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Equivalence of Euclidean and Wightman Field Theories

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Abstract: A new inversion formula for the Laplace transformation of tempered distributions with supports in the closed positive semiaxis is obtained. The inverse Laplace transform of a tempered distribution is defined by means of a limit of a special distribution constructed from this distribution. The weak spectral condition on the Euclidean Green's functions implies that some of the limits needed for the inversion formula exist for any Euclidean Green's function with an even number of variables. We then prove that the initial Osterwalder–Schrader axioms [1] and the weak spectral condition are equivalent with the Wightman axioms.

1. Introduction

In 1973 K. Osterwalder and R. Schrader [1] claimed to have found necessary and sufficient conditions under which Euclidean Green's functions have analytic continuations whose boundary values define a unique set of Wightman distributions. The principal idea of the Osterwalder–Schrader paper [1] was to consider the Euclidean Green's functions to be distributions. Usually the Euclidean Green's functions were considered to be the analytic functions. Later R. Schrader [2] found a counter-example for a central lemma of the paper [1]. In 1975 K. Osterwalder and R. Schrader proposed an additional "linear growth condition" under which Euclidean Green's functions, satisfying the Osterwalder–Schrader axioms [1], define the Wightman theory. But these new extended axioms for the Euclidean Green's functions may not be equivalent with the Wightman axioms. It is possible to restore the equivalence theorem by adding the new condition [2] that the Euclidean Green's functions are Laplace transforms of the tempered distributions with supports in the positive semiaxis with respect to the time variables. The equivalence theorem then becomes trivial [2]. This new condition contradicts the main Osterwalder–Schrader idea to consider the Euclidean Green's functions to be distributions and it is not

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suitable for applications because it seems difficult to check these conditions. This paper is an attempt to understand the mathematical foundation of the Osterwalder–Schrader results. Our aim is to find the additional suitable condition which allows to prove that the extended Osterwalder–Schrader axioms thus obtained are then indeed equivalent with the Wightman axioms.

One of the Osterwalder–Schrader axioms is the positivity condition. If we consider the simplest case and neglect the space variables we can write the positivity condition in the form

$$\int_0^\infty dt \int_0^\infty ds\, f(t+s)\overline{\phi(t)}\phi(s) \geqq 0\,. \tag{1.1}$$

Due to [3, Lemma A] the positivity condition (1.1) for the distribution $f(t) \in D'(\mathbf{R}_+)$, where $\mathbf{R}_+$ is the open positive semiaxis, implies the condition in $\mathbf{R}_+$,

$$\sum_{m,n} a_m \overline{a}_n \frac{d^{m+n} f}{dt^{m+n}}(t) \geqq 0\,, \tag{1.2}$$

for all finite sequences of the complex numbers a_m. Corollary C from [3] implies that the distribution $f(t) \in D'(\mathbf{R}_+)$, satisfying the condition (1.2) for all terminating sequences of complex numbers a_m, is the restriction to the semiaxis of a function $A(x+iy)$ analytic in the tube $\mathbf{R}_+ + i\mathbf{R}$. To explain the difficulties which one encounters in this way in proving the Osterwalder–Schrader theorem we cite here an extract from the remarkable paper [3]: "The Euclidean Green's functions satisfying the Osterwalder–Schrader postulates can be shown to be restrictions of the functions analytic in the whole Wightman causal domain and to satisfy the positivity condition there in a sense to be presently explained. The author has, however, not been able to show the tempered growth of those analytic functions near the real Minkowski space boundary and believes at present that this is impossible to achieve without further assumptions on the growth properties of Schwinger functions s_n with respect to the index n. This is suggested by the fact that in order to reach the real Minkowski space by analytic completion for a given s_n an infinite number of steps are required, each of which involves the other functions s_m via the Schwartz inequality with higher and higher values of m." Our way of proving the equivalence theorem doesn't use the condition that the Euclidean Green's functions are analytic functions. Following the Osterwalder–Schrader idea we consider the Euclidean Green's functions to be distributions.

S. Bernstein [4] called a function exponentially convex if it satisfies the positivity condition (1.1). We shall prove that a tempered distribution $f(t) \in S'(\mathbf{R}_+)$ is exponentially convex iff the tempered distribution $g(t) = f(-t) \in S'(\mathbf{R}_-)$ is absolutely monotonic, i.e. if

$$\frac{d^m g}{dt^m}(t) \geqq 0 \tag{1.3}$$

holds for all $m = 0, 1, \ldots$. The following counter-example $f(t) = \exp\{t\}$ shows that this theorem is wrong for distributions in $D'(\mathbf{R}_+)$. In [4] S. Bernstein also studied the absolutely monotonic functions. It is natural to try to generalize the Bernstein result. We shall prove that if for a distribution $f(t) \in D'(\mathbf{R}_+)$ the distribution $g(t) = f(-t) \in D'(\mathbf{R}_-)$ is absolutely monotonic then

$$f(t) = \int_0^\infty e^{-ts}\, d\mu(s)\,, \tag{1.4}$$

where the positive measure $\mu(s)$ has tempered growth. The measure $\mu(s)$ explicitly depends on the distribution $f(t)$. It is the sum of two limits of the special distributions constructed from the distribution $f(t)$. By using the generalized Bernstein theorem it is possible to obtain a new inversion formula for the Laplace transformation of tempered distributions with supports in the closed positive semiaxis. Our weak spectral condition on the Euclidean Green's functions requires that some of the limits needed for the inversion formula exist for any Euclidean Green's function with even number of variables. We shall prove that the initial Osterwalder–Schrader axioms [1] and the weak spectral condition are equivalent with the Wightman axioms. A new result may be derived immediately from the revised Osterwalder–Schrader theorem. The Wightman axioms are equivalent to the initial Osterwalder–Schrader axioms [1] and one half of the continuity condition [2], namely the condition that any Euclidean Green's function with an even number of variables is a Laplace transform of a tempered distribution with support in the positive semiaxis with respect to the time variables.

In the next section we study absolutely monotonic distributions and prove a generalization of the Bernstein theorem [4]. We then obtain a new inversion formula for the Laplace transformation of the tempered distributions with supports in the closed positive semiaxis. The third section is devoted to a study of exponentially convex tempered distributions and tempered distributions satisfying the Osterwalder–Schrader positivity condition which includes the space variables. In the fourth section the revised Osterwalder–Schrader theorem is proved.

2. Absolutely Monotonic Distributions

Let $D(\overline{\mathbf{R}}_+)$ denote the subspace of $D(\mathbf{R})$ of functions with support in the positive semiaxis $\overline{\mathbf{R}}_+ = [0,\infty)$, given the induced topology. Similarly $D(\overline{\mathbf{R}}_-)$ denotes the subspace of $D(\mathbf{R})$ of functions with support in the negative semiaxis $\overline{\mathbf{R}}_- = (-\infty,0]$, given the induced topology. If the function $\phi(x)$ is in $D(\overline{\mathbf{R}}_-)$ then the function $\phi(-x)$ is in $D(\overline{\mathbf{R}}_+)$.

Since the topology of the space $D(\overline{\mathbf{R}}_+)$ is induced by the topology of the space $D(\mathbf{R})$ for a given distribution $f \in D'(\overline{\mathbf{R}}_+)$ there exists a natural number N such that the estimate

$$|(f,\phi)| \leqq C \sup_{x\in\mathbf{R}_+,\, 0\leqq k<N} \left| \frac{d^k\phi}{dx^k} \right| \tag{2.1}$$

holds for every function $\phi(x) \in D(\mathbf{R})$ with support in the interval $[0,1]$. Using the estimate (2.1) and the identity

$$\frac{d^k}{dx^k}(x^{N_1}\phi(x)) = x^{N_1-k-1} \left[\prod_{j=1}^{k} \left(N_1 - j + x\frac{d}{dx} \right) \right] (x\phi(x)) \tag{2.2}$$

for a natural number k it is easy to show that for any integer $N_1 \geqq N$ the inequality

$$|(x^{N_1} f(x), \phi(x))| \leqq C_1 \sup_{x\in\mathbf{R}_+,\, 0\leqq k<N} \left| \left(x\frac{d}{dx} \right)^k (x\phi(x)) \right| \tag{2.3}$$

holds for every function $\phi(x) \in D(\mathbf{R})$ with support in the interval $[0,1]$. We denote by $N(f)$ the minimal integer N_1 such that the distribution $x^{N_1} f(x) \in D'(\overline{\mathbf{R}}_+)$

satisfies an inequality of type (2.3) for every function $\phi(x) \in D(\mathbf{R})$ with support in the interval $[0,1]$. Inequality (2.3) implies the inequality

$$N(f) \geqq N\left(x\frac{df}{dx}\right) \tag{2.4}$$

for any distribution $f(x) \in D'(\mathbf{R}_+)$.

Let $\theta(x)$ denote the Heaviside step function.

Lemma 2.1. *For every distribution $f(x) \in D'(\mathbf{R}_+)$, for any function $\phi(x) \in D(\mathbf{R})$ and for every integer $N \geqq N(f)$ the limit*

$$\lim_{\alpha \to +0} (x^N f(x), \theta(x)\exp\{-\alpha x^{-1}\}\phi(x)) \tag{2.5}$$

defines an extension $[x^N f](x) \in D'(\mathbf{R})$ with support in the positive semiaxis $\overline{\mathbf{R}}_+$.

Proof. Lemma 2.1 follows from the estimate (2.3) and [5, equality (14)].

The distribution $f(x) \in D'(\mathbf{R}_-)$ is said to be absolutely monotonic if for all natural numbers $m = 0,1,\dots$ the distribution $\frac{d^m f}{dx^m}(x)$ is positive.

If a function $\phi(x) \in D(\mathbf{R})$ then for sufficiently large positive t depending on $\operatorname{supp}\phi$ the function $\phi(x+t) \in D'(\mathbf{R}_-)$.

Lemma 2.2. *Let the distribution $f(x) \in D'(\mathbf{R}_-)$ be absolutely monotonic. There exists a number $L_0^{-1}[f]$ such that for any function $\phi(x) \in D(\mathbf{R})$,*

$$\lim_{t \to -\infty} (f(x), \phi(x-t)) = L_0^{-1}[f] \int_{-\infty}^{\infty} \phi(x)\,dx\,, \tag{2.6}$$

$$\lim_{t \to -\infty} t^k \left(\frac{d^k f}{dx^k}(x), \phi(x-t)\right) = 0, \quad k = 1,2,\dots\,. \tag{2.7}$$

Then the distribution $(-x)^{-1}f(x)$ is well defined as an element of $D'(\mathbf{R}_-)$ and is also absolutely monotonic. Furthermore one has $L_0^{-1}[(-x)^{-1}f(x)] = 0$.

Proof. Let the function $\phi(x) \in D(\mathbf{R})$ be nonnegative and with support in the interval $[a,b]$. The function $f(t;\phi) \equiv (f(x), \phi(x-t))$ is defined on the semiaxis $(-\infty,-a]$. It is infinitely differentiable. Since the distribution $f(x)$ is absolutely monotonic the nonnegativity of the function $\phi(x)$ implies

$$\frac{d^n}{dt^n} f(t;\phi) = \left(\frac{d^n f}{dx^n}(x), \phi(x-t)\right) \geqq 0, \quad n = 0,1,\dots\,. \tag{2.8}$$

Hence for every $n = 1,2,\dots$ we get

$$\begin{aligned} \frac{d^n}{dt^n} f(t;\phi) &\leqq 2|t|^{-1} \int_t^{t/2} \frac{d^n}{dy^n} f(y;\phi)\,dy \\ &\leqq 2|t|^{-1}\left[\frac{d^{n-1}}{dy^{n-1}} f(y;\phi)\,|_{y=t/2} - \frac{d^{n-1}}{dt^{n-1}} f(t;\phi)\right], \end{aligned} \tag{2.9}$$

where $t < \min(0, -a)$. Due to the inequalities (2.8) the function $f(t;\phi)$ is nonnegative and non-decreasing on the semiaxis $(-\infty, -a]$. Therefore the limit (2.6) exists. Then it follows from the inequality (2.9) for $n = 1$ that the limit (2.7) equals zero for $k = 1$. Using induction and inequality (2.9) it is easy to prove the equalities (2.7) for $k = 1, 2, \ldots$ and for any nonnegative function $\phi(x) \in D(\mathbf{R})$.

Let $\phi(x) \in D(\mathbf{R})$ be real and set $M(\phi) = \sup_{x \in \mathbf{R}} |\phi(x)|$. Let $h(x) \in D(\mathbf{R})$ be a nonnegative function equal to one on the support of the function $\phi(x)$. The function $\phi(x)$ is the difference of the nonnegative functions $1/2(M(\phi)h(x) \pm \phi(x))$. This decomposition implies the equalities (2.7) for the function $\phi(x)$ since the equalities (2.7) are valid for nonnegative functions from $D(\mathbf{R})$. The claim (2.7) for arbitrary $\phi(x) \in D(\mathbf{R})$ follows by considering the real and imaginary part of $\phi(x)$ separately.

