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Combinatorial Quantization of the Hamiltonian Chern–Simons Theory II

Anton Yu. Alekseev[1,★], **Harald Grosse**[2,★★], **Volker Schomerus**[3,★★★]

[1] Institute of Theoretical Physics, Uppsala University, Box 803 S-75108, Uppsala, Sweden
[2] Institut für Theoretische Physik, Universität Wien, Austria
[3] Harvard University, Department of Physics, Cambridge, MA 02138, USA

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Abstract: This paper further develops the combinatorial approach to quantization of the Hamiltonian Chern Simons theory advertised in [1]. Using the theory of quantum Wilson lines, we show how the Verlinde algebra appears within the context of quantum group gauge theory. This allows to discuss flatness of quantum connections so that we can give a mathematically rigorous definition of the algebra of observables $\mathscr{A}_{CS}$ of the Chern Simons model. It is a $*$-algebra of "functions on the quantum moduli space of flat connections" and comes equipped with a positive functional ω ("integration"). We prove that this data does not depend on the particular choices which have been made in the construction. Following ideas of Fock and Rosly [2], the algebra $\mathscr{A}_{CS}$ provides a deformation quantization of the algebra of functions on the moduli space along the natural Poisson bracket induced by the Chern Simons action. We evaluate a volume of the quantized moduli space and prove that it coincides with the Verlinde number. This answer is also interpreted as a partition partition function of the lattice Yang–Mills theory corresponding to a quantum gauge group.

1. Introduction

This paper is a second part of the series devoted to combinatorial quantization of the Hamiltonian Chern Simons theory. Here we continue and essentially complete the analysis started in [1].

★ On leave of absence from Steklov Mathematical Institute, Fontanka 27, St. Petersburg, Russia; e-mail: alexeiev@rhea.teorfys.uu.se.
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★★ Part of project P8916-PHY of the 'Fonds zur Förderung der wissenschaftlichen Forschung in Österreich'; e-mail: grosse@pap.univie.ac.at.
★★★ Supported in part by DOE Grant No DE-FG02-88ER25065;e-mail: vschomer@husc.harvard.edu.
Present address: Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany.

To set the stage let us reproduce the well-recognizable landscape of 3D Chern Simons theory. The latter is a 3-dimensional topological theory defined by the action

$$CS(A) = \frac{k}{4\pi}\mathrm{Tr}\int_M \left(\mathrm{Ad}A + \frac{2}{3}A^3\right) . \tag{1.1}$$

Here M is a 3-dimensional manifold, A is a gauge field taking values in some semi-simple Lie algebra and k is a positive integer. In this setting the theory enjoys both gauge and reparametrization symmetry which makes it topological. Elementary observables satisfying the same symmetry conditions may be constructed for each closed contour Γ in M as

$$W_\Gamma = \mathrm{Tr}\,P\exp\left(\int_\Gamma A\right) . \tag{1.2}$$

Choosing the manifold M to be a product of a circle and a 2-dimensional orientable surface Σ, one gets a Hamiltonian formulation of the model. The direction along the circle plays the role of time. Actually, one can relax topological requirements and treat the problem locally. Then such a splitting into time and space directions is always possible. The problem of quantization in the Hamiltonian approach may be stated as follows. One should construct quantum analogoues $\hat{W}_\Gamma$ of the observables (1.2) corresponding to space-like contours. The main questions which arise in this way are the following. We should describe the algebra generated by $\hat{W}_\Gamma$ in terms of commutation or exchange relations. Next, if we are going to use this algebra as a quantum algebra of observables, a $*$-operation and a positive inner product are necessary. The final step is to construct $*$-representations of the algebra of observables. Linear spaces which carry such representations may be used as Hilbert spaces of the corresponding quantum systems.

The Hamiltonian formulation of the Chern Simons theory leads directly to the moduli space of flat connections on a Riemann surface. The latter appears as a phase space of the Chern Simons model. The action (1.1) introduces a natural symplectic form and a Poisson bracket on the moduli space. So, one can look for quantization of this moduli space in the framework of deformation quantization. This is actually a mathematical reformulation of the same problem as Hamiltonian quantization of the Chern Simons model.

In the spirit of deformation quantization one should start with the Poisson bracket on the moduli space. This object was considered for some time in mathematical literature and there are several descriptions of the corresponding Poisson structure. The one suitable for our purposes has been suggested recently by Fock and Rosly [2]. The main idea of this approach is to replace a 2-dimensional surface by a homotopically equivalent fat graph. This gives a finite-dimensional or combinatorial description of the moduli space. The name "combinatorial quantization" originates from this fact. Another important achievement of [2] is that the only object which is used in the description of the Poisson bracket is a classical r-matrix (solution of the classical Yang–Baxter equation). When the Poisson bracket is represented in terms of r-matrices, the quantization procedure is almost straightforward. Roughly speaking, one has to replace solutions of the classical Yang–Baxter equation by the corresponding solutions of the quantum Yang–Baxter equation. In conclusion, the way to deformation quantization of the moduli space was much clarified by [2].

In [1] we have started a description of the quantum algebra of observables. We have introduced such an algebra for any pair of a fat graph and a semi-simple ribbon quasi-Hopf algebra. There we followed the ideas of [2]. The novelties of our approach were introduction of the $*$-operation and of the quantum Haar measure on the algebra of observables. A Haar measure on this type of algebras has been previously considered in [3]. We succeeded to extend our consideration to quasi-Hopf algebras as well. This is motivated by the fact that the most interesting examples – like quantum groups at roots of unity – do not meet the condition of semi-simplicity. After a certain truncation, however, they become semi-simple (weak) quasi-Hopf-algebras [4, 5]. Thus, all the essential technical tools for quantization of the moduli space were introduced in [1]. On the other hand, an important piece of the quantization was still missing there: the quantum analogue of the flatness condition. The "algebra of observables" $\mathscr{A}$ eventually included some field configurations with nonzero curvature. In this paper we overcome this problem and complete the program of quantization.

The quantized algebra of functions on the moduli space (moduli algebra) is expected to provide the description of the algebra of observables in 3-dimensional Chern Simons theory. In principle, we can change the point of view at this point and treat the theory of graph connections with a quantum gauge group as a sort of 2-dimensional lattice gauge theory. As usual, one may be interested in correlation functions of Wilson line observables provided by the trace functional. This 2-dimensional interpretation has its own continuous counterpart. Assuming that in 3-dimensional formulation the moduli algebra reproduces the algebra of observables of the Chern Simons model exactly, one concludes that in the 2-dimensional formulation we obtain an exact lattice counterpart of gauged WZW model or so-called G/G model (for the relation of CS and G/G model, see e.g. [6]). From time to time it is useful to switch from 3-dimensional interpretation to 2-dimensional and back. So, we shall use the vocabulary of both these approaches.

Let us give a short description of the content of each section. Section 2 collects main theorems of [1]. This gives the possibility to understand the results of the paper without referring to [1]. However, we do not give any proofs here and refer the interested reader to the original text. Section 3 is devoted to Wilson line observables $\hat{W}_\Gamma$. In particular, we prove that for Γ being a contractible contour, $\hat{W}_\Gamma$ belongs to the center of the algebra of observables. We study in detail the commutative algebra generated by $\hat{W}_\Gamma$ for a given Γ. It is proved to coincide with the celebrated Verlinde algebra. On the basis of the Verlinde algebra we construct central projectors in the algebra $\mathscr{A}$ and define a quantum analogue of the flatness conditions on the graph. The algebra of observables $\mathscr{A}_{CS}$ with the condition of flatness imposed is our final answer for the quantized algebra of functions on the moduli space of flat connections. In Sect. 4 we prove the correctness of our definition. From the very beginning we replace the surface by a fat graph. This can be done in many ways. We prove that observable algebras $\mathscr{A}_{CS}$ which arise from different graphs are canonically isomorphic to each other. We pick up a particular graph which consists of a bunch of circles intersecting in only one point on the surface and describe the algebra thereon in Sect. 4.2. Then we revisit the "multidimensional Haar measure" in Sect. 5.1, and obtain a graph independent "quantum integration" for $\mathscr{A}_{CS}$. This is used in Sect. 5.2 to determine the volume of the quantum moduli space.

Let us mention here that there is an ambiguity in normalization of the "integration measure." In this paper we use some particular normalization which may be referred to as lattice Yang–Mills measure. The corresponding volume of the quantum moduli space resembles the answer for the partition function in the 2-dimensional Yang–Mills theory. Along with this normalization there exists a canonical one which is fixed by the requirement that the volume of each simple ideal in the moduli algebra should be equal to the square root of its dimension. The volume of the moduli space evaluated by means of the canonical measure reproduces the famous Verlinde formula for the number of conformal blocks in the WZW model [7]. In this way we get a consistency check of our approach. In the 2-dimensional interpretation the Verlinde formula gives the answer for the partition function of our lattice gauge model. It coincides with the experience of the continuous G/G model (see e.g. [8]). Advertising here this result, we postpone a more detailed discussion to the next paper.

For simplicity, we work with ribbon Hopf-algebras throughout most of the text. The generalization to ribbon quasi-Hopf algebras is explained in Sect. 6. Proofs for this section are partly given in a separate appendix of the paper. Let us mention that there is a formal overlap between some of the results in Sects. 3, 5 and the recent work of Buffenoir and Roche [15]. While their work is restricted to real deformation parameters q, we are interested in the "physical" case where the deformation parameter is a root of unity. This causes a number of problems connected e.g. with $*$-structures and semisimplicity. Once they are overcome, the theory at roots of unity has the advantage of involving only finite sums and allowing for the beforementioned physical application to Chern–Simons theories.

After this brief introduction we turn to a more systematic presentation of the main results. There are two basic ingredients used as the input for our construction. The first one is a semi-simple ribbon (weak quasi-) Hopf algebra $\mathscr{G}$. Equivalence classes of irreducible representations of $\mathscr{G}$ are labeled by $I, J, K, \ldots$. Furthermore one needs a compact orientable Riemann surface $\Sigma_{g,m}$ of genus g and with m punctures. These punctures are then marked so that the representation class I_ν is assigned to the ν^{th} point ($\nu = 1, \ldots, m$).

Our combinatorial approach requires to replace $\Sigma_{g,m}$ by a homotopically equivalent fat graph G and to equip G with some extra structure called "ciliation." The ciliated graph will be denoted G_{cil}. In [1] we assigned a $*$-algebra $\mathscr{B}(G_{\text{cil}})$ to this ciliated graph. It is generated by quantum lattice connections and quantum gauge transformations. A $*$-subalgebra $\mathscr{A}(G) \subset \mathscr{B}(G_{\text{cil}})$ generated by invariant quantum lattice connections has to be singled out. Our attempt to implement the flatness condition that will finally lead to Chern–Simons observables, is based on the following theorem.

Theorem I (Fusion algebra and characteristic projectors). *For every contractible plaquette P of the graph (or lattice) G, there is a set of central elements $c^I(P) \in \mathscr{A} \subset \mathscr{B}$ which satisfy the* fusion algebra (*or "Verlinde algebra"*), *i.e.*

$$c^I(P)c^J(P) = \sum_K N_K^{IJ} c^K(P)\,, \tag{1.3}$$

$$(c^I(P))^* = c^I(P)\,. \tag{1.4}$$

If the matrix $S_{IJ} = \mathscr{N}(\mathrm{tr}_q^I \otimes \mathrm{tr}_q^J)(R'R)$, $\mathscr{N}$ being equal to some nonzero real number, is invertible, a set of characteristic projectors *$\chi^I(P)$ can be constructed from the elements $c^I(P)$. They are central orthogonal projectors within $\mathscr{A}$, i.e.*

$$\chi^I(P)^* = \chi^I(P), \quad \chi^I(P)\chi^J(P) = \delta_{I,J}\chi^I(P)\,. \tag{1.5}$$

Explicit formulas for both $c^I(P)$ and $\chi^J(P)$ can be given (see Eqs. (3.19) *and* (3.22) *below).*

Projectors are the analogue of characteristic functions on a manifold in the non-commutative framework. We will show that the "support" of $\chi^0(P)$ consists of a manifold quantum connections, which have trivial monodromy around the plaquette P. So $\chi^0(P)$ plays the role of the δ-function at the group unit. A similar construction has been developed in [3]. This consideration motivates the following construction of an algebra $\mathscr{A}_{CS}^{\{I_\nu\}}$ of Chern–Simons observables,

$$\mathscr{A}_{CS}^{\{I_\nu\}} = \mathscr{A} \prod_{P \in \mathscr{P}_0} \chi^0(P) \prod_{\nu=1}^{m} \chi^{I_\nu}(P_\nu) . \tag{1.6}$$

Here P_ν, $\nu = 1, \ldots, m$, denotes the plaquette containing the ν^{th} puncture on $\Sigma_{g,m}$. It is marked by I_ν. $\mathscr{P}_0$ is the set of all plaquettes on the graph G, which do not contain a marked point.

Actually $\mathscr{A}_{CS}^{\{I_\nu\}}$ comes with some extra structure. First the $*$-operation on $\mathscr{A}$ restricts to $\mathscr{A}_{CS}^{\{I_\nu\}}$. Moreover, the generalized "multidimensional Haar measure" ω on $\mathscr{A}$ which was constructed in [1] furnishes a positive linear functional ω_{CS} on $\mathscr{A}_{CS}^{\{I_\nu\}}$. These data turn out to depend only on the input $(\Sigma_{g,m}, \mathscr{G})$.

Theorem II (Chern Simons observables). *The triple $(\mathscr{A}_{CS}^{\{I_\nu\}}, *, \omega_{CS})$ of an algebra $\mathscr{A}_{CS}^{\{I_\nu\}}$ with $*$-operation $*$ and a positive linear functional $\omega_{CS} : \mathscr{A}_{CS}^{\{I_\nu\}} \mapsto \mathbf{C}$ does not depend on the choice of the fat ciliated group G_{cil} which is used in the construction.*

Positive linear functional generalizes the concept of integration. Having constructed ω_{CS} will allow us to calculate the volume of the quantum moduli space. Actually, it coincides with the Verlinde number assigned to the same Riemann surface with marked points. This may be considered as a representative consistency check of the combinatorial approach.

2. Short Summary of [1]

Before we continue our study of Chern–Simons observables we want to review some notations and results from [1]. We will not attempt to make this section selfcontained but keep our emphasis on formulas and notations frequently used throughout the rest of this paper. Compared with [1], our notations will be slightly changed to adapt them to our new needs.

The theories to be considered here live on a graph (or lattice) G. The latter consists of sites $x, y, z \in S$ and oriented links $\pm i, \pm j, \pm k \in L$. We also introduce a map t from the set of oriented links L to the set of sites S such that $t(i) = x$, if i points towards the site x. Let us assume that two sites on the graph are connected by at most one link (we will come back to this assumption later).

Our models on the graph G will possess a quantum gauge symmetry, which is described by a family of ribbon Hopf-$*$-algebras assigned to the sites $x \in S$. They consist of a $*$-algebra $\mathscr{G}_x$ with co-unit ε_x, co-product Δ_x, antipode $\mathscr{S}_x$, R-matrix R_x and the ribbon element v_x. Let us stress that we deal with structures for which the co-product Δ_x is consistent with the action

$$(\xi \otimes \eta)^* = \eta^* \otimes \xi^* \quad \text{for all } \eta, \xi \in \mathscr{G}_x$$

of the $*$-operation $*$ on elements in the tensor product $\mathcal{G}_x \otimes \mathcal{G}_x$. This case is of particular interest, since it appears for the quantized universal enveloping algebras $U_q(\mathcal{G})$ when the complex parameter q has values on the unit circle [5].

Given the standard expansion of $R_x \in \mathcal{G}_x \otimes \mathcal{G}_x$, $R_x = \sum r^1_{x\sigma} \otimes r^2_{x\sigma}$, one constructs the elements

$$u_x = \sum \mathcal{S}_x(r^2_{x\sigma}) r^1_{x\sigma} \,. \tag{2.1}$$

Among the properties of u_x (cp. e.g. [9]) one finds that the product $u_x \mathcal{S}_x(u_x)$ is in the center of $\mathcal{G}_x$. The ribbon element v_x is a central square root of $u_x \mathcal{S}_x(u_x)$ which obeys the following relations:

$$v_x^2 = u_x \mathcal{S}_x(u_x), \quad \mathcal{S}_x(v_x) = v_x, \quad \varepsilon_x(v_x) = 1 \,, \tag{2.2}$$

$$v_x^* = v_x^{-1}, \quad \Delta_x(v_x) = (R'_x R_x)^{-1}(v_x \otimes v_x) \,. \tag{2.3}$$

The elements u_x and v_x can be combined to furnish a grouplike element $g_x = u_x^{-1} v_x \in \mathcal{G}_x$. It will play an important role throughout the text. So let us list some properties here:

$$g_x^{-1} = \mathcal{S}_x(g_x), \qquad g_x^* = g_x^{-1}, \qquad g_x \mathcal{S}_x(\xi) = \mathcal{S}_x^{-1}(\xi) g_x \,, \tag{2.4}$$

$$\Delta_x(g_x) = (g_x \otimes g_x) \,. \tag{2.5}$$

Examples of ribbon-Hopf-$*$-algebras are given by the quantized enveloping algebras of all simple Lie algebras [9].

The algebras $\mathcal{G}_x$ at different sites x are assumed to be *twist equivalent*, i.e. the Hopf-structure of every pair of symmetry algebras $\mathcal{G}_x, \mathcal{G}_y$ is related by a (unitary) twist in the sense of Drinfel'd [10]. We emphasize that – for the moment – we restrict ourselves to co-associative co-products Δ_x. As in [1], the discussion of the quasi-co-associative case is included at the end of the paper.

The total gauge symmetry is the ribbon Hopf-$*$-algebra $\mathcal{G} = \bigotimes \mathcal{G}_x$, with the induced co-unit ε, co-product Δ, etc. There is a canonic embedding of $\mathcal{G}_x$ into $\mathcal{G}$ and we will not distinguish in notations between the image of this embedding and the algebra $\mathcal{G}_x$, i.e. the symbol $\mathcal{G}_x$ will also denote a subalgebra of $\mathcal{G}$.

Representations of the algebra $\mathcal{G}$ of gauge transformations are obtained as families $(\tau_x)_{x\in S}$ of representations of the symmetries $\mathcal{G}_x$. At this point let us assume that *$\mathcal{G}_x$ are semisimple* and that every equivalence class $[J]$ of irreducible representations of $\mathcal{G}_x$ contains a unitary representative τ_x^J with carrier space V^J. For the moment, the most interesting examples of gauge symmetries, e.g. $U_q(\mathcal{G}), q^p = 1$, are ruled out by this assumption. It was explained in Sect. 7 of [1] how "truncation" can cure this problem once the theory has been extended to quasi-Hopf algebras.

The tensor product $\tau_x^I \boxtimes \tau_x^J$ of two representations τ_x^I, τ_x^J of the semisimple algebra $\mathcal{G}_x$ can be decomposed into irreducibles τ_x^K. This decomposition determines the Clebsch–Gordon maps $C_x^a[IJ|K] : V^I \otimes V^J \mapsto V^K$,

$$C_x^a[IJ|K](\tau_x^I \boxtimes \tau_x^J)(\xi) = \tau_x^K(\xi) C_x^a[IJ|K] \,. \tag{2.6}$$

The same representations τ_x^K in general appears with some multiplicity N_K^{IJ}. The superscript $a = 1, \dots, N_K^{IJ}$ keeps track of these subrepresentations. It is common to call the numbers N_K^{IJ} *fusion rules*. Normalization of these Clebsch Gordon maps is connected with an extra assumption. Notice that the ribbon element v_x is central so that the evaluation with irreducible representations τ_x^I gives complex numbers $v^I = \tau_x^I(v_x)$ (twist equivalence of the gauge symmetries implies that $\tau_x^I(v_x^I)$ does not depend on the site x). We suppose that there exists a set of square roots $\kappa_I, \kappa_I^2 = v^I$, such that

$$C_x^a[IJ|K](R_x')^{IJ}C_x^b[IJ|L]^* = \delta_{a,b}\delta_{K,L}\frac{\kappa_I\kappa_J}{\kappa_K} . \tag{2.7}$$

Here $R_x' = \sum r_{x\sigma}^2 \otimes r_{x\sigma}^1$ and $(R_x')^{IJ} = (\tau_x^I \otimes \tau_x^J)(R_x')$. Let us analyze this relation in more detail. As a consequence of intertwining properties of the Clebsch Gordon maps and the R-element, $\tau^K(\xi)$ commutes with the left-hand side of the equation. So by Schurs' lemma, it is equal to the identity e^K times some complex factor $\omega_{ab}(IJ|K)$. After appropriate normalization, $\omega_{ab}(IJ|K) = \delta_{a,b}\omega(IJ|K)$ with a complex phase $\omega(IJ|K)$. Next we exploit the $*$-operation and relation (2.3) to find $\omega_{ab}(IJ|K)^2 = v^Iv^J/v^K$. This means that (2.7) can be ensured up to a possible sign $\pm$. Here we assume that this sign is always $+$. This assumption was crucial for the positivity in [1]. It is met by the quantized universal enveloping algebras of all simple Lie algebras because they are obtained as a deformation of a Hopf-algebra which clearly satisfies (2.7).

