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Vertex-IRF Correspondence and Factorized L-operators for an Elliptic R-operator

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Abstract: As for an elliptic R-operator which satisfies the Yang–Baxter equation, the incoming and outgoing intertwining vectors are constructed, and the vertex-IRF correspondence for the elliptic R-operator is obtained. The Boltzmann weights of the corresponding IRF model satisfy the star-triangle relation. By means of these intertwining vectors, the factorized L-operators for the elliptic R-operator are also constructed. The vertex-IRF correspondence and the factorized L-operators for Belavin's R-matrix are reproduced from those of the elliptic R-operator.

0. Introduction

In [12, 13, 14] we have introduced an infinite-dimensional R-matrix. It is a new solution of the Yang–Baxter equation. By means of the Fourier transformation of the R-matrix, we defined an R-operator acting on some function space. This R-operator also satisfies the Yang–Baxter equation. Since this operator is deeply linked to analytic properties of an elliptic theta function, we call it the elliptic R-operator. We have shown some properties satisfied by the elliptic R-operator, for example, first inversion relation, fusion procedure, etc. For the trigonometric degenerate case of the elliptic R-operator, we proved that the finite-dimensional, trigonometric R-matrices are constructed from the R-operator through restricting the domain of the R-operator to some finite-dimensional subspaces. Recently Felder and Pasquier [4] showed that Belavin's R-matrix [3, 11] can be obtained through restricting the domain of a modified version of the elliptic R-operator to a suitable finite-dimensional subspace.

In [1], Baxter has introduced the intertwining vectors for the eight-vertex model. Jimbo, Miwa and Okado [8] constructed the outgoing intertwining vectors between Belavin's vertex model and the $A_{n-1}^{(1)}$ face model. We call this relation the vertex-IRF correspondence for Belavin's R-matrix. Hasegawa [6, 7], Quano and Fujii [10] defined the incoming intertwining vectors which are the dual vectors of the outgoing intertwining vectors. Then they constructed the factorized L-operators for Belavin's R-matrix. The vertex-IRF correspondence plays a central role in their methods.

The aim of this paper is to extend the result above to the elliptic R-operator.

Our strategy to construct factorized L-operators for the elliptic R-operator is as follows. At first we define incoming intertwining vectors $\bar{\phi}_\lambda^\kappa$ of the elliptic R-operator $\check{R}(\xi)$ and establish a vertex-IRF correspondence. The vertex-IRF correspondence plays the most important role in this paper. Next we find finite-dimensional subspaces with the following property (cf. Theorem 1.3);

$$\check{R}(\xi_{12})(V_k(\xi_1) \otimes V_k(\xi_2 + \mu)) \subset V_k(\xi_2) \otimes V_k(\xi_1 + \mu),$$

where $\xi_{12} := \xi_1 - \xi_2$. Then we define outgoing intertwining vectors $\phi_{\mathbf{k}}(\xi)_\lambda^\kappa(z) \in V_k(\xi + |\lambda|_{\mathbf{k}})$, which are the duals of $\bar{\phi}_\lambda^\kappa|_{V_k(\xi+|\lambda|_{\mathbf{k}})}$. Making use of the properties of the incoming and outgoing intertwining vectors, we can easily construct factorized L-operators.

This paper is organized as follows. In Sect. 1, we review the properties of the elliptic R-operator $\check{R}(\xi)$ proved in [12, 13, 14, 4]. In Sect. 2, we shall define incoming intertwining vectors $\bar{\phi}_\lambda^\kappa$ and Boltzmann weights $\check{W}\begin{bmatrix} & \kappa' & \\ \lambda & \xi & \nu \\ & \kappa & \end{bmatrix}$ of an IRF model. Then we have the vertex-IRF correspondence for the elliptic R-operator (Theorem 2.1).

Theorem 0.1 (Vertex-IRF Correspondence). *For $\lambda, \kappa, \nu \in \Lambda$,*

$$\bar{\phi}_\lambda^\kappa \otimes \bar{\phi}_\kappa^\nu \check{R}(\xi) = \sum_{\kappa' \in \Lambda} \check{W}\begin{bmatrix} & \kappa' & \\ \lambda & \xi & \nu \\ & \kappa & \end{bmatrix} \bar{\phi}_\lambda^{\kappa'} \otimes \bar{\phi}_{\kappa'}^\nu .$$

Because the elliptic R-operator satisfies the Yang–Baxter equation, we can show that these Boltzmann weights satisfy the star-triangle relation. This IRF model can be regarded as the limiting case $n \to \infty$ of the $A_{n-1}^{(1)}$ face model. In Sect. 3, making use of the results obtained by Felder and Pasquier [4], we shall construct outgoing intertwining vectors in the same way as [6, 7, 10]. We can consequently define factorized L-operators $\check{L}_{\mathbf{k}}(\xi)$ (Theorem 3.4).

Theorem 0.2 (Factorized L-operator). *For $\xi_1, \xi_2 \notin \mathbb{Z} + \mathbb{Z}\tau$,*

$$(1 \otimes \check{R}(\xi_{12}))(\check{L}_{\mathbf{k}}(\xi_1) \otimes 1)(1 \otimes \check{L}_{\mathbf{k}}(\xi_2)) = (\check{L}_{\mathbf{k}}(\xi_2) \otimes 1)(1 \otimes \check{L}_{\mathbf{k}}(\xi_1))(\check{R}(\xi_{12}) \otimes 1).$$

In the last section, after stating the results obtained by Felder and Pasquier [4] more precisely, we show that the vertex-IRF correspondence and the factorized L-operators for the elliptic R-operator imply those for Belavin's R-matrix.

1. Review of the Properties of an Elliptic R-operator

In this section, we review the construction and the properties of an elliptic R-operator [4, 12, 13, 14]. We fix $\tau \in \mathbb{C}$ such that $\operatorname{Im} \tau > 0$ and define an open subset $D \subset \mathbb{C}$ by

$$D = \left\{ z \in \mathbb{C}; |\operatorname{Im} z| < \frac{\operatorname{Im} \tau}{2} \right\} .$$

Let $\mathscr{V}$ be a space of all functions f holomorphic on D and such that

$$f(z+1) = f(z) \qquad \forall z \in D\,.$$

Similarly let $\mathscr{V}\hat{\otimes}\mathscr{V}$ be a space of all functions f holomorphic on $D \times D$ with the property

$$f(z_1+1, z_2) = f(z_1, z_2+1) = f(z_1, z_2) \quad \forall z_1, z_2 \in D\,.$$

Now we define an elliptic R-operator $\check{R}(\xi)$ on $\mathscr{V}\hat{\otimes}\mathscr{V}$. Let μ be a complex number such that $\mu \notin \mathbb{Z} + \mathbb{Z}\tau$ and let $\vartheta_1(z) = \vartheta_1(z,\tau)$ be an elliptic theta function

$$\vartheta_1(z) = \sum_{m\in\mathbb{Z}} \exp\left[\pi\sqrt{-1}\left(m+\frac{1}{2}\right)^2\tau + 2\pi\sqrt{-1}\left(m+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right].$$

The elliptic theta function $\vartheta_1(z)$ satisfies the following properties.

(1) $\vartheta_1(z)$ is entire,
(2) $\vartheta_1(z+1) = -\vartheta_1(z)$,
(3) $\vartheta_1(z+\tau) = -\exp(-2\pi\sqrt{-1}z - \pi\sqrt{-1}\tau)\vartheta_1(z)$,
(4) $\vartheta_1(z)$ has simple zeros at $z \in \mathbb{Z} + \mathbb{Z}\tau$,
(5) $\vartheta_1(z)$ satisfies the three term equation (cf. [15] p. 461);

$$\begin{aligned}
&\vartheta_1(x+y)\vartheta_1(x-y)\vartheta_1(z+w)\vartheta_1(z-w)\\
&\quad + \vartheta_1(x+z)\vartheta_1(x-z)\vartheta_1(w+y)\vartheta_1(w-y)\\
&\quad + \vartheta_1(x+w)\vartheta_1(x-w)\vartheta_1(y+z)\vartheta_1(y-z)\\
&\quad = 0\,,
\end{aligned}$$

(6) $\vartheta_1(-z) = -\vartheta_1(z)$.

