Commun. Math. Phys. 173, 135 – 154 (1995)

Communications in Mathematical Physics

State Sum Models and Simplicial Cohomology

Danny Birmingham[1] *, **Mark Rakowski**[2] **

[1] Universiteit van Amsterdam, Instituut voor Theoretische Fysica, Valckenierstraat 65, NL-1018 XE Amsterdam, The Netherlands

[2] School of Mathematics, Trinity College, Dublin 2, Ireland and Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Received: 1 June 1994 / in revised form: 10 October 1994

Abstract: We study a class of subdivision invariant lattice models based on the gauge group Z_p, with particular emphasis on the four dimensional example. This model is based upon the assignment of field variables to both the 1- and 2-dimensional simplices of the simplicial complex. The property of subdivision invariance is achieved when the coupling parameter is quantized and the field configurations are restricted to satisfy a type of mod-p flatness condition. By explicit computation of the partition function for the manifold $RP^3 \times S^1$, we establish that the theory has a quantum Hilbert space which differs from the classical one.

1. Introduction

A series of Z_p lattice models was introduced in [1] which had the very special property of being subdivision invariant. This means that the partition function is insensitive to successively finer triangulations of the underlying simplicial complex. One should regard this property as the discrete analog of a continuum quantum field theory being metric independent. The formulation of these models involved the assignment of field variables to simplices of various dimensions. In three dimensions, only link based gauge fields are possible and that theory reduced to the abelian Dijkgraaf–Witten model [2]. A new four dimensional model was also introduced which involved fields associated to both 1- and 2-dimensional simplices of the simplicial complex.

A crucial element in securing the property of subdivision invariance was to restrict the allowed field configurations to those satisfying a certain "flatness" condition; in addition, a quantization of the coupling parameter was also necessary. Solutions to the flatness conditions correspond to simplicial cohomology classes of the underlying complex K. The partition function is a sum over these classes of a Boltzmann weight which captures a certain kind of "intersection" of these field

*Supported by Stichting voor Fundamenteel Onderzoek der Materie (FOM)
Email: Dannyb@phys.uva.nl
**Email: Rakowski@maths.tcd.ie

configurations. In our four dimensional model, this intersection is between $H^1(K, Z_p)$ and $H^2(K, Z_p)$.

Our aim here is to develop further the properties of this theory, and specifically to establish "non-triviality" in four dimensions. By this we mean that we have a topological field theory whose quantum Hilbert space differs from the classical one; it is simply a statement about the dependence of the theory on the coupling parameter in the Boltzmann weight. This can be contrasted with the Dijkgraaf-Witten model with gauge group Z_p, where there is no distinction between the classical and quantum Hilbert spaces. Recall that a topological field theory in $d+1$ dimensions associates a Hilbert space to each closed d-manifold. The $d+1$ dimensional theory then governs the topology changing amplitudes between d-manifolds which appear on the boundary. In [2], such a model was constructed in three dimensions and there the dimensions of the quantum Hilbert spaces for various bounding Riemann surfaces were related to conformal field theory. The novelty in our models is that one can study examples in four and higher dimensions as well. These models should also prove useful in the general classification programme of topological field theory.

After reviewing some general properties, we consider in detail the evaluation of the partition function on the manifold $RP^3 \times S^1$ which computes the dimension of the Hilbert space associated to RP^3. This can then be compared with the simple example of $S^3 \times S^1$. We relate our pedestrian formulation of these theories with the Bockstein operator, and finally we present some of the properties associated to 4-manifolds with boundary, including the behaviour under connected sum.

2. General Formalism

A lattice model is based on a simplicial complex which combinatorially encodes the topological structure of some manifold. Let us recall some of the essential ingredients that are required in such a formulation; we refer the reader to [3,4,5] for a more complete account.

Let $V = \{v_i\}$ denote a finite set of N_0 points which we will refer to as the vertices of a simplicial complex. An ordered k-simplex is an array of $k+1$ distinct vertices which we denote by,

$$[v_0, \ldots, v_k] \,. \tag{1}$$

It will usually be convenient to use simply the indices themselves to label a given vertex when no confusion will arise, so the above simplex is denoted more economically by $[0, \ldots, k]$. Pictorially, a k-simplex should be regarded as a point, line segment, triangle, or tetrahedron for k equals zero through three respectively. A simplex which is spanned by any subset of the vertices is called a face of the original simplex. An orientation of a simplex is a choice of ordering of its vertices, where we identify orderings that differ by an even permutation, but for the models described here we will require an ordering of all vertices. One then checks that the invariant we compute is actually independent of the choice made in vertex ordering.

The boundary operator ∂ on the ordered simplex $\sigma = [v_0, \ldots, v_k]$ is defined by,

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_k] \,, \tag{2}$$

where the "hat" indicates a vertex which has been omitted. It is easy to show that the composition of boundary operators is zero; $\partial^2 = 0$.

We model a closed n-dimensional manifold as a collection $K = \{\sigma_i\}$ of n-simplices constructed from the set of vertices V, subject to a few technical conditions. Most importantly, every $(n-1)$-face of any given n-simplex appears as an $(n-1)$-face of precisely two different n-simplices in the collection K. One thinks of the n-simplices then as glued together along $(n-1)$-faces. In order to ensure that the simplicial complex represents a manifold, we require the "link" of each vertex to be a combinatorial $(n-1)$-dimensional sphere. We refer the reader to [3, 5] for a more complete discussion of this condition.

The dynamical variables in the theories we construct will be objects which assign an element in the cyclic group $Z_p = Z/pZ$, which we represent as the set of integers,

$$\{0, \ldots, p-1\}, \tag{3}$$

to ordered simplices of some specified dimension. We call these dynamical variables k-colours with coefficients in Z_p, and denote the evaluation of some k-colour $B^{(k)}$ on the ordered k-simplex $[0, \cdots, k]$ by

$$\langle B^{(k)}, [0, \ldots, k] \rangle = B_{0 \cdots k} \in Z_p \,. \tag{4}$$

The superscript (k) will usually be omitted when its value is clear from context. It is important to note that we are assigning a Z_p element in a way which depends on the ordering of vertices in the simplex; we do not have the rule $B^{(1)}_{01} = -B^{(1)}_{10}$, for example. Instead, we shall assume that,

$$B^{(1)}_{10} = -B^{(1)}_{01} \bmod p\,, \tag{5}$$

and similarly extend this to a k-colour for odd permutations of the vertices. The case closest to conventional lattice gauge theory is where a 1-colour variable is assigned to every 1-simplex in the complex.

