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The Geometry of the Quantum Correction for Topological σ-Models

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Abstract: The ring (Frobenius algebra) of local observables for topological σ-models on $\mathbb{P}^1$ with values in the grassmannian $G(s,n)$ is known to be "the same as" the quotient of the homology ring of the target space by the (inhomogeneous) ideal generated by the so-called quantum correction. While the need for a quantum correction comes from algebraic motivations in field theory, the aim of this paper is to understand its geometric meaning. The simple examples of $\mathbb{P}^1 \to \mathbb{P}^n$ models tell us that the quantum correction comes by restriction on the boundary of the moduli spaces which allows to compute intersections on moduli spaces of lower degrees. We will check this point of view for the case of $\mathbb{P}^1 \to G(s,n)$ models, yielding a proof of the algebraic result from physics in terms of the geometry of the σ-model itself.

1. Introduction

A topological field theory (TFT) is an algebraic object; it is the datum of a Frobenius algebra [D], together with its deformations. There are some TFTs directly connected with geometry (which will be called geometrical TFT or GTFT, for short). These are actually the first examples of topological field theories (see e.g [W, G, I, V]), with concrete realizations in terms of σ-models, topological Yang–Mills theory and topological gravity. The mathematical interest of these examples is that the expectation values of physical interest are actually intersection numbers in some homology rings. The ring of "topological observables" in all known GTFTs is identified in the physical literature with the quotient of the homology ring of suitable moduli spaces by the (inhomogeneous) ideal generated by the so-called "quantum correction."

We immediately have a problem: to understand for a given GTFT the geometrical origin of the quantum correction. The aim of this paper is to give an answer in the case of topological σ-models on the Riemann sphere $\mathbb{P}^1$ with values in

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grassmannian manifolds. In these models the quantum correction has already been computed by means of the algebraic techniques of field theory [I].

The simplest example is the $\mathbb{P}^1 \to \mathbb{P}^1$ models (see [FR] for details). The "moduli" space is the space of holomorphic maps of degree d from the Riemann sphere $\mathbb{P}^1$ to itself, which is isomorphic to $\mathbb{P}^{2d+1}$. Loosely speaking, there is a sequence of inclusions $\mathbb{P}^1 \subset \mathbb{P}^3 \subset \cdots \subset \mathbb{P}^\infty$, thanks to the fact that a degenerate map of degree d is generically the same as the datum of a map of degree $d-1$ (see e.g. [ACGH]) together with the datum of the degeneracy point, which we keep fixed. Now the homology of $\mathbb{P}^\infty$ is generated by the hyperplane class ω, with no relations. According to the rules of TFT, the expectation value $\langle\omega^k\rangle$ gets contributions from all instanton sectors, i.e. we have $\langle\omega^k\rangle = \sum_d \omega^k[\mathbb{P}^{2d+1}]$, where each term of the r.h.s. is the intersection number of ω^k on $\mathbb{P}^{2d+1}$. As this vanishes when $k \neq 2d+1$, we get a single contribution $\langle\omega^{2d+1}\rangle = \omega^{2d+1}[\mathbb{P}^{2d+1}] = 1$ from the intersection of $2d+1$ hyperplanes on $\mathbb{P}^{2d+1}$. The geometrical origin of the quantum correction in this example is as simple as setting $\omega^{2d+1} = \omega^{2(d-1)+1}\omega^2$ and noticing that ω^2 is represented by $\mathbb{P}^{2d-1} \subset \mathbb{P}^{2d+1}$; accordingly we have $\omega^{2d+1}[\mathbb{P}^{2d+1}] = \omega^{2d-1}[\mathbb{P}^{2d-1}]$, which is formally the same as setting $\omega^2 = 1$ in the word ω^{2d+1} and evaluating the quotient word on $\mathbb{P}^{2d-1}$. This procedure can be iterated down to $d = 0$.

More examples are given by the $\mathbb{P}^1 \to \mathbb{P}^{n*}$ models. Here again the "moduli" spaces of maps of degree d are isomorphic to $\mathbb{P}^{(n+1)d+n}$. The chain of inclusions $\mathbb{P}^{(n+1)(d-1)+n} \subset \mathbb{P}^{(n+1)d+n}$ tells us that $\mathbb{P}^{(n+1)(d-1)+n}$ in $\mathbb{P}^{(n+1)d+n}$ represents ω^{n+1} which therefore can be set to 1 as above.

Less obvious is the geometric meaning of the quantum correction for $\mathbb{P}^1 \to G(s;n)$ models (here $G(s;n)$ denotes the grassmannian of the s-planes in an n dimensional complex linear space V). First of the "moduli" spaces $\mathcal{R}_d$ of maps of degree d to $G(s;n)$ are a bit more subtle, bringing into the game certain *Quot* schemes [Gr]. The basic fact for us is that on $X_d = \mathbb{P}^1 \times \mathcal{R}_d$ there is a universal exact sequence [S]

$$0 \to A \to V_{X_d} \to B \to 0\,,$$

where $V_{X_d} = X_d \times V$. Once restricted to the locus $X_d^{(s)}$, where B is locally free, this sequence is the same as the datum of a holomorphic map $f_d : X_d^{(s)} \to G(s;n)$, called in the physical literature the "universal instanton." Since, as we will show, the degeneracy locus $X_d^{s-1} = X_d \backslash X_d^{(s)}$ of B has codimension $r+1 = n-s+1 \geqq 2$ in X_d, f_d can be considered as a rational map on the whole of X_d. The induced map $f_d^* : A(G(s;n)) \to A(X_d)$ of the Chow rings is not a ring homomorphism and will possibly account for the correction to Pieri's formula found in the physical literature [I] on purely algebraic grounds.

In this paper we will concentrate on an example of this phenomenon, which is relevant in understanding the quantum correction. We will study the intersection $\tau = (f_d^{-1}\sigma_r)\cdot(f_d^{-1}\sigma_{1,\dots,1})$, where $\sigma_{a_1,\dots,a_s}$ denotes a Schubert cycle in $G(s;n)$ (see [GH] for notations). This is interesting since, although Pieri's formula tells us that $\sigma_r \cdot \sigma_{1,\dots,1} = 0$ in $A(G(s;n))$, τ does not vanish in $A(X_d)$. By Künneth decomposing $\tau = \{p\} \times a + \mathbb{P}^1 \times b$, we see that the "local observable" b can be represented by a subvariety $\tilde{b}$ of codimension n in $\mathcal{R}_d$. We will prove that

1) there is a rational map $g_p : \mathcal{R}_{d-1} \to \mathcal{R}_d$ whose image covers a component of $\tilde{b}$ (see Proposition 4),

2) given a word $P(s_1,\dots,s_k)$, with $f_d^{-1}\sigma_I = \{p\} \times a_I + \mathbb{P}^1 \times s_I \in A(X_d)$, of codimension $n(d-1)+rs$, the word $P(s_1',\dots,s_k')$ with $f_{d-1}^*\sigma_I = \{p\} \times a_I' +$

$\mathbb{P} \times s'_I \in A(X_{d-1})$ is such that

$$\langle P(s_1,\dots,s_k) \cdot b\rangle_{\mathscr{R}_d} = \langle P(s'_1,\dots,s'_k)\rangle_{\mathscr{R}_{d-1}} \,.$$

(Proposition 9).

