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A Microscopic Derivation of the Critical Magnetic Field in a Superconductor

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Abstract: The propagator for a noninteracting many electron system in a constant magnetic field in three space time dimensions is computed. This formula and the results of [FT1, 2] are used to give a microscopic derivation of a BCS-equation with magnetic field. It is shown that this equation has no solution if the magnetic field is sufficiently large. Perturbation theory in the interaction around the magnetic field propagator is discussed.

I. Introduction

In this paper, we consider the model of a many electron system in a constant magnetic field in three space time dimensions described by the effective potential

$$\mathscr{G}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z} \int e^{-\lambda \mathscr{V}(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)} d\mu_S \,, \tag{I.1}$$

where the interaction

$$\mathscr{V}(\psi, \bar{\psi}) = \sum_{\alpha, \beta \in \{\uparrow, \downarrow\}} \int d\xi d\xi' \bar{\psi}_\alpha(\xi)\psi_\alpha(\xi) V(\xi - \xi')\bar{\psi}_\beta(\xi')\psi_\beta(\xi') \tag{I.2}$$

is assumed to be short range and rotation invariant. Here, $d\mu_S$ is the Grassmann Gaussian measure with covariance S, where S is the exact propagator for a free many electron system in a constant magnetic field,

$$\begin{aligned} S(\mathbf{x}, \tau; \mathbf{x}'\tau') &= e^{i\frac{B}{2}(yx'-xy')} \frac{B}{2\pi} \sum_{n=0}^{\infty} l_n \left(\frac{B|\mathbf{x} - \mathbf{x}'|^2}{2} \right) \\ &\quad \times e^{-\varepsilon_n(\tau-\tau')} [\theta(-\varepsilon_n)\theta(\tau' - \tau) - \theta(\varepsilon_n)\theta(\tau - \tau')] \\ &= e^{i\frac{B}{2}(yx'-xy')} D(\mathbf{x} - \mathbf{x}', \tau - \tau') \,, \end{aligned} \tag{I.3}$$

★ Present address: Institute for Advanced Study, Olden Lane, Princeton, 08540 New Jersey, USA see Lemma II.2. That is, we will consider the interaction as a perturbation, but the magnetic field is treated exactly without using linear response theory.

We will prove (see Theorem II.3 for the notation) that the translation invariant part D of S can be written in momentum space as

$$D(\mathbf{k},\tau) = e^{-\varepsilon_{n_B}\tau}\frac{1}{\mathrm{ch}^2\frac{B\tau}{2}}\int_0^\infty \delta_B(s)e^{-e(\mathbf{k},s)\frac{\mathrm{th}\frac{B\tau}{2}}{\frac{B}{2}}}[\theta(-e(\mathbf{k},s))\theta(-\tau)-\theta(e(\mathbf{k},s))\theta(\tau)]\,ds$$

$$-e^{-\varepsilon_{n_B}\tau}\frac{2}{1+e^{-B\tau}}(-1)^{n_B}l_{n_B}\left(\frac{2\mathbf{k}^2}{B}\right), \tag{I.4}$$

which may be compared to the free electron propagator without magnetic field,

$$C(\mathbf{k},\tau) = e^{-e(\mathbf{k})\tau}[\theta(-e(\mathbf{k}))\theta(-\tau)-\theta(e(\mathbf{k}))\theta(\tau)]\,. \tag{I.5}$$

Using formula (I.4) and the results of [FT1,2], we derive a BCS-equation with magnetic field (III.2.6) from which the existence of a critical magnetic field follows, see also the curves in Sect. III.

As a first step towards rigorously justifying this BCS-equation and its predictions, we consider the perturbation theory of the model defined by (I.1). We show that graphs containing no two or four legged subgraphs are bounded by const^n, four legged subgraphs produce $n!$'s, n being the order of perturbation theory, and two legged subgraphs have to be renormalized. Furthermore, there is convergence graph by graph to the $B=0$ model. The main problem in proving this is to get the correct propagator estimates. Once this is done, one can apply the machinery of [FT1] to get the stated results.

In Sect. II we compute the magnetic field propagator in the symmetric gauge and prove formula (I.4). Since we are interested in small magnetic fields, effects coming from the filling factor of the highest occupied Landau level are neglected. The BCS-equation with magnetic field is derived in Sect. III, and Sect. IV contains a short discussion of perturbation theory.

In this paper, only the main computations are given. The reader who wants to see more is refered to [Le], where all calculations are done in great detail.

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II. The Magnetic Field Free Propagator

The one particle Schrödinger equation for an electron in a constant magnetic field $\bar{B}=(0,0,B)$ in two dimensions without spin is

$$H\psi = \left\{\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - eA\right)^2 - \mu\right\}\psi = \varepsilon\psi\,, \tag{II.1}$$

where μ denotes the chemical potential. The propagator S will be calculated in the symmetric gauge

$$A(x,y) = \left(-\frac{B}{2}y, \frac{B}{2}x, 0\right)\,. \tag{II.2}$$

A computation in a nonsymmetric gauge $\tilde{A}(x,y) = (-By, 0, 0)$ is given in [Le]. In the first case, the eigenfunctions of (II.1) are labelled by two discrete parameters whereas in the second case one gets one discrete and one continuous parameter. So in the first case the calculation of the covariance involves two infinite sums and in the second case there is one integral and one infinite sum which of course yield the same result up to the phase factor

$$S(\xi, \xi') = e^{i(\frac{B}{2}xy - \frac{B}{2}x'y')}\tilde{S}(\xi, \xi'), \tag{II.3}$$

which is due to the gauge transformation

$$A(x,y) = \tilde{A}(x,y) + \nabla\left(\frac{B}{2}xy\right). \tag{II.4}$$

Here $\xi = (\mathbf{x}, \tau) = (x, y, \tau)$.

For simplicity, the computation is done in infinite volume and it is assumed that the highest occupied Landau level is fully occupied with electrons, that is (see (II.6) below) $\varepsilon_n \neq 0\ \forall n \in \mathbb{N}$, so effects coming from the filling factor of the highest Landau level are neglected. This should be no restriction in considering magnetic fields concerning superconductivity, because there the number of Landau levels is of order 10^4 or 10^5 (for BCS superconductors).

We now present the calculation of S in the symmetric gauge. In Sects. III and IV all calculations are done with S. The following lemma summarizes the properties of the eigenfunctions of (II.1) which are well known.

Lemma II.1 (Eigenfunctions). *Put $e, \hbar$ and the electron mass to one and identify (x,y) with $z = x + iy$. Then the normalized eigenfunctions of* (II.1) *in the symmetric gauge* (II.2) *are given by*

$$\phi_{nm}(z) = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}}\left(\frac{n!}{m!}\right)^{\frac{1}{2}}\left(\sqrt{\frac{B}{2}}z\right)^{m-n} L_n^{m-n}\left(\frac{B|z|^2}{2}\right)e^{-\frac{B|z|^2}{4}} \tag{II.5}$$

with energy eigenvalues

$$\varepsilon_{nm} = \varepsilon_n = B\left(n + \frac{1}{2}\right) - \mu, \tag{II.6}$$

where $n, m \in \{0, 1, 2, \ldots\}$ and $m - n$ has the meaning of angular momentum. They have the following properties:

$$\phi_{nm}(z)^* = (-1)^{n-m}\phi_{mn}(z), \tag{II.7a}$$

$$\hat{\phi}_{nm}(k = k_1 + ik_2) = \frac{4\pi}{B}(-1)^n\phi_{nm}\left(-\frac{2}{B}ik\right), \tag{II.7b}$$

$$\sum_{m=0}^{\infty}\phi_{n_1m}(z_1)\phi_{n_2m}(z_2)^* = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}}\phi_{n_1n_2}(z_1 - z_2)e^{i\frac{B}{2}Im(z_1z_2^*)}. \tag{II.7c}$$

Lemma II.2 (Propagator). *The magnetic field free propagator with imaginary time in infinite volume and symmetric gauge is given by*

$$S(\xi,\xi') = \sum_{n,m=0}^{\infty} \phi_{nm}(z)\phi_{nm}(z')^* e^{-\varepsilon_n(\tau-\tau')}[\varepsilon_n, \tau-\tau']$$

$$= e^{i\frac{B}{2}(yx'-xy')} D(\xi-\xi') , \tag{II.8a}$$

where the translation invariant part D of S is given by

$$D(\xi) = \frac{B}{2\pi}\sum_{n=0}^{\infty} l_n\left(\frac{Br^2}{2}\right) e^{-\varepsilon_n\tau}[\varepsilon_n,\tau] . \tag{II.8b}$$

Here, $l_n(v) = L_n(v)e^{-\frac{v}{2}}$ *denotes the Laguerre function,*

$$[\varepsilon_n,\tau] = [\theta(-\varepsilon_n)\theta(-\tau) - \theta(\varepsilon_n)\theta(\tau)] = \begin{cases} -1 & \text{if } \varepsilon_n > 0 \wedge \tau > 0 \\ 1 & \text{if } \varepsilon_n < 0 \wedge \tau < 0 \\ 0 & \text{else} \end{cases} \tag{II.8c}$$

and $\varepsilon_n = B(n+\frac{1}{2}) - \mu$. *The spatial Fourier transform of D is*

$$D(\mathbf{k},\tau) = \sum_{n=0}^{\infty} 2(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right) e^{-\varepsilon_n\tau}[\varepsilon_n,\tau] , \tag{II.8d}$$

where $\mathbf{k} = (k_1, k_2)$.

