Local BRST Cohomology in the Antifield Formalism: II. Application to Yang–Mills Theory

Glenn Barnich[1, *] , **Friedemann Brandt**[2, **] , **Marc Henneaux**[1, ***]

[1] Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

[2] NIKHEF-H, Postbus 41882, NL-1009 DB Amsterdam, The Netherlands; Present address: Instituut voor Theoretische Fysica, K.U. Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium

Abstract: Yang–Mills models with compact gauge group coupled to matter fields are considered. The general tools developed in a companion paper are applied to compute the local cohomology of the BRST differential s modulo the exterior space-time derivative d for all values of the ghost number, in the space of polynomials in the fields, the ghosts, the antifields (=sources for the BRST variations) and their derivatives. New solutions to the consistency conditions $sa + db = 0$ depending non-trivially on the antifields are exhibited. For a semi-simple gauge group, however, these new solutions arise only at ghost number two or higher. Thus at ghost number zero or one, the inclusion of the antifields does not bring in new solutions to the consistency condition $sa + db = 0$ besides the already known ones. The analysis does not use power counting and is purely cohomological. It can be easily extended to more general actions containing higher derivatives of the curvature or Chern–Simons terms.

1. Introduction

In a previous paper [1], referred to as I, we have derived general theorems on the local cohomology of the BRST differential s for a generic gauge theory. We have discussed in particular how it is related to the local cohomology of the Koszul–Tate differential δ and have demonstrated vanishing theorems for the cohomology $H_k(\delta|d)$ under various conditions. In the present paper, we apply the general results of I to Yang–Mills models with compact gauge group and provide the explicit list of all the non-vanishing BRST groups $H^k(s|d)$ for those models.

It has been established on general grounds that the groups $H^k(s)$ and $H^k(s|d)$ are respectively given by

$$H^k(s) \simeq \begin{cases} H^k(\gamma, H_0(\delta)) & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{1.1}$$

* Aspirant au Fonds National de la Recherche Scientifique, Belgium.
** Supported by Deutsche Forschungsgemeinschaft and by the research council (DOC) of the K.U. Leuven.
*** Also at Centro de Estudios Científicos de Santiago, Chile.

and

$$H^k(s|d) \simeq \begin{cases} H^k(\gamma|d, H_0(\delta)) & k \geqq 0 \\ H_{-k}(\delta|d) & k < 0 \end{cases} \tag{1.2}$$

(see [2] and I where this is recalled). Here, γ is the longitudinal exterior derivative along the gauge orbits, denoted by d in [2]. The isomorphisms (1.1) and (1.2) are valid for arbitrary gauge theories and hold when the "cochains" (local q-forms) upon which s acts are allowed to contain terms of arbitrarily high antighost number.

Now, in the case of Yang–Mills models, the BRST differential is just the sum of δ and γ,

$$s = \delta + \gamma \tag{1.3}$$

and so, is *not* an infinite formal series of derivations with arbitrarily high antighost number (as can a priori occur for an arbitrary gauge system). It is thus natural to consider local q-forms that have bounded antighost number, and to wonder whether the equalities (1.1)–(1.2) still hold under this restriction. Our first result, derived in Sect. 3, establishes precisely the validity of (1.1)–(1.2) in the space of local q-forms with bounded antighost number.

The isomorphisms (1.1)–(1.2) are useful in that they indicate how BRST invariance is equivalent to – and can be used as a substitute for – gauge invariance. However, they are not very explicit and a more precise computation of $H^k(s)$ or $H^k(s|d)$ is desired.

It has been shown in [3] that in each cohomological class of s, one can find a representative that does not involve the antifields and which is thus annihilated by γ. It then easily follows that

$$H^k(s) \simeq H^k(\gamma, \mathscr{E})/\mathscr{N} \quad (k > 0), \tag{1.4}$$

where (i) $\mathscr{E}$ is the algebra generated by the vector potential A_μ^a, the ghosts C^a, the matter fields y^i and their derivatives (no antifields); and (ii) $\mathscr{N}$ is the ideal of elements of $\mathscr{E}$ that vanish on-shell. Since the cohomology of γ in $\mathscr{E}$ is well understood in terms of Lie algebra cohomology, Eq. (1.4) provides a more precise characterization of $H^k(s)$ than (1.1) does. The representatives of (1.4) are polynomials in the "primitive forms" on the Lie algebra with coefficients that are invariant polynomials in the field strengths, the matter fields and their covariant derivatives [4–9]. Furthermore, two such objects are in the same class if they coincide on-shell. To get a non-redundant list, one may split the field strengths, the matter fields and their covariant derivatives into "independent" components, which are not constrained by the equations of motion, and "dependent components," which may be expressed on-shell in terms of the independent components. The cocyles may then be chosen to depend only on the independent components. The isomorphism (1.4) is a cohomological reformulation of a theorem proved long ago by Joglekar and Lee [10]. It plays a crucial role in renormalization theory [11, 12].

We derive in this paper an analogous, more precise characterization of the local cohomology $H^k(s|d)$ of s modulo d. For each value of the ghost degree, and in arbitrary spacetime dimension, we provide a constructive procedure for building representatives of each cohomological class. We then list all the solutions, some of which are expressed in terms of non-trivial conserved currents which we assume to have been determined. We find that contrary to what happens for the cohomology of s, there exists cocycles in the cohomology of s modulo d from which the antifields cannot be eliminated by redefinitions. Thus, there are new solutions to

the consistency conditions $sa + db = 0$ besides the antifield independent ones, as pointed out in [13] for a Yang–Mills group with two abelian factors.

However, if the gauge group is semi-simple, these additional solutions do not arise at ghost number zero or one but only at higher ghost number. Accordingly, the conjecture of Kluberg–Stern and Zuber on the renormalization of (local and integrated) gauge invariant operators [14, 15] is valid in that case (in even dimension). Differently put, there is no consistent perturbation of the Yang–Mills Lagrangian of ghost number zero, besides the perturbations by gauge invariant operators (or Chern–Simons terms in odd dimensions). Also, in four dimensions, there is no new candidate gauge anomaly besides the well known Adler–Bardeen one. Our results were partly announced in [16] and do not use power counting. They are purely cohomological.

The BRST differential contains information about the dynamics of the theory through the Koszul–Tate differential δ. Therefore, if one replaces the Yang–Mills Lagrangian $-1/8 \operatorname{tr}(F^{\mu\nu}F_{\mu\nu})$ by a different Lagrangian containing higher order derivatives of the curvature, or Chern–Simons terms in odd dimensions, the local BRST cohomology generically changes even though the gauge transformations remain the same. We show, however, that the procedure for dealing with the Yang–Mills action works also for these more general actions.

2. BRST Differential

We assume throughout that the gauge group G is compact and is thus the direct product of a semi-simple compact group by abelian $U(1)$ factors. As in I, we take all differentials to act from the right. Furthermore we assume the underlying spacetime manifold to be flat and homeomorphic to $R^n (n > 2)$ and use the n-dimensional Minkowski metric to raise and lower Lorentz indices $\mu, \nu, \ldots$.

The BRST differential [17, 18] for Yang–Mills models is a sum of two pieces,

$$s = \delta + \gamma \quad \text{with antigh } \delta = -1 \quad \text{and antigh } \gamma = 0 , \tag{2.1}$$

where δ is explicitly given by

$$\delta A_\mu^a = 0, \qquad \delta C^a = 0, \qquad \delta y^i = 0 ,$$

$$\delta A_a^{*\mu} = -\frac{\delta^L \mathscr{L}_0}{\delta A_\mu^a}, \qquad \delta C_a^* = -D_\mu A_a^{*\mu} + g T_{ai}^j y_j^* y^i, \qquad \delta y_i^* = -\frac{\delta^L \mathscr{L}_0}{\delta y^i} . \tag{2.2}$$

Here, $\mathscr{L}_0 = \mathscr{L}_0^y(y^i, D_\mu^y y^i) - \frac{1}{8} \operatorname{tr}(F^{\mu\nu}F_{\mu\nu})$, where $D_\mu^y y^i = \partial_\mu y^i - g A_\mu^a T_{aj}^i y^j$. We assume for simplicity that the matter fields do not carry a gauge invariance of their own and belong to a linear representation of G. The differential γ is given by

$$\gamma A_\mu^a = D_\mu C^a, \qquad \gamma C^a = -\frac{1}{2} g C_{bc}^a C^b C^c, \qquad \gamma y^i = g T_{aj}^i y^j C^a ,$$

$$\gamma A_a^{*\mu} = g A_c^{*\mu} C_{ab}^c C^b, \qquad \gamma C_a^* = g C_c^* C_{ab}^c C^b, \qquad \gamma y_i^* = -g T_{ai}^j y_j^* C^a . \tag{2.3}$$

There is no term of higher antighost number in s because the gauge algebra closes off-shell. One has

$$\delta^2 = 0, \qquad \gamma^2 = 0, \qquad \gamma\delta + \delta\gamma = 0 . \tag{2.4}$$

As explained in I, Sect. 4, we shall consider x-independent local q-forms that are polynomials in all the variables (Yang–Mills potential A_μ^a, matter fields y^i, ghosts C^a, antifields $A_a^{*\mu}$, y_i^* and C_a^*) and their derivatives. This is natural from the point of view of quantum field theory and implies in particular that the local q-forms under consideration have bounded antighost number. x-dependent solutions are discussed in Sect. 13 below.

Now, the general isomorphism theorems (1.1)–(1.2) have been established under the assumption that the local q-forms may contain terms of arbitrarily high antighost number. Our first task is to refine the theorems to the case where the allowed q-forms are constrained to have bounded antighost number. This is done in the next section.

3. Homological Perturbation Theory and Bounded Antighost Number

Theorem 3.1. *For Yang–Mills models, the isomorphisms*

$$H^k(s) \simeq \begin{cases} H^k(\gamma, H_0(\delta)) & k \geqq 0 \\ 0 & k < 0 \end{cases} \tag{3.1}$$

and

$$H^k(s|d) \simeq \begin{cases} H^k(\gamma|d, H_0(\delta)) & k \geqq 0 \\ H_{-k}(\delta|d) & k < 0 \end{cases} \tag{3.2}$$

also hold in the space of q-forms that are polynomials in all the variables and their derivatives.