Consider a nonnegative function $h(x) \in D(\mathbf{R})$ whose integral is equal to one. Then any function $\phi(x) \in D(\mathbf{R})$ may be rewritten as

$$\phi(x) = h(x) \int_{-\infty}^{\infty} \phi(y)\, dy + \frac{d\psi}{dx}(x)\,, \tag{2.10}$$

where $\psi(x) \in D(\mathbf{R})$. The limit (2.6) exists for any nonnegative function from $D(\mathbf{R})$. Hence the decomposition (2.10) and the equality (2.7) for $k = 1$ imply the equality (2.6).

If the distribution $f(x) \in D'(\mathbf{R}_-)$ is absolutely monotonic the distribution $(-x)^{-1} f(x) \in D'(\mathbf{R}_-)$ is also absolutely monotonic. It follows from the relations (2.6) for the distributions $f(x)$ and $(-x)^{-1} f(x)$ combined with the identity $(-x)^{-1}t = -(-x)^{-1}(x - t) - 1$ that

$$\lim_{t \to -\infty} t((-x)^{-1} f(x), \phi(x - t))$$
$$= -L_0^{-1}[(-x)^{-1} f(x)] \int_{-\infty}^{\infty} x\phi(x)\, dx - L_0^{-1}[f] \int_{-\infty}^{\infty} \phi(x)\, dx\,. \tag{2.11}$$

On the other hand since the limit (2.6) for the distribution $(-x)^{-1} f(x)$ exists the limit (2.11) may only exist when the constant $L_0^{-1}[(-x)^{-1} f(x)]$ equals zero. This concludes the proof of Lemma 2.2.

For a function $\phi(x) \in D(\mathbf{R})$ and $k = 1, 2, \ldots$ we introduce the function

$$\phi^{(-k)}(x) = -((k-1)!)^{-1} \int_{x}^{\infty} (x - y)^{k-1} \phi(y)\, dy\,, \tag{2.12}$$

and for $k = 0$ we define $\phi^{(0)}(x) = \phi(x)$. The infinitely differentiable function (2.12) equals zero on the positive semiaxis whenever $\phi(x) \in D(\mathbf{R}_-)$. Our notation is reasonable since $\frac{d^l}{dx^l}\phi^{(-k)}(x) = \phi^{(l-k)}(x)$ for $l \leqq k$.

Consider a nonnegative function $h(x) \in D(\mathbf{R})$ with the integral equal one and having support in the positive semiaxis. For every $T < 0$ we introduce the infinitely differentiable function with compact support

$$h_T(x) = \int_{x}^{x-T} h(y)\, dy\,. \tag{2.13}$$

Lemma 2.3. *Let the distribution $f(x) \in D'(\mathbf{R}_-)$ be absolutely monotonic. Then for any function $\phi(x) \in D(\mathbf{R}_-)$ and for any integer $k = 1, 2, \ldots,$*

$$\lim_{T \to -\infty} (-1)^k \left(\frac{d^k f}{dx^k}(x), h_T(x) \phi^{(-k)}(x) \right) = (f(x), \phi(x)) - L_0^{-1}[f] \int_{-\infty}^{\infty} \phi(x)\, dx\,, \quad (2.14)$$

where the constant $L_0^{-1}[f]$ is given by the equality (2.6).

Proof. For any integer $k = 1, 2, \ldots$ the definitions (2.12) and (2.13) imply the following relation

$$\left(\frac{d^k f}{dx^k}(x), h_T(x) \phi^{(-k)}(x) \right) = - \left(\frac{d^{k-1} f}{dx^{k-1}}(x), h_T(x) \phi^{(1-k)}(x) \right)$$
$$- \left(\frac{d^{k-1} f}{dx^{k-1}}(x), (h(x-T) - h(x)) \phi^{(-k)}(x) \right) . \quad (2.15)$$

Due to the definitions the supports of the functions $h(x)$ and $\phi^{(-k)}(x)$ are disjoint. Hence we have $h(x)\phi^{(-k)}(x) = 0$. Applying the equality (2.15) k times we get

$$(-1)^k \left(\frac{d^k f}{dx^k}(x), h_T(x) \phi^{(-k)}(x) \right)$$
$$= (f(x), h_T(x) \phi(x)) + \sum_{p=0}^{k-1} (-1)^p \left(\frac{d^p f}{dx^p}(x), h(x-T) \phi^{(-p-1)}(x) \right) . \quad (2.16)$$

Since the functions $h(x)$ and $\phi(x)$ have compact support, for sufficiently large modulus of the negative number T we obtain

$$h(x-T) \int_{-\infty}^{x} (x-y)^{p-1} \phi(y)\, dy = 0\,, \quad (2.17)$$

where the integer $p = 1, 2, \ldots$. It follows from the relations (2.6), (2.7), (2.12) and (2.17) that

$$\lim_{T \to -\infty} \sum_{p=0}^{k-1} (-1)^p \left(\frac{d^p f}{dx^p}(x), h(x-T) \phi^{(-p-1)}(x) \right) = -L_0^{-1}[f] \int_{-\infty}^{\infty} \phi(x)\, dx\,, \quad (2.18)$$

where the constant $L_0^{-1}[f]$ is given by the equality (2.6).

The definition (2.13) implies that the function $h_T(x)$ is equal to one on the support of the function $\phi(x) \in D(\mathbf{R}_-)$ for the sufficiently large modulus of the negative number T, since the integral of the function $h(x) \in D(\mathbf{R}_+)$ is equal to one. Now equality (2.14) is a consequence of the equalities (2.16) and (2.18) concluding the proof of Lemma 2.3.

For a distribution $f(x) \in D'(\mathbf{R}_-)$ we define a functional on the Schwartz space $S(\mathbf{R})$ by the following relation:

$$(L_c^{-1}[f](-x;n,T),\phi(x)) = (f(x), L_c^{-1}[\phi](-x;n,T))$$

$$= (n!)^{-1}((-x)^n \frac{d^{n+1}f}{dx^{n+1}}(x),\ \theta(-x)h_T(x)\phi(-nx^{-1}))\,, \tag{2.19}$$

where n is a positive integer and the function $h_T(x)$ is given by relation (2.13). It is easy to show that the tempered distribution $L_c^{-1}[f](-x;n,T) \in S'(\mathbf{R})$ is positive and its support is contained in the positive semiaxis.

Proposition 2.4. *Let the distribution $f(x) \in D'(\mathbf{R}_-)$ be absolutely monotonic. Then in the topological space $S'(\mathbf{R})$ the limit*

$$\lim_{n\to\infty}\lim_{T\to-\infty} L_c^{-1}[f](-x;n,T) = L_c^{-1}[f](-x) \tag{2.20}$$

exists. The tempered distribution $L_c^{-1}[f](-x) \in S'(\mathbf{R})$ is positive and its support is in the positive semiaxis.

Proof. Let us multiply and divide the function $\phi(x)$ on the right-hand side of the equalities (2.19) by the same polynomial $(1+x)^N$

$$(L_c^{-1}[f](-x;n,T),\phi(x)) = (n!)^{-1}\int dx(-x)^{n+N(f)+1}\frac{d^{n+1}f}{dx^{n+1}}(x)$$

$$\times (-x)^{N-N(f)-1}\theta(-x)h_T(x)\left(\frac{1-nx^{-1}}{n-x}\right)^N \phi(-nx^{-1})\,, \tag{2.21}$$

where we recall that $N(f)$ is the minimal integer N_1 such that the inequality (2.3) is satisfied. The relation (2.2) for $N_1 = 1$ implies

$$(-x)^{n+N(f)+1}\frac{d^{n+1}f}{dx^{n+1}}(x) = (-1)^{n+N(f)+1}(x)^{N(f)+1}\left(\prod_{j=1}^{n}\left(x\frac{d}{dx}+1-j\right)\right)\frac{df}{dx}(x)\,. \tag{2.22}$$

Due to the inequality (2.4) and Lemma 2.1 the positive distribution (2.22) from $D'(\mathbf{R}_-)$ is extended to a positive distribution from $D'(\mathbf{R})$ with support in the negative semiaxis. This extension is defined by the limit analogous to the limit (2.5).

For sufficiently large positive integer n the function $(-x)^{N-N(f)-1}h_T(x)(n-x)^{-N}$ is infinitely differentiable for $N > N(f)$. It is positive on the negative semiaxis. Now the positivity of the extension of the distribution (2.22) implies the following estimate for the integral (2.21):

$$|(L_c^{-1}[f](-x;n,T),\phi(x))| \leqq C_{n,T}(N)\sup_{x\geqq 0}(1+x)^N|\phi(x)|\,. \tag{2.23}$$

Here the constant $C_{n,T}(N)$ depending on f is given as

$$C_{n,T}(N) = (n!)^{-1} \lim_{\alpha \to +0} \int dx(-x)^{n+N(f)+1} \frac{d^{n+1} f}{dx^{n+1}}(x) \times \theta(-x) \exp\{\alpha n x^{-1}\}(-x)^{N-N(f)-1} h_T(x)(n-x)^{-N}$$
$$= \lim_{\alpha \to +0} (L_c^{-1}[f](-x;n,T), \exp\{-\alpha x\}(1+x)^{-N}) \quad (2.24)$$

and $N > N(f)$. We denote by $|\cdot|_N$ the norm on the right-hand side of the inequality (2.23). We define H_N to be the Banach space completion of the space $S(\mathbf{R})$ with respect to this norm. Due to the inequality (2.23) for $N = N(f)+1$ the tempered distribution $L_c^{-1}[f](-x;n,T) \in S'(\mathbf{R})$ is continued to the linear continuous functional $L_c^{-1}[f](-x;n,T)^c$ on the Banach space $H_{N(f)+1}$. Now the relation (2.24) may be rewritten as $C_{n,T}(N) = (L_c^{-1}[f](-x;n,T)^c, (1+x)^{-N})$, where $N > N(f)$.

Let us assume for a moment that for any integer $N > N(f)$ the limit

$$\lim_{n \to \infty} \lim_{T \to -\infty} C_{n,T}(N) \quad (2.25)$$

exists and is finite. This assumption and the inequality (2.23) for $N = N(f)+1$ imply that the family of linear continuous functionals $L_c^{-1}[f](-x;n,T)^c$ parametrized by n and T on the Banach space $H_{N(f)+1}$ is uniformly bounded. In view of relation (2.24) the existence of the limit (2.25) is equivalent to the convergence of the sequence $\{L_c^{-1}[f](-x;n,T)^c\}$ on every function $(1+x)^{-N} \in H_{N(f)+1}$, where $N > N(f)$. We claim that this set is dense in the Banach space $H_{N(f)+1}$. Then by the Banach–Steinhaus theorem [6, Sect. 3.7] the sequence of linear continuous functionals $L_c^{-1}[f](-x;n,T)^c$ on the Banach space $H_{N(f)+1}$ weakly converges to the linear continuous functional on $H_{N(f)+1}$. Therefore the sequence of tempered distributions $L_c^{-1}[f](-x;n,T) \in S'(\mathbf{R})$ weakly converges to a tempered distribution in $S'(\mathbf{R})$. Hence due to [6, Sect. 3.7] the sequence of tempered distributions $L_c^{-1}[f](-x;n,T)$ converges in topology of the space $S'(\mathbf{R})$. Now it follows that the tempered distribution $L_c^{-1}[f](-x) \equiv L_c^{-1}[f](-x;\infty,-\infty) \in S'(\mathbf{R})$ is positive and its support is in the positive semiaxis since these properties are stable under limits.

Let us first prove that the set of the functions $(1+x)^{-N(f)-k-1}$, $k = 0,1,\ldots,$ is dense in the Banach space $H_{N(f)+1}$. For any function $\phi(x) \in S(\mathbf{R})$ the function $\hat{\phi}(t) = \phi(\tan^2 t)(\cos t)^{-2(N(f)+1)}$ is continuous on the open interval $(-\pi/2, \pi/2)$. Since $\phi(x) \in S(\mathbf{R})$ the function $\hat{\phi}(t)$ may be extended to a continuous function on the closed interval $[-\pi/2, \pi/2]$ by setting $\hat{\phi}(-\pi/2) = \hat{\phi}(\pi/2) = 0$. Hence the Weierstrass theorem implies that $\hat{\phi}(t)$ may be approximated by a trigonometric polynomial $\sum b_m \exp\{2mit\}$ on the closed interval $[-\pi/2, \pi/2]$. Since the function $\hat{\phi}(t)$ is even it may be approximated by a trigonometric polynomial $\sum b_m \cos 2mt$. The function $\cos 2mt$ is the polynomial of the variable $\cos^2 t$. Therefore for every positive ε there exists a polynomial $\sum a_m \cos^{2m} t$ such that the modulus of the function $\hat{\phi}(t) - \sum a_m \cos^{2m} t$ on the closed interval $[-\pi/2, \pi/2]$ is less than ε. Due to the relation $\cos^{2m} t = (1 + \tan^2 t)^{-m}$ this implies that

$$\sup_{x \geqq 0} (1+x)^{N(f)+1} |\phi(x) - \sum a_m (1+x)^{-N(f)-m-1}| < \varepsilon\,. \quad (2.26)$$

Thus the set of functions $(1+x)^{-N(f)-k-1}$, $k = 0,1,\ldots,$ is dense in the Banach space $H_{N(f)+1}$, since the space $S(\mathbf{R})$ is dense.