Definition 1.1 (Elliptic R-operator). *For $f \in \mathscr{V}\hat{\otimes}\mathscr{V}$, we define*

$$(\check{R}(\xi)f)(z_1,z_2) := \frac{\vartheta_1(\xi)\vartheta_1(z_{21}-\mu)\vartheta_1'(0)}{\vartheta_1(-\mu)\vartheta_1(z_{21})} f(z_2,z_1) + \frac{\vartheta_1(z_{21}-\xi)\vartheta_1'(0)}{\vartheta_1(z_{21})} f(z_1,z_2)\,,$$

where $z_{21} := z_2 - z_1$, $\vartheta_1'(0) = \frac{\partial\vartheta_1}{\partial z}(z,\tau)|_{z=0}$ and $\xi \in \mathbb{C}$. The complex number ξ is called a spectral parameter.

We set $X = \{(z_1,z_2) \in D \times D;\ z_{21} \in \mathbb{Z}\}$. By the property (4) of the elliptic theta function $\vartheta_1(z)$, the function $\check{R}(\xi)f$ has the singularities at the points $(z_1,z_2) \in X$. The lemma below tells us that all singularities are removable.

Lemma 1.1. *There is a unique function F holomorphic on $D \times D$ and such that $F(z_1,z_2) = (\check{R}(\xi)f)(z_1,z_2)$ for $(z_1,z_2) \in D \times D \backslash X$.*

Proof. For $(z_1,z_2) \in D \times D \backslash X$ and $m \in \mathbb{Z}$,

$$\begin{aligned}
&(\check{R}(\xi)f)(z_1,z_2) \\
&= \frac{\vartheta_1(\xi)\vartheta_1'(0)f(z_2,z_1)}{\vartheta_1(-\mu)} \cdot \frac{\vartheta_1(z_{21}-\mu-m)-\vartheta_1(-\mu)}{z_2-z_1-m} \cdot \frac{z_2-z_1-m}{\vartheta_1(z_{21}-m)} \\
&\quad + \vartheta_1(\xi)\vartheta_1'(0)\frac{f(z_2-m,z_1)-f(z_1,z_1)}{z_2-z_1-m} \cdot \frac{z_2-z_1-m}{\vartheta_1(z_{21}-m)} \\
&\quad + \vartheta_1'(0)\frac{z_2-z_1-m}{\vartheta_1(z_{21}-m)}\left\{f(z_1,z_1)\frac{\vartheta_1(z_{21}-\xi-m)-\vartheta_1(-\xi)}{z_2-z_1-m}\right. \\
&\quad \left. +\vartheta_1(z_{21}-\xi-m)\frac{f(z_1,z_2-m)-f(z_1,z_1)}{z_2-z_1-m}\right\}.
\end{aligned}$$

Thus there is a function F continuous on $D \times D$ and such that $F(z_1,z_2) = (\check{R}(\xi)f)(z_1,z_2)$ for $(z_1,z_2) \in D \times D \backslash X$. In fact, we define

$$F(z_1,z_2) = \begin{cases} \frac{\vartheta_1(\xi)\vartheta_1'(-\mu)+\vartheta_1'(-\xi)\vartheta_1(-\mu)}{\vartheta_1(-\mu)} f(z,z) + \vartheta_1(\xi)\left(\frac{\partial f}{\partial z_1}(z,z) - \frac{\partial f}{\partial z_2}(z,z)\right), & (z_1,z_2) = (z,z+m), \\ (\check{R}(\xi)f)(z_1,z_2), & \text{otherwise}. \end{cases}$$

Making use of the Riemann removable singularity theorem (cf. [5]), this function F is holomorphic on $D \times D$. □

We also denote by $\check{R}(\xi)f$ this holomorphic function F. It is easy to see that

$$(\check{R}(\xi)f)(z_1+1,z_2) = (\check{R}(\xi)f)(z_1,z_2+1) = (\check{R}(\xi)f)(z_1,z_2)$$

for $(z_1,z_2) \in D \times D$. Hence $\check{R}(\xi)f \in \mathscr{V}\hat{\otimes}\mathscr{V}$ for $f \in \mathscr{V}\hat{\otimes}\mathscr{V}$, and $\check{R}(\xi)$ is an operator on $\mathscr{V}\hat{\otimes}\mathscr{V}$ as a result.

Let $\mathscr{V}\hat{\otimes}\mathscr{V}\hat{\otimes}\mathscr{V}$ be a space of all functions f on $D \times D \times D$ and such that

$$f(z_1+1,z_2,z_3) = f(z_1,z_2+1,z_3) = f(z_1,z_2,z_3+1) = f(z_1,z_2,z_3) \ \forall\ z_1,z_2,z_3 \in D.$$

By the three term equation of $\vartheta_1(z)$ (the property (5)), we get the following theorem.

Theorem 1.2 ([12, 13, 14]). *$\check{R}(\xi)$ satisfies the Yang–Baxter equation on $\mathscr{V}\hat{\otimes}\mathscr{V}\hat{\otimes}\mathscr{V}$;*

$$(1 \otimes \check{R}(\xi_{12}))(\check{R}(\xi_{13}) \otimes 1)(1 \otimes \check{R}(\xi_{23})) = (\check{R}(\xi_{23}) \otimes 1)(1 \otimes \check{R}(\xi_{13}))(\check{R}(\xi_{12}) \otimes 1), \tag{1.1}$$

where $\xi_{ij} = \xi_i - \xi_j$.

For $\xi \in \mathbb{C}$ and $n = 1,2,\dots$, let $V_n(\xi)$ be a space of all functions f holomorphic on $\mathbb{C}$ and such that

$$f(z+1) = f(z),$$
$$f(z+\tau) = (-1)^n \exp(2\pi\sqrt{-1}(\xi - nz))f(z).$$

It is well known that $V_n(\xi)$ has dimension n. We easily see that

$$\left\{ \vartheta \begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{n}{2} \end{bmatrix} (\xi - nz, n\tau) \exp(\pi\sqrt{-1}nz) \right\}_{j \in \mathbb{Z}/n\mathbb{Z}} \tag{1.2}$$

is a basis of $V_n(\xi)$. Here $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau)$ is a theta function with rational characteristics;

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{m \in \mathbb{Z}} \exp[\pi\sqrt{-1}(m+a)^2\tau + 2\pi\sqrt{-1}(m+a)(z+b)] .$$

In [4] Felder and Pasquier show the following.

Theorem 1.3 ([4]). $\check{R}(\xi_{12})(V_n(\xi_1) \otimes V_n(\xi_2 + \mu)) \subset V_n(\xi_2) \otimes V_n(\xi_1 + \mu)$.

Remark 1.1. Let $\mathscr{V}^-$ be a space of all functions f holomorphic on D and such that

$$f(z+1) = -f(z) .$$

We set $\mathscr{V}^- \hat{\otimes} \mathscr{V}^-$ and $\mathscr{V}^- \hat{\otimes} \mathscr{V}^- \hat{\otimes} \mathscr{V}^-$ in the same way as $\mathscr{V}$. Then we can define the elliptic R-operator $\check{R}(\xi)$ on $\mathscr{V}^- \hat{\otimes} \mathscr{V}^-$, which is the same as in Definition 1.1. It is easy to see that $\check{R}(\xi)$ on $\mathscr{V}^- \hat{\otimes} \mathscr{V}^-$ satisfies the Yang–Baxter equation (1.1).

We denote $V_n^-(\xi)$ as a space of all functions f holomorphic on $\mathbb{C}$ and such that

$$f(z+1) = -f(z) ,$$

$$f(z+\tau) = (-1)^n \exp 2\pi\sqrt{-1}\left(\xi - nz + \frac{\tau}{2}\right) f(z) .$$

We have

$$\check{R}(\xi_{12})(V_n^-(\xi_1) \otimes V_n^-(\xi_2 + \mu)) \subset V_n^-(\xi_2) \otimes V_n^-(\xi_1 + \mu) .$$

A basis of $V_n^-(\xi)$ is as follows.

$$\left\{ \vartheta \begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{n}{2} \end{bmatrix} (\xi - nz, n\tau) \exp(\pi\sqrt{-1}(n+1)z) \right\}_{j \in \mathbb{Z}/n\mathbb{Z}} .$$

Remark 1.2. Let $\mathscr{M}$ be a space of the meromorphic functions on $\mathbb{C}^2$. Then we note that the elliptic R-operator $\check{R}(\xi)$ can be regarded as an operator on $\mathscr{M}$ and satisfies the Yang–Baxter equation (1.1).

