# An Explicit Construction of the Quantum Group in Chiral WZW-Models

## Matthias R. Gaberdiel[1]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge, CB3 9EW, U.K.

**Abstract:** It is shown how a chiral Wess–Zumino–Witten theory with globally defined vertex operators and a one-to-one correspondence between fields and states can be constructed. The Hilbert space of this theory is the direct sum of tensor products of representations of the chiral algebra and finite dimensional internal parameter spaces. On this enlarged space there exists a natural action of Drinfeld's quasi-quantum group $A_{g,t}$, which commutes with the action of the chiral algebra and plays the rôle of an internal symmetry algebra. The $R$ matrix describes the braiding of the chiral vertex operators and the coassociator $\Phi$ gives rise to a modification of the duality property.

For generic $q$ the quasi-quantum group is isomorphic to the coassociative quantum group $U_q(g)$ and thus the duality property of the chiral theory can be restored. This construction has to be modified for the physically relevant case of integer level. The quantum group has to be replaced by the corresponding truncated quasi-quantum group, which is not coassociative because of the truncation. This exhibits the truncated quantum group as the internal symmetry algebra of the chiral WZW model, which therefore has only a modified duality property. The case of $g = su(2)$ is worked out in detail.

## 1. Introduction

A very important feature of two-dimensional conformal field theory is the fact that the theory "factorises" into a holomorphic and an anti-holomorphic theory. These two subtheories correspond essentially to the left- and right-movers of the original classical theory and are analytic (anti-analytic) in the sense that all correlation functions are meromorphic functions of the analytic (anti-analytic) parameters. Many properties of conformal field theory can be studied separately for the two chiral theories. This is of great importance as it allows the use of the powerful methods of complex analysis for the analysis of conformal field theory.

---

[1] e-mail: M.R.Gaberdiel@damtp.cam.ac.uk

However, the process of breaking up a theory into the two chiral theories is not very well understood. In particular, the naive chiral theory, i.e. the theory, in which the Hilbert space is just the direct sum of the chiral representation spaces, does not possess globally defined vertex operators or a one-to-one correspondence between vertex operators and states.

In this paper I want to show how to construct a chiral theory with a proper Hilbert space formulation for a WZW conformal field theory. The basic idea is to use the Hilbert space formulation of the whole theory and to restrict it to a chiral subtheory by taking a suitable limit. This construction guarantees that the chiral theory has a one-to-one correspondence of states and vertex operators and that the chiral vertex operators are well-defined operators on the whole chiral Hilbert space. This Hilbert space is larger than the "naive chiral Hilbert space," i.e. the direct sum of the irreducible representations of the chiral algebra, and is precisely the Hilbert space Moore and Reshetikhin [35] postulated some years ago. The additional degrees of freedom keep track of the different "chiral vertex operators" of Moore and Seiberg [36] and make sure that the vertex operators are well-defined on the whole Hilbert space. They furthermore retain sufficient information to reconstruct the whole theory from its chiral subtheory.

On the additional degrees of freedom there is a natural action of the quasi-quantum group $A_{g,t}$ of Drinfel'd [13], which commutes with the chiral algebra. Chiral vertex operators transform covariantly under the quasi-quantum group and the braiding of the chiral vertex operators is described by the $R$-matrix, where the deformation parameter $h$ is related to the level of the affine algebra by $h = 2\pi i/(k + h^*)$ with $h^*$ the dual Coxeter number of $g$.

The symmetry algebra is only a *quasi*-Hopf algebra. This means that the algebra is not coassociative, but only coassociative up to conjugation. The operator, by which the two different actions on a triple tensor product have to be conjugated, is called the coassociator $\Phi$. It is an invertible element in the triple tensor product of the quasi-Hopf algebra. The property of the quasi-quantum group to be only quasi-coassociative leads to a modification of the duality property of the chiral vertex operators: the two different ways of writing the operator product are related by the action of $\Phi$ on the internal degrees of freedom.

For generic $q$ the quasi-Hopf algebra is isomorphic to the (coassociative) quantum group $U_q(g)$. I can thus use this isomorphism to regard the internal parameter spaces as representations spaces of this quantum group. The chiral vertex operators then satisfy the (unmodified) duality property and the chiral theory transforms naturally under the action of the quantum group.

The above construction can be extended to the physically relevant case, where $k$ is an integer and $q$ a root of unity. In this case one has to replace the quantum group $U_q(g)$ by the corresponding truncated quantum group. This is the so-called "weak quasi-triangular quasi-Hopf algebra" canonically associated to $U_q(g)$ and has been studied in [31]. It is not a coassociative algebra, as the truncation breaks the coassociativity. This exhibits the truncated quantum group as the internal symmetry algebra of the chiral theory, which thus possesses only a modified duality property.

On the other hand, in contrast to the original quasi-quantum group $A_{g,t}$, the degree by which the truncated quantum group fails to be coassociative can be easily determined. In particular, the $\Phi$-map, which describes the non-coassociativity, is trivial on all triple tensor products, which do not exhibit any truncation.

I would like to mention that the general structure has been conjectured among others by [35, 1, 2] – for a historical review see for example [25]. There have been

attempts to give a construction of the quantum group in conformal field theory in [23, 24] for the minimal models and in [39, 40] for the WZW-models using the Coulomb gas picture. However, in these formulations the quantum group generators do not commute with the Virasoro algebra and the construction relies on a specific realisation of the theory (which only exists for generic $q$).

The construction of the chiral theory (with internal degrees of freedom) was inspired by the recent work of Chu and Goddard [4], in which chiral vertex operators for $su(n)$ at level 1 were constructed. However, applying my construction to the free field realisation of the whole theory (for $su(n)$ at level 1) does *not* reproduce their chiral theory. Indeed, the internal degrees of freedom in my construction are finite dimensional, whereas their internal parameter spaces are infinite dimensional. On the other hand, their construction has the virtue of preserving the (unmodified) duality property for the chiral theory.

It is a priori rather surprising that there should be two different chiral theories, as the quantisation of a classical theory should be somehow unique. On the other hand, this uniqueness property only applies to the whole theory and one would expect that the corresponding reconstructed whole theories agree. Thus the two chiral theories just appear to be two different factorisations of the same whole theory.

It is nevertheless quite remarkable that there exists a chiral theory with the unmodified duality property for the level 1 $su(n)$-theories. However, for non-integer quantum dimensions this is only possible if the internal degrees of freedom are infinite-dimensional. Thus chiral theories with finite dimensional internal degrees of freedom possess in general only a modified duality property.

Finally, I would like to mention that there have been attempts to reconstruct the abstract internal symmetry algebra of a chiral conformal field theory from the structural data of the chiral theory [32, 43, 17, 41, 38], generalising the approach of Doplicher and Roberts [7] to the case of braid group statistics. This approach has been successful in the sense that for every chiral conformal field theory a weak quasi-quantum group can be found whose representation theory reproduces the fusion structure of the chiral theory. However, it has been realised recently [27, 41] that the abstract internal symmetry algebra is not yet uniquely determined by this condition alone, as there exists an infinite choice for the dimensionalities of the representation spaces.

In the above construction of the chiral WZW-model, one of the possible abstract internal symmetry algebras is naturally selected. In addition to satisfying all conditions of an abstract internal symmetry algebra, it is also a subalgebra of the symmetry algebra of the whole theory. The construction therefore seems to suggest that the internal symmetry algebra of a chiral conformal field theory might be uniquely determined by the whole conformal field theory, namely by the condition to be a subalgebra of the whole symmetry algebra. The different possible abstract internal symmetry algebras would then also have a natural interpretation: they would correspond to different whole conformal field theories which possess the same chiral half.

The paper is organised as follows. In Sect. 2, the chiral theory is defined. Section 3 contains a brief review of the quasi-Hopf algebra of Drinfel'd, whose appearance in the chiral theory is proven in Sect. 4. In the following section I explain how the duality property of the chiral vertex operators can be restored for generic $k$ by transforming to the quantum group $U_q(g)$. This is demonstrated explicitly for the case of $g = su(2)$. I then explain how the construction has to be modified for roots of unity. In Sect. 6, I show how the original theory can be

reconstructed from the chiral theory by some sort of gauging procedure. Section 7 contains some conclusions.

## 2. Construction of Chiral Vertex Operators

Let us start by establishing some notation. The physical Hilbert space of the whole (non-chiral) WZW-theory is the finite direct sum

$$\mathscr{H} = \bigoplus_l \mathscr{H}_l \otimes \overline{\mathscr{H}}_l \,, \tag{2.1}$$

where $\mathscr{H}_l(\overline{\mathscr{H}}_l)$ are irreducible (not necessarily conjugate) representations of the chiral algebra $\mathscr{A}(\overline{\mathscr{A}})$. The two chiral algebras $\mathscr{A}$ and $\overline{\mathscr{A}}$ commute and are both isomorphic to the affine algebra $\hat{g}$, generated by

$$[J_m^a, J_n^b] = i\, f_c^{ab} J_{m+n}^c + \frac{1}{2}k\, m\delta_{m,-n}\delta^{ab} \,. \tag{2.2}$$

$k(\overline{k})$ lies in the center of the algebra $\mathscr{A}(\overline{\mathscr{A}})$ and both $k$ and $\overline{k}$ take the same value in all representations. The value of $x = 2k/\psi^2$ is called the level, where $\psi$ is a long root of the Lie algebra $g$. It has to be an integer in a unitary theory. This restriction can also be understood as a quantisation condition, if the WZW model is to be regarded as the quantisation of a classical field theory (see (2.10) below).

