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Asymptotic Completeness for N-Body Stark Hamiltonians

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Abstract: We prove asymptotic completeness for short- and long-range N-body Stark Hamiltonians with local singularities of at most Coulomb type. Our results include the usual models for atoms and molecules.

Section 1. Introduction

In the present paper we will prove asymptotic completeness for short- and long-range N-body Stark Hamiltonians. The results include the usual models for atoms and molecules. The Hamiltonian for N ν-dimensional particles with charges q_i and masses m_i in an external electric field $\mathscr{E}$ is

$$\tilde{H} = \sum_{i=1}^{N} \left(\frac{-\Delta_i}{2m_i} - \mathscr{E} \cdot q_i x_i \right) + \sum_{1 \leqq i < j \leqq N} v_{ij}(x_i - x_j) \quad \text{on } L^2(\mathbb{R}^{\nu N}) \, .$$

By a standard procedure we remove the center of mass motion and obtain the Hamiltonian

$$H = -\Delta - E \cdot x + V \quad \text{on } L^2(X) \, ,$$

where the $\nu(N-1)$ dimensional configuration space X is given by

$$X = \{ x \in \mathbb{R}^{\nu N} : \sum_{i=1}^{N} m_i x_i = 0 \}$$

and the resulting electric field $E \in X$ is given by

$$E = \left\{ \left(\frac{q_1}{2m_1} - \frac{Q}{2M} \right) \mathscr{E}, \ldots, \left(\frac{q_N}{2m_N} - \frac{Q}{2M} \right) \mathscr{E} \right\} \, ,$$

where Q and M stand for the total charge and mass respectively.

We assume $E \neq 0$, that is $\mathscr{E} \neq 0$ and there exist $1 \leqq i < j \leqq N$ such that $\frac{q_i}{m_i} \neq \frac{q_j}{m_j}$.

This paper is a sequel to [HMS1], where absence of bound states and of singular continuous spectrum for H are proved. These results were obtained for a wide class

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of potentials with singularities of at most Coulomb type and the decay assumption at infinity

$$|v_{ij}(y)| + |\nabla v_{ij}(y)| = o(1) \quad \text{for } |y| \to \infty ,$$

plus in addition an assumption on the second order derivatives. Here we shall need more decay assumptions on the potentials. Although we shall not elaborate in this introduction we mention that roughly the short-range case is defined by the condition

$$|v_{ij}(y)| = O(|y|^{-\rho}) \qquad \text{for } |y| \to \infty, \quad \rho > \frac{1}{2} ,$$

while the case $0 < \rho \leqq \frac{1}{2}$ is referred to as the long-range case.

Let a be a decomposition of the particles 1 through N into clusters

$$a = (c_1, c_2, \ldots, c_k), \quad k = \#(a) .$$

We say $a \subset b$ iff each cluster in a is contained in some cluster in b. Let $a_{\min} = ((1), (2), \ldots, (N))$.

A necessary condition for the existence of a channel (defined as for the N-body problem without an electric field) is that the corresponding cluster decomposition a is such that all particles in each cluster have the same charge/mass ratio. This follows from the results in [HMS1]. There exists a largest of these cluster decompositions which we denote by $\tilde{a}$.

We are now ready to phrase our results.

Up to Dollard modifications needed to describe the internal motion (if the internal interaction between particles within clusters of $\tilde{a}$ is long-range in the sense used for $E = 0$, cf. for example [D]) the (short-range) wave operators

$$W_a^{\pm} = \text{s} - \lim_{t \to \pm\infty} \exp(itH) \exp(-itH_a)(P^a \otimes I), \quad a \subset \tilde{a} \tag{1.1}$$

exist and are complete, that is

$$\bigoplus_{a \subset \tilde{a}} \text{Range}(W_a^{\pm}) = L^2(X) .$$

Here H_a is the Hamiltonian H with the intercluster potential (denoted by I_a) removed and P^a is the eigenprojection corresponding to the internal motion within the clusters of a.

In particular for $a = a_{\min}$ the (free channel) wave operators are given by

$$W_{a_{\min}}^{\pm} = \text{s} - \lim_{t \to \pm\infty} \exp(itH) \exp(-itH_0) ,$$

where $H_0 = H_{a_{\min}} = p^2 - E \cdot x$.

In the case

$$v_{ij}(y) = \frac{q_i q_j}{|y|}, \quad \nu \geqq 3 ,$$

there is only the free channel. Thus we obtain that the wave operators corresponding to $a_{\min}$ exist and are unitary. Since generally $\tilde{a} \neq a_{\min}$ we need the internal Dollard modification for this example.

The limits in (1.1) do not exist in general for long-range potentials. Apart from the internal Dollard modifications (suppressed in this presentation) we shall need in this case a phase factor

$$e^{-i\int_0^t I_{\tilde{a}}(Es^2)\,ds} \tag{1.2}$$

as introduced in [Z] for the 2-body case (cf. also [G2] and [A]). (In (1.2) the factor $I_{\tilde{a}}$ should be read as the intercluster potential with the singularities removed.) Up to this additional modification we prove the existence of (1.1) as well as the corresponding completeness result.

In [T1] and [K] asymptotic completeness is proved for 3-body short-range systems. For 3-body long-range systems a (partial) result was obtained in [A]. It required strong fields and the assumption $\tilde{a} = a_{\min}$. (Clearly asymptotic completeness involves only the free channel under the latter assumption.)

In [T2] asymptotic completeness is shown for 4-body short-range systems under the condition $\tilde{a} = a_{\min}$.

In [T3] asymptotic completeness is shown for N-body short-range systems under the condition (on E) that for all cluster decompositions a with $\#(a) \geqq 3$ there exists $1 \leqq i \leqq \#(a)$ such that

$$\frac{\sum_{j\in c_i} q_j}{\sum_{j\in c_i} m_j} \neq \frac{Q}{M} .$$

Finally [K] contains asymptotic completeness for arbitrary N (including Coulomb potentials) assuming a strong field and in addition $\tilde{a} = a_{\min}$.

All the results mentioned above, except for [K], hold for non-singular potentials only. In our paper we prove asymptotic completeness for short- and long-range systems with local singularities of at most Coulomb type without any restrictions on charge/mass ratios (except for $E \neq 0$). Other local singularities of L^p type ($p > \nu$) can be handled using the methods developed in [HMS1] but we prefer to concentrate on the physically relevant Coulomb singularity.

The free classical Stark motion in the center of mass frame is given by

$$x = x_0 + 2\xi_0 t + Et^2 . \tag{1.3}$$

Motivated by this the following two local smoothness results should not be surprising. The first tells us that $|x|$ grows at least as fast as t^2 and the second implies that the motion will concentrate along the field direction.

(1) The multiplication operator $\langle x\rangle^{-\rho}, \rho > \frac{1}{4}$ is locally H-smooth.

(2) The multiplication operator $h\langle x\rangle^{-\frac{1}{4}}$ is locally H-smooth, where h denotes the square root of any non-negative smooth function homogeneous of degree zero outside the unit ball and zero in the direction of E.

Here $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. The statement (1) follows from a resolvent estimate [HMS1, Theorem 6.3] and (2) is verified using a local commutator estimate. The proof of this local commutator estimate relies on a uniform estimate of [HMS1] (Proposition 2.3 in this paper) in conjunction with an idea of Tamura [T1].

In the short-range case asymptotic completeness follows easily from (1) and (2) (cf. [T1] for the 3-body case). In the long-range case a difficulty arises in a step where one proves existence of a certain modified wave operator. One would like to estimate (cf. (1.3))

$$\{x - Et^2\}\exp(-itH)\psi = O(t) ,$$

for a dense set of ψ's. Instead of proving this we introduce an intermediate evolution $U(t)$ satisfying that

$$\mathrm{s}-\lim_{t\to\pm\infty} U^*(t)\exp(-itH)$$

exist and such that one can prove the estimate

$$\{x - Et^2\}U(t)\psi = O(t)\,.$$

This process involves a minimal velocity estimate which is proved using a conjugate operator constructed in [HMS1, Appendix B] and the abstract theory of [Sk].

Another ingredient in our procedure is asymptotic completeness for N-body systems without an electric field [SS, G1 and D]. This statement is directly applicable in the last step of the proof(s).

We find it convenient to consider the problems within the framework of generalized Schrödinger operators although this involves a technical condition (denoted by (C)) which holds in the above physical framework. In Sect. 2 we make the necessary definitions and introduce various conditions to be imposed on the potential. Furthermore Sect. 2 contains (1) and (2) and various results from [HMS1] needed in this paper. In Sect. 3 we prove the local commutator estimate used in the proof of (2) and in Sect. 4 we prove an asymptotic localization result which follows from (2) and can be viewed as an elaboration of (2). In Sects. 5 and 6 we prove existence of some short- and long-range modified wave operators and in Sects. 7 and 8 we use these to prove existence and completeness of the wave operators discussed above. All results in Sects. 3–8 are stated for non-singular potentials only although they all hold with local singularities of at most Coulomb type included. In Sect. 9 we discuss this point and state our most general version of asymptotic completeness. In Appendix A we prove a minimal velocity estimate needed in Sect. 6.

Section 2. Definitions and Preliminary Results

We shall use the framework of generalized Schrödinger operators throughout the whole paper. Let $\{X_a\}_{a\in\mathscr{A}}$ be a finite family of subspaces of a real finite dimensional vector space X, equipped with an inner product. We assume without loss of generality that $X_a = X_b \Rightarrow a = b$. We denote by X^a the orthogonal complement to X_a and we introduce a partial ordering on $\mathscr{A}$ by

$$a \subset b \Leftrightarrow X^a \subset X^b\,.$$

The orthogonal projection onto X^a is denoted by Π^a.

The family of subspaces is assumed to satisfy

(1) $\exists a_{\min}, a_{\max} \in \mathscr{A}$ such that $X_{a_{\min}} = X$ and $X_{a_{\max}} = \{0\}$,
(2) $\forall a, b \in \mathscr{A},\ \exists c \in \mathscr{A}$ such that $X_a \cap X_b = X_c$.

For $x \in X$ we denote by x_a and x^a the orthogonal projections of x on X_a and X^a respectively. Similar notation will be used for components of the momentum operator p.

We use the notation $E \in X\backslash\{0\}$ for the electric field vector.

Since the sets

$$X_a\backslash \bigcup_{b\not\subset a} X_b, \quad a \neq a_{\max}$$

form a disjoint covering of $X\backslash\{0\}$, there exists a unique $\tilde{a} \neq a_{\max}$ such that

$$E \in X_{\tilde{a}} \backslash \bigcup_{b \not\subset \tilde{a}} X_b .$$

Notice that $\tilde{a}$ satisfies $E^a = 0 \Leftrightarrow a \subset \tilde{a}$.