Proof. We extend the action of the even derivation K of Sect. 10 of I on the ghosts as follows:

$$K = N_\partial + A, \tag{3.3}$$

where N_∂ is the operator counting the derivatives of all the variables,

$$N = \sum_{(k)} |k| \left[\frac{\partial^R}{\partial(\partial_{(k)} A_\mu^a)} \partial_{(k)} A_\mu^a + \frac{\partial^R}{\partial(\partial_{(k)} C^a)} \partial_{(k)} C^a + \frac{\partial^R}{\partial(\partial_{(k)} A_a^{*\mu})} \partial_{(k)} A_a^{*\mu} \right.$$
$$\left. + \frac{\partial^R}{\partial(\partial_{(k)} C_a^*)} \partial_{(k)} C_a^* + \frac{\partial^R}{\partial(\partial_{(k)} y^i)} \partial_{(k)} y^i + \frac{\partial^R}{\partial(\partial_{(k)} y_i^*)} \partial_{(k)} y_i^* \right], \tag{3.4}$$

and where A is defined by

$$A = \sum_{(k)} \left[2 \frac{\partial^R}{\partial(\partial_{(k)} A_a^{*\mu})} \partial_{(k)} A_a^{*\mu} + 3 \frac{\partial^R}{\partial(\partial_{(k)} C_a^*)} \partial_{(k)} C_a^* \right.$$
$$\left. + 2 \frac{\partial^R}{\partial(\partial_{(k)} \tilde{y}_i^*)} \partial_{(k)} \tilde{y}_i^* + \frac{\partial^R}{\partial(\partial_{(k)} \bar{y}_i^*)} \partial_{(k)} \bar{y}_i^* - \frac{\partial^R}{\partial(\partial_{(k)} C^a)} \partial_{(k)} C^a \right]. \tag{3.5}$$

The antifields $\tilde{y}_i^*$ are associated with second order differential equations, while the antifields $\bar{y}_i^*$ are associated with first order differential equations. We give A-weight -1 to the ghosts so that γ has only components of non-positive K-degree,

$$\gamma = \gamma^0 + \gamma^{-1}, \tag{3.6}$$

just as δ,

$$\delta = \delta^0 + \delta^{-1} + \delta^{-2} .\tag{3.7}$$

Evidently one has $[K, \partial_\mu] = \partial_\mu$ so that the exterior derivative d increases the eigen-value of N_∂ and K by one unit.

The undifferentiated ghosts are the only variables with negative K-degree ($\partial_\mu C^a$ has degree 0, $\partial_{\mu\nu} C^a$ has degree 1, etc....). Furthermore, because the antifields all carry a strictly positive degree, a form with bounded K-degree k cannot contain terms of antighost number greater than $k + g$, where g is the dimension of the Lie algebra (=number of ghosts). It is thus polynomial in the antifields.

We have indicated in Sect. 10 of I that if a is δ-closed, has positive antighost number and has K-degree bounded by k, then $a = \delta b$, where b has also K-degree bounded by k. Similarly, if a is δ-closed modulo d, has both positive antighost and pure ghost numbers, and has K-degree bounded by k, then $a = \delta b + dc$, where b has K-degree bounded by k and c has K-degree bounded by $k - 1$ ($a = \delta b + dc$ follows from [19]; the bounds on the k-degrees of b and c are then easily derived by expanding the equality according to the K-degree, and using the acyclicity of δ_0, of δ_0 mod d and of d). These properties are crucial in the proof of the theorem.

Let a be an s-cocycle which is polynomial in all the variables and their deriva-tives. Let us expand a according to the antighost number,

$$a = a_0 + a_1 + \cdots + a_m .\tag{3.8}$$

One has

$$\delta a_{i+1} + \gamma a_i = 0, \quad i = 0, 1, 2, \ldots, m - 1\tag{3.9}$$

and

$$\gamma a_m = 0 .\tag{3.10}$$

The isomorphism between $H^k(s)$ and $H^k(\gamma, H_0(\delta))$ is defined by $[a] \mapsto [a_0]$. To prove the theorem, one must verify that this map is injective and surjective. This is done as in [2], by controlling further polynomiality through the K-degree in a man-ner analogous to what is done in I, Sect. 10. For instance, let us prove surjectivity. Let a_0 be a representative of $H^k(\gamma, H_0(\delta))$, i.e., be an antifield independent solution of $\delta a_1 + \gamma a_0 = 0$. Since a_0 and a_1 are polynomials, they have bounded K-degree. We denote this bound by k. To show that a_0 is the image of a polynomial cocycle a of s, one constructs recursively a_2, a_3, etc. by means of (3.9). Because both δ and γ have components of non-negative K-degree, the higher order terms a_2, a_3, etc.... may be chosen to have also K-degree bounded by k. Thus, the recursive construc-tion stops at antighost number $k + g$ (at the latest) and $a = a_0 + a_1 + \cdots + a_{k+g}$ is polynomial. Injectivity, as well as (3.2) are proved along the same lines. $\square$

To conclude, we note that Theorem 3.1 holds for all "normal" theories in the sense of Sect. 10 of I, and, in particular, for Einstein gravity. Moreover, the reader may check that there is some flexibility in the proof of the theorem, in that one may assign different weights to the variables and nevertheless reach the same conclusion.

4. Cohomology of γ

In order to characterize completely $H^*(s|d)$, one needs a few preliminary results. Some of them have been developed already in the literature, while some of them

are new. These results are: the cohomology $H^*(\gamma)$, the invariant cohomology of d and the invariant cohomology of δ modulo d. They are considered in this section and the next two.

The cohomology $H^*(\gamma)$ of γ has been computed completely in [4–9, 3]. The easiest way to describe it is to redefine the generators of the algebra. The new generators adapted to γ are on the one hand A_μ^a, its symmetrized derivatives $\partial_{(\mu_1 \dots \mu_k} A_{\mu_{k+1})}^a$, $(k = 1, 2, \dots)$ and their γ-variations; and on the other hand χ_Δ^u and the undifferentiated ghosts C^a, where the χ_Δ^u stand for the field strengths, the matter fields, the antifields and all their covariant derivatives. (u stands for representation indices; while Δ stands for spacetime or spinorial indices unrelated to the gauge group.) The χ_Δ^u belong to a representation of the Lie algebra $\mathscr{G}$ of the gauge group. Indeed, the field strengths belong to the adjoint representation, the antifields $A_a^{*\mu}$ and C_a^* belong to the co-adjoint representation, while the antifields y_i^* belong to the representation dual to that of the y^i. As a result, the polynomials in the χ's also form a representation of the Lie algebra $\mathscr{G}$ of the gauge group: to any $x \in \mathscr{G}$, there is a linear operator $\rho(x)$ acting in the space of polynomials in the χ's as an even derivation and such that $\rho([x_1, x_2]) = [\rho(x_1), \rho(x_2)]$. The representation ρ is completely reducible. The polynomials belonging to the trivial representation are the invariant polynomials.

The crucial feature in the calculation of $H^*(\gamma)$ is that A_μ^a, its symmetrized derivatives and their γ-variations disappear from $H^*(\gamma)$ since they belong to the "contractile" part of the algebra. More precisely, one has

Theorem 4.1. (i) *The general solution of $\gamma a = 0$ reads*

$$a = \bar{a} + \gamma b, \tag{4.1}$$

where $\bar{a}$ is of the form

$$\bar{a} = \sum \alpha_J(\chi_\Delta^u) \omega^J(C^a). \tag{4.2}$$

Here, the α_J are invariant polynomials in the χ's, while the $\omega^J(C^a)$ belong to a basis of the Lie algebra cohomology of the Lie algebra of the gauge group.

(ii) *$\bar{a}$ is γ-exact if and only if $\alpha_J(\chi_\Delta^u) = 0$ for all J.*

Proof. The proof may be found in [4–9, 3] and will not be repeated here. □

Note that the α_J involve also the spacetime forms dx^μ. This will always be assumed in the sequel, where the word "polynomial" will systematically mean "spacetime form with coefficients that are polynomial in the variables and their derivatives."

5. Invariant Cohomology of d

Let $\alpha(\chi_\Delta^u)$ be an invariant polynomial in the χ's. Assume that α is d-closed, $d\alpha = 0$. Then one knows from the theorem on the cohomology of d that $\alpha = d\beta$ for some β. Can one assume that β is also an invariant polynomial? If α does not contain the antifields, this may not be the case: invariant polynomials in the 2-form $F^a \equiv (1/2)F_{\mu\nu}^a dx^\mu dx^\nu$ are counterexamples (and the only ones) [7, 9]. However, if antigh $\alpha > 0$, one has:

Theorem 5.1. *The cohomology of d in form degree $< n$ is trivial in the space of invariant polynomials in the χ's with strictly positive antighost number. That is,*

the conditions

$$\gamma\alpha = 0, \quad d\alpha = 0, \quad \text{antigh } \alpha > 0, \quad \deg \alpha < n, \quad \alpha = \alpha(\chi_\Delta^u) \qquad (5.1)$$

imply

$$\alpha = d\beta \qquad (5.2)$$

for some invariant $\beta(\chi)$,

$$\gamma\beta = 0. \qquad (5.3)$$

Proof. The proof proceeds as the proof of the proposition on p. 363 in [9]. We shall thus only sketch the salient points.

(i) First, one verifies the theorem in the abelian case with uncharged matter fields. In that case, any polynomial in the χ_Δ^u is invariant since the χ's themselves are invariant. To prove the theorem in the abelian case, one splits the differential d as $d = d_0 + d_1$, where d_1 acts on the antifields only and d_0 on the other fields. Let α be a polynomial in the field strengths, the antifields, the matter fields and their ordinary (= covariant) derivatives. If $d\alpha = 0$, then $d_1\alpha^N = 0$, where α^N is the piece in α containing the maximum number of derivatives of the antifields. But then, $\alpha^N = d_1\beta^{N-1}$, where β^{N-1} is a polynomial in the χ_Δ^u. This implies that $\alpha - d\beta^{N-1}$ ends at order $N - 1$ rather than N. Going on in the same fashion, one removes successively $\alpha^{N-1}, \alpha^{N-2}, \ldots$ until one reaches the desired result.

(ii) Second, one observes that if α is invariant under a global compact symmetry group, then β can be chosen to be also invariant since the action of the group commutes with d.

(iii) Finally, one extends the result to the non-abelian case with coloured matter fields by expanding α according to the number of derivatives of all the fields (see [9], p. 364 for the details). □

What replaces Theorem 5.1 in form degree n is: let $\alpha = \rho dx^0 \cdots dx^{n-1}$ be exact, $\alpha = d\beta$, where ρ is an invariant polynomial of antighost number > 0. [Equivalently, ρ has vanishing variational derivatives with respect to all the fields and antifields.] Then, one may take the coefficients of the $(n-1)$-form β to be also invariant polynomials.