The coboundary operator δ acts on the dynamical variables as follows. Given a $(k-1)$-colour, an application of the coboundary operator produces an integer in Z, when evaluated on an ordered k-simplex, namely

$$\begin{aligned} \langle \delta B^{(k-1)}, [0, \cdots, k] \rangle &= \langle B, \partial[0, \cdots, k] \rangle \\ &= B_{123 \ldots k} - B_{023 \ldots k} + B_{013 \cdots k} - \cdots \,. \end{aligned} \tag{6}$$

We must emphasize that the above sum of integers is not taken with modular p arithmetic; it is simply an element in Z. In cases where we will need to take some combination mod-p, we will put those terms between square brackets, so for example,

$$[a+b] = a + b \bmod p\,. \tag{7}$$

There is also a cup product operation on colours which takes a k-colour $B^{(k)}$ and a l-colour $C^{(l)}$ and gives an integer in Z when evaluated on a $(k+l)$-ordered simplex:

$$\langle B \cup C, [0, \ldots, k+l] \rangle = B_{0 \cdots k} \cdot C_{k \cdots k+l} \,. \tag{8}$$

Note once again that this product is in Z and the value is not taken mod-p.

Let us now put these ingredients together and define our theories. First, we must be given some oriented simplicial complex K which we take to represent a manifold of dimension n. One then has some collection of n-simplices defined up to orientation. Take the vertex set of this complex and give it an ordering. This is done arbitrarily and we will have to show that our construction is independent of this choice, see for example [2, 6, 7]. Now we can write down an ordered collection of the n-simplices; each of the simplices is written in ascending order and a sign in front of that simplex indicates whether that ordering is positively or negatively oriented with respect to the orientation of the complex K. Let us denote this ordered set of n-simplices by K^n,

$$K^n = \sum_i \varepsilon_i \sigma_i , \tag{9}$$

where the index i runs over the ordered n-simplices σ_i and ε_i is a sign which indicates the orientation. We will assign a Boltzmann weight $W[K^n]$ to K^n by taking a product of factors, one for every n-simplex,

$$W[K^n] = \prod_i W[\sigma_i]^{\varepsilon_i} . \tag{10}$$

Each of the individual factors is a nonzero complex number and will be some function of the colours. The details of which colours we use and how the function is defined will depend on the particular model. Finally, the partition function, which we will require to be a combinatorial invariant, is defined to be a quantity which is proportional to the sum of the Boltzmann weights over all colourings,

$$Z = \frac{1}{|G|^{f(N)}} \sum_{\text{colours}} W[K^n] . \tag{11}$$

Here $|G|$ is the order of the gauge group and $f(N)$ is a function of the number of simplices of various dimensions. This function will be fixed for any given theory by scaling considerations. In the four dimensional model to be discussed next, $f(N) = N_1$, where N_1 is the number of 1-simplices in the simplicial complex. In the three dimensional Dijkgraaf-Witten model (based on a single 1-colour field), as formulated in [1], it is equal to the number of vertices N_0.

3. State Sum Model in Four Dimensions

Let us now turn our attention to the four dimensional model of interest. This model is based upon the assignment of field variables to both the 1- and 2- dimensional simplices of the simplicial complex. The Boltzmann weight of an ordered 4-simplex $[0, 1, 2, 3, 4]$ is defined by:

$$\begin{aligned} W[[0,\,1,\,2,\,3,\,4]] &= \exp\{\beta \langle B^{(2)} \cup \delta A^{(1)}, [0,\,1,\,2,\,3,\,4]\rangle\} \\ &= \exp\{\beta B_{012}(A_{23} + A_{34} - A_{24})\} , \end{aligned} \tag{12}$$

where $B^{(2)}$ and $A^{(1)}$ are 2- and 1-colour fields, respectively. Here, β is a complex number which is as yet unrestricted; we shall also find it convenient to use the scale factor $s = \exp[\beta]$. The first item on the agenda is to demonstrate that the Boltzmann weight defines a theory which is subdivision invariant. As we shall see, this requirement will enforce a quantization of the coupling parameter, and lead to a restriction on the allowed colour configurations. In order to establish the property of subdivision invariance, it is sufficient to show that the Boltzmann weight is invariant under a set of moves known as the Alexander moves [8]. Equivalently, for the case of closed manifolds, we can establish invariance by examining the behaviour under a set of (k, l) moves [9], which we now recall.

The (k, l) *Moves*:

In the four dimensional case of interest here, we have five (k, l) moves, with $k = 1, \cdots, 5$, and $k + l = 6$. It suffices to consider the first three cases; the $(4, 2)$ and $(5, 1)$ moves are inverse to the $(2, 4)$ and $(1, 5)$ moves, respectively.

The $(1, 5)$ move:

This is described by adding a new vertex x to the centre of the 4-simplex $[0, 1, 2, 3, 4]$, and linking it to the other 5 vertices. The original 4-simplex is then replaced by an assembly of five 4-simplices, written symbolically as:

$$[0, 1, 2, 3, 4] \rightarrow [x, 1, 2, 3, 4] - [x, 0, 2, 3, 4] + [x, 0, 1, 3, 4] - [x, 0, 1, 2, 4] + [x, 0, 1, 2, 3]\,. \tag{13}$$

This move is also known as an Alexander move of type 4. Note also that we declare the new vertex x to be the first in the total ordering of all vertices.

The $(2, 4)$ move:

In this case, two 4-simplices which share a common 3-simplex $[0, 1, 2, 3]$ are replaced by four 4-simplices sharing a common 1-simplex $[x, y]$:

$$[x, 0, 1, 2, 3] - [y, 0, 1, 2, 3] \rightarrow [x, y, 1, 2, 3] - [x, y, 0, 2, 3] + [x, y, 0, 1, 3] - [x, y, 0, 1, 2]\,. \tag{14}$$

Again, we place the new vertices x, y at the beginning of the vertex list.

The $(3, 3)$ move:

$$[y, z, 0, 1, 2] - [x, z, 0, 1, 2] + [x, y, 0, 1, 2] \rightarrow [x, y, z, 1, 2] - [x, y, z, 0, 2] + [x, y, z, 0, 1]\,. \tag{15}$$

We note that the 2-simplex $[0, 1, 2]$ is common to the left-hand side, with $[x, y, z]$ being common to the right.

For the case of the $(1, 5)$ move, one finds that the Boltzmann weights before and after subdivision are related by:

$$W[[0, 1, 2, 3, 4]]s^{-\langle \delta B \cup \delta A, [x, 0, 1, 2, 3, 4] \rangle} = W[[x, 1, 2, 3, 4]] W[[x, 0, 2, 3, 4]]^{-1} W[[x, 0, 1, 3, 4]] W[[x, 0, 1, 2, 4]]^{-1} W[[x, 0, 1, 2, 3]]\,. \tag{16}$$

It is immediately evident that the Boltzmann weight is not generally invariant under this move, due to the presence of the added "insertion" on the left-hand side of (16). Our task is therefore to trivialize this unwanted insertion factor, and this can indeed be achieved by imposing a restriction on the sum over colourings and on the parameter β. Subdivision invariance of this four dimensional theory is now guaranteed by imposing quantization of the coupling $s^{p^2} = 1$, as well as a restriction of the colourings to those satisfying the conditions

$$[\delta B^{(2)}] = [\delta A^{(1)}] = 0\,. \tag{17}$$

We shall refer to these restrictions as "flatness" conditions. For example, on the 2-simplex $[0, 1, 2]$, we have the restriction on the 1-colour field

$$[\delta A]_{012} \equiv [A_{12} - A_{02} + A_{01}] = 0\,. \tag{18}$$

As a reminder, we note that this particular equation can also be written as

$$[A_{01} + A_{12}] = A_{02}\,. \tag{19}$$

On the 3-simplex $[0, 1, 2, 3]$, the restriction on the 2-colour takes the form:

$$[\delta B]_{0123} \equiv [B_{123} - B_{023} + B_{013} - B_{012}] = 0\,. \tag{20}$$

With these restrictions, the product $\delta B \cup \delta A$ is clearly a multiple of p^2 and the above insertion becomes unity. The resulting identity involving the six Boltzmann weight factors shall be referred to as the $6W$ identity. It is worth pointing out that invariance is achieved here without the necessity of summing over the additional configurations attached to the vertex x.

