On E_{10} and the DDF Construction

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Abstract: An attempt is made to understand the root spaces of Kac Moody algebras of hyperbolic type, and in particular E_{10}, in terms of a DDF construction appropriate to a subcritical compactified bosonic string. While the level-one root spaces can be completely characterized in terms of transversal DDF states (the level-zero elements just span the affine subalgebra), longitudinal DDF states are shown to appear beyond level one. In contrast to previous treatments of such algebras, we find it necessary to make use of a rational extension of the self-dual root lattice as an auxiliary device, and to admit non-summable operators (in the sense of the vertex algebra formalism). We demonstrate the utility of the method by completely analyzing a non-trivial level-two root space, obtaining an explicit and comparatively simple representation for it. We also emphasize the occurrence of several Virasoro algebras, whose interrelation is expected to be crucial for a better understanding of the complete structure of the Kac Moody algebra.

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1. Introduction

Affine Kac Moody algebras (see [29, 25] and references therein), which first appeared in physics in the guise of (two-dimensional) current algebras, have come to play an increasingly important role in string theory and conformal field theory as well as other branches of mathematical physics. By contrast, Kac Moody algebras based on indefinite Cartan matrices have not yet found applications in physics. In view of the scarcity of results about such algebras, it is remarkable that they have nevertheless been suggested as natural candidates for the still elusive fundamental symmetry of string theory (and hence of nature). Being vastly larger than affine Kac Moody algebras, Kac Moody algebras of indefinite type might certainly be "sufficiently big" for a unified and background independent formulation of string (field) theory, but an even more compelling argument supporting such speculations is the intimate link that exists between Kac Moody algebras and the vertex operator construction of string theory (this connection has been known for a long time [1, 28]). More specifically, it has been established that the generators making up a Kac Moody algebra of finite or affine type can be explicitly realized in terms of tachyon and photon emission vertex operators of a compactified open bosonic string [16, 24]. On the basis of these results, it has been conjectured that generalized Kac Moody algebras of indefinite type might not only furnish new symmetries of string theory, but might themselves be understood in terms of string vertex operators associated with the higher excited (massive) states of a compactified bosonic string [24, 14]. Despite its great appeal, however, this idea has not led to a truly satisfactory understanding of these Kac Moody algebras until now.

   Disregarding possible physical applications in string theory, very little is known about indefinite Kac Moody algebras beyond their mere existence and the remarkable result that the Weyl-Kac character formula continues to hold for them [36, 29]. The basic problem here is the proliferation of timelike roots (having *negative* (length)^2) and the concomitant exponential growth in the dimension of the corresponding root spaces. For a limited number of cases, and in particular for roots of level two at most[1], one knows explicit multiplicity formulas counting the dimension of the root spaces [30], but the complete root multiplicities are not known for a single Kac Moody algebra of indefinite type (root multiplicities can be determined in principle from the Peterson recursion formula [31], but this formula quickly becomes too unwieldy for practical use). Unfortunately, the available results have not shed much light on the structure of the corresponding root spaces, and, in contrast to affine Kac Moody algebras, a manageable representation of the root space elements has not been found so far. In an interesting recent development (more concerned with understanding the monster group than with applications in physics), complete and explicit multiplicity formulas were derived for the so-called fake monster Lie algebra based on the 26-dimensional Lorentzian even self-dual lattice $\mathit{II}_{25,1}$ [4]; this algebra is, however, not a conventional Kac Moody algebra in that it has imaginary simple roots beside the

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[1] The notion of "level of a root" is defined in Sect. 4.1.

usual simple roots [3] (the extra simple roots correspond to new Lie algebra elements that cannot be generated by multiple commutators of the conventional Chevalley generators and must therefore be adjoined "by hand"). These results rely heavily on special properties of 26 dimensions such as the no-ghost theorem, and so far no other example has been fully worked out[2].