The generalized Stark Hamiltonian is

$$H = H_0 + V, \qquad H_0 = p^2 - E \cdot x \quad \text{and} \quad V(x) = \sum_{a \in \mathscr{A}} V_a(x^a), \tag{2.1}$$

where the potentials V_a are real functions on X^a (for $a \neq a_{\min}$). We define the cluster Hamiltonians, which are again generalized Stark Hamiltonians, by

$$H_a = H_0 + V^a, \quad V^a(x) = \sum_{b \subset a} V_b(x^b),$$

and the intercluster potentials by

$$I_a = H - H_a = V - V^a = \sum_{b \not\subset a} V_b . \tag{2.2}$$

We can write the cluster Hamiltonians as follows:

$$H_a = H^a \otimes I + I \otimes T_a, \quad \text{on } L^2(X^a) \otimes L^2(X_a), \tag{2.3}$$

where

$$H^a = -\Delta^a - E^a \cdot x^a + V^a \tag{2.4}$$

and

$$T_a = -\Delta_a - E_a \cdot x_a , \tag{2.5}$$

where we denote by Δ^a and Δ_a the Laplacian on $L^2(X^a)$ and $L^2(X_a)$ respectively. If $E^a \neq 0$ the operator H^a is again a generalized Stark Hamiltonian with respect to $\{X^a \ominus X^b\}_{b \subset a}$ as a family of subspaces of the vector space X^a. If $E^a = 0$ we have an ordinary generalized N-body Hamiltonian.

Below we introduce a list of various conditions on the potentials $V_a, a \in \mathscr{A}$. These conditions involve a fixed strictly positive ε and will be combined differently in different contexts in the paper. For notational convenience we assume that $\varepsilon \leqq \frac{1}{2}$.

(V1) $V_a \in C^1(X^a), |V_a(x^a)| + |\nabla V_a(x^a)| = o(1)$.
(V2) $V_a \in C^2(X^a)$ and $|\partial^\alpha V_a| = O(1)$, $|\alpha| \leqq 2$.
(V3) $|V_a(x^a)| = O(|x^a|^{-\frac{1}{2}-\varepsilon})$.
(V4) $|\nabla V_a(x^a)| = O(|x^a|^{-\varepsilon})$.
(V5) $|V_a(x^a)| = O(|x^a|^{-1-\varepsilon})$, when $E^a = 0$.
(V6) $|V_a(x^a)| = O(|x^a|^{-\varepsilon})$ and $|\nabla V_a(x^a)| = O(|x^a|^{-1-\varepsilon})$.
(V7) $|\partial^\alpha V_a(x^a)| = O(|x^a|^{-\frac{1}{2}-\varepsilon})$, $|\alpha| = 2$.
(V8) $|V_a(x^a)| = O(|x^a|^{-(\sqrt{3}-1)-\varepsilon})$ and $|\nabla V_a(x^a)| = O(|x^a|^{-\sqrt{3}-\varepsilon})$, when $E^a = 0$.
(V9) The potential V_a is measurable and obeys

$$|V_a(x^a)| \leqq C_a \sum_{j=1}^{n} |x^a - r_j|^{-1} ,$$

with $r_1, \dots, r_n \in X^a$, and it has distributional derivative satisfying

$$|\nabla V_a(x^a)| \leqq C_a \sum_{j=1}^{n} |x^a - r_j|^{-2} .$$

If C_a is non-zero then $\dim(X^a) \geqq 3$.
(V10) $\operatorname{supp}(V_a)$ is a compact subset of X^a.

We say that the potential is/has/satisfies

(SR) Short-range if V_a satisfies (V1–3) and (V5) for all $a \in \mathscr{A}$.
(LR) Long-range if V_a satisfies (V1–2) and (V6–8) for all $a \in \mathscr{A}$.
(SC) Singularities of at most Coulomb type if V_a satisfies (V9–10) for all $a \in \mathscr{A}$.

We use the corresponding notations V_{short}, V_{long} and V_{sing} respectively. By the condition on the potential

$$V = V_{\text{short}} + V_{\text{long}} + V_{\text{sing}} ,$$

for example, we mean that each term has the indicated form explained above. We find it convenient from time to time to impose some other combinations of the conditions (V1–10). When we write that $V = \sum_{a \in \mathscr{A}} V_a$ satisfies a certain combination of (V1–10) we mean that V_a satisfies the combination of these conditions for all $a \in \mathscr{A}$.

Before we continue we will introduce some notation. Let $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$, $\hat{x} = \frac{x}{\langle x \rangle}$ and $\omega = \frac{E}{|E|}$. Similar notation $\langle A \rangle = (1 + |A|)^{\frac{1}{2}}$ applies to numbers and selfadjoint operators. By $F(\,\cdot\, < R) : \mathbb{R} \to [0,1]$ we will denote the sharp characteristic function for the set $(-\infty, R)$. We will denote by $\chi(\,\cdot\, < R) : \mathbb{R} \to [0,1]$ a smooth characteristic function satisfying $\chi(s < R) = 1,\ s < R$ and $\chi(s < R) = 0,\ s > 2R$. Let $F(\,\cdot\, \geqq R) = 1 - F(\,\cdot\, < R)$ and $\chi(\,\cdot\, > R) = 1 - \chi(\,\cdot\, < R)$. For a given real valued function h we denote by $F(h < R)$ the composition of the functions h and $F(\,\cdot\, < R)$ and likewise for $\chi(h < R)$. For $\delta > 0$ the notation η_δ stands for any smooth function $\eta : \mathbb{R} \to [0,1]$ such that $\eta(t) = 1$ for $|t| \leqq \delta$ and $\eta(t) = 0$ for $|t| > 2\delta$.

In order to include singularities we introduce the technical condition

(C) For all $b \in \mathscr{A}$ for which the singular part of V_b is non-zero and for all $a \neq b$ with $V_a \neq 0$ and $\Pi^a \Pi^b \neq 0$

$$\dim(\operatorname{Range} \Pi^a \Pi^b) \geqq 3 .$$

In the remaining part of this section we assume (C),

$$V = V_1 + V_{\text{sing}}$$

and we will specify V_1 for each result.

We note that the Hamiltonian (2.1) is essentially selfadjoint on $C_0^\infty(X)$ if V_1 satisfies (V1) since by [HMS1, Proposition 6.4] V is H_0-bounded with bound less than one.

This paper depends strongly on the following three results of [HMS1]. The first two follow from [HMS1, Theorem 6.2] and [HMS1, Theorem 6.3].

Theorem 2.1. *Assume V_1 satisfies* (V1–2). *Then the spectrum of H is purely absolutely continuous.*

Theorem 2.2. *Assume* V_1 *satisfies* (V1–2) *and let* $\rho > 0$. *Then the operator of multiplication by* $\langle x\rangle^{-\frac{1}{4}-\rho}$ *is locally* H*-smooth. That is for all* $f \in C_0^\infty(\mathbb{R})$ *there exists* $C > 0$ *such that*

$$\int_{-\infty}^{\infty} \|\langle x\rangle^{-\frac{1}{4}-\rho}\exp(-itH)f(H)\psi\|^2 dt \leqq C\|\psi\|^2 ,$$

for all $\psi \in L^2(X)$.

Proposition 2.3 [HMS1, Corollary 6.7] (cf. [HMS1, Proposition 3.7]). *Assume* V_1 *satisfies* (V1). *Suppose* $R > 0$ *and* $\Pi : X \to X$ *is an orthogonal projection such that* $\Pi E \neq 0$. *Then*

$$\|\eta_\delta(H-\lambda)F(|\Pi x| < R)\| \to 0 \quad \text{for } \delta \to 0 ,$$

uniformly in $\lambda \in \mathbb{R}$.

Assuming in addition to (V1) either (V3) or (V4) we will be able to prove a local commutator estimate (cf. [T1] for a proof in the 3-body case), which implies an improved smoothness result.

Throughout the paper we denote by $\tilde{q}_0$ the function

$$\tilde{q}_0 = \sqrt{|E|(1-\omega\cdot\hat{x})} . \tag{2.6}$$

Proposition 2.4. *Define the observable* $A = \langle x\rangle^{-\frac{1}{4}}(\omega\cdot p)\langle x\rangle^{-\frac{1}{4}}$. *Assume* V_1 *satisfies* (V1) *and either* (V3) *or* (V4). *Then there exists* $\delta > 0$ *such that for all* $\lambda \in \mathbb{R}$ *and real* $f \in C_0^\infty((\lambda-\delta,\lambda+\delta))$, *we have the estimate*

$$f(H)i[H,A]f(H) \geqq \frac{1}{2}f(H)\langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\langle x\rangle^{-\frac{1}{4}}f(H) - f(H)\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}}B\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}}f(H) ,$$

where $B = B(\lambda)$ *is bounded.*

Since A is H-bounded (cf. (3.1) below) this result together with Theorem 2.2 implies

Theorem 2.5. *If* V_1 *satisfies* (V1–2) *and either* (V3) *or* (V4), *then the operator* $\tilde{q}_0\langle x\rangle^{-\frac{1}{4}}$ *is locally* H*-smooth.*

Clearly the above results hold upon replacing H by any cluster Hamiltonian $H_a, a \neq a_{\max}$.

For convenience we define the spaces

$$Q_k(X) = \{q \in C^\infty(X)\colon q \text{ real}, |(\partial^\alpha q)(x)| \leqq C_\alpha\langle x\rangle^{k-|\alpha|}\}, \quad k \in \mathbb{R} .$$

We will by q_k and $\vec{q}_k$ denote elements of $Q_k(X)$ and vectors with entries in $Q_k(X)$ respectively. This shall frequently be tacitly understood.

We now introduce three functions j_1, j_2 and j_3 which will be used extensively throughout the paper. They are assumed to satisfy

(J1) $j_i \in C^\infty(X)$ and is homogeneous of degree 0 for $|x| > 1$.
(J2) $0 \leqq j_i \leqq 1$ and $j_i = 1$ in a neighbourhood of ω.
(J3) The support of j_i satisfies

$$\operatorname{supp}(j_i) \subset X\setminus \bigcup_{b \not\subset \tilde{a}} X_b .$$

(J4) $j_2 j_1 = j_1$ and $j_3 j_2 = j_2$.

Note that (J1) implies $j_i \in Q_0(X)$ and (J1-2) imply $(1 - j_i)(\tilde{q}_0)^k \in Q_0(X)$ for any $k \in \mathbb{R}$.

The property (J3) assures that for $b \not\subset \tilde{a}$ and $x \in \operatorname{supp}(j_i)$ we have

$$|x^b| > C_{b,i}|x| , \tag{2.7}$$

for some constant $C_{b,i} > 0$. Notice that (J4) implies that

$$(1 - j_1)(1 - j_2) = 1 - j_2 \quad \text{and} \quad (1 - j_1)\nabla_{j_2} = \nabla_{j_2} . \tag{2.8}$$

All results in Sects. 3–8 hold for potentials of the form

$$V = V_1 + V_{\text{sing}} ,$$

where V_1 satisfies some combination (depending on the given context) of the conditions (V1–8) and under condition (C). For simplicity we will only state and prove results in the case $V_{\text{sing}} = 0$ leaving the general case to be discussed in Sect. 9.