Proof. By definition (see for example [FW]), the imaginary time propagator is

$$S(\xi,\xi') = \sum_{nm} \phi_{nm}(z)\phi_{nm}(z')^* e^{-\varepsilon_n(\tau-\tau')}[\theta(-\varepsilon_n)\theta(\tau'-\tau) - \theta(\varepsilon_n)\theta(\tau-\tau')] ,$$

where $\theta(v)$ denotes the step function which is one for $v > 0$ and zero otherwise. Use (II.7c) to perform the m-sum:

$$\sum_{m=0}^{\infty} \phi_{nm}(z)\phi_{nm}(z')^* = \frac{B}{2\pi} L_n\left(\frac{B|z-z'|^2}{2}\right) e^{-\frac{B|z-z'|^2}{4}} e^{i\frac{B}{2}(yx'-xy')} .$$

The Fourier transform is computed with (II.7b) for $n = m$:

$$\left[l_n\left(\frac{Br^2}{2}\right)\right]^{\wedge}(\mathbf{k}) = \frac{2\pi}{B} 2(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right) ,$$

thus (II.8c) follows. □

The free $B = 0$ propagator in the mixed representation, that is in $(\mathbf{k},\tau) = (k_1, k_2, \tau)$-space, is given by

$$C(\mathbf{k},\tau) = e^{-e(\mathbf{k})\tau}[e(\mathbf{k}),\tau] \tag{II.9}$$

with

$$e(\mathbf{k}) = \frac{\mathbf{k}^2}{2} - \mu . \tag{II.10}$$

It is hard to see from (II.8d), how (II.9) is obtained as B goes to zero. From the following representation (II.11) of D the limit $B \to 0$ can be read off. The main use of formula (II.11) below is that it simplifies the evaluation of Feynman graphs in momentum space, in particular, the computation of the critical magnetic field in Sect. III.

Theorem II.3 (Propagator). *In $(\mathbf{k}, \tau)$-space, the translation invariant part of the magnetic field free propagator is given by*

$$D(\mathbf{k},\tau) = e^{-\varepsilon_{n_B}\tau}\frac{1}{\mathrm{ch}^2\frac{B\tau}{2}}\int_0^\infty \delta_B(s) e^{-e(\mathbf{k},s)\frac{\mathrm{th}\frac{B\tau}{2}}{\frac{B}{2}}}[e(\mathbf{k},s),\tau]\,ds$$

$$- e^{-\varepsilon_{n_B}\tau}\frac{2}{1+e^{-B\tau}}(-1)^{n_B} l_{n_B}\left(\frac{2\mathbf{k}^2}{B}\right), \tag{II.11}$$

where ch, th *are the hyperbolic cosine, tangent, $l_n(v) = L_n(v)e^{-\frac{v}{2}}$ is the Laguerre function, $e(\mathbf{k},s) = \frac{\mathbf{k}^2}{2} - \mu s$, $\varepsilon_{n_B} = B(n_B + \frac{1}{2}) - \mu$ and*

$$n_B = \left[\frac{\mu}{B} + \frac{1}{2}\right], \tag{II.12}$$

where the square brackets in (II.12) *are Gauss brackets, thus $0 \leqq \varepsilon_{n_B} \leqq B$. Furthermore,*

$$\delta_B(s) = 2\frac{\mu}{B}(-1)^{n_B} l_{n_B}\left(4\frac{\mu}{B}s\right) \tag{II.13}$$

is a δ-sequence with limit $\delta(s-1)$.

Remark. The second term on the right-hand side of (II.11) converges pointwise to zero, since for $s \geqq \varepsilon > 0$ there is the estimate $|l_n(s)| \leqq \frac{\mathrm{const}}{n^{\frac{1}{4}}}$ and n_B goes to infinity if B goes to zero.

Proof. n_B is by definition the smallest natural number such that $\varepsilon_{n_B} > 0$. Thus

$$D(\mathbf{k},\tau) = 2e^{-\frac{B}{2}\tau+\mu\tau}\sum_{n=0}^{n_B-1}(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right)e^{-Bn\tau}\theta(-\tau)$$

$$- 2e^{-\frac{B}{2}\tau+\mu\tau}\sum_{n=n_B}^{\infty}(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right)e^{-Bn\tau}\theta(\tau).$$

We will now prove the formulae:

$$\sum_{k=0}^{n-1}(-1)^k l_k(x)t^k = \frac{t}{(1+t)^2}t^n\int_x^\infty ds(-1)^n l_n(s)e^{-\frac{t-1}{t+1}\frac{s-x}{2}} - \frac{1}{1+t}t^n(-1)^n l_n(x), \tag{II.14}$$

and for $|t| < 1$ it is

$$\sum_{k=n}^{\infty}(-1)^k l_k(x)t^k = \frac{t}{(1+t)^2}t^n\int_0^x ds(-1)^n l_n(s)e^{-\frac{1-t}{1+t}\frac{x-s}{2}} + \frac{1}{1+t}t^n(-1)^n l_n(x). \tag{II.15}$$

The following proof is short, but one has already to know the answer. An alternative proof which makes clear how the above formulae have been found is given in [Le].

For $|t| < 1$, let

$$G(x,t) = \frac{1}{1+t} e^{-\frac{1-t}{1+t}\frac{x}{2}} = \sum_{k=0}^{\infty} l_k(x)(-t)^k$$

be the generating function for the Laguerre functions. We have

$$DG(x,t) = 0 ,$$

where

$$D = \left\{ 2(1+t)\frac{d}{dx} + (1-t) \right\} .$$

Hence

$$2l_0'(x) + l_0 + \sum_{k=1}^{\infty} (2l_k'(x) - 2l_{k-1}'(x) + l_k(x) + l_{k-1}(x))(-t)^k = 0 ,$$

which gives the recursion relation

$$2l_0' + l_0 = 0, \quad 2l_k' - 2l_{k-1}' + l_k + l_{k-1} = 0, \quad k \geqq 1 .$$

Now let $s_n(x,t)$ be the left-hand side and $i_n(x,t)$ be the right-hand side of (II.14). Then

$$\begin{aligned} Ds_n(x,t) =& 2l_0'(x) + l_0(x) + \sum_{k=1}^{n-1} (2l_k'(x) - 2l_{k-1}'(x) + l_k(x) \\ & + l_{k-1}(x))(-t)^k - 2l_{n-1}'(x)(-t)^n + l_{n-1}(x)(-t)^n \\ =& (-2l_{n-1}'(x) + l_{n-1}(x))(-t)^n \end{aligned}$$

and

$$\begin{aligned} Di_n(x,t) &= 2\frac{-t}{1+t}(-t)^n l_n(x) - 2(-t)^n l_n'(x) - \frac{1-t}{1+t}(-t)^n l_n(x) \\ &= -(l_n(x) + 2l_n'(x))(-t)^n = -(2l_{n-1}'(x) - l_{n-1}(x))(-t)^n = Ds_n(x,t) , \end{aligned}$$

where in the last line the recursion relation has been used. Therefore, the difference $\Delta_n(x,t) = s_n(x,t) - i_n(x,t)$ obeys $D\Delta_n(x,t) = 0$ which gives

$$\Delta_n(x,t) = \Delta_n(0,t) e^{-\frac{1-t}{1+t}\frac{x}{2}} .$$

But ([GR],7.414.6)

$$\int_0^{\infty} ds (-1)^n l_n(s) e^{-\frac{t-1}{t+1}\frac{s}{2}} = \frac{1+t}{t}\frac{1}{t^n} ,$$

thus

$$i_n(0,t) = \frac{t}{(1+t)^2} t^n \int_0^\infty ds (-1)^n l_n(s) e^{-\frac{t-1}{t+1}\frac{s}{2}} - \frac{1}{1+t} t^n (-1)^n l_n(0)$$

$$= \frac{1}{1+t} - \frac{1}{1+t}(-t)^n = \sum_{k=0}^{n-1} (-t)^k = s_n(0,t)\,,$$

which proves (II.14). Formula (II.15) is obtained by writing

$$\sum_{k=n}^{\infty} (-1)^k l_k(x) t^k = \frac{1}{1+t} e^{-\frac{1-t}{1+t}\frac{x}{2}} - \sum_{k=0}^{n-1} (-1)^k l_k(x) t^k$$

and

$$\frac{t}{(1+t)^2} t^n \int_0^x ds (-1)^n l_n(s) e^{-\frac{1-t}{1+t}\frac{x-s}{2}}$$