2. Incoming Intertwining Vectors and Vertex-IRF Correspondence

In what follows $\mu \in \mathbb{R} \setminus \mathbb{Z}$, and let Λ be a set of sequences $\lambda = (\lambda_i)$ $(i \in \mathbb{Z})$ such that

$$\lambda_i \in D ,$$

$$\lambda_{ij} := \lambda_i - \lambda_j \notin \mathbb{Z} + \mathbb{Z}\mu \quad \forall i \neq j \in \mathbb{Z} .$$

We take $r \in \mathbb{R}$ such that $r \notin \mathbb{Q} + \mathbb{Q}\mu$, and set

$$\eta_i := ir \quad (i \in \mathbb{Z}) .$$

Then $\eta = (\eta_i) \in \Lambda$. Hence, for any μ, the set Λ is not empty. For $i \in \mathbb{Z}$, we define the sequences $\varepsilon_i = (\delta_{ij})$ $(j \in \mathbb{Z})$, and for $\lambda \in \Lambda$, let $\lambda + \mu\varepsilon_i$ denote the sequence

$$(\lambda + \mu\varepsilon_i)_j = \begin{cases} \lambda_j, & j \neq i, \\ \lambda_i + \mu, & j = i. \end{cases}$$

We note that $\lambda + \mu\varepsilon_i \in \Lambda$ for all $i \in \mathbb{Z}$ by the definition of Λ.

Definition 2.1 (Boltzmann Weight of the IRF Model). *For $\lambda, \kappa, \kappa', \nu \in \Lambda$, Boltzmann weights $\check{W}\begin{bmatrix} & \kappa' & \\ \lambda & \xi & \nu \\ & \kappa & \end{bmatrix} \in \mathbb{C}$ of an interaction-round-a-face* (IRF) *model are given as follows* (cf. [1, 6, 7, 8, 10]). *For $\lambda \in \Lambda$, we put*

$$\check{W}\begin{bmatrix} & \lambda + \mu\varepsilon_i & \\ \lambda & \xi & \lambda + 2\mu\varepsilon_i \\ & \lambda + \mu\varepsilon_i & \end{bmatrix} := \frac{\vartheta_1(\mu - \xi)\vartheta_1'(0)}{\vartheta_1(\mu)},$$

$$\check{W}\begin{bmatrix} & \lambda + \mu\varepsilon_i & \\ \lambda & \xi & \lambda + \mu(\varepsilon_i + \varepsilon_j) \\ & \lambda + \mu\varepsilon_i & \end{bmatrix} := \frac{\vartheta_1(\lambda_{ji} - \xi)\vartheta_1'(0)}{\vartheta_1(\lambda_{ji})} \quad (i \neq j),$$

$$\check{W}\begin{bmatrix} & \lambda + \mu\varepsilon_j & \\ \lambda & \xi & \lambda + \mu(\varepsilon_i + \varepsilon_j) \\ & \lambda + \mu\varepsilon_i & \end{bmatrix} := \frac{\vartheta_1(\xi)\vartheta_1(\lambda_{ji} - \mu)\vartheta_1'(0)}{\vartheta_1(\lambda_{ji})\vartheta_1(-\mu)} \quad (i \neq j),$$

otherwise we set

$$\check{W}\begin{bmatrix} & \kappa' & \\ \lambda & \xi & \nu \\ & \kappa & \end{bmatrix} := 0.$$

Next we define incoming intertwining vectors of the elliptic R-operator.

Definition 2.2 (Incoming Intertwining Vector). *For $\lambda, \kappa \in \Lambda$, define an incoming intertwining vector $\bar{\phi}_\lambda^\kappa \in \mathscr{V}^*$ as follows*:

$$\bar{\phi}_\lambda^\kappa f := \begin{cases} f(\lambda_i), & \exists i \in \mathbb{Z} \text{ s.t. } \kappa = \lambda + \mu\varepsilon_i, \\ 0, & \textit{otherwise}. \end{cases}$$

The incoming intertwining vectors are the Dirac delta functions essentially. By Definition 1.1 we can get a vertex-IRF correspondence for the elliptic R-operator.

Theorem 2.1 (Vertex-IRF Correspondence). *For $\lambda, \kappa, \nu \in \Lambda$,*

$$\bar{\phi}_\lambda^\kappa \otimes \bar{\phi}_\kappa^\nu \check{R}(\xi) = \sum_{\kappa' \in \Lambda} \check{W}\begin{bmatrix} & \kappa' & \\ \lambda & \xi & \nu \\ & \kappa & \end{bmatrix} \bar{\phi}_\lambda^{\kappa'} \otimes \bar{\phi}_{\kappa'}^\nu, \tag{2.1}$$

where both sides are the operators $\mathscr{V} \hat{\otimes} \mathscr{V} \to \mathbb{C}$.

It is to be noted that, by Definition 2.1 and 2.2, both sides of Eq. (2.1) are zero unless there exist $i, j \in \mathbb{Z}$ such that $\kappa = \lambda + \mu\varepsilon_i$, $\nu = \lambda + \mu(\varepsilon_i + \varepsilon_j)$. The other cases are as follows:

$$\bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_i} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_i}^{\lambda+2\mu\varepsilon_i} \check{R}(\xi) = \frac{\vartheta_1(\mu-\xi)\vartheta_1'(0)}{\vartheta_1(\mu)} \bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_i} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_i}^{\lambda+2\mu\varepsilon_i},$$

$$\begin{aligned}\bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_i} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_i}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)} \check{R}(\xi) &= \frac{\vartheta_1(\lambda_{ji}-\xi)\vartheta_1'(0)}{\vartheta_1(\lambda_{ji})} \bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_i} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_i}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)} \\ &\quad + \frac{\vartheta_1(\xi)\vartheta_1(\lambda_{ji}-\mu)\vartheta_1'(0)}{\vartheta_1(\lambda_{ji})\vartheta_1(-\mu)} \bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_j} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_j}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)},\end{aligned}$$

for $i \neq j$.

Since $\check{R}(\xi)$ satisfies the Yang–Baxter equation (1.1), we can show

Proposition 2.2. *The Boltzmann weights of the IRF model satisfy the star-triangle relation*;

$$\begin{aligned}&\sum_{\kappa' \in \Lambda} \check{W}\begin{bmatrix} & \kappa' & \\ \kappa & \xi_{12} & \gamma \\ & \nu & \end{bmatrix} \check{W}\begin{bmatrix} & \alpha & \\ \lambda & \xi_{13} & \kappa' \\ & \kappa & \end{bmatrix} \check{W}\begin{bmatrix} & \beta & \\ \alpha & \xi_{23} & \gamma \\ & \kappa' & \end{bmatrix} \\ &= \sum_{\kappa' \in \Lambda} \check{W}\begin{bmatrix} & \kappa' & \\ \lambda & \xi_{23} & \nu \\ & \kappa & \end{bmatrix} \check{W}\begin{bmatrix} & \beta & \\ \kappa' & \xi_{13} & \gamma \\ & \nu & \end{bmatrix} \check{W}\begin{bmatrix} & \alpha & \\ \lambda & \xi_{12} & \beta \\ & \kappa' & \end{bmatrix}, \qquad (2.2)\end{aligned}$$

for $\lambda, \kappa, \nu, \alpha, \beta, \gamma \in \Lambda$.

Proof. Unless there exist $i,j,k \in \mathbb{Z}$ such that $\kappa = \lambda + \mu\varepsilon_i$, $\nu = \lambda + \mu(\varepsilon_i + \varepsilon_j)$ and $\gamma = \lambda + \mu(\varepsilon_i + \varepsilon_j + \varepsilon_k)$, both sides of Eq. (2.2) are zero. Then we assume that

$$\kappa = \lambda + \mu\varepsilon_i, \qquad \nu = \lambda + \mu(\varepsilon_i + \varepsilon_j), \qquad \gamma = \lambda + \mu(\varepsilon_i + \varepsilon_j + \varepsilon_k) \quad (i,j,k \in \mathbb{Z}).$$

Moreover both sides of Eq. (2.2) are zero unless

$$\alpha = \lambda + \mu\varepsilon_i, \quad \lambda + \mu\varepsilon_j \quad \text{or} \quad \lambda + \mu\varepsilon_k$$

and

$$\beta = \lambda + \mu(\varepsilon_i + \varepsilon_j), \quad \lambda + \mu(\varepsilon_i + \varepsilon_k) \quad \text{or} \quad \lambda + \mu(\varepsilon_j + \varepsilon_k),$$

so it suffices to show Eq. (2.2) in each case.