Section 3. Proof of Proposition 2.4

Put $A = \langle x\rangle^{-\frac{1}{4}}(\omega \cdot p)\langle x\rangle^{-\frac{1}{4}}$ and $D_0 = \hat{x} \cdot p$.

We will need the following two elementary results which hold for V satisfying (V1), cf. [HMS1, Lemma 3.1].

$$\langle x\rangle^{-\frac{1}{2}} \vec{q}_0 \cdot p(H + i)^{-1} \in \mathscr{B}(L^2(X)) , \tag{3.1}$$

and

$$\langle x\rangle^{-1} p^2(H + i)^{-1} \in \mathscr{B}(L^2(X)) . \tag{3.2}$$

Note the A is H-bounded by (3.1) and hence the commutator $i[H, A]$ is naturally defined as a form on $\mathscr{D}(H)$. First we state a result due to Tamura [T1].

Lemma 3.1. *Let V satisfy* (V1). *We have the following inequality*

$$i[H, A] \geqq \langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\langle x\rangle^{-\frac{1}{4}} - \frac{\omega \cdot \nabla V}{\langle x\rangle^{\frac{1}{2}}} + \langle x\rangle^{-\frac{3}{4}}(V - H)\langle x\rangle^{-\frac{3}{4}} + q_{-\frac{7}{2}} .$$

Proof. It is easy to verify that

$$i[E \cdot x, A] = -\langle x\rangle^{-\frac{1}{4}}|E|\langle x\rangle^{-\frac{1}{4}} \tag{3.3}$$

and

$$i[V, A] = -\frac{\omega \cdot \nabla V}{\langle x\rangle^{\frac{1}{2}}} . \tag{3.4}$$

We have the identity

$$\begin{aligned} i[p^2, \langle x\rangle^{-\frac{1}{4}}] &= -\frac{1}{2}\langle x\rangle^{-\frac{5}{4}}D_0 + iq_{-\frac{9}{4}} \\ &= -\frac{1}{2}D_0^*\langle x\rangle^{-\frac{5}{4}} - iq_{-\frac{9}{4}} . \end{aligned}$$

We apply this result to the commutator

$$i[p^2, A] = \langle x\rangle^{-\frac{1}{4}}(\omega \cdot p)i[p^2, \langle x\rangle^{-\frac{1}{4}}] + i[p^2, \langle x\rangle^{-\frac{1}{4}}](\omega \cdot p)\langle x\rangle^{-\frac{1}{4}} ,$$

and obtain after symmetrizing

$$i[p^2, A] = -\frac{1}{2}\langle x\rangle^{-\frac{3}{4}}\{(\omega \cdot p)D_0 + D_0^*(\omega \cdot p)\}\langle x\rangle^{-\frac{3}{4}} + q_{-\frac{7}{2}} .$$

Using the inequality

$$(\omega \cdot p)D_0 + D_0^*(\omega \cdot p) \leqq 2p^2 ,$$

we get

$$i[p^2, A] \geqq -\langle x\rangle^{-\frac{3}{4}} p^2 \langle x\rangle^{-\frac{3}{4}} + q_{-\frac{7}{2}} .$$

By (2.1) we can substitute $p^2 = H + E \cdot x - V$. Combining this with (3.3–4), we conclude the result. □

The next lemma is elementary and its proof is left to the reader.

Lemma 3.2. *Assume V satisfies* (V1). *Let $s, r, k \geqq 0$, $z \in \mathbb{C}$, $\mathrm{Im}(z) \neq 0$ and S be an H-bounded operator. Then we have the following:*

(1) $(H-z)^{-1} : \mathscr{D}(\langle x\rangle^s) \to \mathscr{D}(\langle x\rangle^s)$ *and* $\langle x\rangle^r(H-z)^{-1}\langle x\rangle^{-s} \in \mathscr{B}(L^2(X))$ *for* $r \leqq s$.
(2) $\langle x\rangle^r(H-z)^{-1} : \mathscr{D}(\langle x\rangle^s) \to \mathscr{D}(H)$ *and* $S\langle x\rangle^r(H-z)^{-1}\langle x\rangle^{-s} \in \mathscr{B}(L^2(X))$ *for* $r \leqq s$.
(3) $[(H-z)^{-1}, q_{-k}] : L^2(X) \to \mathscr{D}(\langle x\rangle^{\frac{1}{2}+k})$ *and* $\langle x\rangle^r[(H-z)^{-1}, q_{-k}] \in \mathscr{B}(L^2(X))$ *for* $r \leqq \frac{1}{2}+k$.
(4) $\langle x\rangle^r[(H-z)^{-1}, q_{-k}] : L^2(X) \to \mathscr{D}(H)$ *and* $S\langle x\rangle^r[(H-z)^{-1}, q_{-k}] \in \mathscr{B}(L^2(X))$ *for* $r \leqq \frac{1}{2}+k$.
(5) $S\langle x\rangle^r[(H-z)^{-1}, q_{-k}]\langle x\rangle^s$ *extends from* $\mathscr{D}(\langle x\rangle^s)$ *to a bounded operator on* $L^2(X)$ *for* $r \leqq \frac{1}{2}+k-s$.

Furthermore as $\mathscr{B}(L^2(X))$-valued functions on $\{z \in \mathbb{C} | \mathrm{Im}(z) \neq 0\}$ the operators in (1–5) *are continuous and bounded by polynomials in $|z|$ and $|\mathrm{Im}(z)|^{-1}$.*

We shall use the fact that given $g \in C_0^\infty(\mathbb{R})$, there exists $\tilde{g} \in C_0^\infty(\mathbb{C})$ satisfying

$$g(x) = \tilde{g}_{|\mathbb{R}}(x)$$

and

$$\forall k \in \mathbb{N} \exists C_k \geqq 0 \text{ such that } |\bar{\partial}\tilde{g}(z)| \leqq C_k |\mathrm{Im}\, z|^k . \tag{3.5}$$

Then we have, cf. [Hö, p. 63],

$$g(x) = \frac{1}{\pi}\int_{\mathbb{C}} (\bar{\partial}\tilde{g})(z)(x-z)^{-1} du\, dv, \quad z = u + iv . \tag{3.6}$$

By (3.5–6) and Lemma 3.2 we obtain

Lemma 3.3. *Assume V satisfies* (V1). *Let $s, r, k \geqq 0$, $f \in C_0^\infty(\mathbb{R})$ and S be an H-bounded operator. Then we have the following*:

(1) $f(H) : \mathscr{D}(\langle x\rangle^s) \to \mathscr{D}(\langle x\rangle^s)$ *and* $\langle x\rangle^r f(H)\langle x\rangle^{-s} \in \mathscr{B}(L^2(X))$ *for* $r \leqq s$.
(2) $\langle x\rangle^r f(H) : \mathscr{D}(\langle x\rangle^s) \to \mathscr{D}(H)$ *and* $S\langle x\rangle^r f(H)\langle x\rangle^{-s} \in \mathscr{B}(L^2(X))$ *for* $r \leqq s$.

(3) $[f(H), q_{-k}] : L^2(X) \to \mathscr{D}(\langle x\rangle^{\frac{1}{2}+k})$ *and* $\langle x\rangle^r[f(H), q_{-k}] \in \mathscr{B}(L^2(X))$ *for* $r \leqq \frac{1}{2}+k$.

(4) $\langle x\rangle^r[f(H), q_{-k}] : L^2(X) \to \mathscr{D}(H)$ *and* $S\langle x\rangle^r[f(H), q_{-k}] \in \mathscr{B}(L^2(X))$ *for* $r \leqq \frac{1}{2}+k$.

(5) $S\langle x\rangle^r[f(H), q_{-k}]\langle x\rangle^s$ *extends from* $\mathscr{D}(\langle x\rangle^s)$ *to a bounded operator on* $L^2(X)$ *for* $r \leqq \frac{1}{2}+k-s$.

When we refer to Lemma 3.3(5) without specifying S it is tacitly understood that $S = I$.

Using Lemma 3.3(5) with $k = \frac{3}{4}, s = \frac{1}{2}, r = 0$ and $S = V - H$, we obtain for $g \in C_0^\infty(\mathbb{R})$,

$$(V - H)[g(H), \langle x\rangle^{-\frac{3}{4}}]\langle x\rangle^{\frac{1}{2}} \in \mathscr{B}(L^2(X))\,. \tag{3.7}$$

Let $f \in C_0^\infty(\mathbb{R})$ be given. By choosing $g \in C_0^\infty(\mathbb{R})$ such that $fg = f$ one obtains the following result by applying (3.7) to Lemma 3.1.

Lemma 3.4. *Let* $\lambda \in \mathbb{R}$ *and* $f \in C_0^\infty((\lambda-1, \lambda+1))$ *be real, and suppose* V *satisfies* (V1). *Then we have the local commutator estimate*

$$f(H)i[H, A]f(H) \geqq f(H)\langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\langle x\rangle^{-\frac{1}{4}}f(H) - f(H)\frac{\omega \cdot \nabla V}{\langle x\rangle^{\frac{1}{2}}}f(H)$$
$$+ f(H)\langle x\rangle^{-\frac{1}{2}}B\langle x\rangle^{-\frac{1}{2}}f(H)\,,$$

where $B = B(\lambda)$ *is bounded.*

Lemma 3.5. *Assume the potential satisfies* (V1) *and either* (V3) *or* (V4). *Then there exists* $\delta > 0$ *such that for all* $\lambda \in \mathbb{R}$ *and real* $f \in C_0^\infty((\lambda-\delta, \lambda+\delta))$ *we have*

$$f(H)\frac{\omega \cdot \nabla V}{\langle x\rangle^{\frac{1}{2}}}f(H) \leqq \frac{1}{2}f(H)\langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\langle x\rangle^{-\frac{1}{4}}f(H)$$
$$+ f(H)\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}}B\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}}f(H)\,,$$

where $B = B(\lambda)$ *is bounded.*

Proof. We write

$$\omega \cdot \nabla V = \sum_{b \in \mathscr{A}} \{j_2\omega \cdot \nabla V_b(x^b) + (1 - j_2)\omega \cdot \nabla V_b(x^b)\}\,.$$

First we consider the terms localized away from the field direction.