In this paper, an attempt is made to understand Kac Moody algebras of hyperbolic type, and in particular the maximally extended hyperbolic algebra E_{10}, from a more "physical" (i.e. pedestrian) point of view and to examine the known results as well as the difficulties from what we believe to be a novel perspective. We here (somewhat immodestly) concentrate on E_{10} not only because, in our opinion, it is the most interesting, containing E_8 and E_9 as subalgebras, and because the basic problems are not simplified in any substantial way by considering lower rank hyperbolic algebras instead. Rather, we shall need to make use of two special features of E_{10} not shared by other algebras of this type, namely the self-duality its root lattice, the unique Lorentzian lattice $II_{9,1}$, and secondly the property, crucial for our construction, that the fundamental Weyl chamber of E_{10} contains precisely one null direction (i.e. touches the light-cone in root space only once), in contrast to generic Kac Moody algebras of indefinite type, whose Weyl chambers contain several linearly independent null vectors, and also in contrast to strictly hyperbolic algebras, whose Weyl chambers lie entirely within the light-cone and therefore contain no null vectors at all.

Our key observation is that, beyond level one, there appear *longitudinal* string states and vertex operators, whose significance in this context has so far not been recognized. A central role in our analysis is played by the DDF construction, which provides the most direct and explicit solution of the physical state constraints in string theory [11]. The physical states, which by definition are annihilated by the Virasoro constraints, are simply obtained in this scheme by acting on a tachyonic groundstate with the DDF operators, which commute with all Virasoro generators and form a spectrum generating algebra. For their definition, one must choose a special Lorentz frame, in terms of which one can distinguish transversal and longitudinal DDF operators. As is well known, the longitudinal states (or operators) have zero norm and hence decouple in 26 dimensions by the no-ghost theorem [26, 8, 41]. Above 26 dimensions, no consistent string theories and hence no consistent Kac Moody algebras are expected to exist, as there are always negative norm states. Below 26 dimensions, on the other hand, the longitudinal DDF operators create extra positive norm states (also referred to as Liouville states), and thus modify the spectrum in an essential way. It is therefore hardly surprising that longitudinal states should also play a role in the construction of Lorentzian and hyperbolic Kac Moody algebras of "subcritical" rank. Our results thus suggest a connection between these algebras and Liouville (or subcritical string) theory, the precise nature of which remains to be elucidated, however.

The adaptation of the DDF construction to the present context involves a discretization of the string vertex operator formalism. As is well known [24], the allowed momenta of the string excitations must be elements of the weight lattice of the corresponding (affine or indefinite) Kac Moody algebra. A curious feature of our construction, not encountered in previous studies, is that we are here forced to make use of a rational extension of the self-dual root lattice as an auxiliary device in order to understand the root spaces associated with higher level roots in terms of the DDF

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[2] As an amusing aside, we note that the very notion of what the monster Lie algebra should be has undergone several metamorphoses since it was first proposed in [7].

construction. To be sure, the intermediate states belonging to momenta not on the root lattice do not correspond to elements of the Kac Moody algebra, but they are nonetheless indispensable for the construction of a complete basis for any given root space, which we here obtain in terms of transversal and longitudinal DDF states. The problem of characterizing the root spaces is thereby reduced to finding the "missing states," i.e. those physical states which cannot be reached through multiple commutators of Chevalley generators. To illustrate our method we will completely analyze one non-trivial example of a level-two root space in terms of the DDF decomposition. To appreciate the simplicity of our final result (cf. (4.59)) readers need only contemplate the problem of classifying the states in terms of 75-fold multiple commutators of the Chevalley generators for this example. Notwithstanding eventual refinements which may become necessary at a later stage, we thus believe that our procedure provides a workable method to probe higher level root spaces, yielding a manageable representation of the level-two root space elements for the first time.

Of course, we are aware that vertex operators have been utilized in previous work on indefinite Kac Moody algebras [24, 14]. However, at least to the best of our knowledge, neither the explicit representation of the level-one elements in terms of transversal DDF operators (the level-zero elements just span the affine subalgebra), nor the emergence of longitudinal states and vertex operators at level two and beyond have been exhibited in previous treatments. Our results are couched in the language of vertex algebras [6, 17, 19] (see also [33] for a treatment in the BV formalism), which is entirely equivalent to the formulation of [24, 22], but slightly more convenient for our purposes because of its economy of notation. An important technical point is that the longitudinal vertex operators cannot be associated with definite states, as their action cannot be defined on all of Fock space. Put differently, they do not correspond to summable operators in the sense of [17]; in this respect, vertex algebras encompassing longitudinal states transcend the definition given in [6, 17].