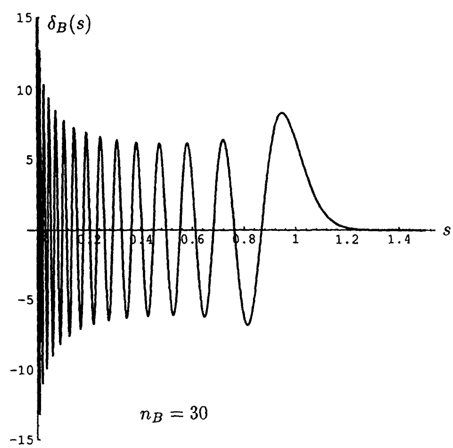
$$= \frac{1}{1+t} e^{-\frac{1-t}{1+t}\frac{x}{2}} - \frac{t}{(1+t)^2} t^n \int_x^\infty ds (-1)^n l_n(s) e^{-\frac{t-1}{t+1}\frac{s-x}{2}}\,.$$

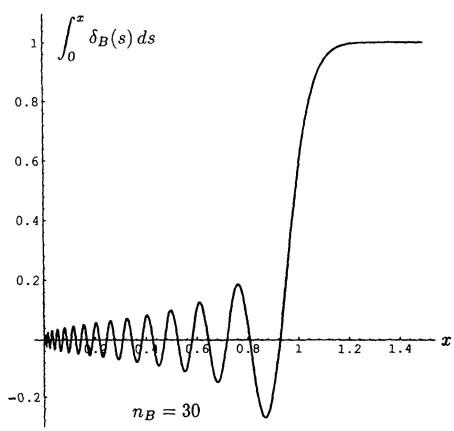
Now put $t = e^{-B\tau}$, $x = \frac{2\mathbf{k}^2}{B}$ and $n = n_B$ in (II.14, 15) and substitute the integration variable s by $v = \frac{B}{4\mu} s$ to obtain the final result (II.11). A detailed discussion of the δ-sequence is given in [Le]. See also the curves below. □

In comparing (II.8, 11) with (II.9), one can see essentially three differences.

(i) The magnetic field propagator is no longer translation invariant. This is due to the fact that linear momentum is no longer an eigenstate but angular momentum.

(ii) In momentum space, there is no longer a sharp fermi surface (or Fermi circle, since we are in two dimensions). Rather, the Fermi surface is smeared out with the delta sequence $\delta_B(s)$ so that the density of states in momentum space $\theta(\mu - \frac{\mathbf{k}^2}{2})$ is substituted by $\int_0^\infty ds \delta_B(s) \theta(\mu s - \frac{\mathbf{k}^2}{2}) = 1 - \int_0^{\frac{\mathbf{k}^2}{2\mu}} ds \delta_B(s)$, see the following curves.





(iii) The imaginary time variable τ is substituted by $\frac{\operatorname{th}\frac{B\tau}{2}}{\frac{B}{2}}$. This is the most important effect (for our purposes), since it changes significantly the values of the Feynman graphs. Graphs containing two legged subgraphs become finite and the flow for the four point function is expected to be convergent for $B \gtrsim e^{-\frac{\text{const}}{\lambda}}$, see Sect. III.3. Furthermore, the factor $\frac{\operatorname{th}\frac{B\tau}{2}}{\frac{B}{2}}$ appears directly in the BCS-equation with magnetic field (III.2.6), and is responsible there for the existence of a critical magnetic field. To see this factor, it is necessary to compute the infinite sum over the Laguerre polynomials, since, mathematically, it comes from the generating function for the Laguerre polynomials. Physically, it expresses the fact that electrons are localized by a magnetic field.

Finally, the factor $e^{-\varepsilon_{n_B}\tau}$ requires a short discussion. The assumption that the highest Landau level is fully occupied with electrons means that there is no $n \in \mathbb{N}$ such that $\varepsilon_n = 0$. n_B is by definition the smallest natural number such that $\varepsilon_n > 0$. Then $0 < \varepsilon_{n_B} < B$, so one can write

$$\varepsilon_{n_B} = \alpha B \tag{II.16a}$$

with some $0 < \alpha < 1$. However, for α arbitrary close to 0 or 1, the τ-decay of the second term in (II.11) becomes arbitrary bad, although it converges pointwise to zero. In order to keep the estimates of the following sections (for example the estimate for the second term in the BCS-equation (III.2.6)) uniform, assume

$$\varepsilon \leqq \alpha \leqq 1 - \varepsilon \tag{II.16b}$$

or equivalently define the set of admitted magnetic fields to be

$$\mathscr{B} = \bigcup_{n \in \mathbb{N}} \left[\frac{\mu}{n + \frac{1}{2} - \varepsilon}, \frac{\mu}{n - \frac{1}{2} + \varepsilon} \right] . \tag{II.16c}$$

Then the measure of the set of neglected fields can be made arbitrarily small since

$$\sum_{n=1}^{\infty} \left(\frac{\mu}{n + \frac{1}{2} - \varepsilon} - \frac{\mu}{n + \frac{1}{2} + \varepsilon} \right) = 2\mu\varepsilon \sum_{n=1}^{\infty} \frac{1}{(n + \frac{1}{2})^2 + \varepsilon^2} \leqq \text{const}\, \varepsilon .$$

III. The Existence of the Critical Field

III.1 The $B = 0$ BCS-Equation. In [FT2], Feldman and Trubowitz obtained the BCS-equation (without magnetic field) in the following way:

Consider the effective potential for an interacting many electron system which is given by

$$\mathscr{G}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z} \int e^{-\lambda \mathscr{V}\,(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)}\, d\mu_C(\psi, \bar{\psi}) , \tag{III.1.1}$$

where C is the free propagator corresponding to the normal ground state

$$C(\xi,\xi') = C(\xi-\xi') = \int \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{-e(\mathbf{k})(\tau-\tau')} [\theta(-e(\mathbf{k}))\theta(-\tau) - \theta(e(\mathbf{k}))\theta(\tau)]$$

$$= \int \frac{d^2k}{(2\pi)^2} \int \frac{dk_0}{2\pi} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-ik_0(\tau-\tau')} \frac{1}{ik_0 - e(\mathbf{k})} \tag{III.1.2}$$

and $e(\mathbf{k}) = \frac{\mathbf{k}^2}{2} - \mu$. The quartic interaction $\mathscr{V}$ is assumed to be short ranged. They started to anlayse $\mathscr{G}$ in perturbation theory [FT1]. It turned out that all graphs containing no two or four legged subgraphs are bounded by const^n, n denoting the order of perturbation theory, that all graphs which contain no two legged subgraphs are bounded by $n!$ const^n, and that graphs containing two legged subgraphs are in general infinite. They introduced a localization operator L, which acts nontrivially only on quadratic and quartic monomials and isolates the singularities and $n!$'s produced by the two and four legged subgraphs. Then all graphs contributing to $(1-L)\mathscr{G}$ are bounded by const^n. In [FMRT] and [FKLT1,2] it is shown that $(1-L)\mathscr{G}$ is indeed an analytic function at $\lambda = 0$ with a fixed, volume independent positive radius of convergence.