I want to assume that there is a one-to-one correspondence between fields and states (for the whole theory) and denote for a given $\psi \otimes \overline{\psi} \in \mathscr{H}_j \otimes \overline{\mathscr{H}}_j$ the corresponding field by $\phi(\psi \otimes \overline{\psi}; z, \overline{z}) : \mathscr{H} \to \mathscr{H}$. I also assume, that if one specifies the "source" and the "range" of the operator – i.e. if one considers only its projected parts – the operator $\phi(\psi \otimes \overline{\psi}; z, \overline{z})$ decouples into a sum of tensor products of operators, each of which depends only on $z$ and $\overline{z}$, respectively, i.e.

$$[\phi(\psi \otimes \overline{\psi}; z, \overline{z})(|\chi\rangle_l \otimes |\overline{\chi}\rangle_l)]_m = \sum_\alpha (V_{ml}^\alpha(\psi, z)\, |\chi\rangle_l) \otimes (\overline{V}_{ml}^\alpha(\overline{\psi}, \overline{z})\, |\overline{\chi}\rangle_l) \,. \tag{2.3}$$

As an aside I would like to point out that (2.3) does not determine the operators $V_{ml}^\alpha(\psi, z)$ and $\overline{V}_{ml}^\alpha(\overline{\psi}, \overline{z})$ uniquely, as there is (at least) the freedom to redefine for each $\alpha$

$$\tilde{V}_{ml}^\alpha(\psi, z) := \lambda V_{ml}^\alpha(\psi, z), \qquad \widetilde{\overline{V}}_{ml}^\alpha(\overline{\psi}, \overline{z}) := \lambda^{-1} \overline{V}_{ml}^\alpha(\overline{\psi}, \overline{z}) \,, \tag{2.4}$$

where $\lambda$ is a non-zero complex number.

In general $\phi(\psi \otimes \overline{\psi})$ does *not* decompose into such a sum of products and thus the whole theory is not simply the product of the holomorphic and the anti-holomorphic theory. However, a chiral theory with a proper Hilbert space formulation can be defined, if one enlarges the (naive chiral) Hilbert space. The internal parameter spaces which are introduced in this way, keep track of the different "chiral vertex operators" of [36] and retain all the necessary information to reconstruct the original theory.

Each irreducible representation $\overline{\mathscr{H}}_j$ of $\overline{\mathscr{A}} = \hat{g}$ contains a subspace $\overline{W}_j$ of highest weight vectors (w.r.t. the Virasoro algebra), which forms a finite dimensional (irreducible) representation of the Lie algebra $g$. This subspace can be interpreted as the space of lowest energy states, as $\overline{L}_0$ is essentially the energy.

The basic idea of the construction is to enlarge the naive chiral Hilbert space by these lowest energy states, i.e. to define the Hilbert space of the holomorphic chiral theory as

$$\mathcal{H}_{\text{chir}} = \bigoplus_l \mathcal{H}_l \otimes \overline{W}_l \ . \tag{2.5}$$

On this chiral Hilbert space one can then define globally well-defined chiral vertex operators. These chiral vertex operators are in one-to-one correspondence with vectors in $\mathcal{H}_{\text{chir}}$ and thus depend on $\psi \otimes \bar{w} \in \mathcal{H}_j \otimes \overline{W}_j$. They can be defined as

$$V(\psi \otimes \bar{w}, z) : \mathcal{H}_{\text{chir}} \to \mathcal{H}_{\text{chir}} \tag{2.6}$$

by setting

$$(V(\psi \otimes \bar{w}, z) \, (|\chi\rangle_l \otimes |\bar{u}\rangle_l))_m := \lim_{\bar{z} \to 0} \bar{z}^{\bar{\Delta}_j + \bar{\Delta}_l - \bar{\Delta}_m} \, (\phi(\psi \otimes \bar{w}; z, \bar{z}) \, (|\chi\rangle_l \otimes |\bar{u}\rangle_l))_m \ , \tag{2.7}$$

where $\bar{\Delta}_i$ is the $\bar{L}_0$-eigenvalue of the highest weight representation $\overline{W}_i$ of $\overline{\mathcal{H}}_i$. A similar construction can be performed for the anti-holomorphic vertex operators.

To prove that the above definition makes sense, I have to show that

$$\lim_{\bar{z} \to 0} \bar{z}^{\bar{\Delta}_j + \bar{\Delta}_l - \bar{\Delta}_m} \, \bar{V}^{\alpha}_{ml}(\bar{w}, \bar{z}) \, |\bar{u}\rangle_l \in \overline{W}_m \ . \tag{2.8}$$

Because of the conformal invariance of the theory, the operator product expansion is given by [22]

$$\bar{V}^{\alpha}_{ml}(\bar{w}, \bar{z}) \, |\bar{u}\rangle_l = \sum_{r \in \mathbb{N}_0} \bar{z}^{r + \bar{\Delta}_m - \bar{\Delta}_l - \bar{\Delta}_l} |\bar{\chi}_r\rangle \ , \tag{2.9}$$

where $|\bar{\chi}_r\rangle \in \overline{\mathcal{H}}_m$ has conformal weight $\bar{\Delta}_m + r$. However, this already implies that (2.8) is satisfied, as I project onto the lowest energy states by taking the limit $\bar{z} \to 0$.

This definition determines well-defined associative operators on $\mathcal{H}_{\text{chir}}$. These operators are in one-to-one correspondence with states in $\mathcal{H}_{\text{chir}}$. They also satisfy certain braid relations and possess a (modified) duality property, as shall be shown in Sect. 4.

It cannot surprise that the properly defined chiral vertex operators depend on these internal parameter spaces. To see this, recall that the WZW-model can be understood as the quantisation of the classical field theory with action

$$\mathcal{S}[g] = k \left[ -\frac{1}{16\pi} \int_{\mathcal{M}} \text{tr}(g^{-1} \partial_\mu g \, g^{-1} \partial^\mu g) \, d^2 x \right.$$
$$\left. + \frac{1}{24\pi} \int_{\mathcal{B}} \varepsilon^{\lambda \mu \nu} \text{tr}(g^{-1} \partial_\lambda g \, g^{-1} \partial_\mu g \, g^{-1} \partial_\nu g) \, d^3 x \right] , \tag{2.10}$$

where $g : \mathcal{M} \to G$ and $\mathcal{M} = S^1 \times \mathbb{R}$, $\mathcal{B}$ is a three-dimensional manifold with $\partial \mathcal{B} = \mathcal{M}$ [5]. Writing $x^{\pm} = t \pm x$, the equations of motion are

$$\partial_-(\partial_+ g g^{-1}) = \partial_+(g^{-1} \partial_- g) = 0 \ , \tag{2.11}$$

which implies that $g(x, t)$ factors into a left- and right-moving part

$$g(x, t) = u(x^+) v(x^-) \ . \tag{2.12}$$

However, this decomposition is only determined up to a constant group element, as one can redefine

$$u \to uh, \qquad v \to h^{-1}v, \tag{2.13}$$

where $h \in G$ is arbitrary. Thus, in order to be able to reconstruct the whole theory from its chiral components, one has to keep track of this ambiguity when defining the chiral theory. Quasi-classically, one therefore expects that one has to retain internal parameter spaces in the chiral theory, which are representations of the Lie algebra $g$. The lowest energy vectors form indeed such spaces and the tensor product of two such representations is just the usual $g$-tensor product, as the comultiplication of the affine algebra [20], restricted to the horizontal Lie algebra, is trivial.

In the (proper) quantum theory one retains the same degrees of freedom. However, these spaces become "quantised," as they should now be regarded as representation spaces of the quasitriangular quasi-Hopf algebra of Drinfel'd $A_{g,t}$ [13]. This is due to the fact that the vertex operators do not commute for finite $k$, but only satisfy braid relations. I shall explain this in more detail in Sect. 4, after I have recalled the definition of this quasi-quantum group. For the moment I only want to point out that this quasi-Hopf algebra is (up to twisting) the unique quantisation of the universal enveloping algebra $U(g)$ [13] and that its appearance here is a typical quantum effect, as only the lowest energy states are involved.

## 3. The Quasitriangular Quasi-Hopf Algebra $A_{g,t}$

As an algebra, $A_{g,t}$ is the universal enveloping algebra $U(g)$ of $g$. The comultiplication is defined to be

$$\Delta(a) = a \otimes \mathbb{1} + \mathbb{1} \otimes a, \tag{3.1}$$

the counit $\varepsilon(a) = 0$, $\varepsilon(\mathbb{1}) = 1$ and the antipode

$$S(a) = -a. \tag{3.2}$$

The $R$-matrix is given by

$$R = \Delta(q^C)(q^{-C} \otimes q^{-C}) = e^{\frac{ht}{2}}, \tag{3.3}$$

where $C$ is the quadratic Casimir of $g$

$$C = \sum_a \tau^a \tau^a, \tag{3.4}$$

and $t$ is twice the split Casimir, namely

$$t = \Delta(C) - C \otimes \mathbb{1} - \mathbb{1} \otimes C, \tag{3.5}$$

which satisfies $m(t) = 2C$, where $m : U(g) \times U(g) \to U(g)$ is the multiplication map. $q$ is a complex number and will turn out to be

$$q = e^{\frac{h}{2}} = e^{\frac{\pi i}{k+h^*}}, \tag{3.6}$$

i.e. $h = \frac{2\pi i}{k+h^*}$, where $h^*$ is the dual Coxeter number of $g$. I have chosen the normalisation $\mathrm{Tr}\, \tau^a \tau^b = \frac{1}{2}\delta^{ab}$, and hence the Casimir (3.4) is half the Casimir operator of Drinfel'd [12].