Accordingly, the quantum correction "is the same as" setting to 1 the homology class b on $\mathscr{R}_d$ and compute expectations at degree $d-1$. Section 3 will be devoted to explain this result in full detail.

There are still open problems to be answered:

1) to show that one recovers the full quantum correction to Pieri's formula in the grassmannian case,

2) to extend the above constructions to algebraic homogeneous spaces G/P, with G semisimple complex Lie group and P a parabolic subgroup,

3) to compute expectation values of non-local observables as intersection numbers, as a clue to set up non-formal perturbation theory.

2. Parametrized Rational Curves on Grassmannians

2.1. We start recalling a natural compactification $\mathscr{R}_d$ of the moduli space $\mathscr{R}_d^{(s)}$ of degree d instantons $f : \mathbb{P}^1 \to G(s;n)$ with values in the grassmannian of s-planes in an n-dimensional linear space V (we will omit the index d whenever convenient). To this end we need to master simultaneously several descriptions of a map to a grassmannian. Let us recall for completeness the following well known facts:

Proposition 1. *There is a one-to-one correspondence between*:

i) *holomorphic maps* $f : \mathbb{P}^1 \to G(s;n)$ *with* $\deg f = d$,

ii) *locally free quotients* B *of* $V_{\mathbb{P}^1}$ $(V_{\mathbb{P}^1} = \mathbb{P}^1 \times V)$ *with* $\operatorname{rank} B = r =: n-s$, $\deg B = d$,

iii) *maximal rank morphisms* $\phi \in H^0(\mathbb{P}^1, A^* \otimes V_{\mathbb{P}^1})$ *of a locally free sheaf* A (*with* $\operatorname{rank} A = s$, $\deg A = -d$) *into* $V_{\mathbb{P}^1}$, *modulo automorphisms of* A.

Proof. We will sketch the proof in terms of the vector bundles corresponding to the locally free sheaves above. Pulling back to $\mathbb{P}^1$ the universal exact sequence on $G(s,n)$,

$$0 \to S \to G(s,n) \times V \to Q \to 0\,,$$

one gets that $B = f^*Q$ is a locally free quotient of $f^*(G(s;n) \times V)$ with the right rank and degree. Clearly, $B_i = f_i^*Q$, $(i = 1,2)$ coincide as quotients iff $f_1 = f_2$. Conversely, given a B as in ii) there are sections $s_1,\dots,s_n \in V \subseteq H^0(\mathbb{P}, B)$ such that the evaluation map $ev_p : V \to B_p$ given by $ev_p(v_1,\dots,v_v) = \sum v_i s_i(p)$ is surjective on all fibres B_p of B. Then setting $f(p) = \ker ev_p \subset V$ gives the desired map to $G(s;n)$. Obviously enough, such a map does not depend on the sections s_i, provided they span all the fibres of B, and therefore there is a unique map f for any such B and moreover $f^*Q = B$. As for iii), notice that setting $A = \ker ev \subset V_{\mathbb{P}}$ there is a canonical exact sequence

$$0 \to A \overset{\phi}{\to} V_{\mathbb{P}} \overset{ev}{\to} B \to 0\,.$$

Now, for any automorphism a of A, we get a morphism $\phi_a = \phi \circ a : A \to V_{\mathbb{P}}$ with the same cokernel B. Then we see that the datum of i) or ii) correspond bijectively to the datum of iii).

Remarks. 1) The set $\mathscr{R}^{(s)}$ is a smooth complex manifold [S]. Therefore its dimension is equal to the dimension of the tangent space to a point f which is isomorphic to $H^0(\mathbb{P}^1, A^* \otimes B)$, A and B being the sheaves associated to f by Propositon 1. This can be computed from the sequence

$$0 \to A^* \otimes A \to A^* \otimes V_{\mathbb{P}} \to A^* \otimes B \to 0\,,$$

which is exact, because A is locally free. The associated cohomology sequence reads

$$0 \to H^0(\mathbb{P}, A^* \otimes A) \to H^0(\mathbb{P}, A^* \otimes V_{\mathbb{P}}) \to H^0(\mathbb{P}, A^* \otimes B) \to H^1(\mathbb{P}, A^* \otimes A) \to 0\,,$$

because $H^1(\mathbb{P}, A^* \otimes V) = 0$. Indeed, by a theorem of Grothendieck's, we know that for any locally free sheaf A on $\mathbb{P}^1$ with $\deg A = -d$ there is a unique non-increasing sequence of integers $d_1 \geqq \cdots \geqq d_s$ such that $A \simeq \bigoplus_1^s \mathcal{O}(-d_\iota)$, with $\sum d_\iota = d_s$, A being a subsheaf of a trivial sheaf, for every i $d_i \geqq 0$. From the cohomology sequence above, it follows that $\dim H^0(\mathbb{P}, A^* \otimes B) = \chi(A^* \otimes V) - \chi(A^* \otimes A)$. From the Riemann–Roch theorem we have:

$$\dim \mathscr{R}^{(s)} = nd + sn - s^2 = nd + r(n-r)\,.$$

2) As $A = \bigoplus_1^s \mathcal{O}(-d_\iota)$ $(d_\iota \geqq 0)$, we have isomorphisms $H^0(\mathbb{P}^1, A^* \otimes V_{\mathbb{P}^1}) \simeq H^0(\mathbb{P}^1, \bigoplus_1^s \mathcal{O}(d_\iota) \otimes V_{\mathbb{P}^1}) \simeq \bigoplus_1^s H^0(\mathbb{P}^1, \mathcal{O}(d_i)^{(n)})$, where the apex (n) denotes the direct sum n times. Accordingly, every morphism $\phi : A \to V_{\mathbb{P}^1}$ can be represented as an s-tuple $(\vec{\phi}_1, \ldots, \vec{\phi}_s)$ of morphisms $\vec{\phi}_i : \mathcal{O}(-d_\iota) \to V$. In other words, if z is an affine coordinate on $\mathbb{P}^1$, $\vec{\phi}_\iota$ is a vector whose entries $\phi_i^k(z)$ are polynomials of degree d_i. If ϕ has maximal rank as in iii) of the proposition above, the whole $s \times n$ matrix $\Phi(z)$ with entries $\phi_\iota^k(z)$ has maximal rank for z in a suitable neighborhood of 0. In particular $\Phi(0)$ will have an $s \times s$ minor $\Phi_K(0)$, with entries $\phi_\iota^{k_j}(0)$ labelled by a multi-index $K = (k_1, \ldots, k_s)$, which is invertible. Then there is a unique $a \in \mathrm{Aut}(A)$ such that the matrix Φ_a corresponding to the morphism $\phi_a = \phi \circ a$ will have the minor $(\Phi_a)_K(0)$ equal to the identity. For instance if this is the first minor (i.e. $K = (1, \ldots, s)$) we can write