Since $\check{R}(\xi)$ satisfies the Yang–Baxter equation (1.1),

$$\begin{aligned}&((1 \otimes \check{R}(\xi_{12}))(\check{R}(\xi_{13}) \otimes 1)(1 \otimes \check{R}(\xi_{23}))f)(z_1,z_2,z_3) \\ &= ((\check{R}(\xi_{23}) \otimes 1)(1 \otimes \check{R}(\xi_{13}))(\check{R}(\xi_{12}) \otimes 1)f)(z_1,z_2,z_3).\end{aligned}$$

Putting $z_1 = \lambda_i$, $z_2 = \lambda_j + \mu\delta_{ij}$ and $z_3 = \lambda + \mu(\delta_{ik} + \delta_{jk})$ in the coefficient of $f(z_1,z_2,z_3)$, we obtain Eq. (2.2) in the case $\alpha = \lambda + \mu\varepsilon_i$ and $\beta = \lambda + \mu(\varepsilon_i + \varepsilon_j)$. We can prove the other cases in the similar way, so we omit the proof. □

Remark 2.1. We define an incoming intertwining vector $\bar{\phi}_{\lambda}^{\kappa} \in (\mathscr{V}^-)^*$ in the same way as Definition 2.2; for $f \in \mathscr{V}^-$,

$$\bar{\phi}_{\lambda}^{\kappa} f := \begin{cases} f(\lambda_i), & \exists i \in \mathbb{Z} \text{ s.t. } \kappa = \lambda + \mu\varepsilon_i, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, we also get a vertex-IRF correspondence; for $\lambda, \kappa, \nu \in \Lambda$,

$$\bar{\phi}_{\lambda}^{\kappa} \otimes \bar{\phi}_{\kappa}^{\nu} \check{R}(\xi) = \sum_{\kappa' \in \Lambda} \check{W} \begin{bmatrix} & \kappa' & \\ \lambda & \xi & \nu \\ & \kappa & \end{bmatrix} \bar{\phi}_{\lambda}^{\kappa'} \otimes \bar{\phi}_{\kappa'}^{\nu} .$$

3. Outgoing Intertwining Vectors and Factorized L-operators

To begin with, we define outgoing intertwining vectors of the elliptic R-operator (cf. [6,7,10]).

Let k_1 and k_2 be integers such that $k_1 \leqq k_2$, and we set $\mathbf{k} := (k_1, k_2)$ and $k = k_2 - k_1 + 1$. For $\lambda, \kappa \in \Lambda$ and $k_1 \leqq j \leqq k_2$, we define $\bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\kappa j} \in \mathbf{C}$ by

$$\bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\kappa j} := \bar{\phi}_{\lambda}^{\kappa} \left(\vartheta \begin{bmatrix} \frac{1}{2} - \frac{j-k_1}{k} \\ \frac{k}{2} \end{bmatrix} (\xi + |\lambda|_{\mathbf{k}} - kz, k\tau) \exp(\pi\sqrt{-1}kz) \right) ,$$

where $|\lambda|_{\mathbf{k}} = \sum_{i=k_1}^{k_2} \lambda_i$.

Proposition 3.1. *For $\lambda \in \Lambda$ and $\xi \notin \mathbf{Z} + \mathbf{Z}\tau$, the $k-by-k$ matrix $(\bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\lambda+\mu\varepsilon_i\, j})_{k_1 \leqq i, j \leqq k_2}$ is invertible.*

Proof. Since

$$(\bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\lambda+\mu\varepsilon_i, j})_{k_1 \leqq i, j \leqq k_2} = \operatorname{diag}(\exp \pi\sqrt{-1}k\lambda_{k_1}, \cdots, \exp \pi\sqrt{-1}k\lambda_{k_2}) \times \left(\vartheta \begin{bmatrix} \frac{1}{2} - \frac{j-k_1}{k} \\ \frac{k}{2} \end{bmatrix} (\xi + |\lambda|_{\mathbf{k}} - k\lambda_i, k\tau) \right)_{k_1 \leqq i, j \leqq k_2} ,$$

it suffices to prove

$$\det \left(\vartheta \begin{bmatrix} \frac{1}{2} - \frac{j-k_1}{k} \\ \frac{k}{2} \end{bmatrix} (\xi + |\lambda|_{\mathbf{k}} - k\lambda_i, k\tau) \right)_{k_1 \leqq i, j \leqq k_2} \neq 0 .$$

The Weyl–Kac denominator formula for $A_{k-1}^{(1)}$ (cf. [9,7]) yields

$$\det \left(\vartheta \begin{bmatrix} \frac{1}{2} - \frac{j}{k} \\ \frac{k}{2} \end{bmatrix} (ku_i, k\tau) \right)_{1 \leqq i, j \leqq k} = (\sqrt{-1}\eta(\tau))^{-\frac{1}{2}(k-1)(k-2)} \vartheta_1 \left(\sum_{i=1}^{k} u_i \right) \prod_{1 \leqq j < i \leqq k} \vartheta_1(u_{ij}) .$$

Here $\eta(\tau)$ is Dedekind's η-function

$$\eta(\tau) = \exp \frac{\pi\sqrt{-1}\tau}{12} \prod_{m=1}^{\infty} (1 - \exp 2\pi\sqrt{-1}m\tau) .$$

Then we obtain

$$\det\left(\vartheta\begin{bmatrix}\frac{1}{2}-\frac{j-k_1}{k}\\ \frac{k}{2}\end{bmatrix}(\xi+|\lambda|_{\mathbf{k}}-k\lambda_i,k\tau)\right)_{k_1\leqq i,j\leqq k_2}$$

$$=(-1)^{k-1}\left(\vartheta\begin{bmatrix}\frac{1}{2}-\frac{j}{k}\\ \frac{k}{2}\end{bmatrix}(\xi+|\lambda|_{\mathbf{k}}-k\lambda_{i+k_1-1},k\tau)\right)_{1\leqq i,j\leqq k}$$

$$=(-1)^{k-1}(\sqrt{-1}\eta(\tau))^{-\frac{1}{2}(k-1)(k-2)}\vartheta_1(\xi)\prod_{k_1\leqq i<j\leqq k_2}\vartheta_1(\lambda_{ij}),$$

thereby completing the proof. □

The proposition above says that for $\lambda,\kappa\in\Lambda$, $k_1\leqq j\leqq k_2$ and $\xi\notin\mathbb{Z}+\mathbb{Z}\tau$, there exist $\phi_{\mathbf{k}}(\xi)^{\kappa}_{\lambda j}\in\mathbb{C}$ which are characterized by the following duality relations;

$$\begin{cases}\sum_{i=k_1}^{k_2}\phi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_i}_{\lambda j}\bar{\phi}_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_i\,l}_{\lambda}=\delta_{jl}\,,\\ \sum_{i=k_1}^{k_2}\bar{\phi}_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_j\,i}_{\lambda}\phi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_l}_{\lambda_i}=\delta_{jl}\,,\end{cases}\tag{3.1}$$

and for $\kappa\neq\lambda+\mu\varepsilon_i\,(k_1\leqq\forall i\leqq k_2)$ we set

$$\phi_{\mathbf{k}}(\xi)^{\kappa}_{\lambda j}:=0\,.$$

Definition 3.1 (Outgoing Intertwining Vector). *For $\lambda,\kappa\in\Lambda$ and $\xi\notin\mathbb{Z}+\mathbb{Z}\tau$, an outgoing intertwining vector $\phi_{\mathbf{k}}(\xi)^{\kappa}_{\lambda}(z)\in V_k(\xi+|\lambda|_{\mathbf{k}})$ of the elliptic R-operator is defined as follows (cf. (1.2)):*

$$\phi_{\mathbf{k}}(\xi)^{\kappa}_{\lambda}(z):=\sum_{j=k_1}^{k_2}\phi_{\mathbf{k}}(\xi)^{\kappa}_{\lambda j}\,\vartheta\begin{bmatrix}\frac{1}{2}-\frac{j-k_1}{k}\\ \frac{k}{2}\end{bmatrix}(\xi+|\lambda|_{\mathbf{k}}-kz,k\tau)\exp(\pi\sqrt{-1}kz)\,.$$

Equation (3.1) is equivalent to

$$\begin{cases}\sum_{i=k_1}^{k_2}\phi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_i}_{\lambda}(z)\bar{\phi}^{\lambda+\mu\varepsilon_i}_{\lambda}=\mathrm{id}\quad\text{on } V_k(\xi+|\lambda|_{\mathbf{k}})\,,\\ \bar{\phi}^{\lambda+\mu\varepsilon_i}_{\lambda}(\phi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_j}_{\lambda})=\delta_{ij}\qquad\text{for } k_1\leqq i,j\leqq k_2\,.\end{cases}\tag{3.2}$$

The outgoing intertwining vectors satisfy the following:

Proposition 3.2. *For $\lambda,\kappa,\nu\in\Lambda$ and $\xi_1,\xi_2\notin\mathbb{Z}+\mathbb{Z}\tau$,*

$$(\check{R}(\xi_{12})\phi_{\mathbf{k}}(\xi_1)^{\kappa}_{\lambda}\otimes\phi_{\mathbf{k}}(\xi_2)^{\nu}_{\kappa})(z_1,z_2)=\sum_{\kappa'\in\Lambda}\phi_{\mathbf{k}}(\xi_2)^{\kappa'}_{\lambda}(z_1)\otimes\phi_{\mathbf{k}}(\xi_1)^{\nu}_{\kappa'}(z_2)\check{W}\begin{bmatrix} & \kappa & \\ \lambda & \xi_{12} & \nu\\ & \kappa' & \end{bmatrix}.$$

Proof. By Definition 2.1 and 3.1, it suffices to show

$$(\check{R}(\xi_{12})\phi_{\mathbf{k}}(\xi_1)^{\lambda+\mu\varepsilon_i}_{\lambda}\otimes\phi_{\mathbf{k}}(\xi_2)^{\lambda+\mu(\varepsilon_i+\varepsilon_j)}_{\lambda+\mu\varepsilon_i})(z_1,z_2)$$

$$=\sum_{l=k_1}^{k_2}\phi_{\mathbf{k}}(\xi_2)^{\lambda+\mu\varepsilon_l}_{\lambda}(z_1)\otimes\phi_{\mathbf{k}}(\xi_1)^{\lambda+\mu(\varepsilon_i+\varepsilon_j)}_{\lambda+\varepsilon_l}(z_2)\check{W}\begin{bmatrix} & \lambda+\mu\varepsilon_i & \\ \lambda & \xi_{12} & \lambda+\mu(\varepsilon_i+\varepsilon_j)\\ & \lambda+\mu\varepsilon_l & \end{bmatrix}$$

for any $\lambda \in \Lambda$ and $k_1 \leqq \forall i, j \leqq k_2$. With the aid of Theorem 2.1 and Eq. (3.2), we obtain for $k_1 \leqq \forall a, b \leqq k_2$,

$$\bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_l} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_l}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)}((\check{R}(\xi_{12})\phi_{\mathbf{k}}(\xi_1)_{\lambda}^{\lambda+\mu\varepsilon_a} \otimes \phi_{\mathbf{k}}(\xi_2)_{\lambda+\mu\varepsilon_a}^{\lambda+\mu(\varepsilon_a+\varepsilon_b)})(z_1, z_2))$$

$$= \sum_{l=k_1}^{k_2} \check{W}\begin{bmatrix} & \lambda+\mu\varepsilon_l & \\ \lambda & \xi_{12} & \lambda+\mu(\varepsilon_i+\varepsilon_j) \\ & \lambda+\mu\varepsilon_i & \end{bmatrix}$$

$$\times (\bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_l} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_l}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)})(\phi_{\mathbf{k}}(\xi_1)_{\lambda}^{\lambda+\mu\varepsilon_a}(z_1) \otimes \phi_{\mathbf{k}}(\xi_2)_{\lambda+\mu\varepsilon_a}^{\lambda+\mu(\varepsilon_a+\varepsilon_b)}(z_2))$$

$$= \check{W}\begin{bmatrix} & \lambda+\mu\varepsilon_a & \\ \lambda & \xi_{12} & \lambda+\mu(\varepsilon_i+\varepsilon_j) \\ & \lambda+\mu\varepsilon_i & \end{bmatrix} \delta_{\lambda+\mu(\varepsilon_i+\varepsilon_j)\ \lambda+\mu(\varepsilon_a+\varepsilon_b)} .$$

Then

$$\sum_{i,j=k_1}^{k_2} (\phi_{\mathbf{k}}(\xi_2)_{\lambda}^{\lambda+\mu\varepsilon_i}(z_1) \otimes \phi_{\mathbf{k}}(\xi_1)_{\lambda+\mu\varepsilon_i}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)}(z_2))$$

$$\times (\bar{\phi}_{\lambda}^{\lambda+\mu\varepsilon_i} \otimes \bar{\phi}_{\lambda+\mu\varepsilon_i}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)})((\check{R}(\xi_{12})\phi_{\mathbf{k}}(\xi_1)_{\lambda}^{\lambda+\mu\varepsilon_a} \otimes \phi_{\mathbf{k}}(\xi_2)_{\lambda+\mu\varepsilon_a}^{\lambda+\mu(\varepsilon_a+\varepsilon_b)})(z_1, z_2))$$

$$= \sum_{i,j=k_1}^{k_2} \phi_{\mathbf{k}}(\xi_2)_{\lambda}^{\lambda+\mu\varepsilon_i}(z_1) \otimes \phi_{\mathbf{k}}(\xi_1)_{\lambda+\mu\varepsilon_i}^{\lambda+\mu(\varepsilon_i+\varepsilon_j)}(z_2)$$

$$\times \check{W}\begin{bmatrix} & \lambda+\mu\varepsilon_a & \\ \lambda & \xi_{12} & \lambda+\mu(\varepsilon_i+\varepsilon_j) \\ & \lambda+\mu\varepsilon_i & \end{bmatrix} \delta_{\lambda+\mu(\varepsilon_i+\varepsilon_j)\ \lambda+\mu(\varepsilon_a+\varepsilon_b)}$$

$$= \sum_{i=k_1}^{k_2} \phi_{\mathbf{k}}(\xi_2)_{\lambda}^{\lambda+\mu\varepsilon_i}(z_1) \otimes \phi_{\mathbf{k}}(\xi_2)_{\lambda+\mu\varepsilon_i}^{\lambda+\mu(\varepsilon_a+\varepsilon_b)}(z_2) \check{W}\begin{bmatrix} & \lambda+\mu\varepsilon_a & \\ \lambda & \xi_{12} & \lambda+\mu(\varepsilon_a+\varepsilon_b) \\ & \lambda+\mu\varepsilon_i & \end{bmatrix} .$$

By virtue of Definition 3.1 and Theorem 1.3 we deduce

$$(\check{R}(\xi_{12})\phi_{\mathbf{k}}(\xi_1)_{\lambda}^{\lambda+\mu\varepsilon_a} \otimes \phi_{\mathbf{k}}(\xi_2)_{\lambda+\mu\varepsilon_a}^{\lambda+\mu(\varepsilon_a+\varepsilon_b)})(z_1, z_2) \in V_k(\xi_2 + |\lambda|_{\mathbf{k}}) \otimes V_k(\xi_1 + |\lambda|_{\mathbf{k}} + \mu) .$$

From Eq. (3.2), we are led to the desired result. □

For $\lambda, \kappa \in \Lambda$ and $\xi \notin \mathbb{Z} + \mathbb{Z}\tau$, we define an operator $\check{L}_{\mathbf{k}}(\xi)_{\lambda}^{\kappa} : \mathscr{V} \to \mathscr{V}$ by

$$(\check{L}_{\mathbf{k}}(\xi)_{\lambda}^{\kappa} f)(z) := \phi_{\mathbf{k}}(\xi)_{\lambda}^{\kappa}(z)\bar{\phi}_{\lambda}^{\kappa} f \quad (f \in \mathscr{V}) .$$

Theorem 2.1 and Proposition 3.2 say

Lemma 3.3. *For $\lambda, \nu \in \Lambda$ and $\xi_1, \xi_2 \notin \mathbb{Z} + \mathbb{Z}\tau$,*

$$\sum_{\kappa \in \Lambda} \check{R}(\xi_{12})\check{L}_{\mathbf{k}}(\xi_1)_{\lambda}^{\kappa} \otimes \check{L}_{\mathbf{k}}(\xi_2)_{\kappa}^{\nu} = \sum_{\kappa \in \Lambda} \check{L}_{\mathbf{k}}(\xi_2)_{\lambda}^{\kappa} \otimes \check{L}_{\mathbf{k}}(\xi_1)_{\kappa}^{\nu}\check{R}(\xi_{12})$$

on $\mathscr{V} \hat{\otimes} \mathscr{V}$.