It requires little extra work to complete the demonstration of subdivision invariance. One first notes that the remaining (k, l) moves also involve six Boltzmann weight factors, and it is easy to see that the $6W$ identity is also a statement of invariance under the $(2, 4)$ and $(3, 3)$ moves.

The subdivision invariant Boltzmann weight for the 4-simplex $[0, 1, 2, 3, 4]$ is given by:

$$W[[0,\ 1,\ 2,\ 3,\ 4]] = \exp\left\{\frac{2\pi i k}{p^2} B_{012}(A_{23} + A_{34} - [A_{23} + A_{34}])\right\}\,, \tag{21}$$

with $k \in \{0,\ 1, \cdots, p-1\}$.

At this point, we can reveal that each of the colour fields enjoys a local gauge invariance. The gauge transformation of the A field defined on the ordered 1-simplex $[0, 1]$ is defined by:

$$A'_{01} = [A - \delta\omega]_{01} = [A_{01} - \omega_1 + \omega_0]\,, \tag{22}$$

where ω is a 0-colour field defined on the vertices of the complex. For the 2-colour field B defined on the ordered 2-simplex $[0, 1, 2]$, we have a gauge transformation given by:

$$B'_{012} = [B - \delta\lambda]_{012} = [B_{012} - \lambda_{12} + \lambda_{02} - \lambda_{01}]\,, \tag{23}$$

where λ is a 1-colour defined on 1-simplices. Our task now is to show that the Boltzmann weight for the case of a closed simplicial complex is invariant with respect to independent gauge transformations of the A and B fields. As we shall see, invariance of the theory under the above transformations is not manifest, but requires both the quantization of the coupling parameter, together with the restriction on the allowed field configurations.

Under the transformation of B, one finds that

$$s^{B' \cup \delta A} = s^{B \cup \delta A} s^{-\delta\lambda \cup \delta A} = s^{B \cup \delta A} s^{-\delta(\lambda \cup \delta A)}\,, \tag{24}$$

where the first equality uses the fact that δA is an integer multiple of p due to the flatness constraint, and that s is a p^2-root of unity. Hence, the Boltzmann weight is invariant up to a total boundary term and the product of all these cancels for a closed oriented complex. To demonstrate invariance under the A field transformation, one first notes the simple identity

$$s^{B\cup\delta A} = s^{-\delta B\cup A} s^{\delta(B\cup A)} . \tag{25}$$

Invariance then follows immediately by the above argument.

As discussed in the previous section, the Boltzmann weight is initially defined for a specific ordering of the vertex set. We recall here a simple argument presented in [10] which can be used to verify that the value of the partition function is independent of this choice.

Let $V = \{v_0, \ldots, v_{N_0-1}\}$ be the vertex set of the complex, I the index set $I = \{0, \ldots, N_0 - 1\}$, and define a vertex ordering to be a map $f : V \to I$. Clearly, if f' is a different vertex ordering, then the composition $f' \circ f^{-1}$ is a permutation on the set I. Furthermore, to each permutation there is a corresponding vertex ordering. Since any permutation of I can be decomposed as a product of transpositions of consecutive numbers, it suffices to show that the Boltzmann weight is invariant when two consecutive values of the ordering f are permuted. Our task is therefore to show that the Boltzmann weights defined with an ordering f, and $f' = \pi \circ f$, coincide. Here, the permutation π is defined by $\pi(j) = j + 1, \pi(j + 1) = j$ for some j, and $\pi(i) = i$ if $i \neq \{j, j + 1\}$.

If j and $j + 1$ label vertices which do not bound a 1-simplex, then the Boltzmann weight is clearly invariant. This follows because j and $j + 1$ are simply dummy variables which can be freely exchanged, without affecting the orientation of any individual 4-simplex in the complex.

In order to establish invariance when the vertices labelled j and $j + 1$ bound a 1-simplex, we recall the definition of an Alexander move of type 1. Given an ordered 4-simplex $[v_0, v_1, v_2, v_3, v_4]$, we introduce an additional vertex x at the centre of the 1-simplex $[v_0, v_1]$, giving rise to the move

$$[v_0, v_1, v_2, v_3, v_4] \to [x, v_1, v_2, v_3, v_4] - [x, v_0, v_2, v_3, v_4] . \tag{26}$$

Since we have shown that the Boltzmann weight is invariant under the (k, l) moves, it is equivalently invariant under all Alexander moves. Thus we are free to perform an Alexander move of type 1 on the 1-simplex with vertices labelled by j and $j + 1$. This has the effect that these vertices no longer bound a 1-simplex, and by the above argument j and $j + 1$ can then be interchanged leaving the Boltzmann weight invariant. In order to recover the original complex with the permuted vertex ordering, one simply performs the inverse Alexander move of type 1.

We have already shown that the Boltzmann weight is invariant under all the (k, l) subdivision moves. However, recall that to achieve subdivision invariance, we are required to restrict the allowed field configurations to those satisfying the appropriate flatness conditions. This is effected in the state sum through the insertion of a set of delta functions which implement the required restrictions. It remains to check the behaviour of these delta functions under the (k, l) moves. As we shall see, the true subdivision invariant partition function is given by including a certain scaling factor, as discussed in relation to Eq. (11). This takes into account the redundancy in the assembly of delta functions which are present under subdivision.

In order to determine the correct scaling factor, we need to examine the behaviour of both the A and B delta functions with respect to the (k,l) moves. If we denote by ΔN_i the increase in the number of i-simplices due to a (k,l) move, then it is straightforward to check that under the $(1,5)$ move we have:

$$\Delta N_0 = 1\,, \qquad \Delta N_1 = 5\,, \qquad \Delta N_2 = 10\,,$$
$$\Delta N_3 = 10\,, \qquad \Delta N_4 = 4\,. \tag{27}$$

The changes under the $(2,4)$ move are given by

$$\Delta N_0 = 0\,, \qquad \Delta N_1 = 1\,, \qquad \Delta N_2 = 4\,,$$
$$\Delta N_3 = 5\,, \qquad \Delta N_4 = 2\,, \tag{28}$$

and of course under the (3,3) move we have $\Delta N_i = 0$, for all i.