Theorem 5.1 can be generalized as follows. Let α be a representative of $H^*(\gamma)$, i.e.,

$$\alpha = \Sigma\alpha_J(\chi_\Delta^u)\omega^J(C^a), \qquad (5.4)$$

where the $\alpha(\chi)$ are invariant polynomials. Because $d\gamma + \gamma d = 0$, d induces a well defined differential on $H^*(\gamma)$. This may be seen directly as follows. The derivative $d\alpha_J = D\alpha_J$ is an invariant polynomial in the χ's since D commutes with the representation, while $d\omega^J = \gamma\hat{\omega}^J(A, C)$ for some $\hat{\omega}^J$. Thus $d\alpha = \pm\Sigma(D\alpha_J)\omega^J + \gamma(\Sigma\alpha_J\hat{\omega}^J)$ defines an element of $H^*(\gamma)$ ($\gamma\alpha_J = 0$), namely the class of $\Sigma(D\alpha_J)\omega^J \equiv \Sigma(d\alpha_J)\omega^J$. What is the cohomology of d on $H^*(\gamma)$? Again, we shall only need the cohomology in form degree $< n$ and antighost number > 0.

Theorem 5.2. $H_k^{g,l}(d, H^*(\gamma)) = 0$ *for* $k \geq 1$ *and* $l < n$. *Here* g *is the ghost number,* l *is the form degree and* k *is the antighost number.*

Proof. Let $\alpha = \Sigma\alpha_J\omega^J$ be such that $d\alpha$ vanishes in $H^*(\gamma)$, i.e., $d\alpha = \gamma\mu$. From the above calculation, it follows that $\Sigma(D\alpha_J)\omega^J = \gamma\mu'$. But $\Sigma(D\alpha_J)\omega^J$ is of the form (4.2). This implies that $D\alpha_J = d\alpha_J = 0$ by (ii) of Theorem 4.1. Thus, by

Theorem 5.1, $\alpha_J = d\beta_J$, where β_J are invariant polynomials in the χ's. It follows that $\alpha = \Sigma(d\beta_J)\omega^J = \pm d(\Sigma\beta_J\omega^J) \mp \gamma(\Sigma\beta_J\hat{\omega}^J)$ is indeed d-trivial in $H^*(\gamma)$. $\square$

Theorem 5.2 is one of the main tools needed for the calculation of $H^*(s|d)$ in Yang–Mills theory. It implies that there is no non-trivial descent [20–22] for $H(\gamma|d)$ in positive antighost number. Namely, if $\gamma a + db = 0$, antigh $a > 0$, one may redefine $a \to a + \gamma\mu + dv = a'$ so that $\gamma a' = 0$. Indeed, the descent $\gamma a + db = 0$, $\gamma b + dc = 0, \dots$ ends with e so that $\gamma e = 0$ and $de + \gamma(something) = 0$. Thus the class of e is trivial and by the redefinition $e \to e + \gamma f + dm$, we may take e to vanish, etc.

6. Invariant Cohomology of δ Modulo d

The final tool needed in the calculation of $H^*(s|d)$ is the invariant cohomology of δ modulo d. We have seen that $H_k(\delta|d)$ vanishes for $k > 2$. Now, let a_p^k be a δ-boundary modulo d with form degree k and antighost number p,

$$a_p^k = \delta\mu_{p+1}^k + d\mu_p^{k-1}, \quad p \geqq 1 . \tag{6.1}$$

Assume that k_p is an invariant polynomial in the χ's (no ghosts). Can one also assume that both μ_{p+1}^k and μ_p^{k-1} are invariant polynomials? The answer is affirmative as we show in this section.

To that end, we associate with Equation (6.1) a tower of equations that starts at form degree n and ends at form degree $k - p + 1$ if $k \geqq p$ or 0 if $k < p$,

$$a_{p+n-k}^n = \delta\mu_{p+n-k+1}^n + d\mu_{p+n-k}^{n-1} \tag{6.2}$$

$$\vdots$$

$$a_p^k = \delta\mu_{p+1}^k + d\mu_p^{k-1} \tag{6.3}$$

$$\vdots$$

$$\begin{cases} a_1^{k-p+1} &= \delta\mu_2^{k-p+1} + d\mu_1^{k-p} \\ &\text{or} \\ a_{p-k}^0 &= \delta\mu_{p-k+1}^0 , \end{cases}$$

where the a's are all invariant polynomials. One goes up the ladder by acting with d and using the fact that if an invariant polynomial is δ-exact in the space of all polynomials, then it is also δ-exact in the space of invariant polynomials (Theorem 2 of [3]). So, for instance, acting with d on (6.1) yields $da_p^k = -\delta d\mu_{p+1}^k$. Since $da_p^k = Da_p^k$ is an invariant polynomial, there exists by Theorem 2 of [3] an invariant polynomial a_{p+1}^{k+1} such that $\delta a_{p+1}^{k+1} = -da_p^k$. The acyclicity of δ implies then that $a_{p+1}^{k+1} = d\mu_{p+1}^k + \delta\mu_{p+2}^{k+1}$ for some μ_{p+2}^{k+1}. One goes down the ladder along the same lines, but by applying δ and using Theorem 5.1.

Using again Theorem 2 of [3] and Theorem 5.1, it is easy to see that if any of the μ_l^j is equal to an invariant polynomial modulo δ or d exact terms, then all of them fulfill that property. That is, if $\mu_i^j = M_i^j + \delta\rho_{i+1}^j + d\rho_i^{j-1}$ for one pair (i,j) $(j - i = k - p - 1)$, then $\mu_l^m = M_l^m + \delta\rho_{l+1}^m + d\rho_l^{m-1}$ for all (l,m). Here, the M_l^m are invariant polynomials. Thus, to prove that the μ's are invariant, it suffices to

establish the property for the top of the ladder, i.e., for the n-forms. It is also clear that one has

Lemma 6.1. *If a_p^k in (6.1) is an n-form of antighost number $p > n$, then the μ's in (6.3) may be taken to be invariant polynomials.*

Proof. The proof is direct. If $a_p^n = \delta\mu_{p+1}^n + d\mu_p^{n-1}$ with $p > n$, one gets at the bottom of the ladder $a_{p-n}^0 = \delta\mu_{p-n+1}^0$. But then, by Theorem 2 of [3], one finds $\mu_{p-n+1}^0 = M_{p-n+1}^0 + \delta\rho_{p-n+2}^0$, where M_{p-n+1}^0 is an invariant polynomial. This implies that all the μ's are of the required form, and in particular that μ_{p+1}^n and μ_p^{n-1} may be taken to be invariant polynomials. $\square$

We are now in a position to establish the following crucial result about the invariant cohomology of δ.

Theorem 6.1. *If the invariant polynomial k_p is a δ-boundary modulo d and has non-vanishing antighost number, $k_p = \delta\mu_{p+1}^k + d\mu_p^{k-1}$ $(p > 0)$, then one may assume that μ_{p+1}^k and μ_p^{k-1} are also invariant polynomials. In particular, $H_k(\delta|d) = 0$ for $k \geq 3$ in the space of invariant polynomials.*

Proof. The proof proceeds as the proof of Theorem 5.1. Namely, one verifies first the theorem in the abelian case with a single gauge field and uncharged free matter fields. One then extends it to the case of many abelian fields with a global symmetry. One finally considers the full non-Abelian case.

Since the last two steps are very similar to those of Theorem 5.1, we shall verify explicitly here only that Theorem 6.1 holds for a single abelian gauge field with uncharged free matter fields. So, let us start with an n-form a_p solution of (6.1) and turn to dual notation,

$$a_p = \delta b'_{p+1} + \partial_\mu j_p^\mu \quad (p \geq 1). \tag{6.4}$$

We shall first prove that if the theorem holds for antighost number $p + 2$, then it also holds for antighost number p. To that end, we take the Euler–Lagrange derivative of (6.4). This yields

$$\frac{\delta^R a_p}{\delta C^*} = \delta Z'_{(p-1)}, \tag{6.5}$$

$$\frac{\delta^R a_p}{\delta A^{*\mu}} = \delta X'_{(p)\mu} + \partial_\mu Z'_{(p-1)}, \tag{6.6}$$

$$\frac{\delta^R a_p}{\delta A_\mu} = \delta Y'^\mu_{(p+1)} - \partial_\nu(\partial^\mu X'^\nu_{(p)} - \partial^\nu X'^\mu_{(p)}), \tag{6.7}$$

$$\frac{\delta^R a_p}{\delta y^i} = D_{ij}X'^j_{(p)} + \delta Y'_{(p+1)i}, \tag{6.8}$$

$$\frac{\delta^R a_p}{\delta y_i^*} = \delta X'^i_{(p)}, \tag{6.9}$$

where $Z'_{(p-1)}$, $X'_{(p)\mu}$, $Y'^\mu_{(p+1)}$, $X'^i_{(p)}$ and $Y'_{(p+1)i}$ are obtained by differentiating b'_{p+1} [$Z' = 0$ if $p = 1$]. The explicit expression of these polynomials will not be needed in the sequel. In (6.8), D_{ji} is the differential operator appearing in the linearized matter equations of motion ($\delta^L \mathscr{L}^{\text{free}}_0 / \delta y^i = D_{ij} y^j$). Because $\delta^R a_p / \delta C^*$, $\delta^R a_p / \delta A^{*\mu}$, $\delta^R a_p / \delta A_\mu$, $\delta^R a_p / \delta y^i$ and $\delta^R a_p / \delta y^*_i$ are invariant polynomials, i.e., involve only the χ's, one may replace in (6.5)–(6.9) the polynomials $Z'_{(p-1)}$, $X'_{(p)\mu}$, $Y'^\mu_{(p+1)}$, $X'^i_{(p)}$ and $Y'_{(p+1)i}$ which may a priori involve symmetrized derivatives of A_μ, by invariant polynomials $Z_{(p-1)}$, $X_{(p)\mu}$, $Y^\mu_{(p+1)}$, $X^i_{(p)}$ and $Y_{(p+1)i}$ depending only on the χ's,

$$\frac{\delta^R a_p}{\delta C^*} = \delta Z_{(p-1)} \,, \qquad (6.10)$$

$$\frac{\delta^R a_p}{\delta A^{*\mu}} = \delta X_{(p)\mu} - \partial_\mu Z_{(p-1)} \,, \qquad (6.11)$$

$$\frac{\delta^R a_p}{\delta A_\mu} = \delta Y^\mu_{(p+1)} - \partial_\nu (\partial^\mu X^\nu_{(p)} - \partial^\nu X^\mu_{(p)}) \,, \qquad (6.12)$$

$$\frac{\delta^R a_p}{\delta y^i} = D_{ij} X^j_{(p)} + \delta Y_{(p+1)i} \,, \qquad (6.13)$$

$$\frac{\delta^R a_p}{\delta y^*_i} = \delta X^i_{(p)} \,. \qquad (6.14)$$

This is obvious for $Z_{(p-1)}$ and $X^i_{(p)}$ (simply set A_μ and its symmetrized derivatives equal to zero in $Z'_{(p-1)}$ and $X'_{(p)}$; this commutes with the action of δ). The assertion is then verified easily for $X^\nu_{(p)}$, $Y_{(p+1)i}$ and $Y^\mu_{(p+1)}$.

