\mu]![2I+\lambda-\mu+1]!} ,$$

giving

$$F^{(s)}_{\lambda\mu} = \frac{[\mu]![2\lambda+1]![2I-\lambda]!}{[\lambda]![2\mu]![2I-\mu]!} \sum_{\nu=0}^{\mu} \frac{(-1)^{2I-\lambda+\nu}[\mu+\nu]![2I-\nu]!^2}{[\nu]!^2[\mu-\nu]![2I-\lambda-\nu]![2I+\lambda-\nu+1]!} . \quad (2.27)$$

We note that, even if we use expressions (2.21) and (2.23) beyond the range of their validity (i.e., for $\mu > m_{kI}$ when $4I > k$) where some of the entries of the transition matrix $S$ and $S^{-1}$ are ill-defined at the value (2.5) of $q$, the $F$ matrix (2.27) is finite in the physical range $0 \leqq \lambda, \mu \leqq m_{kI}$. Moreover, the restricted $(m_{kI}+1)\times(m_{kI}+1)$ matrices $B_1^{(s)}$, $F^{(s)}$, and

$$B_2^{(s)} = F^{(s)} B_1^{(s)} F^{(s)} \quad (F^{(s)2} = \mathbb{1}) \quad (2.28)$$

still satisfy the relations (2.6)–(2.10). This is a non-trivial statement for $4I > k$.

The braid invariance of the two-dimensional Green's function (2.26) implies the relation

$$F^{(s)}_{\lambda\mu} D^{(k,I)}_{\lambda} = D^{(k,I)}_{\mu} F^{(s)}_{\mu\lambda} \quad (2.29)$$

with the positive eigenvalues $D$ of the form $M$ given by (2.20). Hence on the one hand, the $s$-channel $F$ matrix is symmetrizable, and on the other hand, the ratios of amplitudes for the diagonal extension are given by

$$\frac{N_\lambda^2}{N_\mu^2} \equiv \frac{D_\lambda}{D_\mu} = \frac{F^{(s)}_{\mu\lambda}}{F^{(s)}_{\lambda\mu}} . \quad (2.30)$$

## 3. Ratios of Structure Constants for the $E_6$ and the $E_8$ Models

The braid-invariant 4-point functions (2.16), (2.26) give the monodromy free Green's functions for the 2D local extensions of the chiral $su(2)$ current algebras $\mathscr{A}_k$ corresponding to the $A_{k+1}$ series in the $ADE$ classification.

There exists an infinite set of chiral extensions of the $su(2)$ current algebras for level $k$ a multiple of 4, corresponding to the $D_{2n}$ series ($2n = k/2+2$). In these models, the chiral algebras are extended by an $\mathscr{A}_k$-primary simple current: a Bose field of isospin and conformal dimension

$$I = \frac{k}{2} \quad \text{and} \quad \Delta_I = \frac{I(I+1)}{k+2} = \frac{k}{4} \in \mathbb{N} . \quad (3.1)$$

The inclusion of the (nets of) algebras $\mathscr{A}_k$ in the resulting field algebras are well understood: it is of the DHR type [5, 19] with a global $\mathbb{Z}_2$ gauge group which singles out the "observables" $\mathscr{A}_k$ as the gauge invariant elements of the ambient theory [20] (for a recent review and further references see [21]).

Here we shall deal with the more interesting exceptional extensions corresponding to conformal embeddings [12]. These are not of the DHR type, i.e., the $\mathscr{A}_k$ subalgebras are not the gauge invariants with respect to some global gauge group.

*3A. Pairs of Braid Invariant Quadratic Forms for Exceptional Embeddings.* There are just two non-trivial chiral extensions of $\mathscr{A}_k(A_1)$ corresponding to the conformal embeddings

$$\mathscr{A}_{10} = \mathscr{A}_{10}(A_1) \subset \mathscr{A}_1(B_2) = \mathscr{B}_{10} \quad (E_6)\,,$$

$$\mathscr{A}_{28} = \mathscr{A}_{28}(A_1) \subset \mathscr{A}_1(G_2) = \mathscr{B}_{28} \quad (E_8)\,,$$

where the labels $E_6$ and $E_8$ refer to the $E$ series of the $ADE$ classification [3]. The superselection structure of the observables in the "diagonal" representation space of the respective field extensions is encoded in the exceptional partition functions

$$Z(E_6) = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2\,, \tag{3.2a}$$

$$Z(E_8) = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2\,, \tag{3.2b}$$

where the subscripts on the modular characters $\chi$ stand for the dimensions, $2I + 1$, of the $SU(2)$ representations labelling the superselection sectors of $\mathscr{A}_k$. Every term in these sums corresponds to a superselection sector of the extended chiral current algebra $\mathscr{B}$, and every sum of modular characters appearing in each term determines the branching of the corresponding sector upon restriction to $\mathscr{A}_k$. In particular, the first term added to the vacuum character $\chi_1$ in (3.2) corresponds to $\Delta_I = 1$ sector of $\mathscr{A}_k$ generated by the $\mathscr{B}_k$ currents orthogonal to the $\mathscr{A}_k$ currents. These are the (7 component) $I = 3$ primary fields for the $\mathscr{A}_{10}$ theory in the $E_6$ case, and the (11 component) $I = 5$ primary fields for the $\mathscr{A}_{28}$ theory in the $E_8$ case.

The fact that an $\mathscr{A}_k$-primary field $\phi_I$ (with integer dimension $\Delta_I$) is a local Bose field in the extended $\mathscr{B}_k$ theory means that, in particular, there exists a braid-invariant linear combination of 4-point blocks of the associated chiral vertex operators. Namely, the commutation of two fields $\phi_I$ corresponds to a monodromy operation on the conformal block functions. In other words, the representation $\mathfrak{B}_4^{(k,I)}$ must be reducible and have an invariant subspace of joint eigenvectors of $B_i$ with eigenvalue 1.

In the $s$-channel basis of Eq. (2.26), these eigenfunctions are combinations of the form

$$E_6\ (k = 10): \quad s_0^{(3)} + \widetilde{D}_{03} s_3^{(3)}\,, \tag{3.3a}$$

$$E_8\ (k = 28): \quad s_0^{(5)} + \widetilde{D}_{05} s_5^{(5)} + \widetilde{D}_{09} s_9^{(5)}\,, \tag{3.3b}$$

where $\widetilde{D}_{\lambda\mu} \equiv \widetilde{D}_{\lambda\mu}^{(k,I)}$ depend on the model, and $\widetilde{D}_{00} = 1$ is chosen as a normalization. Two-dimensional correlation functions then result as products of two chiral functions (3.3), one for either chiral light-cone. They are thus bilinear in $(\bar{s}_\lambda, s_\mu)$ corresponding to a non-diagonal version of (2.26) with $D$ replaced by $\widetilde{D}$, where

$$\widetilde{D}_{\lambda\mu} = \widetilde{D}_{0\lambda}\widetilde{D}_{0\mu} = \widetilde{D}_{\mu\lambda}\,. \tag{3.4}$$

The expressions (3.3) are $\mathfrak{B}_4^{(k,I)}$-invariant. $B_1$-invariance is automatic since all $s$-channel functions $s_\lambda^{(I)}$ contributing to (3.3) correspond to the same $B_1$ eigenvalue $1\ (=-q^{k+2})$, and it excludes by the same argument all other $s$-channel contributions with $\lambda$ different from 0 or 3 ($E_6$) resp. 0, 5, 9 or 14 ($E_8$). The non-zero elements of $\widetilde{D}$ are determined from $F^{(s)}$ invariance: ${}^tF^{(s)}\widetilde{D} = \widetilde{D}F^{(s)}$. It is sufficient to use the equation

$$({}^tF\widetilde{D})_{0\mu} = (\widetilde{D}F)_{0\mu} = 0 \quad \text{for } \mu = 1,2\,. \tag{3.5}$$

This gives for the isospin $I = 3$ current in the $k = 10$ model:

$$\widetilde{D}_{03} = -\frac{F_{01}^{(s)}}{F_{31}^{(s)}} = -\frac{1}{[5]} = -\frac{1}{2+\sqrt{3}} \quad (k = 10, I = 3)\,, \tag{3.6}$$

and for the isospin $I = 5$ current in the $k = 28$ model:

$$\widetilde{D}_{05} = \frac{F_{02}^{(s)}F_{91}^{(s)} - F_{01}^{(s)}F_{92}^{(s)}}{F_{51}^{(s)}F_{92}^{(s)} - F_{52}^{(s)}F_{91}^{(s)}}\,, \qquad \widetilde{D}_{09} = \frac{F_{01}^{(s)}F_{52}^{(s)} - F_{02}^{(s)}F_{51}^{(s)}}{F_{51}^{(s)}F_{92}^{(s)} - F_{52}^{(s)}F_{91}^{(s)}} \quad (k = 28, I = 5)\,, \tag{3.7}$$

which can be computed from (2.27).

We note that by a change of scale for the $s$-channel basis functions, $D_{\lambda\mu}$ and $\widetilde{D}_{\lambda\mu}$ change by the same factor, hence their ratios are invariant under rescaling. It is remarkable that these invariant ratios turn out to be rational numbers:

$$\frac{\widetilde{D}_{33}}{D_{33}} = 2 \quad (k = 10, I = 3)\,, \tag{3.8}$$

$$\frac{\widetilde{D}_{55}}{D_{55}} = \frac{9}{4}\,, \qquad \frac{\widetilde{D}_{99}}{D_{99}} = \frac{5}{4} \quad (k = 28, I = 5)\,. \tag{3.9}$$

*Remark.* In a unitary basis in which $D_{\lambda\mu} = \delta_{\lambda\mu}$, the matrix $F^{(s)}$ will become symmetric (and unitary) due to (2.29). This unitarized $\widehat{F}$ can be obtained from our $F$ setting

$$\widehat{F}_{\lambda\mu} = (\operatorname{sign} F_{\lambda\mu})\sqrt{F_{\lambda\mu}F_{\mu\lambda}}\,. \tag{3.10}$$

In such a unitary basis, the above ratios will simply coincide with $\widetilde{D}_{\lambda\lambda}$.

*3B. The Braid Group Representation in the Ramond Sector.* The extended model $\mathscr{B}_{10} = \mathscr{A}_1(B_2)$ (see Sect. 3A.) is parallel in many respects to the Ising model and the $su(2)$ level 2 current algebra theory. All three models have three superselection sectors with identical fusion rules, and involve a simple current of dimension $\Delta = \frac{1}{2}$. For $\mathscr{B}_{10}$, this field is the $SO(5)$ vector field $\psi$ which is also an irreducible $\mathscr{A}_{10}$ primary field of isospin 2.