Let us now consider the behaviour of the B delta functions under subdivision. We will first collect some formulas and then put the results together to determine the form of the scaling factor $f(N)$ referred to in Eq. (11). If we denote the additional ten B fields present after a $(1,5)$ move by:

$$I = \{B_{x01}, B_{x02}, B_{x03}, B_{x04}, B_{x12}, B_{x13}, B_{x14}, B_{x23}, B_{x24}, B_{x34}\}\,, \tag{29}$$

then one readily finds that summation over these fields yields the result

$$\begin{aligned}
&\frac{1}{|G|^4}\sum_I \delta([\delta B]_{x012})\delta([\delta B]_{x013})\delta([\delta B]_{x014})\delta([\delta B]_{x023})\delta([\delta B]_{x024})\\
&\qquad \delta([\delta B]_{x034})\delta([\delta B]_{x123})\delta([\delta B]_{x124})\delta([\delta B]_{x134})\delta([\delta B]_{x234})\\
&\qquad \delta([\delta B]_{0123})\delta([\delta B]_{0124})\delta([\delta B]_{0134})\delta([\delta B]_{0234})\delta([\delta B]_{1234})\\
&\qquad\quad = \delta([\delta B]_{0123})\delta([\delta B]_{0124})\delta([\delta B]_{0134})\delta([\delta B]_{0234})\delta([\delta B]_{1234})\,.
\end{aligned} \tag{30}$$

Here, the assembly of delta functions on the right- and left-hand sides above represent the situation before and after subdivision. We specify that the modulo-p delta function is defined by

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \text{ mod-}p \\ 0 & \text{otherwise}\,. \end{cases} \tag{31}$$

For the case of the $(2,4)$ move, one finds that summation over the additional four B fields

$$I = \{B_{xy0}, B_{xy1}, B_{xy2}, B_{xy3}\}\,, \tag{32}$$

produces the result:

$$\begin{aligned}
&\frac{1}{|G|}\sum_I \delta([\delta B]_{xy01})\delta([\delta B]_{xy02})\delta([\delta B]_{xy03})\delta([\delta B]_{xy12})\delta([\delta B]_{xy13})\\
&\qquad \delta([\delta B]_{xy23})\delta([\delta B]_{x012})\delta([\delta B]_{x013})\delta([\delta B]_{x023})\delta([\delta B]_{x123})\\
&\qquad \delta([\delta B]_{y012})\delta([\delta B]_{y013})\delta([\delta B]_{y023})\delta([\delta B]_{y123})\\
&\qquad\quad = \delta([\delta B]_{x012})\delta([\delta B]_{x013})\delta([\delta B]_{x023})\delta([\delta B]_{x123})\delta([\delta B]_{y012})\\
&\qquad\qquad \delta([\delta B]_{y013})\delta([\delta B]_{y023})\delta([\delta B]_{y123})\delta([\delta B]_{0123})\,.
\end{aligned} \tag{33}$$

Turning now to the delta function insertions for the A field, we proceed in a similar manner. The additional A fields present after a $(1,5)$ move are:

$$I = \{A_{x0}, A_{x1}, A_{x2}, A_{x3}, A_{x4}\} . \tag{34}$$

One verifies that the following relation holds:

$$\begin{aligned}
\frac{1}{|G|}\sum_I &\delta([\delta A]_{x01})\delta([\delta A]_{x02})\delta([\delta A]_{x03})\delta([\delta A]_{x04})\delta([\delta A]_{x12}) \\
&\delta([\delta A]_{x13})\delta([\delta A]_{x14})\delta([\delta A]_{x23})\delta([\delta A]_{x24})\delta([\delta A]_{x34}) \\
&\delta([\delta A]_{012})\delta([\delta A]_{013})\delta([\delta A]_{014})\delta([\delta A]_{023})\delta([\delta A]_{024}) \\
&\delta([\delta A]_{034})\delta([\delta A]_{123})\delta([\delta A]_{124})\delta([\delta A]_{134})\delta([\delta A]_{234}) \\
&\quad = \delta([\delta A]_{012})\delta([\delta A]_{013})\delta([\delta A]_{014})\delta([\delta A]_{023})\delta([\delta A]_{024}) \\
&\qquad \delta([\delta A]_{034})\delta([\delta A]_{123})\delta([\delta A]_{124})\delta([\delta A]_{134})\delta([\delta A]_{234}) .
\end{aligned} \tag{35}$$

Finally, we treat the $(2,4)$ move for the A field. There is a single additional A field $I = \{A_{xy}\}$ which is present after subdivision. Summation over this field produces the result:

$$\begin{aligned}
\sum_I &\delta([\delta A]_{xy0})\delta([\delta A]_{xy1})\delta([\delta A]_{xy2})\delta([\delta A]_{xy3})\delta([\delta A]_{012}) \\
&\delta([\delta A]_{013})\delta([\delta A]_{023})\delta([\delta A]_{123})\delta([\delta A]_{x01})\delta([\delta A]_{x02}) \\
&\delta([\delta A]_{x03})\delta([\delta A]_{x12})\delta([\delta A]_{x13})\delta([\delta A]_{x23})\delta([\delta A]_{y01}) \\
&\delta([\delta A]_{y02})\delta([\delta A]_{y03})\delta([\delta A]_{y12})\delta([\delta A]_{y13})\delta([\delta A]_{y23}) \\
&\quad = \delta([\delta A]_{012})\delta([\delta A]_{013})\delta([\delta A]_{023})\delta([\delta A]_{123})\delta([\delta A]_{x01}) \\
&\qquad \delta([\delta A]_{x02})\delta([\delta A]_{x03})\delta([\delta A]_{x12})\delta([\delta A]_{x13})\delta([\delta A]_{x23}) \\
&\qquad \delta([\delta A]_{y01})\delta([\delta A]_{y02})\delta([\delta A]_{y03})\delta([\delta A]_{y12})\delta([\delta A]_{y13}) \\
&\qquad \delta([\delta A]_{y23}) .
\end{aligned} \tag{36}$$

We can now establish the correctly scaled subdivision invariant partition function by combining the previous results. Under the $(1,5)$ move, we see that a factor $|G|^5$ must be accounted for in the combined A and B sectors, and a factor of $|G|^1$ under the $(2,4)$ move. But this is precisely how the number of 1-simplices changes under these moves. If the partition function of (11) is chosen to have $f(N) = N_1$, then it defines a subdivision invariant quantity. Specifically, we have

$$Z = \frac{1}{|G|^{N_1}} \sum_{flat} W[K^n] , \tag{37}$$

where we denote the set of allowed colours satisfying the flatness conditions by *flat*. Clearly, at the trivial $s = 1$ root of unity ($k = 0$ in Eq. (21)), the value of the partition function simply counts the number of solutions to the flatness conditions.

Our main goal is in achieving interesting behaviour at the non-trivial roots of unity where different phase factors can occur.

4. Evaluation of the Partition Function

The models described in the preceding sections require that a space be presented as a simplicial complex for their formulation. It is clear that one can only hope for a non-trivial partition function – one in which the phases (Boltzmann weights) are not all unity – when the field configurations we sum over are sufficiently interesting. This means that we need solutions to the flatness equations (17) which would not be solutions in the "strong sense" if the mod-p brackets had been removed. Perhaps the simplest example in four dimensions is the space $RP^3 \times S^1$, and we will give here a rather detailed exposition of its simplicial description.