$A_{g,t}$ is not a quantum group, but only a quasi-quantum group. By this one means that there exists an invertible element $\Phi \in U(g) \otimes U(g) \otimes U(g)$, $\Phi \neq \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$, such that

$$(id \otimes \varDelta) \circ \varDelta(a) = \Phi(\varDelta \otimes id) \circ \varDelta(a)\Phi^{-1} \tag{3.7}$$

for all $a \in U(g)$. At first sight one might think that $A_{g,t}$ is in fact a quantum group, as the comultiplication is obviously coassociative and thus one might choose $\Phi = \mathbb{1}$. However, the above $R$-matrix (3.3) does not satisfy the consistency condition necessary for an ordinary quantum group, namely

$$(\varDelta \otimes id)(R) = R_{13} R_{23}, \qquad (id \otimes \varDelta)(R) = R_{13} R_{12}, \tag{3.8}$$

but only the weaker condition

$$(\varDelta \otimes id)(R) = \Phi_{312} R_{13} (\Phi_{132})^{-1} R_{23} \Phi, \tag{3.9}$$

$$(id \otimes \varDelta)(R) = (\Phi_{231})^{-1} R_{13} \Phi_{213} R_{12} (\Phi)^{-1}. \tag{3.10}$$

In order for the above to be a well-defined quasitriangular quasi-Hopf algebra, these maps must in addition satisfy a number of consistency conditions. First of all, $R$ must be an $R$-matrix, i.e.

$$\varDelta'(a) = R \varDelta(a) R^{-1}, \tag{3.11}$$

where $\varDelta' = \varDelta$ is the twisted comultiplication. Secondly, the counit must satisfy

$$(\varepsilon \otimes id) \circ \varDelta = id = (id \otimes \varepsilon) \circ \varDelta, \tag{3.12}$$

and $\Phi$ must obey

$$(id \otimes id \otimes \varDelta)(\Phi)(\varDelta \otimes id \otimes id)(\Phi) = (\mathbb{1} \otimes \Phi)(id \otimes \varDelta \otimes id)(\Phi)(\Phi \otimes \mathbb{1}), \tag{3.13}$$

$$m((id \otimes \varepsilon \otimes id)(\Phi)) = \mathbb{1}. \tag{3.14}$$

Finally, in order to be a Hopf-algebra, the antipode must have the following properties: there exist algebra elements $\alpha$ and $\beta$, such that

$$\sum S(\varDelta^{(1)}(a)) \alpha \varDelta^{(2)}(a) = \varepsilon(a)\alpha, \qquad \sum \varDelta^{(1)}(a) \beta S(\varDelta^{(2)}(a)) = \varepsilon(a)\beta, \tag{3.15}$$

where

$$\varDelta(a) = \sum \varDelta^{(1)}(a) \otimes \varDelta^{(2)}(a). \tag{3.16}$$

Furthermore, writing

$$\Phi = \sum \Phi_{(1)} \otimes \Phi_{(2)} \otimes \Phi_{(3)}, \qquad \Phi^{-1} = \sum \Phi_{(1)}^{-1} \otimes \Phi_{(2)}^{-1} \otimes \Phi_{(3)}^{-1}, \tag{3.17}$$

$S$, $\alpha$ and $\beta$ must satisfy

$$\sum \Phi_{(1)} \beta S(\Phi_{(2)}) \alpha \Phi_{(3)} = \mathbb{1}, \qquad \sum S(\Phi_{(1)}^{-1}) \alpha \Phi_{(2)}^{-1} \beta S(\Phi_{(3)}^{-1}) = \mathbb{1}. \tag{3.18}$$

I have given all these consistency conditions for completeness. In the following, I shall not be concerned about the antipode-properties, as these do not seem to play an important rôle in the present application. This is also justified by a theorem of Drinfel'd [13, Proposition 1.4], which – applied to the present situation – asserts, that there exist $\alpha$ and $\beta$ in $A_{g,t}$, such that $S$ defined by (3.2) satisfies (3.15, 3.18).

As far as I am aware of the literature there is no explicit formula for $\Phi$. However, as in [13], one can define $\Phi$ implicitly as follows. Let $\psi_i$, $i = 0, 1, 2, 3$ be four highest weight vectors in irreducible positive energy representations $\pi_i$, $i = 0, 1, 2, 3$ of $\hat{g}$. I want to consider the "chiral four point function"

$$W_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3; z_1, z_2, z_3) = \lim_{z_0 \to \infty} z_0^{2\Delta_0} \langle V(\psi_0, z_0) V(\psi_1, z_1) V(\psi_2, z_2) V(\psi_3, z_3) \rangle$$

$$= \langle \psi_0, V(\psi_1, z_1) V(\psi_2, z_2) V(\psi_3, z_3) \rangle . \tag{3.19}$$

(In the following I shall sometimes suppress the $\psi_i$- or the $z_i$-dependence of $W_{\psi_0}$ in order to stress which dependence I have primarily in mind.) The function $W_{\psi_0}(z_1, z_2, z_3)$ satisfies the Knizhnik–Zamolodchikov equation [29],

$$\frac{\partial}{\partial z_i} W_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3; z_1, z_2, z_3) = -\bar{h} \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} W_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3; z_1, z_2, z_3) ,$$

$$\tag{3.20}$$

where $\bar{h} = \frac{h}{2\pi i} = \frac{1}{k + h^*}$ and $t_{12} = t \otimes \mathbb{1}$, $t_{23} = \mathbb{1} \otimes t$ and similarly for $t_{13}$. (Here, $i \in \{1, 2, 3\}$ and the summation in (3.20) extends over $j = 1, 2, 3$. This equation follows from the KZ-equation for the four point function by letting $z_0 \to \infty$ as above.) These equations are consistent, as the curvature of the corresponding connection is zero [29, 13]. I can write the solution as

$$(z_1 - z_3)^{-\bar{h}(t_{12} + t_{13} + t_{23})} G(\psi_1 \otimes \psi_2 \otimes \psi_3; x) , \tag{3.21}$$

where $x$ is the anharmonic ratio

$$x = \frac{z_1 - z_2}{z_1 - z_3} . \tag{3.22}$$

Then $G$ has to satisfy the equation

$$\frac{d}{dx} G(\psi_1 \otimes \psi_2 \otimes \psi_3; x) = -\bar{h} \left( \frac{t_{12}}{x} + \frac{t_{23}}{x - 1} \right) G(\psi_1 \otimes \psi_2 \otimes \psi_3; x) . \tag{3.23}$$

As shall become clear from the analysis of Sect. 4 the different ways of bracketing a triple tensor product correspond to the different limits in which two of the three points involved are made to coincide. Thus, in particular, the bracketing $((\psi_1 \otimes \psi_2) \otimes \psi_3)$ corresponds to the limit in which $|z_1 - z_2| \ll |z_2 - z_3|$, i.e. to the limit $x \to 0$. In this limit, $W_{\psi_0}$ has the expansion

$$W_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3; z_i) \sim (z_1 - z_2)^{-\bar{h} t_{12}} (z_1 - z_3)^{-\bar{h}(t_{13} + t_{23})} W_{\psi_0}^1(\psi_1 \otimes \psi_2 \otimes \psi_3) ,$$

$$\tag{3.24}$$

and thus $G$ has the expansion

$$G(\psi_1 \otimes \psi_2 \otimes \psi_3; x) \sim x^{-\bar{h} t_{12}} W_{\psi_0}^1(\psi_1 \otimes \psi_2 \otimes \psi_3) . \tag{3.25}$$

Expanding the right-hand-side in terms of the irreducible subrepresentations which are contained in the tensor product $\psi_1 \otimes \psi_2$, it can be rewritten as

$$\sum_p x^{-\bar{h}(C_p - C_1 - C_2)} W_{\psi_0, p}^1(\psi_1 \otimes \psi_2 \otimes \psi_3) , \tag{3.26}$$

where $C_i$ is the value of the quadratic Casimir in the representation $\pi_i$.

Similarly, the bracketing $(\psi_1 \otimes (\psi_2 \otimes \psi_3))$ corresponds to $|z_2 - z_3| \ll |z_1 - z_3|$ and thus to $x \to 1$. In this limit we have

$$W_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3; z_i) \sim (z_2 - z_3)^{-\bar{h} t_{23}} (z_1 - z_3)^{-\bar{h}(t_{12} + t_{13})} W^2_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3),$$
(3.27)

and thus

$$G(\psi_1 \otimes \psi_2 \otimes \psi_3; x) \sim (1 - x)^{-\bar{h} t_{23}} W^2_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3).$$
(3.28)

Again, the right-hand-side can be rewritten as

$$\sum_r (1 - x)^{-\bar{h}(C_r - C_2 - C_3)} W^2_{\psi_0, r}(\psi_1 \otimes \psi_2 \otimes \psi_3),$$
(3.29)

where $r$ parametrises the irreducible subrepresentations in the tensor product $\psi_2 \otimes \psi_3$.