The relevant part $L\mathscr{G}$ of the effective potential is analyzed by a renormalization group flow [FT2]. If one expands the kernel $F^{(h)}$ of the quartic part of $L\mathscr{G}^{(h)}$ into a Fourier series,

$$F^{(h)}(t',s') = F^{(h)}(\cos\theta) = \sum_{l=0}^{\infty} \lambda_l^{(h)} \cos l\theta\,,$$

where $k' = (0, k_F \frac{\mathbf{k}}{\|\mathbf{k}\|})$ denotes the projection onto the Fermi surface, one obtains in the ladder approximation the following flow equation for the coefficients $\lambda_l^{(h)}$, see [FT2]:

$$\lambda_l^{(h-1)} = \lambda_l^{(h)} + \beta^{(h)}(\lambda_l^{(h)})^2, \quad l \geqq 0\,, \tag{III.1.3}$$

where

$$\beta^{(h)} = \int \frac{dp_0 d|\mathbf{p}|}{(2\pi)^3} \frac{|\mathbf{p}|}{k_F} (|C^{(\leq h)}(p)|^2 - |C^{(<h)}(p)|^2)$$

$$= \int \frac{d|\mathbf{p}|}{(2\pi)^2} \frac{|\mathbf{p}|}{k_F} \frac{1}{2|e(\mathbf{p})|} [\rho^2(M^{-2(h+1)}e(\mathbf{p})^2) - \rho^2(M^{-2h}e(\mathbf{p})^2)] \tag{III.1.4a}$$

approaches the limit

$$\beta = \frac{1}{(2\pi)^2 k_F} \int_0^\infty dy \frac{1}{y} [\rho^2(M^{-2}y^2) - \rho^2(y^2)] \tag{III.1.4b}$$

and $\rho(x)$ is some C_0^∞ function, which is one if $x \leqq 1$ and zero if $x \geqq M^2$, M being some constant bigger than one. If one starts with $\lambda_l^{(0)} > 0$, which corresponds to an attractive potential, then the sequence generated by (III.1.3) diverges to infinity which is interpreted as: the normal ground state is not stable. In order to get a well defined effective potential, one introduces a Δ in the following way:

Define the two component Nambu fields

$$\boldsymbol{\Psi}(\xi) = \begin{pmatrix} \psi_\uparrow(\xi) \\ \bar{\psi}_\downarrow(\xi) \end{pmatrix}, \qquad \bar{\boldsymbol{\Psi}}(\xi) = (\bar{\psi}_\uparrow(\xi), \psi_\downarrow(\xi)) \tag{III.1.5a}$$

or in momentum space

$$\boldsymbol{\Psi}(\mathbf{k},k_0) = \begin{pmatrix} \psi_\uparrow(\mathbf{k},k_0) \\ \bar{\psi}_\downarrow(-\mathbf{k},-k_0) \end{pmatrix}, \quad \bar{\boldsymbol{\Psi}}(\mathbf{k},k_0) = (\bar{\psi}_\uparrow(\mathbf{k},k_0), \psi_\downarrow(-\mathbf{k},-k_0)). \tag{III.1.5b}$$

Then the Grassman Gaussian measure becomes $d\mu_C(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}})$ with covariance matrix

$$\mathbf{C}(\xi,\xi') = \langle \boldsymbol{\Psi}(\xi)\bar{\boldsymbol{\Psi}}(\xi')\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-ik_0(\tau-\tau')}(ik_0\mathbf{1} - e(\mathbf{k})\sigma^3)^{-1}, \tag{III.1.6}$$

and the effective potential can be written formally as

$$\mathscr{G}(\boldsymbol{\Psi}^e,\bar{\boldsymbol{\Psi}}^e) = \log \frac{1}{Z}\int e^{-\lambda\mathscr{V}(\boldsymbol{\Psi}+\boldsymbol{\Psi}^e,\bar{\boldsymbol{\Psi}}+\bar{\boldsymbol{\Psi}}^e)} e^{-\int \frac{d^3k}{(2\pi)^3}\bar{\boldsymbol{\Psi}}(k)(ik_0\mathbf{1}-e(\mathbf{k})\sigma^3)\boldsymbol{\Psi}(k)} d(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}}).$$

Then add and subtract $\int \frac{d^3k}{(2\pi)^3}\bar{\boldsymbol{\Psi}}(k)\Delta\sigma^1\boldsymbol{\Psi}(k)$ to get

$$\begin{aligned}
\mathscr{G}(\boldsymbol{\Psi}^e,\bar{\boldsymbol{\Psi}}^e) &= \log \frac{1}{Z}\int e^{-\lambda\mathscr{V}(\boldsymbol{\Psi}+\boldsymbol{\Psi}^e,\bar{\boldsymbol{\Psi}}+\bar{\boldsymbol{\Psi}}^e)-\Delta\int\frac{d^3k}{(2\pi)^3}\bar{\boldsymbol{\Psi}}(k)\sigma^1\boldsymbol{\Psi}(k)} \\
&\quad \times e^{-\int\frac{d^3k}{(2\pi)^3}\bar{\boldsymbol{\Psi}}(k)(ik_0\mathbf{1}-e(\mathbf{k})\sigma^3-\Delta\sigma^1)\boldsymbol{\Psi}(k)} d(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}}) \\
&= \log\frac{1}{Z'}\int e^{-\lambda\mathscr{V}(\boldsymbol{\Psi}+\boldsymbol{\Psi}^e,\bar{\boldsymbol{\Psi}}+\bar{\boldsymbol{\Psi}}^e)-\Delta\int d\xi\bar{\boldsymbol{\Psi}}(\xi)\sigma^1\boldsymbol{\Psi}(\xi)} d\mu_{\mathbf{C}_\Delta},
\end{aligned} \tag{III.1.7}$$

where now

$$\mathbf{C}_\Delta(\xi,\xi') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-ik_0(\tau-\tau')}(ik_0\mathbf{1} - e(\mathbf{k})\sigma^3 - \Delta\sigma^1)^{-1}. \tag{III.1.8}$$

The new covariance is bounded by $\frac{\text{const}}{\Delta}$, which has the consequence that now all graphs are finite. But if Δ is going to zero, graphs containing two legged subgraphs diverge. To produce an expansion uniform in Δ one has to renormalize, that is, one has to add counterterms

$$\delta\mathscr{V} = \delta\mu(\lambda,\mu,\Delta)\int d\xi\bar{\boldsymbol{\Psi}}(\xi)\sigma^3\boldsymbol{\Psi}(\xi), \tag{III.1.9}$$

$$\mathscr{D} = D(\lambda,\mu,\Delta)\int d\xi\bar{\boldsymbol{\Psi}}(\xi)\sigma^1\boldsymbol{\Psi}(\xi), \tag{III.1.10}$$

to the exponent in (III.1.7). But then, to recover the physical effective potential, one has to impose the constraint

$$\Delta = -D(\lambda,\mu,\Delta), \tag{III.1.11}$$

which, in first order, gives the BCS equation:

$\Delta = -D(\lambda,\mu,\Delta) = \frac{\lambda}{2}\mathrm{Tr}\,[\sigma^1($  k $-$  k $)|_{k_0=0,|\mathbf{k}|=k_F}]$

$$\begin{aligned}
&= -\frac{\lambda}{2}\,\mathrm{Tr}\left[\sigma^1\int\frac{d^3p}{(2\pi)^3}\langle k,p|V|p,k\rangle\sigma^3\mathbf{C}_\Delta(p)\sigma^3|_{k_0=0,|\mathbf{k}|=k_F}\right] \\
&= -\lambda\int\frac{d^3p}{(2\pi)^3}\langle k',p|V|p,k'\rangle\frac{\Delta}{p_0^2+e(\mathbf{p})^2+\Delta^2}\,|,
\end{aligned} \tag{III.1.12}$$

where $k' = (0, k_F \frac{\mathbf{k}}{|\mathbf{k}|})$. Taking $\langle k', p|V|p, k'\rangle = \theta(\omega_D - |e(\mathbf{k})|)\theta(\omega_D - |e(\mathbf{p})|)$, one obtains the familiar equation

$$
\begin{aligned}
1 &= -\lambda \operatorname{const} \int d^2p\theta(\omega_D - |e(\mathbf{p})|)\frac{1}{\sqrt{e(\mathbf{p})^2 + \Delta^2}} \\
&= -\lambda \operatorname{const} \int_0^{\omega_D/\Delta} dv \frac{1}{\sqrt{v^2+1}}\,,
\end{aligned}
\tag{III.1.13}
$$

which gives $\Delta = \omega_D e^{-\frac{\mathrm{const}}{\lambda}}$.