We wish to combine the phases κ_I into one element κ_x in the center of $\mathscr{G}_x$, i.e. by definition, κ_x will denote a central element

$$\kappa_x \in \mathscr{G}_x \quad \text{with } \tau_x^J(\kappa_x) = \kappa_J . \tag{2.8}$$

Such an element does exist and is unique. It has the property $\kappa^* = \kappa^{-1}$.

The antipode $\mathscr{S}_x$ of $\mathscr{G}_x$ furnishes a conjugation in the set of equivalence classes of irreducible representations. We use $[\bar{J}]$ to denote the class conjugate to $[J]$. Some important properties of the fusion rules N_K^{IJ} can be formulated with the help of this conjugation. Among them are the relations

$$N_0^{K\bar{K}} = 1, \qquad N_K^{IJ} = N_K^{JI} = N_I^{J\bar{K}} . \tag{2.9}$$

The numbers v^I are symmetric under conjugation, i.e. $v^K = v^{\bar{K}}$. Let us also mention that the trace of the element $\mathscr{S}_x(u_x)v_x^{-1}$ in a given representation τ^I computes the "quantum dimension" d_J of the representation τ^I [9], i.e.

$$d_J \equiv \mathrm{tr}(\tau^I(g_x)) . \tag{2.10}$$

The numbers d_J satisfy the equalities $d_Id_J = \sum N_K^{IJ}d_K$ and $d_K = d_{\bar{K}}$.

We can use the Clebsch Gordon maps $C[K\bar{K}|0]$ to define a "deformed trace" tr_q^K. If $X \in \mathrm{End}(V^K)$, then

$$\mathrm{tr}_q^K(X) = \frac{d_K}{v^K}C_x[\bar{K}K|0]\overset{2}{X}(R_x')^{\bar{K}K}C_x[\bar{K}K|0]^* . \tag{2.11}$$

This definition simplifies with the help of the following lemma which will be applied frequently within the next section.

Lemma 1. *The Clebsch–Gordon maps $C[K\bar{K}|0]$ satisfy the following equations*:

1. *For all $\xi \in \mathscr{G}_x$ they obey the intertwining relations,*

$$C_x[K\bar{K}|0](\tau_x^K(\xi) \otimes \mathrm{id}) = C_x[K\bar{K}|0](\mathrm{id} \otimes \tau_x^K(\mathscr{S}_x(\xi))) ,$$

$$(\tau_x^K(\xi) \otimes \mathrm{id})C_x[K\bar{K}|0]^* = (\mathrm{id} \otimes \tau_x^K(\mathscr{S}_x(\xi)))C_x[K\bar{K}|0]^* . \tag{2.12}$$

2. *With normalization conventions* (2.7) *one finds*

$$d_K \mathrm{tr}^{\bar{K}}(C_x[K\bar{K}|0]^* C_x[K\bar{K}|0]) = e^K ,$$

$$d_K \mathrm{tr}^{\bar{K}}(C_x[\bar{K}K|0]^* C_x[\bar{K}K|0]) = e^K . \qquad (2.13)$$

Here action of the trace $\mathrm{tr}^{\bar{K}}$ *on the first resp. second component is understood and* $e^K = e^K_x$ *is the identity map on* V^K.

Proof. The first two relations are a consequence of the intertwining properties of $C[K\bar{K}|0]$ and the defining relations of an antipode $\mathcal{S}_x$. Using (2.12) one can check that the traces on the left-hand side of Eqs. (2.13) commute with $\tau^K(\xi)$ and hence are proportional to the identity e^K_x. To calculate the normalization, one multiplies with $\tau^K(g)$, evaluates the tr^K of the expression and uses the normalization (2.7) of the Clebsch Gordon maps.

As a consequence of this lemma we find the simple formula

$$\mathrm{tr}^K_q(X) = \mathrm{tr}^K(X\tau^K_x(g_x)) . \qquad (2.14)$$

In particular this implies that $d_K = \mathrm{tr}^K_q(e^K)$.

While the gauge transformations $\xi \in \mathcal{G}$ live in the sites of G, variables $U^I_{ab}(i)$ are assigned to the links of the graph G. They can be regarded as "functions" on the non-commutative space of lattice connections. Together with the quantum gauge transformations $\xi \in \mathcal{G}$ they generate the lattice algebra $\mathcal{B}$ defined in [1]. To write the relations in $\mathcal{B}$, one has to introduce some extra structure on the graph G. The orientation of the Riemann surface Σ determines a canonical cyclic order in the set $L_x = \{i \in L : t(i) = x\}$ of links incident to the vertex x. Writing the relations in $\mathcal{B}$ we were forced to specify a linear order within L_x. To this end one considers ciliated graphs G_{cil}. A ciliated graph can be represented by picturing the underlying graph together with a small cilium c_x at each vertex. For $i, j \in L_x$ we write $i \leqq j$, if (c_x, i, j) appear in a clockwise order.

In contrast to [1], we will write relations in $\mathcal{B}$ in a matrix notation. This means that the generators $U^I_{ab}(I) \in \mathcal{B}$ are combined into one single object

$$U^I(i) \in \mathrm{End}(V^I) \otimes \mathcal{B} .$$

Such algebra valued matrices are widely used in similar contexts and will have many advantages for the calculations to be done later. With this remark we are prepared to review the defining relations of $\mathcal{B}$. The rest of our notations will be explained as we proceed.

$\mathcal{B}$ is characterized by three different types of relations.

1. Covariance properties of the generators $U^I_{ab}(i)$ under gauge transformations are the only relations involving the generators $\xi \in \mathcal{G}$. If $x = t(i)$, $y = t(-i)$, they read

$$\xi U^I(i) = U^I(i)\mu^I_x(\xi) \quad \text{for all } \xi \in \mathcal{G}_x ,$$

$$\mu^I_y(\xi)U^I(i) = U^I(i)\xi \qquad \text{for all } \xi \in \mathcal{G}_y ,$$

$$\xi U^I(i) = U^I(i)\xi \qquad \text{for all } \xi \in \mathcal{G}_z,\ z \notin \{x, y\} . \qquad (2.15)$$

Here we used the symbol $\mu^I_x(\xi) \in \mathrm{End}(V^I) \otimes \mathcal{B}$ which is defined by

$$\mu^I_z(\xi) = (\tau^I_z \otimes \mathrm{id})\Delta_z(\xi) \quad \text{for all } \xi \in \mathcal{G}_z \,. \tag{2.16}$$

The covariance relations (2.15) make sense as relations in $\mathrm{End}(V^I) \otimes \mathcal{B}$, if ξ is regarded as an element $\xi \in \mathrm{End}(V^I) \otimes \mathcal{B}$ with trivial entry in the first component. We will not distinguish in notation between elements $\xi \in \mathcal{G}$ and their image in $\mathrm{End}(V^I) \otimes \mathcal{G}$.

2. *Functoriality* for elements $U^I(i)$ on a fixed link i confirms again that $U^I(i)$ is a covariant object with respect to two copies of the Hopf algebra acting from the left and from the right,

$$\overset{1}{U}{}^I(i)\overset{2}{U}{}^J(i) = \sum_{K,a} C^a_y[IJ|K]^* U^K(i) C^a_x[IJ|K] \,, \tag{2.17}$$

$$U^I(i)U^I(-i) = e^I_y, \qquad U^I(-i)U^I(i) = e^I_x \,. \tag{2.18}$$

The Clebsch Gordon maps $C^a_x[IJ|K]$, $C^a_y[IJ|K]^*$ have been introduced in the last section. One can view their appearance in the multiplication rule as a consequence of the generalized (for Hopf algebras) Wigner–Eckhart theorem. It is important that the left and right indices of $U^I(i)$ always belong to conjugate representations. In this respect the subalgebra formed by the matrix elements of $U^I(i)$ resembles an algebra of functions on the quantum group[1]. To explain the small numbers on top of the U, one has to expand $U^I(i) \in \mathrm{End}(V^I) \otimes \mathcal{B}$ according to $U^I(i) = \sum m^I_\sigma \otimes u^I_\sigma$. Then

$$\overset{1}{U}{}^I = \sum m^I_\sigma \otimes e^J \otimes u^I_\sigma$$

and similarly for $\overset{2}{U}{}^J(i)$. Here and in the following, e^J denotes the identity map on V^J.

3. *Braid relations* between elements $U^I(i)$, $U^J(j)$ assigned to different links have to respect the gauge symmetry and locality of the model. These principles require

$$\overset{1}{U}{}^I(i)\overset{2}{U}{}^J(j) = \overset{2}{U}{}^J(j)\overset{1}{U}{}^I(i) \quad \text{for all } i,j \in L \text{ without common endpoints}\,, \tag{2.19}$$

$$\overset{1}{U}{}^I(i)\overset{2}{U}{}^J(j) = \overset{2}{U}{}^J(j)\overset{1}{U}{}^I(i)R^{IJ}_x \quad \text{for all } i,j \in L \text{ with } t(i) = x = t(j) \text{ and } i < j\,. \tag{2.20}$$

R^{IJ}_x is the matrix $((\tau^I_x \otimes \tau^J_x)(R_x) \otimes e) \in \mathrm{End}(V^I) \otimes \mathrm{End}(V^J) \otimes \mathcal{B}$.

Braid relations for other configurations of the links i,j can be derived. As an example we consider a case where $j, -i$ point towards the same site x and $-i > j$. Then

$$\overset{1}{U}{}^J(j)(R_x)^{IJ}\overset{2}{U}{}^I(i) = \overset{2}{U}{}^I(i)\overset{1}{U}{}^J(j)\,. \tag{2.21}$$

In this form, braid relations will be widely used throughout the text.

Let us briefly describe some of the results obtained in [1]. The lattice algebra $\mathcal{B}$ *allows for a* $*$*-operation*. Its definition uses the elements κ_x introduced in (2.8), or

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[1] More precisely, there exist symmetry valued matrices $F^I_x \in \mathrm{End}(V^I) \otimes \mathcal{G}_x$ such that the $U^I(i)F^I_x$'s generate a subalgebra of $\mathcal{B}$ which is isomorphic to the dual of the symmetry Hopf-algebra $\mathcal{G}_*$. (see also [1]).

rather the element $\kappa \in \mathscr{G}$ they determine in $\mathscr{G} = \prod \mathscr{G}_x$. Before we can explain how $*$ acts on $\mathscr{B}$, we need some more notations. Let $\sigma_\kappa : \mathscr{B} \mapsto \mathscr{B}$ be the automorphism of $\mathscr{B}$ obtained by conjugation with the unitary element $\kappa \in \mathscr{G}$, i.e.

$$\sigma_\kappa(F) = \kappa^{-1} F \kappa \quad \text{for all } F \in \mathscr{B} .$$

σ_κ extends to an automorphism of $\mathrm{End}(V) \otimes \mathscr{B}$ with trivial action on $\mathrm{End}(V)$. Suppose furthermore that $B \in \mathrm{End}(V) \otimes \mathscr{B}$ has been expanded in the form $B = m_\sigma \otimes B_\sigma$ with $m_\sigma \in \mathrm{End}(V)$ and $B_\sigma \in \mathscr{B}$. If V is a Hilbert space and m_σ^* the usual adjoint of the linear map m_σ, the $*$-operation on $\mathscr{B}$ induces a $*$-operation on $\mathrm{End}(V) \otimes \mathscr{B}$ by means of the standard formula $\mathscr{B}^* = m_\sigma^* \otimes B_\sigma^*$. With these notations, the definition for $*$ in [1] becomes

$$(U^I(i))^* = \sigma_\kappa(R_x^I U^I(-i)(R_y^{-1})^I) . \tag{2.22}$$

Again i is supposed to point from $y = t(-i)$ towards $x = t(i)$ and $R_z^I \equiv (\tau_z^I \otimes \mathrm{id})(R_z) \in \mathrm{End}(V^I) \otimes \mathscr{B}$, etc.

Another ingredient in the theory of the lattice algebra $\mathscr{B}$ is the *functional* $\omega : \mathscr{B} \mapsto \mathbf{C}$. It can be regarded as the quantum analog of a multidimensional Haar measure. If we assume that the links i_ν, $\nu = 1, \ldots, n$, are pairwise different, i.e. $i_\nu \neq \pm i_\mu$ for all $\nu \neq \mu$, then

$$\omega(U^{I_1}(i_1) \cdots U^{I_n}(i_n)\xi) = \varepsilon(\xi)\delta_{I_1,0} \cdots \delta_{I_n,0} \tag{2.23}$$

for all $\xi \in \mathscr{G}$ and every set of labels I_ν. Details and examples of explicit calculations with ω can be found in [1].

It is interesting to consider the quantum analog of functions on the space of lattice connections. They form a subset $\langle U \rangle$ in $\mathscr{B}$. More precisely, $\langle U \rangle$ is generated by the matrix elements $U_{ab}^I(i) \in \mathscr{B}$ of quantum lattice connections $U^I(i)$ with the labels i, I running through all their possible values. Here the word "generate" refers to the operations of addition and multiplication in $\mathscr{B}$, while the action of $*$ is not included. So – except from special cases – $\langle U \rangle$ will not be a $*$-subalgebra of $\mathscr{B}$. This is one of the reasons, why we prefer to call $\langle U \rangle$ a subset (as opposed to subalgebra) of $\mathscr{B}$. The other reason is related to the case of quasi-Hopf symmetries $\mathscr{G}$ which will be discussed below. One of the main results in [1] is

Theorem 1 (Positivity) [1]. *Suppose that all the quantum dimensions d_J are positive and that relation* (2.7) *is satisfied. Then*

$$\omega(F^* F) \geqq 0 \quad \text{for all } F \in \langle U \rangle$$

and equality holds only for $F = 0$.

In this theorem, the argument F^*F of the functional ω is in $\mathscr{B}$ rather than in $\langle U \rangle$. Since ω was defined on the whole lattice algebra $\mathscr{B}$ the evaluation of $\omega(F^*F)$ is possible nevertheless.

Invariants within the subset $\langle U \rangle$ are the quantum analog of invariant functions on the space of lattice connections. They form a subset $\mathscr{A}$,

$$\mathscr{A} \equiv \{A \in \langle U \rangle \subset \mathscr{B} | \xi A = A \xi \text{ for all } \xi \in \mathscr{G}\} .$$

Actually, $\mathscr{A} \subset \langle U \rangle$ is also a subalgebra of $\mathscr{B}$ and the $*$-operation on $\mathscr{B}$ does restrict to a $*$-operation on $\mathscr{A}$. The positivity result of Theorem 1 implies that ω restricts to a positive linear functional on the $*$-algebra $\mathscr{A}$ (under the conditions of the theorem). Let us finally mention that $\mathscr{A}$ is independent of the position of cilia which entered the theory when we defined $\mathscr{B}$.

3. The Quantum-Curvature and Chern Simons Observables

Observables of Chern–Simons theories are obtained from the algebra $\langle U \rangle \subset \mathscr{B}$ of "functions" on the space of the quantum lattice connection in a two-step procedure. The restriction to invariants was described in [1]. The second step is to impose the flatness condition. This will be achieved in Sect. 3.2 below after some preparation in the first subsection.

3.1. Monodromy Around Plaquettes. To begin with let us consider a single plaquette P on the graph G. We assume that all cilia at sites on the boundary ∂P of this plaquette lie outside of P. In more mathematical terms we can describe this as follows: suppose that i,j are two links on ∂P and that $t(i) = x = t(j)$. Without any restriction we can take $i \leqq j$. If $k \in L$ is a third link on G with $t(k) = x$ and $i \leqq k \leqq j$, then $i = k$ or $i = j$, which means that in the situation encountered here there can be no link in between i, j.

Next let $\mathscr{C}$ be a curve on ∂P, i.e. a set of links $\{i_\nu\}_{\nu=1,\dots,n}$ with $t(i_\nu) = t(-i_{\nu+1})$, $\nu = 1,\dots,n-1$. Its inverse $-\mathscr{C}$ is the ordered set $-\mathscr{C} = \{-i_{n+1-\nu}\}_{\nu=1,\dots,n}$. On the set of curves $\mathscr{C}$ one can introduce a weight $w(\mathscr{C})$ according to

$$w(\mathscr{C}) = \sum_{\nu=1}^{n-1} \operatorname{sgn}(i_\nu, i_{\nu+1}), \quad \text{where}$$

$$\operatorname{sgn}(i,j) = \begin{cases} -1 & \text{if } i < -j \\ +1 & \text{if } i > -j \end{cases} .$$

Obviously, $w(\mathscr{C})$ changes the sign, if the orientation of $\mathscr{C}$ is inverted, i.e. $w(\mathscr{C}) = -w(-\mathscr{C})$.

From now on we will assume that $\mathscr{C}$ moves in a strictly counter-clockwise direction on ∂P, i.e. $-i_{\nu+1} > i_\nu$ for all $\nu = 1,\dots,n-1$. The starting point $t(-C) \equiv t(-i_1)$ of $\mathscr{C}$ will be called y while we use x to denote the endpoint $x = t(\mathscr{C}) \equiv t(i_n)$.

The *quantum-holonomy along* $\mathscr{C}$ is the family $\{U^I(i)\}_I$ of elements $U^I(i) \in \operatorname{End}(V^I) \otimes \mathscr{B}$ defined by

$$U^I(\mathscr{C}) \equiv \kappa_I^{w(\mathscr{C})} U^I(i_1) \cdots U^I(i_n) . \tag{3.1}$$

Here κ_I are the complex numbers which have been postulated in relation (2.7). Let us gather some of the properties of the holonomies in the following proposition.

Proposition 2 (Properties of $U^I(\mathscr{C})$). *If $\mathscr{C}$ satisfies the requirements described above and $\mathscr{C}$ is not closed (i.e. $t(\mathscr{C}) \neq t(-\mathscr{C})$), the holonomies $U^I(\mathscr{C})$ have the following properties*:

1. *They commute with gauge transformations $\xi \in \mathscr{G}_{x_\nu}$ for all $x_\nu = t(i_\nu)$, $\nu = 1,\dots,n-1$. In other words, the holonomies are gauge invariant except for their endpoints.*

2. *They commute with $U^J(i)$ if the endpoints of i and the endpoints of $\mathscr{C}$ are disjoint,*

$$\overset{1}{U}{}^I(\mathscr{C})\overset{2}{U}{}^J(i) = \overset{2}{U}{}^J(i)\overset{1}{U}{}^I(\mathscr{C})$$

whenever $\{t(i), t(-i)\} \cap \{t(\mathscr{C}), t(-\mathscr{C})\} = \emptyset$.

3. *They satisfy the following "functoriality on curves":*

$$\overset{1}{U}{}^I(\mathscr{C})\overset{2}{U}{}^J(\mathscr{C}) = \sum C_y^a[IJ|K]^* U^K(\mathscr{C}) C_x^a[IJ|K]\,, \tag{3.2}$$

$$U^I(\mathscr{C})U^I(-\mathscr{C}) = e_y^I\,, \qquad U^I(-\mathscr{C})U^I(\mathscr{C}) = e_x^I\,, \tag{3.3}$$

and behave under the action of $*$ *as*

$$(U^I(\mathscr{C}))^* = \sigma_\kappa(R_x^I U^I(-\mathscr{C})(R_y^{-1})^I)\,. \tag{3.4}$$

4. *The elements $U^I(\mathscr{C})$ and $U^I(-\mathscr{C})$ are related by*

$$U^K(-\mathscr{C}) = d_K \mathrm{tr}^{\bar{K}}(C_x[\bar{K}K|0]^* C_y[\bar{K}K|0] \overset{1}{g}{}_y^{\bar{K}} \overset{1}{U}{}^{\bar{K}}(\mathscr{C}))\,, \tag{3.5}$$

where $\overset{1}{g}{}_y^{\bar{K}} = (\tau_y^{\bar{K}}(g_y) \otimes e^K)$.