Obviously, there is some freedom in the normalisation of the four point functions. However, once we have chosen a normalisation for say $W^1_{\psi_0}$, $W^2_{\psi_0}$ is uniquely determined by analytically continuing the solution corresponding to $W^1_{\psi_0}$. In particular, because of (3.20) and (3.23), we can find an invertible element

$$\Phi \in U(g) \otimes U(g) \otimes U(g)$$
(3.30)

such that

$$W^1_{\psi_0}(\psi_1 \otimes \psi_2 \otimes \psi_3) = W^2_{\psi_0}(\Phi(\psi_1 \otimes \psi_2 \otimes \psi_3))$$
(3.31)

independent of $\psi_0$ and $z_i$. (Formally, one can write $\Phi$ as an ordered exponential in the Lie algebra generators, integrating up (3.23).) $\Phi$ is invertible and it can be shown – using conformal field theory arguments – that it satisfies all of the above consistency conditions [13].

I would like to point out that for generic $q$ the matrix elements of $\Phi$ can be expressed in terms of the coupling constants of the theory. I shall use this fact implicitly in Sect. 5.


## 4. The Rôle as Internal Symmetry

In Sect. 2, I have shown that the (holomorphic) chiral theory corresponding to a WZW-model is given as

$$\mathscr{H}_{\text{chir}} = \bigoplus_l \mathscr{H}_l \otimes \overline{W}_l.$$
(4.1)

The chiral algebra $\mathscr{A} = \hat{g}$ acts on $\mathscr{H}_{\text{chir}}$ as

$$a \mapsto a \otimes \mathbb{1}$$
(4.2)

and the vacuum representation is the summand in $\mathscr{H}_{\text{chir}}$ corresponding to

$$\mathscr{A}\Omega \otimes \overline{\Omega}.$$
(4.3)

I would like to remark that the whole vertex operators corresponding to this representation contain the holomorphic currents

$$J^a(z) = \phi(J^a_{-1}\Omega \otimes \overline{\Omega}; z, \bar{z}),$$
(4.4)

as the right-hand-side is in fact independent of $\bar{z}$. Thus the holomorphic currents do belong to the (holomorphic) chiral theory.

There is a natural action of the quasi-quantum group $A_{g,t}$ on this Hilbert space, given by

$$\bar{a} \mapsto \mathbb{1} \otimes \bar{a}, \tag{4.5}$$

where $\bar{a}$ is regarded as an element of the horizontal Lie algebra $g$ of the affine algebra $\mathscr{A} = \hat{g}$. As mentioned before, the comultiplication formula [20] restricted to this subalgebra is trivial and thus agrees with (3.1). Furthermore, the braiding of the chiral vertex operators is described by the $R$-matrix of the quasi-quantum group and thus the universal enveloping algebra of the horizontal Lie algebra is $A_{g,t}$ rather than $U(g)$. To explain this in more detail, recall that the whole vertex operators commute, i.e.

$$\phi(\psi \otimes \overline{\psi}; z, \bar{z}) \, \phi(\chi \otimes \overline{\chi}; \zeta, \bar{\zeta}) = \phi(\chi \otimes \overline{\chi}; \zeta, \bar{\zeta}) \, \phi(\psi \otimes \overline{\psi}; z, \bar{z}) \tag{4.6}$$

upon analytic continuation of $z$ and $\zeta$, and correspondingly "anti-analytic continuation" of $\bar{z}$ and $\bar{\zeta}$. [2] Since I have explicitly constructed chiral vertex operators as a certain limit of whole vertex operators, I can calculate the braiding as follows:

**Theorem 1.** *Upon anticlockwise analytic continuation of the left-hand side one has*

$$V(\psi \otimes \bar{v}, z) V(\chi \otimes \bar{w}, \zeta)(\Omega \otimes \overline{\Omega}) = \sum_{\bar{w}', \bar{v}'} R^{\bar{v}\,\bar{w}}_{\bar{w}'\,\bar{v}'} \, V(\chi \otimes \bar{w}', \zeta) V(\psi \otimes \bar{v}', z)(\Omega \otimes \overline{\Omega}), \tag{4.7}$$

*where $R^{\bar{v}\,\bar{w}}_{\bar{w}'\,\bar{v}'}$ is the matrix element of the $R$-matrix (3.3) with $q = e^{\frac{\pi i}{k+h^*}}$ and the sum extends over a basis of the corresponding finite dimensional internal parameter spaces.*

*Proof.* To calculate the braiding of the analytic continuation, consider the scalar product with $(\varphi \otimes \bar{u}) \in \mathscr{H}_l \otimes \overline{W}_l$. The correlation function of the corresponding whole vertex operators satisfies

$$\langle (\varphi \otimes \bar{u}), \phi(\psi \otimes \bar{v}; z, \bar{z}) \, \phi(\chi \otimes \bar{w}; \zeta, \bar{\zeta})(\Omega \otimes \overline{\Omega}) \rangle = (\bar{z} - \bar{\zeta})^{\bar{\Delta}_l - \bar{\Delta}_{\bar{v}} - \bar{\Delta}_{\bar{w}}} F_{\psi \chi}(z, \zeta). \tag{4.8}$$

Upon anticlockwise analytic continuation of $z$ around $\zeta$ (which corresponds to clockwise analytic continuation of $\bar{z}$ around $\bar{\zeta}$) we obtain

$$e^{-\pi i (\bar{\Delta}_l - \bar{\Delta}_{\bar{v}} - \bar{\Delta}_{\bar{w}})} (\bar{\zeta} - \bar{z})^{\bar{\Delta}_l - \bar{\Delta}_{\bar{v}} - \bar{\Delta}_{\bar{w}}} \tilde{F}_{\psi \chi}(z, \zeta), \tag{4.9}$$

which must equal

$$(\bar{\zeta} - \bar{z})^{\bar{\Delta}_l - \bar{\Delta}_{\bar{v}} - \bar{\Delta}_{\bar{w}}} F_{\chi \psi}(\zeta, z), \tag{4.10}$$

as the whole vertex operators satisfy (4.6).

We obtain the correlation function of the corresponding chiral vertex operators by multiplying the above expression with $(\bar{\zeta} - \bar{z})^{-(\bar{\Delta}_l - \bar{\Delta}_{\bar{v}} - \bar{\Delta}_{\bar{w}})}$—we do not have to take the limit $\bar{\zeta} \to \bar{z}$, as we have already projected onto the lowest $\bar{L}_0$-eigenspace by taking the scalar product. Thus the summand in the operator product

---

[2] Because of (2.3) analytic and anti-analytic continuation is a well-defined concept for correlation functions. The above assertion means that this is true for all correlation functions.

expansion (of the two chiral vertex operators) corresponding to $\mathcal{H}_l \otimes \overline{W}_l$ exhibits the braiding phase

$$e^{\pi i (\bar{\Delta}_l - \bar{\Delta}_{\bar{v}} - \bar{\Delta}_{\bar{w}})} . \tag{4.11}$$

However, because of [20], the fusion of $\overline{W}_{\bar{v}}$ and $\overline{W}_{\bar{w}}$ is just the ordinary tensor product, since the comultiplication is trivial. Thus, to pick out the summand $\mathcal{H}_l \otimes \overline{W}_l$, we only have to decompose the tensor product of $\overline{W}_{\bar{v}}$ and $\overline{W}_{\bar{w}}$ into the direct sum of irreducible representations and project onto the summand $\overline{W}_l$. However, by construction of the $R$-matrix, the matrix element of $R$ in this subspace is exactly (4.11) and thus the theorem is proved. □

I would like to remark that the tensor product of $\overline{W}_{\bar{v}}$ and $\overline{W}_{\bar{w}}$ contains in general irreducible representations, which do not appear in the tensor product of the corresponding current algebra because of truncation. If $\overline{W}_l$ is such a representation, $F_{\psi, \chi}$ in (4.8) is identically zero, as the $\phi$'s are vertex operators of a well-defined conformal field theory. Hence the operator product expansion of the two chiral vertex operators does not contain the corresponding conformal family and in particular the three-point-function of both sides of (4.7) vanishes identically. This also implies that the $R$-matrix given above is in general not uniquely determined.

The above theorem describes the braiding in a rather special case, namely when the product of vertex operators acts on the vacuum. The general case can be derived from this special case, once the duality properties of the chiral theory have been established. In general, as we shall see, the chiral theory does not satisfy the duality property of the whole theory

$$\phi(\psi \otimes \bar{v}; z, \bar{z}) \, \phi(\chi \otimes \bar{w}; \zeta, \bar{\zeta}) = \phi(\phi(\psi \otimes \bar{v}; z - \zeta, \bar{z} - \bar{\zeta})(\chi \otimes \bar{w}); \zeta, \bar{\zeta}) , \tag{4.12}$$

but only a modified version. (This is due to the fact that the different limit procedures (in the definition of the chiral vertex operators) do not commute.) In particular, this implies, that the braiding is not really local. Indeed, the braiding of two chiral vertex operators will turn out to depend also on the state the product of vertex operators is acting on (4.20).

The fact that the duality property has to be modified is intimately related to the fact that the $R$-matrix given above does not satisfy the quantum group consistency condition (3.8), but only the weaker conditions (3.9, 3.10). To complete my argument that the universal enveloping algebra of the horizontal Lie algebra is really $A_{g,t}$, I have to show that the chiral vertex operators satisfy a weaker duality property related to $\Phi$.