III.2 The BCS-Equation with Magnetic Field. In the case with magnetic field, one can proceed in an analogous way. However, because the Hamiltonian is diagonal in (n,m)-space, one has to work in this space rather than in momentum space. In (n,m,k_0)-space, the covariance is

$$
S(n,m,k_0) = \frac{1}{ik_0 - \varepsilon_n}\,.
\tag{III.2.1}
$$

Introduce the two component fields (III.1.5). Since $\phi_{nm}(z)^* = (-1)^{n-m}\phi_{mn}(z)$, the two component fields in (n,m)-space are given by

$$
\begin{aligned}
\boldsymbol{\Psi}(n,m,k_0) &= \begin{pmatrix} \psi_\uparrow(n,m,k_0) \\ (-1)^{n-m}\bar{\psi}_\downarrow(m,n,-k_0) \end{pmatrix}, \\
\bar{\boldsymbol{\Psi}}(n,m,k_0) &= (\bar{\psi}_\uparrow(n,m,k_0), (-1)^{n-m}\psi_\downarrow(m,n,-k_0))\,,
\end{aligned}
\tag{III.2.2}
$$

and the covariance matrix becomes

$$
\mathbf{S}(n,m,k_0) = (ik_0\mathbf{1} - \varepsilon_n\sigma^3)^{-1}\,.
\tag{III.2.3}
$$

Then writing the integration measure formally as

$$
\exp\left\{-\sum_{n,m=0}^{\infty}\int\frac{dk_0}{2\pi}\bar{\boldsymbol{\Psi}}(n,m,k_0)(ik_0\mathbf{1} - \varepsilon_n\sigma^3)\boldsymbol{\Psi}(n,m,k_0)\right\} d(\boldsymbol{\Psi},\bar{\boldsymbol{\Psi}})\,,
$$

and adding and subtracting the term

$$
\sum_{n,m=0}^{\infty}\int\frac{dk_0}{2\pi}\bar{\boldsymbol{\Psi}}(n,m,k_0)\Delta\sigma^1\boldsymbol{\Psi}(n,m,k_0)\,,
\tag{III.2.4}
$$

one obtains the new covariance

$$
\mathbf{S}_\Delta(n,m,k_0) = \mathbf{S}_\Delta(n,k_0) = (ik_0\mathbf{1} - \varepsilon_n\sigma^3 - \Delta\sigma^1)^{-1}\,,
$$

which becomes in coordinate and momentum space

$$
\begin{aligned}
\mathbf{S}_\Delta(\xi,\xi') &= \sum_{n,m=0}^{\infty}\phi_{nm}(z)\phi_{nm}(z')\int\frac{dk_0}{2\pi}e^{-ik_0(\tau-\tau')}(ik_0\mathbf{1} - \varepsilon_n\sigma^3 - \Delta\sigma^1)^{-1} \\
&= e^{i\frac{B}{2}\mathbf{x}\mathbf{x}'^{\perp}}\mathbf{D}_\Delta(\xi-\xi')\,,
\end{aligned}
$$

where $\mathbf{xx}'^{\perp} = yx' - xy'$,

$$\mathbf{D}_\Delta(\xi) = \frac{B}{2\pi}\sum_{n=0}^{\infty} l_n\left(\frac{Br^2}{2}\right)\int\frac{dk_0}{2\pi}e^{-ik_0(\tau-\tau')}(ik_0\mathbf{1} - \varepsilon_n\sigma^3 - \Delta\sigma^1)^{-1},$$

$$\mathbf{D}_\Delta(\mathbf{k},k_0) = \sum_{n=0}^{\infty} 2(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right)(ik_0\mathbf{1} - \varepsilon_n\sigma^3 - \Delta\sigma^1)^{-1}.$$

Then the BCS-equation with magnetic field is given by the first order approximation to the constraint

$$\Delta = -D(\lambda,\mu,B,\Delta). \tag{III.2.5}$$

Theorem III.2.1 (BCS-Equation with Magnetic Field). *Let* $\langle k,p|V|p,k\rangle = \theta(\omega_D - |e(\mathbf{k})|)\,\theta(\omega_D - |e(\mathbf{p})|)$. *Then, using the approximation* $\delta_B(s) \approx \delta(s-1)$, *the first order approximation to the constraint* (III.2.5) *is given by the equation*

$$1 = \text{const}\,\lambda\int_0^\infty dt\,J_0(t)\left\{\text{ch}\alpha\frac{B}{\Delta}t\frac{\frac{B}{\Delta}}{\text{sh}\frac{B}{\Delta}t}\left(1 - e^{-2\frac{\omega_D}{B}\text{th}\frac{Bt}{2\Delta}}\right) + \frac{1}{2}\frac{B}{\Delta}\frac{\text{sh}[(\frac{1}{2}-\alpha)\frac{B}{\Delta}t]}{\text{ch}\frac{1}{2}\frac{B}{\Delta}t}\right\}, \tag{III.2.6}$$

where J_0 *denotes the zeroth Bessel function and* α *is defined by* $\varepsilon_{n_B} = \alpha B$, $\varepsilon \leqq \alpha \leqq 1-\varepsilon$, *see* (II.16).

Remarks. 1) Substituting $\frac{1}{\sqrt{v^2+1}} = \int_0^\infty J_0(t)e^{-vt}dt$ in (III.1.13), the BCS-equation without magnetic field reads

$$1 = \text{const}\,\lambda\int_0^\infty dt\,J_0(t)\frac{1}{t}\left(1 - e^{-\frac{\omega_D}{\Delta}t}\right) \tag{III.2.7}$$

and is the $B \to 0$ limit of the above equation.

2) For zero magnetic field, one has a pairing between $(\mathbf{k},\uparrow)$ and $(-\mathbf{k},\downarrow)$. With magnetic field, linear momentum $\mathbf{k}$ is no longer an eigenstate, but angular momentum $l = m-n$ is. Then the $\Delta\sigma^1$-term in (III.2.4) gives a pairing between $(l,\uparrow)$ and $(-l,\downarrow)$.

Proof. In the case with magnetic field, the graphs contributing to $D(\lambda,\mu,B,\Delta)$ have to be evaluated in (n,m)-space at $n = n_B$ and $k_0 = 0$. A two legged graph $\bar{G}$ is expanded as follows

$$\bar{G}(\xi,\xi') = e^{i\frac{B}{2}\mathbf{xx}'^{\perp}}G(\xi-\xi') = \sum_{n,m=0}^{\infty}\phi_{nm}(z)\phi_{nm}(z')\int\frac{dk_0}{2\pi}e^{-ik_0(\tau-\tau')}G(n,k_0),$$

$$G(n,k_0) = \int dx\,dy\,d\tau\,l_n\left(\frac{Br^2}{2}\right)e^{ik_0\tau}G(x,y,\tau)$$

$$= \int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{2\pi}{B}2(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right)G(\mathbf{k},k_0).$$

With $\langle k,-k|V|p,-p\rangle = \theta(\omega_D - |e(\mathbf{k})|)\theta(\omega_D - |e(\mathbf{p})|)$, one obtains to first order

$$D(\lambda,\mu,B,\Delta) = -\frac{\lambda}{2}\mathrm{Tr}\,[\sigma^1\ ($$

$$-$$

$$)\ |_{n=n_B,k_0=0}]$$

$$= \frac{\lambda}{2}\mathrm{Tr}\left[\sigma^1 \int d|\mathbf{k}|\,|\mathbf{k}|\frac{1}{B}2(-1)^n l_n\left(\frac{2\mathbf{k}^2}{B}\right)\right.$$

$$\left.\times \int \frac{d^3p}{(2\pi)^3}\langle k,p|V|p,k\rangle\sigma^3\mathbf{D}_\Delta(p)\sigma^3|_{n=n_B,k_0=0}\right]$$

$$= \lambda\int_0^\infty ds\delta_B(s)\int\frac{d^3p}{(2\pi)^3}\theta(\omega_D - |\mu s - \mu|)\theta\left(\omega_D - \left|\frac{\mathbf{p}^2}{2} - \mu\right|\right)$$

$$\times\sum_{n=0}^\infty 2(-1)^n l_n\left(\frac{2\mathbf{p}^2}{B}\right)\frac{\Delta}{p_0^2+\varepsilon_n^2+\Delta^2}\,.$$

Now, using $\delta_B(s)\approx\delta(s-1)$ and performing the p_0-integral, one gets

$$D(\lambda,\mu,B,\Delta) = \mathrm{const}\,\lambda\int_{-\omega_D}^{\omega_D} d\left(\frac{\mathbf{p}^2}{2}-\mu\right)$$

$$\times\sum_{n=0}^\infty 2(-1)^n l_n\left(\frac{2\mathbf{p}^2}{B}\right)\frac{\Delta}{\sqrt{\varepsilon_n^2+\Delta^2}}\,. \tag{III.2.8}$$