Proof. 1. is essentially trivial. 2. is an application of the braid relations for composite elements (Proposition 6, [1]). If i has no endpoint on $\mathscr{C}$ the assertion is trivial. Let us suppose that i has one endpoint $z \in S$ on $\mathscr{C}$ and $z \neq x, y$. Without loss of generality we assume $z = t(i)$. We decompose the curve $\mathscr{C}$ into two parts $\mathscr{C}_z^1 = \mathscr{C}^1$, $\mathscr{C}_z^2 = \mathscr{C}^2$ such that $\mathscr{C}^1(\mathscr{C}^2)$ ends (starts) at z. The corresponding elements $U^I(\mathscr{C}^\nu)$ satisfy standard braid relations with $U^J(i)$, i.e.

$$\overset{1}{U}{}^I(\mathscr{C}^\nu)\overset{2}{U}{}^J(i) = \overset{2}{U}{}^J(i)\overset{1}{U}{}^I(\mathscr{C}^\nu)R_z^{IJ}$$

if $\mathscr{C}^1, -\mathscr{C}^2 < i$. Similar relations with $(R_z'^{-1})$ instead of R_z hold if $C^1, -\mathscr{C}^2 > i$. Because of the assumptions on $\mathscr{C}$, other possibilities on the order of $\mathscr{C}^1, -\mathscr{C}^2, i$ do not exist. In the first case, braid relations for composite elements imply that

$$\overset{1}{U}{}^I(\mathscr{C}^1)\overset{1}{U}{}^I(\mathscr{C}^2)\overset{2}{U}{}^J(i) = \overset{2}{U}{}^J(i)\overset{1}{U}{}^I(\mathscr{C}^1)\overset{1}{U}{}^I(\mathscr{C}^2)(\tau_z^0 \otimes \tau_z^J)(R_z)\,.$$

Again, R has to be substituted by R'^{-1} in case that $\mathscr{C}^1, -\mathscr{C}^2 > i$. The representation τ^0 appears because $U^I(\mathscr{C}^1)U^I(\mathscr{C}^2)$ is invariant in z. Since $(\tau_z^0 \otimes \tau_z^J)(R_z) = (\varepsilon_z \otimes \tau_z^J)(R_z) = e_z^J$, we obtain the desired commutation relation. The last case in which both endpoints of i lie on the curve $\mathscr{C}$ is treated in a similar fashion.

3. We prove the first relation by induction on the length n of the curve $\mathscr{C}$. For $n = 1$, $\mathscr{C} = i_1$ and the relation holds because of functoriality on the link i_1. So let us assume that the equation is correct for curves of length $n - 1$. We decompose $\mathscr{C}$ into a curve $\mathscr{C}'$ of length $n - 1$ and one additional link i_n. Using the definition of $U^I(\mathscr{C})$, the braid relations (2.21) and functoriality for curves of length less than n

we obtain

$$\begin{aligned}
&\overset{1}{U}{}^I(\mathscr{C})\overset{2}{U}{}^J(\mathscr{C}) \\
&= (\kappa_I\kappa_J)^{-1}\overset{1}{U}{}^I(\mathscr{C}')\overset{1}{U}{}^I(i_n)\overset{2}{U}{}^J(\mathscr{C}')\overset{2}{U}{}^J(i_n) \\
&= (\kappa_I\kappa_J)^{-1}\overset{1}{U}{}^I(\mathscr{C}')\overset{2}{U}{}^J(\mathscr{C}')(R'_z)^{IJ}\overset{1}{U}{}^I(i_n)\overset{2}{U}{}^J(i_n) \\
&= (\kappa_I\kappa_J)^{-1}\sum_{KL} C^a_y[IJ|K]^* U^K(\mathscr{C}')C^a_z[IJ|K](R'_z)^{IJ}C^b_z[IJ|L]^* U^L(i_n)C^b_x[IJ|L] \\
&= \sum_K C^a_y[IJ|K]^* U^K(\mathscr{C})C^a_x[IJ|K]\,.
\end{aligned}$$

Here $z = t(\mathscr{C}')$ and we used relation (2.7) for the last equality. The other two formulas in 3. are obvious. 4. is a generalization of the formula (4.8) in [1] within our new notations. We want to justify it here. It follows from the functoriality of holonomies (3.2) and relation (2.7) that

$$C_y[\bar{K}K|0](R'_y)^{\bar{K}K}\overset{1}{U}{}^{\bar{K}}(\mathscr{C}) = v_K C_x[\bar{K}K|0]\overset{2}{U}{}^K(-\mathscr{C})\,.$$

Here we also applied $\kappa_K\kappa_{\bar{K}} = v_K$ With the intertwining relation (2.12) of the Clebsch Gordon maps $C[\bar{K}K|0]$ and the definition of $\overset{1}{g}{}^{\bar{K}}_y = (\tau^K_y(g_y)\otimes e)$, this can be rewritten in the form

$$C_y[\bar{K}K|0]\overset{1}{g}{}^{\bar{K}}_y\overset{1}{U}{}^{\bar{K}}(\mathscr{C}) = v_K C_x[\bar{K}K|0]\overset{2}{U}{}^K(-\mathscr{C})\,.$$

Multiplication with $C_x[\bar{K}K|0]^*$ and taking the trace $\mathrm{tr}^{\bar{K}}$ results in the desired expression for $U^K(-\mathscr{C})$ as a consequence of Eq. (2.13).

Let us now turn to the definition of monodromies. This corresponds to the case of closed curves $\mathscr{C}$ which was excluded in the preceding proposition. $\mathscr{C}$ starts and ends in the point x on ∂P. For such holonomies we introduce the new notation

$$M^I(\mathscr{C}) \equiv U^I(\mathscr{C}) \quad \text{for } \mathscr{C} \text{ closed}\,. \tag{3.6}$$

Proposition 3 (Properties of the monodromies). *If $\mathscr{C}$ is a closed curve which satisfies the requirements described above, the monodromies $M^I(\mathscr{C})$ have the following properties*:

1. *They commute with all gauge transformations $\xi \in \mathscr{G}$ with $\xi \notin \mathscr{G}_x$. Their transformation behavior under elements $\xi \in \mathscr{G}_x$ is described by*

$$\mu^I(\xi)M^I(\mathscr{C}) = M^I(\mathscr{C})\mu^I(\xi)\,.$$

2. *They commute with $U^J(i)$ if x is not among the endpoints of i, i.e. $x \notin \{t(i), t(-i)\}$.*

3. *They satisfy the following "functoriality on loops"*:

$$\overset{1}{M}{}^I(\mathscr{C})R^{IJ}_x\overset{2}{M}{}^J(\mathscr{C}) = \sum C^a_x[IJ|K]^* M^K(\mathscr{C})C^a_x[IJ|K]\,, \tag{3.7}$$

$$M^I(\mathscr{C})M^I(-\mathscr{C}) = e^I_x\,, \qquad M^I(-\mathscr{C})M^I(\mathscr{C}) = e^I_x\,, \tag{3.8}$$

and behave under the action of $*$ *as*

$$(M^I(\mathscr{C}))^* = \sigma_\kappa(R_x^I M^I(-\mathscr{C})(R_x^{-1})^I) . \tag{3.9}$$

4. *The elements* $M^I(\mathscr{C})$ *and* $M^I(-\mathscr{C})$ *are related by*

$$M^K(-\mathscr{C}) = d_K \mathrm{tr}^{\bar{K}}(C_x[\bar{K}K|0]^* C_x[\bar{K}K|0] \overset{1}{g}{}_x^{\bar{K}} \overset{1}{M}{}^{\bar{K}}(\mathscr{C}) R_x^{\bar{K}K}) . \tag{3.10}$$

Proof. 1, 2. are obvious from the proof of Proposition 2. For the proof of 3. one breaks $\mathscr{C}$ into two non-closed parts $\mathscr{C}^1, \mathscr{C}^2$ such that $t(\mathscr{C}_1) = y = t(-\mathscr{C}_2)$ and exploits the simple braid relation

$$\overset{1}{U}{}^I(\mathscr{C}^2) R_x^{IJ} \overset{2}{U}{}^J(\mathscr{C}^1) = \overset{2}{U}{}^J(\mathscr{C}^1)(R_y')^{IJ} \overset{1}{U}{}^I(\mathscr{C}^2) . \tag{3.11}$$

Using the functoriality of the holonomies $U^I(\mathscr{C}^\nu)$ derived before, the functoriality on loops follows. Giving more details on the proof of 3, 4 would amount to a repetition of the proof of Proposition 2.

Remark. From the functoriality relations (3.7) of the monodromies one derives the following quadratic relations:

$$\overset{1}{M}{}^I(\mathscr{C}) R_x^{IJ} \overset{2}{M}{}^J(\mathscr{C}) R_x'^{IJ} = R_x^{IJ} \overset{2}{M}{}^J(\mathscr{C}) R_x'^{IJ} \overset{1}{M}{}^I(\mathscr{C}) . \tag{3.12}$$

Relations of this form were found to describe the quantum enveloping algebras of simple Lie algebras [11].

From the monodromies one can prepare new elements $c^I \in \mathscr{A} \subset \mathscr{B}$. For the closed curve $\mathscr{C}$ on the boundary ∂P of the plaquette P we define

$$c^I \equiv c^I(P) \equiv \kappa_I \mathrm{tr}_q^I(M^I(\mathscr{C})) = \kappa_I \mathrm{tr}^I(M^I(\mathscr{C})\tau_x^I(g_x)) . \tag{3.13}$$

We recall that $\tau_x^I(g_x) \equiv g_x^I = \tau_x^I(u_x^{-1} v_x)$ and the last equality is a consequence of Lemma 1. The elements c^I have a number of beautiful properties. They will turn out to be central elements in the algebra $\mathscr{A}$ of invariants in $\langle U \rangle$ and satisfy the defining relations of the *fusion algebra.*

Proposition 4 (Properties of c^I). *If all eyelashes at sites on the boundary of the plaquette* P *lie outside of* P, *the elements* $c^I = c^I(P)$ *have the following properties*:

1. *They are independent of the choice of the start- and endpoint* x *of the closed curve* $\mathscr{C}$.
2. *They are central in the lattice algebra* $\mathscr{B}$. *In particular* c^I *are invariant elements in* $\langle U \rangle$ *and hence central in* $\mathscr{A}$.
3. *They satisfy the* fusion algebra

$$c^I c^J = \sum_K N_K^{IJ} c^K , \tag{3.14}$$

$$(c^I)^* = c^{\bar{I}} . \tag{3.15}$$

Proof. 1. We break the curve $\mathscr{C}$ at an arbitrary point y on ∂P and start again from the braid relations of the holonomies on the two pieces $\mathscr{C}^1, \mathscr{C}^2$ of $\mathscr{C}$,

$$\overset{1}{U}{}^I(\mathscr{C}^1) R_y^{II} \overset{2}{U}{}^I(\mathscr{C}^2) = \overset{2}{U}{}^I(\mathscr{C}^2)(R_x')^{II} \overset{1}{U}{}^I(\mathscr{C}^1) . \tag{3.16}$$

Now multiply with $\overset{1}{g}{}^I_y$ from the right and with $\overset{1}{g}{}^I_x$ from the left. Usage of $\tau^I_x(\xi)g^I_x = g^I_x\tau^I_x(\mathscr{S}^2_x(\xi))$ and the expansion $R_z^{-1} = \sum s^1_{z\sigma} \otimes s^2_{z\sigma}$ result in

$$\overset{1}{g}{}^I_x \overset{1}{U}{}^I(\mathscr{C}^1)\overset{1}{g}{}^I_y \overset{1}{\tau}{}^I_y(\mathscr{S}_y(s^1_{y\sigma}))\overset{2}{\tau}{}^I_y(s^2_{y\sigma})\overset{2}{U}{}^I(\mathscr{C}^2)$$
$$= \overset{2}{U}{}^I(\mathscr{C}^2)\overset{2}{\tau}{}^I_x(s^1_{x\sigma})\overset{1}{\tau}{}^I_x(\mathscr{S}_x^{-1}(s^2_{x\sigma}))\overset{1}{g}{}^I_x \overset{1}{U}{}^I(\mathscr{C}^1)\overset{1}{g}{}^I_y \,. \tag{3.17}$$

Here we made use of the formula $(\mathscr{S}_y \otimes \mathrm{id})(R_y) = R_y^{-1}$. We will insert this formula frequently in the following without further mention. After multiplying the two matrix components in the last equation we take the trace tr^I. With $u^I_z = (v^I)^2\tau^I_z(\mathscr{S}_z(s^1_{z\sigma})s^2_{z\sigma}) = (v^I)^2\tau^I_z(s^1_{z\sigma}\mathscr{S}_z^{-1}(s^2_{z\sigma}))$ the result is

$$\mathrm{tr}^I(g^I_xU^I(\mathscr{C}^I)g^I_yu^I_yU^I(\mathscr{C}^2)) = \mathrm{tr}^I(U^I(\mathscr{C}^2)u^I_xg^I_xU^I(\mathscr{C}^I)g^I_y)\,.$$

Finally, we insert $g^I_zu^I_z = v^I$, Definitions (3.6, 2.11) of the monodromy $M^I(\mathscr{C})$ and the q-trace tr^I_q to end up with

$$\mathrm{tr}^I_q(M^I(\mathscr{C})) = \mathrm{tr}^I_q(M^I(\mathscr{C}'))\,.$$

where $\mathscr{C}'$ starts in the site y and runs along $\mathscr{C}_2$ and $\mathscr{C}_1$ to end up in y again. This means that instead of x one can choose any other site y on the boundary of P to define c^I.

2. is a simple consequence of the properties of the monodromy (Proposition 3) and 1.

3. The first relation is easily obtained from the "operator products" (3.7) of the monodromy. One just multiplies the latter from the right with $\overset{1}{g}{}^I_x\overset{2}{g}{}^J_x(R_x^{-1})^{IJ}$ $\kappa_I\kappa_J$, uses the relation $C^a_x[IJ|K]\overset{1}{g}{}^I_x\overset{2}{g}{}^I_x = g^K_xC^a_x[IJ|K]$ (this is (2.5)) and takes the trace of both matrix-components of the equation. For the right-hand side of (3.7) this leads to

$$\text{r.h.s.} \equiv \kappa_I\kappa_J(\mathrm{tr}^I \otimes \mathrm{tr}^J)[C^a_x[IJ|K]^*M^K(\mathscr{C})C^a_x[IJ|K]\overset{1}{g}{}^I_x\overset{2}{g}{}^J_x(R_x^{-1})^{IJ}]$$
$$= \sum_{K,a}\kappa_K\mathrm{tr}^K(M^K(\mathscr{C})g^K_x) = \sum_K N^{IJ}_K c^K\,,$$

where we also inserted the normalization (2.7) of the Clebsch–Gordon maps and the definition of the numbers N^{IJ}_K. The evaluation of the left-hand side is equally simple. After application of the intertwining relation of $\overset{1}{g}{}^J_x$ and the trace property for tr^J one obtains

$$\text{l.h.s.} = \kappa_I\kappa_J(\mathrm{tr}^I \otimes \mathrm{tr}^J)[\overset{1}{M}{}^I(\mathscr{C})\overset{1}{\tau}{}^I_x(r^1_{x\sigma}\mathscr{S}_x^{-1}(r^1_{x\tau}))\overset{1}{g}{}^I_x\overset{2}{\tau}{}^J(r^2_{x\tau}r^2_{x\sigma})\overset{2}{M}{}^J(\mathscr{C})\overset{2}{g}{}^J_x]\,.$$

The simple calculation $r^1_{x\sigma}\mathscr{S}_x^{-1}(r^1_{x\tau}) \otimes r^2_{x\tau}r^2_{x\sigma} = \mathscr{S}_x^{-1}(r^1_{x\tau}s^1_{x\sigma}) \otimes r^2_{x\tau}s^2_{x\sigma} = e \otimes e$ shows that

$$\text{l.h.s.} = \kappa_I\kappa_J(\mathrm{tr}^I \otimes \mathrm{tr}^J)[\overset{1}{M}{}^I(\mathscr{C})\overset{1}{g}{}^I_x\overset{2}{M}{}^J(\mathscr{C})\overset{2}{g}{}^J_x] = c^Ic^J\,.$$

Let us turn to the behavior of c^I under the action of $*$. The relation (3.9) implies that

$$\begin{aligned}
(c^I)^* &= \kappa_I^{-1} \mathrm{tr}^I[R_x^I M^I(-\mathscr{C})(R_x^{-1})^I \tau_x^I(g_x^{-1})] \\
&= \kappa_I^{-3} \mathrm{tr}^I[\tau_x^I(\mathscr{S}_x(s^1_{x\sigma} u_x)) s^2_{x\sigma} M^I(-\mathscr{C})(R_x^{-1})^I] \\
&= \kappa_I \mathrm{tr}^I[\tau_x^I(\mathscr{S}_x(s^1_{x\sigma})\mathscr{S}_x(s^1_{x\tau}) s^2_{xJ}) s^2_{x\sigma} M^I(-\mathscr{C})(R_x^{-1})^I] \\
&= \kappa_I \mathrm{tr}^I[\mu_x^I(s^2_{x\sigma}) M^I(-\mathscr{C})(R_x^{-1})^I \tau_x^I(\mathscr{S}_x(s^1_{x\sigma}))] \\
&= \kappa_I \mathrm{tr}^I[M^I(-\mathscr{C})\tau_x^I(s^2_{x\sigma}\mathscr{S}_x(s^1_{x\sigma}))] \\
&= \kappa_I \mathrm{tr}^I[M^I(-\mathscr{C})\tau_x^I(u_x^{-1})] = \kappa_I^{-1} \mathrm{tr}^I(M^I(-\mathscr{C}) g_x^I)\,. \qquad (3.18)
\end{aligned}$$

Here $s^i_{x\sigma}$ are still defined by the expansion $R_x^{-1} = \sum s^1_{x\sigma} \otimes s^2_{x\sigma}$ and we used the relations $u_x = \mathscr{S}_x(s^1_{x\sigma}) s^2_{x\sigma} v^2$ and $u_x^{-1} = s^2_{x\sigma}\mathscr{S}_x(s^1_{x\sigma})$ (sum over σ is understood). For the fourth equality we inserted the quasi-triangularity of the element R_x. $\mu_x^I(\xi)$ was defined in (2.16). It also appears in the transformation law of the monodromies

$$\mu_x^I(\xi) M^I(\mathscr{C}) = M^I(\mathscr{C})\mu_x^I(\xi) \quad \text{for all } \xi \in \mathscr{G}_x\,.$$

The latter was used in the above calculation to shift the factor $\mu_x^I(s^2_{x\sigma})$ from the left to the right of $M^I(\mathscr{C})$. After this step, another application of the quasi-triangularity leads to the final result of the above calculation.

At this point we can insert the relation (3.10) and apply Lemma 1 several times:

$$\begin{aligned}
(c^I)^* &= \kappa_I^{-1} d_I(\mathrm{tr}^{\bar{I}} \otimes \mathrm{tr}^I)[C_x[\bar{I}I|0]^* C_x[\bar{I}I|0] \overset{1}{g}{}_x^{\bar{I}} \overset{1}{M}{}^{\bar{I}}(\mathscr{C}) R_x^{\bar{I}I} \overset{2}{g}{}_x^I] \\
&= \kappa_I^{-1} \mathrm{tr}^{\bar{I}}(\tau_x^{\bar{I}}(\mathscr{S}_x^{-1}(r^2_{x\sigma}) M^{\bar{I}}(\mathscr{C})\tau_x^{\bar{I}}(r^1_{x\sigma})) \\
&= \kappa_I^{-1} tr^{\bar{I}}(M^{\bar{I}}(\mathscr{C})\tau_x^{\bar{I}}(\mathscr{S}_x(u_x))) = \kappa_{\bar{I}} \mathrm{tr}^{\bar{I}}(M^{\bar{I}}(\mathscr{C}) g_x^{\bar{I}}) = c^{\bar{I}}\,.
\end{aligned}$$

3.2. The Algebra $\mathscr{A}_{CS}$ of Chern Simons Observables. The results of the preceding subsection show that for every plaquette P on the graph G there is a family $\{c^I(P)\}_I$ of elements in the center of $\mathscr{A}(G)$ with the properties

$$c^I(P) c^J(P) = \sum_K N_K^{IJ} c^K(P), \quad c^I(P)^* = c^{\bar{I}}(P)\,.$$

These elements are obtained as follows: suppose that $\mathscr{A}(G)$ has been constructed with some fixed ciliation on G. The corresponding ciliated graph will be denoted by G_{cil}. Now choose an arbitrary plaquette P and some ciliation on G such that no cilium lies inside of P. We call this ciliated graph $G_{\mathrm{cil}'}$. By Proposition 12 in [1] we know that there is an isomorphism $E : \mathscr{A}(G_{\mathrm{cil}'}) \mapsto \mathscr{A}(G_{\mathrm{cil}})$. We can now use the expressions in the first subsection to construct the elements c^I explicitly in $\mathscr{A}(G_{\mathrm{cil}'})$. Their images $c^I(P) = E(c^I)$ in $\mathscr{A}(G_{\mathrm{cil}})$ will be central and generate the fusion algebra. The automorphism E has been constructed in [1]. From the general action of E and the definition (3.13) for c^I one can obtain the explicit expression

$$c^I(P) = (\kappa_I)^{w(\partial P)+2} \mathrm{tr}_q^I(U^I(i_1)U^I(i_2)\dots U^I(i_n)) , \tag{3.19}$$

where $\{i_1, i_2, \dots, i_n\}$ is a closed curve that surrounds P once and $w(\partial P) = w(\{i_1, i_2, \dots, i_n\}) \pm 1$ if the cilium at $x = t(i_n)$ lies $\substack{\text{outside}\\ \text{inside}}$ the plaquette P. Formula (3.19) no longer depends on the position of cilia.