A remarkable property of the vertex operator realization of hyperbolic Kac Moody algebras, which has not received due attention so far, is the occurrence of several Virasoro algebras in the construction. The first is the usual one with central charge $c = d$ (where d is the rank of the algebra, or, equivalently, the dimension of the root lattice), by means of which physical states can be distinguished from unphysical ones. Furthermore, there are two sets of longitudinal DDF operators, called A_m^- and $\mathfrak{L}_m$ in this paper, which generate Virasoro algebras of central charge $c = 26 - d$ and (independently of the rank) $c = 24$, respectively; the first choice is the standard one used in proofs of the no-ghost theorem (see e.g. [27]). These two sets are related through a GKO construction [23]. Since all DDF operators depend on the tachyon momentum, there are actually infinitely many "longitudinal Virasoro algebras," related to one another by Weyl rotations on the (Lorentzian) root lattice (the Virasoro generators defining the physical states are, of course, invariant under the Weyl group). Thirdly, there is a Sugawara-type Virasoro algebra, whose relevance for the general representation theory of affine Kac Moody algebras was already known [29]. Our results indicate a subtle entanglement of the affine subalgebra and the longitudinal Virasoro algebras, as well as a deep connection between the longitudinal and Sugawara-type Virasoro algebras. The action of the corresponding Sugawara generators of level $\ell > 1$ on the states remains to be fully worked out, however. We believe that a proper understanding of these Virasoro algebras is one of the keys to unlocking the secrets of indefinite Kac Moody algebras.

Since this paper brings together various different concepts and ideas, not all of which may be universally familiar, we have tried to make it self-contained as far as

possible. In Sect. 2, the formalism of vertex algebras is briefly reviewed (for more details, the reader should consult [17, 19]). Section 3 is devoted to a rather detailed exposition of the discrete DDF construction in the framework of vertex (operator) algebras with particular emphasis on the longitudinal states and operators; we stress once more the new feature that these operators are non-summable and have not been considered in this context before. In Sect. 4, we apply the formalism to the maximal hyperbolic Kac Moody algebra E_{10}, first recalling some known results. We then define what we call the DDF decomposition of a given root vector of arbitrary level ℓ; this is the point where we are forced to admit fractional momenta $\frac{1}{\ell}\mathbf{r}$ with $\mathbf{r} \in \mathit{\Pi}_{9,1}$. For the level-one root space elements, which are known to form the so-called basic representation of E_9 [29], we exhibit a simple and explicit realization in terms of DDF states. Finally, we give a complete analysis of a non-trivial level-two root space and construct an explicit basis for it, whereas only its dimension was known so far [30]. This example displays the appearance of longitudinal and the disappearance of certain transversal states, which we conjecture to be generic for higher level root spaces of hyperbolic Kac Moody algebras. Of course, our results should be regarded only as a tiny step into the *terra incognita* of hyperbolic Kac Moody algebras, but we hope at least that they will give the reader a flavor of their complexity. Explicit formulas for transversal and longitudinal DDF states as well as a number of Lie algebra commutators giving level-two root space elements are collected in the appendix.

2. General Setup

2.1. Vertex algebras. We shall provide a short primer to formal calculus and vertex algebras following closely [19]. In [17] the subject is treated thoroughly.