Let us begin by presenting an economical simplicial complex for the manifold RP^3. A complex with a minimal number of 11 vertices has been given in [11], and we label its vertices by elements in the set $\{0, 1, \ldots, 9, a\}$. The complex is fully determined by specifying the 3-simplices; these are 40 in number and are given explicitly by,

$$\begin{aligned}
&+[0,2,9,a]+[0,2,3,9]-[0,2,3,7]-[0,2,7,a]+[0,5,7,a]\\
&-[0,4,5,7]+[0,1,4,5]+[0,1,3,4]-[0,1,3,9]+[0,1,6,9]\\
&+[0,1,5,6]-[0,5,6,a]-[0,6,9,a]+[4,6,9,a]+[4,6,7,9]\\
&-[4,5,7,9]+[5,7,8,9]-[5,7,8,a]+[1,7,8,a]-[1,7,8,9]\\
&-[1,6,7,9]+[1,2,6,7]+[1,2,5,6]+[1,2,4,5]-[1,2,4,a]\\
&+[1,3,4,a]-[1,3,8,a]+[1,3,8,9]+[3,5,8,9]-[2,3,5,9]\\
&+[2,4,5,9]+[2,4,9,a]-[3,5,8,a]+[3,5,6,a]-[3,4,6,a]\\
&+[3,4,6,7]+[2,3,6,7]+[2,3,5,6]+[1,2,7,a]-[0,3,4,7]\,,
\end{aligned} \tag{38}$$

where the signs denote the relative orientations of each simplex. Of course, the lower dimensional simplices are given by all those which appear as subsimplices in the above list. This complex contains 51 1-simplices and 80 2-simplices in addition to the 11 vertices and 40 3-simplices already tabulated. The Euler number is zero as required for a closed 3-manifold. One also easily checks that the boundary of the above complex vanishes and that these 3-simplices are glued together along paired 2-simplices.

Constructing the complex for $RP^3 \times S^1$ is straightforward. We begin by imagining the above complex of 40 3-simplices displayed horizontally. To each of those we add a new vertex and construct a vertical tower beneath which contains a total stack of 12 4-simplices; this is the S^1 direction which gets glued to the top along a common 3-simplex. So, for example, the tower beneath $+[0,1,3,4]$ is given explicitly by,

$$\begin{aligned}
&+[0,1,3,4,0']\\
&-[1,3,4,0',1']\\
&+[3,4,0',1',3']\\
&-[4,0',1',3',4']\\
&+[0',1',3',4',0'']\\
&-[1',3',4',0'',1'']\\
&+[3',4',0'',1'',3'']\\
&-[4',0'',1'',3'',4'']\\
&+[0'',1'',3'',4'',0]\\
&-[1'',3'',4'',0,1]\\
&+[3'',4'',0,1,3]\\
&-[4'',0,1,3,4]\,.
\end{aligned} \tag{39}$$

One sees that for each vertex x in the original complex for RP^3, two new vertices x' and x'' are required in this presentation of $RP^3 \times S^1$. The total vertex set now is then,

$$\{0,\ldots,a,0',\ldots,a',0'',\ldots,a''\} \tag{40}$$

and contains 33 elements. It is straightforward, though tedious, to enumerate all simplices in this complex. The number of each simplex type in this simplicial complex for $RP^3 \times S^1$ is,

$$\begin{aligned}
&0-\textit{simplices} && 33\\
&1-\textit{simplices} && 339\\
&2-\textit{simplices} && 1026\\
&3-\textit{simplices} && 1200\\
&4-\textit{simplices} && 480\,.
\end{aligned} \tag{41}$$

In order to compute the partition function in these theories, we need to first determine the admissible field configurations. This means finding the gauge inequivalent solutions to the equations,

$$\begin{aligned}
[\delta A]_{012} &= [A_{12} - A_{02} + A_{01}] = 0\,,\\
[\delta B]_{0123} &= [B_{123} - B_{023} + B_{013} - B_{012}] = 0\,,
\end{aligned} \tag{42}$$

where gauge transformations are given by,

$$\begin{aligned}
A'_{01} &= [A_{01} - \omega_1 + \omega_0]\,,\\
B'_{012} &= [B_{012} - \lambda_{12} + \lambda_{02} - \lambda_{01}]\,.
\end{aligned} \tag{43}$$

We remind the reader that the brackets in these equations denote that the quantity inside is to be taken mod-p.

The number of gauge inequivalent solutions to these equations is in correspondence with the first and second cohomology groups of the complex K with coefficients in Z_p. Beginning with the well known homology groups with integer coefficients for RP^3 and S^1 [3, 4],

$$H_0(RP^3) = H_3(RP^3) = H_0(S^1) = H_1(S^1) = Z\,,$$
$$H_1(RP^3) = Z_2, \qquad H_2(RP^3) = 0\,, \tag{44}$$

the Eilenberg-Zilber theorem,

$$H_n(X \times Y) = \sum_{i+j=n} H_i(X) \otimes H_j(Y) \oplus \sum_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y))\,, \tag{45}$$

computes the homology of the product $K = RP^3 \times S^1$, and one finds

$$H_0(K) = H_3(K) = H_4(K) = Z, \qquad H_1(K) = Z \oplus Z_2,\ H_2(K) = Z_2\,. \tag{46}$$

The universal coefficient theorem for cohomology,

$$H^n(X, G) = \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G)\,, \tag{47}$$

then gives the required cohomology groups which for $G = Z_p$ are,

$$H^0(K, Z_p) = H^4(K, Z_p) = Z_p\,,$$
$$H^1(K, Z_p) = H^3(K, Z_p) = \begin{cases} Z_p & \text{for } p \text{ odd} \\ Z_p \oplus Z_2 & \text{for } p \text{ even}\,, \end{cases}$$
$$H^2(K, Z_p) = \begin{cases} 0 & \text{for } p \text{ odd} \\ Z_2 \oplus Z_2 & \text{for } p \text{ even}\,. \end{cases} \tag{48}$$

Now, the partition function in this four dimensional model essentially amounts to a sum over field configurations which represent inequivalent classes in the cohomology groups $H^1(K, Z_p)$ and $H^2(K, Z_p)$. While the scale factor we have introduced, $1/G^{N_1}$, defines a subdivision invariant quantity, one could also adopt a different normalization where the partition function is precisely proportional – in a way independent of the simplicial complex – to a sum over these classes. To relate our original partition function to the latter, we need to count carefully gauge equivalent copies of all field configurations. Also, knowing the number of allowed gauge constraints is important for the purposes of explicitly finding all solutions.