Let us describe next, how one obtains "characteristic projectors" $\chi^I(P)$ from the Casimirs $c^J(P)$. In fact this step is quite standard, but it requires an additional assumption on the gauge symmetries $\mathcal{G}_x$. From now on we *suppose that the matrix*

$$S_{IJ} \equiv \mathcal{N}(\mathrm{tr}_q^I \otimes \mathrm{tr}_q^J)(R'R) \quad \text{with } \mathcal{N} \equiv \left(\sum_K d_K^2\right)^{-1/2} < \infty \tag{3.20}$$

is invertible. A number of standard properties of S can be derived from the invertibility (and properties of the ribbon Hopf-$*$-algebra). We list them here without further discussion. Proofs can be found e.g. in [12].

$$\begin{aligned} S_{IJ} &= S_{JI} , \qquad & S_{0J} &= \mathcal{N} d_J , \\ \sum_J S_{IJ}\overline{S_{KJ}} &= \delta_{IK} , \qquad & \sum_J S_{IJ}S_{JK} &= C_{IK} , \\ \sum_K N_K^{IJ} S_{KL} &= S_{JL}S_{IL}(\mathcal{N} d_L)^{-1} \end{aligned} \tag{3.21}$$

with $C_{IJ} = N_0^{IJ}$. For the relations in the second line, the existence of an inverse of S is obviously necessary. Invertibility of S is also among the defining features of a modular Hopf-algebra in [13].

Theorem 5 (Characteristic projectors). *Suppose that the matrix $S = (S_{IJ})$ defined in Eq.* (3.20) *is invertible so that it has the properties stated in* (3.21)*. Then the elements $\chi^J(P) \in \mathcal{A}$ defined by*

$$\chi^I(P) \equiv \mathcal{N} d_I (SC)_{IK} c^K(P) = \mathcal{N} d_I S_{IK} c^{\bar{K}}(P) \tag{3.22}$$

are central orthogonal projectors in $\mathcal{A}$, i.e. they satisfy the following relations:

$$\chi^I(P)^* = \chi^I(P), \qquad \chi^I(P)\chi^J(P) = \delta_{I,J}\, \chi^I(P) . \tag{3.23}$$

Proof. The simple calculation needs no further comments,

$$\begin{aligned} \chi^I\chi^J &= \mathcal{N}^2 d_I d_J S_{I\bar{K}} S_{JL} c^K c^{\bar{L}} \\ &= \mathcal{N}^2 d_I d_J S_{I\bar{K}} S_{JL} N_L^{KM} c^{\bar{M}} \\ &= \mathcal{N}^2 d_I d_J S_{I\bar{K}} S_{MJ} S_{KJ} (\mathcal{N} d_J)^{-1} c^{\bar{M}} \\ &= d_I S_{I\bar{K}} S_{KJ} (d_J)^{-1} \chi^J \\ &= \delta_{I,J} \chi^J . \end{aligned}$$

Let us also determine the action of $*$ on the projectors χ^I,

$$(\chi^I)^* = \mathcal{N} d_I \overline{S_{IJ}} (c^{\bar{J}})^* = \mathcal{N} d_I S_{\bar{J}I} c^J = \chi^I .$$

This concludes the proof of Theorem 5. The result is quite remarkable and central for our final step in constructing Chern Simons observables.

Consider once more the graph G that we have drawn on the punctured Riemann surface Σ. Suppose that G has M plaquettes P, m of which contain a marked point. The latter will be denoted by P_v, $v = 1, \dots, m$. Let us use $\mathscr{P}$ for the set of all plaquettes on G and $\mathscr{P}_0$ for the subset of plaquettes which do not contain a marked point. By construction, the plaquettes P_v contain at most one puncture which is marked by a label I_v. To every family of such labels I_v, $v = 1, \dots, m$, we can assign a central orthogonal projector in $\mathscr{A}$.

$$\chi(\{I_v\}) = \prod_{P \in \mathscr{P}_0} \chi^0(P) \prod_{v=1}^{m} \chi^{I_v}(P_v) . \tag{3.24}$$

Since all elements $\chi^I(P)$ commute with each other, the order of multiplication is irrelevant.

Definition 6 (Chern Simons observables). *The algebra* $\mathscr{A}_{CS}^{\{I_v\}}$ *of* Chern Simons observables *on a Riemann surface* Σ *with* m *punctures marked by* I_v, $v = 1, \dots, m$, *is given through*

$$\mathscr{A}_{CS}^{\{I_v\}} \equiv \mathscr{A} \chi(\{I_v\}) = \chi(\{I_v\}) \mathscr{A} . \tag{3.25}$$

Remark. Notice that the $*$-operation on $\mathscr{A}$ restricts to a $*$-operation on $\mathscr{A}_{CS}$. The same is true for the positive linear functional in [1].

To call elements in $\mathscr{A}_{CS}$ "Chern Simons observables" has certain aspects of a conjecture. A full justification of this name needs a detailed comparison with other approaches to quantized Chern–Simons theories. This is discussed at length in a forthcoming paper [14]. Some remarks are also made in the last section of this paper. At this point we can only give a more "physical" argument by showing that elements in $\mathscr{A}_{CS}$ have "their support on the space of lattice connections which are flat everywhere except for the marked points." So whenever we multiply an element $A \in \mathscr{A}_{CS}$ with the matrix $M^I(\mathscr{C})$ and $\mathscr{C}$ wraps around a plaquette $P \in \mathscr{P}_0$, only the contributions from flat connections survive in $M^J(\mathscr{C})$. Since flat connections have trivial monodromy, this means that for all $A \in \mathscr{A}_{CS}$, $AM^J(\mathscr{C}) \sim Ae^J$ up to a complex factor which depends on the conventions. This will follow from the next proposition.

Proposition 7 (Flatness). *The elements* $\chi^0(P)$ *and* $M^I(\mathscr{C})$ *satisfy the following relation*:

$$\chi^0(P) M^J(\mathscr{C}) = (\kappa_J)^{-1} \chi^0(P) e_x^J . \tag{3.26}$$

Here $\mathscr{C}$ *is a closed curve on the boundary* ∂P *of the plaquette* P *which starts and ends in the site* x.

Proof. The point of departure is the operator product of the monodromies (Eq. (3.7)),

$$\overset{1}{M}{}^I(\mathscr{C})(R_x)^{IJ} \overset{2}{M}{}^J(\mathscr{C}) = \sum C_x^a[IJ|K]^* M^K(\mathscr{C}) C_x^a[IJ|K] .$$

As in the proof of Proposition 4.3 we obtain

$$\begin{aligned} c^I M^J(\mathscr{C}) &= \kappa_I \sum \mathrm{tr}^I_q[(R_x^{-1})^{IJ} C_x^a[IJ|K]^* M^K(\mathscr{C}) C_x^a[IJ|K]] \\ &= \kappa_I \sum \frac{d_I}{v_I} C_x[\bar{I}I|0](R_x^{-1})^{IJ} C_x^a[IJ|K]^* M^K(\mathscr{C}) \\ &\quad \cdot C_x^a[IJ|K](R_x')^{\bar{I}I} C_x[\bar{I}I|0]^* \\ &= \kappa_I \sum \frac{d_K^2}{v_J v_K d_I} C_x[\bar{K}K|0](R_x')^{J\bar{K}} C_x^a[J\bar{K}|\bar{I}]^* C_x^a[J\bar{K}|\bar{I}] \\ &\quad \cdot \overset{3}{M}{}^K(\mathscr{C})(R_x')^{\bar{K}K} C_x[\bar{K}K|0]^* . \end{aligned}$$

While the second equality follows from the definition (2.11) of the q-trace, the third equality is a consequence of the following lemma.

Lemma 2. *The Clebsch–Gordon maps satisfy the following two relations*:

$$\begin{aligned} C_x^a[IJ|K](R_x')^{\bar{I}I} C_x[\bar{I}I|0]^* &= \frac{d_K v_I}{d_I v_K} A_b^a C_x^b[J\bar{K}|\bar{I}](R_x')^{\bar{K}K} C_x[\bar{K}K|0]^* , \\ C_x[\bar{I}I|0](R_x')^{IJ} C_x^a[IJ|K]^* &= \frac{d_K v_I}{d_I v_K}(A^{-1})_b^a C_x[\bar{K}K|0](R_x')^{J\bar{K}} C_x^b[J\bar{K}|\bar{I}]^* , \end{aligned}$$

with an invertible, complex matrix A.

The proof of the lemma relies on intertwining properties and normalizations of Clebsch Gordon maps. Since it is somewhat similar to the proof of Lemma 1 – though certainly more sophisticated – we leave details to the reader.

As we continue to calculate $\chi^0 M^J(\mathscr{C})$, we will use the completeness of Clebsch Gordon maps, i.e. the relation

$$\sum_{\bar{I},a} \frac{\kappa_{\bar{I}}}{\kappa_J \kappa_{\bar{K}}} (R_x')^{J\bar{K}} C_x^a[J\bar{K}|\bar{I}]^* C_x^a[J\bar{K}|\bar{I}] = e^J \otimes e^{\bar{K}} . \tag{3.27}$$

With $\chi^0 = \sum \mathscr{N}^2 d_I c^I$ it follows that

$$\begin{aligned} \chi^0 M^J(\mathscr{C}) &= \mathscr{N}^2 \sum_I \frac{d_K^2 \kappa_I}{(\kappa_J \kappa_K)^2} C_x[\bar{K}K|0](R_x')^{J\bar{K}} C_x^a[J\bar{K}|\bar{I}]^* C_x^a[J\bar{K}|\bar{I}] \\ &\quad \cdot \overset{3}{M}{}^K(\mathscr{C})(R_x')^{\bar{K}K} C_x[\bar{K}K|0]^* \\ &= \mathscr{N}^2 \sum \frac{d_K^2}{\kappa_J \kappa_K} e_x^J C_x[\bar{K}K|0] \overset{2}{M}{}^K(\mathscr{C})(R_x')^{\bar{K}K} C_x[\bar{K}K|0]^* \\ &= \mathscr{N}^2 \sum \frac{d_K \kappa_K}{\kappa_J} e_x^J \mathrm{tr}_q^K(M^K(\mathscr{C})) \\ &= (\kappa_J)^{-1} e_x^J \sum \mathscr{N}^2 d_K c^K = (\kappa_J)^{-1} \chi^0 e_x^J . \end{aligned}$$

4. Changing the Graph G

The quantum algebra $\mathscr{A}_{CS}^{\{I_\nu\}}$ is shown to depend only on the marked Riemann surface $\Sigma_{g,m}$ with punctures labeled by I_ν and the quantum symmetry $\mathscr{G}$. Then a particular graph is described which allows for a relatively simple presentation of $\mathscr{A}_{CS}$. This presentation will be useful for explicit calculations (see e.g. Sect. 5.2) and the discussion of representation theory in a forthcoming paper [14].

4.1. Independence of the Graph. We plan to prove some fundamental isomorphisms within this section. The algebra $\mathscr{A}$ of invariants can be constructed in many different ways. One first chooses a fat graph G on the punctured Riemann surface $\Sigma_{g,m}$ and equips it with cilia at all the sites. Then one constructs the lattice algebra $\mathscr{B}$ for this ciliated graph and considers the algebra $\mathscr{A}$ of invariants in $\langle U\rangle \subset \mathscr{B}$. Even though $\mathscr{B}$ depends on the position of eyelashes, the algebra $\mathscr{A}$ that is obtained following these steps does not (Proposition 12, [1]). We will see now that the concrete choice of the graph G is also irrelevant once we restrict ourselves to the subalgebra $\mathscr{A}_{CS}$ of "functions" on the quantum moduli space of flat connections (as long as the graph G is homotopically equivalent to the punctured Riemann surface $\Sigma_{g,m}$).

Proposition 8 (Dividing a link). *Let G_1 be a graph and construct a second graph G_2 from G_1 by choosing an arbitrary link i on G_1 and introducing an additional site x on i so that i is divided into two links i_1, i_2 on G_2 with $t(i_2) = t(i)$, $t(-i_1) = t(-i)$ and $t(i_1) = t(-i_2) = x$. Then the algebras $\mathscr{A}_1 = \mathscr{A}(G_1)$ and $\mathscr{A}_2 = \mathscr{A}(G_2)$ are isomorphic as $*$-algebras.*

Proof. We know already that the algebras $\mathscr{A}$ do not depend on the ciliations (Proposition, [1]). So choose an arbitrary ciliation for G_2 and introduce the same cilia at the corresponding sites of G_1. Generators in $\mathscr{B}_1$ and $\mathscr{B}_2$ will be distinguished by a subscript, i.e. $U_1^I(i) \in \mathscr{B}_1$ and $U_2^I(i) \in \mathscr{B}_2$. Looking at the proof of the properties of holonomies we see immediately that the product $\kappa_I^{\mp 1} U_2^I(i_1) U_2^I(i_2)$ satisfies precisely the same relations in $\mathscr{B}_2 = \mathscr{B}(\mathscr{G}_{2,\mathrm{cil}})$ as $U_1^I(i)$ does in $\mathscr{B}_1 = \mathscr{B}(G_{1,\mathrm{cil}})$ (the sign depends on the position of the cilium at the new point x). This establishes an isomorphism of $\langle U_1\rangle$ with

$$\langle U_2\rangle_x \equiv \{U \in \langle U_2\rangle | \xi U = U\xi \text{ for all } \xi \in \mathscr{G}_x\} . \tag{4.1}$$

This isomorphism is consistent with the $*$-operation $*$ and clearly induces a $*$-isomorphism between $\mathscr{A}_1$ and $\mathscr{A}_2$.

Let us remark that this simple proposition shows how to define a lattice algebra $\mathscr{B}(G)$ and the corresponding algebra $\mathscr{A}(G)$ on a multigraph G, on which two given sites may be connected by more than one link. Our original definition in [1] did not include this case. If G is a multigraph, one can always construct a graph G' (which has at most one link connecting two given sites) simply by dividing some of the links on G. Even though the resulting graph G' is certainly not unique, the algebra $\mathscr{A}(G')$ is. This makes $\mathscr{A}(G) \equiv \mathscr{A}(G')$ well defined for every multigraph G. The idea can be extended to the lattice algebra $\mathscr{B}(G)$. Since we will need this in some of the proofs to come, let us briefly explain the details. Suppose that the link i on G has been divided into two links i_1 and i_2 on G'. Then we define the element $U^I(i)$ by ($\pm$ depending on the ciliation)

$$U^I(i) \equiv \kappa_I^{\mp 1} U^I(i_1) U^I(i_2) .$$

Observe that the right-hand side is meaningful since the arguments i_1, i_2 are links on a graph G' (whereas i is a link on a multigraph so that $U^I(i)$ was previously not defined). If we identify the set S of sites on G and the corresponding subset of sites on G', we can set $\mathscr{B}(G)$ to be the subalgebra of $\mathscr{B}(G')$ which is generated by components of $U^I(i)$, $i \in L$ and the elements $\xi \in \mathscr{G}_x$, $x \in S$. Along these lines, even graphs with loops (i.e. links which start and end in the same site) can be admitted. Needless to say that it would have been possible to give a direct definition of $\mathscr{B}(G)$ for all these types of graphs similar to the definition of $\mathscr{B}$ in [1]. But the more complicated the type of graph becomes the more cases have to be distinguished in writing the defining relations of $\mathscr{B}$. For many proofs this would have been an enormous inconvenience. On the other hand, our results for algebras on graphs G imply corresponding results for algebras on multigraphs G since the latter have been identified as subalgebras of the former. After this excursion we can give up our strict distinction between graphs and multigraphs.

Proposition 9 (Contraction of a link). *Let G_1 be a graph and construct a second graph G_2 from G_1 by contracting an arbitrary link i on G_1. This means that on the subgraph $G_1 - i$ which is obtained from G_1 by removing the link $\pm i$, the end-points $t(i) = x$ and $t(-i) = y$ of i are identified to get G_2. The resulting algebras $\mathscr{A}_1 = \mathscr{A}(G_1)$ and $\mathscr{A}_2 = \mathscr{A}(G_2)$ are isomorphic as $*$-algebras.*

Remark. Observe that G_2 can be a multigraph even if G_1 is a graph. So objects on G_2 have to be understood in the sense of our general discussion preceding this proposition.

Proof. To prove the proposition we will adopt the following conventions. The site on G_2 that corresponds to the pair (x, y) of sites on G_1 will be denoted by z. We will use the same letters for a link $k \in L_1$ on G_1 and its "partner" $k \in L_2$ on G_2. In addition to i, the site x is the endpoint of n other links $j_1, \ldots, j_n$. We assume that they all point away from x, i.e. $t(-j_\nu) = x$ for all $\nu = 1, \ldots, n$. Next we introduce a ciliation on G_1 such that i becomes the largest link at x and $-i$ is the smallest at y. In a canonical way, this induces a ciliation for G_2.

As in the proof of the preceding proposition, the desired isomorphism will be obtained by restricting an isomorphism between $\langle U_1 \rangle_x$ and $\langle U_2 \rangle$. From definition (4.1) it is obvious that $\langle U_1 \rangle_x$ is generated by components of $U_1^K(k)$, $k \neq \pm i, \pm j_\nu$, and

$$\overset{1}{U}{}_1^{J_1}(j_1) \cdots \overset{n}{U}{}_1^{J_n}(j_n) C[J_1 \cdots J_n | J] U_1^J(i) , \tag{4.2}$$

where the maps $C[J_1 \ldots J_n | J]$ are only restricted by the property

$$(\tau_x^{J_1} \boxtimes \cdots \boxtimes \tau_x^{J_n})(\xi) C[J_1 \cdots J_n | J] = C[J_1 \cdots J_n | J] \tau_x^J(\xi) \quad \text{for all } \xi \in \mathscr{G}_x .$$

This guarantees that components of the elements (4.2) are invariant at x. We define a map $\Phi : \langle U_1 \rangle_x \mapsto \langle U_2 \rangle$ by an action on these generators.

$$\Phi(U_1^K(k)) = U_2^K(k) \quad \text{for all } k \neq \pm i, \pm j_\nu \in L_1 , \tag{4.3}$$

$$\begin{aligned} &\Phi(\overset{1}{U}{}_1^{J_1}(j_1) \cdots \overset{n}{U}{}_1^{J_n}(j_n) C[J_1 \cdots J_n | J] U_1^J(i) \\ &\quad = \overset{1}{U}{}_2^{J_1}(j_1) \cdots \overset{n}{U}{}_2^{J_n}(j_n) C[J_1 \cdots J_n | J] . \end{aligned} \tag{4.4}$$

Indeed Φ can be extended to an isomorphism. Let us start to establish the consistency with the multiplication. It is clear that Φ respects all relations between generators $U_1^K(k)$, $k \neq \pm i, \pm j_\nu$, and between such $U_1^K(k)$ and elements (4.2). Suppose next that we have two elements in $\langle U_2 \rangle$ which are both of the form of the right-hand side in Eq. (4.4). Their multiplication defines maps C_a'', F by

$$\overset{11}{U}{}_2^{J_1}(j_1)\cdots \overset{1n}{U}{}_2^{J_n}(j_n)C[J_1\cdots J_n|J]\overset{21}{U}{}_2^{K_1}(j_1)\cdots \overset{2n}{U}{}_2^{K_n}(j_n)C'[K_1\cdots K_n|K]$$
$$= F\overset{1}{U}{}_2^{L_1}(j_1)\cdots \overset{n}{U}{}_2^{L_n}(j_n)C_a''[L_1\cdots L_n|L]C_x^a[JK|L]\,. \tag{4.5}$$

F is just a simple combination of Clebsch–Gordon maps, but since it will be of no concern to us, we do not want to spell this out. On the other hand we may multiply elements in $\langle U_1 \rangle_x$ of the form (4.2). When dealing with the product

$$\overset{11}{U}{}_1^{J_1}(j_1)\cdots \overset{1n}{U}{}_1^{J_n}(j_n)C[J_1\cdots J_n|J]\overset{1}{U}{}_1^{J}(i)$$
$$\cdot\, \overset{21}{U}{}_1^{K_1}(j_1)\cdots \overset{2n}{U}{}_1^{K_n}(j_n)C'[K_1\cdots K_n|K]\overset{1}{U}{}_1^{K}(i)\,, \tag{4.6}$$

we first apply the proposition on braid relations of composites to move $U_1^J(i)$ to the right. The elements on links j_ν can then be rearranged precisely as in $\langle U_2 \rangle$ before. The result of these manipulations is

$$= F\overset{1}{U}{}_1^{L_1}(j_1)\ldots \overset{n}{U}{}_1^{L_n}(j_n)C_a''[L_1\ldots L_n|L]C_x^a[JK|L](R_x^{-1})^{JK}\overset{1}{U}{}_1^{J}(i)\overset{2}{U}{}_1^{K}(i)$$
$$= F\overset{1}{U}{}_1^{L_1}(j_1)\ldots \overset{n}{U}{}_1^{L_n}(j_n)C_a''[L_1\ldots L_n|L]U_1^I(i)C_x^a[JK|L]\,,$$

with the same F, C_a'' as in Eq. (4.5). For the second equality we used functoriality on the link i. We see that Φ maps products (4.6) to the element on the right-hand side of Eq. (4.5). This shows consistency of Φ with multiplication. Consistency of Φ with the $*$-operation is proved in a similar way. We leave this as an exercise. Since elements in the image of Φ generate $\langle U_2 \rangle$, we established Proposition 9.