The state space of the fermionic field $\psi$ splits into two irreducible representations with respect to the extended "super current algebra" generated by $\psi(z)$: the Neveu–Schwarz sector $\mathscr{H}_1 \oplus \mathscr{H}_5$, and the Ramond sector $\mathscr{H}_2$, where $\mathscr{H}_d$ denote the level 1 $spin(5)$ current algebra representations labelled by the dimension $d$ of their lowest energy subspace. The correlation functions of $\psi$ are single-valued in the Neveu–Schwarz sector, and double valued in the Ramond sector.

Furthermore, in all three models, the primary dimension in the Ramond sector is related to the Virasoro central charge

$$\Delta = \frac{1}{8}c\,, \tag{3.11}$$

$c$ being given as $\frac{1}{2}$ times the number of components of $\psi$ ($c = \frac{5}{2}$ for $\mathscr{B}_{10}$).

We proceed to compute the $4 \times 4$ braid matrices in the $s$-channel basis of all $\mathscr{A}_{10}$ conformal blocks of four fields of isospin $I = \frac{3}{2}$ and dimension $\Delta = \frac{5}{16}$ which belong to the Ramond sector of $\mathscr{B}_{10}$ (see Eq. (3.2a)). Then we determine the subrepresentation acting in the subspace of conformal blocks of the extended theory $\mathscr{B}_{10}$ which constitute the 2D local Ramond 4-point functions.

Applying (2.24) and (2.27) for $I = \frac{3}{2}$, we obtain

$$B_1^{(s)} = q^{\frac{9}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q^2 & 0 & 0 \\ 0 & 0 & q^6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(k = 10,\, q = e^{\frac{i\pi}{12}},\, I = \frac{3}{2}\right)\,, \tag{3.12}$$

and

$$F^{(s)} = \begin{pmatrix} \frac{1-[3]}{3[2]} & \frac{[3]-1}{3} & -\frac{[3]}{3[2]} & \frac{1}{3} \\ \frac{4-[3]}{3} & -\frac{1}{[2]} & 0 & \frac{2}{3[2]} \\ -\frac{[3]}{2[2]} & 0 & \frac{[3]}{2[2]} & \frac{[3]-1}{6} \\ 1 & \frac{[3]}{[2]} & \frac{[3]-1}{3} & \frac{4-[3]}{6[2]} \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{3}}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{3} & \frac{1}{3} \\ \frac{\sqrt{3}-1}{\sqrt{3}} & \frac{1-\sqrt{3}}{\sqrt{2}} & 0 & \sqrt{2}\frac{\sqrt{3}-1}{3} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{3}} \\ 1 & \sqrt{2} & \frac{1}{\sqrt{3}} & \frac{2-\sqrt{3}}{\sqrt{6}} \end{pmatrix}. \tag{3.13}$$

The first matrix displayed here was computed with $q$-number identities valid for every Galois transform of $q$. Evaluating $[3] = \sqrt{2}[2] = 1 + \sqrt{3}$ at $q = e^{\frac{i\pi}{12}}$, one obtains the second matrix (3.13).

We are now looking for an $E_6$-type braid invariant $s$-channel quadratic form $\widetilde{D} \equiv \widetilde{D}^{(10,2)}$,

$$\widetilde{D} = \begin{pmatrix} 1 & 0 & 0 & \widetilde{N}_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \widetilde{N}_2^2 & 0 \\ \widetilde{N}_3 & 0 & 0 & \widetilde{N}_3^2 \end{pmatrix} \quad \text{where } \widetilde{N}_\lambda = \frac{\widetilde{N}_{\frac{3}{2}\frac{3}{2}\lambda}}{\widetilde{N}_{\frac{3}{2}\frac{3}{2}0}}\,. \tag{3.14}$$

The equality of the first and the last eigenvalue of $B_1^{(s)}$ (Eq. (3.12)) ensures $B_1$-invariance of $\widetilde{D}$. The real parameters $\widetilde{N}_\lambda$ can be determined from $F$-invariance ${}^tF\widetilde{M} = \widetilde{M}F$ of the quadratic form $\widetilde{M} = {}^tS\widetilde{D}S$, which implies

$$F_{01}^{(s)} + F_{31}^{(s)}\widetilde{N}_3 = 0\,, \qquad F_{20}^{(s)}\widetilde{N}_2^2 = F_{02}^{(s)} + \widetilde{N}_3 F_{32}^{(s)}\,.$$

This yields $\widetilde{N}_3 = -1/\sqrt{6}$ and $\widetilde{N}_2^2 = 1$ for $I = \frac{3}{2}$. We obtain the invariant ratios with the structure constants $D_\lambda \equiv N_\lambda^2$ of the diagonal theory given by (2.20) or by (2.30):

$$\frac{\widetilde{N}_3^2}{N_3^2} = \frac{F_{10}^{(s)}F_{01}^{(s)}}{F_{13}^{(s)}F_{31}^{(s)}} = \frac{1}{2}\,, \qquad \frac{\widetilde{N}_2^2}{N_2^2} = 1 - \frac{F_{01}^{(s)}F_{32}^{(s)}}{F_{02}^{(s)}F_{31}^{(s)}} = \frac{3}{2} \quad \left(k = 10,\, I = \frac{3}{2}\right)\,. \tag{3.15}$$

The same result is obtained for the invariant ratio of structure constants for the isospin $I = \frac{7}{2}$ field, as expected since the latter is the "partner" of the isospin $I = \frac{3}{2}$ field in the partition function (3.2a), related by the simple current of isospin 5. Indeed, according to (2.3),

$$\Delta\left(\frac{7}{2}\right) - \Delta\left(\frac{3}{2}\right) = \frac{21}{16} - \frac{5}{16} = 1 ,$$

and hence the matrices $B_1^{(s)}$ (projected into the physical subspace of $s$-channel blocks $s_\lambda$, $0 \leqq \lambda \leqq m_{kI}$) coincide for $I = \frac{3}{2}$ and $\frac{7}{2}$. It is instructive to verify that, although the $s$-channel $F$-matrices do not coincide for $I = \frac{7}{2}$ and $\frac{3}{2}$, the invariant ratios (3.15) are the same.

In computing $F^{(s)} = US^{-1}$ for $I = \frac{7}{2}$ in terms of the $s$- and $u$-channel transition matrices $S$ and $U$ (see Sect. 2), one encounters the problem of the reduction from the 8-dimensional space of KZ solutions to the 4-dimensional physical subspace. It is simplified by the observation that due to the triangular form of $S$ and $U$, the reduced matrix $F^{(s)}$ for $4I > k$ is obtained by just taking the first $m_{kI} + 1 = k - 2I + 1$ rows and columns of both $U$ and $S^{-1}$. In particular, for $I = \frac{7}{2}$ we observe that the symmetrized (unitary) matrices (3.10) corresponding to $I = \frac{3}{2}$ (Eq. (3.13)) and to $I = \frac{7}{2}$ coincide.

The 2-dimensional braid invariant subspace comprising the conformal blocks of local Ramond fields of the $\mathscr{B}_{10}$ model is spanned by the pair of vectors

$$v_0 = {}^t\left(-\frac{2}{3}, 0, 0, \sqrt{\frac{2}{3}}\right) , \qquad v_2 = {}^t(0, 0, 1, 0) , \tag{3.16}$$

which are ortho-normalized with respect to the metric (3.14):

$${}^t v_a \widetilde{D} v_b = \delta_{ab} \quad (a, b = 0, 2) . \tag{3.17}$$

In this basis, we have the following reduced form of the $s$-channel generators:

$$\hat{B}_1^{(s)} = q^{\frac{3}{2}} \begin{pmatrix} q^3 & 0 \\ 0 & -q^{-3} \end{pmatrix} , \quad \hat{F}^{(s)} = -\frac{[2]}{[3]} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \left(k = 10, I = \frac{3}{2}\right) . \tag{3.18}$$

(At $q = e^{\frac{i\pi}{12}}$, one has $[3] = \sqrt{2}[2]$). Identical expressions are obtained for the reduced generators acting in the invariant subspace of conformal blocks for $I = \frac{7}{2}$.

The resulting 2-dimensional representation of $\mathfrak{B}_4$ is a finite matrix group. It is a central extension of the 24-element 2-fold covering of the tetrahedron group. This is worth noticing, since the appearance of finite matrix groups among the monodromy representations of $\mathfrak{B}_4$ is rather exceptional [22].

## 4. Subfactors for Field Extensions

We turn now to the treatment of the same problem: the determination of relative amplitudes like (3.8), (3.9), in the algebraic (DHR) framework of quantum field theory. A theory $\mathscr{A}$ is described by a local net of von Neumann algebras, i.e., the association $\mathscr{O} \mapsto \mathscr{A}(\mathscr{O})$ of observables with the space-time region $\mathscr{O}$ in which they are localized. The algebras $\mathscr{A}(\mathscr{O})$ generate the global $C^*$ algebra $\mathscr{A}$. The regions may be double cones ($\mathcal{O}$) in two and more dimensions, or intervals ($\mathcal{I}$) on the light-cone in chiral conformal theories.

In the following, we consider a pair of local quantum field theories given by the nets of local von Neumann algebras $\mathcal{A}(\mathcal{O})$ and $\mathcal{B}(\mathcal{O})$ such that for every region,

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O}) \tag{4.1}$$

are irreducible inclusions with common unit. Our terminology will be "observables" for $a \in \mathcal{A}$ and "charged fields" for $b \in \mathcal{B}$, following the notions of the case when $\mathcal{A}$ are the gauge invariants within $\mathcal{B}$ and taking into account the fact that fields from $\mathcal{B}$ will interpolate inequivalent representations (superselection sectors) of $\mathcal{A}$. Nevertheless, the ambient net $\mathcal{B}$ being itself local, it may also be regarded as a net of observables on its own right (a local extension of $\mathcal{A}$).