For link based fields, the counting of gauge copies is the same as in lattice gauge theory where a different copy of the gauge group is assigned to each vertex. The gauge transformation here is,

$$A' = [A - \delta\omega]\,, \tag{49}$$

and what we seek is the dimension of the image of the map

$$\delta^0\colon C^0(K) \to C^1(K)\,. \tag{50}$$

Here we have explicitly attached a superscript to δ to denote the restriction to C^0. But $\delta^0(C^0)$ is isomorphic to,

$$C^0/\mathrm{Ker}(\delta^0) \tag{51}$$

and the kernel of δ^0 is 1 dimensional for a connected complex (same as $H^0(K,Z_p)$). One then sees that the image of this map has dimension $N_0 - 1$, and this is the number of links we can gauge fix.

For the 2-simplex field B, the counting is only slightly more difficult. As a gauge field, we are assigning an element in the gauge group to each link, and we seek the dimension of the image of the map,

$$\delta^1 : C^1(K) \to C^2(K), \tag{52}$$

which will tell us how many of the 2-simplex fields can be gauged away. Let us restrict the following discussion to the case of p a prime number so Z_p is a field. The kernel of δ^1, the 1-cocycles, is then parametrized by the image of δ^0 together with $H^1(K,Z_p)$. When p is prime, the latter cohomology group is then a sum of copies of Z_p; let h^1 denote the number of these copies, i.e. the dimension of $H^1(K,Z_p)$ as a vector space over Z_p. Hence the image of δ^1 has dimension,

$$N_1 - (N_0 - 1) - h^1\,, \tag{53}$$

Putting these numbers for the maximal trees together, we can, for $|G| = p$ a prime number, write the partition function (37) as

$$Z = \frac{1}{|G|^{h_1}} \sum_{flat'} W[K]\,, \tag{54}$$

where *flat'* indicates the sum is over the gauge inequivalent configurations which are the cohomology classes.

In the case of the complex for $RP^3 \times S^1$, we have 1026 equations for the A field and 1200 for B. These are highly redundant due to the Bianchi identities, but nevertheless, the number is quite large and one needs to make maximal use of the gauge freedom. For the link based gauge field A, one is allowed to set to zero (gauge fix) the fields on a maximal tree. A maximal tree is any maximal set of links which contains no closed loops, and as we saw above, that number is always one less than the number of veritces. It is trivial to pick such a set by inspection, and in this case we can gauge fix the A field on 32 1-simplices.

For the 2-colour field B, the situation is somewhat more intricate. In practice, it is not easy to identify such a set by inspection of the complex. Instead, we solved this problem by associating a vector of length 339 (one place for each link field λ) to each of the 1026 2-simplices; this vector then represents a gauge transformation. By using a program in Mathematica [12], we could find a maximal number of linearly independent vectors which was found to be 306 for our complex. There is one additional complication however. The RowReduce routine in Mathematica gives vectors which are linearly independent over the real numbers, and we seek a set which is linearly independent over Z_p. One indeed finds a single vector in that set which is not linearly independent for all p. Our gauge choice then amounts to 305 conditions. This is the number given by (53) for our complex (41) when $p = 2$. While it is not a maximal tree in general, it is an allowed choice for all p.

Another check on the gauge choice we have made here is to nominate one link variable as the independent one for each of the gauge conditions we seek to impose. One then shows that the 305 choices found previously can be made with no duplications.

At this stage, solving the equations subject to a maximal gauge choice is not difficult, though it is somewhat tedious. Typically, repeated use of the gauge conditions forced most other fields to vanish. For the A field, we found that the nonzero pieces could be parametrized in terms of two mod-p variables a and x, where $[2\,a] = 0$ and x was unconstrained. The same is also true for the B field, where we parametrize the solution in terms of b and y, with $[2\,b] = 0$ and y unconstrained.

For each of these field configurations, one then computes the Boltzmann weight which is a product of 480 factors, one for each 4-simplex in the complex. We find the Boltzmann weight

$$\exp\left[\frac{2\pi ik}{p^2} 2\,b\,a\right] . \tag{55}$$

One notices immediately that the Boltzmann weight is independent of the x and y parameters in the general solution. Since the p odd case has no non-zero solutions for a and b, there are no non-trivial phases. For p even, the sum over a and b yields

$$3 + (-1)^k = 2.2^{\delta_2(k)} , \tag{56}$$

and the partition function at $s = \exp[2\pi ik/p^2]$ is given by,

$$Z[RP^3 \times S^1] = \begin{cases} 2.2^{\delta_2(k)} & \text{for } p \text{ even} \\ 1 & \text{for } p \text{ odd} . \end{cases} \tag{57}$$

The symbol $\delta_p(k)$ denotes the mod-p delta function; its value is 1 if $k = 0$ mod-p, and 0 otherwise. In detail the calculation for p even takes the form,

$$Z \frac{1}{p^{339}} p^{305} p^{32} p^2 (3 + (-1)^k) . \tag{58}$$

The number 339 comes from the number of 1-simplices in the complex, the factors with 305 and 32 take into account the gauge equivalent copies of the solutions to the flatness equations, and the remaining factors come from summing the solutions over a, x, b, and y.

One can compare this result to that obtained for the 4-sphere S^4. In this case, a complex is easily given as the boundary of a single 5-simplex; one has the following data,

$$\begin{array}{ll} 0-\mathit{simplices} & 6 \\ 1-\mathit{simplices} & 15 \\ 2-\mathit{simplices} & 20 \\ 3-\mathit{simplices} & 15 \\ 4-\mathit{simplices} & 6 . \end{array} \tag{59}$$

The calculation of the partition function is easily seen to take the form,

$$Z[S^4] = \frac{1}{p^{15}} p^{10} p^5 1 = 1 , \tag{60}$$

though there are no interesting solutions to the flatness equations and hence no possibility of non-trivial phases in this case. Hence, the value of the partition function is independent of k.

One triangulates the space $S^3 \times S^1$ in the same manner as for the projective space we have already considered. There is no possibility of any phases and a straightforward calculation gives,

$$Z[S^3 \times S^1] = 1 . \tag{61}$$

It is also straightforward to carry out calculations on the spaces $L(p,q) \times S^1$, where $L(p,q)$ is a lens space [3], though the triangulations [11] get progressively larger. In this series, RP^3 appears as $L(2,1)$. We have also done the analogous computation of the partition function for $L(5,1) \times S^1$. We find that the Boltzmann weight depends on two variables a and b which must satisfy $[5\,a] = [5\,b] = 0$, so there are no non-unit phases when p is not a multiple of 5. When p is a multiple of 5, the Boltzmann weight takes the form

$$\exp\left[\frac{2\pi i k}{p^2} 5\,b\,a\right] . \tag{62}$$

The partition function becomes,

$$Z[L(5,1) \times S^1] = \begin{cases} 5.5^{\delta_5(k)} & \text{for } p \text{ a multiple of } 5 \\ 1 & \text{otherwise} . \end{cases} \tag{63}$$

From these examples, we see that there is generally a dependence of the partition function on the coupling parameter. Since the partition function on $M_3 \times S^1$ gives the dimension of the Hilbert space [13, 14], we see that the quantum Hilbert space associated to M_3 differs from the classical one ($k = 0$).