Proposition 10 (Erasure of a link). *Let G be a graph and P be a plaquette of G. Suppose that the link i lies on the boundary ∂P of this plaquette and that $G' = G - i$ is the subgraph of G obtained by removing the link $\pm i$. Then the $*$-algebras $\chi^0(P)\mathscr{A}(G)$ and $\mathscr{A}(G')$ are isomorphic. Denote the other plaquette incident to the link i in G by $\tilde{P}$ and the resulting plaquette which replaces P and $\tilde{P}$ in G' by $\tilde{P}'$. The $*$-subalgebras $\chi^0(P)\chi^I(\tilde{P})\mathscr{A}(G)$ and $\chi^I(\tilde{P}')\mathscr{A}(G')$ are isomorphic.*

Proof. The proof is obtained as a reformulation of Proposition 7 above. We choose cilia to be outside of P and $\mathscr{C} = \{-i, j_1, \ldots, j_n\}$ such that it surrounds P in clockwise direction. With the decomposition $\mathscr{C} = \mathscr{C}_o \circ \{-i\}$, Eq. (3.26) can be restated as

$$\chi^0(P)U^J(\mathscr{C}_o) = \chi^0(P)U^J(i)\,.$$

Since $\mathscr{A}(G')$ can be identified with the subalgebra of elements $A \in \mathscr{A}(G)$ which do not contain $U^I(\pm i)$, the formula means that $\chi^0(P)\mathscr{A}(G) = \chi^0(P)\mathscr{A}(G') = \mathscr{A}(G')$. Now choose $\tilde{\mathscr{C}} = \{i, j_{n+1}, \ldots, j_{n+\tilde{n}}\}$ to surround $\tilde{P}$ in the clockwise direction and decompose it according to $\tilde{\mathscr{C}} = \{i\} \circ \tilde{\mathscr{C}}_o$. It follows immediately that

$$\begin{aligned}\chi^0(P)c^I(\tilde{\mathscr{C}}) &= \chi^0(P)\mathrm{tr}_q^I(U^I(i)U^I(\tilde{\mathscr{C}}_o)) \\ &= \chi^0(P)\mathrm{tr}_q^I(U^I(\mathscr{C}_o)U^I(\tilde{\mathscr{C}}_o)) \\ &= \chi^0(P)c^I(\tilde{P}') \,.\end{aligned}$$

This implies the second statement of the proposition.

The last proposition reflects the topological nature of the Chern Simons theory. Since elements in $\mathscr{A}_{CS}$ have a factor $\chi^0(P)$ for every plaquette $P \in \mathscr{P}_0$ which does not contain a marked point, the proposition implies that such plaquettes can be arbitrarily added or removed from the graph G without any effect on $\mathscr{A}_{CS}$.

Contracting and erasing links and making inverse operations one can obtain from any admissible graph on a punctured Riemann surface any other admissible graph. The algebra of observables does not change when we contract and erase links. So, we can conclude that this algebra is actually graph-independent as a $*$-algebra. Let us note that such strategy of proving graph independence has been applied in [2] to the Poisson algebra of functions on the moduli space.

4.2. Theory on the Standard Graph $G_{g,m}$. Since the algebra $\mathscr{A}_{CS}$ does not depend on the graph G one may choose any graph on the Riemann surface to construct it. This section is devoted to a special example of such a graph called the "standard graph." It is also the basis for the representation theory of the moduli algebra considered in a forthcoming paper [14].

The standard graph is one of the simplest possible graphs which is homotopically equivalent to a Riemann surface $\Sigma_{g,m}$ of genus g and with m marked points. It has $m+1$ plaquettes, $m+2g$ links and only one vertex. To give a precise definition we consider the fundamental group $\pi_1(\Sigma_{g,m})$ of the marked Riemann surface. Let us choose a set generators $l_\nu,\ \nu = 1,\dots,m;\ a_i, b_i,\ i = 1,\dots,g$ in $\pi_1(\Sigma_{g,n})$ so that

1. l_ν is homologous to a small circle around the ν^{th} marked point,
2. a_i, b_i are a- and b-cycles winding around the i^{th} handle, which means in terms of intersections

$$l_\nu \# l_\mu = l_\nu \# a_j = l_\nu \# b_j = 0, \qquad a_i \# b_j = \delta_{i,j} \,, \tag{4.7}$$

3. the only relation between generators in $\pi_1(\Sigma_{g,n})$ is

$$l_1 \cdots l_m [a_1, b_1] \cdots [a_g, b_g] = 1 \,, \tag{4.8}$$

where we use the notation $[x,y] = yx^{-1}y^{-1}x$ for elements x, y of the group $\pi_1(\Sigma_{g,m})$.

We call such a basis in $\pi_1(\Sigma_{g,m})$ a *standard basis*. Having a standard basis, one can draw a standard graph on the Riemann surface $\Sigma_{g,m}$.

Definition 11 (Standard graph $G_{g,m}$). *Given a standard basis in $\pi_1(\Sigma_{g,n})$, a standard graph $G_{g,m}$ corresponding to this basis is a collection of circles on the surface, representing the generators l_ν, a_i, b_i in such a way that they intersect only in one "base point" p.*

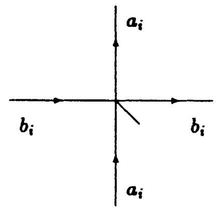

Fig. 1. Position of the cilium at the only vertex of the standard graph. The letters a_i, b_i mark loopends corresponding to the a- and b-cycles.

Any standard graph may be supplied with a canonical ciliation which orders the link ends such that $l_\nu < l_\mu < (a_i, b_i) < (a_j, b_j)$ for all $\nu < \mu$ and $i < j$. Within the i^{th} pair (a_i, b_i) of a- and b-cycles we assume the order of Fig. 1. The notion $l_\mu < (a_i, b_i)$ means for example that the elements in the triple (cilium, ends of l_ν, ends of a_i and b_i) appear in a clockwise order with respect to a fixed orientation of $\Sigma_{g,m}$.

Still we have a big choice as there are infinitely many standard graphs. In principle, we should describe how the formalism behaves when we pass from one standard graph to another one. However, the algebraic content of the theory is identically the same for any standard graph. So, we forget for a moment about this ambiguity and turn to the corresponding graph algebra $\mathcal{S}_{g,m} = \mathcal{B}(G_{g,m})$. To write the defining relations of $\mathcal{S}_{g,m}$ one simply follows the general rules discussed above. So in principle $\mathcal{S}_{g,m} \equiv \mathcal{B}(G_{g,m})$ suffices as a definition of $\mathcal{S}_{g,m}$. In view of the central role, the graph algebra $\mathcal{S}_{g,m}$ will play for the representation theory of $\mathcal{A}_{CS}$ we would like to give a completely explicit definition here.

Definition 12 (Graph algebra $\mathcal{S}_{g,m}$). *The* graph-algebra $\mathcal{S}_{g,m}$ *is a $*$-algebra which is generated by matrix elements of $M^I(l_\nu)$, $M^I(a_i)$, $M^I(b_i) \in \mathrm{End}(V^I) \otimes \mathcal{S}_{g,m}$, $\nu = 1, \ldots, m$, $i = 1, \ldots, g$ together with elements ξ in a quasi-triangular Hopf algebra $\mathcal{G}_* = \mathcal{G}$ with R-element $R_* = R$ and co-product $\Delta_* = \Delta$. As usual, the superscript I runs through the set of equivalence classes of irreducible representations of $\mathcal{G}$. Elements in $\mathcal{S}_{g,m}$ are subject to the following relations*:

$$\overset{1}{M}{}^I(l_\nu) R^{IJ} \overset{2}{M}{}^J(l_\nu) = \sum C^a[IJ|K]^* M^K(l_\nu) C^a[IJ|K] ,$$

$$\overset{1}{M}{}^I(a_i) R^{IJ} \overset{2}{M}{}^J(a_i) = \sum C^a[IJ|K]^* M^K(a_i) C^a[IJ|K] ,$$

$$\overset{1}{M}{}^I(b_i) R^{IJ} \overset{2}{M}{}^J(b_i) = \sum C^a[IJ|K]^* M^K(b_i) C^a[IJ|K] ,$$

$$(R^{-1})^{IJ} \overset{1}{M}{}^I(a_i) R^{IJ} \overset{2}{M}{}^J(b_i) = \overset{2}{M}{}^J(b_i)(R')^{IJ} \overset{1}{M}{}^I(a_i) R^{IJ} ,$$

$$(R^{-1})^{IJ} \overset{1}{M}{}^I(l_\nu) R^{IJ} \overset{2}{M}{}^J(l_\mu) = \overset{2}{M}{}^J(l_\mu)(R^{-1})^{IJ} \overset{1}{M}{}^I(l_\nu) R^{IJ} \quad \textit{for } \nu < \mu ,$$

$$(R^{-1})^{IJ} \overset{1}{M}{}^I(l_\nu) R^{IJ} \overset{2}{M}{}^J(a_j) = \overset{2}{M}{}^J(a_j)(R^{-1})^{IJ} \overset{1}{M}{}^I(l_\nu) R^{IJ} \quad \forall\ \nu, j ,$$

$$(R^{-1})^{IJ} \overset{1}{M}{}^I(l_\nu) R^{IJ} \overset{2}{M}{}^J(b_j) = \overset{2}{M}{}^J(b_j)(R^{-1})^{IJ} \overset{1}{M}{}^I(l_\nu) R^{IJ} \quad \forall\ \nu, j ,$$

$$(R^{-1})^{IJ} \overset{1}{M}{}^I(a_i) R^{IJ} \overset{2}{M}{}^J(a_j) = \overset{2}{M}{}^J(a_j)(R^{-1})^{IJ} \overset{1}{M}{}^I(a_j) R^{IJ} \quad \textit{for } i < j ,$$

$$(R^{-1})^{IJ}\overset{1}{M}{}^I(a_i)R^{IJ}\overset{2}{M}{}^J(b_j) = \overset{2}{M}{}^J(b_j)(R^{-1})^{IJ}\overset{1}{M}{}^I(a_j)R^{IJ} \quad \textit{for } i < j\,,$$

$$(R^{-1})^{IJ}\overset{1}{M}{}^I(b_i)R^{IJ}\overset{2}{M}{}^J(b_j) = \overset{2}{M}{}^J(b_j)(R^{-1})^{IJ}\overset{1}{M}{}^I(b_j)R^{IJ} \quad \textit{for } i < j\,,$$

$$(R^{-1})^{IJ}\overset{1}{M}{}^I(b_i)R^{IJ}\overset{2}{M}{}^J(a_j) = \overset{2}{M}{}^J(a_j)(R^{-1})^{IJ}\overset{1}{M}{}^I(b_i)R^{IJ} \quad \textit{for } i < j\,,$$

$$\mu^J(\xi)M^J(l_v) = M^J(l_v)\mu^J(\xi)\,,$$

$$\mu^J(\xi)M^J(a_i) = M^J(a_i)\mu^J(\xi)\,, \quad \mu^J(\xi)M^J(b_i) = M^J(b_i)\mu^J(\xi)\,,$$

where $\mu^I(\xi) \equiv (\tau^I \otimes \mathrm{id})(\Delta(\xi)) \in \mathrm{End}(V^I) \otimes \mathcal{G}$ *as before. With* $M^I(-l_v)$, $M^I(-a_i)$ *and* $M^I(-b_i)$ *being constructed from* $M^I(l_v)$, $M^I(a_i)$ *and* $M^I(b_i)$ *with the help of formula* (3.10) (*so that* $M^I(l_v)M^I(-l_v) = e^I$ *etc.*) *the action of the* $*$*-operation on* $\mathcal{S}_{g,m}$ *is given through*

$$(M^I(l_v))^* = \sigma_\kappa(R^I M^I(-l_v)(R^{-1})^I)\,,$$

$$(M^I(a_i))^* = \sigma_\kappa(R^I M^I(-a_i)(R^{-1})^I)\,,$$

$$(M^I(b_i))^* = \sigma_\kappa(R^I M^I(-b_i)(R^{-1})^I)\,,$$

where σ_κ *means conjugation by* κ (*see Sect.* 2 *for details*).

This definition requires some comments. All links of the standard graph are closed ("loops"). This explains why all the functoriality relations have the form (3.7). The relations between generators on different loops reflect the particular ciliation described above and follow strictly from the rules given in Sect. 2. To verify this, one should recall that quantum lattice connections on closed links were defined in Sect. 4 as special elements in a larger lattice algebra $\mathcal{B}(G)$. Here G is a graph on which all loops have been divided into two (non-closed) links by introducing additional sites on the loops. After one has gained some experience with this type of exchange relations, the rather pedantic procedure of dividing links will become superfluous.

There is one more remark we need in order to prepare for a calculation in the next section. We saw in Proposition 2 that holonomies which are made up from products of lattice connections assigned to different links satisfy the same type of functoriality as the lattice connection $U^I(i)$ themselves. A similar property holds for the lattice connections $M^I(l)$ on loops l. We demonstrate this at the example of $M^I(a_i)$, $M^I(b_i) \in \mathrm{End}(V^I) \otimes \mathcal{S}_{g,m}$,

$$\kappa_I^{-1}\overset{1}{M}{}^I(b_i)\overset{1}{M}{}^I(a_i)R^{IJ}\kappa_J^{-1}\overset{2}{M}{}^J(b_i)\overset{2}{M}{}^J(a_i)$$

$$= (\kappa_I\kappa_J)^{-1}\overset{1}{M}{}^I(b_i)R^{IJ}\overset{2}{M}{}^J(b_i)(R')^{IJ}\overset{1}{M}{}^I(a_i)R^{IJ}\overset{2}{M}{}^J(a_i)$$

$$= (\kappa_I\kappa_J)^{-1}\sum C^a[IJ|K]^* M^K(b_i)C^a[IJ|K]$$

$$\cdot\, (R')^{IJ}C^b[IJ|L]^* M^L(a_i)C^b[IJ|L]$$

$$= \sum \kappa_K^{-1}C^a[IJ|K]^* M^K(b_i)M^K(a_i)C^b[IJ|K]\,.$$

From this one can easily derive the following formula, which is similar to Eq. (3.10):

$$\kappa_I^{-1} M^I(b_i) M^I(a_i) = d_I \operatorname{tr}^I[R^{I\bar{I}} \kappa_I \overset{2}{M}{}^{\bar{I}}(-a_i) \overset{2}{M}{}^{\bar{I}}(-b_i) \overset{2}{g}{}^{\bar{I}} C[I\bar{I}|0]^* C[I\bar{I}|0]] \,. \quad (4.9)$$

It will be used in Subsect. 5.2.

5. Quantum Integration

The "multidimensional Haar measure" $\omega : \mathscr{A} \mapsto \mathbf{C}$ mentioned in Theorem 1 restricts to a positive functional on the algebra $\mathscr{A}_{CS}$ of Chern Simons observables. When the latter is properly normalized, it does not depend on the choice of the graph and thus furnishes a distinguished functional $\omega_{YM} : \mathscr{A}_{CS} \mapsto \mathbf{C}$. This functional is a generalization of the integration measure in the lattice Yang–Mills theory. We use this to calculate the volume of the quantum moduli space in the second subsection.

5.1. The Yang–Mills Functional ω_{YM}. To define the functional ω_{YM} we use the same notations as in Sect. 3.2. In particular, the graph G which we have drawn on the punctured Riemann surface Σ is supposed to possess M plaquettes. With the finite real constant $\mathscr{N} = (\sum (d_I)^2)^{-1/2}$ introduced in relation (3.20) we define

$$\omega_{YM}(A) \equiv \mathscr{N}^{-2M} \omega(A) \quad \text{for all } A \in \mathscr{A}_{CS}^{\{I_\nu\}} \,. \quad (5.1)$$

Obviously, ω_{YM} inherits its positivity from the positivity of ω.

We have seen at the end of the preceding section that the algebra $\mathscr{A}_{CS} = \mathscr{A}_{CS}^{\{I_\nu\}}$ does not depend on the graph G. The main purpose of this subsection is to establish the graph-independence for ω_{YM}.

Proposition 13 (Graph-independence of ω_{YM}). *Let G be a graph and suppose that G' is a second graph so that either*

1. *G' is obtained from G by dividing one link i on G into links i_1, i_2 on G' by adding one additional site x on i,*
2. *G' is obtained from G by contracting one link i on G so that i is removed and its endpoints are identified on G',*
3. *G' is obtained from G by erasure of a link i on the common boundary of two plaquettes P, P', with P containing no marked point.*

Then the Yang–Mills functional ω_{YM} for the graph G is equal to the Yang–Mills functional ω'_{YM} assigned to the graph G' (since in all three cases the corresponding algebras $\mathscr{A}_{CS} = \mathscr{A}_{CS}^{\{I_\nu\}}(G)$ are isomorphic to $A'_{CS} = \mathscr{A}_{CS}^{\{I_\nu\}}(G')$, equality of the functionals is well defined).

Proof. The first case is essentially trivial. It follows directly from Definition (2.23) of ω. 2. can be derived by combining the first and the last case. The simple argument is left to the reader. In turning to the proof of 3., let $\mathscr{C}, \mathscr{C}'$ denote two curves on the boundary of P, P' such that $\{\mathscr{C}, i\}$ and $\{-i, \mathscr{C}'\}$ are closed. Both are assumed to move counter-clockwise. By definition, an element A in $\mathscr{A}_{CS}$ is of the form $A = e_{CS} \hat{A}$ with

$$e_{CS} \equiv e_{CS}^{\{I_\nu\}} \equiv \prod_{P \in \mathscr{P}_0} \chi^0(P) \prod_{\nu=1}^{m} \chi^{I_\nu}(P_\nu) \quad (5.2)$$

being the unit element in the moduli algebra $\mathscr{A}_{CS}$. According to Proposition 10, the element $\hat{A} \in \mathscr{A}$ can be written without usage of $U^I(\pm i)$. Given an arbitrary presentation of $\hat{A}$ one simply has to replace $U^I(\pm i)$ by $U^I(\mp\mathscr{C})$. In the following we will assume that this replacement has been made. The image of A under the isomorphism between $\mathscr{A}_{CS}$ and $\mathscr{A}'_{CS}$ is $A' = e'_{CS}\hat{A}$ (e'_{CS} is the unit element in $\mathscr{A}'_{CS}$). With these notations, the statement of the proposition,

$$\omega_{YM}(A) = \omega'_{YM}(A')\,,$$

is equivalent to

$$\mathscr{N}^{-2M}\omega(e_{CS}\hat{A}) = \mathscr{N}^{-2(M-1)}\omega(e'_{CS}\hat{A})\,. \tag{5.3}$$

The different powers of $\mathscr{N}$ are due to the fact that G' has one plaquette less than G. This equation is a consequence of the following lemma.

Lemma 3. *Suppose that $\hat{F} \in \mathscr{A} = \mathscr{A}(G)$ does not contain elements $U^I(\pm i)$ with i being on the common boundary of two arbitrary plaquettes P, P' of G. After i is removed, the plaquettes P, P' merge into a single plaquette $P \cup P'$ on $G' = G - i$. We have*

$$\omega(c^I(P)c^J(P')\hat{F}) = (d_I)^{-1}\omega(c^I(P \cup P')\hat{F})\delta_{I,J}\,. \tag{5.4}$$

Here $c^I(P), c^J(P')$ and $c^I(P \cup P')$ are given by Eq. (3.19) (*which holds for arbitrary but fixed ciliations*).