Now, the invariant polynomial $Y^\mu_{(p+1)}$ is δ-closed modulo d by (6.12) since $\delta a_p / \delta A_\mu$ is of the form $\partial_\nu W^{\mu\nu}$ for some $W^{\mu\nu} = -W^{\nu\mu}$ (this follows from the fact that a_p depends on A_μ only through its field strength). Thus, it is δ-exact modulo d because $H^{n-1}_{p+1}(\delta|d) \simeq H^n_{p+2}(\delta|d)$ is zero ($p + 2 \geqq 3$). This means that $Y^\mu_{(p+1)}$ can be written as $\delta T^\mu_{(p+2)} + \partial_\nu S^{\mu\nu}_{(p+1)}$, where $T^\mu_{(p+2)}$ and $S^{\mu\nu}_{(p+1)}$ are both invariant polynomials since we assume that the theorem holds for antighost number $p + 2$ in form degree n, or, what is the same, by our general discussion above, for antighost number $p + 1$ in form degree $n - 1$.

If one injects relations (6.10)–(6.14) in the identity

$$a_p = \int dt \left[\frac{\delta^R a_p}{\delta C^*} C^* + \frac{\delta^R a_p}{\delta A^{*\mu}} A^{*\mu} + \frac{\delta^R a_p}{\delta A_\mu} A_\mu + \frac{\delta^R a_p}{\delta y^i} y^i + \frac{\delta^R a_p}{\delta y^*_i} y^*_i \right] + \partial_\mu \rho'^\mu \quad (6.15)$$

one gets, using $Y^\mu_{(p+1)} = \delta T^\mu_{(p+2)} + \partial_\nu S^{\mu\nu}_{(p+1)}$ and making integrations by parts, that

$$a_p = \delta b_{p+1} + \partial_\mu \rho^\mu \,, \qquad (6.16)$$

where b_{p+1} is *manifestly invariant*. This proves that the theorem holds in antighost number p if it holds in antighost number $p + 2$ (ρ^μ may also be chosen to be invariant by Theorem 5.1). But we know by Lemma 6.1 that the theorem is true for antighost number $> n$. Accordingly, the theorem is true for all (strictly) positive values of the antighost number. $\quad \square$

7. Calculation of $H^*(s\,|\,d)$–General Method

We can now turn to the calculation of $H^*(s\,|\,d)$ itself. The strategy for computing $H^*(s\,|\,d)$ adopted here [16] is to relate as mush as possible elements of $H^*(s\,|\,d)$ to the known elements of $H^*(\gamma\,|\,d)$ [23, 4–9, 3]. To that end, one controls the antifield dependence through Theorems 5.2 and 6.1. This is done by expanding the cocycle condition $sa + db = 0$ according to the antighost number. At maximum antighost number k, one gets $\gamma a_k + db_k = 0$. Theorem 5.2 and its consequences for the descent equations for γ in the presence of antifields then implies, for $k \geq 1$, that one can choose b_k equal to zero. Thus $\gamma a_k = 0$, and by Theorem 4.1, $a_k = \Sigma \alpha_J(\chi_\Delta^u)\omega^J(C)$ up to γ-exact terms. [The redefinition $a_k \rightarrow a_k + \gamma m_k + dn_k$ can be implemented through $a \rightarrow a + sm_k + dn_k$, which does not change the class of a in $H(s\,|\,d)$.] The equation at antighost number $k - 1$ reads $\delta a_k + \gamma a_{k-1} + db_{k-1} = 0$. Acting with γ, we get $d\gamma b_{k-1} = 0$, which implies $\gamma b_{k-1} + dc_{k-1} = 0$.

If $k - 1 \geq 1$, Theorem 5.2 implies again that one can choose $\gamma b_{k-1} = 0$ with $b_{k-1} = \Sigma \beta_J(\chi_\Delta^u)\omega^J(C)$. Inserting the forms of a_k and b_{k-1} into the equation at antighost number $k - 1$ gives $\Sigma(\delta \alpha_J + d\beta_J)\omega^J(C) = \gamma(something)$ which implies $\delta \alpha_J + d\beta_J = 0$ by part (ii) of Theorem 4.1, i.e. α_J is a δ-cycle modulo d. Suppose that α_J is trivial, $\alpha_J = \delta \mu_J + dv_J$. Theorem 6.1 then implies that μ_J and v_J can be chosen to be invariant polynomials. The redefinition $a \rightarrow a \pm s(\Sigma\mu_J\omega^J - \Sigma v_J\hat\omega_J) - d(\Sigma v_J\omega^J)$ allows one to absorb a_k. [Recall that $\gamma\hat\omega^J = d\omega^J$. The corresponding redefinition of b is $b \rightarrow b - s(\Sigma v_J\omega^J)$, which leaves b_k equal to zero since $\gamma v_J = 0$.] Consequently, we have learned (i) that for $k \geq 1$, the last term a_k in any s-cocycle modulo d may be chosen to be of the form $\Sigma\alpha_J\omega^J(C)$, where the α_J are invariant polynomials; and (ii) that for $k \geq 2$, α_J define δ-cycles modulo d which must be non-trivial since otherwise, a_k can be removed from a by adding to a an s-coboundary modulo d.

We can classify the elements of $H^*(s\,|\,d)$ according to their last non-trivial term in the antighost number expansion. The results on the cohomology of $H_*(\delta\,|\,d)$ show that only three cases are possible.

Class I. a stops at antighost number 2,

$$a = a_0 + a_1 + a_2 \tag{7.1}$$

(with $a_0 = 0$ if gh $a = -1$, or $a_0 = a_1 = 0$ if gh $a = -2$). The last term a_2 is invariant,

$$a_2 = \sum \alpha_J(\chi_\Delta^u)\omega^J(C) \tag{7.2}$$

and the $\alpha_J(\chi_\Delta^u)$ define non-trivial elements of $H_2(\delta\,|\,d)$.

Class II. a stops at antighost number one,

$$a = a_0 + a_1 \tag{7.3}$$

(with $a_0 = 0$ if gh $a = -1$). The last term a_1 is invariant,

$$a_1 = \sum \alpha_J(\chi_\Delta^u)\omega^J(C) . \tag{7.4}$$

We shall see in Sect. 9 below that the $\alpha_J(\chi_\Delta^u)$ must also be non-trivial δ-cycles modulo d.

Class III. a does not contain the antifields,

$$a = a_0 . \tag{7.5}$$

Then, of course, gh $a \geqq 0$,

8. Solutions of Class *I*

The solutions of Class *I* arise only when $H_2(\delta|d)$ is non-trivial, i.e., when there are free abelian gauge fields. This is a rather academic context from the point of view of realistic Lagrangians, but the question turns out to be of interest in the construction of consistent couplings among free, massless vector particles [24].

   One can divide the solutions of Class *I* into three different types, according to whether they have total ghost number equal to -2 (type I_a), -1 (type I_b) or $\geqq 0$ (type I_c).

Type I_a. If gh $a = -2$, then *a* reduces to a_2 and cannot involve the ghosts. The solutions of Type I_a have form degree n and are exhausted by Theorem 13.1 of *I*, in agreement with the isomorphism $H^{-2}(s|d) \simeq H_2(\delta|d)$. They read explicitly

$$a \equiv a_2 = f^\alpha C_\alpha^* , \quad f^\alpha = \text{constant} , \tag{8.1}$$

where C_α^* are the antifields conjugate to the ghosts of the free abelian gauge fields. We switch back and forth between the form notation and the dual notation. The C_α^* should thus be viewed alternatively as n-forms or as densities.

Type I_b. If gh $a = -1$, then a_2 must involve one ghost C^A. This ghost must be abelian since one must have $\gamma C^A = 0$. Thus,

$$a_2 = f_{A\alpha} C^{*\alpha} C^A , \quad f_{A\alpha} = \text{const.}, \tag{8.2}$$

where the sum over A runs a priori over *all* abelian ghosts. The equation in antighost number one yields for a_1,

$$a_1 = f_{A\alpha} A_\mu^A A^{*\alpha\mu} \tag{8.3}$$

up to a solution m_1 of $\gamma m_1 + dn_1 = 0$ which however is not relevant for a_1, cf. discussion of Class I_c-solutions given below (m_1 turns out to be a solution of Class II_a up to a trivial contribution). The next equation $\delta a_1 + db_0 = 0$ is then equivalent to

$$f_{A\alpha} F_{\mu\nu}^A F^{\alpha\mu\nu} = \partial_\rho k^\rho \tag{8.4}$$

for some k^ρ. This equality can hold only if the variational derivatives of the left-hand side vanish identically, which implies $f_{A\alpha} = 0$ for $A \neq \beta$ and $f_{\alpha\beta} = -f_{\beta\alpha}$. Thus, one gets finally

$$a = f_{\alpha\beta}(A_\mu^\alpha A^{*\beta\mu} + C^\alpha C^{*\beta}) , \quad f_{\alpha\beta} = -f_{\beta\alpha} . \tag{8.5}$$

Type I_c. If gh $a \geqq 0$, then all three terms a_0, a_1, and a_2 are in principle present. The term a_2 reads

$$a_2 = f_{\alpha J} C^{*\alpha} \omega^J(C) , \tag{8.6}$$

where $\omega^J(C)$ belongs to a basis of the Lie algebra cohomology. The $\omega^J(C)$ can be written as polynomials in the so-called "primitive forms." The primitive forms are of degree one (C^A) for the abelian factors and of degree $\geqq 3$ ($\text{tr}C^3, \text{tr}C^5, \ldots$) for each simple factor [25].

It will be useful in the sequel to isolate explicitly the abelian ghosts in (8.6). Thus, we write

$$a_2 = \sum_k \frac{1}{k!} f_{\alpha \Gamma A_1 \ldots A_k} \omega^\Gamma(C) C^{A_1} \cdots C^{A_k} C^{*\alpha} , \qquad (8.7)$$

where $\omega^\Gamma(C)$ involve only the ghosts of the simple factors. The pure ghost numbers of the terms appearing in (8.7) must of course add up to $2 + q$, where q is the total ghost number of a. The factors $\omega^\Gamma(C)$ have the useful property of belonging to a chain of descent equations [20–22] involving at least two steps

$$\partial_\mu \omega^\Gamma(C) = \gamma \hat{\omega}_\mu^\Gamma , \qquad (8.8)$$

$$\partial_{[\mu} \hat{\omega}_{\nu]}^\Gamma = \gamma \hat{\omega}_{[\mu\nu]}^\Gamma . \qquad (8.9)$$

For instance,

$$\hat{\omega}_\mu^\Gamma = \frac{\partial^R \omega^\Gamma}{\partial C^a} A_\mu^a \qquad (8.10)$$

(see [23, 7]). By contrast, the abelian ghosts belong to a chain that stops after the first step. One has $\partial_\mu C^A = \gamma A_\mu^A$, but there is clearly no $f_{\mu\nu}$ such that $\partial_{[\mu} A_{\nu]} = \gamma f_{\mu\nu}$. Since it will be necessary below to "lift" twice the elements $\omega^J(C)$ of the basis through equations of the form (8.8) and (8.9), the abelian factors play a distinguished role.