Let $\lambda \in \mathbb{R}$ and $f \in C_0^\infty((\lambda-\delta, \lambda+\delta))$ for some $\delta > 0$ to be chosen later. In the following computations we abbreviate

$$q_{-\frac{1}{4}} = \langle x\rangle^{-\frac{1}{4}}\tilde{q}_0(1 - j_1)$$

and

$$h_b = (1 - j_2)\frac{\omega \cdot \nabla V_b(x^b)}{\tilde{q}_0^2}\,.$$

Notice that h_b is bounded. We write using (2.8),

$$\begin{aligned} & f(H)\langle x\rangle^{-\frac{1}{4}}(1-j_2)\omega\cdot\nabla V_b(x^b)\langle x\rangle^{-\frac{1}{4}}f(H) \\ & = f(H)\eta_\delta(H-\lambda)q_{-\frac{1}{4}}h_b q_{-\frac{1}{4}}\eta_\delta(H-\lambda)f(H) \\ & = T_1+T_2+T_3\,, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} T_1 &= f(H)q_{-\frac{1}{4}}\eta_\delta(H-\lambda)h_b\eta_\delta(H-\lambda)q_{-\frac{1}{4}}f(H)\,, \\ T_2 &= 2\mathrm{Re}\{f(H)[\eta_\delta(H-\lambda),q_{-\frac{1}{4}}]h_b\eta_\delta(H-\lambda)q_{-\frac{1}{4}}f(H)\}\,, \\ T_3 &= f(H)[\eta_\delta(H-\lambda),q_{-\frac{1}{4}}]h_b[q_{-\frac{1}{4}},\eta_\delta(H-\lambda)]f(H)\,. \end{aligned}$$

Consider

$$\begin{aligned} \eta_\delta(H-\lambda)h_b\eta_\delta(H-\lambda) &= \eta_\delta(H-\lambda)F(|x^b|<R)h_b\eta_\delta(H-\lambda) \\ &\quad + \eta_\delta(H-\lambda)F(|x^b|>R)h_b\eta_\delta(H-\lambda)\,. \end{aligned}$$

We start by estimating the second term. Since by assumption $|\nabla V_b(y)|=o(1)$ we can choose R large enough such that

$$|F(|x^b|>R)h_b| \leqq \frac{1}{4|\mathscr{A}|}\,.$$

Fix such an R.

To estimate the first term we can assume $E^b \neq 0$, since otherwise we will have $\omega\cdot\nabla V_b(x^b)=0$. This observation assures that we can use Proposition 2.3. Thus there exists $\delta=\delta(b)>0$ such that for all $\lambda\in\mathbb{R}$ we have

$$\begin{aligned} & \eta_\delta(H-\lambda)F(|x^b|<R)h_b\eta_\delta(H-\lambda) \\ & \leqq \eta_\delta(H-\lambda)F(|x^b|<R)\left\{\sup_{x\in X}|h_b(x)|\right\}\eta_\delta(H-\lambda) \leqq \frac{1}{4|\mathscr{A}|}I\,. \end{aligned}$$

We can now estimate

$$T_1 \leqq \frac{1}{2|\mathscr{A}|}f(H)\langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\langle x\rangle^{-\frac{1}{4}}f(H)\,, \tag{3.9}$$

for all $f\in C_0^\infty((\lambda-\delta,\lambda+\delta))$.

We can estimate T_3 using Lemma 3.3(5) twice, with $s=\frac{1}{2}$, $r=0$ and $k=\frac{1}{4}$, obtaining

$$T_3 = f(H)\langle x\rangle^{-\frac{1}{2}}B\langle x\rangle^{-\frac{1}{2}}f(H)\,. \tag{3.10}$$

To estimate T_2 we use Lemma 3.3(5) with $s=k=\frac{1}{4}$ and $r=\frac{1}{2}$ and Lemma 3.3(1) with $r=s=\frac{1}{4}$ and obtain

$$T_2 = f(H)\langle x\rangle^{-\frac{1}{2}}B\langle x\rangle^{-\frac{1}{2}}f(H)\,, \tag{3.11}$$

where B is bounded.

Combining (3.8–11) we have obtained a $\delta = \delta(b) > 0$ such that for all $\lambda \in \mathbb{R}$ and $f \in C_0^\infty((\lambda - \delta, \lambda + \delta))$,

$$f(H)\langle x\rangle^{-\frac{1}{4}}(1 - j_2)\omega \cdot \nabla V_b(x^b)\langle x\rangle^{-\frac{1}{4}} f(H)$$
$$\leqq \frac{1}{2|\mathscr{A}|} f(H)\langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\langle x\rangle^{-\frac{1}{4}} f(H) + f(H)\langle x\rangle^{-\frac{1}{2}} B\langle x\rangle^{-\frac{1}{2}} f(H), \quad (3.12)$$

where $B = B(\lambda)$ is bounded.

Clearly (by choosing the smallest) we can assume that δ does not depend on b.

Now we consider the terms localized along the field direction.

If $b \subset \tilde{a}$ we have $\omega \cdot \nabla V_b(x^b) = 0$. If $b \not\subset \tilde{a}$, we have by (2.7) that

$$f(H) j_2 \frac{\omega \cdot \nabla V_b(x^b)}{\langle x\rangle^{\frac{1}{2}}} f(H) = f(H)\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}} B\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}} f(H), \quad (3.13)$$

if V satisfies (V4).

For V satisfying (V3) we obtain using (3.1)

$$j_2 \frac{\omega \cdot \nabla V_b(x^b)}{\langle x\rangle^{\frac{1}{2}}} = \langle x\rangle^{-\frac{1}{4}} j_2 \omega \cdot i[p, V_b]\langle x\rangle^{-\frac{1}{4}} = \langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}} S\langle x\rangle^{-\frac{1}{4}-\frac{\varepsilon}{2}},$$

where S is H-bounded. Using Lemma 3.3(5), with S as the H-bounded operator, $k = \frac{1}{4} + \frac{\varepsilon}{2}, s = \frac{1}{4} + \frac{\varepsilon}{2}$ and $r = 0$, we can derive (3.13) for this case also. Combining (3.12) and (3.13) we now conclude the lemma. □

Proposition 2.4 is a direct consequence of Lemmas 3.4 and 3.5. □

Section 4. Asymptotic Localization

In this section we prove that asymptotically any state will be localized in an arbitrarily small conic neighbourhood of the field.

We will introduce the notation

$$\psi(t) \sim \phi(t) \Leftrightarrow \psi(t) = \phi(t) + o(1), \quad t \to +\infty$$

for families $\psi(t)$ and $\phi(t)$ of functions in $L^2(X)$. It will be used in this and subsequent sections.

We will need the following lemma.

Lemma 4.1. *Let $i \in \{1, 2, 3\}$ and $g \in C(\mathbb{R})$ such that $g(x) \to 0$ for $|x| \to \infty$. Then the following operators are compact for V satisfying* (V1).

(1) $j_i(g(H) - g(H_{\tilde{a}}))$,

(2) $[g(H_a), j_i]$ *for any $a \in \mathscr{A}$.*

Proof. Clearly it is enough to consider $g \in C_0^\infty(\mathbb{R})$. Because of (3.5–6) it is sufficient to show that the operators

$$j_i((H - z)^{-1} - (H_{\tilde{a}} - z)^{-1}) \quad \text{and} \quad [(H_a - z)^{-1}, j_i]$$

are compact for fixed z with $\mathrm{Im}(z) \neq 0$. First we note that the operator $f(x)(H_a - z)^{-1}$ is compact if $f(x) = o(1)$ (cf. (3.1–2)). This gives the result for the commutator. The compactness of the first operator is seen from the identity

$$j_i((H_{\tilde{a}} - z)^{-1} - (H - z)^{-1}) = j_i(H_{\tilde{a}} - z)^{-1} I_{\tilde{a}}(H - z)^{-1} ,$$

using the compactness of the commutator above and (2.7). □

The following fact which holds for V satisfying (V1–2) follows from Theorem 2.1 and will be used in conjunction with Lemma 4.1,

$$\mathrm{w} - \lim_{t \to \pm\infty} \exp(-itH) = 0 . \tag{4.1}$$

Proposition 4.2. *Let V satisfy* (V1–2) *and either* (V3) *or* (V4). *Then we have for any $\psi \in L^2(X)$,*

$$\lim_{t \to \pm\infty} (1 - j_2) \exp(-itH)\psi = 0 .$$

Notice that j_2 could be chosen with support in an arbitrarily small conic neighbourhood of the field direction.

Proof. The first step will be to verify existence of the modified wave operators

$$\tilde{W}_\pm = \mathrm{s} - \lim_{t \to \pm\infty} \tilde{W}(t) ,$$

where

$$\tilde{W}(t) = \exp(itH)(1 - j_2)\exp(-itH) . \tag{4.2}$$

Next we will show that $\tilde{W}_\pm = 0$ thereby concluding the proposition.

We will only consider the case $t \to +\infty$ since the case $t \to -\infty$ is similar.

Let $\psi = f(H)\psi_0$, where $f \in C_0^\infty(\mathbb{R})$ and $\psi_0 \in L^2(X)$. By Lemma 4.1 and (4.1) we have

$$\tilde{W}(t)\psi \sim g(H)\tilde{W}(t)\psi, \quad g \in C_0^\infty(\mathbb{R}), gf = f .$$

Hence we can write

$$\tilde{W}(t)\psi \sim g(H)(1 - j_2)\psi + \int_0^t g(H)\exp(isH)i[H, 1 - j_2]\exp(-isH)\psi\, ds . \tag{4.3}$$

We compute the commutator using (2.8) and (3.1),

$$i[H, 1 - j_2] = O(\langle x\rangle^{-2}) - 2\nabla j_2 \cdot p = O(\langle x\rangle^{-2}) + q_{-\frac{1}{4}} S q_{-\frac{1}{4}} ,$$

where $q_{-\frac{1}{4}} = \langle x\rangle^{-\frac{1}{4}} \tilde{q}_0(1 - j_1)$ and S is H-bounded. Consider

$$g(H) q_{-\frac{1}{4}} S q_{-\frac{1}{4}} f(H) = T_1 + T_2 ,$$

where

$$T_1 = g(H) q_{-\frac{1}{4}} S g(H) q_{-\frac{1}{4}} f(H) \quad \text{and} \quad T_2 = g(H) q_{-\frac{1}{4}} S [q_{-\frac{1}{4}}, g(H)] f(H) .$$

Applying Lemma 3.3(5) to T_2 with the above $S, k = \frac{1}{4}, s = \frac{1}{2}$ and $r = 0$ we obtain the existence of the limit of (4.3) by Theorems 2.2 and 2.5. We have thus proved existence of the modified wave operator.

We shall now prove that $\tilde{W}_+$ is zero. Consider

$$\psi = f(H)\langle x\rangle^{-\frac{1}{4}}\psi_0, \quad \psi_0 \in L^2(X), f \in C_0^\infty(\mathbb{R}) .$$

We will use the following notation to denote the expectation of an observable A in a state ψ,

$$\langle A\rangle_\psi = \langle \psi, A\psi\rangle \, .$$

We abbreviate

$$\psi(t) = \exp(-itH)\psi \, ,$$

and estimate using the Cauchy Schwarz inequality

$$\int_1^\infty \|\langle x\rangle^{-\frac{1}{4}}\tilde{q}_0^2\psi(t)\|^2 dt \geqq \int_1^\infty \frac{\|\tilde{q}_0\psi(t)\|^4}{\langle\langle x\rangle^{\frac{1}{2}}\rangle_{\psi(t)}} dt \, . \tag{4.4}$$

By Theorem 2.5 the left-hand side is finite. (We used that $\tilde{q}_0$ is bounded.) By the choice of ψ and (3.1) we can estimate the expectation of $\langle x\rangle^{\frac{1}{2}}$,

$$\begin{aligned}\langle\langle x\rangle^{\frac{1}{2}}\rangle_{\psi(t)} &= \langle \psi(t), \langle x\rangle^{\frac{1}{2}}\psi(t)\rangle \\ &= \langle\langle x\rangle^{\frac{1}{2}}\rangle_\psi + \int_0^t \langle \psi, \exp(isH)i[p^2, \langle x\rangle^{\frac{1}{2}}]\exp(-isH)\psi\rangle ds \\ &= O(t) \, . \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) we can find a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$ for $n \to \infty$ and

$$\lim_{n\to\infty} \langle \tilde{q}_0^2\rangle_{\psi(t_n)} = 0 \, . \tag{4.6}$$

Estimating

$$\|\tilde{W}(t)\psi\|^2 = \langle (1-j_2)^2\rangle_{\psi(t)} \leqq C\langle \tilde{q}_0^2\rangle_{\psi(t)} \, ,$$

we obtain by (4.6) that

$$\lim_{n\to\infty} \|\tilde{W}(t_n)\psi\| = 0 \, .$$

Thus $\tilde{W}_+$ is zero. □

The simple sequence argument applied at the end of the above proof was also used in [E].