Proof of the Lemma. We want to show first that the left-hand side of Eq. (5.4) is nonzero only for $I = J$. The formula (2.23) for ω reveals that the value of ω can be nonzero, only if the argument has a component which contains the factor $U^0(i)$. The product $c^I(P)c^J(P')\hat{F}$ contains $U^I(i)$ and $U^J(-i)$ and these are the only elements associated with the link i. Now $U^J(-i)$ can be expressed as a linear combination of $U^{\bar{J}}(i)$. The "operator product" of $U^I(i)$ and $U^{\bar{J}}(i)$ has components proportional to $U^0(i)$, if and only if I is the conjugate of $\bar{J}$, i.e. iff $I = J$. So we can set $I = J$ for the rest of the proof. For simplicity we will also assume that the cilia at the sites $x = t(i)$ and $y = t(-i)$ lie outside of both P and P'. For different positions of eyelashes, the proof contains some additional phases $(v_I)^{\pm 1}$ which cancel in the end. By Eqs. (3.19) and (3.18) we have

$$c^I(P) = \mathrm{tr}^I_q(U^I(\mathscr{C})U^I(i))\,, \tag{5.5}$$

$$c^I(P') = \mathrm{tr}^{\bar{I}}_q(U^{\bar{I}}(i)U^{\bar{I}}(-\mathscr{C}'))\,. \tag{5.6}$$

Functoriality on the link i gives

$$\overset{1}{U}{}^I(\mathscr{C})\overset{1}{U}{}^I(i)\overset{2}{U}{}^{\bar{I}}(i)\overset{2}{U}{}^{\bar{I}}(-\mathscr{C}') = \sum \overset{1}{U}{}^I(\mathscr{C})C^a_y[I\bar{I}|K]^* U^K(i) C^a_x[I\bar{I}|K]\overset{2}{U}{}^{\bar{I}}(-\mathscr{C}')\,.$$

If we apply $\mathrm{tr}^I_q \otimes \mathrm{tr}^{\bar{I}}_q$ to this relation, we multiply with $\hat{F}$ and evaluate the resulting expression with ω to obtain

$$\begin{aligned}
&\omega(c^I(P)c^J(P')\hat{F})\\
&\quad = \omega\left((\mathrm{tr}^I_q \otimes \mathrm{tr}^{\bar{I}}_q)\left[\sum \overset{1}{U}{}^I(\mathscr{C})C^a_y[I\bar{I}|0]^* C^a_x[I\bar{I}|0]\overset{2}{U}{}^{\bar{I}}(-\mathscr{C}')\right]\hat{F}\right)\,.
\end{aligned}$$

A formula similar to Eq. (3.5) allows us to rewrite the right-hand side so that it becomes

$$= d_{\bar{I}}^{-1}\omega(\mathrm{tr}_q^I(U^I(\mathscr{C})g_x^I U^I(\mathscr{C}')(g_x^I)^{-1})\hat{F})$$
$$= d_{\bar{I}}^{-1}\omega(c^I(P \cup P')\hat{F})\,.$$

This proves formula (5.4) and thus Lemma 3.

With the explicit expressions (3.22) for the characteristic projectors and Lemma 3 we can calculate

$$\begin{aligned}\omega(\chi^K(P)\chi^L(P')\hat{F}) &= \mathscr{N}^2 d_K d_L S_{K\bar{I}} S_{L\bar{J}}\,\omega(c^I(P)c^J(P')\hat{F}) \\ &= \mathscr{N}^2 d_K d_L S_{K\bar{I}} S_{L\bar{I}} (d_{\bar{I}})^{-1}\omega(c^I(P \cup P')\hat{F}) \\ &= \mathscr{N}^2 \mathscr{N} d_K d_L N_R^{KL} S_{R\bar{I}}\,\omega(c^I(P \cup P')\hat{F}) \\ &= \frac{d_K d_L}{d_R} N_R^{KL} \mathscr{N}^2 \omega(\chi^R(P \cup P')\hat{F})\,. \end{aligned} \tag{5.7}$$

Proof of Proposition 13 (3.) (continued). For Proposition 13, P was assumed not to contain a marked point so that it contributed with a factor $\chi^0(P)$ to e_{CS}. P' was arbitrary and so is the associated factor $\chi^L(P')$. In the calculation leading to (5.7) we can set $K = 0$ and use

$$\hat{F} = \prod_{P \in \mathscr{P}_0}{}' \chi^0(P) \prod_{\nu=1}^{m}{}' \chi^{I_\nu}(P_\nu)\hat{A}$$

with $'$ meaning that the product is restricted to plaquettes nonequal to P, P'. With $d_0 = 1$ and $N_R^{0L} = \delta_{L,R}$ this gives the formula (5.3) and hence proves the proposition.

5.2. Volume of the Quantum Moduli Space. To demonstrate how computations can be performed within the framework of this paper, the volume of quantum moduli space of flat connections on a marked Riemann surface Σ is calculated[2]. In practice we define the volume of the quantum space as an integral or trace of the characteristic projector. In the framework of the 2-dimensional lattice gauge model one can interpret this result as a partition function of the system. As there is no Hamiltonian involved, we shall get just a number.

The "characteristic function" for the quantum moduli space is the projector

$$e_{CS} \equiv e_{CS}^{\{I_\nu\}} \equiv \prod_{P \in \mathscr{P}_0} \chi^0(P) \prod_{\nu=1}^{m} \chi^{I_\nu}(P_\nu)\,, \tag{5.8}$$

which contains one factor for every plaquette of the graph which we have drawn on the marked Riemann surface $\Sigma_{g,m}$. The I_ν, $\nu = 1, \dots, m$, are the labels sitting at the m punctures. $\mathscr{P}_0$ denotes the set of plaquettes without marked point. If a characteristic function is integrated, this gives the volume of the corresponding space. In our case, integration is defined with the help of the Yang–Mills functional ω_{YM} and this means that the volume of the moduli space is $\omega_{YM}(e_{CS}^{\{I_\nu\}})$. Using the graph independence of the algebra $\mathscr{A}_{CS}^{\{I_\nu\}}$ and the functional ω_{YM}, we fix a

[2] A similar calculation was also done recently by Buffenoir and Roche [15].

particular graph from the very beginning. Let us use the standard graph discussed in Sect. 4.2 for this purpose. This is certainly not necessary for the computations to follow, but it simplifies the presentation and can help to make it as concrete as possible. Before we give the general result, we would like to discuss two examples.

Example 1. Genus 0. Recall that the standard graph $G_{0,m}$ on a Riemann sphere with m marked points consists of m loops which start and end at the same vertex. The standard graph has $m+1$ plaquettes, one of which does not contain a marked point. So the characteristic projector is

$$\chi^0(P_0)\prod_{\nu=1}^{m}\chi^{K_\nu}(P_\nu)\,.$$

To calculate its expectation value, one should recall the formula (5.7). It allows to relate the expectation value of the characteristic projector $e_{CS}^{\{I_\nu\}}$ on the standard graph G_m to a similar expectation value on a simpler graph, from which one link and one plaquette has been removed. One can actually iterate this procedure to get

$$\omega_{YM}(e_{CS}^{\{I_\nu\}}) = \left(\prod_{\nu=1}^{m-1} d_{I_\nu}\right)\sum_{K_\mu} N_{K_1}^{I_1I_2}N_{K_2}^{K_1I_3}\dots N_{K_{m-2}}^{K_{m-3}I_{m-1}}$$
$$\cdot\, d_{K_{m-2}}^{-1}\omega_{YM}(\chi^{K_{m-2}}(P)\chi^{I_m}(P_m))\,.$$

Here $P=\bigcup_{\nu=0}^{m-1}P_\nu$ and we used $d_0=1$ and $N_K^{0J}=\delta_{J,K}$. We stopped the calculation before we integrate over the last link on the graph which separates the two plaquettes P and P_m. From the definition of characteristic projectors and Eq. (3.22) one infers $\chi^K(P)=\chi^{\bar K}(P_m)$. Using the property $\chi^K(P_m)\chi^L(P_m)=\delta_{K,L}\chi^K(P_m)$ we can treat the remaining expectation value as follows:

$$\begin{aligned} d_{K_{m-2}}^{-1}\omega_{YM}(\chi^{K_{m-2}}(P)\chi^{I_m}(P_m)) &= \delta_{I_m,\bar K_{m-2}}d_{I_m}^{-1}\omega_{YM}(\chi^{I_m}(P_m))\\ &= \delta_{I_m,\bar K_{m-2}}d_{I_m}^{-1}\mathcal{N}^{-4}\omega(\mathcal{N}d_{I_m}S_{I_mJ}c^J)\\ &= N_0^{K_{m-2}I_m}\mathcal{N}^{-2}d_{I_m}\,. \end{aligned}$$

We made use of the normalization of ω_{YM} on a graph with two plaquettes, the definition (3.22) of characteristic projectors, the property $\omega(c^I)=\delta_{I,0}$ and properties of the S-matrix. The result implies for the volume

$$\omega_{YM}(e_{CS}^{\{I_\nu\}}) = \left(\prod_{\nu=1}^{m} d_{I_\nu}\right)\sum_{K_\mu} N_{K_1}^{I_1I_2}N_{K_2}^{K_1I_3}\cdots N_0^{K_{m-2}I_m}\mathcal{N}^{-2}\,.$$

We want to rewrite this using properties of the matrix S. To this end we insert $N_0^{K_{m-2}I_m}=\sum S_{K_{m-2}J}S_{JI_m}$ and move the first S through the product of fusion matrices. This results is

$$\omega_{YM}(e_{CS}^{\{I_\nu\}}) = \sum_J d_J^{2-m}\left(\prod_{\nu=1}^{m}\frac{d_{I_\nu}}{\mathcal{N}}S_{JI_\nu}\right)\,. \tag{5.9}$$

Example 2. Genus 1. The standard graph $G_{1,m}$ has again $m+1$ plaquettes. But this time the plaquette P_0 is bounded by the links l_ν as well as by a- and b-cycles on the torus. Let us merge step by step all plaquettes into one and call it P. In this way we erase all l_ν links so that the boundary of P looks as $ba^{-1}b^{-1}a$. Using the same arguments as in the first example we see that

$$\omega_{YM}(e_{CS}^{\{I_\nu\}}) = \left(\prod_{\nu=1}^{m} d_{I_\nu}\right) \sum_{K_\mu} N_{K_1}^{I_1 I_2} N_{K_2}^{K_1 I_3} \ldots N_{K_{m-1}}^{K_{m-2} I_m} d_{K_{m-1}}^{-1} \omega_{YM}(\chi^{K_{m-1}}(P)) . \quad (5.10)$$

Observe that the boundary of P contains every link twice so that the evaluation of $\omega(\chi^K(P))$ is quite nontrivial. Before one can integrate over the degrees of freedom assigned to a particular link on the graph, one has to ensure that this link appears only once and only in one orientation in the integrand. This can be done with the help of exchange relations and functoriality. Let us calculate the expectation value of $c^J(P)$ first. To this end we insert the formula (3.19) for $c^J(P)$ and invert the orientation of the $\{-a, -b\}$ in the middle. This is done with the help of Eq. (4.9). Next functoriality can be applied on the link b which then allows to perform the integration on b. In formulas this is

$$\begin{aligned}
\omega(c^J(P)) &= \kappa_J^4 \omega(\mathrm{tr}_q^J(M^J(b)M^J(-a)M^J(-b)M^J(a))) \\
&= d_J \omega((\mathrm{tr}^J \otimes \mathrm{tr}^{\bar{J}})[\overset{1}{M}{}^J(b) R^{J\bar{J}} \overset{2}{M}{}^{\bar{J}}(b) \overset{2}{M}{}^{\bar{J}}(a) (R')^{J\bar{J}} C[J\bar{J}|0]^* \\
&\quad \cdot C[J\bar{J}|0] \overset{1}{M}{}^J(a) \overset{1}{g}{}^J]) \\
&= d_J \omega((\mathrm{tr}^J \otimes \mathrm{tr}^{\bar{J}})[C[J\bar{J}|0]^* C[J\bar{J}|0] \overset{2}{M}{}^{\bar{J}}(a) (R')^{J\bar{J}} C[J\bar{J}|0]^* \\
&\quad \cdot C[J\bar{J}|0] \overset{1}{M}{}^J(a) \overset{1}{g}{}^J]) \\
&= \kappa_J^2 \omega((\mathrm{tr}^J \otimes \mathrm{tr}^{\bar{J}})[C[J\bar{J}|0]^* \mathrm{tr}_q^{\bar{J}}(M^{\bar{J}}(a)) C[J\bar{J}|0] \overset{1}{M}{}^J(a) \overset{1}{g}{}^J]) .
\end{aligned}$$

In this expression we inserted the definition (2.11) of the q-trace. Now we can integrate on the link a. The formula

$$\mathrm{tr}_q^{\bar{J}}(M^J(a)) M^J(a) = \sum \mathrm{tr}_q^{\bar{J}}[(R^{-1})^{\bar{J}J} C[\bar{J}J|K]^* M^K(a) C[\bar{J}J|K]]$$

follows from functoriality and was derived earlier in Sect. 3. It shows that

$$\omega(\mathrm{tr}_q^{\bar{J}}(M^J(a)) M^J(a)) = \mathrm{tr}_q^{\bar{J}}[v_J^{-1} \overset{2}{g}{}^J C[\bar{J}J|K]^* C[\bar{J}J|K]] = v_J^{-1} d_J^{-1} .$$

We may insert this into our expression for $\omega(c^J(P_0))$ which then becomes

$$\begin{aligned}
\omega(c^J(P)) &= d_J^{-1} (\mathrm{tr}^J \otimes \mathrm{tr}^{\bar{J}})[C[J\bar{J}|0]^* C[J\bar{J}|0] \overset{1}{g}{}^J] \\
&= d_J^{-2} \mathrm{tr}_q^J(e^J) = d_J^{-1} .
\end{aligned} \quad (5.11)$$

With the normalization of ω_{YM} on a graph having only one plaquette, we find

$$\omega_{YM}(\chi^{K_{m-1}}(P)) = \mathcal{N}^{-1} \sum_J d_{K_{m-1}} S_{K_{m-1}J} (d_J)^{-1} .$$

When this is finally plugged into the formula above, we can write an expression for the volume,

$$\begin{aligned}\omega_{YM}(e_{CS}^{\{I_\nu\}}) &= \sum_J \left(\prod_{\nu=1}^m d_{I_\nu}\right) \sum_{K_\mu} N_{K_1}^{I_1 I_2} N_{K_2}^{K_1 I_3} \dots N_{K_{m-1}}^{K_{m-2} I_m} S_{K_{m-1} J} (\mathcal{N} d_J)^{-1} \\ &= \sum_J d_J^{-m} \left(\prod_{\nu=1} \frac{d_{I_\nu}}{\mathcal{N}} S_{J I_\nu} \right) , \end{aligned} \tag{5.12}$$

where the last line employs the same type of algebra described in the first example.

Now we are sufficiently prepared to deal with the general case.

Proposition 14 (Volume of the moduli space). *The volume of the quantum moduli space of flat connections on a compact Riemann surface $\Sigma_{g,m}$ of genus g and with m punctures marked by I_ν, $\nu = 1,\dots,m$ evaluated with the Yang–Mills measure is given through*

$$\omega_{YM}(e_{CS}^{\{I_\nu\}}) = \sum_J d_J^{2-2g-m} \left(\prod_{\nu=1} \frac{d_{I_\nu}}{\mathcal{N}} S_{J I_\nu} \right) . \tag{5.13}$$

Proof. The proof of this formula is again done with a calculation on the standard graph $G_{g,m}$. The latter has $m+1$ plaquettes. When we erase all links l_ν we are left with the plaquette P which is bounded by a combination of a- and b-cycles which corresponds to Eq. (4.8). We designed the proof for the $g = 1$ case in such a way, that it can be applied directly to the higher genus. We leave this to the reader. Let us just do the power counting for d_J. A generalization of formula (5.11) for the value of $\omega(c^J(P))$ shows that every pair (a_i, b_i) of a- and b-cycles contributes with a factor d_J^{-2} until only $\mathrm{tr}_q^J(e^J)$ is left. So the g pairs (a_i, b_i) together with $\mathrm{tr}_q^J(e^J) = d_J$ give rise to d_J^{1-2g}, i.e.

$$\omega(c^J(P)) = d_J^{1-2g} .$$

Compared to the result for the torus, this gives an extra factor of d_J^{2-2g} in the final formula for the volume of the moduli space.

5.3. Canonical Measure and Verlinde Formula. The functional ω_{YM} that we have discussed so far had the fundamental properties of being gauge invariant and graph independent. These invariances fix the functional only up to a coefficient which may depend on the genus g, the number m of marked points, the labels I_ν, $\nu = 1,\dots,m$ at the punctures and on the deformation parameter q. Actually, there exists a canonical normalization of the measure. This is used in the Chern Simons theory and we call it ω_{CS}. The different normalizations of ω_{CS} and ω_{YM} can be encoded in the following relation:

$$\omega_{CS} = \lambda(g, m, I_1, \dots, I_m, q)\omega_{YM} . \tag{5.14}$$

The aim of this subsection is to explain the choice of the positive coefficients λ.

One can define the canonical normalization in two different ways. The first approach is through the representation theory of the moduli algebra. Assume for a moment that the latter is finite dimensional (this is indeed the case for q being a root of unity). As a $*$-algebra with positive inner product, the moduli algebra is semi-simple and splits into a direct sum of matrix algebras. One can fix the canonical functional ω_{CS} by the requirement that – when restricted to a simple summand – it coincides with the usual matrix trace. This approach furnishes a proper definition for ω_{CS} which is fundamental for the considerations to follow below. On the other hand, a direct computation of the canonical normalization from this definition requires the full information about the representation theory. This is the subject of the subsequent paper [14] where the representation theory of the moduli algebra is considered in detail.

Another approach refers to the theory of deformation quantization. Treating the deformation parameter q as an exponent $q = \exp(h)$ of the Planck constant, one can expand the commutation relations of the moduli algebra into formal power series in h and identify this picture with deformation quantization of the moduli space (for more details see [24]). According to the theorem of Tsygan and Nest [23] there exists a unique canonical trace in the framework of deformation quantization. We conjecture that this trace coincides with an expansion into the formal power series in h of the canonical functional ω_{CS} defined via the representation theory.

Both ways to normalize the functional ω include some complicated analysis which is beyond the scope of this paper. Instead, we plan to describe a different way to determine the coefficients $\lambda(g, m, I_1, \ldots, I_m, q)$ which characterize the canonical functional ω_{CS}. This computation is based on several suggestive properties of the moduli algebra which are more natural to prove in the context of the paper [14]. We formulate these properties as theorems labeled by latin letters. Assuming validity of these theorems, our method gives a derivation of the canonically functional ω_{CS}. Actually, we are going to combine the ideology of the Topological Field Theory and the algebraic approach of this paper.

Let us introduce some new notations first. Using the graph algebra corresponding to some standard graph, one can assign a bunch of matrix generators $M^I(x)$ to each cycle x on a Riemann surface. As we have discussed above, they furnish elements $c^I(x)$ – one for every label I – when they are evaluated with the q-trace. For the homologically trivial cycle x, the corresponding elements $c^I(x)$ belong to the center of the graph algebra. If the cycle x is nontrivial, they are no longer central. Their algebraic relations, however, remain those of a fusion algebra, i.e.

$$c^I(x)c^J(x) = \sum_K N^{IJ}_K c^K(x), \qquad (c^I(x))^* = c^{\bar{I}}(x)\,, \tag{5.15}$$

for arbitrary cycle x. We denote this algebra by $\mathscr{V}(x)$.

Suppose that $\mathscr{X}$ be a subalgebra of an algebra $\mathscr{Y}$. The (relative) commutant of $\mathscr{X} \in \mathscr{Y}$ will be denoted by $\mathscr{C}(\mathscr{X}, \mathscr{Y})$.

Let $\mathscr{A}^{\{I_1,\ldots,I_m\}}_{g,m}$ be a moduli algebra corresponding to a Riemann surface of genus g and with m marked points. Consider the a-cycle a_g of some standard graph and the fusion algebra $\mathscr{V}(a_g)$. Without proof we state

Theorem A (Induction in the genus g). *The commutant $\mathscr{C}(\mathscr{V}(a_g), \mathscr{A}^{\{I_1,\ldots,I_m\}}_{g,m})$ splits into the direct sum of moduli algebras of genus $g-1$ with $m+2$ marked points*

$$\mathscr{C}(\mathscr{V}(a_g), \mathscr{A}^{\{I_1,\ldots,I_m\}}_{g,m}) \cong \bigoplus_I \mathscr{A}^{\{I_1,\ldots,I_m,I,\bar{I}\}}_{g-1,m+2}\ . \tag{5.16}$$

Here the sum runs over all classes of irreducible representations of the symmetry Hopf algebra.