5. The Bockstein Operator

At first sight, the construction of this general class of models that we are considering in this paper may seem mathematically unorthodox. The motivation stemmed from a desire to realize discrete Chern–Simons and BF theories from a concrete point of view. The intuition was that one should make use of the coboundary operator on simplicial cochains in some fashion, but we did so with a cup product which differed from the usual one in so far as we did not take the product mod-p. However, we can, in retrospect, make an observation which brings the whole construction into orthodoxy, and this is the connection with the Bockstein operator [15]; the homotopy-type nature of these models is then transparent. We shall restrict attention here to the case of closed manifolds, so that the fundamental class exists.

Let x be an element of the simplicial cohomology group $H^q(K,Z_p)$, and let $\bar{x} \in C^q(K,Z)$ denote a representative of x as an integral cochain. Since $[\delta x] = 0$, this means $\delta\bar{x} = p\,u$, for some integral $(q+1)$-cochain u. The Bockstein operator

$$\beta : H^q(K,Z_p) \to H^{q+1}(K,Z_p) \tag{64}$$

is defined by

$$\beta(x) = \left[\frac{1}{p}\delta\bar{x}\right] = [u] . \tag{65}$$

In terms of the Bockstein operator and normal cup product, we can then rewrite the Boltzmann weight (21) quite simply as,

$$\exp\left[\frac{2\pi ik}{p}\langle B\cup\beta(A),\ \sigma_4\rangle\right], \tag{66}$$

where σ_4 is a 4-simplex. Of course, the Boltzmann weight for the abelian Dijkgraaf–Witten [2] theory can also be so written;

$$\exp\left[\frac{2\pi ik}{p}\langle A\cup\beta(A),\ \sigma_3\rangle\right], \tag{67}$$

where the connection with Chern–Simons theory is striking. The key observation is that the Bockstein operator and not the simple coboundary operator is what is relevant in the construction of these models with gauge group Z_p.

The extension of this theory to 5 and higher dimensions is then transparent, and one takes the Boltzmann weight,

$$\exp\left[\frac{2\pi ik}{p}\langle A\cup\beta(A)\cup\cdots\cup\beta(A),\ \sigma_{2m+1}\rangle\right], \tag{68}$$

where we have m factors of $\beta(A)$. It is worth noting that theories whose partition functions lead to Gauss sums such as the 3d Dijkgraaf–Witten theory will also appear in 7d where the link field A in (67) becomes a 3-colour field. In this case, partition functions will generally be complex valued unlike the class of $B\cup\beta(A)$ theories where (for p prime at least) they are real.

6. Manifolds with Boundary

A general axiomatic framework was presented for topological quantum field theory (*TQFT*) in [13]; see also [14, 16, 17]. As we have seen, the general class of abelian models considered here can be expressed in terms of standard modulo-p cohomological operations, and one expects the axioms of *TQFT* to be satisfied by these models.

Let K be a 4-manifold with boundary ∂K. The partition function of these models is well defined, and represents a transition amplitude when we specify the field configurations on the boundary components. We take the field configurations on the boundary to the flat, and define the allowed field configuration on K to be all flat configurations which extend those specified on the boundary. The partition function remains subdivision invariant as long as we keep the triangulation on the boundary fixed [17]. It is convenient, however, to rescale the partition function by

$$Z'[K] = w^{N_1(\partial K)}Z[K], \tag{69}$$

where $w=\sqrt{|G|}$, and $N_1(\partial K)$ is the number of 1-simplices on the boundary ∂K. This scaling gives the following gluing rule,

$$Z'[K,\tau_1,\tau_2] = \sum_{\tau_3} Z'[K_1,\tau_1,\tau_3]\cdot Z'[K_2,\tau_3,\tau_2]\,. \tag{70}$$

Here K_1 is a cobordism between boundary manifolds Σ_1 and Σ_3 with fixed flat field configurations τ_1 and τ_3, and similarly for K_2. K represents a composition of K_1 and K_2 and the above sum is over all intermediate flat field configurations.

Consider now a gauge transformation of the B field, when a boundary ∂K is present. From (24), we see that the Boltzmann weights are related by a phase factor depending only on the boundary values of the fields, namely:

$$s^{\langle B' \cup \delta A, K\rangle} = s^{\langle B \cup \delta A, K\rangle} s^{-\langle \lambda \cup \delta A, \partial K\rangle} . \tag{71}$$

When computing the partition function on K, we sum over all allowed field configurations with fixed boundary data, and thus we see that the partition function also transforms with this phase factor. It is equally simple to determine the behaviour under a gauge transformation of the A field.

For the purposes of illustration let us consider the case of a 4-manifold of the form $M_3 \times I$, where I is the unit interval, and M_3 is some bounding 3-manifold. The value of the partition function then gives a transition amplitude between the two copies of M_3. Given that $H^*(M_3 \times I, Z_p) = H^*(M_3, Z_p)$, we know that the transition matrix $Z'_{if}[M_3 \times I]$ must be diagonal. Moreover, because of subdivision invariance,

$$Z'_{if}[M_3 \times I] = \sum_j Z'_{ij}[M_3 \times I] \cdot Z'_{jf}[M_3 \times I] , \tag{72}$$

which shows that any diagonal element can only be 0 or 1. Since the two copies of the bounding manifold appear with opposite orientation, the behaviour of the partition function under changes of cohomology representatives (gauge transformations) is given by

$$Z'_{i'f'} = \exp[i\alpha_i] Z'_{if} \exp[-i\alpha_f] , \tag{73}$$

where Z'_{if} denotes the transition amplitude between the initial and final copy of M_3. It is then clear, for example, that the diagonal elements along with the trace of Z'_{if} are gauge invariant quantities. In particular, we have the result

$$\sum_i Z'_{ii}[M_3 \times I] = Z[M_3 \times S^1] , \tag{74}$$

which we can interpret as the dimension of the Hilbert space associated to M_3. In taking the above trace, one must take into account the gauge equivalent copies of the fixed boundary data.

Indeed, the value of the partition function for the manifold $M_3 \times S^1$ can be obtained more easily by first performing the computation on $M_3 \times I$, and then taking the trace. In this way, one sees that a vertical tower (see (39)) of only four 4-simplices is required.

Up to gauge equivalence, field configurations on any single boundary component M_3 are in one to one correspondence with the set $H^1(M_3, Z_p) \times H^2(M_3, Z_p)$. We can define a complex vector space $V(M_3)$ associated to M_3 by taking it to be the vector space freely generated by this set of field configurations. This we might call the classical Hilbert space of M_3 [2]. However, the partition function on the cylinder represents a transition amplitude, and the map $Z'_{if}[M_3 \times I]$ may well have a non-zero kernel. The quantum Hilbert space $H(M_3)$ is defined to be,

$$H(M_3) = V(M_3)/\mathrm{Ker}(Z'_{if}[M_3 \times I]) , \tag{75}$$

and its dimension is given by the above trace (74). Thus, our computations show that the quantum Hilbert space of these models is generically different from the classical Hilbert space. By classical Hilbert space one simply refers to the situation where $k = 0$ and the kernel of Z'_{if} on the cylinder always vanishes.