**Theorem 2.**

$$V \left( V(\psi \otimes \bar{v}, z - \zeta)(\chi \otimes \bar{w}), \zeta \right)(\varphi \otimes \bar{u})$$
$$= \sum_{\bar{v}', \bar{w}', \bar{u}'} \Phi^{\bar{v} \, \bar{w} \, \bar{u}}_{\bar{v}' \, \bar{w}' \, \bar{u}'} \, V(\psi \otimes \bar{v}', z) V(\chi \otimes \bar{w}', \zeta)(\varphi \otimes \bar{u}') , \tag{4.13}$$

*where $\Phi^{\bar{v} \, \bar{w} \, \bar{u}}_{\bar{v}' \, \bar{w}' \, \bar{u}'}$ are the matrix elements of $\Phi$ and the sum extends over a basis of the finite dimensional internal parameter spaces.*

*Proof.* As above it is sufficient to consider the case, where we take the scalar product with $(\omega \otimes \bar{x}) \in \mathcal{H}_l \otimes \overline{W}_l$. For the corresponding whole vertex operators we have

$$\langle (\omega \otimes \bar{x}), \phi(\phi(\psi \otimes \bar{v}; z - \zeta, \bar{z} - \bar{\zeta})(\chi \otimes \bar{w}); \zeta, \bar{\zeta})(\varphi \otimes \bar{u}) \rangle$$
$$= \sum_p T_p(\omega, \psi, \chi, \varphi; z, \zeta) \widehat{W}^1_{\bar{x}, \bar{p}}(\bar{v} \otimes \bar{w} \otimes \bar{u}; \bar{z}, \bar{\zeta}, 0) , \tag{4.14}$$

and

$$\langle(\omega \otimes \bar{x}), \phi(\psi \otimes \bar{v}; z, \bar{z})\phi(\chi \otimes \bar{w}; \zeta, \bar{\zeta})(\varphi \otimes \bar{u})\rangle$$
$$= \sum_r S_r(\omega, \psi, \chi, \varphi; z, \zeta)\widehat{W}^2_{\bar{x}, \bar{r}}(\bar{v} \otimes \bar{w} \otimes \bar{u}; \bar{z}, \bar{\zeta}, 0), \qquad (4.15)$$

where the indices $p, \bar{p}$ and $r, \bar{r}$ indicate the different conformal families, which contribute in the $t$- and $s$-channel, respectively. Equation (4.14) is defined for $|\zeta| > |z - \zeta|$, (4.15) for $|z| > |\zeta|$ and the right-hand sides of (4.14) and (4.15) are the same analytic function in the four (independent) variables $z, \zeta, \bar{z}$ and $\bar{\zeta}$. The "equality" of (4.14) and (4.15) is the duality property of the whole theory (4.12), which can be established from first principles using locality [22].

To obtain the left-hand-side of (4.13) we have to take the limit $(\bar{z} - \bar{\zeta}) \to 0$ and $\bar{\zeta} \to 0$ in (4.14). Using the above equality this is the same as

$$\sum_p T_p(\omega, \psi, \chi, \varphi; z, \zeta) \lim_{\bar{\zeta} \to 0} \bar{\zeta}^{\Delta_{\bar{u}} + \Delta_{\bar{p}} - \Delta_v} \lim_{\bar{z} \to \bar{\zeta}} (\bar{z} - \bar{\zeta})^{\Delta_{\bar{v}} + \Delta_{\bar{w}} - \Delta_{\bar{p}}} \widehat{W}^1_{\bar{x}, \bar{p}}(\bar{z}, \bar{\zeta}, 0). \qquad (4.16)$$

In the notation of Sect. 3 we obtain thus

$$\sum_p T_p(\omega, \psi, \chi, \varphi; z, \zeta) W^1_{\bar{x}, \bar{p}}(\bar{v} \otimes \bar{w} \otimes \bar{u}). \qquad (4.17)$$

The right-hand-side of (4.13) is obtained by taking the limit $\bar{\zeta} \to 0$ and $\bar{z} \to 0$ in (4.15). Using again the above equality, this is the same as

$$\sum_r S_r(\omega, \psi, \chi, \varphi; z, \zeta) \lim_{\bar{z} \to 0} \bar{z}^{\Delta_{\bar{v}} + \Delta_{\bar{r}} - \Delta_{\bar{x}}} \lim_{\bar{\zeta} \to 0} \bar{\zeta}^{\Delta_{\bar{w}} + \Delta_{\bar{u}} - \Delta_{\bar{r}}} \widehat{W}^2_{\bar{x}, \bar{r}}(\bar{z}, \bar{\zeta}, 0), \qquad (4.18)$$

and thus in the notation of Sect. 3 equals

$$\sum_r S_r(\omega, \psi, \chi, \varphi; z, \zeta) W^2_{\bar{x}, \bar{r}}(\bar{v} \otimes \bar{w} \otimes \bar{u}). \qquad (4.19)$$

For fixed $z$ and $\zeta$, the function (4.14) and (4.15) satisfies the KZ-equation for $\bar{z}$ and $\bar{\zeta}$. Therefore, the two expressions (4.17) and (4.19) are related as in Sect. 3, and thus, by definition of $\Phi$, the theorem holds.                                    $\square$

Putting the information of Theorem 1 and Theorem 2 together, we can now determine the braiding of two chiral vertex operators in the general case. One easily finds that the anticlockwise analytic continuation of $z$ around $\zeta$ on the left-hand-side equals

$$V(\psi \otimes \bar{v}, z)V(\chi \otimes \bar{w}, \zeta)(\varphi \otimes \bar{u}) = \sum_{\bar{w}', \bar{v}', \bar{u}'} (\Phi_{213}R_{12}\Phi^{-1})^{\bar{v}\bar{w}\bar{u}}_{\bar{v}'\bar{w}'\bar{u}'} V(\chi \otimes \bar{w}', z)$$
$$\times V(\psi \otimes \bar{v}', \zeta)(\varphi \otimes \bar{u}'), \qquad (4.20)$$

where $(\Phi_{213}R_{12}\Phi^{-1})^{\bar{v}\bar{w}\bar{u}}_{\bar{v}'\bar{w}'\bar{u}'}$ is the matrix element of the composition of the three maps. Thus in particular the braiding of two chiral vertex operators also depends on the state the operator product is acting on.

I have thus shown that the universal enveloping algebra of the anti-holomorphic horizontal Lie algebra forms indeed $A_{g,t}$. Its physical significance is that it plays the

rôle of an internal symmetry for the chiral WZW-theory, as it satisfies the following properties:

it commutes with the chiral algebra $\mathscr{A} = \hat{g}$, which can be interpreted to be some sort of observable algebra,

it annihilates the vacuum $\Omega \otimes \overline{\Omega}$, as $\bar{a}\,\overline{\Omega} = 0$ for all $\bar{a} \in g \subset \overline{\mathscr{A}}$,

as a consequence of the comultiplication formula for the horizontal subalgebra [20] chiral vertex operators transform covariantly under $A_{g,t}$, i.e.

$$\bar{a}\,V(\psi \otimes \bar{w}, z) = V(\psi \otimes \bar{w}, z)\,\bar{a} + V(\psi \otimes \bar{a}\bar{w}, z) \quad \text{for all } \bar{a} \in g \subset \overline{\mathscr{A}}, \quad (4.21)$$

the braiding (Theorem 1) and the duality (Theorem 2) of the chiral vertex operators are described by the $R$-matrix and $\Phi$ of $A_{g,t}$, respectively.

The quasi-Hopf algebra is not coassociative and therefore the chiral vertex operators only satisfy a modified duality property. For generic $q$, the duality property of the chiral vertex operators can be restored, using the isomorphism between $A_{g,t}$ and the coassociative quantum group $U_q(g)$. However, for the physically relevant case of integer $k$ this construction has to be modified. In particular, the duality property of the chiral theory cannot be completely restored, as the quantum group has to be replaced by the corresponding truncated quasi-quantum group. This seems to indicate that a chiral theory possesses in general only a weaker version of duality.

## 5. The Quantum Group $U_q(g)$

It has been shown by Drinfel'd [12], that the quasi-quantum group defined above and the (well-known) quantum group $U_q(g)$ are isomorphic for generic $q$, where $h$ and $q$ are related as in Sect. 2,

$$q = e^{h/2} = e^{\frac{\pi t}{k+h^*}} . \quad (5.1)$$

Here I choose the conventions as in [33], i.e.

$$[H_i, X_{\pm j}] = \pm(\alpha_i, \alpha_j)X_{\pm j}, \qquad [X_{+i}, X_{-j}] = \delta_{ij}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \quad (5.2)$$

and

$$\Delta_q(H_i) = H_i \otimes \mathbb{1} + \mathbb{1} \otimes H_i, \qquad \Delta_q(X_{\pm i}) = X_{\pm i} \otimes q^{\frac{H_i}{2}} + q^{-\frac{H_i}{2}} \otimes X_{\pm i}. \quad (5.3)$$

By this I mean that there exists an invertible map $\varphi : U_q(g) \to A_{g,t}$, $\varphi = id(\mathrm{mod}\,h)$ and an invertible element $F \in U(g) \otimes U(g)$, $F = \mathbb{1} \otimes \mathbb{1}$ (mod $h$), such that

$$F\,\Delta(\varphi(a)) = (\varphi \otimes \varphi)\Delta_q(a)\,F \quad (5.4)$$

for all $a \in U_q(g)$. Furthermore the $R$-matrices are related by

$$(\varphi \otimes \varphi)(R_q) = F'\,R\,F^{-1}, \quad (5.5)$$

where $F' = F_{21}$ in the usual notation, and

$$\Phi_q = (\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}) = (\mathbb{1} \otimes F)(id \otimes \Delta)(F)\,\Phi((\Delta \otimes id)(F))^{-1}(F \otimes \mathbb{1})^{-1}. \quad (5.6)$$

In particular this implies that the quasi-quantum group with non-trivial $\Phi$ is in fact isomorphic to a quantum group with $\Phi_q = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$.