The idea of utilizing a smoothness estimate (as the one given by Theorem 2.5) to obtain a pointwise estimate (cf. Proposition 4.2) was used in a different but somewhat related context in [He]. (See also [E].)

Section 5. Modified Short-Range Wave Operators

In this section we prove existence of some modified short-range wave operators which will be used in Sect. 7 to prove existence of wave operators and asymptotic completeness in the short-range case.

Proposition 5.1. *Assume V satisfies* (V1–3). *Let*

$$\bar{W}(t) = \exp(itH)j_2 \exp(-itH_{\tilde{a}}) \, .$$

Then the modified wave operators

$$\bar{W}_\pm = \text{s} - \lim_{t\to\pm\infty} \bar{W}(t)$$

and

$$\bar{W}_{\pm}^{*} = \text{s} - \lim_{t \to \pm\infty} \bar{W}^{*}(t)$$

exist.

Remark. Since both limits exist $\bar{W}_{+}^{*}$ is in fact the adjoint of $\bar{W}_{+}$ (motivating the notation). A similar remark applies to $\bar{W}_{-}^{*}$.

Proof. We will only prove the existence of $\bar{W}_{+}^{*}$ since existence of the other three wave operators are proven similarly. The proof goes along the line of the first part of the proof of Proposition 4.2.

Let $\psi = f(H)\psi_0$, where $f \in C_0^{\infty}(\mathbb{R})$ and $\psi_0 \in L^2(X)$. By (4.1) and Lemma 4.1 we can write

$$\bar{W}^{*}(t)\psi \sim g(H_{\tilde{a}})\bar{W}^{*}(t)\psi, \quad g \in C_0^{\infty}(\mathbb{R}),\, gf = f\,.$$

Then we can compute using (V3) and (2.7–8),

$$\bar{W}^{*}(t)\psi \sim g(H_{\tilde{a}})j_2\psi + \int_0^t g(H_{\tilde{a}}) \exp(isH_{\tilde{a}}) i(H_{\tilde{a}}j_2 - j_2H) \exp(-isH)\psi\, ds$$

and

$$\begin{aligned} i(H_{\tilde{a}}j_2 - j_2H) &= O(\langle x\rangle^{-2}) + q_{-\frac{1}{4}}Sq_{-\frac{1}{4}} - ij_2I_{\tilde{a}} \\ &= O(\langle x\rangle^{-\frac{1}{2}-\varepsilon}) + q_{-\frac{1}{4}}Sq_{-\frac{1}{4}}\,, \end{aligned}$$

where $q_{-\frac{1}{4}} = \langle x\rangle^{-\frac{1}{4}}\tilde{q}_0(1 - j_1)$ and S is H-bounded. We can now apply Lemma 3.3(5) (cf. the proof of Proposition 4.2) and conclude the existence of the limit by Theorems 2.2 and 2.5. □

Section 6. Modified Long-Range Wave Operators

In order to prove asymptotic completeness in the long-range case we will need some modified long-range wave operators introduced in [Z] for the 2-body case (cf. also [G2]).

The modified wave operators are

$$W_Z^{\pm} = \text{s} - \lim_{t \to \pm\infty} W_Z(t)\,,$$

$$W_Z^{\pm *} = \text{s} - \lim_{t \to \pm\infty} W_Z^{*}(t)\,,$$

where

$$W_Z(t) = \exp(itH)j_2U_{\tilde{a}}(t)$$

and

$$U_{\tilde{a}}(t) = e^{-i\int_0^t I_{\tilde{a}}(Es^2)\,ds} \exp(-itH_{\tilde{a}})\,. \tag{6.1}$$

Notice that $U_{\tilde{a}}(t)$ is the time evolution corresponding to the time dependent Hamiltonian

$$H_{\tilde{a}}(t) = H_{\tilde{a}} + I_{\tilde{a}}(Et^2)\,. \tag{6.2}$$

All results in this section hold for V satisfying (V1–2) and (V6–7).

Proposition 6.1. *The modified long-range wave operators* $W_Z^{\pm}$ *and* $W_Z^{\pm *}$ *exist.*

The existence of $W_Z^{\pm}$ will be proved in a simpler and more direct way than the existence of $W_Z^{\pm *}$. For the latter problem we shall need an intermediate evolution $U(t)$ generated by a time dependent Hamiltonian $H(t)$.

To define $H(t)$ we let θ be as in Proposition A.1 in Appendix A. Let $\kappa = \frac{1}{4}\min\{\theta, |E|\}$. The time dependent Hamiltonian generating $U(t)$ is

$$H(t) = H_{\tilde{a}} + I(t,x),$$

where

$$I(t,x) = I_{\tilde{a}}(x)\chi\left(\frac{|x|}{\langle t\rangle^2} > \kappa\right) j_3 . \tag{6.3}$$

From the definition of $I(t,x)$ one obtains for $|\alpha| \leqq 2$, $0 \leqq \rho \leqq \frac{m(|\alpha|)}{2} + \varepsilon$ and $n \in \mathbb{N} \cup \{0\}$

$$(\partial_t^n \partial_x^\alpha I)(t,x) = \langle x\rangle^{-\rho} O(\langle t\rangle^{-m(|\alpha|)-2\varepsilon-n+2\rho}), \tag{6.4}$$

where $m(0) = 0$, $m(1) = 2$ and $m(2) = 1$.

To handle this intermediate evolution we shall need two smoothness results.

Lemma 6.2. *There exists* $C > 0$ *such that for all* $\rho > 0$ *and* $\psi \in L^2(X)$ *we have*

$$\int_0^\infty \|\langle x\rangle^{-\frac{1}{4}-\rho}(H_{\tilde{a}} - i)^{-1} U(t)\psi\|^2 \, dt \leqq C\|\psi\|^2 .$$

Lemma 6.3. *There exists* $C > 0$ *such that for all* $\psi \in L^2(X)$ *we have*

$$\int_0^\infty \|\tilde{q}_0 \langle x\rangle^{-\frac{1}{4}}(H_{\tilde{a}} - i)^{-2} U(t)\psi\|^2 \, dt \leqq C\|\psi\|^2 .$$

Proof of Lemma 6.2. We can assume that $\rho \leqq \frac{1}{4}$. Since

$$\langle x\rangle^{-\frac{1}{2}}(H_{\tilde{a}} - i)^{-1}\langle E \cdot p\rangle = \{\langle x\rangle^{-\frac{1}{2}}(H_{\tilde{a}} - i)^{-1}(E \cdot p - i)\}\{(E \cdot p - i)^{-1}\langle E \cdot p\rangle\},$$

we obtain by (3.1) and Lemma 3.2(2) with $S = \langle x\rangle^{-\frac{1}{2}}(E \cdot p + i)$ and $s = r = \frac{1}{2}$ that $\langle x\rangle^{-\frac{1}{2}}(H_{\tilde{a}} - i)^{-1}\langle E \cdot p\rangle$ is bounded. By an interpolation argument it is thus sufficient to find a $C > 0$ such that for all $\psi \in L^2(X)$,

$$\int_0^\infty \|\langle E \cdot p\rangle^{-\frac{1}{2}-2\rho} U(t)\psi\|^2 \, dt \leqq C\|\psi\|^2 . \tag{6.5}$$

Let

$$A(t) = U^*(t)\{E \cdot p\}U(t)$$

and

$$A_0(t) = \exp(itH_{\tilde{a}})\{E \cdot p\}\exp(-itH_{\tilde{a}}) = E \cdot p + |E|^2 t . \tag{6.6}$$

Then the difference

$$A_0(t) - A(t) = \int_0^t U^*(s)E \cdot \nabla I(s,x)U(s)\,ds = O(1),$$

by (6.4). This result implies

$$\|\langle A(t)\rangle^{-1}\langle A_0(t)\rangle\| = O(1).$$

By an interpolation argument we thus obtain

$$\|\langle A(t)\rangle^{-s}\langle A_0(t)\rangle^{s}\| = O(1)$$

for $0 \leqq s \leqq 1$.

We can now estimate

$$\|\langle E \cdot p\rangle^{-\frac{1}{2}-2\rho}U(t)\psi\|^2 = \|\langle A(t)\rangle^{-\frac{1}{2}-2\rho}\psi\|^2 \leqq C\|\langle A_0(t)\rangle^{-\frac{1}{2}-2\rho}\psi\|^2.$$

Applying this estimate to the left-hand side of (6.5) and by using (6.6) we obtain

$$\begin{aligned}\int_0^\infty \|\langle E \cdot p\rangle^{-\frac{1}{2}-2\rho}U(t)\psi\|^2\,dt &\leqq C\int_0^\infty \|\langle A_0(t)\rangle^{-\frac{1}{2}-2\rho}\psi\|^2\,dt\\ &= C\int_0^\infty \|\langle E \cdot \xi + |E|^2t\rangle^{-\frac{1}{2}-2\rho}\hat{\psi}\|^2\,dt\\ &= C|E|^{-2}\int_X \int_{\xi \cdot E}^\infty |\langle s\rangle^{-\frac{1}{2}-2\rho}\hat{\psi}(\xi)|^2\,ds\,d\xi\\ &\leqq C|E|^{-2}\int_{-\infty}^\infty \langle s\rangle^{-1-4\rho}\,ds\|\psi\|^2,\end{aligned}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. We are done since $\rho > 0$. $\square$

Proof of Lemma 6.3. Define the observable

$$\tilde{A} = (H_{\tilde{a}} + i)^{-2}A(H_{\tilde{a}} - i)^{-2},$$

where A is the observable from Sect. 3. By (6.4) with $|\alpha| = 1$, $\rho = \frac{1}{2}$ and $n = 0$ we obtain

$$i[H(t),(H_{\tilde{a}} - i)^{-1}] = (H_{\tilde{a}} - i)^{-1}i[p^2, I(t,x)](H_{\tilde{a}} - i)^{-1} = O(\langle t\rangle^{-1-2\varepsilon}).$$