At last let us prove the existence of the limit (2.25) for every integer $N > N(f)$. Since by assumption $h(x) \in D(\mathbf{R}_+)$ for sufficiently large modulus of the negative number T we have $h(x-T) \in D(\mathbf{R}_-)$. Using the definition (2.13), the identity (2.2), the estimate (2.3) and in analogy to the derivation (2.15) we can rewrite the expression (2.24) for sufficiently large modulus of the negative number T in the following form:

$$C_{n,T}(N) = B_{n,T}(N) - (n!)^{-1} \sum_{k=0}^{n-1} (-1)^{n-k} \times \left(\frac{d^{n-k} f}{dx^{n-k}}(x), h(x-T) \frac{d^k}{dx^k}(x^{n+N}(x-n)^{-N}) \right), \quad (2.27)$$

where the constant

$$B_{n,T}(N) = \lim_{\alpha \to +0} \left(\frac{df}{dx}(x), \theta(-x) \exp\{\alpha x^{-1}\} h_T(x) \chi_{n,N}(-x^{-1}) \right) \quad (2.28)$$

and the function

$$\chi_{n,N}(x) = (n!)^{-1} \frac{d^n}{dy^n}(y^{n+N}(y-n)^{-N})|_{y=-x^{-1}} . \quad (2.29)$$

It follows from the identity (2.2) that

$$x^{k-n} \frac{d^k}{dx^k}(x^{n+N}(x-n)^{-N}) = \left[\prod_{j=1}^{k} \left(n+1-j-y\frac{d}{dy} \right) \right] (1+y)^{-N}|_{y=-nx^{-1}} . \quad (2.30)$$

Now it is easy to show that the expression (2.30) is bounded on the closed negative semiaxis. Hence the absolutely monotonicity of the distribution $f(x) \in D'(\mathbf{R}_-)$ and the relations (2.7), (2.27) imply that

$$\begin{aligned} &\lim_{T \to -\infty} \left| \left(\frac{d^{n-k} f}{dx^{n-k}}(x), h(x-T) \frac{d^k}{dx^k}(x^{n+N}(x-n)^{-N}) \right) \right| \\ &\leqq C \lim_{T \to -\infty} \left(\frac{d^{n-k} f}{dx^{n-k}}(x), h(x-T)(-x)^{n-k} \right) \\ &= C \sum_{m=0}^{n-k} \frac{(n-k)!}{m!(n-k-m)!} \lim_{T \to -\infty} (-T)^m \\ &\quad \times \left(\frac{d^{n-k} f}{dx^{n-k}}(x), h(x-T)(T-x)^{n-k-m} \right) = 0 , \end{aligned} \quad (2.31)$$

$$\lim_{T \to -\infty} C_{n,T}(N) = \lim_{T \to -\infty} B_{n,T}(N) . \quad (2.32)$$

Let us prove that the numbers $B_{n,T}(N)$ for $N > N(f)$ form a Cauchy sequence when $n \to \infty, T \to -\infty$. In view of the equality (2.32) it implies the existence of the limit (2.25). If $T_2 \leqq T_1$, then for the sufficiently large modulus of the negative number T_1 the nonnegative function $h_{T_2}(x) - h_{T_1}(x) \in D(\mathbf{R}_-)$ and by the positivity of the distribution $\frac{df}{dx}(x) \in D'(\mathbf{R}_-)$ the relation (2.28) implies the following estimate:

$$|B_{n,T_2}(N) - B_{n,T_1}(N)| \leqq (f(x), h(x-T_1) - h(x-T_2)) \sup_{x \geqq 0} |\chi_{n,N}(x)| . \quad (2.33)$$

In virtue of relation (2.6) the first multiplier in the right-hand side of the inequality (2.33) converges to zero when $T_1, T_2 \to -\infty$. Due to the positivity of the distribution $\frac{df}{dx}(x) \in D'(\mathbf{R}_-)$ we find from the relation (2.28),

$$|B_{n_2,T}(N) - B_{n_1,T}(N)| \leqq \lim_{\alpha\to+0}\left(\frac{df}{dx}(x), \theta(-x)\exp\{\alpha x^{-1}\}h_T(x)(1-x^{-1})^{-N(f)-1}\right) \times \sup_{x\geqq 0}(1+x)^{N(f)+1}|\chi_{n_2,N}(x) - \chi_{n_1,N}(x)| \,. \tag{2.34}$$

We denote the first factor on the right-hand side of the inequality (2.34) by $A(T)$. The numbers $A(T)$ form a Cauchy sequence when $T \to -\infty$ and consequently the numbers $A(T)$ are uniformly bounded on the negative semiaxis. The proof of this is exactly analogous to that of the inequality (2.33). If we can prove that the functions (2.29) converge in norm $|\cdot|_{N(f)+1}$ to the function

$$\chi_{\infty,N}(x) = ((N-1)!)^{-1}\int_0^{x^{-1}} t^{N-1}e^{-t}dt \tag{2.35}$$

when $n \to \infty$, then the inequalities (2.33) and (2.34) show that the numbers $B_{n,T}(N)$ form a Cauchy sequence when $n \to \infty, T \to -\infty$.

By the definition (2.29) we get

$$\chi_{n,N}(x) = \frac{(n+N)!}{n!(N-1)!}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!(n-k)!}\frac{(1+nx)^{-n-N+k}}{n+N-k} = \frac{(n+N)!}{n^N n!(N-1)!}\int_0^{(n^{-1}+x)^{-1}} t^{N-1}(1-tn^{-1})^n dt \,. \tag{2.36}$$

The inequality $t^{N-1}e^{-t} \leqq x^{-N-1}t^{-2}$ holds on the interval $[(n^{-1}+x)^{-1}, x^{-1}]$, where $x, n > 0$. It implies the following estimate:

$$\sup_{1\leqq x<\infty}(1+x)^{N(f)+1}\int_{(n^{-1}+x)^{-1}}^{x^{-1}} t^{N-1}e^{-t}dt \leqq n^{-1}2^{N(f)+1} \,. \tag{2.37}$$

Here we use the inequality $(1+x)^{N(f)+1}x^{-N-1} \leqq 2^{N(f)+1}$ valid on the semiaxis $[1,\infty)$ for $N \geqq N(f)$. The maximal value of the function $t^{N+1}e^{-t}$ on the positive semiaxis is equal to $((N+1)e^{-1})^{N+1}$. Hence by using the inequality $(1+x)^{N(f)+1} \leqq 2^{N(f)+1}$ for $0 \leqq x \leqq 1$ we obtain

$$\sup_{0\leqq x\leqq 1}(1+x)^{N(f)+1}\int_{(n^{-1}+x)^{-1}}^{x^{-1}} t^{N-1}e^{-t}dt \leqq n^{-1}2^{N(f)+1}((N+1)e^{-1})^{N+1} \,. \tag{2.38}$$

For any positive number x and for a natural number $n > 0$ we have the estimate $(n^{-1}+x)^{-1} \leqq n$. Then the equalities (2.35), (2.36) and the estimate (2.38) imply that

$$\sup_{0\leqq x\leqq 1}(1+x)^{N(f)+1}\left|\chi_{\infty,N}(x)-\frac{n^N n!}{(n+N)!}\chi_{n,N}(x)\right|$$
$$\leqq 2^{N(f)+1}((N-1)!)^{-1}\left[n^{-1}((N+1)e^{-1})^{N+1}+\int_0^\infty(1+s^2)^{-1}ds\right.$$
$$\left.\times\sup_{0\leqq t<\infty}(1+t^2)t^{N-1}|e^{-t}-(1-tn^{-1})^n_+|\right], \tag{2.39}$$

where the function x^n_+ is equal to x^n for $x\geqq 0$ and it is equal to zero otherwise. It follows from the equalities (2.35), (2.36) and the estimate (2.37) for $N>N(f)$ that

$$\sup_{1\leqq x<\infty}(1+x)^{N(f)+1}\left|\chi_{\infty,N}(x)-\frac{n^N n!}{(n+N)!}\chi_{n,N}(x)\right|$$
$$\leqq 2^{N(f)+1}((N-1)!)^{-1}\left[n^{-1}+N^{-1}\sup_{0\leqq t<\infty}|e^{-t}-(1-tn^{-1})^n_+|\right]. \tag{2.40}$$

For any natural number $n>1$ the function x^n_+ is differentiable everywhere. The maximal value of the function $\exp\{-t\}-(1-tn^{-1})^n_+$ on the positive semiaxis is at the point a_0 satisfying the equation $\exp\{-a_0\}=(1-a_0n^{-1})^{n-1}_+$. This equation implies

$$\sup_{0\leqq t<\infty}|e^{-t}-(1-tn^{-1})^n_+|=|e^{-a_0}-(1-a_0n^{-1})^n_+|=n^{-1}a_0e^{-a_0}\leqq(ne)^{-1}. \tag{2.41}$$

For natural numbers k and $n>1$ the maximal value of the function $t^k(e^{-t}-(1-tn^{-1})^n_+)$ on the positive semiaxis is at the point a_k satisfying the following equation:

$$ka_k^{k-1}(e^{-a_k}-(1-a_kn^{-1})^n_+)=a_k^k(e^{-a_k}-(1-a_kn^{-1})^{n-1}_+). \tag{2.42}$$

If $a_k\leqq k+1$ the inequality (2.41) implies the estimate

$$\sup_{0\leqq t<\infty}t^k|e^{-t}-(1-tn^{-1})^n_+|\leqq(k+1)^k(ne)^{-1}. \tag{2.43}$$

If $a_k\geqq k+1$ by using the identity (2.42) we get

$$\sup_{0\leqq t<\infty}t^k|e^{-t}-(1-tn^{-1})^n_+|=n^{-1}((1+n^{-1}k)a_k-k)^{-1}a_k^{k+2}e^{-a_k}$$
$$<n^{-1}a_k^{k+2}e^{-a_k}\leqq n^{-1}((k+2)e^{-1})^{k+2}. \tag{2.44}$$

In order to get the first inequality in (2.44) we used the inequality $(1+n^{-1}k)a_k-k>1$ valid for $a_k\geqq k+1$. If in addition one uses the fact that

$$\lim_{n\to\infty}\frac{n^N n!}{(n+N)!}=1,$$

it follows from the estimates (2.39), (2.40), (2.41), (2.43) and (2.44) that the functions (2.36) converge in norm $|\cdot|_{N(f)+1}$ to the function (2.35). Therefore Proposition 2.4 is proved.

By the definitions (2.19) and (2.20) the tempered distribution $L_c^{-1}[f](-x) \in S'(\mathbf{R})$ is positive and its support is contained in the positive semiaxis. Now Theorem 2 from [7, Chapter 2, Sect. 2.2] implies that a positive tempered distribution is given by a positive measure. This positive measure has tempered growth and its support is in the positive semiaxis.

Theorem 2.5. *For any absolutely monotonic distribution* $f(x) \in D'(\mathbf{R}_-)$ *the following representation*

$$(f(x), \phi(x)) = L_0^{-1}[f] \int_{-\infty}^{\infty} \phi(x)\,dx + \int_{-\infty}^{\infty} L_c^{-1}[f](-p)\,dp \int_{-\infty}^{\infty} e^{px}\phi(x)\,dx \qquad (2.45)$$

is valid for any function $\phi(x) \in D(\mathbf{R}_-)$. *Here the positive number* $L_0^{-1}[f]$ *is given by the relation* (2.6), *the positive measure* $L_c^{-1}[f](-p)$ *with tempered growth and support in the positive semiaxis is defined by the relations* (2.19), (2.20).

Proof. Let the distribution $f(x) \in D'(\mathbf{R}_-)$ be absolutely monotonic. Then in view of Lemma 2.3 the relation (2.14) holds for any function $\phi(x) \in D(\mathbf{R}_-)$ and for any integer $k = n + 1$, where n is a natural number. Note that

$$(-1)^{n+1}\left(\frac{d^{n+1}f}{dx^{n+1}}(x), h_T(x)\phi^{(-n-1)}(x)\right)$$
$$= (n!)^{-1}\left((-x)^n \frac{d^{n+1}f}{dx^{n+1}}(x), h_T(x)L_n[\phi](-nx^{-1})\right) \qquad (2.46)$$

with

$$L_n[\phi](x) = \int_{-nx^{-1}}^{0} (1 + n^{-1}xy)^n \phi(y)\,dy\,. \qquad (2.47)$$

Here we have taken into account that the function $\phi(x)$ has support in the negative semiaxis.