Proof. For $f \in \mathscr{V} \hat{\otimes} \mathscr{V}$,

$$\sum_{\kappa \in \Lambda} (\check{R}(\xi_{12}) \check{L}_{\mathbf{k}}(\xi_1)^{\kappa}_{\lambda} \otimes \check{L}_{\mathbf{k}}(\xi_2)^{\nu}_{\kappa} f)(z_1, z_2)$$

$$= \sum_{\kappa \in \Lambda} (\check{R}(\xi_{12}) \phi_{\mathbf{k}}(\xi_1)^{\kappa}_{\lambda} \otimes \phi_{\mathbf{k}}(\xi_2)^{\nu}_{\kappa})(z_1, z_2) \cdot (\bar{\phi}^{\kappa}_{\lambda} \otimes \bar{\phi}^{\nu}_{\kappa}) f$$

$$= \sum_{\kappa, \kappa' \in \Lambda} \phi_{\mathbf{k}}(\xi_2)^{\kappa'}_{\lambda}(z_1) \otimes \phi_{\mathbf{k}}(\xi_1)^{\nu}_{\kappa'}(z_2) \check{W} \begin{bmatrix} & \kappa & \\ \lambda & \xi_{12} & \nu \\ & \kappa' & \end{bmatrix} (\bar{\phi}^{\kappa}_{\lambda} \otimes \bar{\phi}^{\nu}_{\kappa}) f$$

$$= \sum_{\kappa' \in \Lambda} \phi_{\mathbf{k}}(\xi_2)^{\kappa'}_{\lambda}(z_1) \otimes \phi_{\mathbf{k}}(\xi_1)^{\nu}_{\kappa'}(z_2) (\bar{\phi}^{\kappa'}_{\lambda} \otimes \bar{\phi}^{\nu}_{\kappa'} \check{R}(\xi_{12}) f)$$

$$= \sum_{\kappa \in \Lambda} (\check{L}_{\mathbf{k}}(\xi_2)^{\kappa}_{\lambda} \otimes \check{L}_{\mathbf{k}}(\xi_1)^{\nu}_{\kappa} \check{R}(\xi_{12}) f)(z_1, z_2) .$$

We have thus proved the lemma. □

Now we are in the position to construct factorized L-operators for the elliptic R-operator. Let $\mathscr{W}$ be a space of all $\mathbb{C}$-valued functions on Λ, and let $\mathscr{V} \hat{\otimes} \mathscr{W}$ (resp. $\mathscr{W} \hat{\otimes} \mathscr{V}$) be a space of all functions $g : D \times \Lambda \to \mathbb{C}$ (resp. $\Lambda \times D \to \mathbb{C}$) such that $g(\,\cdot\,, \lambda) \in \mathscr{V}$ (resp. $g(\lambda, \,\cdot\,) \in \mathscr{V}$) for any $\lambda \in \Lambda$. We define a factorized L-operator $\check{L}_{\mathbf{k}}(\xi) : \mathscr{V} \hat{\otimes} \mathscr{W} \to \mathscr{W} \hat{\otimes} \mathscr{V}$ as follows [2,6,7,10]. For $g \in \mathscr{V} \hat{\otimes} \mathscr{W}$ and $\xi \notin \mathbb{Z} + \mathbb{Z}\tau$,

$$(\check{L}_{\mathbf{k}}(\xi) g)(\lambda, z) := \sum_{\kappa \in \Lambda} (\check{L}_{\mathbf{k}}(\xi)^{\kappa}_{\lambda} g(\,\cdot\,, \kappa))(z) . \tag{3.3}$$

For $\lambda \in \Lambda$ we set $\delta^{\lambda} \in \mathscr{W}$ as follows:

$$\delta^{\lambda}(\kappa) = \delta_{\lambda\kappa} .$$

We note that $\mathscr{W} = \prod_{\kappa \in \Lambda} \mathbb{C} \delta^{\kappa}$ (cf. [6]). Then, for $f \in \mathscr{V}$,

$$(\check{L}_{\mathbf{k}}(\xi)(f \otimes \delta^{\kappa}))(\lambda, z) = (\check{L}_{\mathbf{k}}(\xi)^{\kappa}_{\lambda} f)(z) ,$$

and Eq. (3.3) is hence equivalent to

$$\check{L}_{\mathbf{k}}(\xi)(f \otimes \delta^{\kappa}) = \sum_{\lambda \in \Lambda} \delta^{\lambda} \otimes \check{L}_{\mathbf{k}}(\xi)^{\kappa}_{\lambda} f .$$

We define $\mathscr{V} \hat{\otimes} \mathscr{V} \hat{\otimes} \mathscr{W}$ (resp. $\mathscr{W} \hat{\otimes} \mathscr{V} \hat{\otimes} \mathscr{V}$) by a space of all functions $g : D \times D \times \Lambda \to \mathbb{C}$ (resp. $\Lambda \times D \times D \to \mathbb{C}$) such that $g(\,\cdot\,, \,\cdot\,, \lambda) \in \mathscr{V} \hat{\otimes} \mathscr{V}$ (resp. $g(\lambda, \,\cdot\,, \,\cdot\,) \in \mathscr{V} \hat{\otimes} \mathscr{V}$) for any $\lambda \in \Lambda$. By means of Lemma 3.3, we immediately obtain the following theorem.

Theorem 3.4 (Factorized L-operator). *For* $\xi_1, \xi_2 \notin \mathbb{Z} + \mathbb{Z}\tau$,

$$(1 \otimes \check{R}(\xi_{12}))(\check{L}_{\mathbf{k}}(\xi_1) \otimes 1)(1 \otimes \check{L}_{\mathbf{k}}(\xi_2)) = (\check{L}_{\mathbf{k}}(\xi_2) \otimes 1)(1 \otimes \check{L}_{\mathbf{k}}(\xi_1))(\check{R}(\xi_{12}) \otimes 1) ,$$

where both sides are the operators $\mathscr{V} \hat{\otimes} \mathscr{V} \hat{\otimes} \mathscr{W} \to \mathscr{W} \hat{\otimes} \mathscr{V} \hat{\otimes} \mathscr{V}$.

Remark 3.1. In the same way as this section, we can construct factorized L-operators for $\check{R}(\xi)$ on $\mathscr{V}^- \hat{\otimes} \mathscr{V}^-$ by using $V_n^-(\xi)$ instead of $V_n(\xi)$ (cf. Remark 1.1 and 2.1).

In this case, outgoing intertwining vectors are characterized by the following duality relation:

$$\begin{cases} \sum_{i=k_1}^{k_2} \phi_{\mathbf{k}}(\xi)_{\lambda j}^{\lambda+\mu\varepsilon_i} \bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\lambda+\mu\varepsilon_i \ell} = \delta_{j\ell} , \\ \sum_{i=k_1}^{k_2} \bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\lambda+\mu\varepsilon_j i} \phi_{\mathbf{k}}(\xi)_{\lambda i}^{\lambda+\mu\varepsilon_\ell} = \delta_{j\ell} . \end{cases}$$

Here, for $\lambda, \kappa \in \Lambda$, $k_1 \leqq j \leqq k_2$, we define $\bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\kappa j} \in \mathbb{C}$ as follows (cf. Remark 1.1):

$$\bar{\phi}_{\mathbf{k}}(\xi)_{\lambda}^{\kappa j} := \bar{\phi}_{\lambda}^{\kappa} \left(\vartheta \begin{bmatrix} \frac{1}{2} - \frac{j-k_1}{k} \\ \frac{k}{2} \end{bmatrix} (\xi + |\lambda| - kz, k\tau) \exp(\pi\sqrt{-1}(k+1)z) \right) .$$

4. Vertex-IRF Correspondence and Factorized L-operators for Belavin's R-matrix

In this section, we apply Theorem 2.1 to the R-matrix obtained through restricting the domain of the elliptic R-operator to some finite-dimensional subspace. Then we will show that the vertex-IRF correspondence for Belavin's R-matrix proved by Baxter [1], Jimbo, Miwa and Okado [8] is obtained from Theorem 2.1. Moreover we will construct the factorized L-operators for Belavin's R-matrix obtained by Hasegawa [6], Quano and Fujii [10]. First let us state the results proved by Felder and Pasquier [4] more precisely.