$$\Phi(z) = \begin{pmatrix} 1 + p_1^1(z) & p_2^1(z) & \cdots & p_s^1(z) \\ p_1^2(z) & 1 + p_2^2(z) & \cdots & p_s^2(z) \\ \vdots & \vdots & \cdots & \vdots \\ p_1^s(z) & p_2^s(z) & \cdots & 1 + p_s^s(z) \\ q_1^1(z) & q_2^1(z) & \cdots & q_s^1(z) \\ \vdots & \vdots & \cdots & \vdots \\ q_1^r(z) & q_2^r(z) & \cdots & q_s^r(z) \end{pmatrix}\,,$$

where the $p_k^i(z)$, $q_j^\iota(z)$ are polynomials of degree d_i, the $p_k^\iota(z)$ vanishing at $z = 0$. We can now count directly the number of free parameters in such a matrix; for each $i = 1, \ldots, s$ there are sd_ι parameters for the $p_k^i(z)$ $(k = 1, \ldots, s)$ and $r(d_\iota + 1)$ parameters for the $q_j^\iota(z)$ $(j = 1, \ldots, r)$ adding to a total of $(s + r)d + rs = \dim R^{(s)}$. This numerical coincidence comes from the fact that, whenever the polynomials occurring in Φ are generic enough, the closure of the image of the map $f : \mathbb{C} \subset \mathbb{P} \to G(s; n)$ corresponding to Φ is actually an instanton of degree d. If we write

$\Phi(z) = (\Phi_K(z)|\Phi_{K'}(z))^t$, $K = (1,\ldots,s)$, $K' = (s+1,\ldots,s+r)$, the minor $\Phi_K(z)$, being invertible at $z = 0$, is invertible for $|z| < \varepsilon$ and we have

$$\Phi(z) = \begin{pmatrix} \mathbf{1} \\ T(z) \end{pmatrix} \Phi_K(z)^{-1} ,$$

the rs entries of the matrix $T(z)$ giving a local representation of the map f in terms of rational functions.

2.2. It is clear that $\mathscr{R}^{(s)}$ is not compact, as a family of maximal rank morphisms $\phi_t : A \to V_{\mathbb{P}^1}$, $(t \in \mathbb{C},\ 0 \leqq |t| \leqq 1)$, may degenerate to a morphism ϕ_0 of lower rank. Equivalently, a family B_t of locally free quotients may degenerate to a coherent sheaf B_0 which is no longer locally free. Of course, to get a sensible compactification of $\mathscr{R}^{(s)}$ we will consider flat families only. Notice that such degenerate quotients do not give rise any more to holomorphic maps to $G(s;n)$; nevertheless they will play a crucial rôle in the following.

The obvious thing to do is to extend $\mathscr{R}^{(s)}$ to the set $\mathscr{R} \supset \mathscr{R}^{(s)}$ of coherent quotients B of $V_{\mathbb{P}^1}$ of degree d and rank r, that is with Hilbert polynomial $(m+1)r+d$. The set $\mathscr{R}$ is actually a *Quot* scheme [G], which turns out to be [S] an (irreducible, rational) smooth projective variety, giving a projective compactification of $\mathscr{R}^{(s)}$.

We next summarize some of the ideas in [S]. By Proposition 1, the datum of a morphism $f : \mathbb{P}^1 \to G(s;n)$ is equivalent to the datum of a locally free quotient B of $V_{\mathbb{P}^1}$ of degree d and rank $r = n - s$. The Hilbert polynomial of B is defined as the asymptotical expression of the Hilbert function $\dim(H^0(B \otimes O(m)) - \dim(H^1(B \otimes O(m)) \simeq (m+1)r + d$. Such a polynomial is constant on flat families of quotients [H], but such quotients may degenerate to a quotient which is no longer locally free. The *Quot* scheme $R_d = Quot^{(m+1)r+d}_{V,\mathbb{P}^1}$ [Gr] is a scheme parametrizing all the quotients with fixed Hilbert polynomial in an universal way. It gives rise to an universal short exact sequence on $X_d = \mathbb{P}^1 \times \mathscr{R}_d$,

$$0 \to A \to V_{X_d} \to B \to 0 ,$$

where B is flat on R_d (with respect to the projection $p_2 : X_d \to R_d$), has Hilbert polynomial $\chi(B(m)) = (m+1)r + d$ on the fibres of p_2. The universality property is that for every flat family of quotients of $V_{\mathbb{P}^1}$ parametrized by a scheme T,

$$0 \to E \to V_{\mathbb{P}^1 \times T} \to F \to 0 ,$$

with Hilbert polynomial $(m+1)r + d$ on the fibres of $p_2 : \mathbb{P}^1 \times T \to T$, there is a morphism $\xi : T \to R_d$ such that the last sequence on T is the pull back via ξ of the universal one. In other words R_d "represents" the functor of flat families of such quotients. It turns out that the scheme R_d is an irreducible, rational, non-singular, projective variety with dimension $nd + r(n-r) = nd + \dim G(s;n)$.

One can give an explicit description of the scheme R_d. Following [S] let us denote B_m (A_m) the coherent sheaves $p_{2*}E(m)$, $E(m) = E \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(m)$ for $E = B$, ($E = A$ resp.). B_m is locally free with rank $(m+1)r + d$ whereas A_m is locally free with rank $(m+1)s - d$ if $m \geqq d - 1$, as follows from the Grauert and Riemann–Roch theorems plus the obvious fact that $H^1(A(d-1)|_{\mathbb{P}\times\rho}) = 0$ for all $\rho \in \mathscr{R}$.