A direct calculation shows that

$$\delta a_2 = -\gamma \left[\left(\sum \frac{1}{(k-1)!} \omega^\Gamma f_{\alpha \Gamma A_1 \ldots A_k} C^{A_1} \cdots C^{A_{k-1}} A_\mu^{A_k} \right. \right.$$

$$\left. \left. + \sum \frac{1}{k!} (-)^k \hat{\omega}_\mu^\Gamma f_{\alpha \Gamma A_1 \ldots A_k} C^{A_1} \cdots C^{A_k} \right) A^{*\alpha\mu} \right] + \partial_\mu V^\mu \qquad (8.11)$$

for some V^μ. This fixes a_1 to be

$$a_1 = \left[\sum f_{\alpha \Gamma A_1 \ldots A_k} \left(\frac{1}{(k-1)!} \omega^\Gamma C^{A_1} \cdots C^{A_{k-1}} A_\mu^{A_k} \right. \right.$$

$$\left. \left. + \frac{1}{k!} (-)^k \hat{\omega}_\mu^\Gamma C^{A_1} \cdots C^{A_k} \right) \right] A^{*\alpha\mu} \qquad (8.12)$$

up to a solution m_1 of $\gamma m_1 + d n_1 = 0$. Using again the absence of a non-trivial descent in positive antighost number, we may assume $n_1 = 0$ and $m_1 = \Sigma_J \mu_J (\chi_\Delta^u) \omega^J$ (C) by a redefinition $m_1 \to m_1 + d\alpha + \gamma\beta$ that would only affect a_0 as $a_0 \to a_0 + \delta\beta$ (if it exists). That is, a_1 takes the form (8.12) modulo an invariant object of antighost number one.

Compute now δa_1. One finds

$$\delta a_1 = -\frac{1}{2} \sum \frac{1}{(k-1)!} \omega^\Gamma f_{\gamma \Gamma A_1 \ldots A_k} C^{A_1} \cdots C^{A_{k-1}} F_{\mu\nu}^{A_k} F^{\alpha\mu\nu} + \delta m_1$$

$$+ \gamma (M_{\mu\nu\alpha} F^{\alpha\mu\nu}) + \delta_\mu \tilde{V}^\mu \qquad (8.13)$$

for some $\tilde{V}^\mu$. Here, $M_{\mu\nu\alpha}$ is explicitly given by

$$M_{\mu\nu\alpha} = \sum \left[\frac{1}{2(k-2)!} f_{\alpha\Gamma A_1 \dots A_k} \omega^\Gamma C^{A_1} \cdots C^{A_{k-2}} A_\mu^{A_{k-1}} A_\nu^{A_k} \right.$$

$$- \frac{1}{(k-1)!} (-)^k \hat{\omega}_{[\mu}^\Gamma f_{\alpha\Gamma A_1 \dots A_k} C^{A_1} \cdots C^{A_{k-1}} A_{\nu]}^{A_k}$$

$$\left. - \frac{1}{k!} \hat{\omega}_{[\mu\nu]}^\Gamma f_{\alpha\Gamma A_1 \dots A_k} C^{A_1} \cdots C^{A_k} \right]. \qquad (8.14)$$

At antighost number zero, $sa + db = 0$ requires $\delta(a_1 - m_1)$ to be γ-exact modulo d. Hence, a_0 exists if and only if the first term on the right-hand side of (8.13) is weakly γ-exact modulo d, i.e.,

$$-\frac{1}{2} \sum \frac{1}{(k-1)!} \omega^\Gamma f_{\alpha\Gamma A_1 \dots A_k} C^{A_1} \cdots C^{A_{k-1}} F_{\mu\nu}^{A_k} F^{\alpha\mu\nu} + \delta m_1 = \gamma m_0 + \partial_\mu n_0^\mu \qquad (8.15)$$

for some m_0 and n_0^μ of antighost number zero. This forces this first term to vanish, as we now show.

By acting with γ on (8.15), one gets $d\gamma n_0 = 0$ and thus $\gamma n_0 + dn_0' = 0$. Accordingly, n_0 is an antifield independent solution of the γ-cocycle condition modulo d. This equation has been completely solved in the literature [4, 7–9] and the solutions fall into two classes: those that are annihilated by γ and are therefore invariant objects (up to redefinitions); and those that lead to a non-trivial descent, i.e. those for which no redefinition can make n_0' equal to zero. This second class involves only the forms $A^a = A_\mu^a dx^\mu$, $F^a = (1/2)F_{\mu\nu}^a dx^\mu dx^\nu$, their exterior products, and the ghosts. Thus, $n_0 = \bar{n}_0 + \bar{\bar{n}}_0$, where $\bar{n}_0$ belongs to the first class and $\bar{\bar{n}}_0$ belongs to the second class.

The solutions of the second class are obtained by considering the descent $\gamma\bar{\bar{n}}_0 + d\bar{\bar{n}}_0' = 0$, $\gamma\bar{\bar{n}}_0' + d\bar{\bar{n}}_0'' = 0$, etc. ... One successively lifts the last term of the descent, which is annihilated by γ all the way to $\bar{\bar{n}}_0$. The term $d\bar{\bar{n}}_0$ itself can be written as a γ-exact term, unless there is an "obstruction." This obstruction is an invariant polynomial which involves $\omega^J(C)$ and the components $F_{\mu\nu}^\alpha$ but only through the forms F^a and their exterior products, but no other combination [23]. In particular, the dual of F^a cannot occur. Accordingly, the obstruction cannot be written as a term involving $F_{\mu\nu}^A F^{\alpha\mu\nu}$ plus a term involving the equations of motion, plus a term of the form $d\bar{n}_0$, with $\bar{n}_0$ invariant. This means that the obstruction must be zero if a_0 is to exist, so that $d\bar{\bar{n}}_0 = \gamma\mu_0$ by itself. By adding to a_0 a solution of Type III if necessary, we may assume $\bar{\bar{n}}_0$ to be absent.

If n_0 reduces to the invariant piece $\bar{n}_0$, Eq. (8.15) and Theorem 4.1 imply that

$$-\frac{1}{2} \sum \frac{1}{(k-1)!} \omega^\Gamma f_{\alpha\Gamma A_1 \dots A_k} C^{A_1} \cdots C^{A_{k-1}} F_{\mu\nu}^{A_k} F^{\alpha\mu\nu} + \delta m_1 - \sum (D_\mu n_J^\mu) \omega^J = 0 \qquad (8.16)$$

with $\bar{n}_0 = \sum n_J^\mu \omega^J$. If we set in this equality the covariant derivatives of $F_{\mu\nu}^a$ equal to zero, one gets the desired result that $f_{\alpha\Gamma A_1 \dots A_k} C^{A_1} \cdots C^{A_{k-1}} F_{\mu\nu}^{A_k} F^{\alpha\mu\nu}$ should vanish. This implies that $f_{\alpha\Gamma A_1 \dots A_k}$ (i) has as non-vanishing components only $f_{\alpha\Gamma\alpha_1 \dots \alpha_k}$; and (ii) is completely antisymmetric in $(\alpha, \alpha_1, \dots, \alpha_k)$. The solutions of Class I_c are

consequently exhausted by

$$
\begin{aligned}
a = \sum f_{\alpha \Gamma \alpha_1 \ldots \alpha_k} & \left[\left(-\frac{1}{2(k-2)!} \omega^\Gamma C^{\alpha_1} \cdots C^{\alpha_{k-2}} A_\mu^{\alpha_{k-1}} A_\nu^{\alpha_k} \right. \right. \\
& + \frac{1}{(k-1)!} (-)^k \hat{\omega}^\Gamma_{[\mu} C^{\alpha_1} \cdots C^{\alpha_{k-1}} A_{\nu]}^{\alpha_k} + \frac{1}{k!} \hat{\hat{\omega}}^\Gamma_{[\mu\nu]} C^{\alpha_1} \cdots C^{\alpha_k} \right) F^{\alpha\mu\nu} \\
& + \left(\frac{1}{(k-1)!} \omega^\Gamma C^{\alpha_1} \cdots C^{\alpha_{k-1}} A_\mu^{\alpha_k} + \frac{1}{k!} (-)^k \hat{\omega}^\Gamma_\mu C^{\alpha_1} \cdots C^{\alpha_k} \right) A^{*\alpha\mu} \\
& \left. + \frac{1}{k!} \omega^\Gamma C^{\alpha_1} \cdots C^{\alpha_k} C^{*\alpha} \right]
\end{aligned}
\tag{8.17}
$$

(modulo solutions of Class *II*). This ends our discussion of the solutions of Class *I*, corresponding to elements of $H_2(\delta|d)$.

[The analysis has been performed explicitly for spacetime dimensions greater than or equal to three. In two spacetime dimensions, there are further solutions. The solutions of ghost number -2 read $(\partial f/\partial F_{01}^a) C_a^* + (1/2)(\partial^2 f/\partial F_{01}^b \partial F_{01}^a) \varepsilon_{\mu\nu} A_a^{*\mu} A_b^{*\nu}$, where f is an invariant polynomial in those field strengths $F_{\mu\nu}^a$ that obey $D_\mu F_{01}^a = 0$ on-shell. The solutions of ghost number -1 and higher are constructed as above, by multiplying the solutions of ghost number -2 with the γ-invariant polynomials $\omega^J(C)$, and then solving successively for a_1 and a_0. There are possible obstructions in the presence of abelian factors which restrict the coefficients of ω^J. We leave the details to the reader.]

9. Solutions of Class *II*

The next case to consider is given by a cocycle a whose expansion stops at antighost number 1. Again, we may consider two subcases: Type II_a, with $\mathrm{gh}\, a = -1$; and Type II_b, with $\mathrm{gh}\, a \geqq 0$.