To compute the infinite sum, use the fact that the Laplace transform of the zeroth Bessel function is $(s^2+1)^{-\frac{1}{2}}$, that is

$$\int_0^\infty dt\,J_0(\Delta t)e^{-|\varepsilon_n|t} = \frac{1}{\sqrt{\varepsilon_n^2+\Delta^2}}\,. \tag{III.2.9}$$

The resulting sum can be computed using Theorem II.3. One obtains

$$\sum_{n=0}^\infty 2(-1)^n l_n\left(\frac{2\mathbf{p}^2}{B}\right)e^{-|\varepsilon_n|t} = D(\mathbf{p},-t) - D(\mathbf{p},t)$$

$$= \int_0^\infty dv\,\delta_B(v)\frac{1}{\mathrm{ch}^2\frac{Bt}{2}}e^{-\frac{\mathrm{th}\frac{Bt}{2}}{\frac{B}{2}}\left|\frac{\mathbf{p}^2}{2}-\mu v\right|}\left(e^{\varepsilon_{n_B}t}\theta\left(\mu v - \frac{\mathbf{p}^2}{2}\right) + e^{-\varepsilon_{n_B}t}\theta\left(\frac{\mathbf{p}^2}{2}-\mu v\right)\right)$$

$$+2\left(\frac{e^{-\varepsilon_{n_B}t}}{1+e^{-Bt}} - \frac{e^{\varepsilon_{n_B}t}}{1+e^{Bt}}\right)(-1)^{n_B}l_{n_B}\left(\frac{2\mathbf{p}^2}{B}\right)\,. \tag{III.2.10}$$

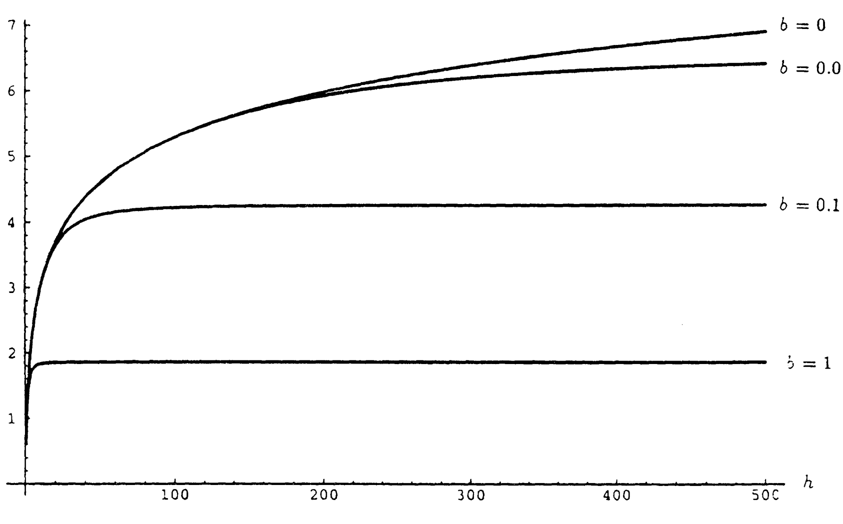
Write $\varepsilon_{n_B} = \alpha B$ with $\varepsilon < \alpha < 1 - \varepsilon$ (see (II.16), and again neglect the smearing in the chemical potential. Then (III.2.8, 10) yield

$$
\begin{aligned}
D(\lambda, \mu, B, \Delta) =& \operatorname{const} \lambda \Delta \int_0^\infty dt\, J_0(\Delta t) \int_{-\omega_D}^{\omega_D} d\left(\frac{\mathbf{p}^2}{2} - \mu\right) \\
& \times \left\{ \frac{1}{\operatorname{ch}^2 \frac{Bt}{2}} e^{-\frac{\operatorname{th}\frac{Bt}{2}}{\frac{B}{2}} \left|\frac{\mathbf{p}^2}{2} - \mu\right|} \left(e^{\alpha B t} \theta\left(\mu - \frac{\mathbf{p}^2}{2}\right) + e^{-\alpha B t} \theta\left(\frac{\mathbf{p}^2}{2} - \mu\right)\right)\right. \\
& \left. + \frac{\operatorname{sh}[(\frac{1}{2} - \alpha) Bt]}{\operatorname{ch}\frac{1}{2} Bt} 2(-1)^{n_B} l_{n_B}\left(\frac{2\mathbf{p}^2}{B}\right)\right\} \\
=& \operatorname{const} \lambda \Delta \int_0^\infty dt\, J_0(t) \left\{ \operatorname{ch} \alpha \frac{B}{\Delta} t \frac{\frac{B}{\Delta}}{\operatorname{sh}\frac{B}{\Delta} t} \left(1 - e^{-2 \frac{\omega_D}{B} \operatorname{th} \frac{Bt}{2\Delta}}\right)\right. \\
& \left. + \frac{c_B}{2} \frac{B}{\Delta} \frac{\operatorname{sh}[(\frac{1}{2} - \alpha) \frac{B}{\Delta} t]}{\operatorname{ch}\frac{1}{2} \frac{B}{\Delta} t}\right\}
\end{aligned}
$$

and again $c_B = \int_{1-\frac{\omega_D}{\mu}}^{1+\frac{\omega_D}{\mu}} ds\, \delta_B(s)$ may be approximated by one, which gives the stated equation. □

In order to have the $B = 0$ BCS-equation (III.2.7) a solution Δ, the exponent $\frac{\omega_D}{\Delta}$ must be chosen large. In (III.2.6) however, the magnitude of the exponent is determined by the ratio $\frac{\omega_D}{B}$ because the Δ appears in the hyperbolic tangents which is always bounded by one. Thus in order to get a solution Δ, B has to be sufficiently small. This becomes clear in considering the following curves:

$$\int_0^\infty d\tau\, J_0\left(\frac{\tau}{bh}\right) \operatorname{ch}\frac{\tau}{2}\, \frac{1 - e^{-\frac{2}{b}\operatorname{th}\frac{\tau}{2}}}{\operatorname{sh}\tau}, \quad b = \frac{B}{\omega_D}, \quad h = \frac{\omega_D}{\Delta}, \quad \alpha = \frac{1}{2}$$



For the computation of the critical field, let $b = \frac{B}{\omega_D}$, $h = \frac{\omega_D}{\Delta}$ and let $\Delta \to 0$ or $h \to \infty$. (III.2.6) becomes

$$1 = \text{const}\,\lambda \int_0^\infty d\tau\, J_0\left(\frac{\tau}{bh}\right)\left\{\operatorname{ch}\alpha\tau \frac{1 - e^{-\frac{2}{b}\operatorname{th}\frac{\tau}{2}}}{\operatorname{sh}\tau} + \frac{1}{2}\frac{\operatorname{sh}[(\frac{1}{2} - \alpha)\tau]}{\operatorname{ch}\frac{1}{2}\tau}\right\}$$

$$\overset{h\to\infty}{\longrightarrow} \text{const}\,\lambda \int_0^\infty d\tau \left\{\operatorname{ch}\alpha\tau \frac{1 - e^{-\frac{2}{b}\operatorname{th}\frac{\tau}{2}}}{\operatorname{sh}\tau} + \frac{1}{2}\frac{\operatorname{sh}[(\frac{1}{2} - \alpha)\tau]}{\operatorname{ch}\frac{1}{2}\tau}\right\}$$

and one computes

$$B_c = \text{const}\,\omega_D e^{-\frac{1}{\text{const}\,\lambda}} = \text{const}\,\Delta\,. \tag{III.2.12}$$

III.3 The Flow of the Four Point Function. In the preceding paragraph, it has been shown that the appearance of the hyperbolic tangents $\frac{2}{B}\operatorname{th}\frac{B\tau}{2}$ instead of τ in the magnetic field free propagator is responsible for the existence of a critical field. Thus, one would expect that the flow of the four legged part of the effective potential behaves differently than the $B = 0$ flow (III.1.3, 4) if one takes the approximation

$$D(\mathbf{k},\tau) \approx \frac{1}{\operatorname{ch}^2\frac{B\tau}{2}} e^{-e(\mathbf{k})\frac{\operatorname{th}\frac{B\tau}{2}}{\frac{B}{2}}}[e(\mathbf{k}),\tau] \equiv \tilde{D}(\mathbf{k},\tau) \tag{III.3.1}$$

for the exact propagator (II.8, 11) and neglects the phase factor in (II.8a). This is indeed the case.