Due to the inequality (2.23) for $N = N(f) + 1$ the tempered distribution $L_c^{-1}[f]$ $(-x; n, T)$, defined by the equality (2.19), may be continued to a linear continuous functional $L_c^{-1}[f](-x; n, T)^c$ on the Banach $H_{N(f)+1}$. Let us prove that the function (2.47) converges as $n \to \infty$ in norm $|\cdot|_{N(f)+1}$ to the Laplace transform of the function $\phi(-x)$. It is straightforward to show that

$$\left|\int_{-\infty}^{0} e^{xy}\phi(y)\,dy - L_n[\phi](x)\right|_{N(f)+1}$$
$$\leqq \int_{-\infty}^{0} |\phi(y)|\,dy \sup_{0 \leqq x < \infty} (1 + x)^{N(f)+1} |e^{xy} - (1 + n^{-1}xy)_+^n|\,. \qquad (2.48)$$

The right-hand side of the inequality (2.48) is majorized by the sum

$$\sum_{k=0}^{N(f)+1} \frac{(N(f)+1)!}{k!(N(f)+1-k)!} \int_{-\infty}^{0} |y|^{-k}|\phi(y)|\,dy \sup_{0 \leqq t < \infty} t^k |e^t - (1 + n^{-1}t)_+^n|\,. \qquad (2.49)$$

Due to the estimates (2.43), (2.44) the sum (2.49) converges to zero as $n \to \infty$. Thus we have proved that the right-hand side of the equality (2.46) converges to

$$(L_c^{-1}[f](-x), \int_{-\infty}^{0} e^{xy}\phi(y)\,dy)\,. \tag{2.50}$$

Now the relation (2.14) implies the equality (2.45), concluding the proof of Theorem 2.5.

For O an open set in $\mathbf{R}^n, S(O)$ denotes the subspace of $S(\mathbf{R}^n)$ of functions with support in the closure $\overline{O}$, given the induced topology. For example $S(\mathbf{R}_-)$ denotes the subspace of $S(\mathbf{R})$ of functions with support in the negative semiaxis, given the induced topology. Similarly $S(\mathbf{R}_+)$ denotes the subspace of $S(\mathbf{R})$ of functions with support in the positive semiaxis, given the induced topology. It follows from Theorem 2.5 that every absolutely monotonic distribution $f(x) \in D'(\mathbf{R}_-)$ may be extended to a tempered distribution in $S'(\mathbf{R}_-)$.

We recall that a tempered distribution $g(t) \in S'(\mathbf{R}_-)$ is called absolutely monotonic if it satisfies the conditions (1.3) for all $m = 0, 1, \ldots$.

Corollary 2.6. *The tempered distribution $f(x) \in S'(\mathbf{R}_+)$ is the Laplace transform of a tempered distribution with support in the positive semiaxis if and only if there exists a natural number k such that*

$$(-x)^{-k} f(-x) = g_1(x) - g_2(x)\,, \tag{2.51}$$

where the tempered distributions $g_j(x) \in S'(\mathbf{R}_-)$, $j = 1,2$, are absolutely monotonic.

Proof. Due to Theorem from [6, Sect. 3.8] a tempered distribution with support in the positive semiaxis may be written as

$$g(x) = \sum_{m=0}^{k-1} \frac{d^m}{dx^m} \mu_m(x)\,, \tag{2.52}$$

where $\mu_m(x)$ are measures with tempered growth and with supports in the positive semiaxis.

It is well known that $(m!)^{-1}(\frac{d}{dx})^{m+1} x_+^m = \delta(x)$. Hence the relation (2.52) implies

$$g(x) = \frac{d^k}{dx^k}\left[\sum_{m=0}^{k-1}((k-m-1)!)^{-1} x_+^{k-m-1} * \mu_m(x)\right]\,, \tag{2.53}$$

where $*$ denotes the convolution of two tempered distributions with supports in the positive semiaxis. If we represent the measure in the right-hand side of the equality (2.53) as the difference of two positive measures with tempered growth and with supports in the positive semiaxis we get

$$g(x) = \frac{d^k}{dx^k}[\nu_1(x) - \nu_2(x)]\,. \tag{2.54}$$

Taking the Laplace transform of the equality (2.54) and dividing by x^k we obtain the equality (2.51), where x is replaced by $-x$.

It is straightforward to show that the equality (2.51) and Theorem 2.5 imply that the tempered distribution $f(x)$ is the Laplace transform of a tempered distribution with support in the positive semiaxis.

Let x denote a point in $\mathbf{R}^4$ with coordinates $(x^0,x^1,x^2,x^3) = (x^0,\mathbf{x})$. A point in $\mathbf{R}^{4n}$ will be written as $\underline{x} = (x_1,\dots,x_n)$, $x_\iota \in \mathbf{R}^4$. We will use the following open set $\mathbf{R}^{4n}_+ = \{\underline{x} \in \mathbf{R}^{4n} | x^0_j > 0,\ j = 1,\dots,n\}$.

Theorem 2.7. *Let the tempered distribution $f(x) \in S'(\mathbf{R}^{4n}_+)$ be the Laplace transform with respect to the time variables of a tempered distribution with support in the closure $\overline{\mathbf{R}}^{4n}_+$. Then there is the natural number K such that for any integers $k > K$, $1 \leqq j_1 < \cdots < j_l \leqq n$, $1 \leqq l \leqq n$ and for all test functions $\phi_j(x) \in S(\mathbf{R}^4)$, $j \in \{j_1,\dots,j_l\}$; $\phi_i(x) \in S(\mathbf{R}^4_+)$, $1 \leqq i \leqq n$, $i \neq j_1,\dots,j_l$, the following limit exists*:

$$\int d^{4n}x L_c^{-1}\left[\left(\prod_{m=1}^{l} x^0_{j_m}\right)^{-k} f\right]_{\underline{x}^0_{\underline{l}}}(\underline{x}) \prod_{i=1}^{n} \phi_i(x_i)$$

$$= \lim_{n_1,\dots,n_l \to \infty, n_\iota \in \mathbf{Z}} \ \lim_{T_1,\dots,T_l \to -\infty, T_\iota \in \mathbf{R}} \int d^{4n}x \left(\prod_{m=1}^{l} x^0_{j_m}\right)^{-k} f(\underline{x})$$

$$\times \left(\prod_{i=1, i \neq j_1,\dots,j_l}^{n} \phi_i(x_i)\right) \prod_{m=1}^{l} L_c^{-1}[\phi_{j_m}]_{x^0_{j_m}}(x_{j_m}; n_m, T_m) \tag{2.55}$$

with $x^0_{\underline{l}} = (x^0_{j_1},\dots,x^0_{j_m})$ *and*

$$L_c^{-1}[\phi]_{x^0}(x; n, T) = (n!)^{-1} \left(\frac{\partial}{\partial x^0}\right)^{n+1} ((x^0)^n \theta(x^0) h_T(-x^0) \phi(n(x^0)^{-1},\mathbf{x}))\,, \tag{2.56}$$

where the function $h_T(x^0)$ is given by the equality (2.13).

The limit (2.55) *defines the inversion formula for the Laplace transformation*:

$$\left(f(\underline{x}), \prod_{i=1}^{n} \phi_i(x_i)\right) = \int d^{4n}x \left(\left(\prod_{m=1}^{l} \frac{\partial}{\partial x^0_{j_m}}\right)^{k} L_c^{-1}\left[\left(\prod_{m=1}^{l} x^0_{j_m}\right)^{-k} f\right]_{x^0_{\underline{l}}}(\underline{x})\right)$$

$$\times \int_0^\infty dy^0_{j_1} \dots \int_0^\infty dy^0_{j_m} \exp\left\{-\sum_{m=1}^{l} x^0_{j_m} y^0_{j_m}\right\}$$

$$\times \left(\prod_{m=1}^{l} \phi_{j_m}(y^0_{j_m}, \mathbf{x}_{j_m})\right) \prod_{\iota=1, \iota \neq j_1,\dots,j_l}^{n} \phi_i(x_\iota) \tag{2.57}$$

for all functions $\phi_i(x) \in S(\mathbf{R}^4_+)$, $i = 1,\dots,n$, and for any integer $k > K$.

Proof. Theorem from [6, Sect. 3.8] implies that a tempered distribution with support in the closure $\overline{\mathbf{R}}^{4n}_+$ has the following form:

$$g(\underline{y}) = \sum_{|\underline{m}| \leqq K-1} \left(\frac{\partial}{\partial \underline{y}}\right)^{\underline{m}} \mu_{\underline{m}}(\underline{y})\,, \tag{2.58}$$

where we use the standard multiindex notations and $\mu_{\underline{m}}(\underline{y})$ are measures with tempered growth and supports in $\overline{\mathbf{R}}^{4n}_+$. In analogy to the proof of Corollary 2.6 the relation (2.58) may be written as

$$g(\underline{y}) = \left(\prod_{s=1}^{l} \frac{\partial}{\partial y^0_{j_s}}\right)^K \sum_{|\underline{m}| \leqq K-1,\, m^0_{j_1} = \cdots = m^0_{j_l} = 0} \left(\frac{\partial}{\partial \underline{y}}\right)^{\underline{m}} v_{\underline{m}}(\underline{y}) , \tag{2.59}$$

where $v_{\underline{m}}(\underline{y})$ the measures with tempered growth and supports in $\overline{\mathbf{R}}^{4n}_{+}$. Let us integrate the distribution (2.59) with a test function which is the product of the functions: $\exp\{-x^0_j y^0_j\}\phi_j(\mathbf{y}_j)$, $\phi_j(\mathbf{y}) \in S(\mathbf{R}^3)$, $j \in \{j_1,\ldots,j_l\}$, and $\exp\{-x^0_i y^0_i\}$ $\phi_i(x^0_i, \mathbf{y}_i)$, $\phi_i(x) \in S(\mathbf{R}^4_+)$, $1 \leqq i \leqq n$, $i \neq j_1,\ldots,j_l$. Let us divide the resulting integral by $(x^0_{j_1}\ldots x^0_{j_l})^K$. If the distribution $f(\underline{x})$ is the Laplace transform of the tempered distribution (2.59) with respect to time variables we get

$$\left(\prod_{m=1}^{l} x^0_{j_m}\right)^{-K} \int \left(\prod_{i=1, i \neq j_1,\ldots,j_l}^{n} dx^0_i\right) d^{3n}\mathbf{x} f(\underline{x}) \left(\prod_{m=1}^{l} \phi_{j_m}(\mathbf{x}_{j_m})\right) \prod_{i=1, i \neq j_1,\ldots,j_l}^{n} \phi_i(x_i)$$
$$= \int dv(y_1,\ldots,y_l) \exp\left\{-\sum_{m=1}^{l} x^0_{j_m} y_m\right\} , \tag{2.60}$$

where the measure $v(y_1,\ldots,y_l)$ with support in $(\overline{\mathbf{R}}_{+})^{\times l}$ is defined by

$$\int dv(y_1,\ldots,y_l)\psi(y_1,\ldots,y_l)$$
$$= \sum_{|\underline{m}| \leqq K-1,\, m^0_{j_1} = \cdots = m^0_{j_l} = 0} \int dv_{\underline{m}}(\underline{y})\psi(y^0_{j_1},\ldots,y^0_{j_l}) \left(-\frac{\partial}{\partial \underline{y}}\right)^{\underline{m}}$$
$$\times \left[\left(\prod_{m=1}^{l} \phi_{j_m}(\mathbf{y}_{j_m})\right) \left(\prod_{i=1, i \neq j_1,\ldots,j_l}^{n} \int dx^0_i \exp\{-x^0_i y^0_i\}\phi_i(x^0_i, \mathbf{y}_i)\right)\right] . \tag{2.61}$$

The measure $v(y_1,\ldots,y_l)$ is the difference $v_1(y_1,\ldots,y_l) - v_2(y_1,\ldots,y_l)$ of two positive measures with tempered growth and supports in $(\overline{\mathbf{R}}_{+})^{\times l}$. Then the left-hand side of the equality (2.60) is equal to

$$\int [dv_1(y_1,\ldots,y_l) - dv_2(y_1,\ldots,y_l)] \exp\left\{-\sum_{m=1}^{l} x^0_{j_m} y_m\right\} . \tag{2.62}$$

The expression (2.62), considered as the function on $(\overline{\mathbf{R}}_{-})^{\times l}$, is the difference $f_1(x^0_{j_1},\ldots,x^0_{j_l}) - f_2(x^0_{j_1},\ldots,x^0_{j_l})$ of two absolutely monotonic with respect to each variable distributions from $S'((\mathbf{R}_{-})^{\times l})$.