It is interesting to go further and identify precisely the zero modes in the examples we have computed. For the $p = 2$, $k = 1$ theory with RP^3 boundaries, the calculation of the partition function on $RP^3 \times S^1$ indicates a Hilbert space of dimension 2, compared to the classical $k = 0$ result of 4. Hence there must be two zero modes in the "propagator" on the cylinder. One finds that these zero modes correspond to the non-trivial $H^1(RP^3, Z_2)$ configuration which means that the Hilbert space is in correspondence with $H^2(RP^3, Z_2)$. Whether this is a general phenomenon, or is something peculiar to the lens spaces, is not known.

Let us now examine the behaviour of the partition function with respect to the connected sum of manifolds. For manifolds M_1 and M_2, the connected sum is denoted by $M = M_1 \# M_2$. The manifold M is produced by first excising a 4-ball from each of the components M_1 and M_2, which are then identified along their common S^3 boundary. According to the tenets of the axiomatic approach, the partition function on a closed manifold can be computed by cutting the manifold along a common boundary, and then taking the pairing between the state vectors in the dual Hilbert spaces.

Now, in order to obtain a relationship between the partition function $Z[M]$ and that of its components $Z[M_1]$ and $Z[M_2]$, we recall the value of the partition function $Z[S^3 \times S^1] = 1$. We thus see that the S^3 Hilbert space is 1-dimensional. It is then a simple consequence of 1-dimensional linear algebra [18] to see that the following relationship holds:

$$Z[M_1 \# M_2] Z[S^4] = Z[M_1] Z[M_2] \,. \tag{76}$$

Since $Z[S^4] = 1$ for the four dimensional model under consideration, the partition function behaves multiplicatively under connected sum.

7. Concluding Remarks

The four dimensional model we have considered here is part of a generic construction available in all dimensions [1]. The cornerstone of these state sum models is a partition function which is a sum over simplicial cohomology classes of a certain Boltzmann weight. This phase factor arises from the modulo-p valued intersection form between those classes. In particular, we have seen that the relevant "kinetic" operator used to define these actions is provided by the Bockstein coboundary operator. We remark that one can also consider the "non-kinetic" type models [19] as providing observables for the theories presented here. In three dimensions, it reduces to the Dijkgraaf–Witten model [2] which is related to group cohomology. Such a connection is not transparent in general.

We have shown here that the four dimensional model is indeed non-trivial in the sense that interesting phases can be obtained; without them the model only counts cohomology classes. In this regard, one sees that the dimensions of the quantum Hilbert spaces are in general different from the classical dimensions. By construction, these models lead to piecewise linear invariants; however, with the insight that they can be formulated (in the closed case) in terms of the Bockstein operator, one sees that the models yield homotopy-type invariants.

In [7], a four dimensional subdivision invariant model was described in terms of combinatorial data. Subsequently, it was established in [20, 21], that the partition function was expressible in terms of the Euler and Pontryagin numbers, and as such encoded classical topological data. On the other hand, this model could then be viewed as describing classical invariants in terms of a quantum state sum. The models presented here can similarly be viewed as providing a quantum state sum formulation of classical modulo-p cohomological data. Perhaps it is also worth remarking on how these models differ from the structures presented in [22]. Quite apart from having to address issues of regularizing the formally divergent path integrals of those models, one is also dealing with cohomology with real coefficients; as such, the models are insensitive to the presence of any torsion subgroups. However, as we have seen for the models discussed here, the essence of non-triviality lies in the presence of torsion in the cohomology groups.

Acknowledgements. M.R. would like to thank Ron Kantowski for computer access at the University of Oklahoma.

References

1. Birmingham, D., Rakowski, M.: On Dijkgraaf–Witten Type Invariants. University of Amsterdam preprint, ITFA-94-07, February 1994, hepth/9402138
2. Dijkgraaf, R., Witten, E.: Topological Gauge Theories and Group Cohomology. Commun. Math. Phys. **129**, 393 (1990)
3. Munkres, J.: Elements of Algebraic Topology. Menlo Park, CA: Addison-Wesley, 1984
4. Rotman, J.: An Introduction of Algebraic Topology. New York: Springer, Berlin, Heidelberg, 1988.
5. Stillwell, J.: Classical Topology and Combinatorial Group Theory. New York: Springer, Berlin, Heidelberg, 1980
6. Yetter, D.N.: State-sum Invariants of 3-Manifolds Associated to Artinian Semisimple Tortile Categories. Top. and its App. **58**(1), 47 (1994)
7. Crane, L., Yetter, D.N.: Categorical Construction of 4D Topological Quantum Field Theories In: Quantum Topology. L.H. Kauffman, R.A. Baadhio, (eds.), Singapore World Scientific, 1993, p. 120
8. Alexander, J.W.: The Combinatorial Theory of Complexes. Ann. Math. **31**, 292 (1930)
9. Pachner, U.: P.L. Homeomorphic Manifolds are Equivalent by Elementary Shelling. Eur. J. Comb. **12**, 129 (1991)
10. Felder, G., Grandjean, O.: On Combinatorial Three-Manifold Invariants. In: Low-Dimensional Topology and Quantum Field Theory. ed. H. Osborn, New York: Plenum Press, 1993
11. Brehm, U., Swiatkowski, J.: Triangulations of Lens Spaces with Few Simplices. T. U. Berlin preprint, 1993
12. Wolfram, S.: Mathematica. New York: Addison-Wesley, 1988
13. Atiyah, M.F.: Topological Quantum Field Theories. Publ. Math. IHES **68**, 175 (1989)
14. Turaev, V.G.: Quantum Invariants of 3-Manifolds. Publ. de l'Institute de Recherche Mathématique Avancée 509/P-295 CNRS, Strasbourg, France (1992).
15. Dubrovin, B., Fomenko, A., Novikov, S.: Modern Geometry-Methods and Applications, Part III. New York: Springer-Verlag, 1990
16. Yetter, D.N.: Triangulations and TQFT's In: Quantum Topology. L.H. Kauffman and R.A. Baadhio, (eds), Singapore: World Scientific, 1993 p. 354
17. Wakui, M.: On Dijkgraaf–Witten Invariant for 3-Manifolds. Osaka J. Math. **29**, 675 (1992)
18. Witten, E.: Quantum Field Theory and the Jones Polynomial. Commun. Math. Phys. **121**, 351 (1989)
19. Birmingham, D., Rakowski, M.: Discrete Quantum Field Theories and the Intersection Form. Mod. Phys. Lett. **A9**, 2265 (1994)

20. Roberts, J.D.: Skein Theory and Turaev-Viro Invariants. Cambridge University preprint, 1993
21. Crane, L., Kauffman, L.H., Yetter, D.: Evaluating the Crane–Yetter Invariant. In: Quantum Topology. L. H. Kauffman and R. A. Baadhio, (eds), Singapore: World Scientific, 1993, p. 131
22. Schwarz, A.S.: The Partition Function of a Degenerate Quadratic Functional and the Ray–Singer Invariants. Lett. Math. Phys. **2**, 247 (1978)

Communicated by G. Felder