To be more specific I want to construct this isomorphism explicitly for the case of $su(2)$. I define the map (analogously to [6])

$$\varphi : U_q(su(2)) \to A_{su(2),t} , \tag{5.7}$$

by

$$\varphi(H) = H, \qquad \varphi(X_\pm) = X_\pm P_\pm , \tag{5.8}$$

where

$$P_\pm = \sqrt{\frac{[j \mp m]_q \, [j \pm m + 1]_q}{(j \mp m)(j \pm m + 1)}} , \tag{5.9}$$

and

$$[p]_q := \frac{q^p - q^{-p}}{q - q^{-1}} . \tag{5.10}$$

Here, $j(j+1)$ is the eigenvalue of the quadratic Casimir

$$C := \frac{1}{4}H^2 + \frac{1}{2}(X_+X_- + X_-X_+) , \tag{5.11}$$

and $m$ is the eigenvalue of $\frac{1}{2}H$. Thus I can express $j$ and $m$ in terms of elements of $A_{su(2),t}$. This map is well-defined for arbitrary $q$. For generic $q$ it is also invertible.

Next I want to relate the action of the two algebras on tensor products. To this end I define the map $\widetilde{F}$

$$\widetilde{F} : V_{j_1} \otimes V_{j_2} \to V_{j_1} \otimes_q V_{j_2} \tag{5.12}$$

by

$$\widetilde{F}(|j_1, m_1\rangle \otimes |j_2, m_2\rangle) := \sum_{m_1', m_2'} \sum_{J} (d_{j_1, j_2}^J)^{-1} \langle j_1 j_2 m_1' m_2' | J, m_1 + m_2 \rangle_q$$
$$\times \langle j_1 j_2 m_1 m_2 | J, m_1 + m_2 \rangle |j_1, m_1'\rangle \otimes_q |j_2, m_2'\rangle , \tag{5.13}$$

where $d_{j_1, j_2}^J$ are constants to be specified shortly, $\langle j_1 j_2 m_1 m_2 | J, M \rangle$ and $\langle j_1 j_2 m_1 m_2 | J, M \rangle_q$ are the Clebsch–Gordon coefficients of $su(2)$ and $U_q(su(2))$, respectively, and the subscript $q$ of the tensor product indicates that it is to be regarded as a representation of the quantum group via the action $(\varphi \otimes \varphi)\Delta_q(a)$. Then, by construction, $\widetilde{F}$ satisfies

$$\widetilde{F} \Delta(\varphi(a)) = (\varphi \otimes \varphi)\Delta_q(a) \widetilde{F} . \tag{5.14}$$

Thus, I can use the Racah formula [34] for the Clebsch–Gordon coefficients of $su(2)$ and a similar formula for $U_q(su(2))$ [28] to rewrite (for generic $q$) $\widetilde{F}$ as $F \in U(g) \otimes U(g)$. $F$ satisfies then by construction (5.4) and (5.5). Furthermore, $F$ is the identity modulo $h$, if $d_{j_1, j_2}^J \to 1$ as $k \to \infty$.

A priori it is not clear whether $F$ satisfies (5.6). However, by Schur's lemma, $F$ can differ from the invertible element relating $A_{su(2),t}$ and $U_q(su(2))$ (for generic $q$) at most by a scalar function of $q$ for each irreducible subrepresentation in the tensor product of two (irreducible) representations. I can therefore choose the "coupling constants" $d_{j_1, j_2}^J$ so that $F$ satisfies (5.6).

To fix these constants explicitly, I shall use results obtained from the Coulomb gas representation of the $su(2)$ conformal field theory [10, 3, 14, 8, 9]. In particular, the coupling constants for the whole theory turn out to be [9]

$$D^{j_3,m_3,\overline{m}_3}_{j_1,m_1,\overline{m}_1;j_2,m_2,\overline{m}_2} = \sqrt{\frac{(2j_1+1)(2j_2+1)}{(2j_3+1)}} \langle j_1 j_2 m_1 m_2 | j_3,m_3 \rangle \langle j_1 j_2 \overline{m}_1 \overline{m}_2 | j_3,\overline{m}_3 \rangle d^{j_3}_{j_1,j_2},$$

(5.15)

where

$$d^{j_3}_{j_1,j_2} = \frac{a_{j_1} a_{j_2}}{a_{j_3}} \prod_{l=1}^{j_1+j_2-j_3} \frac{\Gamma(\frac{l}{k+2})}{\Gamma(-\frac{l}{k+2})} \prod_{l=0}^{j_1+j_2-j_3-1} \frac{\Gamma(-\frac{2j_1-l}{k+2})\Gamma(-\frac{2j_2-l}{k+2})\Gamma(\frac{2j_3+2+l}{k+2})}{\Gamma(\frac{2j_1-l}{k+2})\Gamma(\frac{2j_2-l}{k+2})\Gamma(-\frac{2j_3+2+l}{k+2})}$$

(5.16)

and

$$a_j = \left[ \prod_{l=1}^{2j} \frac{\Gamma(\frac{l}{k+2})\Gamma(-\frac{1+l}{k+2})}{\Gamma(-\frac{l}{k+2})\Gamma(\frac{1+l}{k+2})} \right]^{\frac{1}{2}}.$$

(5.17)

The square root in (5.15) is a consequence of the normalisation convention of [11]. $d^{j_3}_{j_1,j_2}$ is the $k$-dependent part and tends to 1 as $k \to \infty$.

Using the explicit expression for the coupling constants of the whole theory, we can calculate the analytic continuation of the left-hand-side of (4.13) and thus determine the matrix elements of $\Phi$. This fixes the constants $d^J_{j_1,j_2}$ in (5.13) to be equal to (5.16).

I would like to point out that the results of the Coulomb gas representation have only been derived for generic (irrational) $k$. On the other hand, the matrix elements of the $R$-matrix and the $\Phi$-map of the quasi-quantum group $A_{g,t}$ depend continuously on $k$. We shall use this argument below to show that the formula for the constant $d^J_{j_1,j_2}$ (for suitably restricted $j_1, j_2$ and $J$) extends to the case of a root of unity.

The above formula for $\varphi$ is only invertible and $\widetilde{F}$ is only well defined on all tensor products if $q$ is generic. The breakdown of the formulae at a root of unity is mirrored by the fact that the quantum group ceases to be semisimple at roots of unity, as not all representations of the quantum group are completely reducible. Thus, at a root of unity, the symmetry algebra will not be the original quantum group, but only some modification.

In fact, at a root of unity, the quantum group has to be replaced by its truncated version. By this I mean that one restricts the action of the quantum group to the so-called physical representations and considers only the projection of the tensor product onto its completely decomposable part. This truncated version of the quantum group has been studied in [31], where it was called the canonically associated "weak quasi-triangular quasi-Hopf algebra." It is important to note that the truncation breaks the coassociativity and thus that the resulting algebra is only a *quasi*-Hopf algebra. However, the corresponding $\Phi$-map can be easily determined (a formula is given in [31]) and it is trivial on all triple tensor products, which do not exhibit any truncation[3].

The physical representations are those which correspond to the unitary positive energy representations of the affine algebra at level $x = 2k/\psi^2$. For $su(2)$, $x = k$

---

[3] It should be noted that the coassociator $\Phi$ of a weak quasi-quantum group is not invertible, but rather possesses only a quasi-inverse. The same applies to the $R$ matrix.

and the unitary positive energy representations are characterised by $j \leqq k/2$. It is easy to see that $\varphi$ is indeed invertible on these representations.

To project the tensor product of physical representations onto its completely decomposable part one restricts the sum over $J$ in (5.13) to $J \leqq k - j_1 - j_2$. This truncation is necessary to give a well-defined meaning to $\widetilde{F}$, since $[j_1 + j_2 + J + 1]_q! = 0$ for $j_1 + j_2 + J + 1 \geqq k + 2$ and the Clebsch–Gordon coefficient for $U_q(su(2))$ becomes singular. The truncation is precisely the truncation of the fusion rules of the corresponding WZW-model [44, 19, 18].[4] On this truncated tensor product the map $\widetilde{F}$ is invertible.

The modified maps do indeed relate the so truncated quasi-quantum group $A_{g,t}$ and the truncated quantum group. This follows from the fact that the matrix elements of $R$ and $\Phi$ and the map $\widetilde{F}$ depend continuously on $k$. Therefore on the truncated tensor product the $R$ matrix and the $\Phi$ map of the truncated quantum group agree with the right-hand sides of (5.5) and (5.6). In particular, the constants $d^J_{j_1, j_2}$ are still given by (5.16).