In contrast to conformal field theory (see e.g. [2] or [20]), in the vertex algebra approach we use *formal* variables z, z_0, z_1, z_2, etc. The objects we will work with are formal power series. For a vector space W, we set

$$W[\![z, z^{-1}]\!] = \left\{ \sum_{n \in \mathbb{Z}} w_n z^n \,\middle|\, w_n \in W \right\}, \tag{2.1}$$

$$W[\![z]\!] = \left\{ \sum_{n \in \mathbb{N}} w_n z^n \,\middle|\, w_n \in W \right\}, \tag{2.2}$$

$$W[z, z^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} w_n z^n \,\middle|\, w_n \in W, \text{ almost all } w_n = 0 \right\} \quad \text{(Laurent pol.)}, \tag{2.3}$$

$$W[z] = \left\{ \sum_{n \in \mathbb{N}} w_n z^n \,\middle|\, w_n \in W, \text{ almost all } w_n = 0 \right\} \quad \text{(polynomials)}, \tag{2.4}$$

where "almost all" means "all but finitely many." These sets are $\mathbb{C}$-vector spaces under obvious pointwise operations. We can generalize above spaces in a straightforward way to the case of several commuting formal variables, e.g. $W[\![z_1, z_2^{-1}]\!] = \{\sum_{m,n \in \mathbb{N}} w_{mn} z_1^m z_2^{-n} \mid w_{mn} \in W\}$.

Since we will often multiply formal series or add up an infinite number of series, it is necessary to introduce the notion of algebraic summability. Let $(x_i)_{i \in I}$ be a family in $\operatorname{End} W$, the vector space of endomorphisms of W (I is an index set). We say that $(x_i)_{i \in I}$ is **summable** if for every $w \in W$, $x_i w = 0$ for all but a finite number

of $i \in I$. Then the operator $\sum_{i \in I} x_i$ is well-defined. In general an algebraic limit or a product of formal series is defined if and only if the coefficient of *every* monomial in the formal variables in the formal expression is summable.

If we define

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \quad \in \mathbb{C}[\![z, z^{-1}]\!], \tag{2.5}$$

then, formally, this is the Laurent expansion of the classical δ-function at $z = 1$. Indeed, $\delta(z)$ enjoys the following fundamental properties:
Let $w(z) \in W[z, z^{-1}]$, $a \in \mathbb{C}^\times$. Then

$$w(z)\delta(az) = w(a^{-1})\delta(az). \tag{2.6}$$

Let $X(z_1, z_2) \in (\text{End } W)[\![z_1, z_1^{-1}, z_2, z_2^{-1}]\!]$ be such that $\lim_{z_1 \to z_2} X(z_1, z_2)$ exists (algebraically) and let $a \in \mathbb{C}^\times$. Then

$$\begin{aligned}
X(z_1, z_2)\delta\left(a\frac{z_1}{z_2}\right) &= X(a^{-1}z_2, z_2)\delta\left(a\frac{z_1}{z_2}\right) \\
&= X(z_1, az_1)\delta\left(a\frac{z_1}{z_2}\right).
\end{aligned} \tag{2.7}$$

Note that $w(z)$ must be a Laurent polynomial to ensure existence of the product with the δ-series. For explicit calculations it is useful to keep in mind that the substitutions in the arguments correspond formally to $az = 1$ and $az_1/z_2 = 1$, respectively.

Now we want to introduce the tools for formal calculus which correspond to contour integrals and residues for complex variables. We will need quotients of formal power series which will often be expressed by analytic functions of z and z^{-1}, respectively. They are understood as formal Taylor or Laurent expansions. E.g., for $a \in \mathbb{C}$ we have

$$(1 + z)^a = \sum_{n \in \mathbb{N}} \binom{a}{n} z^n \in \mathbb{C}[\![z]\!], \tag{2.8}$$

$$(1 + z^{-1})^a = \sum_{n \in \mathbb{N}} \binom{a}{n} z^{-n} \in \mathbb{C}[\![z^{-1}]\!]. \tag{2.9}$$

In the following we will always (though sometimes not explicitly stated) refer to the **binomial convention** which says that *all binomial expressions are to be expanded in nonnegative integral powers of the second variable.* This is the only point in explicit calculations at which one must not be too sloppy. For example, for $a \in \mathbb{C}$ the following expressions are in general not the same:

$$\left(\frac{z_1 - z_2}{z_0}\right)^a = \sum_{n \in \mathbb{N}} \binom{a}{n}(-1)^n z_0^{-a} z_1^{a-n} z_2^n, \tag{2.10}$$

$$\left(\frac{-z_2 + z_1}{z_0}\right)^a = \sum_{n \in \mathbb{N}} \binom{a}{n}(-1)^{a-n} z_0^{-a} z_1^n z_2^{a-n}. \tag{2.11}$$