The arguments of the proof of Proposition 2.4 lead to the existence of the limit (2.55) for any test function which is the product of the functions: $\psi_j(x^0_j)\phi_j(\mathbf{x}_j)$, where $\psi_j(x^0) \in S(\mathbf{R})$, $\phi_j(\mathbf{x}) \in S(\mathbf{R}^3)$, $j = j_1,\ldots,j_l$, and $\phi_i(x_i) \in S(\mathbf{R}^4_+)$, $1 \leqq i \leqq n$, $i \neq j_1,\ldots,j_l$. Since the weak convergence in S' implies the convergence in the topology of the space S' (see [6, Sect. 3.7]) the limit is a multilinear functional, continuous in each variable. Now the nuclear theorem [7, Chapter 1, Sect. 1, Theorem 6] implies the existence of the distribution $L_c^{-1}[(\prod_{m=1}^{l} x^0_{j_m})^{-K} f]_{\underline{x}^0_l}(\underline{x})$ and the equality (2.55) for any test function which is the product of the functions $\phi_j(x_j) \in S(\mathbf{R}^4)$, $j = j_1,\ldots,j_l$, and $\phi_i(x_i) \in S(\mathbf{R}^4_+)$, $1 \leqq i \leqq n$, $i \neq j_1,\ldots,j_l$.

We use this set of the functions in order to avoid cumbersome notations. For the special case $l = n$ the notations are simple: $L_c^{-1}[(\prod_{m=1}^{l} x_m^0)^{-K} f]_{\underline{x}^0}(\underline{x}) \in S'(\mathbf{R}^{4n})$ and its support is in $\overline{\mathbf{R}}_+^{4n}$. The equality (2.55) is valid in this case for any test function $\phi(\underline{x}) \in S(\mathbf{R}^{4n})$.

For the absolutely monotonic tempered distribution $g(x) \in S'(\mathbf{R}_-)$ the tempered distribution $(-x)^{-m} g(x) \in S'(\mathbf{R}_-)$ is absolutely monotonic for any natural number m. It is therefore possible to divide the expression (2.62) by $(x_{j_1}^0 \ldots x_{j_l}^0)^{k-K}$ and to prove the above results for any integer $k > K$. For the integer $k > K$ Lemma 2.2 implies that the limits of type (2.6) are equal to zero. We now apply the arguments of the proof of Theorem 2.5 to obtain the equality (2.57), concluding the proof of the theorem.

3. Exponentially Convex Distributions

Motivated by the discussion of S. Bernstein [4] for functions we call a tempered distribution $f(x) \in S'(\mathbf{R}_+)$ exponentially convex if it satisfies the positivity condition (1.1) for any function $\phi(x) \in S(\mathbf{R}_+)$.

Proposition 3.1. *For any exponentially convex tempered distribution $f(x) \in S'(\mathbf{R}_+)$ the tempered distribution $f(-x) \in S'(\mathbf{R}_-)$ is absolutely monotonic.*

Proof. Let us introduce the convolution function

$$\overline{\phi} * \phi(x) = \int_{-\infty}^{\infty} \overline{\phi(x-y)} \phi(y)\, dy\,. \tag{3.1}$$

For $\phi(x) \in S(\mathbf{R}_+)$ the function $\overline{\phi} * \phi(x) \in S(\mathbf{R}_+)$. The condition (1.1) may be rewritten as $(f(x), \overline{\phi} * \phi(x)) \geqq 0$. Let $F[\phi](p)$ be the Fourier transform of the function $\phi(x) \in S(\mathbf{R}_+)$. The definition (3.1) implies the equality $F[\overline{\phi} * \phi](p) = \overline{F[\phi](-p)} F[\phi](p)$.

By $FS(\mathbf{R}_+)$ we denote the space of all functions $\psi(z)$ analytic in the open upper half plane and infinitely differentiable in the closed upper half plane such that the seminorms of the form

$$\sup_{\mathrm{Im}\, z \geqq 0} (1 + |z|)^n \left| \frac{d^m \psi}{dz^m}(z) \right| \tag{3.2}$$

are finite for all positive integers m and n. The topology of $FS(\mathbf{R}_+)$ is given by the set of seminorms (3.2).

Let us prove that the Fourier transformation defines an isomorphism between two topological spaces: $S(\mathbf{R}_+)$ and $FS(\mathbf{R}_+)$. The Fourier transform $F[\phi](x)$ of a function $\phi(x) \in S(\mathbf{R}_+)$ has an analytical continuation $F[\phi](z)$ into the open upper half plane. The function $F[\phi](z)$ is infinitely differentiable in the closed upper half plane. The inequality $y^k \exp\{-yt\} \leqq C(k) t^{-k}$, valid for $t > 0$, $y \geqq 0$ implies the following estimate:

$$\sup_{x \in \mathbf{R},\, y \geqq 0} \left| x^{k_1} y^{k_2} \frac{d^m F[\phi]}{dz^m}(x + iy) \right| \leqq C \sup_{t > 0,\, 0 \leqq l \leqq k_1} (1 + t^2) t^{-k_2 - l} \left| \frac{d^{k_1 - l}}{dt^{k_1 - l}} (t^m \phi(t)) \right| . \tag{3.3}$$

Therefore the Fourier transformation defines a continuous mapping of the space $S(\mathbf{R}_+)$ into the space $FS(\mathbf{R}_+)$. For a function $\psi(z) \in FS(\mathbf{R}_+)$ its restriction $\psi(x)$ to the real axis belongs to the Schwartz space $S(\mathbf{R})$. A straightforward application of Cauchy's theorem shows that the inverse Fourier transform $F^{-1}[\psi](p)$ of the function $\psi(x)$ may be rewritten for any $y > 0$ as

$$F^{-1}[\psi](p) = (2\pi)^{-1} e^{py} \int_{-\infty}^{\infty} \exp\{-ipx\}\psi(x+iy)\,dx\,. \tag{3.4}$$

Since any seminorm (3.2) is finite we get $F^{-1}[\psi](p) = 0$ for $p < 0$ by letting y in (3.4) tend to infinity. Hence $F^{-1}[\psi](p) \in S(\mathbf{R}_+)$. The inverse Fourier transformation is a topological isomorphism of the Schwartz space $S(\mathbf{R})$. Any seminorm of the Schwartz space $S(\mathbf{R})$ on the subspace $FS(\mathbf{R}_+)$ is majorized by a corresponding seminorm of the type (3.2). Thus the inverse Fourier transformation is a mapping of the space $FS(\mathbf{R}_+)$ into the space $S(\mathbf{R}_+)$.

For any natural number k and for any number $\alpha > 0$ we define the function

$$F[\chi_\alpha](z) = ((\alpha + z/i)^{1/2})^k \exp\{-\alpha(1+z/i)^{1/2}\} \tag{3.5}$$

with

$$(a+z/i)^{1/2} = (x^2 + (a+y)^2)^{1/4} \exp\{-i/2 \arctan[x(a+y)^{-1}]\}\,,$$

which is holomorphic in the open upper half plane. Due to the estimate $\mathrm{Re}(1 + z/i)^{1/2} \geqq (|z|/2)^{1/2}$, valid in the closed upper half plane, the function (3.5) belongs to the space $FS(\mathbf{R}_+)$. Hence its inverse Fourier transform $\chi_\alpha(x)$ belongs to the space $S(\mathbf{R}_+)$, as does the function $\chi_\alpha(x-t)$ for any positive number t. Therefore the convolution function $\overline{\chi}_\alpha * \chi_\alpha(x-2t) = \overline{\chi_\alpha(\,\cdot\, - t)} * \chi_\alpha(\,\cdot\, - t)(x)$ belongs to the space $S(\mathbf{R}_+)$ for any $t \geqq 0$. Now the positivity condition (1.1) for the exponentially convex tempered distribution $f(x) \in S'(\mathbf{R}_+)$ implies the inequality $(f(x), \overline{\chi}_\alpha * \chi_\alpha(x-2t)) \geqq 0$ for any $t \geqq 0$. Integrating this inequality with a nonnegative function $1/2\phi(2t) \in S(\mathbf{R}_+)$ we obtain

$$(f(x), (\overline{\chi}_\alpha * \chi_\alpha) * \phi(x)) \geqq 0\,. \tag{3.6}$$

In view of the definition (3.5) we get $\overline{F[\chi_\alpha](-x)} = F[\chi_\alpha](x)$. Now it is easy to show that

$$(\overline{\chi}_\alpha * \chi_\alpha) * \phi(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} dp e^{-ipx} F[\phi](p)(F[\chi_\alpha](p))^2\,. \tag{3.7}$$

When $\alpha \to +0$ the functions $F[\phi](z)(F[\chi_\alpha](z))^2$ converge to the function $(z/i)^k F[\phi](z)$ in the topology of the space $FS(\mathbf{R}_+)$. Then the left-hand side of the equality (3.7) converges to the function $\frac{d^k\phi}{dx^k}(x)$ in the topology of the space $S(\mathbf{R}_+)$ when $\alpha \to +0$. In the limit $\alpha \to +0$ inequality (3.6) therefore gives

$$(-1)^k \left(\frac{d^k f}{dx^k}(x), \phi(x)\right) \geqq 0\,. \tag{3.8}$$

It follows from the inequalities (3.8) for arbitrary natural numbers k and for arbitrary nonnegative functions $\phi(x) \in S(\mathbf{R}_+)$ that the tempered distribution $f(-x) \in S'(\mathbf{R}_-)$ is absolutely monotonic.

A distribution $f(x) \in S'(\mathbf{R}^4_+)$ is said to satisfy the Osterwalder–Schrader positivity condition, if for any function $\phi(x) \in S(\mathbf{R}^4_+)$,

$$\int d^4xd^4yf(x^0+y^0,\mathbf{x}-\mathbf{y})\overline{\phi(x)}\phi(y) \geqq 0\,. \tag{3.9}$$

Therefore the Osterwalder–Schrader positivity condition (3.9) is the condition of exponential convexity with respect to the time variable and it is the condition of positive definiteness with respect to the space variables. By using the proof of Proposition 3.1 and the proof of Theorem 1 from [7, Chapter 2, Sect. 3.1] it is possible to show that the Fourier transform $F_{\mathbf{x}}[f](x^0,\mathbf{x})$ with respect to the space variables of the distribution $f(x) \in S'(\mathbf{R}^4_+)$, satisfying the condition (3.9), satisfies the following condition:

$$\left(-\frac{\partial}{\partial x^0}\right)^n F_{\mathbf{x}}[f](x^0,\mathbf{x}) \geqq 0 \tag{3.10}$$

for any natural number $n = 0, 1, \ldots$.

Lemma 3.2. *Let the tempered distribution $f(x) \in S'(\mathbf{R}^4_+)$ satisfy the Osterwalder–Schrader positivity condition* (3.9). *Then for any function $\phi(x) \in S(\mathbf{R}^4_-)$,*

$$\lim_{t\to-\infty}(f(x),\phi(-x^0-t,\mathbf{x})) = \int_{\mathbf{R}^3} L_0^{-1}[f]_{x^0}(\mathbf{x})\,d^3\mathbf{x} \int_{-\infty}^{\infty} \phi(x^0,\mathbf{x})\,dx^0\,, \tag{3.11}$$

where the tempered distribution $L_0^{-1}[f]_{x^0}(\mathbf{x}) \in S'(\mathbf{R}^3)$ is positively definite, and for any natural number $k = 1,2,\ldots$,

$$\lim_{t\to-\infty} t^k\left(\left(\frac{\partial}{\partial x^0}\right)^k f(x),\phi(-x^0-t,\mathbf{x})\right) = 0\,. \tag{3.12}$$

If the tempered distribution $f(x) \in S'(\mathbf{R}^4_+)$ satisfies the Osterwalder–Schrader positivity condition (3.9), *then the tempered distribution $(x^0)^{-1}f(x) \in S'(\mathbf{R}^4_+)$ satisfies the inequalities* (3.10) *and the limits* (3.11), (3.12) *for this distribution are equal to zero.*

Proof. Let the Fourier transform $F_{\mathbf{x}}[\phi](x^0,\mathbf{x})$ with respect to the space variables of a function $\phi(x) \in S(\mathbf{R}^4_-)$ be a positive function. Then by the straightforward application of the inequalities (3.10) and by the proof of Lemma 2.2 we can prove the relations (3.12) and the existence of the limit (3.11).