In the language of Topological Field Theory, evaluation of the commutant corresponds to shrinking the cycle a_g so that we get a surface of lower genus. It has two marked points at the place where the handle is pinched – one on either side of the cut. Shrinking all the a-cycles one after another, one produces spheres with $m+2g$ marked points.

We apply a similar technique to reduce the number of marked points. Consider the moduli algebra $\mathscr{A}_{0,m}^{\{I_1,\dots,I_m\}}$ corresponding to a sphere with m marked points. Pick up a cycle $l = l_{m-1} \circ l_m$ (i.e. the product of the two elementary loops l_{m-1} and l_m) and construct the fusion algebra $\mathscr{V}(l)$. As before, we investigate the commutant of $\mathscr{V}(l)$ in $\mathscr{A}_{0,m}^{\{I_1,\dots,I_m\}}$. The result is given by the following theorem.

Theorem B (Induction in the number m of punctures). *The commutant of $\mathscr{V}(l)$ in $\mathscr{A}_{0,m}^{\{I_1,\dots,I_m\}}$ splits into the direct sum of products of moduli algebras corresponding to $m-1$ and 3 marked points*

$$\mathscr{C}(\mathscr{V}(l), \mathscr{A}_{0,m}^{\{I_1,\dots,I_m\}}) \cong \bigoplus_I \mathscr{A}_{0,m-1}^{\{I_1,\dots,I_{m-2},I\}} \otimes \mathscr{A}_{0,3}^{\{I,I_{m-1},I_m\}} . \tag{5.17}$$

Here the sum runs over all classes of irreducible representations of the symmetry Hopf algebra.

In Topological Field Theory the evaluation of the commutant should be interpreted as a fusion of two marked points into one. As before, one can imagine that we create a long neck which separates these two points from the rest of the surface. When we cut the neck, the surface splits into two pieces. The "main part" carries the rest of the marked points and a new one created by the cut. The other piece has only three punctures, two of them are those that we wish to fuse and the new one appears because of the cut. Iteration of this procedure results in a product of 3-punctured spheres.

In the following we will need two simple consequences of Theorems A and B. Namely, the decomposition of the commutant of the fusion algebra determines the following decompositions of the unit:

$$e_{g,m}^{\{I_1,\dots,I_m\}} = \sum_I e_{g-1,m+2}^{\{I_1,\dots,I_m,I,\bar{I}\}} ,$$

$$e_{0,m}^{\{I_1,\dots,I_m\}} = \sum_I e_{0,m-1}^{\{I_1,\dots,I_m,I\}} \otimes e_{0,3}^{\{\bar{I},I_{m-1},I_m\}} .$$

Here $e_{g,m}^{\{I_\nu\}}$ denotes the unit of the moduli algebra $\mathscr{A}_{g,m}^{\{I_\nu\}}$. The equations are a simple consequence of the completeness of characteristic projectors, i.e. of $\sum_I \chi^I(x) = \mathrm{id}$.

Now we can turn back to the discussion of the canonical functional ω_{CS}. By definition, it is supposed to coincide with a usual matrix trace. When the standard properties of a matrix trace are combined with the decomposition formulas for the units $e_{g,m}^{\{I_1,\dots,I_m\}}$, one arrives at

$$\omega_{CS}(e_{g,m}^{\{I_1,\dots,I_m\}}) = \sum_I \omega_{CS}(e_{g-1,m+2}^{\{I_1,\dots,I_m,I,\bar{I}\}}) ,$$

$$\omega_{CS}(e_{0,m}^{\{I_1,\dots,I_m\}}) = \sum_I \omega_{CS}(e_{0,m-1}^{\{I_1,\dots,I_m,I\}}) \omega_{CS}(e_{0,3}^{\{\bar{I},I_{m-1},I_m\}}) .$$

By iterations, these equations reduce the evaluation of $\omega_{CS}(e_{g,m}^{\{I_\nu\}})$ to the knowledge of $\omega_{CS}(e_{0,3}^{I,J,K})$. This has a simple geometric interpretation. The algebraic information in Theorem A and B about the decomposition of the commutant and the definition of the canonical functional ω_{CS} as the standard matrix trace mean that the *volume of the quantum moduli space does not change under pinching* when it is evaluated with the canonical measure ω_{CS}. Using this invariance, one can first decompose the moduli space according to the decomposition of a punctured Riemann surface into 3-punctured spheres and then calculate the volume directly as a function of the volume of the quantum moduli-space assigned to the 3-punctured sphere, i.e. as a function of $\omega_{CS}(e_{0,3}^{I,J,K})$.

In the algebraic context, the numbers $\omega_{CS}(e_{0,3}^{I,J,K})$ have to be determined from the representation theory. For this we anticipate the following simple result (a proof will be given in [14]).

Theorem C (3-punctured sphere). *The moduli algebra $\mathscr{A}_{0,3}^{I,J,K}$ corresponding to a 3-punctured sphere is isomorphic to a full matrix algebra of the dimension $d = N_K^{IJ}$.*

This means that the canonical functional ω_{CS} obeys

$$\omega_{CS}(e_{0,3}^{I,J,K}) = N_K^{IJ} \ .$$

Putting all this together, the normalization of ω_{CS} is completely determined, i.e. $\omega_{CS}(e_{g,m}^{\{I_\nu\}})$ can be calculated. After rewriting everything in terms of the matrix S, the result is

$$\omega_{CS}(e_{CS}^{\{I_\nu\}}) = \mathscr{N}^{2-2g} \sum_J d_J^{2-2g-m} \left(\prod_{J=1}^{m} \frac{1}{\mathscr{N}} S_{JI_\nu} \right) \ . \tag{5.18}$$

In particular, for a surface of genus g without marked points one recovers the famous Verlinde formula,

$$\omega_{CS}(e_g) = \mathscr{N}^{2-2g} \sum_J d_J^{2-2g-m} \ . \tag{5.19}$$

These formulas have to be compared with the corresponding formula for ω_{YM} in Proposition 14. From this comparison one infers

$$\lambda(g, m, I_1, \dots, I_m, q) = \mathscr{N}^{2-2g} \left(\prod_i^m d_{I_i} \right)^{-1} \ . \tag{5.20}$$

Let us note that the Verlinde formula has been designed to compute the number of conformal blocks in the WZW model. Witten related the space of conformal bolcks to the space of holomorphic sections of the quantum line bundle over the moduli space of flat connections. The latter is a natural Hilbert space associated to quantization of the moduli space by geometric quantization. We deal with another quantization scheme which associates to the moduli space a quantized algebra of functions. In this approach, the Hilbert space arises as a representation space of the moduli algebra. Actually, the canonical trace functional evaluated at the unit element of the algebra provides a dimension provided by the Verlinde formula. Equation (5.19) proves that this is indeed the case.

We get another interpretation of formula (5.19) if we switch to the point of view of the two-dimensional lattice model. Then the canonical trace gives a partition function of the lattice gauge model with the quantum gauge group. As this partition function coincides with the Verlinde number, it is natural to conjecture that the lattice gauge model at hand is an exact lattice approximation of the gauged WZW model (also called the G/G model). This is supported by the fact that the gauged WZW model is equivalent to the CS model on the manifold which is a product of a Riemann surface and a circle. To establish the equivalence one should consider arbitrary correlation functions. The partition function is the simplest among them. We do not go into detailed consideration in this paper.

6. Generalization to Quasi-Hopf Symmetries

All the theory developed above was valid under the assumption that the symmetry algebras $\mathcal{G}_x$ are semisimple. It is well known that this requirement is not satisfied for the quantum group algebras $U_q(\mathcal{G})$ when q is a root of unity. To treat this important case we proposed (cp. [1]) to use the semisimple truncation of $U_q(\mathcal{G})$, $q^p = 1$, which has been constructed in [5]. In this truncation, semisimplicity is gained in exchange for co-associativity, i.e. the truncated $U_q^T(\mathcal{G})$ of [5] are only quasi-co-associative. In addition, the co-product Δ of these truncated structures is not unit preserving (i.e. $\Delta(e) \neq e \otimes e$). This leads to a generalization of Drinfeld's axioms [10] which was called "weak quasi-Hopf-algebra" in [5]. To fulfill our program, we have to explain how the theory of Sects. 2 to 5 generalises to (weak) quasi-Hopf algebras. In order not to clutter the presentation, we decided to outline proofs without conceptual significance in a separate appendix.

In the quasi-Hopf context, there appear three additional distinguished elements associated with the local gauge symmetry $\mathcal{G}_x$. These are the elements $\alpha_x, \beta_x \in \mathcal{G}_x$ and the re-associator $\varphi_x = \mathcal{G}_x \otimes \mathcal{G}_x \otimes \mathcal{G}_x$. The re-associator φ_x satisfies the fundamental relation

$$\varphi_x(\Delta_x \otimes \mathrm{id})\Delta_x(\xi) = (\mathrm{id} \otimes \Delta_x)\Delta_x(\xi)\varphi_x \quad \text{for all } \xi \in \mathcal{G}_x$$

and is quasi-invertible in the sense that

$$\varphi_x\varphi_x^{-1} = (\mathrm{id} \otimes \Delta_x)\Delta_x(e), \qquad \varphi_x^{-1}\varphi_x = (\Delta_x \otimes \mathrm{id})\Delta_x(e)\,.$$

Following Drinfel'd, α_x, β_x are required to obey

$$\mathcal{S}_x(\xi_\sigma^1)\alpha_x\xi_\sigma^2 = \alpha_x\varepsilon_x(\xi), \qquad \xi_\sigma^1\beta_x\mathcal{S}_x(\xi_\sigma^2) = \beta_x\varepsilon_x(\xi)\,,$$

$$\varphi_{x\sigma}^1\beta_x\mathcal{S}_x(\varphi_{x\sigma}^2)\alpha_x\varphi_{x\sigma}^3 = e, \qquad \mathcal{S}_x(\phi_{x\sigma}^1)\alpha_x\phi_{x\sigma}^2\beta_x\mathcal{S}_x(\phi_{x\sigma}^3) = e\,,$$

with $\xi_\sigma^l, \varphi_{x\sigma}^l, \phi_{x\sigma}^l$ being defined through the expansions of $\Delta_x(\xi), \varphi_x, \phi_x = \varphi_x^{-1}$ as usual. As remarked in [1], consistency with the $*$-operation means

$$\alpha_x^* = \beta_x, \qquad \varphi_x^* = \varphi_x\,.$$

Details and further relations can be found elsewhere (see e.g. [1] and references therein).

An element $u_x \in \mathcal{G}_x$ is defined by relations similar to (2.1),

$$\mathcal{S}_x(\alpha_x)u_x = \mathcal{S}_x(r_{x\sigma}^2)\alpha_x r_{x\sigma}^1, \qquad \beta_x u_x = r_{x\sigma}^2\mathcal{S}_x^{-1}(r_{x\sigma}^1\beta_x)\,. \tag{6.1}$$

Notice that the second equation follows from the first by taking adjoints. u_x and the ribbon element $v_x \in \mathcal{G}_x$ continue to satisfy $v_x^2 = u_x\mathcal{S}_x(u_x)$. All other relations (2.2, 2.3) of the ribbon element remain true as well. As for ribbon-Hopf algebras, the product $g_x = u_x^{-1} v_x$ is unitary and enjoys the intertwining relation $g_x \mathscr{S}_x(\xi) = \mathscr{S}_x^{-1}(\xi) g_x$. On the other hand, g_x is no longer grouplike. The correct generalization of Eq. (2.5) can be found in Appendix A.

The representation theoretic statements and notations outlined Sect. 2 carry over to the more general situation. In particular, Clebsch Gordon maps $C_x^a[IJ|K]$ are defined and normalized by the relations (2.6, 2.7). Starting with the definition (2.10) of quantum dimensions, the theory is again subject to changes. It is possible to describe them in a very economic way. Indeed, the whole theory in Sects. 2 and 3 can be rewritten for quasi-Hopf-algebras with the help of a small number of "substitution rules." We collect these rules in the following table:

$$R_x^{IJ} \to \mathscr{R}_x^{IJ} = (\tau_x^I \otimes \tau_x^J \otimes \mathrm{id})(\mathscr{R}_x) = (\tau_x^I \otimes \tau_x^J \otimes \mathrm{id})((\varphi_{213} R_{12} \varphi^{-1})_x) ,$$

$$C_x^a[IJ|K] \to C_x^a[IJ|K](\varphi_x^{-1})^{IJ} = C_x^a[IJ|K](\tau_x^I \otimes \tau_x^J \otimes \mathrm{id})(\varphi_x^{-1}) ,$$

$$C_y^a[IJ|K]^* \to (\varphi_y')^{IJ} C_y^a[IJ|K]^* \text{with} \quad \omega' = \omega_{213} \quad \text{for all} \quad \omega \in \mathscr{G}_y \otimes \mathscr{G}_y \otimes \mathscr{G}_y ,$$

$$d_K \to d_K \equiv \mathrm{tr}^K(\tau_x^K(g_x \mathscr{S}_x(\beta_x)\alpha_x)) ,$$

$$R_x^I \to R_x^I = (\tau_x^1 \otimes \mathrm{id}) R_x, \qquad (R_y^{-1})^I \to (R_y^{-1})^I . \tag{6.2}$$

Before we give some examples of formulas we obtain with these rules, we have to release a *warning*. Some of the formulas of the preceding sections were obtained with the help of Lemma 1. A similar lemma holds for the quasi-Hopf case, but it is *not* obtained from Lemma 1 and the substitutions (6.2). Lemma 1 was used above to simplify a number of expressions. As a result of these simplifications, factors g_x appeared in several formulas. The substitution rules (6.2) should never be used in equations containing a factor g_x. Instead one has to depart from the ancestors of such relations (notice that we gave no substitution rule for g_x!).

Let us consider the definition of the "deformed trace" (2.11) as a first example of the substitution rules (6.2). In our present context the definition becomes

$$\mathrm{tr}_q^K(X) = \frac{d^K}{v^K} C_x[\bar{K}K|0](\varphi_x^{-1})^{\bar{K}K} \overset{2}{X} (\mathscr{R}_x')^{\bar{K}K} (\varphi_x')^{\bar{K}K} C_x[\bar{K}K|0]^* . \tag{6.3}$$

In Sect. 2 this formula was rewritten with the help of Lemma 1. A generalization of this lemma has been announced already.

Lemma 4. *The map* $\hat{C}_x[K\bar{K}|0] \equiv C_x[K\bar{K}|0] \overset{2}{\tau}{}_x^{\bar{K}}(\alpha_x^{-1})$ *and its adjoint* $\hat{C}_x[K\bar{K}|0]^* = \overset{2}{\tau}{}_x^{\bar{K}}(\beta_x^{-1}) C_x[K\bar{K}|0]^*$ *satisfy the following equations*:

1. *For all* $\xi \in \mathscr{G}_x$ *they obey the intertwining relations*

$$\hat{C}_x[K\bar{K}|0](\tau_x^K(\xi) \otimes \mathrm{id}) = \hat{C}_x[K\bar{K}|0](\mathrm{id} \otimes \tau_x^K(\mathscr{S}_x(\xi))) ,$$

$$(\tau_x^K(\xi) \otimes \mathrm{id}) \hat{C}_x[K\bar{K}|0]^* = (\mathrm{id} \otimes \tau_x^K(\mathscr{S}_x(\xi))) \hat{C}_x[K\bar{K}|0]^* . \tag{6.4}$$

2. *With the normalization conventions* (2.7) *one finds*

$$d_K \mathrm{tr}^{\bar{K}}(\hat{C}_x[K\bar{K}|0]^* \hat{C}_x[K\bar{K}|0]) = e_x^K ,$$

$$d_K \mathrm{tr}^{\bar{K}}(\hat{C}_x[\bar{K}K|0]^* \hat{C}_x[K\bar{K}|0]) = e_x^K . \tag{6.5}$$

With Lemma 4, Definition (6.3) of tr_q^K simplifies to

$$\mathrm{tr}_q^K(X) = d_K \mathrm{tr}_q^K(m_x^K X w_x^K g_x^K) \text{ with } \quad m_x^K = \tau_x^K(\mathcal{S}_x(\phi_{x\sigma}^1)\alpha_x \phi_{x\sigma}^2)\phi_{x\sigma}^3$$

$$\text{and} \quad w_x^K = \tau_x^K(\varphi_{x\sigma}^2 \mathcal{S}_x^{-1}(\varphi_{x\sigma}^1 \beta_x))\varphi_{x\sigma}^3 \, . \tag{6.6}$$

Here $\varphi_x^1 = \sum \phi_{x\sigma}^1 \otimes \phi_{x\sigma}^2 \otimes \phi_{x\sigma}^3$ is used in the second line. Equation (6.6) should be regarded as the analogue of formula (2.14).

The substitution rules for the defining relations of $\mathcal{B}$ are straightforward to implement. The resulting algebra is identical to the one introduced in [1]. Comparison with our formulation in [1] is mostly obvious. One has to recall, however, that we used the generators $\hat{U}^I(i) \equiv m_y^I U^I(i)$– with m_y^I given by the expression in (6.6) – instead of generators $U^I(i)$ (cp. the remark after Proposition 17 in [1]). Let us make some specific remarks concerning functoriality on the link. Within our present formulation it becomes

$$\overset{1}{U}{}^I(i)\overset{2}{U}{}^J(i) = \sum_{K,a} (\varphi'_y)^{IJ} C_y^a[IJ|K]^* U^K(i) C_x^a[IJ|K](\varphi_x^{-1})^{IJ} \, ,$$

$$U^I(i)U^I(-i) = e_y^I, \qquad U^I(-i)U^I(i) = e_x^I \, .$$

An argument similar to the proof of Proposition 2.4 reveals the following relation between $U^I(i)$ and $U^I(-i)$:

$$m_x^I U^I(-i) = d_I \mathrm{tr}^{\bar{I}}[\hat{C}[\bar{I}I|0]^* \hat{C}[\bar{I}I|0] \overset{1}{g}{}^{\bar{I}} \overset{1}{m}{}_y^{\bar{I}} \overset{1}{U}{}^{\bar{I}}(i)] \, . \tag{6.7}$$

This shows that $\hat{U}^I(-i) \equiv m_x^I U^I(-i)$ and $\hat{U}^{\bar{I}}(i)$ are complex linear combinations of each other. The latter fact was used in [1] to implement functoriality on the link i.

In spite of its compact appearance, the matrix formulation for $\mathcal{B}$ has a major drawback. This becomes apparent when one tries to construct elements in the algebra $\mathcal{A}$. It is crucial to notice that the algebra $\mathcal{A}$ (as it was defined e.g. in the case of Hopf-algebras) does not contain all invariants in $\mathcal{B}$ but only invariants in a special subset $\langle U \rangle \subset \mathcal{B}$. This subset was easily described in the situation of Hopf-algebras, but a similar matrix-description for the quasi-Hopf case does not exist. Our strategy is now as follows: for a moment we will switch to the "vector notations" of [1] so that we can construct elements in $\mathcal{A}$ by the prescription given there. Applied to elements on curves $\mathcal{C}$, the general prescription will indeed reduce to the ordinary matrix product of generators $U^I(i)$. Let us first recall the general procedure: We fix a basis within every representation space V^I. Then:

1. Regard $\mathcal{B}$ to be generated by elements $\xi \in \mathcal{G}$ and $\hat{U}^I(i) \equiv m_y^I U^I(i)$. The elements $\hat{U}^I(i)$ transform according to $\xi \hat{U}^I(i) = \hat{U}^I(i)(\bar{\tau}_y^I \otimes \mathrm{id})(\Delta_y(\xi))$ for all $\xi \in \mathcal{G}_y$ and as in Eq. (2.15) for all other elements in $\mathcal{G}$. Here $\bar{\tau}_x^I(\xi) = {}^t\, \tau_x^I(\mathcal{S}_x(\xi))$ with ^{t} being the transpose w.r.t. the fixed basis in V^I.

2. Construct the (linear) set of all covariant products obtained from $\hat{U}^I(i)$. Covariant products were defined as follows ([5,4]). Suppose that $F^\nu \in \mathrm{End}(V^\nu) \otimes \mathcal{B}$, $\nu = 1, 2$, transform covariantly according to the representations τ^ν of $\mathcal{G}$, i.e.

$\xi F^v = F^v(\tau^v \otimes \mathrm{id})(\Delta(\xi))$ for all $\xi \in \mathcal{G}$. Then their covariant product is an element $F^1 \times F^2 \in \mathrm{End}(V^1) \otimes \mathrm{End}(V^2) \otimes \mathcal{B}$ defined by

$$F^1 \times F^2 \equiv F^1 F^2 (\tau^1 \otimes \tau^2 \otimes \mathrm{id})(\varphi) .$$

3. With the help of Clebsch Gordon maps one can finally build invariants within the set covariant products. These invariants are the elements of $\mathcal{A}$.