I can thus use these maps to regard the internal parameter spaces of the chiral vertex operators as representation spaces of the quantum group. By this I mean that I define new chiral vertex operators

$$V^q(\psi \otimes \bar{w}, z) : \mathscr{H}_{\mathrm{chir}} \to \mathscr{H}_{\mathrm{chir}} , \qquad (5.18)$$

whose action on a state in $\mathscr{H}_{\mathrm{chir}}$ is given as

$$V^q(\psi \otimes \bar{w}, z)(\chi \otimes \bar{v}) := \sum_{\bar{w}', \bar{v}'} (\widetilde{F}^{-1})^{\bar{w}, \bar{v}}_{\bar{w}', \bar{v}'} V(\psi \otimes \bar{w}', z)(\chi \otimes \bar{v}') , \qquad (5.19)$$

where $\widetilde{F}^{-1}$ is the inverse of $\widetilde{F}$ on the truncated tensor product and $V(\psi \otimes \bar{w}', z)$ is the original chiral vertex operator. Similarly, the original chiral vertex operators can be expressed in terms of the new chiral vertex operators using $\widetilde{F}$. It should be noted that the two descriptions are indeed in one-to-one correspondence as the Clebsch–Gordon series of the truncated tensor product and the fusion rules of the WZW-model agree.

All arguments of the previous section can be easily adapted. Thus, in particular, Theorem 1 and Theorem 2 hold, where $R$ and $\Phi$ are now the $R$-matrix and the $\Phi$-map of the truncated quantum group, and the covariance property (4.21) becomes

$$a V^q(m \otimes \bar{w}, z) = \sum V^q(m \otimes \Delta^{(1)}_q(a)\bar{w}, z)\Delta^{(2)}_q(a) \qquad (5.20)$$

for all $a \in U_q(g)$, where I have adapted the notation

$$\Delta_q(a) = \sum \Delta^{(1)}_q(a) \otimes \Delta^{(2)}_q(a) . \qquad (5.21)$$

This exhibits the truncated quantum group as the internal symmetry algebra of the WZW-model for integer $k$.

Having specified $\widetilde{F}$ explicitly, I can also calculate the braiding matrix for all vertex operators. As an example, let us consider the braiding matrix of two spin-$\frac{1}{2}$ vertex operators. In the usual notation for chiral vertex operators introduced in [36],

---

[4] For the case of $su(n)$, this follows already from the fact that there is only one associative fusion ring, which is a certain truncation of the Clebsch–Gordon series of the corresponding Lie algebra [26].

the braiding matrix is given for $0 < j < \frac{k}{2}$ as

$$
B_{p,p'} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ j & j \end{bmatrix} = \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix} R_{m,m'} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ j & j \end{bmatrix} \begin{pmatrix} d_+^{-1} & 0 \\ 0 & d_-^{-1} \end{pmatrix} ,
\tag{5.22}
$$

and for $j = 0$ or $j = \frac{k}{2}$ as

$$
B_{p,p'} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = -q^{-\frac{3}{2}} \delta_{p,p'} = B_{p,p'} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{k}{2} & \frac{k}{2} \end{bmatrix} .
\tag{5.23}
$$

Here

$$
R_{m,m'} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ j & j \end{bmatrix} = \begin{pmatrix} \dfrac{-q^{-(2j+\frac{3}{2})}}{[2j+1]_q} & \dfrac{\sqrt{q^{-1}[2j]_q[2j+2]_q}}{[2j+1]_q} \\[3mm] \dfrac{\sqrt{q^{-1}[2j]_q[2j+2]_q}}{[2j+1]_q} & \dfrac{q^{2j+\frac{1}{2}}}{[2j+1]_q} \end{pmatrix}
\tag{5.24}
$$

is the braiding matrix of $U_q(su(2))$ and

$$
d_\pm = d_{j,\frac{1}{2}}^{j\pm\frac{1}{2}} .
\tag{5.25}
$$

The diagonal matrices in (5.22) relate the coupling constants of the original chiral theory and the new chiral theory. The explicit form follows directly from (5.13). Equation (5.23) is just a phase, as only the vacuum representation in the tensor product of the two spin-$\frac{1}{2}$ representations contributes.

The expressions agree with the result given in [42]. This is immediate for (5.23). For the case of (5.22) this follows from

$$
\frac{d_+}{d_-} = \frac{\Gamma(-\frac{2j+1}{k+2})}{\Gamma(\frac{2j+1}{k+2})} \sqrt{\frac{\Gamma(\frac{2j}{k+2})\Gamma(\frac{2j+2}{k+2})}{\Gamma(-\frac{2j}{k+2})\Gamma(-\frac{2j+2}{k+2})}} = \frac{\gamma_-}{\gamma_+} ,
\tag{5.26}
$$

where $\gamma_\pm$ is defined in [42, p. 349].

## 6. Reconstruction

In Sect. 2, I explained how the chiral subtheory corresponding to a WZW conformal field theory can be constructed. The chiral Hilbert space, which was obtained in this way, is larger than the direct sum of the representations of the chiral algebra, and similarly – as there exists a correspondence between vertex operators and states – the chiral vertex operators depend on internal degrees of freedom.

In this section I want to describe how one can reconstruct the original conformal field theory from its chiral subtheory. I will show that there exists some sort of "identity operator" (related to the diagonal theory, i.e. to the theory where $\mathscr{H}_I$ is conjugate to $\overline{\mathscr{H}_I}$), the action of which on a chiral theory reconstructs the corresponding original theory. At first this might seem a bit surprising, as naively, the whole theory should be obtained by putting together the holomorphic and the anti-holomorphic theory. However, as I have defined the (holomorphic) chiral subtheory to contain the zero modes of the anti-holomorphic representation spaces, it already retains sufficient information about the original theory.

As a rough analogy this is similar to the process of reconstructing a component of a (multi-component) group from a single point is this component and the action of the identity component of the group.

To explain this in more detail, recall that for a given affine algebra there always exists a modular invariant diagonal theory, which is the unique WZW-theory corresponding to the simply connected group [21, 15, 16, 30]. This theory plays the rôle of the identity operator, and we denote the (anti-holomorphic) chiral theory corresponding to this theory as

$$\overline{\mathcal{H}}^0 = \bigoplus_m U_m \otimes \overline{\mathcal{H}}^0_m \,, \tag{6.1}$$

where the $U_m$ are representations spaces of the corresponding quantum group $U_q(g)$. By construction, the subspace of lowest energy vectors in $\overline{\mathcal{H}}^0_m$ is conjugate to $U_m$ as a representation of $U_q(g)$.

Consider now a (not necessarily diagonal) WZW-model corresponding to the same affine algebra $\hat{g}_k$. We denote the Hilbert space of this theory by

$$\mathcal{H} = \bigoplus_l \mathcal{H}_l \otimes \overline{\mathcal{H}}_l \,, \tag{6.2}$$

and the corresponding (holomorphic) chiral theory by

$$\mathcal{H}_{\text{chir}} = \bigoplus_l \mathcal{H}_l \otimes \overline{W}_l \,. \tag{6.3}$$

We want to reconstruct the original theory $\mathcal{H}$ from its chiral subtheory $\mathcal{H}_{\text{chir}}$. To this end we consider the product space

$$\begin{aligned}
\widetilde{\mathcal{H}} &:= \mathcal{H}_{\text{chir}} \otimes \overline{\mathcal{H}}^0 \\
&= \bigoplus_l \bigoplus_m \mathcal{H}_l \otimes \overline{W}_l \otimes U_m \otimes \overline{\mathcal{H}}^0_m \,,
\end{aligned} \tag{6.4}$$

on which there is a natural action of the quantum group $U_q(g)$, given by

$$U_q(g) \ni a \mapsto \sum \mathbb{1} \otimes \Delta_q^{(1)}(a) \otimes \Delta_q^{(2)}(a) \otimes \mathbb{1} \,, \tag{6.5}$$

where we have used the same notation as in (5.21). We can reconstruct the original Hilbert space $\mathcal{H}$ by restricting the product space $\widetilde{\mathcal{H}}$ to the subspace, which is invariant under the action of the quantum group $U_q(g)$ (6.5). It is easy to see that this space has the right size.

To reconstruct the original vertex operator corresponding to $\psi \otimes \overline{\psi} \in \mathcal{H}_j \otimes \overline{\mathcal{H}}_j$, we consider the tensor product of vertex operators

$$V^q(\psi \otimes \overline{w}, z) \otimes V^q(w \otimes \overline{\psi}, \overline{z}) \,, \tag{6.6}$$

where $\overline{w} \in \overline{W}_j$, $w \in U_{r(j)}$ and we regard $\overline{\psi}$ as an element of $\overline{\mathcal{H}}^0_{r(j)}$. Here, $r(j)$ is defined by the condition that $\overline{\mathcal{H}}^0_{r(j)}$ is the (unique) representation isomorphic to $\overline{\mathcal{H}}_j$. Then $\overline{W}_j$ and $U_{r(j)}$ are conjugate representations of the quantum group. These tensor products act naturally on $\widetilde{\mathcal{H}}$. Furthermore, there exists a unique (up to scalar multiple) linear combination of these tensor products of vertex operators, which

leaves the subspace $\mathscr{H}$ of $\widetilde{\mathscr{H}}$ invariant. (In fact, this linear combination corresponds to the unique vacuum vector in the tensor product of the two $U_q(g)$-representations.) We reconstruct the whole vertex operator by restricting the action of this linear combination to the subspace $\mathscr{H}$.

This determines the vertex operators up to scalar multiples, which can be absorbed into the normalisation of the fields. For generic $q$, the reconstructed theory satisfies the unmodified duality property, is local and therefore agrees with the original theory. This remains true in the truncated case. It should be noted that the construction preserves the one-to-one correspondence between states and vertex operators at all stages.