Although we are going to develop a general theory of such extensions, we have in mind as illustrations of our method two specific such nets, namely

1. the $E_6$ extension studied in Sects. 2 and 3, i.e., the conformal inclusion [14, 23] of the chiral $su(2)$ current algebra at level 10 into the chiral $sp(4)$ current algebra at level 1, denoted by

$$\mathcal{A}_{\mathrm{ch}}(\mathcal{I}) \subset \mathcal{B}_{\mathrm{ch}}(\mathcal{I}) , \tag{4.2}$$

where $\mathcal{I}$ are intervals on the circle (= compactified conformal light-cone). and

2. the tensor product of two chiral $su(2)$ current algebras at level 10 (on both light-cones) contained in the algebra of the two-dimensional WZNW model [13]:

$$\mathcal{A}^{(2)}(\mathcal{O}) \equiv \mathcal{A}_{\mathrm{ch}}(\mathcal{I}) \otimes \mathcal{A}_{\mathrm{ch}}(\bar{\mathcal{I}}) \subset \mathcal{B}^{(2)}(\mathcal{O}) , \tag{4.3}$$

where a two-dimensional double cone $\mathcal{O} = \mathcal{I} \times \bar{\mathcal{I}}$ is the Cartesian product of two chiral light-cone intervals.

The local von Neumann algebras in (4.2) can be defined according to [10] as $\pi_0^{(k)}(L_{\mathcal{I}}G)''$, where $LG$ is the loop group over the respective compact Lie group $G$ and $L_{\mathcal{I}}G$ the subgroup of loops with support in the interval $\mathcal{I}$. $\pi_0^{(k)}$ is the vacuum representation at level $k$ in the Segal construction, and the double primes stand for the von Neumann closure. These nets of algebras satisfy the axioms of the DHR theory. The current $j^a(x)$ are operator-valued distributions affiliated with the local von Neumann algebras.

We shall obtain the local von Neumann algebras $\mathcal{B}^{(2)}(\mathcal{O})$ in (4.3) by adjoining to their chiral subalgebras $\mathcal{A}^{(2)}(\mathcal{O})$ a single "characteristic" isometry. The latter is determined implicitly by Proposition 5.2 below along with the general theory [9] of finite-index extensions of local nets, see Proposition 4.3. It is, however, beyond the scope of this paper to show that the local Wightman fields of the WZNW model are affiliated with these algebras.

Let us anticipate here that the model (4.3) is the one described by the standard diagonal form $D$ in the previous sections, while the form $\widetilde{D}$ corresponds to a combination of (4.2) and (4.3):

$$\mathcal{A}^{(2)}(\mathcal{O}) \equiv A_{\mathrm{ch}}(\mathcal{I}) \otimes \mathcal{A}_{\mathrm{ch}}(\bar{\mathcal{I}}) \subset \mathcal{B}_{\mathrm{ch}}(\mathcal{I}) \otimes \mathcal{B}_{\mathrm{ch}}(\bar{\mathcal{I}}) \subset \widetilde{\mathcal{B}}^{(2)}(\mathcal{O}) .$$

Here, the first inclusion is the tensor product of the chiral extensions (4.2) and the second inclusion is the standard diagonal contraction of chiral vertex operators for

$\mathscr{B}_{\mathrm{ch}}$. (We shall say more about these "standard" constructions in Sect. 5; see also [24, 9].)

A subfactor $A \subset B$ is irreducible if the relative commutant is trivial: $A' \cap B = \mathbb{C}$. This requirement excludes from our analysis all chiral current subalgebras associated with subgroups unless the embedding is "conformal" [12], since the coset stress-energy tensor is contained in the relative commutant. However, including the coset stress-energy tensor into the observables (which then have the structure of a tensor product of two chiral theories), would again yield irreducible inclusions [10, 11].

We have to recall some subfactor theory. First, we note that we are dealing with type $\mathrm{III}_1$ subfactors, since under very general conditions, the local von Neumann algebras in quantum field theory are hyperfinite type $\mathrm{III}_1$ factors [25, 26]. Associated with an (irreducible) type III subfactor $A \subset B$ is a *canonical endomorphism* $\gamma \in \mathrm{End}(B)$ such that $\gamma(B) \subset A$ is a dual subfactor [8, 27]. $A \subset B$ has finite index if and only if [8] there is a pair of isometries $W \in A$ and $V \in B$ such that the following operator identities hold:

$$\text{(a) } Wa = \varrho(a)W \quad (a \in A,\ \varrho := \gamma|_A \in \mathrm{End}(A)),$$

$$\text{(b) } Vb = \gamma(b)V \quad (b \in B),$$

$$\text{(c) } W^*V = \lambda^{-1/2}\mathbb{1} = W^*\gamma(V). \tag{4.4}$$

The real number $\lambda$ is called the *index* of the subfactor $A \subset B$. These relations express the duality between $A \subset B$ and $\gamma(B) \subset A$. They also state that $B$ is the Jones extension [1] of $A$ by its subfactor $\gamma(B)$. The Jones projection is $E = VV^*$, satisfying the Jones–Temperley–Lieb relation with its dual $F = WW^*$:

$$EFE = \lambda^{-1}E, \qquad FEF = \lambda^{-1}F.$$

Associated with these data, there is a *conditional expectation* $\mu: B \to A$ given by

$$\mu(b) = W^*\gamma(b)W \quad (b \in B), \tag{4.5}$$

and conversely the canonical endomorphism can be expressed in the form

$$\gamma(b) = \lambda \cdot \mu(VbV^*) \quad (b \in B). \tag{4.6}$$

$\mu$ is a positive and $A$-linear map which generalizes the Haar average over a compact group acting on $B$ with fixpoints $A$. It satisfies the Pimsner–Popa bound

$$\mu(b) \geqq \lambda^{-1} \cdot b \quad (b \in B,\ b \geqq 0) \tag{4.7}$$

as an operator estimate for every positive operator $b \in B$. This lower bound for conditional expectations was first introduced in [28] to define the index. It is optimal since it is saturated by

$$\mu(VV^*) = \lambda^{-1}\mathbb{1}.$$

We note also that $W = \lambda^{-1/2} \cdot \mu(V)$. The physical relevance of these objects will become clear in due context.

The following results on quantum field theoretical nets of subfactors as in Eq. (4.1) will be proven (and qualified) elsewhere [9]. Let us just state the essentials. Let the vacuum vector $\Omega$ be cyclic and separating for every local von Neumann algebra $\mathscr{B}(\mathscr{O})$ of the theory $\mathscr{B}$, i.e. $\pi^0(\mathscr{B}(\mathscr{O}))\Omega$ are dense subspaces of the vacuum representation space $\mathscr{H}^0$. This property holds, by the Reeh–Schlieder

Theorem, quite generally for covariant quantum field theories with positive energy. Let also $\mathscr{H}_0 = \overline{\pi^0(\mathscr{A})\Omega} \subset \mathscr{H}^0$ be the vacuum representation space of $\mathscr{A}$ such that $\Omega$ is also cyclic and separating in $\mathscr{H}_0$ for every $\mathscr{A}(\mathscr{O})$. Let furthermore the conditional expectation $\mu$ preserve localizations, i.e., map $\mathscr{B}(\mathscr{O})$ onto $\mathscr{A}(\mathscr{O})$. If the local subfactors are irreducible and therefore possess a unique conditional expectation, then $\mu$ must commute with the translations (= the rotations of the circle in the case of a chiral conformal theory). If the vacuum state $\omega = \langle \Omega, \pi^0(\,\cdot\,)\Omega \rangle$ on $\mathscr{B}$ is the unique translation invariant state, then it must also be invariant under $\mu$, i.e.,

$$\omega \circ \mu = \omega \quad \text{on } \mathscr{B}\,. \tag{4.8}$$

We shall assume the invariance property (4.8) in the sequel. The underlying structure admits the interpretation as a generalized global unbroken gauge symmetry with $\mu$ generalizing the gauge group average [9, 11].

Under these circumstances, the canonical endomorphism $\gamma$ defined above for a fixed local subfactor $\mathscr{A}(\mathscr{O}_0) \subset \mathscr{B}(\mathscr{O}_0)$ extends to an endomorphism of the global $C^*$ algebra $\mathscr{B}$, and maps $\mathscr{B}$ into the global $C^*$ algebra of observables $\mathscr{A}$ [9]. Restricted to the observables, $\gamma|_A$ turns out to be a localized endomorphism with localization in $\mathscr{O}_0$, denoted by $\varrho$ in the sequel. It therefore describes a (reducible) superselection sector [5] of the theory $\mathscr{A}$. Its physical significance is given by the following

**Proposition 4.1** [10, 9]. *Let $\pi_0$ denote the vacuum representation of $\mathscr{A}$ on $\mathscr{H}_0$, and $\pi^0$ the vacuum representation of $\mathscr{B}$ on $\mathscr{H}^0$. Then $\pi^0$ considered as a reducible representation of the subalgebra $\mathscr{A}$ is unitarily equivalent to the representation $\pi_0 \circ \varrho$ of $\mathscr{A}$, where $\varrho$ is the restriction to $\mathscr{A}$ of the canonical endomorphism $\gamma\colon \mathscr{B} \to \mathscr{A}$.*

In other words: the reducible superselection sector $\varrho$ comprises all the charged sectors of $\mathscr{A}$ which are interpolated from the vacuum by fields in $\mathscr{B}$. If, as endomorphisms, $\varrho \simeq \bigoplus_s N_s \varrho_s$, then as representations,

$$\pi^0|_{\mathscr{A}} \simeq \pi_0 \circ \varrho \simeq \bigoplus_s N_s \pi_s\,, \tag{4.9}$$

where $N_s$ are finite multiplicities, and $\pi_s \equiv \pi_0 \circ \varrho_s$. If the observables $\mathscr{A}$ are the gauge invariants under a compact gauge symmetry group of $\mathscr{B}$, then the decomposition (4.9) is given by the representation of the gauge group, with multiplicities $N_s$ given by the dimensions of the latter. In the case of current algebras, the branching rules (4.9) are read of the Kac characters.

Equation (4.9) allows to compute the index $\lambda$ of the subfactor. It is given by the formula

$$\lambda = d(\varrho) = \sum_s N_s d(\varrho_s) \tag{4.10}$$

in terms of the statistical dimensions $d(\varrho_s) \equiv d_s$ of the superselection sectors [5, 8] contained in $\varrho$. In the gauge group case, $d(\varrho_s) = N_s$, and the index equals the order of the group.