Lemma III.3.1. *Substituting the $B = 0$ propagator C by $\tilde{D}$, the flow equation in the ladder approximation* (III.1.3) *becomes*

$$\lambda_l^{(h-1)}(B) = \lambda_l^{(h)}(B) + \beta_B^{(h)}(\lambda_l^{(h)}(B))^2\,, \tag{III.3.2}$$

where contrary to (III.1.4b) *the $\beta_B^{(h)}$'s satisfy*

$$\sum_{h=-\infty}^{0} \beta_B^{(h)} = \text{const}\left(\log\frac{1}{B} + \text{const}\right)\,. \tag{III.3.3}$$

Proof. The $\beta^{(h)}$'s become

$$\beta_B^{(h)} = \int \frac{dp_0 d|\mathbf{p}|}{(2\pi)^3}\frac{|\mathbf{p}|}{k_F}(|\tilde{D}^{(\leqq h)}(p)|^2 - |\tilde{D}^{(<h)}(p)|^2)\,, \tag{III.3.4}$$

where

$$\tilde{D}^{(\leqq h)}(p_0,\mathbf{p}) = \int d\tau\, e^{ip_0\tau}\frac{1}{\operatorname{ch}^2\frac{B\tau}{2}} e^{-e(\mathbf{p})\frac{\operatorname{th}\frac{B\tau}{2}}{\frac{B}{2}}}[e(\mathbf{p}),\tau]\rho^2(M^{-2(h+1)}e(\mathbf{p})^2)$$

and ρ as in (III.1.4). One computes

$$\int \frac{dp_0}{2\pi}(|\tilde{D}^{(\leqq h)}(p)|^2 - |\tilde{D}^{(<h)}(p)|^2)$$

$$= \int_0^{2/B} dv\left(1 - \frac{B^2}{4}v^2\right)e^{-2|e(\mathbf{p})|v}\{\rho^2(M^{-2(h+1)}e(\mathbf{p})^2) - \rho^2(M^{-2h}e(\mathbf{p})^2)\}\,.$$

Hence (III.3.4) gives

$$\beta_B^{(h)} = \text{const} \int_0^\infty dy \int_0^{2/B} dv \left(1 - \frac{B^2}{4} v^2\right) e^{-2|y-\mu|v} \times \left\{\rho^2(M^{-2(h+1)}(y-\mu)^2) - \rho^2\left(M^{-2h}(y-\mu)^2\right)\right\} . \tag{III.3.5}$$

These $\beta_B^{(h)}$'s show a different behaviour than (III.4a, b) since

$$\sum_{h=-\infty}^{0} \beta_B^{(h)} = \text{const} \int_0^\infty dy \int_0^{2/B} dv \left(1 - \frac{B^2}{4} v^2\right) e^{-2|y-\mu|v} \rho^2(M^{-2}(y-\mu)^2)$$

$$= \text{const} \int_0^1 du (1-u^2) \frac{1 - e^{-4\frac{M^2}{B}u}}{u} = \text{const}\left(\log\frac{1}{B} + \text{const}\right)$$

in contrast to $\lim_{h\to-\infty} \beta^{(h)} = \beta$. □

It has been shown in [FT2], that if

$$\sup_{l\geqq 0}\{|\lambda_l^{(0)}|\} \sum_{h=-\infty}^{0} \beta_B^{(h)} \leqq \gamma < 1 , \tag{III.3.6}$$

then all sequences of $\lambda_l^{(h)}(B)$'s generated by the flow equation (III.3.2) converge irrespective of the sign of $\lambda_l^{(0)}$. That is, if (III.3.6) is satisfied, then the normal ground state is stable, no matter whether the potential is attractive or repulsive provided λ is small enough. Since $\lambda_l^{(0)}$ is proportional to λ and because of (III.3.3), condition (III.3.6) implies

$$\left|\lambda\left(\text{const}\log\frac{1}{B} + \text{const}\right)\right| < 1 \quad \text{or} \quad B > \text{const}\, e^{\frac{-\text{const}}{|\lambda|}}$$

in agreement with (III.2.12).

IV. Perturbation Theory

In this section, we summarize without proof (for details, see [Le]) the results concerning perturbation theory of the model (I.1, 2, 3), where V is assumed to be a rotation invariant potential in $L^1(R^3)$. Spin indices are neglected. Since one is interested in bounds which are uniform for small B, the strategy is the same as in the zero magnetic field case. For $B = 0$, it is proven in [FT1] that

- the ultraviolet parts is irrelevant, that is each graph is bounded by const^n in the ultraviolet regime;

  For the infrared part, one obtains
- two legged graphs are in general infinite, they have to be renormalized;
- four legged graphs produce $n!$'s.

In the case with magnetic field, one obtains the same results uniform in $0 \leqq B \leqq B_0$, that is

- the ultraviolet part is irrelevant, each graph is bounded by const^n with a B-independent constant;

For the infrared part, one obtains

- two legged graphs are finite, but they blow up for small B, so they have to be renormalized;
- the values of all graphs without two legged subgraphs converge to the corresponding values of the $B = 0$ graphs as distributions. The same holds for renormalized graphs if there are two legged subgraphs. Graphs containing four legged subgraphs may be bounded by const$_B^n$, but in the limit this jumps up to const$^n n!$ which is the uniform bound.

The ultraviolet and infrared part of the model are defined by the decomposition

$$S(\xi, \xi') = e^{i\frac{B}{2}\mathbf{x}\mathbf{x}'^{\perp}} \left(D^{(0)}(\xi - \xi') + \sum_{j=j_B}^{-1} D^{(j)}(\xi - \xi') \right) ,$$

where the ultraviolet part is given by

$$D^{(0)}(\xi) = \frac{B}{2\pi} \sum_{n=0}^{\infty} l_n \left(\frac{Br^2}{2} \right) e^{-\varepsilon_n \tau} [\varepsilon_n, \tau] h(\varepsilon_n^2) \tag{IV.1}$$

and the infrared part at scale j is

$$D^{(j)}(\xi) = \frac{B}{2\pi} \sum_{n=0}^{\infty} l_n \left(\frac{Br^2}{2} \right) e^{-\varepsilon_n \tau} [\varepsilon_n, \tau] f(M^{-2j} \varepsilon_n^2) , \tag{IV.2}$$

where h is a smooth monotone function obeying

$$h(x) = \begin{cases} 0 & \text{if } x \leqq 1 \\ 1 & \text{if } x \geqq M^2 , \end{cases}$$

M is a real number bigger than one and

$$f(x) = h(x)(1 - h(M^{-2}x)) = \begin{cases} h(x) & \text{if } x \leqq M^2 \\ 1 - h(M^{-2}x) & \text{if } x \geqq M^2 , \end{cases}$$

has support in $[1, M^4]$, thus $f(M^{-2j}x)$ forces $M^{2j} \leqq x \leqq M^4 M^{2j}$ and

$$1 = h(x) + \sum_{j=-\infty}^{-1} f(M^{-2j}x), \quad x > 0 . \tag{IV.3}$$

j_B is determined by $M^{j_B+2} = \varepsilon B$, $\varepsilon B \leqq \varepsilon_{n_B} \leqq (1-\varepsilon)B$. The basic estimates are given in the following

Lemma IV.1 (Covariance Estimates).
a) *There is the decomposition* $D^{(0)}(\xi) = D^{(0)}_{\text{reg}}(\xi) + D^{(0)}_{\text{sing}}(\xi)$, *where*

$$|D^{(0)}_{\text{reg}}(\xi)| \leqq \text{const} \frac{1}{1+r^4} \frac{1}{1+\tau^2}, \tag{IV.4}$$

$$D^{(0)}_{\text{sing}}(\xi) = -\rho(\xi) \frac{1}{2\pi} \frac{B/2}{\text{sh}\frac{B\tau}{2}} e^{\mu\tau} e^{-\frac{\frac{B}{2}}{\text{th}\frac{B\tau}{2}} \frac{r^2}{2}} \theta(\tau), \tag{IV.5}$$