## 7. Conclusions

In this paper I have explained how a chiral theory with a proper Hilbert space formulation can be defined for the WZW conformal field theory. The Hilbert space of the (holomorphic) chiral theory is larger than the direct sum of the chiral representations spaces, the extra degrees of freedom being the lowest energy states of the anti-holomorphic representation spaces. I have shown that there is a natural action of the truncated quantum group $U_q(g)$ on these internal degrees of freedom, such that the truncated quantum group plays the rôle of an internal symmetry algebra for the chiral theory (for integer $k$). As a consequence of the truncation, the truncated quantum group is only quasi-coassociative and the chiral theory possesses only a modified duality property. The original theory can be recovered from the chiral theory by some sort of gauging procedure.

The construction of the internal symmetry algebra is explicit and explains the significance of the quantum group symmetry independent of any specific construction of the theory. It has furthermore the virtue – in contrast to some earlier attempts – of fixing the $q$-parameter in terms of the level of $\hat{g}$.

It should be possible to generalise the construction to arbitrary conformal field theories. In the general case the rôle of the internal parameter spaces is played by the "special subspaces," recently introduced by Nahm [37]. (For the WZW-models these spaces are just the lowest energy states). This should provide a constructive way of finding the internal symmetry of a conformal field theory; details remain to be worked out.

## References

1. Alvarez-Gaumé, L., Gomez, C., Sierra, G.: Quantum group interpretation of some conformal field theories. Phys. Lett. **B 220**, 142–152 (1989)
2. Alvarez-Gaumé, L., Gomez, C., Sierra, G.: Duality and quantum groups. Nucl. Phys. **B 330**, 347–398 (1990)
3. Christe, P., Flume, R.: The four-point correlations of all primary operators of the $d = 2$ conformally invariant $SU(2)$ $\sigma$-model with Wess–Zumino term. Nucl. Phys. **B 282**, 466–494 (1987)

4. Chu, M., Goddard, P.: Quantisation of the $SU(N)$ WZW model at level $k$. Cambridge preprint, DAMTP-94-42 (1994)

5. Chu, M., Goddard, P., Halliday, I., Olive, D., Schwimmer, A.: Quantisation of the Wess–Zumino–Witten model on a circle. Phys. Lett. **B 266**, 71–81 (1991)

6. Curtright, T.L., Zachos, C.K.: Deforming maps for quantum algebras. Phys. Lett. **B 243**, 237–244 (1990)

7. Doplicher, S., Roberts, J.E.: Why there is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics. Commun. Math. Phys. **131**, 51–107 (1990)

8. Dotsenko, V.S.: The free field representation of the $su(2)$ conformal field theory. Nucl. Phys. **B 338**, 747–758 (1990)

9. Dotsenko, V.S.: Solving the $su(2)$ conformal field theory using the Wakimoto free field representation. Nucl. Phys. **B 358**, 547–570 (1991)

10. Dotsenko, V.S., Fateev, V.A.: Conformal algebra and multipoint correlation functions in 2D statistical models. Nucl. Phys. **B 240** [FS12], 312–348 (1984)

11. Dotsenko, V.S., Fateev, V.A.: Operator algebra of two-dimensional conformal field theories with central charge $c \leqq 1$. Phys. Lett. **B 154**, 291–295 (1985)

12. Drinfel'd, V.G.: On almost cocommutative Hopf algebras. Len. Math. J. **1**, 2, 321–342 (1990) (Engl. translation)

13. Drinfel'd, V.G.: Quasi-Hopf algebras. Len. Math. J. **1**, 6, 1419–1457 (1990) (Engl. translation)

14. Fateev, V.A., Zamolodchikov, A.B.: Operator algebra and correlation functions in the two-dimensional $SU(2) \times SU(2)$ chiral Wess–Zumino model. Yad. Fiz. **43**, 1031–1044 [Engl. translation: Sov. J. Nucl. Phys. **43**, 657–664] (1986)

15. Felder, G., Gawedzki, K., Kupiainen, A.: The spectrum of Wess–Zumino–Witten models. Nucl. Phys. **B 299**, 355–366 (1988)

16. Felder, G., Gawedzki, K., Kupiainen, A.: Spectra of Wess–Zumino–Witten models with arbitrary simple groups. Commun. Math. Phys. **117**, 127–158 (1988)

17. Fuchs, J., Ganchev, A., Vecsernyés, P.: Towards a Classification of rational Hopf algebras. Preprint NIJHEF-H/94-05, KL-TH-94/4, hep-th/9402153; Fuchs, J., Ganchev, A., Vecsernyés, P.: On the Quantum Symmetry of rational field theories. Preprint KL-TH-94/14, hep-th/9407013

18. Fuchs, J., van Driel, P.: WZW fusion rules, quantum groups, and the modular matrix $S$. Nucl. Phys. **B 346**, 632–648 (1990)

19. Furlan, P., Ganchev, A.C., Petkova, V.B.: Quantum groups and fusion rules multiplicities. Nucl. Phys. **B 343**, 205–227 (1990)

20. Gaberdiel, M.: Fusion in conformal field theory as the tensor product of the symmetry algebra. Int. J. Mod. Phys. **A 9**, 4619–4636 (1994)

21. Gepner, D., Witten, E.: String Theory on Group Manifolds. Nucl. Phys. **B 278**, 493–549 (1986)

22. Goddard, P.: Meromorphic conformal field theory. In: Kac V.G. (ed.) Infinite dimensional Lie Algebras and Lie Groups. Proceedings of CIRM-Luminy Conference 1988, p. 556. Singapore, New Jersey, Hong Kong: World Scientific, 1989

23. Gomez, C., Sierra, G.: Quantum group meaning of the Coulomb gas. Phys. Lett. **B 240**, 149–157 (1990)

24. Gomez, C., Sierra, G.: The quantum symmetry of rational conformal field theories. Nucl. Phys. **B 352**, 791–828 (1991)

25. Gomez, C., Sierra, G.: A brief history of hidden quantum symmetries in conformal field theories. Lecture presented in the XXI DGMTP-conference (Tianjin, 1992) and in the NATO-seminar (Salamanca, 1992)

26. Goodman, F.M., Nakanishi, T.: Fusion algebras in integrable systems in two dimensions. Phys. Lett. **B 262**, 259–264 (1991)

27. Kratz, H.: Weak quasi-quantum group symmetries in all minimal conformal models. Phys. Lett. **B 317**, 60–63 (1993)

28. Kirillov, A.N., Reshetikhin, N.: LOMI preprint E-9-88

29. Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess–Zumino model in two dimensions. Nucl. Phys. **B 247**, 83–103 (1984)

30. Loke, T.: Inequivalent quantisations and modular invariance in the WZW model. Unpublished notes (September 1993)

31. Mack, G., Schomerus, V.: Quasi Hopf quantum symmetry in quantum theory. Nucl. Phys. **B 370**, 185–230 (1992)
32. Majid, S.: Quasi-Quantum Groups as Internal Symmetries of Topological Quantum Field Theories. Lett. Math. Phys. **22**, 83–90 (1991); Majid, S.: Braided Groups and Algebraic Quantum Field Theories. Lett. Math. Phys. **22**, 167–175 (1991); Majid, S.: Reconstruction Theorems and rational conformal field theories. Int. J. Mod. Phys. **A 6**, 4359–4374 (1991)
33. Majid, S.: Foundations of quantum group theory. In preparation
34. Messiah, A.: Quantum Mechanics. Volume II. Amsterdam: North Holland, 1970
35. Moore, G., Reshetikhin, N.: A comment on quantum group symmetry in conformal field theory. Nucl. Phys. **B 328**, 557–574 (1989)
36. Moore, G., Seiberg, N.: Classical and Quantum Conformal Field Theory. Commun. Math. Phys. **123**, 177–254 (1989)
37. Nahm, W.: Quasi-rational fusion products. Bonn preprint (1994)
38. Nill, F.: Fusion Structures from Quantum Groups: I. Charge Conjugation, Markov Traces and Towers of Intertwiner Algebras. Berlin preprint, Sfb 288, 106 (revised version, January 1994); Nill, F.: Fusion Structures from Quantum Groups: II. Why Truncation is necessary. Lett. Math. Phys. **29**, 83–90 (1993)
39. Ramirez, C., Ruegg, H., Ruiz-Altaba, M.: Explicit quantum symmetries of WZNW theories. Phys. Lett. **B 247**, 499–508 (1990)
40. Ramirez, C., Ruegg, H., Ruiz-Altaba, M.: The contour picture of quantum groups: Conformal field theories. Nucl. Phys. **B 364**, 195–233 (1991)
41. Schomerus, V.: Construction of Field Algebras with Quantum Symmetry from Local Observables. Harvard preprint, hep-th/9401042
42. Tsuchiya, A., Kanie, Y.: Vertex operators in conformal field theory on $P^1$ and monodromy representations of braid group. Advanced Studies in Pure Mathematics **16**, 297–372 (1988); Erratum: Advanced Studies in Pure Mathematics **19**, 675–682 (1989)
43. Vecsernyés, P.: On the quantum symmetry of the chiral Ising model. Nucl. Phys. **B 415** [FS], 557–588 (1994)
44. Walton, M.A.: Fusion rules in Wess–Zumino–Witten models. Nucl. Phys. **B 340**, 777–790 (1990)

Communicated by R.H. Dijkgraaf