We compute using this estimate in conjunction with (6.4), Lemma 3.1 and the fact that $\omega \cdot \nabla V^{\tilde{a}} = 0$,

$$\begin{aligned}i[H(t),\tilde{A}] &\geqq (H_{\tilde{a}} + i)^{-2}\frac{\tilde{q}_0^2}{\langle x\rangle^{\frac{1}{2}}}(H_{\tilde{a}} - i)^{-2}\\ &\quad + (H_{\tilde{a}} + i)^{-2}\{\langle x\rangle^{-\frac{3}{4}}(V - H)\langle x\rangle^{-\frac{3}{4}} + q_{-\frac{7}{2}}\}(H_{\tilde{a}} - i)^{-2} + O(\langle t\rangle^{-1-2\varepsilon})\\ &= (H_{\tilde{a}} + i)^{-2}\frac{\tilde{q}_0^2}{\langle x\rangle^{\frac{1}{2}}}(H_{\tilde{a}} - i)^{-2}\\ &\quad + (H_{\tilde{a}} + i)^{-1}\langle x\rangle^{-\frac{3}{4}}B\langle x\rangle^{-\frac{3}{4}}(H_{\tilde{a}} - i)^{-1} + O(\langle t\rangle^{-1-2\varepsilon}),\end{aligned}$$

where B is bounded. In the last step we used Lemma 3.2(5). The result now follows from Lemma 6.2. □

Lemma 6.4. *For $\psi = \langle x\rangle^{-1} f(p_{\tilde a})(H^{\tilde a} - i)^{-1}\varphi$, $f \in C_0^\infty(X_{\tilde a})$ and $\varphi \in L^2(X)$ we have the estimates*

$$\{x - Et^2\}U_{\tilde a}(t)\psi = O(\langle t\rangle)$$

and

$$\{x - Et^2\}U(t)\psi = O(\langle t\rangle)\,.$$

Proof. For ψ of the above form we compute (using (6.4))

$$\begin{aligned} U^*(t)\{x_{\tilde a} - Et^2\}U(t)\psi &= (x_{\tilde a} + 2p_{\tilde a}t)\psi - 2\int_0^t\int_0^s U^*(r)\nabla_{\tilde a} I(r,x)U(r)\psi\, dr\, ds \\ &= O(\langle t\rangle)\,. \end{aligned}$$

For $U_{\tilde a}$ the same procedure applies. We are thus left with proving

$$x^{\tilde a}U(t)\psi = O(\langle t\rangle)$$

and

$$x^{\tilde a}U_{\tilde a}(t)\psi = O(\langle t\rangle)\,.$$

We represent $U^*(t)x^{\tilde a}U(t)\psi - x^{\tilde a}\psi$ as an integral and use the fact that $p^{\tilde a}(H^{\tilde a} + I(t,x) - i)^{-1}$ is uniformly bounded in t to conclude that it is sufficient to prove

$$(H^{\tilde a} + I(t,x) - i)U(t)\psi = O(1)\,. \tag{6.7}$$

This statement for $U(t)$ replaced by $U_{\tilde a}(t)$ follows from the identity

$$(H^{\tilde a} - i)U_{\tilde a}(t)\psi = U_{\tilde a}(t)(H^{\tilde a} - i)\psi\,.$$

To prove (6.7) we compute

$$\begin{aligned} &U^*(t)(H^{\tilde a} + I(t,x) - i)U(t)\psi \\ &\quad = (H^{\tilde a} + I(0,x) - i)\psi + \int_0^t U^*(s)\left\{ i[T_{\tilde a}, I(s,x)] + \frac{\partial}{\partial s} I(s,x)\right\} U(s)\psi\, ds \\ &\quad = O(1) + 2\int_0^t U^*(s)\nabla_{\tilde a} I(s,x)\cdot p_{\tilde a}U(s)\psi\, ds\,. \end{aligned}$$

By another application of (6.4) we compute

$$U^*(t)p_{\tilde a}U(t)\psi = p_{\tilde a}\psi + \int_0^t U^*(s)\{E - \nabla_{\tilde a} I(s,x)\}U(s)\psi\, ds = O(\langle t\rangle)\,.$$

Combining these statements we conclude (6.7) and hence the lemma. □

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. Throughout the proof we will only consider limits at plus infinity. To prove existence of W_Z^{+*} it is sufficient to prove existence of the strong limits of

$$U_{\tilde a}^*(t)U(t)$$

and

$$U^*(t)j_2\exp(-itH).$$

Let ψ be of the form considered in Lemma 6.4. We compute using (J2), (6.3) and the choice of κ,

$$\begin{aligned}U_{\tilde{a}}^*(t)U(t)\psi &= U_{\tilde{a}}^*(t_0)U(t_0)\psi + \int_{t_0}^{t} U_{\tilde{a}}^*(s)i\{I_{\tilde{a}}(Es^2) - I(s,x)\}U(s)\psi\,ds\\ &= U_{\tilde{a}}^*(t_0)U(t_0)\psi + \int_{t_0}^{t} U_{\tilde{a}}^*(s)i\{I(s,Es^2) - I(s,x)\}U(s)\psi\,ds,\end{aligned}$$

where $t > t_0 = \max\{1, \frac{1}{\sqrt{|E|}}\}$. By Taylor's formula, (6.4) and Lemma 6.4 the right-hand side has a limit as $t \to +\infty$.

Let $\psi = (H+i)^{-2}\psi_0, \psi_0 = f(H)\langle x\rangle^{-1}\varphi, f \in C_0^\infty(\mathbb{R})$ and $\varphi \in L^2(X)$. We write using Lemma 4.1, (4.1) and Proposition A.1,

$$\begin{aligned}U^*(t)j_2\exp(-itH)\psi &\sim U^*(t)(H_{\tilde{a}}+i)^{-2}j_2\exp(-itH)\psi_0\\ &\sim U^*(t)(H_{\tilde{a}}+i)^{-2}\chi\left(\frac{|x|}{\langle t\rangle^2} > 2\kappa\right)j_2\exp(-itH)\psi_0\\ &= \psi_1 + \int_0^t U^*(s)((H_{\tilde{a}}+i)^{-2}\{T_1(s)+T_2(s)+T_3(s)\} + T_4(s))\exp(-isH)\psi_0\,ds,\end{aligned}$$

where $\psi_1 = (H_{\tilde{a}}+i)^{-2}\chi(|x| > 2\kappa)j_2\psi_0$ and

$$T_1(s) = i\left[p^2, \chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right)j_2\right],$$

$$T_2(s) = i\chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right)j_2(I(s,x) - I_{\tilde{a}}),$$

$$T_3(s) = j_2\frac{d}{ds}\chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right),$$

$$T_4(s) = i[I(s,x), (H_{\tilde{a}}+i)^{-2}]\chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right)j_2.$$

We treat the terms one by one. In the following we denote by $B(s)$ and $S(s)$ families of bounded and H-bounded operators such that $\|B(s)\|$ and $\|S(s)(H-i)^{-1}\|$ are uniformly bounded in time, respectively.

We abbreviate $q_{-\frac{1}{4}} = \langle x\rangle^{-\frac{1}{4}}\tilde{q}_0(1-j_1)$ and calculate using (3.1) and (2.8),

$$T_1(s) = q_{-\frac{1}{4}}S_1(s)q_{-\frac{1}{4}} + \langle s\rangle^{-1}S_2(s)F\left(\frac{|x|}{\langle s\rangle^2} < 4\kappa\right). \tag{6.8}$$

The contribution from the first term can be shown to have a limit by Lemma 3.3(5) with $k = \frac{1}{4}$, $s = \frac{1}{2}$ and $r = 0$, Lemma 6.3, Theorems 2.2 and 2.5. As for the contribution from the second term we invoke Proposition A.1.

From the definition of $I(t,x)$ and (J4) we obtain

$$T_2(s) = -i\chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right)\chi\left(\frac{|x|}{\langle s\rangle^2} < \kappa\right)j_2 I_{\tilde{a}} = 0\,.$$

We compute

$$T_3(s) = \chi'\left(\frac{|x|}{\langle s\rangle^2} < 2\kappa\right)\frac{-2s|x|}{\langle s\rangle^4}j_2 = O(\langle s\rangle^{-1})F\left(\frac{|x|}{\langle s\rangle^2} < 4\kappa\right), \tag{6.9}$$

and thus we obtain integrability by Proposition A.1.

Using (3.1) and (6.4) with $|\alpha| = 1$, $\rho = \frac{1}{2}$ and $n = 0$ we get

$$T_4(s) = \langle s\rangle^{-1-2\varepsilon}B(s)\,.$$

We will now show the existence of W_Z^+.

Let $\psi = g(H_{\tilde{a}})\psi_0$, $\psi_0 = \langle x\rangle^{-1}f(p_{\tilde{a}})(H^{\tilde{a}}+i)^{-1}\varphi$, where $f \in C_0^\infty(X_{\tilde{a}})$, $g \in C_0^\infty(\mathbb{R})$ and $\varphi \in L^2(X)$. Let $h \in C_0^\infty(\mathbb{R})$ satisfy that $hg = g$. Using Lemma 4.1, (4.1) and Proposition A.1 (with H replaced by $H_{\tilde{a}}$) we obtain

$$\exp(itH)j_2U_{\tilde{a}}(t)\psi \sim h(H)\exp(itH)\chi\left(\frac{|x|}{\langle t\rangle^2} > 2\kappa\right)j_2U_{\tilde{a}}(t)\psi$$

$$= \psi_2 + \int_{t_0}^{t} h(H)\exp(isH)\{T_1(s) + T_3(s) + T_5(s)\}U_{\tilde{a}}(s)\psi\,ds\,,$$

where $\psi_2 = h(H)\exp(it_0H)\chi(\frac{|x|}{\langle t_0\rangle^2} > 2\kappa)j_2U_{\tilde{a}}(t_0)\psi$ and $t > t_0 = \max\{1, \frac{1}{\sqrt{|E|}}\}$. The terms $T_1(s)$ and $T_3(s)$ are as above and

$$T_5(s) = i\chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right)j_2(I_{\tilde{a}} - I_{\tilde{a}}(Es^2))\,.$$

By (6.8–9) and similar arguments we can show the existence of the limit of the contributions from $T_1(s)$ and $T_3(s)$. We write using (J2), (J4), (6.3) and the choice of κ,

$$T_5(s) = i\chi\left(\frac{|x|}{\langle s\rangle^2} > 2\kappa\right)j_2(I(s,x) - I(s,Es^2))\,.$$

This tells us that the contribution from $T_5(s)$ is integrable by Taylor's formula, (6.4) and Lemma 6.4 provided that we can prove

$$[\{x - Es^2\},\, g(H_{\tilde{a}})]U_{\tilde{a}}(s)\psi_0 = O(\langle s\rangle)\,.$$

By Lemma 6.4 and a simple interpolation it is sufficient to verify

$$[x, g(H_{\tilde{a}})]\langle x\rangle^{-\frac{1}{2}} \in \mathscr{B}(L^2(X))\,. \tag{6.10}$$

To do that we compute for $\mathrm{Im}(z) \neq 0$,

$$[x, (H_{\tilde{a}} - z)^{-1}]\langle x\rangle^{-\frac{1}{2}} = -i2(H_{\tilde{a}} - z)^{-1}S\langle x\rangle^{\frac{1}{2}}(H_{\tilde{a}} - z)^{-1}\langle x\rangle^{-\frac{1}{2}}, \quad S = p\langle x\rangle^{-\frac{1}{2}}\,.$$

This formula in conjunction with (3.1), Lemma 3.2(2) and (3.6) implies (6.10). □

Section 7. Short-Range Systems

In this section we will prove existence of wave operators and asymptotic completeness for short-range systems.