Let us set $V_m = V_{\mathbb{P}^1} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m))$ and $V_m = p_{2*}V(m)$. The push forward on R_d of the universal sequence reads

$$0 \to A_m \to V_m \to B_m \to 0 .$$

This in turn induces a morphism $R_d \to G((m+1)s - d;\, n(m+1)) = G_m$ in such a way that it is isomorphic to the pull back of the tautological sequence on G_m,

$$0 \to S_m \to V_{m,G_m} \to Q_m \to 0 .$$

For each $m \geqq 0$ there is [S] a natural exact sequence on $\mathbb{P}^1$,

$$0 \to V_{m-1}(-1) \to V_m \to V(m) \to 0 ,$$

inducing a map $j_m : V_{m-1} \to V_m \otimes H^0(\mathcal{O}(1))$, and a diagram on $G = G_{m-1} \times G_m$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (S_{m-1})_G & \xrightarrow{i_{m-1}} & (V_{m-1})_G & \longrightarrow & (Q_{m-1})_G & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle j_m} & & & & \\ 0 & \longrightarrow & (S_m)_G \otimes \mathbb{C}^2 & \longrightarrow & (V_m)_G \otimes \mathbb{C}^2 & \xrightarrow{p_m} & (Q_m)_G \otimes \mathbb{C}^2 . & & \end{array}$$

It is a very nice fact ([S] Theorem 4.1) that there exists an isomorphism between R_d and the zero locus of the composite map $\sigma = p_m \circ j_m \circ i_{m-1}$. In particular the class of R_d in G is the square of the top Chern class of the bundle $S^*_{m-1} \otimes Q_m$. We will need the following simple remark.

Proposition 2. *Let A, B be the universal subsheaf and the universal quotient of V_{X_d} in the sequence above. Then*

i) *A is locally free of* rank s,

ii) *B has a torsion subsheaf with support on the locus X_d^{s-1}, where ϕ has rank $\leqq s-1$.*

Proof. As for i), as a subsheaf of a locally free sheaf, A is torsion free. By flatness, the injection of A_{X_d} into V_{X_d} restricts to an injection on $\mathbb{P}^1$. Since torsion free sheaves on $\mathbb{P}^1$ are locally free, the rank of A is constant on $\mathbb{P}^1 \times \rho$ for every $\rho \in \mathcal{R}$. By base change, the restriction of A to $p \times \mathcal{R}$ is flat on $\mathcal{R}$ and hence locally free. Therefore the rank is constant on X_d and by Nakayama's lemma A is locally free. Finally ii) is a direct consequence of i) and Proposition 1, iii).

3. The Quantum Correction for Grassmannians

3.1. As we learned from the example of the $\mathbb{P}^1$ model recalled in the introduction, the quantum correction is actually a device to reduce the computation of intersection numbers on the moduli space of degree d instantons to a similar computation on the moduli space of degree $d-1$. To see how this goes in the present case, we need to find a map $g_p : \mathcal{R}_{d-1} \to \mathcal{R}_d$ and to compute the homology class of its image.

We will be actually a bit more general and work on X_d. This will help in keeping control of the "non-local observables" as well as of the local ones. To construct g_p, we will adopt the following procedure:

i) We will blow up an open subvariety $\tilde{X}_d \subset X_d$ on a suitable locus, getting a holomorphic map $\tilde{f} : Bl(\tilde{X}_d) \to G(s;n)$ from this blow up to the target grassmannian (Sect. 3.2).

ii) We will study the cycles $\tilde{f}^{-1}\sigma_r$ and $\tilde{f}^{-1}\sigma_{1,\dots,1}$ in $Bl(\tilde{X}_d)$, which are Poincaré dual to the top Chern class and to the top Segre class on $G(s;n)$. We will find that, although σ_r and $\sigma_{1,\dots,1}$ do not intersect on $G(s;n)$ and therefore $\tilde{f}^{-1}\sigma_r$ and $\tilde{f}^{-1}\sigma_{1,\dots,1}$ do not intersect in $Bl(\tilde{X}_d)$, both these last two cycles intersect the exceptional divisor in $Bl(\tilde{X}_d)$. So, setting $S_s = \pi_*(\tilde{f}^{-1}\sigma_{1,\dots,1}) = f^{-1}(\sigma_{1,\dots,1})$, $C_r = \pi_*(\tilde{f}^{-1}\sigma_r) = f^{-1}(\sigma_r)$, where $\pi : Bl(\tilde{X}_d) \to \tilde{X}_d$ is the blow-down map, we get that the intersection $S_s \cdot C_r$ is not empty and has codimension $n = r + s$ in $\tilde{X}_d$ (see Sect. 3.3).

iii) We will next show that, for any $p \in \mathbb{P}^1$, there is a birational map $g_p : \{p\} \times R_{d-1} \to C_r \cdot S_s \cdot (\{p\} \times \mathscr{R}_d)$ (Sect. 3.4).

We will first give an intuitive clue to these facts. The proofs require a slightly different set up and will be given at the end of this section.

3.2. We will denote by X_d^k the locus in X_d where $\operatorname{rank}\phi \leqq k$, while as above $X_d^{(k)}$ denotes the locus where ϕ has rank strictly equal to k. Given $(p, \rho_0) \in X_d^{(s-1)}$, the morphism $\phi_0 = \phi(\,\cdot\,, \rho_0)$ will have rank s but at isolated points in $\mathbb{P}^1$, including of course the point p itself. If not, $B|_{\rho_0}$ would have a rank strictly larger than r. It will be enough for our aims to study the generic case in which ϕ_0 is non-degenerate on $\mathbb{P}^1 - \{p\}$ and degenerates "of order 1" (see below) at p. Indeed the non-generic locus has codimension $\geqq 2$ in $X_d^{(s-1)}$ and we can forget about it while studying rational maps. Since $\operatorname{rank}\phi_0(p) = s - 1$, its image at p will determine a point $Q \in G(s-1;n)$. Let us consider the set of all s-planes $W \in G(s;n)$ containing Q; this set is clearly isomorphic to the projective space $\mathbb{P}^r$, as the choice of such a W is the same as fixing a vector in $V/Q - \{0\} \simeq \mathbb{C}^{r+1} - \{0\}$ modulo homotheties.