$\rho \in C_0^\infty$ *being one for* $|\xi| < 1$ *and zero for* $|\xi| < 2$.
b) *Let* $j_B \leqq j \leqq -1$ *and* $N, N' \in \mathbb{N}$ *arbitrary. Then there are* μ *dependent constants* $c_2 > c_1 > 0$ *and a constant* $\text{const} = \text{const}_{N,N',M,\mu}$ *such that*

$$D^{(j)}(\xi) \leqq \text{const} \max_{\frac{c_1}{B} \leqq n \leqq \frac{c_2}{B}} \left\{ \left| l_n \left(\frac{Br^2}{2} \right) \right| \right\} M^j [1 + (M^j r)^N]^{-1} [1 + (M^j |\tau|)^{N'}]^{-1}. \tag{IV.6}$$

c) *There are the pointwise limits*

$$\lim_{B \to 0} D^{(0)}(\xi) = C^{(0)}(\xi), \qquad \lim_{B \to 0} D^{(j)}(\xi) = C^{(j)}(\xi), \tag{IV.7}$$

where

$$C^{(0)}(\xi) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i(\mathbf{kx} - k_0\tau)} e^{-e(\mathbf{k})\tau} [e(\mathbf{k}), \tau] h(e(\mathbf{k})^2) \tag{IV.8}$$

is the ultraviolet part and

$$C^{(j)}(\xi) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i(\mathbf{kx} - k_0\tau)} e^{-e(\mathbf{k})\tau} [e(\mathbf{k}), \tau] f(M^{-2j} e(\mathbf{k})^2) \tag{IV.9}$$

is the scale j *infrared part of the* $B = 0$ *propagator.*

Inequality (IV.4) is the same as the $B = 0$ bound and (IV.5) is smaller than the $B = 0$ bound which is $-\rho(\xi)\frac{1}{2\pi\tau} e^{\mu\tau} e^{-\frac{r^2}{2\tau}} \theta(\tau)$. Thus the fact that the ultraviolet part of (I.1) is irrelevant is an immediate consequence of the corresponding result of the $B = 0$ model. Equation (IV.6) differs from the $B = 0$ bound only in the factor $\max_{\frac{c_1}{B} \leqq n \leqq \frac{c_2}{B}} \left\{ \left| l_n \left(\frac{Br^2}{2} \right) \right| \right\}$ which would be substituted in the latter case by $(1+r)^{-\frac{1}{2}}$. However,

$$\max_{\frac{c_1}{B} \leqq n \leqq \frac{c_2}{B}} \left\{ \left| l_n \left(\frac{Br^2}{2} \right) \right| \right\} \not\leqq \text{const} \frac{1}{(1+r)^{\frac{1}{2}}}, \tag{IV.10}$$

since the estimate fails near the turning point of the Laguerre function where the decay is only $r^{-\frac{1}{3}}$, so the decomposition

$$\frac{1}{(1+r)^{\frac{1}{2}}} \leqq \sum_{k=-\infty}^{-1} \text{const}\, M^{\frac{1}{2}k} e^{-M^k(1+r)}, \tag{IV.11}$$

which is done to estimate the $B = 0$ graphs, has to be substituted by a suitable decomposition of the Laguerre function. This can be done (see [Le], Lemma IV.1.3). The net effect is, that, as in the $B = 0$ model, the bound on a labelled graph, that is a graph with scales on all lines, in $2 + 1$ dimensions can be reduced to the one dimensional case where the covariance $C_l^{(j_l)}$ at scale j_l, l being some line of the graph, obeys

$$|C_l^{(j_l)}(y)| \leqq M^{\frac{1}{2}j_l} g(M^{j_l} y), \quad g \in L^1(R) \cap L^\infty(R) . \tag{IV.12}$$

The power counting of such graphs is given by the following

Lemma IV.2 (Power Counting). *Let G_{2q} be a connected amputated graph with $2q$ external legs build up from generalized vertices or subgraphs I_{2q_v} obeying*

$$|||I_{2q_v}|||_\emptyset \equiv \sup_l \sup_{x_l} \left\{ \left(\prod_{j \neq l} \int d^d x_j \right) |I_{2q_v}(x_1, \ldots, x_{2q_v})| \right\} < \infty .$$

For $S \subset \{1, \ldots, 2q\} \neq \emptyset$ and testfunctions $f_k \in L^1(R^d) \cap L^\infty(R^d)$, introduce the norm

$$|||G_{2q}|||_S = \int \prod_{l=1}^{2q} d^d x_l \prod_{k \in S} |f_k(x_k)| |G_{2q}(x_1, \ldots, x_{2q})| .$$

Suppose each line of the graph has a covariance $C^{(j)}$ with

$$|C^{(j)}(x)| \leqq M^{\frac{d}{2}j} g(M^j x), \quad g \in L^1(R^d) \cap L^\infty(R^d) .$$

Then there are the following bounds

$$|||G_{2q}|||_\emptyset \leqq c^{\sum_v q_v - q} \prod_{v \in V} (|||I_{2q_v}|||_\emptyset M^{\frac{d}{4}(2q_v - 4)j}) M^{-\frac{d}{4}(2q-4)j} , \tag{IV.13}$$

$$|||G_{2q}|||_S \leqq c^{\sum_v q_v - q} \prod_{v \in V_{\text{int}}} (|||I_{2q_v}|||_\emptyset M^{\frac{d}{4}(2q_v - 4)j})$$
$$\times \prod_{v \in V_{\text{ext}}} (|||I_{2q_v}|||_{S_v} M^{\frac{d}{4}(2q_v - |S_v|)j}) M^{-\frac{d}{4}(2q - |S|)j} , \tag{IV.14}$$

where $c = \max\{\|g\|_{L^1}, \|g\|_{L^\infty}\}$. Thereby a vertex is called external, if at least one of its legs is integrated against a testfunction.

Iterating (IV.13, 14) for different scales, one gets a summable decay for $q_v \geqq 3$, a marginal situation which produce $n!$'s for $q_v = 2$ and an exploding factor for $q_v = 1$, that is, in the case of two legged subgraphs. They have to be renormalized. Since the magnetic field propagator $(ik_0 - \varepsilon_n)^{-1}$ has its maximum at $k_0 = 0$ and $n = n_B$, the local part of a two legged diagram $\bar{G}(\xi_1, \xi_2) = e^{i\frac{B}{2}x_1 x_2^\perp} G(\xi_1 - \xi_2)$ is given by

$$\mathbf{L} \int d\xi_1 d\xi_2 \bar{G}(\xi_1, \xi_2) \bar{\psi}(\xi_1) \psi(\xi_2) = G(n = n_B, k_0 = 0) \int d\xi \, \bar{\psi}(\xi) \psi(\xi) , \tag{IV.15}$$

where

$$G(n, k_0) = \int d\tau \, e^{ik_0 \tau} \int d^2 r l_n \left(\frac{Br^2}{2} \right) G(r, \tau) . \tag{IV.16}$$

Then a renormalized graph $(1-\mathbf{L})\int d\xi_1 d\xi_2 \bar{G}(\xi_1,\xi_2)\bar{\psi}(\xi_1)\psi(\xi_2)$ has indeed an improved power counting since ([Le], Lemma IV.3.6, 7)

$$|G(n,k_0)-G(n_B,0)| \leqq (|k_0|+|\varepsilon_n-\varepsilon_{n_B}|)\|\,|\xi|G(\xi)\|_{L^1} \leqq M^j M^{-i_G} M^{\frac{4}{3}i_G} \quad \text{(IV.17)}$$

which is an improvement of M^{j-i_G} since $j<i_G$, i_G being the lowest scale of G, because the renormalized tree expansion produces renormalized subgraphs RG only with scale $i_{RG}>j$ whereas counterterm subgraphs LG have scale $i_{LG}\leqq j$. To review the formalism of renormalization, see for example [FT1] or [FKLT1, 2], where an inductive treatment is given.

Using (IV.17), one can prove ([Le], Lemma IV.3.8) as in the $B=0$ case ([FT2], Lemma II.2), that a string of two legged subgraphs (renormalized and counterterm) may be substituted by a single covariance. Then one can apply the power counting lemma without having $q_v=1$ to obtain

Theorem IV.3. *Let $G=G(B)$ be a (necessarily connected and amputated) n^{th} order graph with $2q$ external legs contributing to the renormalized effective potential*

$$\mathscr{G}(\psi^e,\bar{\psi}^e)=\log\frac{1}{Z}\int e^{-\lambda\mathscr{V}(\psi+\psi^e,\bar{\psi}+\bar{\psi}^e)+\delta\mu(\lambda,B)\int d\xi(\psi+\psi^e)(\xi)(\bar{\psi}+\bar{\psi}^e)(\xi)}d\mu_S(\psi,\bar{\psi}).$$

Then there is a constant independent of B such that

$$|||G(B)|||_{\{1,\dots,2q\}} \leqq n!\,\text{const}^n|\lambda|^n \prod_{v\in V_{\text{int}}}|||V|||_{\emptyset}\prod_{v\in V_{\text{ext}}}|||V|||_{S_v}.$$

Furthermore,

$$\lim_{B\to 0}|||G(B)|||_{\{1,\dots,2q\}}=|||G(0)|||_{\{1,\dots,2q\}}.$$

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