Type II_a. If $\mathrm{gh}\, a = -1$, then a reduces to a_1 and does not involve the ghosts. It is clearly an invariant element of $H_1(\delta|d)$, by the equations $\gamma a_1 = 0$ and $\delta a_1 + db_0 = 0$. The groups $H_1^k(\delta|d)$ are non-zero in form degree n (conserved currents) and $n-1$ (if there are uncoupled abelian fields). Thus, given a complete set of invariant non-trivial conserved currents, one may construct $H^{-1}(s|d)$ explicitly.[1]

Let j_Δ^μ be such a complete set and let $X_{\mu\Delta}^a, X_\Delta^i$ be the corresponding global symmetries of the fields, $\delta_\Delta A_\mu^a = X_{\mu\Delta}^a, \delta_\Delta y^i = X_\Delta^i$. One has

$$
\delta(X_{\mu\Delta}^a A_a^{*\mu} + X_\Delta^i y_i^*) = \partial_\mu j_\Delta^\mu .
\tag{9.1}
$$

In order for a_1 to be invariant, we impose $X_{\mu\Delta}^a A_a^{*\mu} + X_\Delta^i y_i^*$ to be invariant i.e., to be annihilated by γ. Because the equations of motion involve derivatives of the field strengths and are not invariant polynomials in the forms F^a, there is no obstruction to taking j_Δ^μ annihilated by γ as well. [It turns out that the condition

---

[1] Since we are working in the space of local forms which do not depend explicitly on the x^μ, it is understood that we have in mind here x-independent conserved currents which are non-trivial in this space, cf. remarks in Sect. 13.

$\gamma(X_{\mu\Delta}^a A_a^{*\mu} + X_\Delta^i y_i^*) = 0$ can always be fulfilled, in the absence of free abelian fields, by a suitable redefinition of the global symmetry within its equivalence class. This actually follows from the relationship between $H(s|d)$ and $H(\delta|d)$ and will be spelled out in detail elsewhere [26].]

One gets for the BRST cohomology $H^{-1}(s|d)$:

In form degree $n - 1$:

$$a = f^\alpha A_\alpha^{*\mu}, \quad f^\alpha = \text{constant} . \tag{9.2}$$

In form degree n:

$$a = f^\Delta(X_{\mu\Delta}^a A_a^{*\mu} + X_\Delta^i y_i^*), \quad f^\Delta = \text{constant} . \tag{9.3}$$

Turn now to the solutions of Type II_b.

Type II_b. We must solve $\gamma a_0 + \delta a_1 + d b_0 = 0$ with $a_1 = \sum \alpha_J \omega^J$. The derivation above does not imply that b_0 is annihilated by γ and thus, it is not clear at this stage that the α_J belong to $H(\delta|d)$. However, by acting with γ on $\delta a_1 + \gamma a_0 + d b_0 = 0$, one gets again that $\gamma b_0 + dc_0 = 0$. The analysis proceeds then in a manner similar to that of Type II_c. As mentioned above, the general solution to $\gamma b_0 + dc_0 = 0$ is known [4, 7–9] and takes the form $b_0 = \bar{b}_0 + \bar{\bar{b}}_0$, where (i) $\bar{b}_0$ is annihilated by γ and thus given by $\bar{b}_0 = \Sigma \beta_{0J}(\chi) \omega^J(C)$ (up to irrelevant γ-exact terms) with β_{0J} invariant polynomials in the χ's; and (ii) $\bar{\bar{b}}_0$ is γ closed only modulo a non-trivial d exact term and involves the forms $A^a = A_\mu^a dx^\mu$, $F^a = (1/2)F_{\mu\nu}^a dx^\mu dx^\nu$, and C^a. The obstruction [23] to writing $d\bar{\bar{b}}_0$ as a γ exact term involves the forms F^a and $\omega^J(C)$. It cannot be written as the sum of a term proportional to the equations of motion and a term of the form $d\bar{b}_0$ and $\bar{b}_0$ invariant since such terms involve unavoidably the covariant derivatives of the field strengths. Thus, the obstruction must be absent and $d\bar{\bar{b}}_0 = -\gamma \bar{\bar{a}}_0$, for some $\bar{\bar{a}}_0$. The equation $\delta a_1 + \gamma a_0 + d b_0 = 0$ splits therefore into two separate equations $\gamma \bar{\bar{a}}_0 + d\bar{\bar{b}}_0 = 0$ and $\gamma \bar{a}_0 + d\bar{b}_0 + \delta a_1 = 0$.

The first equation defines a solution of Class *III*. We need only consider in this section the second equation. Because $\bar{b}_0$ is annihilated by γ, we may follow the procedure of Sect. 7 to find again that the invariant polynomials α_J in a_1 define elements of $H_1(\delta|d)$. One gets explicitly:

In form degree $n - 1$:

$$a = f_J^\alpha(\hat{\omega}_\nu^J F_\alpha^{\mu\nu} + \omega^J(C) A_\alpha^{*\mu}), \quad f_J^\alpha = \text{constant} . \tag{9.4}$$

In form degree n:

$$a = f_J^\Delta[\hat{\omega}_\mu^J j_\Delta^\mu + \omega^J(C)(X_{\mu\Delta}^a A_a^{*\mu} + X_\Delta^i y_i^*)], \quad f_J^\Delta = \text{constant} . \tag{9.5}$$

(with $\gamma \hat{\omega}_\mu^J = \partial_\mu \omega^J$).

[In two dimensions, there are further solutions obtained by taking $f_J^a = \partial f_J / \partial F_{01}^a$, where f_J are arbitrary invariant polynomials in the F_{01}^a. We leave the details to the reader.]

The solutions of Class *I* exist only if there are free, abelian gauge fields. For a semi-simple gauge group, Class *I* is empty. By constrast, the solutions of Class *II* in form degree n exist whenever ther are non-trivial gauge invariant (x-independent) conserved currents, or, equivalently, corresponding non-trivial global symmetries.

They occur at ghost number -1, or $-1 + l_J$, where l_J is the ghost number of the element ω^J of the chosen basis of the Lie algebra cohomology. For a semi-simple gauge group, l_J is greater than or equal to three. Thus, the solutions of Class *II* occur at ghost number $-1, 2$, and higher, but not at ghost number 0 or 1. The solutions at ghost number 2 are given by (9.5) with $\omega^J = \mathrm{tr} C^3$ and $\hat{\omega}^J_\mu = 3\mathrm{tr} C^2 A_\mu$.

We close this section by pointing out that one may regroup the conserved currents j_A (viewed as $(n-1)$-forms) and the coefficients X^i_A into a single object

$$\bar{G}_A = d^n x (X^a_{\mu A} A^{*\mu}_a + X^i_A y^*_i) + j_A \ , \tag{9.6}$$

which has the remarkable property of being annihilated by the sum $\bar{s} = s + d$,

$$\bar{s}\bar{G}_A = 0 \ . \tag{9.7}$$

This equation is the analog of a similar equation holding for $\bar{q}^*_\alpha$,

$$\bar{q}^*_\alpha = C^*_\alpha + A^*_\gamma + *F_\alpha \ , \tag{9.8}$$

where the C^*_α are viewed as n-forms, the A^*_γ are viewed as $(n-1)$-forms and the dual $*F_\gamma$ to the uncoupled free abelian field strength are $(n-2)$-forms. One has also

$$\bar{s}\bar{q}^*_\alpha = 0 \ . \tag{9.9}$$

In verifying these relations, one must use explicitly the fact that the spacetime dimension is n through $d(n\text{-form}) = 0$.

10. Non-Triviality of Solutions of Classes *I* and *II*

We verify in this section that the solutions of Types *I* and *II* are all non-trivial.

Theorem 10.1. *Any BRST cocycle a modulo d belonging to Class I or to Class II is necessarily non-trivial, $a \neq sc + de$.*

Proof. The idea of the proof is to show that if $a = sc + de$, then, the $\alpha_J(\chi^u_A)$ all define trivial elements of $H_2(\delta|d)$ or $H_1(\delta|d)$. So, assume that $a = sc + de$. Expand this equation according to the antighost number. One gets

$$a_0 = \gamma c_0 + \delta c_1 + de_0, \qquad a_1 = \gamma c_1 + \delta c_2 + de_1 \tag{10.1}$$

and

$$0 = \gamma c_i + \delta c_{i+1} + de_i \quad (i \geq 2) \tag{10.2}$$

(we assume a to belong to Class *II* for definiteness; the argument proceeds in the same way for Class *I*). Let c stop at antighost number M, $c = c_0 + c_1 + \cdots + c_M$. Then, one may assume that e stops also at antighost number M. Indeed, the higher order terms can be removed from e by adding a d-exact term since $H^k(d) = 0$ for $k < n$. Now Eq. (10.2) for $i = M$ reads $\gamma c_M + de_M = 0$ and is precisely of the form analysed above. Since $M \geq 2$, one may assume $e_M = 0$ and then, by adding to c_M an s-exact modulo d-term (which does not modify a), that c_M is of the form $c_M = \sum \gamma_J(\chi^u_A)\omega^J(C)$. Next, the equation at order $M-1$ shows that c_M can actually be removed, unless $M = 2$. Thus, we may assume without loss of generality that $c = c_0 + c_1 + c_2$, $c_2 = \sum \gamma_J(\chi^u_A)\omega^J(C)$ and $e = e_0 + e_1$. It follows that the equation

for a_1 reads

$$\sum \alpha_J(\chi_\Delta^u)\omega^J(C) = \gamma c_1 + \sum \delta\gamma_J(\chi_\Delta^u)\omega^J(C) = de_1 . \tag{10.3}$$

By acting with γ on this equation, we obtain as above that e_1 may also be chosen to be invariant, $e_1 = \sum \varepsilon_J(\chi_\Delta^u))\omega^J(C)$. Accordingly, (10.3) reads

$$\sum (\alpha_J(\chi_\Delta^u) - \delta\gamma_J(\chi_\Delta^u) - d\varepsilon_J(\chi_\Delta^u))\omega^J(C) = \gamma c_1' , \tag{10.4}$$

from which one infers, using Theorem 4.1, that

$$\alpha_J(\chi_\Delta^u) - \delta\gamma_J(\chi_\Delta^u) - d\varepsilon_J(\chi_\Delta^u) = 0 . \tag{10.5}$$

This shows that all the α_J are δ-exact modulo d, in contradiction to the fact that they define non-trivial elements of $H_*(\delta|d)$. Therefore, the cocycle a cannot be s-exact modulo d. $\quad\square$

11. Solutions of Class *III*

The solutions of Class *III* do not depend on the antifields and fulfill $\gamma a_0 + db_0 = 0$. As we have recalled, these equations have been extensively studied previously and their general solution is known [4, 23, 7–9]. For this reason, we refer the reader to the existing literature for their explicit construction.

The solutions are classified according to whether b_0 can be removed by redefinitions or not.

Type III$_a$. $\gamma a_0 = 0$.

Type III$_b$. $\gamma a_0 + db_0 = 0$, with b_0 non-trivial. In that case, a_0 and b_0 may be assumed to depend only on the forms A^a, F^a, C^a and their exterior products.

The elements of $H(\gamma|d)$ not involving the antifields are non-trivial as elements of $H(s|d)$ if and only if they do not vanish on-shell modulo d. Thus, the non-trivial elements of $H(\gamma|d)$ of Type *III$_b$* remain non-trivial as elements of $H(s|d)$ since the forms A^a and F^a are unrestricted by the equations of motion. However, the solutions of Type *III$_a$* may become trivial even if they are non-trivial as elements of $H(\gamma|d)$.