In the models (4.2), (4.3), the branching of the vacuum sector of $\mathscr{B}$ is well known, leading to $\varrho \simeq \varrho_0 \oplus \varrho_3$ for the inclusion (4.2) and $\varrho \simeq \bigoplus_I \varrho_I \otimes \varrho_I$ for the inclusion (4.3), where $\varrho_I$ are the isopin $I$ sectors of the chiral $su(2)$ current algebra. $\varrho_0 \equiv id$ corresponds to the vacuum representation. In the former case, the formula (4.10) yields the index $\lambda = d_0 + d_3 = 1 + \sin\frac{7\pi}{12}/\sin\frac{\pi}{12} = 3 + \sqrt{3}$. (For the coincidence of statistical dimensions and "quantum dimensions" $d(\varrho_I) = [2I+1]$ for $su(2)$ current algebras see [10].)

The formulae (4.4)–(4.7) remain valid for $\gamma$ considered as an endomorphism of $\mathscr{B}$ and for $\varrho$ as an endomorphism of $\mathscr{A}$. Note that the isometries $W$ and $V$ are local operators $W \in \mathscr{A}(\mathscr{O}_0)$ and $V \in \mathscr{B}(\mathscr{O}_0)$. We shall refer to the intertwining properties expressed by Eqs. (4.4(a, b)) by the notation $V: id \to \gamma$ and $W: id \to \varrho$ in the sequel. The latter implies that $\pi_0(WW^*)$ is the projection in the representation space of $\pi_0 \circ \varrho$ which corresponds to the vacuum subrepresentation contained in (4.9).

For every other subsector $\pi_s$ contained in (4.9) there are corresponding projections of the form $\pi_0(W_{s,i}W_{s,i}^*)$, where $W_{s,i}: \varrho_s \to \varrho$ are orthonormal isometric intertwiners in $\mathscr{A}(\mathscr{O}_0)$; the multiplicity index $i$ runs from 1 to $N_s$. For simplicity, we shall in the following consider only multiplicities $N_s = 1$ (covering abelian gauge groups, as well as our models above). One has the orthogonality relation $W_s^* W_t = \delta_{st}$ (because otherwise, the intertwiner $W_s^* W_t: \varrho_t \to \varrho_s$ would contradict the inequivalence of the representations $\pi_s$ and $\pi_t$), and the completeness relation $\sum_s W_s W_s^* = \mathbb{1}$. Clearly, $W_0 \equiv W$.

Putting

$$\psi_s := W_s^* V$$

we obtain charged intertwiners, i.e., elements of $\mathscr{B}$ which satisfy the commutation relations with the observables

$$\psi_s a = \varrho_s(a)\psi_s \quad (a \in \mathscr{A}) . \tag{4.11}$$

This equation means that $\psi_s \in \mathscr{B}$ make transitions (in the vacuum representation of $\mathscr{B}$) between the vacuum representation $\pi_0$ of $\mathscr{A}$ and the charged representations $\pi_s$.

Conversely

$$V = \sum_s W_s \psi_s , \tag{4.12}$$

and the commutation relation

$$Va = \varrho(a)V \quad (a \in \mathscr{A}) \tag{4.13}$$

gives to $V$ the physical interpretation as a "master field" carrying the reducible charge $\varrho$ from which the charged intertwiners $\psi_s$ are projected out by means of $W_s$.

A particularly interesting object is the observable operator

$$X := \gamma(V) \in \mathscr{A}(\mathscr{O}_0) . \tag{4.14}$$

From the definitions it is clear that $X$ is an isometric intertwiner $X: \varrho \to \varrho^2$. Indeed, we can compute

$$\begin{aligned} X = \gamma(V) = \lambda\mu(VVV^*) &= \lambda \sum_{stu} \mu(W_t\psi_t W_s\psi_s\psi_u^* W_u^*) \\ &= \lambda \sum_{stu} \varrho(W_s)W_t \cdot \mu(\psi_t\psi_s\psi_u^*) \cdot W_u^* , \end{aligned}$$

where the expressions $\mu(\psi_t\psi_s\psi_u^*)$ are observable intertwiners $T: \varrho_u \to \varrho_t\varrho_s$. They are therefore multiples of isometric basis intertwiners $T_e$ which project onto the subrepresentations $\pi_u$ contained in the DHR composition product $\pi_t \times \pi_s = \pi_0 \circ (\varrho_t\varrho_s)$;

$$\lambda\mu(\psi_t\psi_s\psi_u^*) = \lambda(e) \cdot T_e \tag{4.15}$$

with numerical coefficients

$$\lambda(e)\mathbb{1} = \lambda \cdot T_e^* \mu(\psi_t\psi_s\psi_u^*) . \tag{4.16}$$

(The multi-index $e$ stands here and in the sequel for the fusion channel $\pi_u \prec \pi_t \times \pi_s$.)

Denoting by $\tilde{T}_e = \varrho(W_s)W_t \cdot T_e \cdot W_u^*$ the "lifts" of intertwiners $T_e : \varrho_u \to \varrho_t\varrho_s$ to intertwiners $\tilde{T}_e : \varrho \to \varrho^2$, we obtain the expansion

$$X = \sum_e \lambda(e)\tilde{T}_e \,. \tag{4.17}$$

We note that only channels $e$ contribute to (4.17) for which $\varrho_s, \varrho_t, \varrho_u$ are all subsectors of canonical endomorphism $\varrho$, in spite of the fact that in general $\varrho_t\varrho_s$ will also contain subsectors which are not contained in $\varrho$. We shall relate this observation to the "truncated fusion rules" in the next section.

The importance of the isometry $X$ is due to the following result, while the relevance of its expansion coefficients $\lambda(e)$ will reveal itself in the sequel.

**Proposition 4.2** [29]. *The irreducible subfactor $A \subset B$ is uniquely characterized (up to unitary equivalence) by the triple $(\varrho, W, X)$, where $\varrho \in \mathrm{End}(A)$ and $W : id \to \varrho$ and $X : \varrho \to \varrho^2$ are isometric intertwiners in $A$, satisfying the following identities:*

$$\begin{aligned} &(\mathrm{i})\ W^*X = \lambda^{-1/2}\mathbb{1} = \varrho(W^*)X \quad \textit{with } \lambda = d(\varrho)\,, \\ &(\mathrm{ii})\ XX^* = \varrho(X^*)X\,, \\ &(\mathrm{iii})\ XX = \varrho(X)X\,. \end{aligned} \tag{4.18}$$

Clearly, the identities (4.18) follow from (4.4). Conversely, given a triple as in Proposition 4.2, one recovers $B$ as follows. Put $A_1 := X^*\varrho(A)X$ and $B :=$ the Jones extension of $A$ by $A_1$. This extension is of the form $B = AV$, where $V$ is an isometry with $VV^* = E$, the Jones projection. Define $\gamma \in \mathrm{End}(B)$ by $\gamma(aV) := \varrho(a)X$. Then $\gamma$, satisfying (4.4), is the canonical endomorphism for $A \subset B$ and $\varrho = \gamma|_A$, $A_1 = \gamma(B)$. The conditional expectation is $\mu = W^*\gamma(\,\cdot\,)W$.

In our present context, $A = \mathcal{A}(\mathcal{O})$ and $B = \mathcal{B}(\mathcal{O})$, the point about this characterization of extensions (4.1) is that it entirely refers to the observables $\mathcal{A}$ and their superselection sectors. Finding such a triple in a given theory $\mathcal{A}$ amounts to find a field extension $\mathcal{B}$ of the observables of the form (4.1). The problem involves the knowledge of the "fusion coefficients" of the theory $\mathcal{A}$, i.e., the coefficients of expressions like $\varrho_v(T_e)$ (entering $\varrho(X)$) in terms of a basis $T_gT_hT_f^*$. These are the solutions to the Moore–Seiberg "pentagon identities" [30] which are intrinsically determined by the DHR theory of superselection sectors [7] (but often tedious to compute).

Let us briefly sketch the "reverse program" of construction and classification of (local) field extensions of finite index [9].

The main step is to decide which combinations $\varrho \simeq \bigoplus_s N_s\varrho_s$ of the irreducible localized endomorphisms (sectors) of $\mathcal{A}$ are *canonical* endomorphisms of the local von Neumann algebra $A \equiv \mathcal{A}(\mathcal{O}_0)$ with respect to some subfactor $A_1 \subset A$. This amounts [29] to verify the existence of a pair of isometric intertwiners $W : id \to \varrho$ and $X : \varrho \to \varrho^2$ in $\mathcal{A}(\mathcal{O}_0)$ solving (4.18). If the desired inclusion is required to be irreducible, then $id \prec \varrho$ with multiplicity $N_0 = 1$, and if the index is finite, then one can prove the bound $N_s \leqq d_s$. Therefore, if $\mathcal{A}$ is a "rational" theory, i.e., it has only finitely many sectors of finite statistics, then the classification problem is a finite problem in the form of a non-linear system of the unknown coefficients $\lambda(e)_{ij}^k$ (with multiplicities).

If we are interested in *local* field extensions, then we have to require in addition (see below) that the solution $X$ satisfies

$$\varepsilon_\varrho X = X \,, \tag{4.19}$$

where $\varepsilon_\varrho \in \varrho^2(\mathscr{A})' \cap \mathscr{A}(\mathscr{O}_0)$ is the statistics operator for the localized endomorphism $\varrho$ [5]. $\varepsilon_\varrho = U^*\varrho(U)$ can be computed in terms of a charge transporting unitary interwiner $U: \varrho \to \hat{\varrho}$, where $\hat{\varrho}$ is an equivalent endomorphism localized at space-like distance from $\varrho$.