Due to Theorem 2 from [7, Chapter 2, Sect. 2.2] the inequalities (3.10) imply the following estimates for any function $\phi(x) \in S(\mathbf{R}^4_-)$:

$$\left|\left(\left(\frac{\partial}{\partial x^0}\right)^n f(x),\phi(-x^0,\mathbf{x})\right)\right| \leqq C \sup_{x^0\leqq 0,\mathbf{x}\in\mathbf{R}^3} (1+|x|^2)^p|F_{\mathbf{x}}[\phi](x^0,\mathbf{x})|\,, \tag{3.13}$$

where the numbers C and p depend on the natural number $n = 0,1,2,\ldots$. Let $\alpha(\mathbf{x})$ be an infinitely differentiable nonnegative function with compact support and let it be equal to one into some neighborhoods of zero. For a function $\phi(x) \in S(\mathbf{R}^4_-)$

we define $M(\phi) = \sup |F_{\mathbf{x}}[\phi](x)|$. The difference of the two following nonnegative functions from $S(\mathbf{R}^4_-)$:

$$F_{\mathbf{x}}[\phi_{1m}](x) = \exp\{m^{-1}(x^0 + (x^0)^{-1})\}\theta(-x^0)\alpha(m^{-1}\mathbf{x})M(\phi),$$

$$F_{\mathbf{x}}[\phi_{2m}](x) = \exp\{m^{-1}(x^0 + (x^0)^{-1})\} \times \theta(-x^0)\alpha(m^{-1}\mathbf{x})(M(\phi) - F_{\mathbf{x}}[\phi](x)) \tag{3.14}$$

converges as $m \to \infty$ to the function $F_{\mathbf{x}}[\phi](x) \in S(\mathbf{R}^4_-)$ in the norm (3.13). Due to the inequality (3.13) this now implies the relations (3.12) and the existence of the limit (3.11) for any function $\phi(x) \in S(\mathbf{R}^4_-)$.

Let $h(x^0) \in S(\mathbf{R}_-)$ be nonnegative with integral equal to one. Any function $\phi(x) \in S(\mathbf{R}^4_-)$ may be represented as

$$\phi(x) = h(x^0) \int_{-\infty}^{\infty} \phi(y^0, \mathbf{x})\, dy^0 + \frac{\partial \psi}{\partial x^0}(x), \tag{3.15}$$

where the function $\psi(x) \in S(\mathbf{R}^4_-)$. The existence of the limit (3.11) and the equalities (3.15) and (3.12) for $k = 1$ provide the equality (3.11) for any function $\phi(x) \in S(\mathbf{R}^4_-)$. By virtue of the inequality (3.10) for $n = 0$ the tempered distribution $L_0^{-1}[f]_{x^0}(\mathbf{x})$ is positive definite. The proof of the last part of Lemma 3.2 follows the arguments of Lemma 2.2.

In analogy to (2.12) for any function $\phi(x) \in S(\mathbf{R}^4_-)$ and for any integer $k = 1, 2, \ldots$, we define

$$\phi_{x^0}^{(-k)}(x) = -((k-1)!)^{-1} \int_{x^0}^{\infty} (x^0 - y^0)^{k-1} \phi(y^0, \mathbf{x})\, dy^0 \tag{3.16}$$

and $\phi_{x^0}^{(0)}(x) = \phi(x)$. The infinitely differentiable function (3.16) equals zero for $x^0 > 0$. It is easy to see that $(\frac{\partial}{\partial x^0})^l \phi_{x^0}^{(-k)}(x) = \phi_{x^0}^{(l-k)}(x)$ for $l \leqq k$. Let the nonnegative function $h_T(x^0)$ be given by the relation (2.13) for some nonnegative function $h(x^0) \in D(\mathbf{R}_-)$ having the integral equal one.

Lemma 3.3. *Let the tempered distribution $f(x) \in S'(\mathbf{R}^4_+)$ satisfy the Osterwalder–Schrader positivity condition* (3.9). *Then for any function $\phi(x) \in S(\mathbf{R}^4_-)$ and for any integer $k = 1, 2, \ldots,$*

$$\lim_{T \to -\infty} \left(\left(\frac{\partial}{\partial x^0} \right)^k f(x), h_T(-x^0) \phi_{x^0}^{(-k)}(-x^0, \mathbf{x}) \right)$$
$$= (f(x), \phi(-x^0, \mathbf{x})) - \int_{\mathbf{R}^3} L_0^{-1}[f]_{x^0}(\mathbf{x}) \int_{-\infty}^{\infty} \phi(x^0, \mathbf{x})\, dx^0, \tag{3.17}$$

where the positive definite tempered distribution $L_0^{-1}[f]_{x^0}(\mathbf{x}) \in S'(\mathbf{R}^3)$ is defined by the equality (3.11).

The proof of Lemma 3.3 is exactly analogous to that of Lemma 2.3 and can be omitted.

For a tempered distribution $f(x) \in S'(\mathbf{R}^4_+)$ satisfying the Osterwalder–Schrader positivity condition (3.9) we define a functional on the space $S(\mathbf{R}^4)$ by the following relation:

$$(L_c^{-1}[f]_{x^0}(x;n,T), \phi(x)) = (f(x), L_c^{-1}[\phi]_{x^0}(x;n,T))$$

$$= (n!)^{-1}\left((x^0)^n\left(-\frac{\partial}{\partial x^0}\right)^{n+1} f(x^0,\mathbf{x}), \theta(x^0)h_T(-x^0)\phi(n(x^0)^{-1},\mathbf{x})\right), \quad (3.18)$$

where n is a positive integer and the function $h_T(x^0)$ is given by the equality (2.13). It is easy to prove that the tempered distribution $F_{\mathbf{x}}[L_c^{-1}[f]_{x^0}(\,\cdot\,;n,T)](x)$ is positive and its support is in the closure of $\mathbf{R}^4_+$.

Proposition 3.4. *Let the tempered distribution* $f(x) \in S'(\mathbf{R}^4_+)$ *satisfy the Osterwalder–Schrader positivity condition* (3.9). *Then in the topological space* $S'(\mathbf{R}^4)$ *the following limit exists*:

$$\lim_{n\to\infty}\lim_{T\to-\infty} L_c^{-1}[f]_{x^0}(x;n,T) = L_c^{-1}[f]_{x^0}(x)\,. \quad (3.19)$$

The tempered distribution $F_{\mathbf{x}}[L_c^{-1}[f]_{x^0}(\,\cdot\,)](x) \in S'(\mathbf{R}^4)$ *is positive and its support is in the closure of* $\mathbf{R}^4_+$.

Proof. The number $N(f)$ is defined by means of the estimate similar to the estimate (2.3). A result analogous of Lemma 2.1 is valid for the tempered distribution $f(x) \in S'(\mathbf{R}^4_+)$. By using the inequalities (3.10) and Theorem 2 from [7, Chapter 2, Sect. 2.2] it is possible to prove the estimate similar to (2.23), namely

$$|(L_c^{-1}[f]_{x^0}(x;n,T), \phi(x))|$$

$$\leqq C_{n,T}(N(f)+1)\sup_{x^0\geqq 0,\mathbf{x}\in\mathbf{R}^3}(1+x^0)^{N(f)+1}(1+|\mathbf{x}|)^p|F_{\mathbf{x}}[\phi](x)|\,. \quad (3.20)$$

The arguments analogous to those of the proof of Proposition 2.4 allow us to replace the constant $C_{n,T}(N(f)+1)$ by a constant independent of the numbers n and T.

The inequalities (3.10) and Proposition 2.4 imply that the limit (3.19) exists on every test function from $S(\mathbf{R}^4)$ whose Fourier transform with respect to the space variables is a nonnegative function. Let $\alpha(x) \in D(\mathbf{R}^4)$ be nonnegative and let it be equal to one into some neighborhoods of zero. For a function $\phi \in S(\mathbf{R}^4)$ we define $M(\phi) = \sup|F_{\mathbf{x}}[\phi](x)|$. The difference of the two nonnegative functions from $S(\mathbf{R}^4)$

$$F_{\mathbf{x}}[\phi_{1m}](x) = \alpha(m^{-1}x)M(\phi)\,,$$

$$F_{\mathbf{x}}[\phi_{2m}](x) = \alpha(m^{-1}x)(M(\phi) - F_{\mathbf{x}}[\phi](x)) \quad (3.21)$$

converges as $m \to \infty$ to the function $F_{\mathbf{x}}[\phi](x) \in S(\mathbf{R}^4)$ in the norm (3.20). Since the constant $C_{n,T}(N(f)+1)$ in the estimate (3.20) may be replaced by the constant independent of the numbers n and T this implies the existence of the limit (3.19) for every test function from $S(\mathbf{R}^4)$. Therefore due to [6, Sect. 3.7] the sequence of tempered distribution $L_c^{-1}[f]_{x^0}(x;n,T)$ converges in topology of the space $S'(\mathbf{R}^4)$. It follows now from the inequalities (3.10) and the definition (3.18) that the tempered distribution $F_{\mathbf{x}}[L_c^{-1}[f]_{x^0}(\,\cdot\,)](x) \in S'(\mathbf{R}^4)$ is positive and its support is in the closure of $\mathbf{R}^4_+$.

A straightforward application of the arguments of the proof of Theorem 2.5 leads to the following theorem.

Theorem 3.5. *Let the tempered distribution* $f(x) \in S'(\mathbf{R}^4_+)$ *satisfy the Osterwalder–Schrader positivity condition* (3.9). *Then the following representation*:

$$(f(x), \phi(x)) = \int_{\mathbf{R}^3} d^3\mathbf{x} L_0^{-1}[f]_{x^0}(\mathbf{x}) \int_{-\infty}^{\infty} dp^0 \phi(p^0, \mathbf{x}))$$

$$+ \int_{\mathbf{R}^4} d^4x L_0^{-1}[f]_{x^0}(x) \int_{-\infty}^{\infty} dp^0 \exp\{-x^0 p^0\} \phi(p^0, \mathbf{x})) \tag{3.22}$$

is valid for any function $\phi(x) \in S(\mathbf{R}^4_+)$. *Here the positive definite tempered distribution* $L_0^{-1}[f]_{x^0}(\mathbf{x}) \in S'(\mathbf{R}^3)$ *is defined by the equality* (3.11) *and the tempered distribution* $L_c^{-1}[f]_{x^0}(x) \in S'(\mathbf{R}^4)$ *is defined by the relations* (3.18), (3.19). *The distribution* $F_{\mathbf{x}}[L_c^{-1}[f]_{x^0}(\,\cdot\,)](x) \in S'(\mathbf{R}^4)$ *is the positive measure with tempered growth and support in the closure of* $\mathbf{R}^4_+$.

4. Revised Osterwalder–Schrader Theorem

We deal with the theory of one Hermitian scalar field. By using the results below and Chapter 6 of the paper [1] it is possible to formulate the extended Osterwalder–Schrader axioms and to prove the revised Osterwalder–Schrader theorem for theories containing spinor fields.

We use some notation from the papers [1] and [2]. We define the following open sets in $\mathbf{R}^{4n}$: $\mathbf{R}^{4n}_< = \{\underline{x} \in \mathbf{R}^{4n} | x^0_{j+1} > x^0_j,\ j = 1, \dots, n-1\}$ and $\mathbf{R}^{4n}_0 = \{\underline{x} \in \mathbf{R}^{4n} | x_i \neq x_j,\ 1 \leqq i < j \leqq n\}$. For O an open set in $\mathbf{R}^{4n}$, the space $S(O)$ is defined above. On $S(\mathbf{R}^{4n})$ we define two involutions

$$f^*(x_1, \dots, x_n) = \bar{f}(x_n, \dots, x_1)\,,$$

$$\theta f(x_1, \dots, x_n) = f(\theta x_1, \dots, \theta x_n)\,, \tag{4.1}$$

where $\theta x = (-x^0, \mathbf{x})$ and $\bar{f}$ means complex conjugation. The space $S(\mathbf{R}^{4n}_<)$ is invariant under the involution $f \to \theta f^*$. Let $f \in S(\mathbf{R}^{4n})$, $R \in SO_4$ be an element in the rotation group, $a \in \mathbf{R}^4$ and $\pi \in P_n$ be an element in the group of all permutations of n objects (the letter S_n will be used elsewhere). Then we define $f_{(a,R)}$ and f^π by $f_{(a,R)}(x_1, \dots, x_n) = f(Rx_1 + a, \dots, Rx_n + a)$ and $f^\pi(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$.

We recall the Osterwalder–Schrader axioms [1] for the Schwinger functions (Euclidean Green's functions). The set of the Schwinger functions $\{s_n\}$ is a sequence of distributions $s_n(x_1, \dots, x_n)$ with the following properties:

E0. Distributions.

$$s_0 \equiv 1,\ s_n \in S'(\mathbf{R}^{4n}_0)$$

and

$$\overline{(s_n, f)} = (s_n, \theta f^*) \tag{4.2}$$

for all functions $f \in S(\mathbf{R}^{4n}_<)$.