The solutions of direct interest are those of ghost number zero and one. At ghost number zero, Class *III$_a$* contains the invariant polynomials in the field strengths, the matter fields and their covariant derivatives. The Yang–Mills Lagrangian belongs to Class *III$_a$*. Class *III$_b$* contains non-trivial solutions at ghost number zero only in odd spacetime dimensions $2k + 1$ if we require these solutions to be Lorentz invariant (see also Sect. 13 for remarks on this point). These non-trivial solutions are the Chern–Simons terms, given by

$$\mathscr{L}_{CS} = \mathrm{tr}(AF^k + \cdots) , \tag{11.1}$$

where the dots denote polynomials in A^a and F^a whose degree in F is smaller than k and whose form degree equals $2k + 1$.

At ghost number one, Type *III$_a$* contains solutions of the form "abelian ghost times invariant polynomial." The abelian anomaly $CF_{\mu\nu}dx^\mu dx^\nu$ in two dimensions belongs to this class. Type *III$_a$* contains no solution with ghost number one if the group is semi-simple. Type *III$_b$* contains the famous Adler–Bardeen anomaly.

12. More General Lagrangians

In the previous discussion, we have assumed that the Lagrangian was the standard Yang–Mills Lagrangian. This assumption was explicitly used in the calculation since the dynamics enters the BRST differential through the Koszul–Tate differential.

It turns out, however, that for a large class of Lagrangians, one can repeat the analysis and get similar conclusions. These Lagrangians are gauge invariant up to a total derivative and thus read

$$\mathscr{L} = \mathscr{L}_0(y, F_{\mu\nu}, D_\mu y, D_\rho F_{\mu\nu}, \ldots) + \mathscr{L}_{CS} , \qquad (12.1)$$

where $\mathscr{L}_0$ is an invariant polynomial in the matter fields, the fields strengths and their covariant derivatives, and where the Chern–Simons term $\mathscr{L}_{CS}$ is available only in odd dimensions if we insist on Lorentz invariance. We shall assume that the Yang–Mills gauge symmetry exhausts all the gauge symmetries. We shall also impose that the Lagrangian $\mathscr{L}$ defines a normal theory in the sense of Sect. 10 of I. The calculation of $H(s|d)$ can then be performed along the lines of this paper.

(i) First, one verifies that the γ-invariant cohomology $H_k(\delta|d)$ is described as before: $H_k(\delta|d)$ is zero for k strictly greater than 2; for $k = 2$, it is non-zero only if there are uncoupled abelian gauge fields, in which case it is spanned by C^*_α; and for $k = 1$, it is isomorphic to the set of non-trivial global symmetries with invariant a_1. Thus, the dynamics enters explicitly $H_k(\delta|d)$ only at $k = 1$, through the conserved currents.

(ii) The solutions of Class I make a further use of the dynamics through the study of the obstructions of the existence of a_0. A case by case analysis, which proceeds as in Sect. 8, is in principle required. Recall, however, that Class I exists only in the academic situation where there are uncoupled abelian gauge fields.

(iii) Class II also uses the equations of motion in the proof that a_1 should define elements of $H_1(\delta|d)$. It must be verified whether the equations of motion can or cannot remove obstructions given by polynomials in the forms F^a. Again, the analysis proceeds straightforwardly as in Sect. 9.

(iv) Class III is obviously unchanged since it does not involve the antifields (only the coboundary condition is modified, since the concept of "on-shell trivial" is changed).

The analysis is particularly simple for the pure Chern–Simons theory in three dimensions, without the Yang–Mills part. We take a semi-simple gauge group. Class I is then empty. Class II is empty as well since there is no non-trivial a_1 annihilated by γ. Only Class III is present. Among the solutions of Class III, those that are of the Subtype III_a turn out to be trivial since the field strengths and their covariant derivatives vanish on-shell. Thus, we are left with Class III_b. These solutions are obtained from the standard descent, with bottom given by the elements ω^J of the basis of the Lie algebra cohomology ($\mathrm{tr}C^3, \mathrm{tr}C^5$ etc). with constant coefficients (no F since $F = 0$ on-shell). This agrees with the analysis of [27].

13. Discussion of x-Dependent Solutions

We point out again that the analysis has been carried out in the case of local forms which do not depend explicitly on the spacetime coordinates x^μ. This is natural in the quantum field theoretical context. Nevertheless one may ask how the results

change if performed in the larger space of local forms which can also depend on the x^μ. Of particular interest are the cohomology classes of $H^{-1}(s|d)$ in that larger space since they provide all non-trivial global symmetries.

Our analysis goes through step by step even in the larger space of x-dependent local forms[2] until one arrives at Eq. (8.3) resp. (8.12) in the cases I_b resp. I_c. The discussion of these equations however yields now a different result since additional (x-dependent) contributions m_1 to a_1 are available (recall that a_1 was determined by (8.3) resp. (8.12) only up to γ-invariant contributions m_1 which turn out to be irrelevant in the space of x-independent forms). One finds e.g. that in $n \neq 4$ dimensions not only the antisymmetric part of the constants $f_{\alpha\beta}$ occurring in (8.2) provides solutions of Type I_b but the symmetric part too. The latter are given by

$$f_{\alpha\beta}\left(\frac{n-4}{2}C^{*\alpha}C^\beta + A^{*\alpha\mu}\left[x^\nu F^\beta_{\nu\mu} + \frac{n-4}{2}A^\beta_\mu\right]\right), \quad f_{\alpha\beta} = f_{\beta\alpha} \tag{13.1}$$

and are clearly of Type I_b for $n \neq 4$ (in the case $n = 4$ they reduce to solutions of Type II_a). The piece $A^{*\alpha\mu}x^\nu F^\beta_{\nu\mu}f_{\alpha\beta}$ occurring in (13.1) is the contribution m_1 mentioned above. In Class I_c one finds analogous x-dependent solutions which we do not spell out.

The solutions of Class II_a correspond as before to non-trivial global symmetries $\delta_\Delta A^a_\mu = X^a_{\mu\Delta}$, $\delta_\Delta y^i = X^i_\Delta$ such that $X^a_{\mu\Delta}A^{*\mu}_a + X^i_\Delta y^*_i$ is γ-invariant. However, now Δ labels global symmetries which are non-trivial in the space of x-dependent local forms (both $X^a_{\mu\Delta}$ and X^i_Δ can involve the x^μ). Class II contains therefore solutions which were not present before (as, e.g., those involving the Lorentz transformations if $\mathscr{L}_0$ is Lorentz invariant). Furthermore, it can (and does) happen that some symmetries which are non-trivial in the space of x-independent forms become trivial in the space of x-dependent forms. An example is provided by the global symmetry $\delta_\Delta A^\alpha_\mu = \xi^\alpha_\mu = const.$ which becomes trivial since it can be written as a gauge transformation $\delta_\Delta A^\alpha_\mu = \partial_\mu \varepsilon^\alpha$ with x-dependent parameter $\varepsilon^\alpha = x^\mu \xi^\alpha_\mu$.

There is another subtlety arising in the analysis of solutions of Class II in the space of x-dependent forms. We shall discuss it in more detail now since it concerns in particular the structure of x-dependent conserved Noether currents and thus has a direct physical relevance. Namely, recall that the j^μ_Δ occurring in (9.1) are conserved Noether currents corresponding to x-independent solutions of Type II_a. As we have pointed out in Sect. 9, these currents can be always taken to be gauge invariant, i.e. to satisfy $\gamma j^\mu_\Delta = 0$. This property of the currents is however less obvious for x-dependent solutions of Type II_a. Nevertheless it still holds, at least if we restrict the investigation to local forms depending polynomially on the x^μ, as we shall prove in the following. For simplicity we consider only one free abelian gauge field, i.e. the very simple case $\mathscr{L}_0 = -(1/4)F_{\mu\nu}F^{\mu\nu}$ (the reasoning can be adopted in the general case straightforwardly). In that case the solutions of Type II_a are (non-trivial) solutions of

$$\delta a = dj, \quad a = d^n x \, X_\mu(x,[F])A^{*\mu}, \tag{13.2}$$

where $X_\mu(x,[F])$ is a polynomial in the x^μ and the $\partial_{\mu_1\ldots\mu_k}F_{\rho\sigma}$, and where j, the conserved current, is a local $(n-1)$-form which generally depends polynomially on the x^μ as well. According to (13.2), dj is a gauge invariant n-form. Hence, j

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[2] One has of course to add the x's as arguments of functions where necessary in the preceding steps; in particular, invariant polynomials $\alpha_J(\chi)$ have to be replaced by $\alpha_J(x,\chi)$ where they occur.

itself is gauge invariant up to a monomial of degree $\frac{n}{2}$ in the field strength 2-form [7,9]. Since such a monomial cannot occur in odd dimensions, we can reduce the investigation to even dimensions and conclude in this case:

$$n = 2r : \quad dj = db + \kappa F^r, \quad F = dx^\mu dx^\nu F_{\mu\nu}, \tag{13.3}$$

where κ is a constant and b, the gauge invariant part of j, is a polynomial in the x^μ and the $\partial_{\mu_1...\mu_k} F_{\rho\sigma}$. We have to show $\kappa = 0$.

To this end we use $\mathrm{SL}(2r, R)$-transformations in spacetime, whose infinitesimal form reads

$$\delta_A x^\mu = -\Lambda^\mu_\nu x^\nu, \quad \delta_A dx^\mu = -\Lambda^\mu_\nu dx^\nu, \quad \delta_A A_\mu = \Lambda^\nu_\mu A_\nu, \quad [\delta_A, \partial_\mu] = \Lambda^\nu_\mu \partial_\nu$$

with $\Lambda \in sl(2r, R)$ (i.e. the Λ are real traceless $2r \times 2r$-matrices). The conclusion $\kappa = 0$ can be reached from the fact that F^r is $\mathrm{SL}(2r, R)$-invariant whereas δa is not (as a consequence of the fact that the Lagrangian $\mathcal{L}_0$ is not $\mathrm{SL}(2r, R)$-but only Lorentz-invariant). To make the argument precise we note that (13.2) can be decomposed into parts transforming according to irreducible representations under $\mathrm{SL}(2r, R)^3$. Hence, each of these parts has to satisfy (13.2) separately. F^r occurs only in the $\mathrm{SL}(2r, R)$-invariant part which reads

$$[\delta a]_0 = d[b]_0 + \kappa F^r, \tag{13.4}$$

where $[\delta a]_0$ and $[b]_0$ denote the $\mathrm{SL}(2r, R)$-invariant parts of (δa) and b respectively. In (13.4) we have used already that d, unlike δ, commutes with the $\mathrm{SL}(2r, R)$-transformations which implies $[db]_0 = d[b]_0$. We can assume $[\delta a]_0$ and $[b]_0$ to have total degree r in the $F_{\mu\nu}$ and their derivatives since this holds also for F^r (all other parts of $[\delta a]_0$ and $[b]_0$ must cancel separately). Furthermore a simple scaling argument ($x^\mu \to \lambda^{-1} x^\mu, \partial_\mu \to \lambda \partial_\mu, A_\mu \to \lambda A_\mu$) shows that we can assume $[\delta a]_0$ and $d[b]_0$ to contain only monomials whose total degree in the x^μ equals the total number of derivatives acting on the $F_{\mu\nu}$.