We denote by P^a the eigenprojection of H^a for $a \neq a_{\min}$. If this projection is non-zero Theorem 2.1 implies $a \subset \tilde{a}$.

Theorem 7.1 (Existence of short-range wave operators). *Assume V satisfies* (SR). *Then the wave operators*

$$W_a^{\pm} = \mathrm{s} - \lim_{t \to \pm\infty} W_a(t), \quad a \subset \tilde{a}\,,$$

where

$$W_a(t) = \exp(itH)\exp(-itH_a)(P^a \otimes I)$$

exist. (*Here the tensor symbol refers to the decomposition $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. For $a = a_{\min}, P^a \otimes I = I$.*) *Their ranges are closed and mutually orthogonal.*

Proof. We will only prove the existence of $W_a^+, a \subset \tilde{a}$, since the others are verified to exist similarly. Let $\psi = \psi^{\tilde{a}} \otimes \psi_{\tilde{a}} \in L^2(X^{\tilde{a}}) \otimes L^2(X_{\tilde{a}})$ and $a \subset \tilde{a}$. We write

$$\begin{aligned} W_a(t)\psi &= \exp(itH)\exp(-itH_{\tilde{a}})\exp(itH_{\tilde{a}})\exp(-itH_a)(P^a \otimes I)\psi \\ &= \exp(itH)\exp(-itH_{\tilde{a}})(\{\exp(itH^{\tilde{a}})\exp(-itH_a^{\tilde{a}})(P^a \otimes I)\} \otimes I)(\psi^{\tilde{a}} \otimes \psi_{\tilde{a}}), \end{aligned}$$

where

$$H_a^{\tilde{a}} = (p^{\tilde{a}})^2 + V^a \quad \text{on} \quad L^2(X^{\tilde{a}})\,.$$

By the existence of the wave operators for the usual N-body problem [SS, G1, D], there exists

$$\varphi^{\tilde{a}} = \lim_{t \to +\infty} \exp(itH^{\tilde{a}})\exp(-itH_a^{\tilde{a}})(P^a \otimes I)\psi^{\tilde{a}}\,.$$

We can thus write using Propositions 4.2 and 5.1,

$$\begin{aligned} W_a(t)\psi &\sim \exp(itH)\exp(-itH_{\tilde{a}})(\varphi^{\tilde{a}} \otimes \psi_{\tilde{a}}) \\ &\sim \bar{W}(t)(\varphi^{\tilde{a}} \otimes \psi_{\tilde{a}}) \sim \bar{W}_+(\varphi^{\tilde{a}} \otimes \psi_{\tilde{a}})\,, \end{aligned}$$

which implies the existence of the wave operator.

Since the wave operators are partial isometries their ranges are closed. Mutual orthogonality follows from mutual orthogonality for the usual N-body problem and the above calculation. □

Theorem 7.2 (Asymptotic completeness for short-range systems). *Assume V satisfies* (SR). *Then the wave operators are complete. That is*

$$\bigoplus_{a \subset \tilde{a}} \mathrm{Range}(W_a^{\pm}) = L^2(X)\,.$$

Proof. We will only prove completeness for $t \to +\infty$ since the other case is proved similarly. Let $\delta > 0$ and $\psi \in L^2(X)$. Choose $\varphi = \sum_{j=1}^{n} \psi_j^{\tilde{a}} \otimes \psi_{\tilde{a},j} \in L^2(X^{\tilde{a}}) \otimes L^2(X_{\tilde{a}})$

such that $\|\bar{W}_+^*\psi - \varphi\| \leqq \delta$. Then by Propositions 4.2 and 5.1

$$\exp(-itH)\psi \sim \exp(-itH_{\tilde{a}})\bar{W}_+^*\psi = (\exp(-itH^{\tilde{a}}) \otimes \exp(-itT_{\tilde{a}}))\varphi + O_t(\delta),$$

where $O_t(\delta) \in L^2(X)$ satisfies $\|O_t(\delta)\| \leqq \delta$ uniformly in time. We can now apply asymptotic completeness [SS, G1, D], for the usual N-body problem to obtain the existence of $\psi_{a,j}^{\tilde{a}} \in L^2(X^{\tilde{a}})$, $a \subset \tilde{a}$ and $1 \leqq j \leqq n$, such that

$$\exp(-itH)\psi \sim \sum_{a \subset \tilde{a}} \exp(-itH_a)(P^a \otimes I)\psi_a + O_t(\delta),$$

where $\psi_a = \sum_{j=1}^n \psi_{a,j}^{\tilde{a}} \otimes \psi_{\tilde{a},j}$. Since δ is arbitrary Theorem 7.1 now implies the result. □

Section 8. Long-Range Systems

In this section we use the modified long-range wave operators from Sect. 6 to prove existence and completeness of the long-range wave operators $W_{D,a}^{\pm}$ to be defined below.

Let

$$I_a^{\tilde{a}} = I_a - I_{\tilde{a}} = \sum_{\substack{b \subset \tilde{a} \\ b \not\subset a}} V_b, \quad a \subset \tilde{a}. \tag{8.1}$$

We introduce the Dollard modifications corresponding to the usual generalized N-body system defined by $\tilde{a}$,

$$s_a^{\tilde{a}}(\xi, t) = (\xi^{\tilde{a}} - \xi^a)^2 + I_a^{\tilde{a}}(2t(\xi^{\tilde{a}} - \xi^a)), \quad a \subset \tilde{a}.$$

The Dollard type Hamiltonians are

$$H_{D,a}(t) = H^a \otimes I \otimes I + I \otimes s_a^{\tilde{a}}(p,t) \otimes I + I \otimes I \otimes T_{\tilde{a}} + I_{\tilde{a}}(Et^2), \quad a \subset \tilde{a},$$

where we have decomposed $L^2(X) = L^2(X^a) \otimes L^2(X^{\tilde{a}} \ominus X^a) \otimes L^2(X_{\tilde{a}})$. The corresponding evolutions are

$$U_{D,a}(t) = e^{-i\int_0^t I_{\tilde{a}}(Es^2)ds}\{\exp(-itH^a) \otimes \exp(-iS_a^{\tilde{a}}(p,t)) \otimes \exp(-itT_{\tilde{a}})\}, \quad a \subset \tilde{a},$$

where

$$S_a^{\tilde{a}}(\xi, t) = t(\xi^{\tilde{a}} - \xi^a)^2 + \int_0^t I_a^{\tilde{a}}(2s(\xi^{\tilde{a}} - \xi^a))\, ds.$$

(This choice of $S_a^{\tilde{a}}$ is not unique.)

The Dollard type wave operators are

$$W_{D,a}^{\pm} = \mathrm{s} - \lim_{t \to \pm\infty} \exp(itH)U_{D,a}(t)(P^a \otimes I), \quad a \subset \tilde{a},$$

where P^a are the eigenprojections of H^a.

Using [D, Theorem 3.6], Propositions 4.2 and 6.1 we can apply exactly the same procedure as in Sect. 7 to obtain the following results.

Theorem 8.1 (Existence of long-range wave operators). *Let V satisfy* (LR). *Then the wave operators $W_{D,a}^{\pm}$, $a \subset \tilde{a}$ exist and their ranges are closed and mutually orthogonal.*

Theorem 8.2 (Asymptotic completeness for long-range systems). *Let V satisfy* (LR). *Then the wave operators $W^{\pm}_{D,a}$, $a \subset \tilde{a}$ are complete, that is*

$$\bigoplus_{a\subset\tilde{a}} \mathrm{Range}\,(W^{\pm}_{D,a}) = L^2(X)\,.$$

Section 9. Inclusion of Singularities

In this section we will explain how to include singularities of at most Coulomb type in our results of Sects. 3–8.

We consider potentials satisfying (C) and

$$V = V_1 + V_{\mathrm{sing}}\,,$$

where V_1 satisfies a combination of the conditions (V1–8) depending on the context.

The only situation where inclusion of singularities cannot be handled using only the H-boundedness of the Coulomb potential [HMS1, Proposition 6.4] is the estimate (3.9) in the proof of Lemma 3.5.

To obtain (3.9) with singularities we have to prove that

$$\|\eta_\delta(H-\lambda)h_b\eta_\delta(H-\lambda)\|$$

is small when δ is small, where h_b is given as in the proof of Lemma 3.5. A similar statement was an ingredient in the proof of [HMS1, Theorem 6.2]. We can assume V_b has a singular part (otherwise we can use Proposition 2.3 as before) and without loss of generality that $n = 1$ and $r_1 = 0$ in (V9). As before we can assume $E^b \neq 0$. Let $\rho > 0$ be given. Then by (V9) it is enough to find $\delta > 0$ such that

$$\left\|\eta_\delta(H-\lambda)\frac{1}{|x^b|}\right\| \leqq \rho\,. \tag{9.1}$$

To do this we use [HMS1, Lemma 6.8] to obtain $\delta_1, \delta_2 > 0$ such that

$$\left\|\eta_{\delta_1}(H-\lambda)\frac{1}{|x^b|}F(|x^b| < \delta_2)\right\| \leqq \frac{\rho}{2}\,.$$

Since $E^b \neq 0$ it follows from Proposition 2.3 that

$$\left\|\eta_\delta(H-\lambda)\frac{1}{|x^b|}F(|x^b| \geqq \delta_2)\right\| \leqq \frac{\rho}{2}$$

for $\delta > 0$ small enough. Thus by choosing $\delta < \frac{\delta_1}{2}$ small one obtains (9.1).

We have the following generalization of the results in Sect. 7.

Theorem 9.1. *Assume* (C) *and $V = V_{\mathrm{short}} + V_{\mathrm{sing}}$. Then the wave operators $W^{\pm}_a, a \subset \tilde{a}$ exist, their ranges are closed and mutually orthogonal. Furthermore they are complete.*

As for the results in Sects. 6 and 8 these hold under the condition (C) and

$$V = V_l + V_{\text{short}} + V_{\text{sing}} ,$$

where V_l is assumed to satisfy a combination of the conditions (V1–2) and (V6–8) depending on the section.

To obtain the existence of the modified long-range wave operators in Sect. 6 we replace $I_{\tilde{a}}$ by $I_{\tilde{a},l}$ in (6.1–3), where $I_{\tilde{a},l}$ is given in terms of V_l which satisfies (V1–2) and (V6–7). As for the results of Sect. 8 one should define $I_a^{\tilde{a}}$ (cf. (8.1)) in terms of $V_l = V_{\text{long}}$. (Notice that the wave operators depend on the splitting of the potential which is non-canonical).