E1. Euclidean invariance.

$$(s_n, f_{(a,R)}) = (s_n, f) \tag{4.3}$$

for all $R \in SO_4$, $a \in \mathbf{R}^4$ and $f \in S(\mathbf{R}_0^{4n})$.

E2. Positivity.

$$\sum_{n,m} (s_{n+m}, \theta f_n^* \otimes f_m) \geqq 0 \tag{4.4}$$

for all finite sequences of the functions $f_n \in S(\mathbf{R}_<^{4n} \cap \mathbf{R}_+^{4n})$, where the function $(f \otimes g)(x_1,\dots,x_{n+m}) = f(x_1,\dots,x_n)g(x_{n+1},\dots,x_{n+m})$ is defined for all functions $f \in S(\mathbf{R}^{4n})$ and $g \in S(\mathbf{R}^{4m})$. The relation (4.2) is a consequence of inequality (4.4).

E3. Symmetry.

$$(s_n, f^\pi) = (s_n, f) \tag{4.5}$$

for all permutations $\pi \in P_n$ and for all functions $f \in S(\mathbf{R}_0^{4n})$.

E4. Cluster property.

$$\lim_{t\to\infty} (s_{n+m}, \theta f_n^* \otimes (g_m)_{(ta,1)}) = (s_n, \theta f_n^*)(s_m, g_m) \tag{4.6}$$

for all $f_n \in S(\mathbf{R}_<^{4n} \cap \mathbf{R}_+^{4n})$, $g_m \in S(\mathbf{R}_<^{4m} \cap \mathbf{R}_+^{4m})$, $a = (0, \mathbf{a})$, $\mathbf{a} \in \mathbf{R}^3$.

Let us consider the restriction of the distribution $s_n \in S'(\mathbf{R}_0^{4n})$ to test functions from the space $S(\mathbf{R}_<^{4n})$. Then the translation invariance (4.3) implies

$$s_n(x_1,\dots,x_n) = S_{n-1}(x_2 - x_1,\dots,x_n - x_{n-1}), \tag{4.7}$$

where the distribution $S_{n-1}(\underline{x}) \in S'(\mathbf{R}_+^{4(n-1)})$. We note that for a function g of the form $g(x_1,\dots,x_{n+1}) = f(x_2 - x_1,\dots,x_{n+1} - x_n)$ the definitions (4.1) imply the equality $\theta g^*(x_1,\dots,x_{n+1}) = (\theta_p f^*)(x_2 - x_1,\dots,x_{n+1} - x_n)$, where the involution

$$\theta_p f(x_1,\dots,x_n) = f(-\theta x_1,\dots,-\theta x_n) \tag{4.8}$$

leaves the space $S(\mathbf{R}_+^{4n})$ invariant.

Into the inequality (4.4) we substitute the sequence consisting of a single function

$$f_{m+1}(x_1,\dots,x_{m+1}) = \begin{cases} \overline{\phi}_1(x_1)\phi_m(x_2 - x_1,\dots,x_{m+1} - x_m), & m > 0 \\ \overline{\phi}_1(x_1), & m = 0, \end{cases} \tag{4.9}$$

where the functions $\phi_1 \in S(\mathbf{R}_+^4)$ and $\phi_m \in S(\mathbf{R}_+^{4m})$. Then by using the definitions (4.7) and (4.8) we can rewrite the inequality (4.4) for $m = 0$ in the form (3.9) and for $m > 0$ in the following form:

$$\int d^4x d^4y S_{2m+1}(\theta_p \phi_m^*, x - \theta y, \phi_m)\overline{\phi}_1(x)\phi_1(y) \geqq 0. \tag{4.10}$$

Here we have introduced the distribution

$$\int d^4x S_{n+m+1}(f_n, x, f_m) f_1(x) = \int d^{4(n+m+1)}x S_{n+m+1}(\underline{x}) f_n \otimes f_1 \otimes f_m(\underline{x}), \tag{4.11}$$

constructed from the distribution $S_{n+m+1}(\underline{x}) \in S'(\mathbf{R}_+^{4(n+m+1)})$ and the test functions $f_n \in S(\mathbf{R}_+^{4n})$, $f_m \in S(\mathbf{R}_+^{4m})$. For $n = 0$ or $m = 0$ the distribution (4.11) is defined in an obvious way.

The inequalities (4.10) show that the distributions $S_{2m+1}(\theta_p \phi_m^*, x, \phi_m)$ are extremely significant. We formulate the new axiom exactly for these distributions.

E5. Weak spectral condition.

Let $S_{2m+1}(\underline{x}) \in S'(\mathbf{R}_+^{4(2m+1)})$, $m = 1, 2, \ldots$, be any distribution defined by the relation (4.7). Then there is the natural number K such that for any integers $k > K$, $1 \leqq l \leqq m$ and for all test functions $\psi_i(x) \in S(\mathbf{R}^4)$, $i = 1, \ldots, l$, $\psi_j(x) \in S(\mathbf{R}_+^4)$, $j = l+1, \ldots, m+1$, the following limit exists:

$$\lim_{n_1,\ldots,n_l \to \infty, n_i \in \mathbf{Z}} \lim_{T_1,\ldots,T_l \to -\infty, T_i \in \mathbf{R}} \int d^{4(2m+1)}x \left(\prod_{i=1}^{l} x_i^0 x_{2m+2-i}^0 \right)^{-k} S_{2m+1}(\underline{x})$$
$$\times [(L_c^{-1}[\psi_1 \otimes \cdots \otimes \psi_m]_{x_1^0,\ldots,x_l^0}(\,\cdot\,; \underline{n}, \underline{T})) \otimes \psi_{m+1}$$
$$\otimes\, \theta_p(L_c^{-1}[\psi_1 \otimes \cdots \otimes \psi_m]_{x_1^0,\ldots,x_l^0}(\,\cdot\,; \underline{n}, \underline{T}))^*](\underline{x})\,, \tag{4.12}$$

with

$$(L_c^{-1}[\psi_1 \otimes \cdots \otimes \psi_m]_{x_1^0,\ldots,x_l^0}(\underline{x}; \underline{n}, \underline{T})) = \left(\prod_{i=1}^{l} L_c^{-1}[\psi_i]_{x_i^0}(x_i; n_i, T_i) \right) \left(\prod_{i=l+1}^{m} \psi_i(x_i) \right), \tag{4.13}$$

and where the function $L_c^{-1}[\psi]_{x^0}(x; n, T)$ is given by the equality (2.56).

Theorem 2.7 clarifies the relevance of this weak spectral condition.

Let us recall the Wightman axioms [8] for Wightman distributions. The set of Wightman distributions $\{w_n\}$ is a sequence of distributions with the following properties:

R0. Temperedness.

$w_0 \equiv 1$, $w_n \in S'(\mathbf{R}^{4n})$ and

$$\overline{(w_n, f)} = (w_n, f^*) \tag{4.14}$$

for all $f \in S(\mathbf{R}^{4n})$.

R1. Relativistic invariance.

$$(w_n, f_{(a,\Lambda)}) = (w_n, f) \tag{4.15}$$

for all vectors $a \in \mathbf{R}^4$, for all Lorentz transformations $\Lambda \in L_+^\uparrow$ and for all functions $f \in S(\mathbf{R}^{4n})$, where the function $f_{(a,\Lambda)}(\underline{x}) = f(\Lambda^{-1}(x_1 - a), \ldots, \Lambda^{-1}(x_n - a))$.

R2. Positivity.

$$\sum_{n,m} (w_{n+m}, f_n^* \otimes f_m) \geqq 0 \tag{4.16}$$

for all finite sequences of the functions $f_n \in S(\mathbf{R}^{4n})$. The relation (4.14) is a consequence of inequality (4.16).

R3. Local commutativity.

For all natural numbers $n > 0$ and $j = 1, \ldots, n-1$,

$$w_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n) = w_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) \tag{4.17}$$

if the vector $x_{j+1} - x_j \in \mathbf{R}^4$ is spacelike:

$$(x_{j+1} - x_j, x_{j+1} - x_j) \equiv (x_{j+1}^0 - x_j^0)^2 - \textstyle\sum_{i=1}^3 (x_{j+1}^i - x_j^i)^2 < 0.$$

R4. Cluster property.

$$\lim_{\lambda\to\infty} w_{n+m}(x_1,\dots,x_n,x_{n+1}+\lambda a,\dots,x_{n+m}+\lambda a) = w_n(x_1,\dots,x_n)w_m(x_{n+1},\dots,x_{n+m}) \tag{4.18}$$

for all natural numbers $n, m > 0$ and for all spacelike vectors $a \in \mathbf{R}^4$.

R5. Spectral condition.
For all natural numbers $n > 1$ there exists a tempered distribution $W^{\sim}_{n-1} \in S'(\mathbf{R}^{4(n-1)})$ with support in $\overline{V}_+^{\times n}$, where $\overline{V}_+$ is the closed forward light cone, such that

$$w_n(\underline{x}) = \int d^{4(n-1)}p W^{\sim}_{n-1}(\underline{p}) \exp\left\{ i \sum_{j=1}^{n-1} (p_j, (x_{j+1}-x_j)) \right\} . \tag{4.19}$$

Now we are able to formulate the revised Osterwalder–Schrader theorem.

Theorem 4.1. *To a given sequence of Wightman distributions satisfying* R0–R5, *there corresponds a unique sequence of Schwinger functions with the properties* E0–E5. *To a given sequence of Schwinger functions satisfying* E0–E5, *there corresponds a unique sequence of Wightman distributions with the properties* R0–R5.

Proof. We start from a relativistic field theory given by a sequence of Wightman distributions, satisfying the axioms R0–R5. Due to Theorem 3.5 from [8] the Wightman distribution w_n is the boundary value of the Wightman function $w_n(z_1,\dots,z_n) = W_{n-1}(z_2-z_1,\dots,z_n-z_{n-1})$, where the function $W_{n-1}(z_1,\dots,z_{n-1})$ is analytic in the tube $T_{n-1} = \{z_1,\dots,z_{n-1} | \operatorname{Im} z_i \in V_+,\ i = 1,\dots,n-1\}$. The Wightman function $w_n(z_1,\dots,z_n)$ is Lorentz invariant (Lorentz covariant for the theories of arbitrary spinor fields). The Bargmann–Hall–Wightman theorem [8, Theorem 2.11] implies that the function $W_n(z_1,\dots,z_n)$ allows a single valued $L_+(\mathbf{C})$ invariant ($L_+(\mathbf{C})$ covariant for the theories of arbitrary spinor fields) analytic continuation into the extended tube $T'_n = \cup_{A\in L_+(\mathbf{C})} AT_n$. Using Theorem 3.6 in [8] we conclude that the function $w_n(z_1,\dots,z_n)$ has an $L_+(\mathbf{C})$ invariant, single valued, symmetric under the permutations analytic continuation into the domain $IT_n^p = \{z_1,\dots,z_n | (z_{\pi(2)} - z_{\pi(1)},\dots,z_{\pi(n)} - z_{\pi(n-1)}) \in T'_{n-1}$ for some permutation $\pi(1),\dots,\pi(n)$ of the numbers $1,\dots,n\}$. (For the theories of arbitrary spinor fields this function has an $L_+(\mathbf{C})$ covariant, single valued analytic continuation into the domain IT_n^p with obvious symmetry properties under the permutations.) The set IT_n^p contains the set of Euclidean points $E_n = \{z_1,\dots,z_n | \operatorname{Re} z_k^0 = 0,\ \operatorname{Im} \mathbf{z}_k = 0, z_k \neq z_j$ for all $1 \leqq k, j \leqq n, k \neq j\}$. The restriction of the Wightman functions to Euclidean points defines the Schwinger functions

$$s_n(x_1,\dots,x_n) = w_n((ix_1^0,\mathbf{x}_1),\dots,(ix_n^0,\mathbf{x}_n)) . \tag{4.20}$$

The derivation of the extended Osterwalder–Schrader axioms E0–E5 from the Wightman axioms follows the arguments given in [1] and Theorem 2.7.