Assume now that we can show $[\delta a]_0 = 0$. Then we can conclude $\kappa = 0$ from $d[b]_0 = -\kappa F^r$ since otherwise we would obtain a contradiction to the results of [7,9] stating that the F^p with $p \leq n/2$ are just those forms which are closed but not exact in the space of local gauge invariant forms.

Therefore, to complete the argument, we need to show that (13.4) implies $[\delta a]_0 = 0$ or $n = 2$. Note that $[\delta a]_0$ is an n-form which is (i) $\mathrm{SL}(2r, R)$-invariant and (ii) weakly zero. We show in the following that (i) and (ii) contradict each other unless $[\delta a]_0 = 0$ or $n = 2$. To this end we determine first all n-forms satisfying (i) and the above mentioned restrictions on the total degrees in the x^μ, $F_{\mu\nu}$ and the derivatives acting on them. Note that the volume element $d^n x$ is $\mathrm{SL}(2r, R)$-invariant and that $\varepsilon^{\mu_1...\mu_{2r}}$, $\varepsilon_{\mu_1...\mu_{2r}}$ and δ^ν_μ are the only $\mathrm{SL}(2r, R)$-invariant tensors which are available to contract the indices of the x^μ and $\partial_{\mu_1...\mu_k} F_{\rho\sigma}$ in a $\mathrm{SL}(2r, R)$-invariant way. Hence, (i) requires that all indices of the x^μ are contracted with indices of the $\partial_{\mu_1...\mu_k} F_{\rho\sigma}$. The number of remaining "free" indices of the latter (i.e. those indices which are not contracted with x's) equals then $2r = n$ by the counting and scaling arguments given above. Hence, these free indices have to be contracted with $\varepsilon^{\mu_1...\mu_{2r}}$ and thus are totally antisymmetrized. Taking into account the Bianchi identities, it is then straightforward to verify that the $2r$ free indices stem soley from monomials

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$$G^m_{\mu\nu} = x^{\rho_1} \cdots x^{\rho_m} \partial_{\rho_1 \dots \rho_m} F_{\mu\nu}, \quad H^{m+1}_\mu = x^\nu G^m_{\mu\nu} ,$$

where $m = 0, 1, \dots$. The $SL(2r, R)$-invariants constructable from the G's and H's and one $\varepsilon^{\mu_1 \dots \mu_{2r}}$ are linear combinations of functions of the form

$$G^{m_1}_{\mu_1\nu_1} \cdots G^{m_k}_{\mu_k\nu_k} H^{n_1}_{\mu_{k+1}} \cdots H^{n_{r-k}}_{\mu_r} H^{n_{r-k+1}}_{\nu_{k+1}} \cdots H^{n_{2(r-k)}}_{\nu_r} \varepsilon^{\mu_1\nu_1\dots\mu_r\nu_r} . \tag{13.5}$$

Recall now that we are only interested in functions which have total degree r in the $F_{\mu\nu}$ and their derivatives. The only functions (13.5) satisfying this constraint are those with $k = r$, i.e. those which do not depend on the H's at all. They also satisfy the constraint imposed by the scaling argument. Hence, the functions we are looking for are linear combinations of

$$P(m_1, \dots, m_r) = G^{m_1}_{\mu_1\nu_1} \cdots G^{m_r}_{\mu_r\nu_r} \varepsilon^{\mu_1\nu_1\dots\mu_r\nu_r} . \tag{13.6}$$

As indicated by the notation, the $P(m_i)$ are characterized and distinguished completely by their arguments $m_i = (m_1, \dots, m_r)$. Notice that (a) $P(m_i)$ is totally symmetric in all its arguments, (b) the sum of these arguments is the total degree of $P(m_i)$ in the x^μ and (c) the arguments m_i indicate the order of the derivatives of the $F_{\mu\nu}$ occurring in the $P(m_i)$. We conclude that there is only one $P(m_i)$ for fixed total degree in the x^μ and fixed orders of derivatives of the $F_{\mu\nu}$.

Since the equations are homogeneous in the derivatives of F, different $P(m_i)$ can never combine to weakly vanishing terms unless each $P(m_i)$ itself vanishes weakly. For $r > 1$ (i.e., for $n \neq 2$) one readily checks that no $P(m_i)$ vanishes weakly since the equations of motion constrain only those derivatives $\partial_{\mu_1 \dots \mu_k} F_{\rho\sigma}$ for which at least one of the μ_i equals ρ or σ. One easily makes sure however that $P(m_i)$ contains monomials which do not involve these derivatives at all if $n \neq 2$. In contrast, in two dimensions all $P(m_i) \equiv P(m)$ vanish weakly apart from $P(0) = F_{\mu\nu}\varepsilon^{\mu\nu}$ since the equations of motion set to zero all derivatives of the field strength. Therefore the case $n = 2$, which we have anyhow, excluded, provides the only counterexample to the result $\kappa = 0$. This counterexample is obtained for $\delta a = d^2 x P(1)$ since $P(1) = x^\rho \partial_\rho F_{\mu\nu}\varepsilon^{\mu\nu} = 2x^\rho \varepsilon_{\rho\nu}\partial_\mu F^{\mu\nu}$ implies

$$n = 2 : \delta(2\varepsilon_{\rho\nu} x^\rho A^{*\nu}) = \partial_\rho(x^\rho F_{\mu\nu}\varepsilon^{\mu\nu}) - 2F_{\mu\nu}\varepsilon^{\mu\nu} ,$$

which is the dual version of (13.4).

Hence, assuming polynomiality in the x^μ (and $n \geqq 2$), one still can take the conserved currents corresponding to solutions of Type II_a to be gauge invariant and the classification of all solutions of Type II in the space of local forms depending polynomially on the x^μ can be performed as in Sect. 9.

For the sake of completeness we finally note that, in the space of x-dependent forms, Class III_a consists of solutions of the form "invariant polynomial in the χ^u_Δ" times "function of the x^μ." Among the solutions of Type III_b, present in the space of x-independent forms, only those "survive" in the space of x-dependent forms which are Lorentz invariant. The others become either trivial or can be "shifted" to Class III_a. This follows immediately from an inspection of the descent equations

$$s a^p + d a^{p+1} = 0, \quad s a^{p-1} + d a^{p-2} = 0, \dots, \quad s a^{p-k} = 0$$

associated with these solutions (the superscript of the a's denotes their form degree; one has $k > 0$ since solutions of Type III_b have by definition a non-trivial descent). Namely the BRST invariant form a^{p-k} occurring at the last equation is a linear combination of forms

$$\hat{a}^{p-k} = \alpha^{p-k-i}(F)\omega^J(C)\eta^i(dx) ,$$

where the $\eta^i(dx)$ are i-forms involving only the differentials, $\omega^J(C)$ denote the polynomials in the ghosts introduced in Sect. 4 and $\alpha^{p-k-i}(F)$ are invariant homogeneous polynomials of degree $(p - k - i)/2$ in the curvature 2-forms $F^a = 1/2dx^\mu dx^\nu F_{\mu\nu}^a$. Now, in the space of x-dependent forms, $\hat{a}^{p-k}$ is trivial unless $\eta^i(dx)$ is a 0-form. Namely for $i > 0$ one has $\eta^i(dx) = d\eta^{i-1}(x, dx)$ which implies, using (8.8),

$$i \neq 0 : \hat{a}^{p-k} = s\hat{b}^{p-k} + d\hat{b}^{p-k-1} ,$$

where

$$\hat{b}^{p-k-1} = \alpha^{p-k-i}\omega^J\eta^{l-1}, \qquad \hat{b}^{p-k} = \alpha^{p-k-i}\hat{\omega}_\mu^J\eta^{l-1}dx^\mu .$$

By standard arguments one verifies that trivial contributions can be removed from a^{p-k}. Hence, we can assume $i = 0$ and a^{p-k} to be Lorentz invariant (but note that this reasoning is not possible in the space of x-independent forms since there the forms $\eta^{l-1}(x, dx)$ are not available). As mentioned above, this implies eventually that the parts of a^p which are not Lorentz invariant become trivial or equivalent to solutions of Class III_a in the space of x-dependent forms. A simple example for the latter case is provided by the 4-form $a^4 = f_{\mu AB}dx^\mu A^A F^B$, where $f_{\mu AB} = f_{\mu BA}$ are constants and A^A and F^B denote abelian connection 1-forms and curvature 2-forms respectively. Namely one has $a^4(f_{\mu AB}x^\mu A^A F^B) - \tilde{a}4$, where $\tilde{a}4 = f_{\mu AB}x^\mu F^A F^B$ is an x-dependent solution of Type III_a.

14. Conclusion

In this paper, we have explicitly computed the cohomology groups $H^k(s|d)$ for Yang–Mills theory. Our work goes beyond previous analyses on the subject [17, 20, 4, 28–31, 23, 22, 5, 7, 9], in that (i) we do not use power counting, and (ii) we explicitly include the antifields (=sources for the BRST variations). We have shown that new cohomological classes depending on the antifields appear whenever there are conserved currents, but for a semi-simple gauge group they occur only at ghost numbers $g = -1$ and $g \geq 2$. Our results confirm previous conjectures in the field. [The existence of antifield-dependent solutions of the consistency equation at ghost number one for a theory with abelian factors was anticipated in [28]. The structure of these solutions was partly elucidated and an argument was given that they cannot occur as anomalies.]

The central feature behind our analysis is a key property of the antifield formalism, namely, that the antifields provide a resolution of the stationary surface through the Koszul–Tate differential [2]. It is by attacking the problem from that angle that we have been able to carry out the calculation to completion, while previous attempts following different approaches turned out to be unsuccessful. Thus, even in the familiar Yang–Mills context, the formal ideas of the antifield formalism prove to be extremely fruitful.

Our results can be extended in various directions. First, one can repeat the Yang–Mills calculation for Einstein gravity with or without matter. This will be done explicitly in [32]. Second, at a more theoretical level, one can analyze further the connection between the local BRST cohomology, the characteristic cohomology and the variational bicomplex [33]. This will be pursued elsewhere.

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