Every solution $(\varrho, W, X)$ to the system (4.18) defines a field net $\mathscr{B}$ extending $\mathscr{A}$ with finite index $\lambda = d(\varrho)$ as follows. If $\varrho$ is localized in $\mathscr{O}_0$, one reconstructs $B = \mathscr{B}(\mathscr{O}_0)$ and $\gamma \in \mathrm{End}(B)$ from $A_1 \subset A = \mathscr{A}(\mathscr{O}_0)$ as in the paragraph after Proposition 4.2. Thus $\mathscr{B}(\mathscr{O}_0) = \mathscr{A}(\mathscr{O}_0)V$ for an isometry $V \in \mathscr{B}(\mathscr{O}_0)$ satisfying (4.4). Next, one considers the invariant state $\omega_0\mu = \omega(W^*\gamma(\,\cdot\,)W)$ on $B$. The GNS representation $\pi^0$ it induces extends to $\mathscr{A}$ and satisfies (4.9). In $\pi^0$, one defines $\mathscr{B}(\mathscr{O}) := \mathscr{A}(\mathscr{O})UV$ with the help of charge transporters $U \in \mathscr{A}$, i.e., unitary intertwiners $U: \varrho \to \hat{\varrho}$, where $\hat{\varrho}$ is localized in $\mathscr{O}$. Note that $\mathscr{B}(\mathscr{O})$ thus defined contains the identity operator $\mathbb{1} \propto W^*V = \hat{W}^*UV$, since $\hat{W} = UW : id \to \hat{\varrho}$ is in $\mathscr{A}(\mathscr{O})$. Consequently, $\mathscr{B}(\mathscr{O})$ contain and extend $\mathscr{A}(\mathscr{O})$. This construction yields a net $\mathscr{B}$ which is relatively local with respect to $\mathscr{A}$, since $\varrho$ is localized; namely if $\mathscr{O}$ is at space-like distance from $\mathscr{O}_0$, then $\mathscr{A}(\mathscr{O}_0)$ commutes with $\mathscr{B}(\mathscr{O})$:

$$UV \cdot a = U\varrho(a)V = \hat{\varrho}(a)UV = a \cdot UV \quad (a \in \mathscr{A}(\mathscr{O}_0))\,.$$

The field extension $\mathscr{B}$ turns out to be local if and only if the solution $X$ satisfies also (4.19). Namely, the commutativity of $V \in \mathscr{B}(\mathscr{O}_0)$ with $UV \in \mathscr{B}(\mathscr{O})$ at space-like distance is equivalent to $VV = U^*VUV$, and hence to

$$XV = \gamma(V)V = VV = U^*VUV = U^*\varrho(U)VV = \varepsilon_\varrho\gamma(V)V = \varepsilon_\varrho XV\,.$$

Let us summarize the previous discussion:

**Proposition 4.3** [9]. *Let $\mathscr{A}$ be a Haag–Kastler net of observables and $\varrho$ a DHR endomorphism of $\mathscr{A}$ which contains $id \prec \varrho$ precisely once. Let $W: id \to \varrho$ and $X: \varrho \to \varrho^2$ be a pair of isometries in $\mathscr{A}$ satisfying the identities* (4.18). *Then the triple $(\varrho, W, X)$ defines a unique irreducible extension of $\mathscr{A}$ into a field net $\mathscr{B}$ such that $\varrho$ is the restriction of the canonical endomorphism $\gamma: \mathscr{B} \to \mathscr{A}$, and $W$ and $X = \gamma(V)$ the associated pair of isometries as in* (4.4). *The field net $\mathscr{B}$ is relatively local w.r.t. $\mathscr{A}$, and it is itself local if and only if, in addition, $X$ satisfies* (4.19). *The index of the extension equals the statistical dimension $d(\varrho)$.*

Further details of the proof of Proposition 4.3 beyond the above sketch are found in [9].

We observe that the system (4.18) alone will have many solutions. e.g., those of the form $\varrho = \bar{\sigma}\sigma$, $X = \sigma(\bar{W})$, where $\sigma$ is any irreducible localized endomorphism of the theory $\mathscr{A}$ with finite statistics, $\bar{W} : id \to \sigma\bar{\sigma}$ an isometry. These solutions will, however, violate the condition (4.19), and will therefore not give rise to *local* field extensions.

Note that, actually, locality of the field net was not required for the general analysis in the first part of this section, as long as it has the Reeh–Schlieder property, and fields commute with observables at space-like distance. However, since it is not clear which physical principles should determine a "good choice" of a non-local and therefore *a priori* unobservable field algebra except that it generates the superselection sectors of the observables, we prefer to consider only local field extensions which offer the option to be regarded as observable theories of their own right.

If $\mathscr{A}$ are the gauge invariants under a gauge group acting on $\mathscr{B}$, then the system (4.18) has a solution with multiplicities $N_s$ given by the dimensions of the representations of the gauge group. The corresponding coefficients $\lambda(e)^k_{ij}$ in the expansion (4.17) of $X$ are precisely the group-theoretical Clebsch–Gordan coefficients. Indeed, one may rephrase the content of the Doplicher–Roberts (DR) reconstruction theorem [19] as follows: every system of sectors of the observables which have finite permutation group statistics among each other, closed under composition, reduction, and conjugation, admits a solution to (4.18) with $X$ given by (4.17) in terms of Clebsch–Gordan coefficients of some compact gauge group. The DR solution is distinguished by the validity of (4.19) if there are only bosonic sectors of $\mathscr{A}$, and a graded variant of (4.19) in the presence of fermionic sectors.

We emphasize that, while our general theory above comprises the case of a compact gauge symmetry group, the models (4.2), (4.3) we are actually interested in are not given by a gauge symmetry group. The sectors $\pi_s$ contained in the restriction $\pi^0|_{\mathscr{A}}$ are not closed under composition, and their multiplicities differ from their statistical dimensions. Although the fields are local, the sectors $\pi_s$ have braid group statistics. None of these features could hold with a gauge group.

Displayed in terms of the coefficients $\lambda(e)^k_{ij}$, the system (4.18) is converted into a system of identities well-known to hold for Clebsch–Gordan coefficients (with the $6j$ symbols as fusion coefficients). The absence of a completeness property in (4.18) is related to the truncated fusion rules discussed in the next section.

## 5. Truncated Fusion Rules and Partial Wave Decomposition

Let us now study multiplicative properties of the charged fields $\psi_s$ ("operator product expansions"). For a generic charged operator $b \in \mathscr{B}$ one has the expansion formula (generalizing the harmonic analysis in the gauge symmetry case) implied by (4.4), (4.5),

$$b = \lambda\mu(bV^*)V = \lambda \sum_s \mu(b\psi_s^*)\psi_s \quad (b \in \mathscr{B}) . \tag{5.1}$$

In particular, by (4.15),

$$\psi_t\psi_s = \sum_u \lambda(e)\, T_e \psi_u , \tag{5.2}$$

where as before, $e$ is the channel $\varrho_u \prec \varrho_t\varrho_s$. We observe, that only charged fields with charge $\varrho_u \prec \varrho$ contribute to this operator product expansion, even if there are other sectors present in the DHR sector decomposition of $\varrho_t\varrho_s$. That this "truncation of the fusion rules" is consistent, can be retraced, e.g., to the identity (4.18(iii)) as follows.

Obviously, $\psi_t\psi_s$ is a charged intertwiner : $id \to \varrho_t\varrho_s$, so one might expect that all charges $\varrho_v$ contained in $\varrho_t\varrho_s$ are interpolated by this composite field. But, in order to project a field carrying charge $\varrho_v$ out of $\psi_t\psi_s$, we have to multiply the latter with $T_e^*$, where $T_e : \varrho_v \to \varrho_t\varrho_s$. Now, computing $T_e^*\psi_t\psi_s$, or rather its image under $\gamma$, we get

$$\gamma(T_e^*\psi_t\psi_s) = \gamma(T_e^* W_t^* V W_s^* V) = \gamma(T_e^* W_t^* \varrho(W_s^*) \cdot VV) = \varrho(T_e^* W_t^* \varrho(W_s^*)) \cdot XX .$$

Using $XX = \varrho(X)X$, we obtain an expression involving $\varrho[T_e^* W_t^* \varrho(W_s^*)X]$, where the argument in square brackets is an intertwiner : $\varrho \to \varrho_v$ in $\mathscr{A}$ which must vanish unless $\varrho_v \prec \varrho$. In other words, since the expansion (4.17) of $X$ contains only $T_e$ for fusion channels which are already contained in $\varrho$, it is annihilated by all $T_e$ leading to other channels. Therefore, the identity $T_e^* \psi_t \psi_s = 0$ following from identity (iii) precisely describes in the operator product expansion for charged fields the suppression of channels $\varrho_v$ not contained in $\varrho$, i.e., the truncated fusion rules.

We now turn to our main result, the decomposition of correlation functions of charged fields into "partial wave" contributions, and the decomposition of charged fields $\psi_s$ into "chiral exchange fields."

Applying the expansion (5.2) (and (4.11)) repeatedly, we find the following expansion for vacuum correlations of generic charged fields of the form $\varphi = \psi_s^* a$,

$$\langle \Omega, \varphi_n \cdots \varphi_1 \Omega \rangle = \sum_\xi \prod_i \overline{\lambda(e_i)} \cdot \langle \Omega, T_{e_n}^* \varrho_{t_n}(a_n) \cdots T_{e_2}^* \varrho_{t_2}(a_2) T_{e_1}^* a_1 \Omega \rangle \,, \tag{5.3}$$

where $T_{e_i} : \varrho_{u_i} \to \varrho_{t_i} \varrho_{s_i}$ and the sum extends over all vacuum-to-vacuum "channels" of successive fusion $\xi = e_n \circ \cdots \circ e_1$ such that $t_i = u_{i-1}$ and $u_n = 0 = t_1$. The last step in this computation, the evaluation of a single charged field of the form $\psi_s^* a$ in the vacuum state, exploits the invariance of the vacuum state

$$\omega(\psi_s^* a) = \omega(\mu(\psi_s)^* a) = \delta_{s0} \lambda^{-1/2} \omega(a) \,,$$

since $\mu(\psi_s) = W^* \gamma(W_s^* V) W = W_s^* W^* X W = \delta_{s0} \lambda^{-1/2} \mathbb{1}$. The factor $\lambda^{-1/2}$ is absorbed in the product in (5.3) in the guise of $\lambda(e_1)$ (note that for $\varrho_t = id$, $T_e = \mathbb{1}$, and $\tilde{T}_e = W W_s W_s^*$, one obtains $\lambda(e) = W_s^* W^* X W_s = \lambda^{-1/2}$).

We conclude

**Proposition 5.1.** *The (local) n-point functions of charged fields from a field extension $\mathscr{B}$ of $\mathscr{A}$ have the partial wave expansions $\sum_\xi N_\xi \mathscr{F}_\xi$ where the "partial wave" contributions*

$$\mathscr{F}_\xi = \langle \Omega, T_{e_n}^* \varrho_{t_n}(a_n) \cdots T_{e_2}^* \varrho_{t_2}(a_2) T_{e_1}^* a_1 \Omega \rangle \tag{5.4}$$

*are kinematically distinguished correlation functions which depend only on the subtheory $\mathscr{A}$ and its superselection structure, while only the coefficients*

$$N_\xi = \prod_i \overline{\lambda(e_i)} \,, \tag{5.5}$$

*involving the factors $\lambda(e_i)$ for every single transition in the channel of successive fusions, bear reference to the extension $\mathscr{B}$.*

The kinematical distinction of the partial waves is exhibited by their response to variations of the charged fields at intermediate positions; in the models of $su(2)$ chiral current algebras, these could be Möbius transformations and global $SU(2)$ transformations, which are sensitive to the spectra of $L_0$ and $Q^a = \int j^a(x)dx$ on the corresponding intermediate state vectors. Since these spectra are dictated by the fusion rules via the intertwiners $T_e$ in (5.4), it is clear that the partial waves at hand are precisely the bounded-operator versions of $s$-channel conformal block functions.