For $k = 1, 2, \ldots$, let $\tilde{V}_k(\xi)$ be a space of entire functions f of one variable such that

$$f(z+1) = (-1)^k f(z) ,$$

$$f(z+\tau) = (-1)^k \exp(-2\pi\sqrt{-1}(kz - \xi + \tfrac{k\tau}{2})) f(z) .$$

We note that $\tilde{V}_k(\xi) \subset \mathscr{V}$ if k is even and that $\tilde{V}_k(\xi) \subset \mathscr{V}^-$ if k is odd. In the same fashion as Theorem 1.3 and Remark 1.1, we obtain

$$\check{R}(\xi_{12})(\tilde{V}_k(\xi_1) \otimes \tilde{V}_k(\xi_2 + \mu)) \subset \tilde{V}_k(\xi_2) \otimes \tilde{V}_k(\xi_1 + \mu) .$$

The space $\tilde{V}_k(\xi)$ is of k dimensions and a basis is given by

$$\{e_j(\xi)(z) := \vartheta \begin{bmatrix} \frac{1}{2} - \frac{j}{k} \\ \frac{k}{2} \end{bmatrix} (\xi - kz, k\tau)\}_{j \in \mathbb{Z}/k\mathbb{Z}} .$$

For $k = 1, 2, \ldots$, define a translation operator $T_k(\xi)$ on the space of all holomorphic functions on $\mathbb{C}$ [4] by

$$(T_k(\xi)f)(z) := f\left(z - \frac{\xi}{k}\right) .$$

$T_k(\xi)$ maps isomorphically $\tilde{V}_k := \tilde{V}_k(0)$ onto $\tilde{V}_k(\xi)$. We modify the elliptic R-operator as

$$\check{R}_k(\xi_{12}) := T_k(\xi_2)^{-1} \otimes T_k(\xi_1 + \mu)^{-1} \check{R}(\xi_{12}) T_k(\xi_1) \otimes T_k(\xi_2 + \mu)\big|_{\tilde{V}_k \otimes \tilde{V}_k} .$$

We note that $\check{R}_k(\xi_{12})$ is determined by the difference ξ_{12}. In fact,

$$(\check{R}_k(\xi)f)(z_1,z_2) = \frac{\vartheta_1(\xi)\vartheta_1(z_{21}+\frac{\xi+\mu}{k}-\mu)\vartheta_1'(0)}{\vartheta_1(-\mu)\vartheta_1(z_{21}+\frac{\xi+\mu}{k})} f\left(z_2+\frac{\mu}{k}, z_1-\frac{\mu}{k}\right)$$

$$+\frac{\vartheta_1(z_{21}+\frac{\xi+\mu}{k}-\xi)\vartheta_1'(0)}{\vartheta_1(z_{21}+\frac{\xi+\mu}{k})} f\left(z_1-\frac{\xi}{k}, z_2+\frac{\xi}{k}\right) .$$

Felder and Pasquier prove

Theorem 4.1 ([4]). *$\check{R}_k(\xi)$ preserves $\tilde{V}_k \otimes \tilde{V}_k$ and obeys the Yang–Baxter equation* (1.1).

Let $\{e^j\}_{j\in\mathbb{Z}/k\mathbb{Z}} \subset \tilde{V}_k^*$ be the dual basis of $\{e_j := e_j(0)\} \subset \tilde{V}_k$;

$$e^i(e_j) = \delta_{ij} .$$

Now we define an operator $\check{R}_k(\xi)^*$ on $\tilde{V}_k^* \otimes \tilde{V}_k^*$, the transpose of $\check{R}_k(\xi)$ on $\tilde{V}_k \otimes \tilde{V}_k$.

$$(\check{R}_k(\xi)^* e^\gamma \otimes e^\delta)(e_\alpha \otimes e_\beta) := (e^\delta \otimes e^\gamma)(\check{R}_k(\xi) e_\beta \otimes e_\alpha) .$$

Proposition 4.2 (cf. [4]). *The R-matrix $\check{R}_k(\xi)^*$ is Belavin's R-matrix up to constant.*

Proof. Let A and B be operators on the space of all holomorphic functions on $\mathbb{C}$ as

$$(Af)(z) = -f\left(z+\frac{1}{k}\right) ,$$

$$(Bf)(z) = -\exp\left(2\pi\sqrt{-1}\left(z+\frac{\tau}{2k}\right)\right) f\left(z+\frac{\tau}{k}\right) .$$

The space $\tilde{V}_k$ is invariant under the actions of A and B. In fact, A and B are expressed on $\tilde{V}_k$ as

$$Ae_j = e_j \exp\frac{2\pi\sqrt{-1}j}{k} ,$$
$$Be_j = e_{j+1} .$$

We define operators A^* and B^* on $\tilde{V}_k^*$ to be the transposes of A and B on $\tilde{V}_k$, respectively;

$$A^* e^j = e^j \exp\frac{2\pi\sqrt{-1}j}{k} ,$$
$$B^* e^j = e^{j-1} .$$

To prove this proposition, it is enough to show the following [3, 6, 7].

(1) $\check{R}_k(\xi)^*$ is an entire $\mathrm{End}(\tilde{V}_k^* \otimes \tilde{V}_k^*)$-valued function in ξ.

(2) $\check{R}_k(\xi)^* x \otimes x = x \otimes x \check{R}_k(\xi)^* \quad x = A^*, B^*$.

(3) $\check{R}_k(\xi+1)^* = (1 \otimes A^*)^{-1}\check{R}_k(\xi)^*(A^* \otimes 1) \times (-1)$.

(4) $\check{R}_k(\xi+\tau)^* = (1 \otimes B^*)^{-1}\check{R}_k(\xi)^*(B^* \otimes 1) \times (-\exp 2\pi\sqrt{-1}(\xi+\frac{\tau}{2}-\frac{\mu}{k}))^{-1}$.

(5) $\check{R}_k(0)^* = \vartheta_1'(0)\,\mathrm{id}$.

The operator $\check{R}_k(\xi)$ on $\tilde{V}_k \otimes \tilde{V}_k$ has the properties below, which imply the properties (2), (3), (4), and (5) above, respectively.

(2) $\check{R}_k(\xi)x \otimes x = x \otimes x\check{R}_k(\xi) \quad x = A, B.$

(3) $\check{R}_k(\xi + 1) = (1 \otimes A)\check{R}_k(\xi)(A \otimes 1)^{-1} \times (-1).$

(4) $\check{R}_k(\xi + \tau) = (1 \otimes B)\check{R}_k(\xi)(B \otimes 1)^{-1} \times (-\exp 2\pi\sqrt{-1}(\xi + \frac{\tau}{2} - \frac{\mu}{k}))^{-1}.$

(5) $\check{R}_k(0) = \vartheta_1'(0)\,\mathrm{id}.$

The proof is quite straightforward, so we omit it.

To prove (1), it suffices to show that $\check{R}_k(\xi)$ is an entire $\mathrm{End}(\tilde{V}_k \otimes \tilde{V}_k)$-valued function in ξ. Let us introduce another basis of $\tilde{V}_k$ (cf. [4]);

$$\left\{\tilde{e}_j(z) := (-1)^j \vartheta \begin{bmatrix} \frac{k}{2} \\ \frac{1}{2} - \frac{j}{k} \end{bmatrix} \left(z, \frac{\tau}{k}\right)\right\}_{j \in \mathbb{Z}/k\mathbb{Z}} .$$

In the same way as [4], we can calculate the matrix coefficients of $\check{R}_k(\xi)$ on $\tilde{V}_k \otimes \tilde{V}_k$ with respect to the basis $\{\tilde{e}_i \otimes \tilde{e}_j\}$ and can check that all matrix coefficients are entire in ξ. This completes the proof. $\square$

For $\lambda, \kappa \in \Lambda$, we put $\phi(\xi)_\lambda^\kappa := \bar{\phi}_\kappa^\lambda \circ T_k(\xi + |\lambda|_{\mathbf{k}} - k\mu)|_{\tilde{V}_k}$. Since

$$\bar{\phi}_\kappa^\lambda \circ T_k(\xi + |\lambda|_{\mathbf{k}} - k\mu)(e_j)$$
$$= \begin{cases} \vartheta \begin{bmatrix} \frac{1}{2} - \frac{j}{k} \\ \frac{k}{2} \end{bmatrix} (\xi + |\lambda|_{\mathbf{k}} - k\lambda_i, k\tau) & \text{if } \kappa = \lambda - \mu\varepsilon_i \ (k_1 \leqq \exists i \leqq k_2)\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

we get

$$\phi(\xi)_\lambda^\kappa = \sum_{j=0}^{k-1} \bar{\phi}_\kappa^\lambda \circ T_k(\xi + |\lambda|_{\mathbf{k}} - k\mu)(e_j)e^j$$
$$= \begin{cases} \sum_{j=0}^{k-1} \vartheta \begin{bmatrix} \frac{1}{2} - \frac{j}{k} \\ \frac{k}{2} \end{bmatrix} (\xi + |\lambda|_{\mathbf{k}} - k\lambda_i, k\tau)e^j, & \text{if } \kappa = \lambda - \mu\varepsilon_i \ (k_1 \leqq \exists i \leqq k_2)\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Hence the vector $\phi(\xi)_\lambda^\kappa$ is nothing but the outgoing intertwining vector of Belavin's R-matrix [6, 7], which was first discovered by Baxter [1], Jimbo, Miwa and Okado [8].