Lemma 3. *For every $x_0 = (p, \rho) \in X_d^{(s-1)}$, the fibre at x_0 of the normal bundle $N_{X_d/X_d^{(s-1)}}$ to $X_d^{(s-1)}$ in X_d is isomorphic to V/Q, with $Q = \operatorname{Im}\phi(p, \rho_0)$.*

Proof. Set $x_0 = (p, \rho_0)$, fix $W \supset Q$ and choose $\vec{w} \in V$, $\vec{w} \notin Q$ such that $W = \mathbb{C}\{\vec{w}\} \oplus Q$. It is enough to construct a curve $x_t(\vec{w}) \subset X_d^{(s)}$ for $t \neq 0$ with $x_0(\vec{w}) = x_0$. Assume $A|_{\mathbb{P}^1 \times \{\rho_0\}} \simeq \oplus\mathcal{O}(-d_i)$ and choose a basis of V such that the first $s-1$ vectors span Q and $\vec{e}_s = \vec{w}$. There are automorphisms a of A such that $\phi_0 \circ a = (\vec{\phi}_1, \dots, \vec{\phi}_s)$ has $\vec{\phi}_s = 0$. If z is a local coordinate centered at p, $\phi_0 \circ a$ will have a matrix representation

$$\Phi(z) = \begin{pmatrix} 1 + p_1^1 & p_1^1 & \cdots & p_{s-1}^1 & p_s^1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_1^{s-1} & p_2^{s-1} & \cdots & 1 + p_{s-1}^{s-2} & p_s^{s-1} \\ p_1^s & p_2^s & \cdots & p_{s-1}^s & z(1 + p_s^s) \\ q_1^1 & q_2^1 & \cdots & q_{s-1}^1 & zq_s^1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ q_1^r & q_2^r & \cdots & q_{s-1}^r & zq_s^r \end{pmatrix}$$

whose columns are the $\vec{\phi}_i$ $(i = 1, \dots, s)$ and the polynomials p_j^i vanish at z of order 1 while the q_j^i are generic (by this we mean that ϕ are degenerate of order 1). Then the morphism $\phi_t(w)$ with matrix representation $\Phi_t(\vec{w}) = (\vec{\phi}_1, \dots, \vec{\phi}_s + t\vec{w})$ is non-degenerate for $t \neq 0$. The proof then follows by dimensional reasons.

From the matrix representation above, it is clear that X_d^{s-1} has the "correct" codimension $r+1$ in X_d. Moreover [ACGH] the singular locus of X_d^{s-1} is contained in X_d^{s-2}, hence $X_d^{(s-1)} = X_d^{s-1} \backslash X_d^{s-2}$ is smooth. Set $\tilde{X}_d =: X_d \backslash X_d^{s-2}$. The holomorphic map $f : X_d^{(s)} \to G(s;n)$ uniquely defines a holomorphic map $\tilde{f} : Bl(\tilde{X}_d) \to G(s;n)$, where $Bl(\tilde{X}_d)$ denotes the blow up of $\tilde{X}_d$ on $X_d^{(s-1)}$, with the exceptional divisor the projective normal bundle $\mathbb{P}N_{\tilde{X}_d/X_d^{(s-1)}}$. If $\pi : Bl(\tilde{X}_d) \to \tilde{X}_d$ is the blow-down map, $\tilde{f}$ is given by $\tilde{f}(\tilde{x}) = f(x)$ for $x = \pi(\tilde{x}) \notin X^{(s-1)}$ and $\tilde{f}(x, \vec{w}) = \lim_{t \to 0} \operatorname{Im}(\vec{\phi}_1(p), \ldots, \vec{\phi}_s(p) + t\vec{w})$ for $x = (p, \rho) \in X^{(s-1)}$, where $\vec{\phi}_i$ are the components of the morphism corresponding to ρ with $\vec{\phi}_s(p) = 0$. Notice that since

$$(\vec{\phi}_1(p), \ldots, 0 + t\vec{w}) = (\vec{\phi}_1(p), \ldots, \vec{w}) \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & t \end{pmatrix},$$

the image does not depend on t for all $t \neq 0$ and the limit above exists.

3.3. We want next to study the preimages of the top Segre and Chern classes on $G(s;n)$ under the map $\tilde{f}_p$. Recall that

- a Segre cycle $\sigma_{1,\ldots,1} \subset G(s;n)$ is the set of s-dimensional subspaces of V contained in a given hyperplane $H \subset V$. It is therefore isomorphic to $G(s;n-1)$ and has codimension s,
- a Chern cycle $\sigma_r \subset G(s;n)$ is given by the subspaces containing a given line L of V; it is isomorphic to $G(s-1;n-1)$ and has codimension r.

Since generically $L \not\subset H$, one sees that they do not intersect, i.e. $\sigma_r \cdot \sigma_{1,\ldots,1} = 0$ on $G(s;n)$, nor will their preimages intersect under $\tilde{f}$, i.e. $(\tilde{f}^{-1}c_r) \cdot (\tilde{f}^{-1}s_s) = 0$ on $Bl(\tilde{X})$. However, setting $S_s = \pi_*(\tilde{f}^{-1}\sigma_{1,\ldots,1})$ and $C_r = \pi_*(f^{-1}c_r)$ (where π is the blow-down map), we see that the locus $C_r \cdot S_s \subset \tilde{X}$ is given by the set of points $x \in \tilde{X}$ such that both $\tilde{f}^{-1}\sigma_r$ and $\tilde{f}^{-1}\sigma_{1,\ldots,1}$ intersect the exceptional fibre $\pi^{-1}(x) \subset Bl(\tilde{X})$. Since the last two cycles do not intersect outside of the exceptional components, the generic point $x \in S_s \cdot C_r$ must belong to $X^{(s-1)}$. Conversely, given $x_0 = (p, \rho_0) \in X^{(s-1)}$ corresponding to the morphism $\phi_0 = \phi(\,\cdot\,, \rho_0)$ with $\operatorname{Im}\phi_0 = Q$, look at f on the exceptional fibre $\pi^{-1}(x_0) \simeq \mathbb{P}^r$. We obviously have $\tilde{f}(x_0, \vec{w}) = \operatorname{Im}(\vec{\phi}_1(p), \ldots, \vec{w})$, where $\vec{w} \pmod{Q}$ is now considered as a set of homogeneous coordinates on $\mathbb{P}^r$. Take $H \subset V$ to be the span of the last $n-1$ basis vectors for V and L the span of the first, $\vec{e}_1$ say. Taking $\vec{w} = \vec{e}_1$ we have that $\tilde{f}(x_0, \vec{e}_1) \supset L$, and hence $\tilde{f}(x_0, \vec{e}_1) \in c_r$, but in general there will be no w such that $\tilde{f}(p, w) \subset H$. This will happen precisely when $Q \subset H$, i.e. when $\langle \vec{\phi}_i(p), \vec{e}_1 \rangle = 0$ for all $i = 2, \ldots, s$ (since $\phi_1(p) = 0$ identically) giving us $s-1$ extra conditions. So we see that $S_s \cdot C_r$ has codimension $s-1$ in $X^{(s-1)}$ and codimension $s-1+r+1 = n$ in X.