Indeed, the partial waves (5.4) are recognized by inspection as correlation functions of "reduced field bundle" operators $F(e,a)$, which are defined on the Hilbert space of the representation $\pi^0|_{\mathscr{A}} \equiv \bigoplus_s \pi_s$ (cf. Proposition 4.1) as a bounded operator version of chiral vertex operators [6, 7]: If $e$ is the channel $\varrho_u \prec \varrho_t \varrho_s$, then

$F(e,a) \equiv F(e,\mathbb{1})\pi^0(a)$ interpolates from the subspace for $\pi_t$ to the subspace for $\pi_u$ by the formula

$$F(e,a)|t;\Psi\rangle := |u;\pi_0(T_e^* \varrho_t(a))\Psi\rangle \, .$$

The algebra spanned by finite linear combinations of operators $F(e,a)$ is closed under multiplication (involving fusion coefficients for the superselection sectors) and under the adjoint operation. The operators satisfy "exchange algebra" commutation relations at spacelike separation (whence the name "exchange fields" [31, 24]) involving matrix elements of the relevant statistics operators (braid matrices). The corresponding representation of the braid group on the partial waves (5.4) coincides, for chiral current algebra models, with the representation acting on the "physical" solution space of the KZ equation, see, e.g., [6].

Due to the identification of the partial wave contributions (5.4) as vacuum expectation values of products of reduced field bundle operators $F(e,a)$, the expansion (5.3) implies the identification

$$\psi_s^* a = \sum_e \overline{\lambda(e)} F(e,a) \, , \tag{5.6}$$

where the sum extends over all fusion channels with fixed charge label $s$. This formula is remarkable since the charged fields in $\mathscr{B}$ which satisfy local commutation relations and truncated fusion rules as discussed above, arise as specific *linear* combinations of reduced field bundle operators which satisfy exchange algebra commutation relations and do not exhibit truncation. Similarly, while every single partial wave contribution (5.4) is non-local, the sum (5.3) is a manifestly local $n$-point function. This is possible due to cancellations among the relevant fusion coefficients, which can be seen independently to follow from the system (4.18), (4.19) if written as a nonlinear system involving fusion coefficients and braid matrices along with the Clebsch–Gordan coefficients $\lambda(e)$. A similar statement applies to the identities

$$\psi_s^* = d_s^{1/2} R_s^* \psi_{\bar{s}} \quad (R_s : id \to \bar{\varrho}_s \varrho_s \text{ isometric})$$

and

$$\psi_s^* \psi_s = d_s/\lambda \cdot \mathbb{1}$$

valid in $\mathscr{B}$, which we have not discussed here, but which can be proven within the reduced field bundle, with the identification (5.6), along the same lines. We refrain from working out the details here, which are not relevant for the following. Actually, the decomposition (5.6) can also be directly established in terms of the unitary equivalence between $\bigoplus \pi_s$ and $\pi^0|_{\mathscr{A}}$.

In a given model such as (4.2), the decomposition (5.6) has to be interpreted in the sense that bounded functions of smeared currents in the $su(3)$ directions orthogonal to the embedded $su(2)$ are linear combinations of reduced field bundle operators with coefficients $\overline{\lambda(e)}$ determined by the structure of the local subfactors.

In order to compare our present results with those of Sects. 2 and 3, we face the technical problem that the partial waves cannot be directly identified with conformal block functions. The latter may, however, be expected to be pointlike limits of the former when the charged fields are localized in arbitrarily small intervals by Möbius (scale) transformations [7]. This heuristic view is of course supported by the coincidence of the braid group representations upon which a rigorous analysis can be based (with methods as developed in [32]).

Instead, the complicated kinematics of such limits are bypassed if one considers only *relative* amplitudes $N_\xi/N_\eta$. Furthermore, since $s$-channel solutions to the KZ differential equation are determined only up to a normalization, it is also advisable to cancel these normalizations by considering *ratios* of relative amplitudes for *different extensions* of the same algebra $\mathscr{A}$.

We are thus on the safe side (by virtue of Proposition 5.1) if we compare only double ratios of the form

$$\frac{N'_\xi/N'_\eta}{N_\xi/N_\eta}, \tag{5.7}$$

which are completely normalization independent "characteristic" quantities, well-defined even without control of pointlike limits. Here, $N'$ and $N$ distinguish the amplitudes of a given partial wave contributing to correlators of two different local extensions.

Since the double ratios (5.7) are given by (5.5), we have established the desired relation between relative amplitudes of conformal blocks and the data of the relevant local subfactors. This relation is based on the identification of the expansion coefficients $\lambda(e)$ in (4.17) for the characteristic isometry and in (5.2) for operator products of charged fields (reflected also in (5.6) for charged fields as elements of the reduced field bundle).

Let us now compute the amplitudes (5.5) for our first model (4.2) from its characteristic triple $(\varrho, W, X)$. The branching of the vacuum representation of $\mathscr{B}$ upon restriction to $\mathscr{A}$ tells us that $\varrho \simeq \varrho_0 \oplus \varrho_3$ (see Sect. 4). By (4.10), the index is $\lambda = d(\varrho) = d_0 + d_3 = 3 + \sqrt{3}$. Actually, finite index type $\mathrm{III}_1$ subfactors are isomorphic to type $\mathrm{II}_1$ subfactors tensored with a type III factor [33]. The corresponding type $\mathrm{II}_1$ subfactor associated with the model (4.2) is the well known subfactor of index $\lambda = 3 + \sqrt{3}$ constructed in [34].

Choosing $\varrho_0 = id$ in its equivalence class, the isometry $W : id \to \varrho$ is uniquely determined up to an irrelevant phase. The coefficients $\lambda(e)$ for the isometry $X$ can be computed from $X^*X = \mathbb{1}$ and the identity (4.18(i)): there are only five fusion channels $\varrho_u \prec \varrho_t\varrho_s$ with all $\varrho_s, \varrho_t, \varrho_u \prec \varrho$, with which we associate isometric intertwiners as follows:

$$T_a : \varrho_0 \to \varrho_0\varrho_0, \quad T_b : \varrho_3 \to \varrho_3\varrho_0, \quad T_c : \varrho_3 \to \varrho_0\varrho_3, \quad T_d : \varrho_0 \to \varrho_3\varrho_3, \quad T_e : \varrho_3 \to \varrho_3\varrho_3 .$$

Since $\varrho_0 = id$, we may choose $T_a = T_b = T_c = \mathbb{1}$. According to standard notation [5, 6, 7], we call $R$ the isometry $T_d : id \to \varrho_3^2$. We have therefore:

$$X = \lambda(a) \cdot \varrho(W_0)W_0W_0^* + \lambda(b) \cdot \varrho(W_0)W_3W_3^* + \lambda(c) \cdot \varrho(W_3)W_0W_3^*$$
$$+ \lambda(d) \cdot \varrho(W_3)W_3RW_0^* + \lambda(e) \cdot \varrho(W_3)W_3T_eW_3^* ,$$

where $W_0 \equiv W : id \to \varrho$ and $W_3 : \varrho_3 \to \varrho$ are orthonormal isometries, and $E_0 = W_0W_0^*$ and $E_3 = W_3W_3^*$ are complementary projections in the commutant of $\varrho$ onto the two subsectors of $\varrho$. Then (4.18(i)) reads

$$W_0^*X = \lambda(a)E_0 + \lambda(c)E_3 = \lambda^{-1/2}\mathbb{1} ,$$

$$\varrho(W_0^*)X = \lambda(a)E_0 + \lambda(b)E_3 = \lambda^{-1/2}\mathbb{1} ,$$

hence $\lambda(a) = \lambda(b) = \lambda(c) = \lambda^{-1/2}$. We are free to choose the complex phases of $R$ and $T_e$ such that $\lambda(d)$ and $\lambda(e)$ are also positive. Now, the isometricity of $X$

together with the orthogonality of $R$ and $T_e$ (i.e., $R^* T_e = 0$) implies

$$X^* X = [\lambda(a)^2 + \lambda(d)^2] E_0 + [\lambda(b)^2 + \lambda(c)^2 + \lambda(e)^2] E_3 = \mathbb{1} .$$

hence $\lambda(d) = \sqrt{1 - \lambda^{-1}}$ and $\lambda(e) = \sqrt{1 - 2\lambda^{-1}}$. We don't need to verify the remaining identities (4.18), (4.19) since we know that the extension is local and yields a subfactor of index $\lambda = 3 + \sqrt{3}$. (Unfortunately, the computation is much less obvious for the other, $E_8$, extension treated in Sects. 2 and 3.)

For charged fields with charge $I = 3$, only the channels $c \equiv (30)$, $d \equiv (03)$, $e \equiv (33)$ are relevant (here, $(ji)$ labels an intertwiner $T_{(ji)} : \varrho_j \to \varrho_i \varrho_3$ resp. an exchange field of charge 3 acting on $\mathscr{H}_i$ with values in $\mathscr{H}_j$). We have

$$\lambda(30) = \lambda^{-1/2}, \qquad \lambda(03) = \left(\frac{\lambda - 1}{\lambda}\right)^{1/2}, \qquad \lambda(33) = \left(\frac{\lambda - 2}{\lambda}\right)^{1/2} .$$

This gives for the ratio of the amplitudes of the conformal blocks with intermediate $s$-channel $j = 0, 3$ contributing to the 4-point function of the isospin 3 field

$$N_3/N_0 = \frac{\lambda(03)\lambda(33)\lambda(33)\lambda(30)}{\lambda(03)\lambda(30)\lambda(03)\lambda(30)} = \frac{\lambda - 2}{\sqrt{\lambda - 1}} = \sqrt{2} . \tag{5.8}$$

As discussed before, due to uncontrolled normalizations, one has to compute double ratios like (5.7) of relative amplitudes comparing two different field extensions. Indeed, there is always a "standard" extension to compare with, which specializes for chiral current algebras to the $A$ series of modular invariants [3], and therefore yields the diagonal extensions as in our model (4.3).