On the other hand, we put

$$\tilde{W} \begin{bmatrix} & \kappa & \\ \lambda & \xi & \nu \\ & \kappa' & \end{bmatrix} := \check{W} \begin{bmatrix} & \kappa' & \\ \nu & \xi & \lambda \\ & \kappa & \end{bmatrix},$$

and then Theorem 2.1 and Remark 2.1 lead us to

Theorem 4.3 (Vertex-IRF Correspondence for Belavin's R-matrix [1,8]). *For $\lambda,\kappa,\nu\in\Lambda$,*

$$\check{R}_k(\xi_{12})^*\phi(\xi_1)^\kappa_\lambda\otimes\phi(\xi_2)^\nu_\kappa=\sum_{\kappa'\in\Lambda}\phi(\xi_2)^{\kappa'}_\lambda\otimes\phi(\xi_1)^\nu_{\kappa'}\tilde{W}\begin{bmatrix} & \kappa & \\ \lambda & \xi_{12} & \nu \\ & \kappa' & \end{bmatrix}.$$

Next we construct the factorized L-operators for Belavin's R-matrix proved by Hasegawa [6], Quano and Fujii [10]. To begin with, we introduce outgoing intertwining vectors in $\tilde{V}_k(\xi)$ in the same fashion as Definition 3.1. In the sequel, we fix $k_1,k_2\in\mathbb{Z}$ such that $k=k_2-k_1+1$ and assume that $\lambda,\kappa,\nu\in\Lambda$ and the $\xi,\xi_1,\xi_2\notin\mathbb{Z}+\mathbb{Z}\tau$.

For $k_1\leqq j\leqq k_2$, we define $\bar{\varphi}_{\mathbf{k}}(\xi)^{\kappa j}_\lambda\in\mathbb{C}$ by

$$\bar{\varphi}_{\mathbf{k}}(\xi)^{\kappa j}_\lambda:=\bar{\phi}^\kappa_\lambda(e_j(\xi+|\lambda|_{\mathbf{k}}))\,,$$

and also define $\varphi_{\mathbf{k}}(\xi)^\kappa_{\lambda j}\in\mathbb{C}$ by the following duality relations (cf. Proposition 3.1):

$$\begin{cases}\sum_{i=k_1}^{k_2}\varphi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_i}_{\lambda j}\bar{\varphi}_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_i l}_\lambda=\delta_{jl}\,,\\ \sum_{i=k_1}^{k_2}\bar{\varphi}_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_j i}_\lambda\varphi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_l}_{\lambda i}=\delta_{jl}\,.\end{cases}$$

For $\kappa\neq\lambda+\mu\varepsilon_i$ ($k_1\leqq\forall i\leqq k_2$) we set

$$\varphi_{\mathbf{k}}(\xi)^\kappa_{\lambda\,j}:=0\,.$$

Outgoing intertwining vectors $\varphi_{\mathbf{k}}(\xi)^\kappa_\lambda(z)\in\tilde{V}_k(\xi+|\lambda|_{\mathbf{k}})$ of the elliptic R-operator are defined as

$$\varphi_{\mathbf{k}}(\xi)^\kappa_\lambda(z):=\sum_{j=k_1}^{k_2}\varphi_{\mathbf{k}}(\xi)^\kappa_{\lambda j}e_j(\xi+|\lambda|_{\mathbf{k}})(z)\,.$$

Then we define the operators $\check{L}_{\mathbf{k}}(\xi)^\kappa_\lambda$ as follows:

$$(\check{L}_{\mathbf{k}}(\xi)^\kappa_\lambda f)(z):=\varphi_{\mathbf{k}}(\xi)^\kappa_\lambda(z)\bar{\phi}^\kappa_\lambda f\,,$$

where $f\in\mathscr{V}$ if k is even and $f\in\mathscr{V}^-$ if k is odd. In the same way as Sect. 3, these operators satisfy (cf. Lemma 3.3)

$$\sum_{\kappa\in\Lambda}\check{R}(\xi_{12})\check{L}_{\mathbf{k}}(\xi_1)^\kappa_\lambda\otimes\check{L}_{\mathbf{k}}(\xi_2)^\nu_\kappa=\sum_{\kappa\in\Lambda}\check{L}_{\mathbf{k}}(\xi_2)^\kappa_\lambda\otimes\check{L}_{\mathbf{k}}(\xi_1)^\nu_\kappa\check{R}(\xi_{12})\,.$$

We put

$$\tilde{L}_{\mathbf{k}}(\xi)^\kappa_\lambda:=T_k(\xi+|\kappa|_{\mathbf{k}}-k\mu)^{-1}\check{L}_{\mathbf{k}}\,(\xi-k\mu)^\lambda_\kappa T_k(\xi+|\kappa|_{\mathbf{k}}-k\mu)\big|_{\tilde{V}_k}\,,$$

and denote its transpose as $\tilde{L}^*_{\mathbf{k}}(\xi)^\kappa_\lambda:\tilde{V}^*_k\to\tilde{V}^*_k$. Thus, for Belavin's R-matrix $\check{R}_k(\xi)^*$,

$$\sum_{\kappa\in\Lambda}\check{R}_k(\xi_{12})^*\tilde{L}^*_{\mathbf{k}}(\xi_1)^\kappa_\lambda\otimes\tilde{L}^*_{\mathbf{k}}(\xi_2)^\nu_\kappa=\sum_{\kappa\in\Lambda}\tilde{L}^*_{\mathbf{k}}(\xi_2)^\kappa_\lambda\otimes\tilde{L}^*_{\mathbf{k}}(\xi_1)^\nu_\kappa\check{R}_k(\xi_{12})^*\,.$$

We define an operator $\tilde{L}^*_{\mathbf{k}}(\xi) : \tilde{V}^*_k \otimes \mathscr{W} \to \mathscr{W} \otimes \tilde{V}^*_k$ by

$$\tilde{L}^*_{\mathbf{k}}(\xi)(e^i \otimes \delta^\kappa) = \sum_{\lambda \in \Lambda} \delta^\lambda \otimes \tilde{L}^*_{\mathbf{k}}(\xi)^\kappa_\lambda e^i .$$

The theorem below tells us that the operator $\tilde{L}^*_{\mathbf{k}}(\xi)$ is the factorized L-operator for Belavin's R-matrix, which were first constructed by Hasegawa [6], Quano and Fujii [10].

Theorem 4.4 (Factorized L-operator for Belavin's R-matrix). *For $\xi_1, \xi_2 \notin Z + \mathbb{Z}\tau$,*

$$(1 \otimes \check{R}_k(\xi_{12})^*)(\tilde{L}^*_{\mathbf{k}}(\xi_1) \otimes 1)(1 \otimes \tilde{L}^*_{\mathbf{k}}(\xi_2)) = (\tilde{L}^*_{\mathbf{k}}(\xi_2) \otimes 1)(1 \otimes \tilde{L}^*_{\mathbf{k}}(\xi_1))(\check{R}_k(\xi_{12})^* \otimes 1) .$$

Here both sides are the operators $\tilde{V}^*_k \otimes \tilde{V}^*_k \otimes \mathscr{W} \to \mathscr{W} \otimes \tilde{V}^*_k \otimes \tilde{V}^*_k$.

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Note added in proof. We have two remarks about the incoming and outgoing intertwining vectors.

(1) We can add one more parameter to the incoming intertwining vector $\bar{\phi}^\kappa_\lambda$ in Definition 2.2. For $\alpha \in \mathbb{R}$, we set

$$\bar{\phi}^\kappa_\lambda(\alpha) f := \begin{cases} f(\lambda_i + \alpha), & \exists i \in \mathbb{Z} \text{ s.t. } \kappa = \lambda + \mu\varepsilon_i , \\ 0, & \text{otherwise} . \end{cases}$$

These incoming intertwining vectors also satisfy the vertex-IRF correspondence (Theorem 2.1). Making use of the incoming intertwining vectors $\bar{\phi}^\kappa_\lambda(\alpha)$ instead of $\bar{\phi}^\kappa_\lambda$, we can construct the factorized L-operators (Theorem 3.4).

(2) By means of the Weyl–Kac denominator formula (cf. Proposition 3.1), we obtain the explicit form of the outgoing intertwining vector in Definition 3.1. For $k_1 \leqq i \leqq k_2$,

$$\phi_{\mathbf{k}}(\xi)^{\lambda+\mu\varepsilon_i}_\lambda(z) = \exp(\pi\sqrt{-1}k(z-\lambda_i))\frac{\vartheta_1(\xi+\lambda_i-z)}{\vartheta_1(\xi)} \prod_{k_1 \leqq j \leqq k_2, j \neq i} \frac{\vartheta_1(z-\lambda_j)}{\vartheta_1(\lambda_{ij})} .$$

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