3.4. Next we want to study the locus $C_r \cdot S_s \subset X$. First of all, let us decompose the relevant cycles as follows:

$$C_r = \{q\} \times v_{r-1} + \mathbb{P}^1 \times c_r\,,$$

$$S_s = \{q\} \times u_{s-1} + \mathbb{P}^1 \times s_s\,.$$

The locus $u_{s-1} \subset \mathscr{R}$ corresponds to the set of morphisms $\phi(\cdot, \rho)$ such that the polynomials ϕ_i^1 $(i = 1, \dots, s)$ have a common zero. It is therefore a component of the set determined by the $s-1$ conditions $\Sigma(\phi_1^1, \phi_i^1) = 0$ $(i = 2, \dots, s)$, where Σ is the Sylvester determinant. Recall that [ACGH] two polynomials $a(z) = \sum_1^l a_i z^i$, $b(z) = \sum_1^m b_i z^i$ have a common zero if and only if

$$\Sigma(a,b) = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_l & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_l & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & a_l \\ b_0 & \cdots & b_m & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & \cdots & b_m & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_0 & \cdots & b_m \end{pmatrix} .$$

Generically, a common root is simple at q, say. The same holds for $v_{r-1} \subset \mathscr{R}$; this is the set of morphisms such that ϕ_1^i $(i \neq 1,\ i = i_1, \dots, i_r)$ have a common root, i.e. it is a component of the locus determined by the $r-1$ conditions $\Sigma(\phi_1^{i_1}, \phi_1^{i_k}) = 0$ $(k = 2, \dots, r)$. Again generically the common root is simple. The cycle s_s arises as follows; for any $p \in \mathbb{P}^1$, consider the set H_p of ϕ's such that $\operatorname{Im} \phi(p) \subset H$, i.e. $\phi_j^1(p) = 0$ $(j = 1, \dots, s)$ and notice that $\bigcup_{p \in \mathbb{P}^1} H_p \simeq \mathbb{P}^1 \times t_s$. In the same way consider the set L_p of ϕ's such that $\operatorname{Im} \phi(p) \supset L$, whence $\bigcup_{p \in \mathbb{P}^1} L_p \simeq \mathbb{P}^1 \times c_r$. Let us now look at the intersection

$$C_r \cdot S_s = \{q\} \times (v_{r-1} \cdot s_s + c_r \cdot u_{s-1}) + \mathbb{P}^1 \times c_r \cdot s_s .$$

This has clearly two components, the second one corresponds to a local observable in the topological σ-model, while the first corresponds to a non-local observable (the "second descendant" of the local observable, according to the physical terminology).

From now on we will concentrate on the study of the "local component" and give a geometrical description of the locus $s_s \cdot c_r \subset \mathscr{R}^{(s-1)}$. For any fixed $p \in \mathbb{P}^1$, let $f_p : \{p\} \times \mathscr{R} \to G(s,n)$ be the universal instanton evaluated at p. Obviously $f_p^{-1}\sigma_r \simeq c_r$ and $f_p^{-1}\sigma_{1,\dots,1} \simeq s_s$ and $c_r \cdot s_s = C_r \cdot S_s \cdot (\{p\} \times \mathscr{R}_d)$. For a generic point $\rho \in \mathscr{R}_{d-1}$ corresponding to the morphism $\phi = \phi(\cdot, \rho)$, we consider the morphism $\phi_0 = \phi(\cdot, \rho_0) =: (z_p \vec{\phi}_1, \vec{\phi}_2, \dots, \vec{\phi}_s)$, where z_p is a section of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing at p (modulo homotheties). Notice that ϕ_0 is clearly of degree $d - 1 + 1 = d$ and degenerates at p. On the other hand, since ρ is generic, one can always assume that $\phi_i^1(p) \neq 0$ $(i \geqq 2)$ and therefore $\operatorname{Im} \phi_0(p) \subset H$. We get in this way a map $g_p : \mathscr{R}_{d-1} \to \mathscr{R}_d$.

Proposition 4. 1) $\operatorname{Im} g_p \simeq c_r \cdot s_s$, 2) *$g_p$ is a birational isomorphism.*

Proof. 1) Easily follows because the matrix representing $\phi_0(p)$ has both the first row and the first column vanishing. So, as in the proof of Lemma 3, there are two vectors $\vec{e}_1, \vec{w}$ such that $\operatorname{Im} \phi_t(\vec{e}_1) \in \sigma_r$ and $\operatorname{Im} \phi_t(\vec{w}) \in \sigma_{1,\dots,1}$.

2) The map $h_p : c_r \cdot s_s \to \mathscr{R}_{d-1}$ given by $h_p(\phi_0) = \phi$ with $\phi(q) = \phi_0(q)$, $(q \neq p)$ and $\operatorname{Im} \phi(p) = \lim_{q \to p} \operatorname{Im} \phi_0(q)$ is an inverse of g_p, wherever defined. Indeed, if $\phi(\cdot, g_p(\rho)) = \phi_0(\cdot)$ and $\phi(\cdot, h_p \circ g_p(\rho)) = \phi_1(\cdot)$ then

$$\operatorname{Im} \phi_1(p) = \lim_{q \to p} \operatorname{Im} \phi_0(q) = \operatorname{Im} \phi(p) .$$

Remark. It is easy to show that s_s and c_r intersect transversally where g_p is well defined. Indeed the codimension of $s_s \cdot c_r$ is $n(=\dim \mathscr{R}_d - \dim \mathscr{R}_{d-1})$ and we can find n vectors normal to this locus as in Lemma 1. The vector $\vec{w}_1 = \langle \phi(p, \rho_0), \vec{e}_1 \rangle \vec{e}_1$ is tangent to c_r, while $w_k = \langle \phi(p, \rho_0), \vec{e}_k \rangle \vec{e}_k$ with $k = 2, \ldots, n$ give $n-1$ vectors tangent to s_s.

3.5. We have now in our hands the basic tools to understand the quantum correction for the case under consideration. First of all, let us recall some results of [S] about the Chow ring $\mathscr{A}(\mathscr{R})$ of $\mathscr{R}$. Every cycle $S_i \in \mathscr{A}(\mathbb{P}^1 \times \mathscr{R})$ can be decomposed as $S_i = \{p\} \times u_{i-1} + \mathbb{P}^1 \times s_i$. It is clear that the cycles s_i represent the local observables of the TFT. Recall also [S] that there is a surjection of the graded ring $\mathbb{Z}(s_1, \ldots, s_s, u_1, \ldots, u_{s-1})$ onto $\mathscr{A}(\mathscr{R})$ which is an isomorphism up to degree d.