**Proposition 5.2** [35, 9]. *For rational chiral theories $\mathscr{A}_{\text{ch}}$ (i.e., theories with only a finite number of superselection sectors $\pi_s$ with finite statistics), $\varrho \simeq \bigoplus_s \varrho_s \otimes \bar{\varrho}_s$ is a canonical endomorphism of $\mathscr{A}^{(2)} \equiv \mathscr{A}_{\text{ch}} \otimes \mathscr{A}_{\text{ch}}$ satisfying the conditions of Proposition* 4.3, *and therefore defines a local two-dimensional field extension $\mathscr{B}^{(2)}$ with $\mathscr{A}^{(2)}(\mathscr{O}) \equiv \mathscr{A}_{\text{ch}}(\mathscr{I}) \otimes \mathscr{A}_{\text{ch}}(\bar{\mathscr{I}}) \subset \mathscr{B}^{(2)}(\mathscr{O})$ for $\mathscr{O} = \mathscr{I} \times \bar{\mathscr{I}}$.*

This result is a corollary to the computation in [35] of the associated characteristic isometry $X^{(2)}$ satisfying the system of identities (4.18), (4.19). The vacuum representation of this extension contains all "diagonal" sectors of $\mathscr{A}^{(2)}$ of the form $\pi_s \otimes \pi_{\bar{s}}$ precisely once.

It is more convenient to deviate from the basis conventions in [35] and choose a $CPT$ conjugate pair of bases of isometric intertwiners $T_e$ and $T_{\bar{e}} = j(T_e)$ on the two chiral light-cones (cf. [9]). The anti-linear $CPT$ conjugation $j$ is an appropriate Tomita–Takesaki modular conjugation [26, 36]. It acts geometrically like a reflection $x \leftrightarrow -x$ on the algebras of chiral intervals, and relates conjugate sectors $\varrho \leftrightarrow \bar{\varrho} = j \circ \varrho \circ j$. In such a basis, the isometry $X^{(2)}$ is simply

$$X^{(2)} = \Lambda^{-1/2} \sum_e \sqrt{\frac{d_t d_s}{d_u}}\, \tilde{T}_e \otimes \tilde{T}_{\bar{e}} , \tag{5.9}$$

where $\tilde{T}_e$ are local intertwiners in $\mathscr{A}_{\text{ch}}$ as in (4.17) corresponding to the fusion channels $\varrho_u \prec \varrho_t \circ \varrho_s$ as before, $\tilde{T}_{\bar{e}} = j(\tilde{T}_e)$ correspond to the $CPT$ conjugate channel $\bar{\varrho}_u \prec \bar{\varrho}_t \circ \bar{\varrho}_s$, and $d_s$ are the statistical dimensions of $\varrho_s$. The index equals $\Lambda = \sum_s d_s^2$. The fusion channels contributing to the isometry $X^{(2)}$ for the two-dimensional subtheory (4.3) are of the form $e \otimes \bar{e}$, and the coefficients $\lambda^{(2)}(e \otimes \bar{e})$ are read off Eq. (5.9). The fact that the corresponding two-dimensional fields

$$\Phi_s = \sum_{e\otimes\bar{e}} \sqrt{\frac{d_t d_s}{d_u}}\, F(e \otimes \bar{e}, \mathbb{1} \otimes \mathbb{1}) \equiv \sum_e \sqrt{\frac{d_t d_s}{d_u}}\, F(e, \mathbb{1}) \otimes F(\bar{e}, \mathbb{1})$$

contracted from chiral exchange fields of fixed charge $[s]$, $[\bar{s}]$ are indeed local fields acting on the Hilbert space $\mathscr{H}^{(2)} = \bigoplus_t \mathscr{H}_t \otimes \mathscr{H}_{\bar{t}}$, was already established in [24]. Although the diagonal sectors are not closed under composition whenever there are non-simple fusion rules among the chiral sectors $\pi_s$, the operator product of the diagonal fields $\Phi_s$ contains only other diagonal fields due to cancellations among the fusion coefficients. This is another instance of truncated fusion rules.

From

$$\lambda^{(2)}(e \otimes \bar{e}) = \sqrt{\frac{d_s}{\Lambda}} \sqrt{\frac{d_t}{d_u}}$$

it is obvious that the amplitudes for the 2D partial waves contributing to a given $n$-point function of integer isospin fields $\langle \Omega, \Phi_n \cdots \Phi_1 \Omega \rangle = \sum_\xi N^{(2)}_{\xi\otimes\bar{\xi}} \mathscr{F}_\xi \cdot \bar{\mathscr{F}}_{\bar{\xi}}$ are all equal:

$$N^{(2)}_{\xi\otimes\bar{\xi}} = \prod_i \sqrt{d_{s_i}/\Lambda} = const. \tag{5.10}$$

Given the diagonal standard extension, we can predict characteristic invariants for every other extension which can be read off the respective $n$-point functions, independent of all normalizations of partial waves and conformal blocks, by taking double ratios of amplitudes (5.5) and (5.10),

$$\frac{(N_\xi/N_\eta)(N_{\bar{\xi}}/N_{\bar{\eta}})}{N^{(2)}_{\xi\otimes\bar{\xi}}/N^{(2)}_{\eta\otimes\bar{\eta}}} = \prod_i \frac{\lambda(e_i)\bar{\lambda}(\bar{e}_i)}{\lambda(f_i)\bar{\lambda}(\bar{f}_i)} = \prod_i \frac{|\lambda(e_i)|^2}{|\lambda(f_i)|^2}\,. \tag{5.11}$$

Here we have used the fact that the coefficients of $X$ and $j(X)$ in $CPT$ conjugate bases are complex conjugates, $\bar{\lambda}(\bar{e}) = \overline{\lambda(e)}$. E.g., for the 4-point function of the isospin 3 field in the $E_6$ model (4.2), we get

$$\frac{(N_3/N_0)^2}{N_3^{(2)}/N_0^{(2)}} = 2 \tag{5.12}$$

in agreement with the result obtained previously (Eq. (3.8) and [14]) by the analysis of locality in terms of explicit conformal block functions given as solutions to KZ differential equations.

We emphasize that this method works for every "non-diagonal" extension of a given chiral theory without controlling the actual pointlike limits of $F(e,a)$ (or even assuming its existence), since there is always the "diagonal" one which provides a normalization standard for all contributing partial waves. Moreover, the formula (5.11) immediately applies to mixed and higher $n$-point functions.

We conclude this section with another instructive (albeit almost trivial) example giving rise to anyonic field extensions. We consider a local theory $\mathscr{A}$ with $N$ simple superselection sectors $\varrho_s$ with $\mathbb{Z}_N$ fusion rules $[s][t] = [s + t\,(\text{mod}\,N)]$. For simplicity, assume that the automorphisms $\varrho_s$ can be chosen to satisfy $\varrho_s\varrho_t = \varrho_{s+t}$ (understood mod $N$), by which all intertwiners $T_e$ of the general analysis are trivial $= \mathbb{1}$. This choice is always possible for odd $N$, and for even $N$ provided the fractional spin of $\varrho_s$ satisfies $N\Delta_s \in \mathbb{Z}$ (cf. [24]). The sector structure is that of the simple sectors in $su(N)$ current algebras. It also occurs in the models constructed in [37], where, however, the violation of the spin condition leads to a minor complication which we want to ignore here. The case $N = 2$ includes the $D_n$ series of chiral $su(2)$ current algebra extensions.

We choose a complete system of orthonormal isometries $W_s$ and construct the reducible endomorphism $\varrho(a) := \sum_s W_s\varrho_s(a)W_s^*$. Then the triple $(\varrho, W, X)$, where $W = W_0$ and

$$X := N^{-1/2}\sum_{st} \varrho(W_s)W_t W_{s+t}^*$$

(with trivial Clebsch–Gordan coefficients for an abelian group) solves the system (4.18). The charged fields $\psi_s$ are obtained (up to a normalization factor $N^{1/2}$) as the unitary shift operators $|t;\Psi\rangle \mapsto |t+s;\Psi\rangle$ on $\bigoplus_t \mathscr{H}_t$. They satisfy $\psi_s\psi_t = \psi_{s+t}$ and implement the endomorphisms $\varrho_s$ (in the representation $\pi^0 = \bigoplus \pi_s$)

$$\varrho_s(a) = \psi_s a \psi_s^* \quad (a \in \mathscr{A})\,.$$

The gauge group $\mathbb{Z}_N$ acts by $\gamma_n(\psi_s) = e^{2\pi i n s/N}\,\psi_s$ with average $\mu(\psi_s) = \delta_{s0}\mathbb{1}$. Putting

$$V := N^{-1/2}\sum_s W_s\psi_s\,,$$

and defining $\gamma$ by (4.6) with index $\lambda = |\mathbb{Z}_N| = N$, then $\gamma(V) = X$ and the triple $(\gamma, V, W)$ satisfies the identities (4.4). Adjoining the charged fields $\psi_s$ to the local algebras, we obtain an anyonic field extension $\mathscr{B}$ by the simple sectors of $\mathscr{A}$.

## 6. Concluding Remarks

The old hope that the "germ of the observable algebra" generated by the internal symmetry currents and the stress-energy tensor completely determines a local quantum field theory turns out to require some qualifications. Two-dimensional conformal current algebra models tell us that depending on the value of the level $k$ (which characterizes both the algebra $\mathscr{A}_k$ and the vacuum state of the theory), there may be several – one, two, or three for $\mathscr{A}_k(su(2))$ – local conformal field theories corresponding to the same vacuum representation of $\mathscr{A}_k$.

The different theories are distinguished by different maximal local chiral extensions $\mathscr{B}_k$ and by different braid invariant quadratic forms $M$. The primary local chiral fields which extend $\mathscr{A}_k$ obey fusion rules which are majorized by the intrinsic DHR fusion rules of superselection sectors. Both the invariant ratios of structure constants which are characteristic quantities for local field extensions, and the truncated fusion rules are understood and computed in conventional field theoretical terms and in terms of the theory of subfactors applied to a single local subfactor $\mathscr{A}(\mathscr{I}) \subset \mathscr{B}(\mathscr{I})$.

Our field theoretical computation uses a closed expression for the $s$-channel fusion matrix (that is already implicit in [15]) which has the virtue of displaying their invariance under Galois automorphisms (the individual structure constants as well as the matrix elements of the monodromy representation of the mapping class group belonging to the same algebraic number field). The relevance of such arithmetic properties has been recently exhibited in a study of the Schwarz problem ("When is the representation of the braid group a finite matrix group?") for the KZ equation [22].