With the binomial convention we can rewrite the generating function for the derivatives of the δ-series as

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) = z_0^{-1} e^{-z_2 \frac{\partial}{\partial z_1}} \delta \left(\frac{z_1}{z_0} \right). \tag{2.12}$$

As an important exercise one may prove subsequent identities which will be extremely useful for vertex operator calculus:

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) = z_1^{-1} \delta \left(\frac{z_0 + z_2}{z_1} \right), \tag{2.13}$$

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right), \tag{2.14}$$

where all binomial expressions are expanded in nonnegative integral powers of the second variable.

We shall use the following residue notation. For a formal series

$$w(z) = \sum_{n \in \mathbb{Z}} w_n z^n \in W[\![z, z^{-1}]\!], \tag{2.15}$$

we write

$$\mathrm{Res}_z \left[w(z) \right] = w_{-1}, \tag{2.16}$$

so that we may think of $\mathrm{Res}_z [\ldots]$ as the operation $\oint_0 \frac{dz}{2\pi \mathrm{i}} [\ldots]$ in complex analysis. Indeed, formal residue enjoys some properties of contour integration. For $w(z) \in W[\![z, z^{-1}]\!]$ as above and $n \in \mathbb{Z}$ we find that

$$w_n = \mathrm{Res}_z \left[z^{-n-1} w(z) \right]; \tag{2.17}$$

for $v(z), w(z) \in W[\![z, z^{-1}]\!]$ integration by parts reads as

$$\mathrm{Res}_z \left[v(z) \frac{d}{dz} w(z) \right] = -\mathrm{Res}_z \left[w(z) \frac{d}{dz} v(z) \right]. \tag{2.18}$$

We have already used exponentials of derivatives in (2.12) to obtain a formula for the higher derivatives of $\delta(z)$. However, one might also expect $e^{z_0 \frac{d}{dz}}$ to act somehow as a one-parameter group of automorphisms (parametrized by z_0). This turns out to be also true in formal calculus. Let $w(z) = \sum_{m \in \mathbb{Z}} w_m z^m \in W[\![z, z^{-1}]\!]$, $y \in z_0 \mathbb{C}[\![z_0]\!]$ and write $D_n = -z^{n+1} \frac{d}{dz}$, $n \in \mathbb{N}$. Then we have

1. (Translation)

$$e^{-y D_{-1}} w(z) \equiv e^{y \frac{d}{dz}} w(z) = w(z + y), \tag{2.19}$$

2. (Scaling)

$$(e^y)^{-D_0} w(z) \equiv e^{yz \frac{d}{dz}} w(z) = w(e^y z), \tag{2.20}$$

3. (Projective change)

$$e^{y D_n} w(z) = w \left((z^{-n} + ny)^{-1/n} \right) \quad \text{for } n \neq 0, \tag{2.21}$$

with binomial convention.

We shall give a definition of vertex algebra [15] using the notation of [21] which we believe is more accessible to physicists.

A **vertex algebra** is a $\mathbb{Z}$-graded vector space,

$$\mathscr{F} = \bigoplus_{n \in \mathbb{Z}} \mathscr{F}_{(n)}, \tag{2.22}$$

equipped with a linear map $\mathscr{V} : \mathscr{F} \to (\text{End}\,\mathscr{F})[\![z, z^{-1}]\!]$, which assigns to each state $\psi \in \mathscr{F}$ a **vertex operator** $\mathscr{V}(\psi, z)$, and the vertex operators satisfy the following axioms:

1. **(Regularity)** If $\psi, \varphi \in \mathscr{F}$ then

$$\text{Res}_z \left[z^n \mathscr{V}(\psi, z)\varphi \right] = 0 \quad \text{for} \ \ n \ \text{sufficiently large}, \tag{2.23}$$

   and n depending on ψ and φ.