The results in Sect. 8 generalize to

Theorem 9.2. *Assume* (C) *and* $V = V_{\text{long}} + V_{\text{short}} + V_{\text{sing}}$. *Then the wave operators* $W^{\pm}_{D,a}, a \subset \tilde{a}$ *exist, their ranges are closed and mutually orthogonal. Furthermore they are complete.*

Theorem 9.2 implies

Corollary 9.3 (Coulomb systems). *Let* $V_a = \frac{q_a}{|x^a|}$, $q_a \in \mathbb{R}$ *and assume* (C). *Then the conclusions of Theorem* 9.2 *hold.*

Notice that in this case $V_{\text{long}} \neq 0$ if $\tilde{a} \neq a_{\min}$, but we can take $I_{\tilde{a},l} = 0$ in the purely multiplicative phase factor.

By Corollary 9.3 and [B, Si, FH] we obtain the following result for the physical Stark Hamiltonian with Coulomb interactions.

Corollary 9.4 (Coulomb systems). *Let* H *be given as in Sect.* 1 *and suppose* $v_{ij}(y) = \frac{q_i q_j}{|y|}$, $v \geqq 3$. *Then the wave operators* $W^{\pm}_{D,a_{\min}}$ *exist and are unitary operators on* $L^2(X)$.

In a similar way one can prove asymptotic completeness for Born–Oppenheimer molecules.

Appendix A. Minimal Velocity Estimate

In this appendix we will prove the following proposition.

Proposition A.1. *Assume* (C), $V = V_l + V_{\text{sing}}$ *and that* V_l *satisfies* (V1–2). *Then there exist* $\rho, \theta > 0$ *such that for all* $f \in C_0^\infty(\mathbb{R})$ *we have*

$$F\left(\frac{|x|}{\langle t\rangle^2} < \theta\right) \exp(-itH) f(H) \langle x\rangle^{-1} = O(\langle t\rangle^{-\rho}) \quad \text{for } |t| \to \infty .$$

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Consider the observable

$$A = \frac{1}{2}\{E(x) \cdot p + p \cdot E(x)\} = E(x) \cdot p - \frac{i}{2}\nabla \cdot E(x) ,$$

defined in Appendix B in [HMS1]. By [HMS1, Proposition B.4 and Corollary B.6] we can use [Sk, Corollary 2.5] (with $\theta = 1$, $t_0 = 1$, $\kappa_0 = 0$, $n_0 = 2$ and $\beta_0, \alpha_0 \in \mathbb{R}$

such that $0 < \frac{\alpha_0}{2} < \beta_0 < \alpha_0 < 1$) to obtain $\rho, \delta > 0$ such that

$$F\left(\frac{A}{\langle t\rangle} < 2\delta\right) \exp(-itH)f(H)(H+i)\langle A\rangle^{-1} = O(\langle t\rangle^{-\rho}) \quad \text{for } t \to +\infty .$$

Although Corollary 2.5 in [Sk] is not directly applicable since it requires H to be bounded from below the result still holds. To see this we remark that [Sk, Lemma 2.11] can be proved without the lower boundedness assumption by using the representation formula (3.6). For that one needs a slight modification of [Sk, Lemma 2.10].

Since (3.1), Lemma 3.2(2) and [HMS1, (B.5)] imply that $\langle A\rangle(H+i)^{-1}\langle x\rangle^{-1}$ is a bounded operator we thus obtain

$$F\left(\frac{A}{\langle t\rangle} < 2\delta\right) \exp(-itH)f(H)\langle x\rangle^{-1} = O(\langle t\rangle^{-\rho}) . \tag{A.1}$$

We can assume $\rho \leqq 2$.

Let χ and ψ be abbrevations for

$$\chi\left(\frac{|x|}{\langle t\rangle^2} < \theta\right) \quad \text{and} \quad \exp(-itH)f(H)\langle x\rangle^{-1}\varphi, \quad \varphi \in L^2(X) ,$$

respectively. Here $\theta > 0$ will be chosen later.

In the following all estimates are uniform with respect to φ, $\|\varphi\| \leqq 1$. We compute using [HMS1, Proposition 6.4] and (3.1)

$$\begin{aligned}
\|p\chi\psi\|^2 &= \langle H - V + E \cdot x\rangle_{\chi\psi} \\
&\leqq \|\chi\psi\|\{1 + \|(V+i)(H+i)^{-1}\|\}\|(H+i)\chi\psi\| + 2\theta|E|\langle t\rangle^2\|\chi\psi\|^2 \\
&= \sqrt{\theta}\langle t\rangle\|\chi\psi\| \cdot \frac{C}{\sqrt{\theta}\langle t\rangle}\|(H+i)\chi\psi\| + 2\theta|E|\langle t\rangle^2\|\chi\psi\|^2 \\
&\leqq \theta\langle t\rangle^2\|\chi\psi\|^2 + \frac{C^2}{\theta\langle t\rangle^2}\|(H+i)\chi\psi\|^2 + 2\theta|E|\langle t\rangle^2\|\chi\psi\|^2 \\
&\leqq \theta(2|E|+1)\langle t\rangle^2\|\chi\psi\|^2 + \frac{C_{\theta,1}}{\langle t\rangle^2} .
\end{aligned}$$

Using this estimate we calculate

$$\begin{aligned}
\|A\chi\psi\|^2 &\leqq 2\|E(x) \cdot p\chi\psi\|^2 + C_1\|\chi\psi\|^2 \\
&\leqq \left\{2\left(\sup_{x\in X}|E(x)|^2\right)\theta(2|E|+1)\langle t\rangle^2 + \sqrt{\theta}\langle t\rangle\frac{C_1}{\sqrt{\theta}\langle t\rangle}\right\}\|\chi\psi\|^2 + \frac{C_{\theta,2}}{\langle t\rangle^2} \\
&\leqq M_\theta\langle t\rangle^2\|\chi\psi\|^2 + \frac{C_{\theta,3}}{\langle t\rangle^2} ,
\end{aligned} \tag{A.2}$$

where

$$M_\theta = \theta\{1 + (4|E|+2)\sup_{x\in X}|E(x)|^2\} .$$

We can also estimate

$$\|A\chi\psi\|^2 \geqq \delta^2\langle t\rangle^2\left\langle\chi\left(\frac{A^2}{\langle t\rangle^2} > \delta^2\right)\right\rangle_{\chi\psi}$$

$$= \delta^2\langle t\rangle^2\|\chi\psi\|^2 - \delta^2\langle t\rangle^2\left\|\chi\left(\frac{A^2}{\langle t\rangle^2} < \delta^2\right)\chi\psi\right\|^2 . \tag{A.3}$$

Putting (A.2–3) together we obtain

$$\delta^2\langle t\rangle^2\left\|\chi\left(\frac{A^2}{\langle t\rangle^2} < \delta^2\right)\chi\psi\right\|^2 + \frac{C_{\theta,3}}{\langle t\rangle^2} \geqq (\delta^2 - M_\theta)\langle t\rangle^2\|\chi\psi\|^2 . \tag{A.4}$$

We now choose $\theta > 0$ such that $M_\theta < \frac{\delta^2}{2}$. To estimate the left-hand side we notice that the commutator

$$\left[\chi\left(\frac{A^2}{\langle t\rangle^2} < \delta^2\right), \chi\right] = O(\langle t\rangle^{-3}) ,$$

as can be shown by first writing it on the form $[g(\frac{A^2}{\langle t\rangle^2}), \chi]$ for some $g \in C_0^\infty(\mathbb{R})$ and then applying (3.6). We thus obtain

$$\left\|\chi\left(\frac{A^2}{\langle t\rangle^2} < \delta^2\right)\chi\psi\right\|^2 \leqq 2\left\|\chi\left(\frac{A^2}{\langle t\rangle^2} < \delta^2\right)\psi\right\|^2 + C\langle t\rangle^{-6} .$$

This result in conjunction with (A.1) and (A.4) yields

$$\|\chi\psi\| \leqq C_1\langle t\rangle^{-\rho} . \qquad \square$$

Note added in Proof. After the submission of this paper the authors realized how to remove the technical condition (V7) assumed for the main result, Theorem 9.2. This is done by proving a weaker version of Lemma 6.4 using certain differential inequalities.

References

[A] Adachi, T.: Long-range scattering for three-body Stark Hamiltonians. Preprint 1993

[B] Balslev, E.: Absence of positive eigenvalues of Schrödinger operators. Arch. Rational Mech. Anal. **54**, 343–357 (1975)

[D] Derezinski, J.: Asymptotic completeness of long range N-body quantum systems. Ann. Math. **138**, 427–476 (1993)

[E] Enss, V.: Long-range scattering of two- and three-body quantum systems. Proceedings of the conference "Equations aux derivees partielles" Saint Jean de Monts, Ecole Polytechnique, 1989

[FH] Froese, R.G., Herbst, I.: Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators. Commun. Math. Phys. **87**, 429–447 (1982)

[G1] Graf, G.M.: Asymptotic completeness for N-body short-range quantum systems: A new proof. Commun. Math. Phys. **132**, 73–101 (1990)

[G2] Graf, G.M.: A remark on long-range Stark scattering. Helvetica Physica Acta **64**, 1168–1174 (1991)

[He] Herbst, I.: Spectral and scattering theory for Schrödinger operators with potentials independent of $|x|$. Am. J. Math. **113**, 509–565 (1991)

[Hö] Hörmander, L.: The analysis of linear partial differential operators I. Berlin, Heidelberg, New York: Springer, 1983

[HMS1] Herbst, I., Møller, J.S., Skibsted, E.: Spectral analysis of N-body Stark Hamiltonians. Commun. Math. Phys. (1995)

[K] Korotyaev, E.L.: On the scattering theory of several particles in an external electric field. Math. USSR Sb. **60**, 177–196 (1988)

[Sk] Skibsted, E.: Propagation estimates for N-body Schrödinger operators. Commun. Math. Phys. **142**, 67–98 (1991)

[Si] Simon, B.: Absence of positive eigenvalues in a class of multiparticle quantum systems. Math. Ann. **207**, 133–138 (1974)

[SS] Sigal, I.M., Soffer, A.: The N-particle scattering problem: Asymptotic completeness for short-range systems. Ann. Math. **125**, 35–108 (1987)

[T1] Tamura, H.: Spectral and scattering theory for 3-particle Hamiltonians with Stark effect: Asymptotic completeness. Osaka J. Math. **29**, 135–159 (1992)

[T2] Tamura, H.: Spectral and scattering theory for 3-particle Hamiltonian with Stark effect: Non-existence of bound states and resolvent estimate. Preprint 1993

[T3] Tamura, H.: Scattering theory for N-particle systems with Stark Effect: Asymptotic completeness. Publ. RIMS, Kyoto Univ. **29**, 869–884 (1993)

[Z] Zorbas, J.: Scattering theory for Stark Hamiltonians involving long-range potentials. J. Math. Phys. **19**, 577–580 (1978)

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