One can again use a sketchy argument to conjecture that Proposition 9 below holds true. We would like to prove that the intersection of a cycle s_i with the locus $s_s \cdot c_r$ gives exactly the cycle s_i on R_{d-1}. This is a bit delicate because on $s_s \cdot c_r$ the universal instanton f_p is not defined at all. In other words the representatives we just defined for the cycles s_i do not intersect transversally with the cycle $s_s \cdot c_r$. The naive idea is to change the representatives. Consider "deformed" cycles $s_i(p') := f_{p'}^*(\sigma_{1,\ldots,1})$, where, as usual, $f_{p'}$ is the universal instanton evaluated at p'. It is clear that $s_i(p) \simeq s_i(p')$ in $A(R_d)$ for every $p,\ p' \in \mathbf{P}^1$. Moreover if a quotient ρ_0 in $s_s \cdot c_r$ is represented by a couple (H, ρ) with $\rho \in R_{d-1}$, we have $f_{p'}(\rho_0) = \operatorname{Im} \phi(p', \rho)$ if $p \neq p'$, and hence $\lim_{p' \to p} f_{p'}(\rho_0) = \lim_{p' \to p} \operatorname{Im} \phi(p', \rho) = \operatorname{Im} \phi(p, \rho) = f_p'(\rho)$, where $f_p' : \{p\} \times \mathscr{R}_{d-1} \to G(s, n)$ is the universal instanton of degree $d-1$. Notice that, if we set $s_i' = f'^{-1} \sigma_{1,\ldots,1}$ on R_{d-1}, we finally get $\langle P(s_1, \ldots, s_s) \cdot (s_s \cdot c_r) \rangle_{R_d} = \lim_{p' \to p} \langle P(t_1(p'), \ldots, t_s(p')) \cdot (s_s \cdot c_r) \rangle_{R_d} = \langle P(f_p'^{-1}(\sigma_1), \ldots, f_p'^{-1}(\sigma_{1,\ldots,1})) \rangle_{R_{d-1}} = \langle P(s_1', .., s_s') \rangle_{R_{d-1}}$. To give an effective proof of Proposition 9, one needs to be more cautious and check a number of facts which are the content of the following four lemmas.

Lemma 5. *Let $P = \{p_1, \ldots, p_N\}$ be a N-tuple of points $\mathbb{P}^1$. The locus $\mathscr{R}_d^{\mathbf{s}-\mathbf{h}}(P)$ ($\mathbf{s} - \mathbf{h} = (s - h_1, \ldots, s - h_N)$), where* $\operatorname{rank} \phi(p_i) \leqq s - h_i$ *has the expected codimension $\sum_i h_i(r + h_i)$.*

Proof. A standard argument (see e.g. [ACGH]) tells us that the expected codimension is $\sum_i h_i(r + h_i)$. It is then enough to prove that $\mathscr{R}_d^{\mathbf{s}-\mathbf{h}}(P)$ has a codimension larger or equal to the expected one. Now $\mathscr{R}_d^{\mathbf{s}-\mathbf{h}}(P)$ is a finite union of sets of the form $\mathscr{R}_{d,(\mathbf{s}-\mathbf{k}),\mathbf{k}'}$ with $\mathbf{k}, \mathbf{k}' \in \mathbb{N}^N$ and $k_i' \geqq k_i \geqq h_i$, where $\operatorname{Im} \phi(p_i)$ has dimension $s - k_i$. As for the meaning of k_i', at a point $\rho \in \mathscr{R}_{d,(\mathbf{s}-\mathbf{k}),\mathbf{k}'}$ we have an exact sequence on $\mathbb{P}^1$; $0 \to \bigoplus_i \mathscr{S}^{k_i'}(p_i) \to B \to \tilde{B} \to 0$, where $\mathscr{S}^{k_i'}(p_i)$ is the skyscraper sheaf with stalk $\mathbb{C}^{k_i'}$ supported at $\{p_i\} \in \mathbb{P}^1$. The quotient $\tilde{B}$ is locally free around all the p_i's and corresponds to a point in $\mathscr{R}_{d - \sum k_i'}$. Around $\rho \in \mathscr{R}_{d,(\mathbf{s}-\mathbf{k}),\mathbf{k}'}$ we have a map $F : \mathscr{R}_{d,(\mathbf{s}-\mathbf{k}),\mathbf{k}'} \to \mathscr{R}_{d - \sum k_i'}$. We need only to estimate the dimension of its fibres, that is the dimension of the space $\{\psi \in \operatorname{Hom}(A, \oplus \mathscr{S}^{k_i'}(p_i)) | \text{for all } i, \ker \psi \supset \ker \phi(p_i)\}$ of deformations of the sequence $0 \to A \to \tilde{A} \to \oplus \mathscr{S}^{k_i'}(p_i) \to 0$, preserving that the support of the quotient is $P \subset \mathbb{P}^1$. So $\dim \mathscr{R}_{d,(\mathbf{s}-\mathbf{k}),\mathbf{k}'} \leqq \dim \mathscr{R}_{d - \sum k_i'} + \sum k_i'(s - k_i')$ and

$$\operatorname{codim} \mathscr{R}_{d,(\mathbf{s}-\mathbf{k}),\mathbf{k}'} \geqq \sum k_i'(r + k_i') \geqq \sum h_i(r + h_i)\,.$$

Remark. In the proof of Lemma 5 we will use the fact that $Z_d^{\mathbf{s}-\mathbf{h}} =: \bigcup_P \mathscr{R}_d^{\mathbf{s}-\mathbf{h}} \subset \mathscr{R}_d$ has codimension larger than $\sum h_i(r+h_i) - N$. We denote by $\mathscr{R}_{d,\mathrm{sing}}$ the locus of non-locally free quotients.

We want to look at the Chern classes, now denoted by $[m_l]$, of the quotient Q in the sequence $0 \to S \to V \to Q \to 0$ which generate the Chow ring of the grassmannian. We denote by Λ_l the space of the analytic representatives m_l of the classes $[m_l]$ and set $N_l = \dim \Lambda_l$. Let $c_l(p_i)$ be the closure of their preimages under the universal instanton restricted to $\{p_i\} \times \mathscr{R}_d$.

Lemma 6. *There are choices of the data* $(m_{l_i}, p_i) \in \Lambda_{l_i} \times \mathbb{P}^1$, $(i = 1, \dots, N)$ *such that*

$$c_{l_1}(p_1) \cap \cdots \cap c_{l_N}(p_N) \subset \mathscr{R}_d \backslash \mathscr{R}_{d,\mathrm{sing}}$$

whenever $\sum_i l_i = \dim \mathscr{R}_d$.