2. **(Vacuum)** There is a preferred state $\mathbf{1} \in \mathscr{F}$, called the vacuum, satisfying

$$\mathscr{V}(\mathbf{1}, z) = \text{id}_{\mathscr{F}}. \tag{2.24}$$

3. **(Injectivity)** There is a one-to-one correspondence between states and vertex operators,

$$\mathscr{V}(\psi, z) = 0 \quad \Longleftrightarrow \quad \psi = 0. \tag{2.25}$$

4. **(Conformal vector)** There is a preferred state $\omega \in \mathscr{F}$, called the conformal vector, such that its vertex operator

$$\mathscr{V}(\omega, z) = \sum_{n \in \mathbb{Z}} \mathrm{L}_{(n)} z^{-n-2}, \tag{2.26}$$

   a) gives the **Virasoro algebra** with some central charge $c \in \mathbb{R}$,

$$[\mathrm{L}_{(m)}, \mathrm{L}_{(n)}] = (m-n)\mathrm{L}_{(m+n)} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}; \tag{2.27}$$

   b) provides a **translation generator**, $\mathrm{L}_{(-1)}$,

$$\mathscr{V}(\mathrm{L}_{(-1)}\psi, z) = \frac{d}{dz} \mathscr{V}(\psi, z) \quad \text{for every} \ \ \psi \in \mathscr{F}; \tag{2.28}$$

   c) gives the grading of $\mathscr{F}$ via the eigenvalues of $\mathrm{L}_{(0)}$,

$$\mathrm{L}_{(0)}\psi = n\psi \equiv \Delta_\psi \psi \quad \text{for every} \ \ \psi \in \mathscr{F}_{(n)}, n \in \mathbb{Z}, \tag{2.29}$$

   the eigenvalue Δ_ψ is called the **(conformal) weight** of ψ.

5. **(Jacobi identity)** For every $\psi, \varphi \in \mathscr{F}$,

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) \mathscr{V}(\psi, z_1)\mathscr{V}(\varphi, z_2) - z_0^{-1} \delta \left(\frac{-z_2 + z_1}{z_0} \right) \mathscr{V}(\varphi, z_2)\mathscr{V}(\psi, z_1)$$

$$= z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) \mathscr{V}(\mathscr{V}(\psi, z_0)\varphi, z_2), \tag{2.30}$$

where binomial expressions have to be expanded in nonnegative integral powers of the second variable.

We denote the vertex algebra just defined by $(\mathscr{F}, \mathscr{V}, \mathbf{1}, \boldsymbol{\omega})$.

We may think of $\mathscr{F}$ as the space of finite occupation number states in a Fock space so that $\mathscr{F}$ is a dense subspace of the Hilbert space $\mathscr{H}$ of states. The regularity axiom states that, given $\psi, \varphi \in \mathscr{F}$, there is always a high enough power z^n such that $z^n \mathscr{V}(\psi, z)\varphi$ is (at "$z = 0$") a regular formal series. In other words, the regularity axiom ensures that any $\mathscr{V}(\psi, z)\varphi$ contains only a *finite* number of singular (at "$z = 0$") expressions. In terms of creation and annihilation operators it reflects the fact that any finite occupation number state φ is killed by a finite but large enough number of annihilation operators contained in (the normal ordered expression) ψ_n. We also mention that in physical applications the vertex operator of the conformal vector corresponds to the stress–energy tensor of the field theory.

A **vertex operator algebra** is a vertex algebra with the additional assumptions that

1. the spectrum of $L_{(0)}$ is bounded below,
2. the eigenspaces $\mathscr{F}_{(n)}$ of $L_{(0)}$ are finite-dimensional.