Conversely, let $\{s_n\}$ be a sequence of distributions satisfying the extended Osterwalder–Schrader axioms E0–E5. If we substitute into the inequality (4.4) the sequence consisting of a single function (4.9) for $m = 0$ we get the inequality (3.9) for the distribution $S_1(x)$. Due to Theorem 3.5 this distribution is the Laplace transform with respect to the time variable of a tempered distribution with support in the closure $\overline{\mathbf{R}}^4_+$. Let us substitute into the inequality (4.4) the sequence consisting of two functions $f_{n+1}(x)$ and $f_{m+1}(x)$ of type (4.9) with the same function $\phi_1(x)$. Then we obtain the following inequality:

$$\int d^4xd^4yS\{\phi_n,\phi_m\}(x-\theta y)\overline{\phi_1(x)}\phi_1(y) \geqq 0\,, \tag{4.21}$$

where the distribution

$$\begin{aligned} S\{\phi_n,\phi_m\}(x) &= S\{\phi_m,\phi_n\}(x) \\ &= S_{2m+1}(\theta_p\phi_m^*,x,\phi_m) + S_{2n+1}(\theta_p\phi_n^*,x,\phi_n) \\ &\quad + S_{m+n+1}(\theta_p\phi_m^*,x,\phi_n) + S_{m+n+1}(\theta_p\phi_n^*,x,\phi_m)\,. \end{aligned} \tag{4.22}$$

This definition may be easily modified for the case $n=0$ or $m=0$,

$$\begin{aligned} S\{\lambda,\phi_m\}(x) &= S\{\phi_m,\lambda\}(x) \\ &= S_{2m+1}(\theta_p\phi_m^*,x,\phi_m) + |\lambda|^2S_1(x) \\ &\quad + \lambda S_{m+1}(\theta_p\phi_m^*,x) + \overline{\lambda}S_{m+1}(x,\phi_m)\,, \end{aligned} \tag{4.23}$$

where λ is a complex number. The equality (4.22) implies the relation $S\{\phi_n,\phi_n\}(x) = 4S_{2n+1}(\theta_p\phi_n^*,x,\phi_n)$. Hence the inequality (4.10) is the particular case of inequality (4.21) for $m=n$. It follows from the definitions (4.22),(4.23) that the distribution (4.11) is a linear combination of the distributions (4.22) and (4.23)

$$\begin{aligned} S_{m+n+1}(\phi_m,x,\phi_n) &= 1/2S\{\phi_n,\theta_p\phi_m^*\}(x) + i/2S\{\phi_n,i\theta_p\phi_m^*\}(x) \\ &\quad -(1+i)/2S_{2m+1}(\phi_m,x,\theta_p\phi_m^*) - (1+i)/2S_{2n+1}(\theta_p\phi_n^*,x,\phi_n)\,. \end{aligned} \tag{4.24}$$

In particular for $m=0$ or $n=0$ and $\phi_0=1$ we get

$$\begin{aligned} S_{n+1}(x,\phi_n) &= 1/2S\{1,\phi_n\}(x) + i/2S\{i,\phi_n\}(x) \\ &\quad -(1+i)/2S_{2n+1}(\theta_p\phi_n^*,x,\phi_n) - (1+i)/2S_1(x)\,, \end{aligned} \tag{4.25}$$

$$\begin{aligned} S_{n+1}(\phi_n,x) &= 1/2S\{1,\theta_p\phi_n^*\}(x) + i/2S\{1,i\theta_p\phi_n^*\}(x) \\ &\quad -(1+i)/2S_{2n+1}(\phi_n,x,\theta_p\phi_n^*) - (1+i)/2S_1(x)\,. \end{aligned} \tag{4.26}$$

The inequalities (4.21) imply that for any function $\phi_n \in S(\mathbf{R}_+^4)$ every one of the four distributions, depending on the variable x, in the right-hand side of the equality (4.25) is proportional to a distribution from $S'(\mathbf{R}_+^4)$ satisfying the Osterwalder–Schrader positivity condition (3.9). Due to Lemma 3.2 the limits (3.11) and (3.12) are equal to zero for the distribution $(x^0)^{-k}S_{n+1}(x,\phi_n) \in S'(\mathbf{R}_+^4)$ if the integer $k > 0$. It follows from Proposition 3.4 that for the distribution $S_{n+1}(\underline{x}) \in S'(\mathbf{R}_+^{4(n+1)})$ the limit (2.55) exists for the integers $l=1$, $j_1=1$, $K=0$. By definition the support of the distribution $L_c^{-1}[(x_1^0)^{-k}S_{n+1}]_{x_1^0}(x_1,\ldots,x_{n+1})$ with respect to the first variable x_1 is in the closure $\overline{\mathbf{R}}_+^4$. Theorem 3.5 and Lemma 3.2 imply that for all functions

$\psi_i \in S(\mathbf{R}^4_+)$, $i = 1,\ldots,n+1$, for any integer $k > 0$ the following relation holds:

$$\int d^{4(n+1)}x S_{n+1}(\underline{x}) \prod_{i=1}^{n+1} \psi_i(x_i) = \int d^{4(n+1)}x \left(\frac{\partial}{\partial x_1^0}\right)^k L_c^{-1}[(x_1^0)^{-k} S_{n+1}]_{x_1^0}(\underline{x}) \times \int dy_1^0 \exp\{-x_1^0 y_1^0\}\psi_1(y_1^0,\mathbf{x}_1) \prod_{i=2}^{n+1} \psi_i(x_i)\,. \tag{4.27}$$

In view of the equality (4.26) all above results are valid for the distribution $S_{n+1}(\phi_n, x) \in S'(\mathbf{R}^4_+)$, where the function $\phi_n \in S(\mathbf{R}^{4n}_+)$.

The weak spectral condition E5 implies the existence of the limit

$$\lim_{m\to\infty} \lim_{T\to-\infty} \int d^{4(2n+1)}x S_{2n+1}(\underline{x})(x_1^0)^{-k} L_c^{-1}[\psi_1]_{x_1^0}(x_1;m,T) \left(\prod_{i=2}^{n+1} \psi_i(x_i)\right) \theta_p \times \left(\left(\prod_{i=2}^{n} \overline{\psi}_i(x_{2n+2-i})\right)(x_{2n+1}^0)^{-k} L_c^{-1}[\overline{\psi}_1]_{x_{2n+1}^0}(x_{2n+1};m,T)\right) \tag{4.28}$$

for some positive integer k and for all functions $\psi_1(x) \in S(\mathbf{R}^4)$, $\psi_i(x) \in S(\mathbf{R}^4_+)$, $i = 2,\ldots,n+1$. Here the function $L_c^{-1}[\psi_1]_{x^0}(x;m,T)$ is defined by the relation (2.56). The linear functional (4.28) with respect to the function $\psi_{n+1}(x) \in S(\mathbf{R}^4_+)$ is a tempered distribution in the space $S'(\mathbf{R}^4_+)$. It satisfies the Osterwalder–Schrader positivity condition (3.9). Similar arguments may be applied for the distributions $S\{1,\theta_p(\prod_{i=1}^n \psi_i(x_i))^*\}(x)$ and $S\{1,i\theta_p(\prod_{i=1}^n \psi_i(x_i))^*\}(x)$ in the right-hand side of the equality of type (4.26). Thus the limit (2.56) $(L_c^{-1}[(x_1^0)^{-k}S_{n+1}]_{x_1^0}(\underline{x}), (\prod_{i=1}^{n+1} \psi_i(x_i)))$, the existence of which is proved above, for some positive integer k and for all functions $\psi_1(x) \in S(\mathbf{R}^4)$, $\psi_i(x) \in S(\mathbf{R}^4_+)$, $i = 2,\ldots,n+1$, has the decomposition of type (4.26) into four distributions with respect to the function $\psi_{n+1}(x) \in S(\mathbf{R}^4_+)$. These distributions are proportional to the distributions from $S'(\mathbf{R}^4_+)$ satisfying the Osterwalder–Schrader positivity condition (3.9). Now Proposition 3.4 implies that for the distribution $S_{n+1}(\underline{x}) \in S'(\mathbf{R}^{4(n+1)}_+)$ the limit (2.55) exists for $l = 2, j_1 = 1, j_2 = n+1$ and for some positive integer k. Due to the definition the supports of this limiting distribution $L_c^{-1}[(x_1^0 x_{n+1}^0)^{-k} S_{n+1}]_{x_1^0,x_{n+1}^0}(\underline{x})$ with respect to the first and the last variables are in the closure $\overline{\mathbf{R}}^4_+$. Theorem 3.5, Lemma 3.2 and the relation (4.27) imply that for sufficiently large positive integer k and for all functions $\psi_i(x) \in S(\mathbf{R}^4_+)$, $i = 1,\ldots,n+1$, the following relation holds:

$$\int d^{4(n+1)}x S_{n+1}(\underline{x}) \prod_{i=1}^{n+1} \psi_i(x_i) = \int d^{4(n+1)}x \left(\frac{\partial^2}{\partial x_1^0 \partial x_{n+1}^0}\right)^k L_c^{-1}[(x_1^0 x_{n+1}^0)^{-k} S_{n+1}]_{x_1^0,x_{n+1}^0}(\underline{x}) \int dy_1^0 dy_{n+1}^0 \times \exp\left\{-\sum_{i=1,n+1} x_i^0 y_i^0\right\}\left(\prod_{i=1,n+1} \psi_i(y_i^0,\mathbf{x}_i)\right) \prod_{i=2}^{n} \psi_i(x_i)\,. \tag{4.29}$$

By using the weak spectral condition E5 and the equalities (4.24) it is possible to prove step by step that the limit (2.55) exists for the distribution $S_{n+1}(\underline{x}) \in$

$S'(\mathbf{R}_+^{4(n+1)})$, for $l = n+1$ and for some positive integer k. By definition the support of this limiting distribution $L_c^{-1}[(\prod_{l=1}^{n+1} x_l^0)^{-k} S_{n+1}]_{\underline{x}^0}(\underline{x})$ with respect to any variable x_i is in the closure $\overline{\mathbf{R}}_+^4$. Since the weak convergence in the space S' implies the convergence in the topology of the space S' (see [6, Sect. 3.7]) the limit $(L_c^{-1}[(\prod_{l=1}^{n} x_l^0)^{-k} S_{n+1}]_{\underline{x}^0}(\underline{x}), (\prod_{l=1}^{n+1} \psi_l(x_l)))$ is continuous in each function $\psi_l(x) \in S(\mathbf{R}^4)$. Hence the nuclear theorem [7, Chapter 1, Sect. 1, Theorem 6] implies that $L_c^{-1}[(\prod_{i=1}^{n+1} x_i^0)^{-k} S_{n+1}]_{\underline{x}^0}(\underline{x}) \in S'(\mathbf{R}^{4(n+1)})$. Its support is in the closure $\overline{\mathbf{R}}_+^{4(n+1)}$. The application step by step of Theorem 3.5 and Lemma 3.2 gives for sufficiently large positive integer k and for all functions $\psi_i(x) \in S(\mathbf{R}_+^4)$, $i = 1, \ldots, n+1$ the following relation:

$$\begin{aligned}
&\int d^{4(n+1)} x S_{n+1}(\underline{x}) \prod_{l=1}^{n+1} \psi_l(x_l) \\
&= \int d^{4(n+1)} x \left(\frac{\partial^{n+1}}{\partial x_1^0 \ldots \partial x_{n+1}^0} \right)^k L_c^{-1} \left[\left(\prod_{l=1}^{n+1} x_l^0 \right)^{-k} S_{n+1} \right]_{\underline{x}^0} (\underline{x}) \\
&\times \int dy_1^0 \ldots dy_{n+1}^0 \exp\left\{ -\sum_{i=1}^{n+1} x_i^0 y_i^0 \right\} \left(\prod_{i=1}^{n+1} \psi_i(y_i^0, \mathbf{x}_l) \right) . \qquad (4.30)
\end{aligned}$$

Therefore for any $n = 1, 2, \ldots$ a distribution $S_n(\underline{x}) \in S'(\mathbf{R}_+^{4n})$ is the Laplace transform of the tempered distribution from $S'(\mathbf{R}^{4n})$ with support in the closure $\overline{\mathbf{R}}_+^{4n}$.

Now the derivation of the Wightman axioms R0–R5 from the Osterwalder–Schrader axioms E0–E4 follows the arguments of the paper [1].

References

1. Osterwalder, K., Schrader, R.: Axioms for Euclidean Green's Functions. Commun. Math. Phys. **31**, 83–112 (1973)
2. Osterwalder, K., Schrader, R.: Axioms for Euclidean Green's Functions 2. Commun. Math. Phys. **42**, 281–305 (1975)
3. Glaser, V.: On the Equivalence of the Euclidean and Wightman Formulation of Field Theory. Commun. Math. Phys. **37**, 257–272 (1974)
4. Bernstein, S.: Sur les fonctions absolument monotones. Acta Math. **52**, 1–66 (1929)
5. Zinoviev, Yu.M.: Inversion formulas for the Laplace transform. 1. The octant. Teor. Mat. Fiz. **65**, 16–23 (1985); English transl. in Theor. Math. Phys. **65**, (1985)
6. Vladimirov, V.S.: Methods of Theory of Functions of Many Complex Variables. Cambridge, MA: MIT Press, 1966
7. Gel'fand, I.M., Vilenkin, N.Ya.: Generalized Functions. Vol. 4. New York: Academic Press, 1964
8. Streater, R.F., Wightman, A.S.: PCT, spin and statistics and all that. New York, Amsterdam: Benjamin, 1964

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