Proof. We use again a dimensional argument. Let $U = \times_{i=1}^N (\Lambda_{l_i} \times \mathbb{P}^1)$. For every $u \in U$ let $V(u) = c_{l_1}(p_1) \cap \cdots \cap c_{l_N}(p_N)$ be the intersection of the analytic representatives parametrized by u. Suppose that, for all $u \in U$. $V(u) \cap \mathscr{R}_{d,\mathrm{sing}} =: S(u) \neq \emptyset$ and call $Y = \bigcup_{u \in U} S(u) \subset \mathscr{R}_{d,\mathrm{sing}}$. Look at a component Y_0 of Y and suppose that the generic point of Y_0 belongs to $Z =: Z_d^{\mathbf{s}-\mathbf{h}}$ for some degeneracy multi-index $\mathbf{h}$. Consider the set $\tilde{Z} = \{(u,z) \in U \times Z | z \in S(u)\}$ which, by hypothesis, cannot be empty. We need to estimate the dimension D of the fibres of the second projection $\pi_2 : \tilde{Z} \to Z$. Take $z \in \operatorname{Im} \pi_2$ and notice that the points of $\pi_2^{-1}(z) = \{u \in U | z \in V(u)\}$ correspond to all the choices of points $p_1, \dots, p_N$ and analytic representatives $c_{l_1}, \dots, c_{l_N}$ such that $\phi(p_i) \in c_{l_i}$. Then $D \leqq D_1 =: \sum_1^N (N_{l_j} - l_j + h_i r)$; in fact we can move the c_{l_j} in a N_{l_j}-dimensional variety and asking that the c_{l_j}'s pass through a point of the target flag manifold imposes l_j conditions. The extra contribution to the codimension comes from the freedom of choosing subspaces of V which contains the degenerate image of $\phi(p_j)$. This is the estimate from above of D. On the other hand $D \geqq D_2 = \dim U - \dim Z = N + \sum_1^N N_{l_j} - \dim \mathscr{R}_d + \sum h_i(r+h_i) - N$. Now, since $\sum l_j = \dim \mathscr{R}_d$, we have $D_2 - D_1 = \sum h_i^2$ which is strictly positive leading to an absurd.

Let us now look at the intersection $W_d =: s_s(0) \cdot c_r(0) \subset \mathscr{R}_d$. The next lemma we need is

Lemma 7. W_d *is an irreducible subvariety of* $\mathscr{R}_d$ *of codimension* n.

Proof. The codimension is obviously smaller or equal to the stated one. We prove that it is also greater or equal to n. Notice that $W_d \subset \mathscr{R}_d^{s-1}$ and that, since s_s is given by the locus where $\operatorname{Im} \phi(0)$ is contained in a fixed hyperplane $H \subset V$, the same is true for all $\rho \in W_d$. Suppose now that $\rho \in W_d \cap \mathscr{R}_{d,(s-h),h'}(0)$ with $h' \geqq h \geqq 1$. As in Lemma 5, we have that $B|_{\{\rho\} \times \mathbb{P}^1} = \tilde{B}|_{\{\tilde{\rho}\} \times \mathbb{P}^1} \oplus \mathscr{S}^{h'}(0)$, where the quotient $\tilde{B}$ gives a point $\tilde{\rho} \in \mathscr{R}_{d-h'}$. It is a simple computation to check that the condition $\rho \in s_s$ imposes $s-h$ conditions on the torsion part $\mathscr{S}^{h'}(0)$. Thus $\operatorname{codim} W_d \geqq \operatorname{codim} \mathscr{R}_d^{s-h} + s - h = h(r+h) + s - h \geqq hr + s \geqq n$, the equality holding if and only if $\rho \in \mathscr{R}_d^{(s-1)} \cap W_d := W_{d,(s-1)}$. Summing up, we see that this last set is dense in W_d and that there is a rational dominant map $G : W_d \to \mathscr{R}_{d-1}$ given by "forgetting" the torsion part of the quotients in $W^{d,(s-1)}$. Since the two varieties have the same dimension, the general fibre of G is zero dimensional. It actually consists of a single point, in fact for a generic quotient $\tilde{\rho} \in \mathscr{R}_{d-1}$ the quotient $\rho \in W_d$ such that $G(\rho) = \tilde{\rho}$ is uniquely determined by the conditions,

$$\operatorname{Im} \phi|_{\rho\times\mathbb{P}^1}(p) = \operatorname{Im} \phi|_{\tilde{\rho}\times\mathbb{P}^1}(p), \quad \text{if } p \neq 0\,,$$

$$\operatorname{Im} \phi|_{\rho\times\mathbb{P}^1}(0) = \operatorname{Im} \phi|_{\rho\times\mathbb{P}^1}(0) \cap H\,.$$

In conclusion, G is a birational map to an irreducible variety and hence W_d is irreducible as well.

Remarks. a) A direct inspection of the conditions imposed on the Zariski tangent space shows that the varieties s_s and c_r intersect transversally at a generic point. We can then suppose that W_d is the scheme-theoretic intersection of the two varieties giving a representative of the class $[s_s \cdot c_r]$.

b) As usual, for every $\rho \in W_d^{(s-h)} =: W_d \cap \mathscr{R}_d^{(s-h)}$, forgetting the torsion part we get a quotient in $W_{d'}$, with $d' \leqq d - h$. Now the codimensions of W_d in $\mathscr{R}_d$ and of $W_{d'}$ in $\mathscr{R}_{d'}$ are the same and the number of parameters of the torsion part does not change; hence the degeneracy loci of the bundle map ϕ on $\mathbb{P}^1 \times W_d$ have the expected codimensions and the same proof of Lemma 7 shows that the following also holds;

Lemma 8. *There are choices of the data* $(c_{l_j}, p_j) \in \Lambda_{l_j} \times \mathbb{P}^1$, $(j = 1,\dots,N)$ *such that, whenever* $\sum_j l_j = \dim W_d$,

$$W_d \cap c_{l_1}(p_1) \cap \cdots \cap c_{l_N}(p_N) \subset \tilde{W}_d\,,$$

where the quotients B are locally free but at $\{0\}$ *with the minimal degeneration.*

From these lemmas it follows that

Proposition 9. *Given a word* $P(s_1,\dots,s_N)$ *in* $A(\mathscr{R}_d)$ *of degree equal to* $\dim \mathscr{R}_{d-1}$, *we have that*

$$\langle P(s_1,\dots,s_N) \cdot s_s \cdot c_r \rangle_{\mathscr{R}_d} = \langle P(s'_1,\dots,s'_N) \rangle_{\mathscr{R}_{d-1}}\,.$$

Proof. Obviously on $\tilde{W}_d$ the birational map G is well defined. Thanks to Lemma 6, we can suppose that both the intersections live in the locus where the map G is an isomorphism and where $\mathscr{R}_{d-1}$ represent maps to grassmannian.

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