This procedure is now applied to elements which live on curves $\mathscr{C}$. Since the construction is "local," i.e. it can be performed independently at all the sites x on the curve, it suffices to consider one site x. As usual, we have two links i, j with $t(i) = x = t(-j)$. In the following calculation we will omit the subscripts z. It is understood that all objects which come with the gauge symmetry are assigned to the site x,

$$\begin{aligned} &\hat{U}^I_{ab}(i)\hat{U}^I_{ce}(j)(\tau^I_{bf} \otimes \bar{\tau}^I_{cg} \otimes \mathrm{id})(\varphi)\bar{\tau}^I_{gf}(\mathcal{S}^{-1}(\beta)) \\ &\sim U^I_{ab}(i) m^I_{cd} U^I_{de}(j)(\tau^I_{bf} \otimes \bar{\tau}^I_{cg} \otimes \mathrm{id})(\varphi)\bar{\tau}^I_{gf}(\mathcal{S}^{-1}(\beta)) \\ &= U^I_{ab}(i) m^I_{cd} U^I_{de}(j)\bar{\tau}^I_{cb}(\varphi^2_\tau \mathcal{S}^{-1}(\varphi^1_\tau \beta))\varphi^3_\tau \\ &= U^I_{ab}(i) m^I_{cd} U^I_{de}(j) w^I_{cb} = U^I_{ab}(i) U^I_{be}(j) . \end{aligned}$$

Here $\sim$ means "up to contributions at sites $y \neq x$." The coefficients $\bar{\tau}^I_{gf}(\mathcal{S}^{-1}(\beta))$ ensure invariance of the expression at the site x and Proposition 17.2 of [1] was employed for the last equality. From the previous calculation we learn that the matrix products of elements $U^I(i)$ furnish the right prescription to construct Wilson line observables within our theory. So we define $U^I(\mathscr{C})$ as before by

$$U^I(\mathscr{C}) \equiv \kappa_I^{w(\mathscr{C})} U^I(i_1) \cdots U^I(i_n) ,$$

and use $M^I(\mathscr{C})$ instead of $U^I(\mathscr{C})$ whenever $\mathscr{C}$ is closed (and satisfies the other assumptions specified in Sect. 3). Properties of $U^I(\mathscr{C})$ and $M^I(\mathscr{C})$ are obtained from Propositions 2 and 3 together with the substitution rules (6.2). The algebra of monodromies $M^I(\mathscr{C})$ on the loop $\mathscr{C}$ reads for example

$$\overset{1}{M}{}^I(\mathscr{C}) \mathscr{R}^{IJ}_x \overset{2}{M}{}^J(\mathscr{C}) = \sum (\varphi'_y)^{IJ} C^a_x[IJ|K]^* M^K(\mathscr{C}) C^a_x[IJ|K](\varphi_x^{-1})^{IJ} \tag{6.8}$$

$$M^I(\mathscr{C}) M^I(-\mathscr{C}) = e^I_x, \qquad M^I(-\mathscr{C}) M^I(\mathscr{C}) = e^I_x , \tag{6.9}$$

$$(M^I(\mathscr{C}))^* = \sigma_\kappa(R^I_x M^I(-\mathscr{C})(R^{-1}_x)^I) . \tag{6.10}$$

Under reversal of $\mathscr{C}$, the monodromies behave as

$$m^I M^I(-\mathscr{C}) = d_I \mathrm{tr}^{\bar{I}}[\hat{C}[\bar{I}I|0]^* \hat{C}[\bar{I}I|0] \overset{1}{g}{}^{\bar{I}} \overset{1}{m}{}^{\bar{I}} \overset{1}{M}{}^{\bar{I}}(\mathscr{C}) \mathscr{R}^{\bar{I}I}] . \tag{6.11}$$

This algebra can be regarded as a generalization of quantum enveloping algebras of simple Lie algebras (within the formulation of [11]).

The definition (3.13), our substitution rules (6.2) and formula (6.6) combine into the following expression for the elements c^I:

$$c^I \equiv \kappa_I \mathrm{tr}^I_q(M^I(\mathscr{C})) = \kappa_I \mathrm{tr}^I(m^I M^I(\mathscr{C}) w^I g^I) . \tag{6.12}$$

As before, the c^I do not depend on the starting point x of $\mathscr{C}$ and they satisfy the defining relations (3.14, 3.15) of a fusion algebra. The derivation of these properties is sketched in Appendix A. Once the fusion algebra is established, we can proceed exactly as in Sect. 3.2 to build the characteristic projectors $\chi^I(P)$. $\chi^0(P)$ continues to implement flatness (cp. Appendix A). In a generalization of Propositions 7 and 10 this shows that the theory does not depend on the choice of the graph G. The same holds true for the Chern–Simons functional ω_{CS}.

7. Appendix A: Proofs for Section 6

This Appendix contains some material which is used to prove the statements of Sect. 6. The derivation of the fusion algebra and the generalization of Proposition 7 are discussed in some detail.

To begin with we have to recall a number of results on quasi-Hopf algebras. For (co-associative) Hopf-algebras it is well known that $\Delta(\xi) = (\mathscr{S} \otimes \mathscr{S})\Delta'(\mathscr{S}^{-1}(\xi))$. A generalization of this fact was already noticed by Drinfel'd [10]. To state his observation we introduce the following notations:

$$\begin{aligned} \gamma &= \sum \mathscr{S}(U_\sigma)\alpha V_\sigma \otimes \mathscr{S}(T_\sigma)\alpha W_\sigma \\ &\quad \text{with} \quad \sum T_\sigma \otimes U_\sigma \otimes V_\sigma \otimes W_\sigma = (\varphi \otimes e)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi^{-1}) , \\ f &= \sum (\mathscr{S} \otimes \mathscr{S})(\Delta'(\phi^1_\sigma))\gamma\Delta(\phi^2_\sigma \beta \mathscr{S}(\phi^3_\sigma)) , \\ &\quad \text{with} \quad \varphi^{-1} = \sum \phi^1_\sigma \otimes \phi^2_\sigma \otimes \phi^3_\sigma \end{aligned} \tag{7.1}$$

Drinfel'd proved in [10] that the element f satisfies

$$\begin{aligned} f\Delta(\xi)f^{-1} &= (\mathscr{S} \otimes \mathscr{S})(\Delta'(\mathscr{S}^{-1}(\xi)) \quad \text{for all } \xi \in \mathscr{G} , \\ \gamma &= f\Delta(\alpha) . \end{aligned} \tag{7.2}$$

This remains true in the presence of truncation. The first equation asserts that f "intertwines" between the co-product Δ and the combination of Δ and $\mathscr{S}$ on the right-hand side.

When we perform this construction for the algebras $\mathscr{G}_x$ we end up with elements $f_x \in \mathscr{G}_x \otimes \mathscr{G}_x$. f_x appears in the expression for $\Delta_x(g_x) = \Delta_x(u_x^{-1}v_x)$,

$$\Delta_x(g_x) = g_x \otimes g_x(\mathscr{S}_x \otimes \mathscr{S}_x)(f_x'^{-1})f_x . \tag{7.3}$$

This is a generalization of Eq. (2.5) and shows that g_x is no longer grouplike.

Without proof we state a number of useful relations which follow from the basic axioms of a weak quasi-Hopf algebras (cp. [16]). With $\mathscr{R} = \varphi_{213}R_{12}\varphi^{-1}$ we have

$$[(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\varphi)]_{2314}(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\mathscr{R})(e \otimes \varphi^{-1}) = \mathscr{R}_{134}(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\mathscr{R} , \tag{7.4}$$

$$\varphi_{124}(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\mathscr{R})(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\varphi^{-1}) = [(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\mathscr{R})]_{1324}\mathscr{R}_{234} . \tag{7.5}$$

These two relations are in fact equivalent to Drinfeld's pentagon and hexagon equations. If we use $m = \mathcal{S}(\phi^1_\sigma)\alpha\phi^2_\sigma \otimes \phi^3_\sigma$ and $w = \varphi^2_\sigma \mathcal{S}^{-1}(\varphi^1_\sigma \beta) \otimes \varphi^3_\sigma$ as before then

$$\begin{aligned}&(\mathrm{id} \otimes \Delta)\Delta(\phi^3_\sigma)(\mathrm{id} \otimes \Delta)(w)(e \otimes w)(\mathcal{S}^{-1} \otimes \mathcal{S}^{-1})(f'(\phi^2_\sigma \otimes \phi^1_\sigma))\\ &\quad = \varphi(\Delta \otimes \mathrm{id})(w)\,,\end{aligned} \tag{7.6}$$

$$\begin{aligned}&f^{-1}(\mathcal{S} \otimes \mathcal{S})(\varphi^2_\sigma \otimes \varphi^1_\sigma)(e \otimes m)(\mathrm{id} \otimes \Delta)(m)(\mathrm{id} \otimes \Delta)\Delta(\varphi^3_\sigma)\\ &\quad = (\Delta \otimes \mathrm{id})(m)\varphi^{-1}\,.\end{aligned} \tag{7.7}$$

Here $f = \sum f^1_\sigma \otimes f^2_\sigma$ is the element (7.1) and $f' = \sum f^2_\sigma \otimes f^1_\sigma$. Of course, all these relations hold also "locally" at the sites x of the graph.

We are now prepared to prove the basic properties of the elements $c^I \in \mathcal{A}$ defined in (6.12).

Independence of x . The definition of c^I depends on the curve $\mathcal{C}$ on the boundary of the plaquette P through the site x at which $\mathcal{C}$ starts and ends. We want to show that c^I does not change, if we use the site $y \in \partial P$ instead of x. As in the proof of Proposition 4 we break the curve $\mathcal{C}$ at an arbitrary point y on ∂P and use the braid relations

$$\overset{1}{U}{}^I(\mathcal{C}^1)\mathcal{R}_y{}^{II}\overset{2}{U}{}^I(\mathcal{C}^2) = \overset{2}{U}{}^I(\mathcal{C}^2)(\mathcal{R}'_x)^{II}\overset{1}{U}{}^I(\mathcal{C}^1)\,. \tag{7.8}$$

From Eq. (7.4) one obtains

$$e \otimes m_z = (e \otimes \mathcal{S}_z(\rho^2_{z\sigma}) \otimes e)[(\mathrm{id} \otimes \Delta_z)(m_z)]_{213}\mathcal{R}_z(\rho^1_{z\sigma} \otimes \Delta_z(\rho^3_{z\sigma})) \tag{7.9}$$

with $\mathcal{R}_z = \sum \rho^1_{z\sigma} \otimes \rho^2_{z\sigma} \otimes \rho^3_{z\sigma}$. This relation and the covariance properties (2.15) furnish

$$\begin{aligned}&[(\mathcal{S}_x(\rho^2_{x\sigma}) \otimes e \otimes e)(m_x)_{13}]^{II}\overset{1}{U}{}^I(\mathcal{C}^1)\overset{2}{m}{}^I_y\overset{2}{U}{}^I(\mathcal{C}^2)[e \otimes \rho^1_{x\sigma} \otimes \rho^3_{x\sigma}]^{II}\\ &\quad = [(\mathcal{S}_y(\rho^2_{y\sigma}) \otimes e \otimes e)(m_y)_{13}]^{II}_{213}\overset{2}{U}{}^I(\mathcal{C}^2)\overset{1}{m}{}^I_x\overset{1}{U}{}^I(\mathcal{C}^1)[\rho^1_{y\sigma} \otimes e \otimes \rho^3_{y\sigma}]^{II}\,.\end{aligned}$$

Now we use the equation

$$\sum (\mathcal{S}_z(w^1_{z\sigma}) \otimes e)m_z\Delta_z(w^2_{z\sigma}) = \Delta_z(e)$$

to cancel the factors m_z in between the two factors U^I. Then one multiplies the two components of Eq. (7.10) and evaluates the trace tr^I of the product. To simplify the resulting expression, the formula

$$\sum \rho^1_{z\sigma}\mathcal{S}_z(\rho^2_{z\sigma}w^1_{z\tau}) \otimes \rho^3_{z\tau}w^2_{z\sigma} = w_z(\mathcal{S}_z(u_z) \otimes e)$$

is inserted. This leads to

$$\mathrm{tr}^I(m^I_x U^I(\mathcal{C}^1)U^I(\mathcal{C}^2)w^I_x g^I_x) = \mathrm{tr}^I(m^I_y U^I(\mathcal{C}^2)U^I(\mathcal{C}^1)w^I_y g^I_y)\,,$$

which establishes the independence of c^I on the choice of the site x on ∂P.

Fusion algebra. The derivation of the fusion algebra in the quasi-Hopf case departs from the "operator products" of monodromies

$$\overset{1}{M}{}^{I}(\mathscr{C})\mathscr{R}_x^{IJ}\overset{2}{M}{}^{J}(\mathscr{C}) = \sum (\varphi_x')^{IJ} C_x^a[IJ|K]^* M^K(\mathscr{C}) C_x^a[IJ|K](\varphi_x^{-1})^{IJ} . \quad (7.10)$$

Let us omit the subscript x for the rest of this section. Together with the covariance of monodromies, $\mu^I(\xi)M^I(\mathscr{C}) = M^I(\mathscr{C})\mu^I(\xi)$ for all $\xi \in \mathscr{G}_x$, the formula (7.9) can be used to convert the factor $\mathscr{R}$ on the left-hand side of the operator product into a factor m,

$$\begin{aligned}\overset{1}{M}{}^{I}(\mathscr{C})\,\overset{1}{m}{}^{J}\,\overset{2}{M}{}^{J}(\mathscr{C}) &= \sum [(\mathscr{S}(\rho_\sigma^2)\otimes\varepsilon\otimes e)(\mathrm{id}\otimes\Delta)(m)\varphi]_{213}^{IJ} C^a[IJ|K]^* \\ &\quad \cdot M^K(\mathscr{C}) C^a[IJ|K](\varphi^{-1}(\rho_\sigma^1\otimes\Delta(\rho_\sigma^3))^{IJ} .\end{aligned}$$

From here we can calculate $c^I c^J$ using the relation (6.12) and

$$(\rho_\sigma^1\otimes\Delta(\rho_\sigma^3))(e\otimes w)(e\otimes\mathscr{S}^{-1}(\rho_\sigma^2)\otimes e) = \mathscr{R}^{-1}[(\mathrm{id}\otimes\Delta)w]_{213}$$

together with Eqs. (7.6, 7.7). The result of a short calculation is

$$\begin{aligned}c^I c^J &= \kappa_I\kappa_J \sum (\mathrm{tr}^I\otimes\mathrm{tr}^J)[(f')^{IJ} C^a[IJ|K]^* m^K M^K(\mathscr{C}) w^K \\ &\quad \cdot C^a[IJ|K](R^{-1}(\mathscr{S}^{-1}\otimes\mathscr{S}^{-1})(f^{-1}))^{IJ}\,\overset{1}{g}{}^{I}\,\overset{2}{g}{}^{J}] \\ &= \kappa_I\kappa_J \sum (\mathrm{tr}^I\otimes\mathrm{tr}^J)[C^a[IJ|K]^* m^K M^K(\mathscr{C}) w^K g^K C^a[IJ|K](R^{-1})^{IJ}] .\end{aligned}$$

The last equality is a consequence of Eq. (7.3). The fusion algebra $c^I c^J = \sum N_K^{IJ} c^J$ finally follows from the normalization (2.7).

Evaluation of $(c^I)^*$ is left as an exercise. As an intermediate result one shows

$$(c^I)^* = \kappa_I^{-1}\mathrm{tr}^I[m^I M^I(-\mathscr{C}) w^I g^I] .$$

Then relation (6.11) is inserted. After application of Lemma 4 one uses Eq. (6.1) and writes $\mathscr{R}$ as a product $\varphi_{213}R_{12}\varphi^{-1}$. This gives the result $(c^I)^* = c^{\bar{I}}$.

Flatness. Every line in the prrof of Proposition (7) can be "translated" with the substitution rules (6.2). There is only one problem. In the second part of the proof (i.e. after Lemma 2) we exploit the completeness (3.27) of Clebsch Gordon maps, which fails to hold in the case of truncation. Even though completeness was the fastest way to get the desired results, it is not necessary. In fact, (quasi)-associativity of the co-product does suffice. We want to show this with $\varphi = e\otimes e\otimes e$. The case of nontrivial φ is again obtained with the rules (6.2). From associativity of the tensor product of representation it follows that

$$e^J C_x[\bar{K}K|0] = \sum_{I,a,b} F_{ab}(I\ J\ K) C_x^b[\bar{I}K|J] C_x^a[J\bar{K}|\bar{I}] .$$

The complex coefficients are a subset of $6j$-symbols. We may use the normalization (2.7) to rewrite this as

$$\frac{\kappa_{\bar{I}}}{\kappa_J \kappa_{\bar{K}}}(R'_x)^{J\bar{K}} C^a_x[J\bar{K}|\bar{I}]^* = \sum_b F_{ba}(I\ J\ K) C^b_x[\bar{I}K|J]\ .$$

These two formulas can be inserted into the first equation after relation (3.27) and furnish an alternative calculation of $\chi^0 M^J$. It does not use the completeness (3.27) and remains valid in the case of truncation.

8. Discussion and Outlook

In this paper we have introduced a new quantum algebra. It is natural to interpret it as a quantized algebra of functions on the moduli space of flat connections (moduli algebra). This algebra plays the role of the observable algebra in the Hamiltonian Chern Simons theory. The construction of the moduli algebra requires a quasi-triangular ribbon $*$-Hopf algebra to be used as a gauge symmetry. A lot of examples of such symmetry algebras are provided by quantized universal enveloping algebras of simple Lie algebras. Given a Riemann surface of genus g with m marked points, a simple Lie algebra, a c-number q being some root of unity and a set of m representation-classes $[I_\nu]$ of the corresponding quantized universal enveloping algebra, one can construct the moduli algebra $\mathscr{A}_{CS}^{\{I_\nu\}}$. This means that we have completed the program of deformation quantization of the moduli space and now we are going to discuss perspectives of the combinatorial approach to quantization of the Chern Simons model.

The main question which arises naturally is the comparison to other quantization schemes already applied to Chern Simons theory. Among them we pick up two approaches which are the most suitable for comparison. These are geometric quantization [17, 18] and the Conformal Field Theory approach which was used originally to solve the Chern Simons model [19]. Both these approaches use the Hamiltonian picture of quantization. So, their results may be easily compared to the results of combinatorial approach.

Opening the list of unsolved questions we start with

1. *Compare results of geometric quantization and the Conformal Field Theory approach to combinatorial quantization.*

Geometric quantization as well as Conformal Field Theory produces the Hilbert space of the Chern Simons model rather than the observable algebra. In the Conformal Field Theory approach, vectors in the Hilbert space are identified with conformal blocks of the WZW model. More precisely, they come as solutions of a certain system of linear differential equations. In the case of the Riemann sphere this system of equations was discovered in [20] and called the Knizhnik–Zamolodchikov equation. For higher genera it was considered in [21, 17] etc. Geometric quantization provides a more abstract picture of the Hilbert space. There it appears as a space of holomorphic sections of the quantum line bundle over the moduli space. It is possible to find a contact between these two pictures. of the Hilbert space. The key observation is that each complex structure on the underlying Riemann surface provides a complex structure on the moduli space. Thus, the $\bar{\partial}$-operator in the quantum line bundle depends on this complex structure. One can think that for each complex structure on the surface we get its own geometric quantization and its own Hilbert space. One needs a projectively flat connection on the space of complex structures in order to identify these bunch of Hilbert spaces with the unique Hilbert space of the quantum theory. Here one makes a bridge with the Conformal Field Theory approach. It appears [17] that the linear system of equations for conformal blocks may be reinterpreted as a covariance conditions with respect to some projectively flat connection.

While these approaches to quantization provide Hilbert spaces, our combinatorial quantization provides an algebra of observables. The natural route for comparison is to realise the algebra of observables in given Hilbert spaces. This is the second point in our list.

2. *Represent the moduli algebra in the Hilbert spaces of the Chern–Simons theory provided by geometric quantization and Conformal Field Theory.*

In principle, this is the central question of the whole program and – provided it is done – one can stop here. However, there are a couple of questions that one should add to the list. The first one concerns the action of the mapping class group. It is known that the projective representation of the mapping class group acts in the Hilbert space of the Chern–Simons theory. This representation proves to be useful in constructing invariants of 3-manifolds. One can try to relate this idea to the moduli algebra.

3. *Construct the action of the mapping class group on the moduli algebra.*

The last question which we would like to mention here concerns a very particular application of the machinery that we have developed. It has been recently proven that the relativistic analogue of the Calogero–Moser integrable model may be naturally realized on the moduli space of flat connections on a torus with a marked point [22]. It would be interesting to develop this idea from the point of view of the moduli algebra.

4. *Work out details for the example of a torus with a marked point.*

We are going to consider the listed problems in the forthcoming paper [14].

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