On the other hand, the application of the theory of finite index subfactors to local field extensions gives a natural interpretation of the field theoretical structures in terms of a generalized "harmonic analysis." The "irreducible tensor operators" of this analysis are the quantum field theoretical charged intertwiners. This approach is very close to the spirit of Ocneanu who first considered subfactors as "generalized groups," but gives more evidence to this view than the combinatorial description in terms of bi-partite graphs and connections [38]. Part of Ocneanu's induction-restriction graph is reflected in the "truncated fusion rules" which in turn derive from harmonic analysis in the form of operator product expansions for charged fields. Through Longo's theorem relating the truncation to the depth of the inclusion [29], it is nicely exhibited that the generalized symmetry associated with conformal embeddings is not given by a Hopf $C^*$ algebra in general. Longo's characterization of a subfactor in terms of a triple $(\varrho, W, X)$ gives rise to a notion of generalized Clebsch–Gordan coefficients which does not refer to any assumed linear transformation law of the irreducible tensor operators. We note that the interpretation of these structures as a generalized symmetry is not imposed but emerges naturally from the theory of subfactors.

When one compares our two different approaches, one can also observe some unbalance. E.g., the role of the Galois automorphisms is not yet understood in terms of the subfactor approach. In particular, the Galois group acting on the structure constants does not map a unitary theory into another unitary theory, nor are there any "Galois relatives" of a subfactor. Indeed, the characteristic ratios of structure constants like (3.8), (3.9), (3.15) resp. (5.12) turn out to be rational numbers and are, therefore, Galois invariants.

The characterization of a local extension in terms of a triple $(\varrho, W, X)$ as in Proposition 4.2 logically proceeds in two steps: first, one has to solve the system (4.18) which, among other things, controls the consistent truncated operator product expansions. This already yields field extensions which, however, may be non-local. E.g., a fermionic field theory as an extension of its even (bosonic) subtheory arises in this way. The locality condition (4.19) is only imposed in a second step. On the other hand, in the conformal block approach the locality condition seems to be the only vital step. In fact, we consider the analogue of the first step to be hidden in the KZ equation, whose solutions automatically give rise to a consistent fusion.

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## References


1. Jones, V.F.R.: Index for subfactors. Invent. Math. **72**, 1–25 (1983); Braid groups, Hecke algebras, and type $II_1$ factors. In: Geometric Methods in Operator Algebras Proc. of US-Japan Seminar, Kyoto 1983, Pitman Research Notes in Math. **123**, (1986), pp. 242–273
2. Pasquier, V.: Operator content of the ADE lattice models. J. Phys. A: Math. Gen. **20**, 5707–5717 (1987)
3. Cappelli, A., Itzykson, C., Zuber, J.-B.: The ADE classification of minimal and $A_1^{(1)}$ conformal invariant theories. Commun. Math. Phys. **113**, 1–26 (1987)
4. Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys. **5**, 848–861 (1964)
5. Doplicher, S., Haag, R., Roberts, J.E.: Local observables and particle statistics. I + II. Commun. Math. Phys. **23**, 199–230 (1971) and **35**, 49–85 (1974); Haag, R.: Local Quantum Physics. Texts and Monographs in Physics, Berlin-Heidelberg-New York: Springer, 1992
6. Fredenhagen, K., Rehren, K.-H., Schroer, B.: Superselection sectors with braid group statistics and exchange algebras. I. Commun. Math. Phys. **125**, 201–226 (1989)
7. Fredenhagen, K., Rehren, K.-H., Schroer, B.: Superselection sectors with braid group statistics and exchange algebras. II. Rev. Math. Phys. Special issue, 113–157 (1992)
8. Longo, R.: Index of subfactors and statistics of quantum fields. I + II. Commun. Math. Phys. **126**, 217–247 (1989) and Commun. Math. Phys. **130**, 285–309 (1990)
9. Longo, R., Rehren, K.-H.: Nets of subfactors. Rev. Math. Phys. **7**, 567–597 (1995) a related program was announced by A. Wassermann, cf. ref. [10] and [32]; for some of the general ideas see [11]
10. Wassermann, A.: Subfactors arising from positive energy representations of some infinite dimensional groups. Unpublished notes 1990; Operator algebras and conformal field theory. Contribution to the ICM conference Zürich 1994, Cambridge 1994
11. Rehren, K.-H.: Subfactors and coset models. In: Generalized Symmetries in Physics, V. Dobrev, et al. eds., Singapore: World Scientific, 1994, pp. 338–356
12. Schellekens, A., Warner, N.: Conformal subalgebras of Kac–Moody algebras. Phys. Rev. **D34**, 3092–3101 (1986); Bouwknegt, P., Nahm, W.: Realization of the exceptional modular invariant $A_1^{(1)}$ partition functions. Phys. Lett. **B184**, 359–362 (1987); Bais, F.A., Bouwknegt, P.G.: A classification of subgroup truncations of the bosonic string. Nucl. Phys. **B279**, 561–570 (1987)
13. Witten, E.: Non-Abelian bosonization in two dimensions. Commun. Math. Phys. **92**, 445–472 (1984)
14. Fuchs, J., Klemm, A.: The computation of the operator algebra in non-diagonal conformal field theories. Ann. Phys. (N.Y.) **194**, 303–335 (1989); Fuchs, J., Klemm, A., Scheich, C.: The operator algebra of the $E_8$ type $SU_2$ WZW theory. Zeitschr. f. Phys. **C46**, 71–74 (1990); Furlan, P., Ganchev, A.Ch., Petkova, V.B.: Fusion matrices and $c < 1$ (quasi) local conformal theories. Int. J. Mod. Phys. **A5**, 2721–2736 (1990)
15. Stanev, Ya.S., Todorov, I.T., Hadjiivanov, L.K.: Braid invariant rational conformal models with a quantum group symmetry. Phys. Lett. **B276**, 87–94 (1992)
16. Stanev, Ya.S., Todorov, I.T., Hadjiivanov, L.K.: Braid invariant chiral conformal models with a quantum group symmetry. In: Quantum Symmetries, H.-D., Doebner, V.K., Dobrev, eds., Singapore: World Scientific, 1993, pp. 24–40
17. Tsuchiya, A., Kanie, Y.: Vertex operators in conformal field theory on $\mathbb{P}^1$ and monodromy representations of braid group. In: Conformal Field Theory and Solvable Lattice Models, Adv. Stud. Pure Math. **16**, 297–372 (1988); and II: Lett. Math. Phys. **13**, 303–312 (1987)
18. Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess–Zumino model in two dimensions. Nucl. Phys. **B247**, 83–103 (1984)
19. Doplicher, S., Roberts, J.E.: Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. Commun. Math. Phys. **131**, 51–107 (1990)
20. Felder, G., Gawędzki, K., Kupiainen, A.: The spectrum of Wess–Zumino–Witten models. Nucl. Phys. **B299**, 355–366 (1988)
21. Todorov, I.T.: What are we learning from 2-dimensional conformal models. In: Mathematical Physics Towards the 21st Century, R.N. Sen, A. Gersten, eds., Beer Sheva, Israel: Ben Gurion University of the Negev Press, 1994, pp. 160–176

22. Stanev, Ya.S., Todorov, I.T.: On Schwarz problem for the $\widehat{su}_2$ Knizhnik–Zamolodchikov equation. Vienna preprint ESI 121 (1994), Lett. Math. Phys. (To appear)
23. Michel, L., Stanev, Ya.S., Todorov, I.T.: D-E classification of the local extensions of $su_2$ current algebras. Teor. Mat. Fiz. **92**, 507–521 (1992) (Theor. Math. Phys. **92**, 1063 (1993))
24. Rehren, K.-H.: Space-time fields and exchange fields. Commun. Math. Phys. **132**, 461–483 (1990)
25. Buchholz, D., D'Antoni, C., Fredenhagen, K.: The universal structure of local algebras. Commun. Math. Phys. **111**, 123–135 (1987)
26. Gabbiani, F., Fröhlich, J.: Operator algebras and conformal field theory. Commun. Math. Phys. **155**, 569–640 (1993)
27. Kosaki, H.: Extension of Jones' theory on index to arbitrary factors. J. Funct. Anal. **66**, 123–140 (1986)
28. Pimsner, M., Popa, S.: Entropy and index for subfactors. Ann. Sci. Éc. Norm. Sup. **19**, 57–106 (1986)
29. Longo, R.: A duality for Hopf algebras and for subfactors. I. Commun. Math. Phys. **159**, 133–150 (1994)
30. Moore, G., Seiberg, N.: Polynomial equations for rational conformal field theories. Phys. Lett. **212B**, 451–460 (1988)
31. Rehren, K.-H., Schroer, B.: Einstein causality and Artin braids. Nucl. Phys. **B312**, 715–750 (1989)
32. Loke, T.: Operator algebras and conformal field theory of the discrete series representations of Diff($S^1$). PhD thesis Trinity College, Cambridge, UK, 1994
33. Popa, S.: Classification of subfactors of finite depth of the hyperfinite type $\mathrm{III}_1$ factor. UCLA preprint (1994); KHR is indebted to Wassermann, A. who informed him of this result
34. Goodman, F., de la Harpe, P., Jones, V.: Coxeter graphs and towers of algebras. MSRI Publ. **14**, Berlin-Heidelberg-New York: Springer, 1989, Chap. 4.5
35. Rehren, K.-H.: Field operators for anyons and plektons. Commun. Math. Phys. **145**, 123–148 (1992); cf. in particular Proposition 2.6. and the 'Notes added in proof'
36. Brunetti, R., Guido, D., Longo, R.: Modular structure and duality in conformal quantum field theory. Commun. Math. Phys. **156**, 201–219 (1993); Wiesbrock, H.-W.: Conformal quantum field theory and half-sided modular inclusions of von Neumann algebras. Commun. Math. Phys. **158**, 537–544 (1993)
37. Buchholz, D., Mack, G., Todorov, I.: The current algebra on the circle as a germ for local field theories. Nucl. Phys. B (Proc.Suppl.) **5B**, 20–56 (1988)
38. Ocneanu, A.: Quantized groups, string algebras, and Galois theory for algebras. In: Operator Algebras and Applications. Vol. **2**, D.E. Evans, et al. (eds.) London Math. Soc. Lecture Notes Series **136**, Cambridge, 1988, pp. 119–172
39. Petkova, V.B., Zuber, J.-B.: On structure constants of sl(2) theories. Nucl. Phys. **B438**, 347–372 (1995)


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