The first condition is an immediate consequence of a physical postulate. As we will see $L_{(0)}$ generates scale transformations. Recalling that the variable z in conformal field theory has its origin in e^{t+ix} (cf. [20]) one finds that $L_{(0)}$ corresponds to time translations. Thus it may be identified with the energy which should be bounded below in any sensible quantum field theory. In fact, vertex operator algebras can be regarded as a rigorous mathematical definition of chiral algebras in physics [38]. Then the formal variable z can be thought of as a local complex coordinate and the above relations (2.17) and (2.17) can be realized by contour integrals. The vertex operators $\mathscr{V}(\psi, z)$ correspond to holomorphic chiral fields, i.e. they can be viewed as operator-valued distributions on a local coordinate chart of a Riemann surface. In this context the three terms of the Jacobi identity are geometrically interpreted as the three ways of cutting the Riemann sphere with four punctures into two spheres with three punctures [18, 43].

Since vertex operators are operator valued formal Laurent series we can give an alternative formulation (see [5], e.g.) of the axioms of a vertex algebra using the mode expansion

$$\mathscr{V}(\psi, z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}. \tag{2.31}$$

One has

1. (Regularity)

$$\psi_n \varphi = 0 \quad \text{for } n \text{ sufficiently large,} \tag{2.32}$$

2. (Vacuum)

$$\mathbf{1}_n \psi = \delta_{n+1,0} \psi, \tag{2.33}$$

3. (Injectivity)

$$\psi_n = 0 \quad \forall n \in \mathbb{Z} \quad \Longleftrightarrow \quad \psi = 0, \tag{2.34}$$

4. (Conformal vector)

$$\boldsymbol{\omega}_{n+1} = L_{(n)}, \tag{2.35}$$

5. (Jacobi identity)

$$\sum_{i\geq 0}(-1)^i \binom{l}{i}\left(\psi_{l+m-i}(\varphi_{n+i}\xi) - (-1)^l\varphi_{l+n-i}(\psi_{m+i}\xi)\right)$$

$$= \sum_{i\geq 0}\binom{m}{i}(\psi_{l+i}\varphi)_{m+n-i}\xi, \qquad (2.36)$$

for all $\psi, \varphi, \xi \in \mathscr{F}$, $l, m, n \in \mathbb{Z}$.

In what follows we will frequently make use of two important formulas which are the special cases $m = 0$ and $l = 0$, respectively, of Eq. (2.36):
(Associativity formula)

$$(\psi_l\varphi)_n = \sum_{i\geq 0}(-1)^i\binom{l}{i}\left(\psi_{l-i}\varphi_{n+i} - (-1)^l\varphi_{l+n-i}\psi_i\right), \qquad (2.37)$$

(Commutator formula)

$$[\psi_m, \varphi_n] = \sum_{i\geq 0}\binom{m}{i}(\psi_i\varphi)_{m+n-i}, \qquad (2.38)$$

for all $\psi, \varphi \in \mathscr{F}$, $l, m, n \in \mathbb{Z}$.

To get a feeling of the formalism and the axioms it is instructive to derive some important properties of vertex algebras. Iterating (2.28) and using translation (2.19) we find that $L_{(-1)}$ indeed generates translations,

$$\mathscr{V}\left(e^{z_0 L_{(-1)}}\psi, z\right) = \mathscr{V}(\psi, z + z_0). \qquad (2.39)$$

Moreover, the vacuum is translation invariant because (2.28) for $\psi = \mathbf{1}$ together with the vacuum axiom (2.24) and injectivity (2.25) gives

$$L_{(-1)}\mathbf{1} = 0. \qquad (2.40)$$

Taking $\mathrm{Res}_{z_0}\left[\mathrm{Res}_{z_1}\left[z_1^n(\text{Jacobi identity})\right]\right]$ in the special case $\psi = \omega$ we obtain

$$[L_{(n)}, \mathscr{V}(\varphi, z)] = \sum_{i\geq -1}\binom{n+1}{i+1}\mathscr{V}(L_{(i)}\varphi, z)z^{n-i}. \qquad (2.41)$$

In particular,

$$[L_{(-1)}, \mathscr{V}(\varphi, z)] = \frac{d}{dz}\mathscr{V}(\varphi, z), \qquad (2.42)$$

$$[L_{(0)}, \mathscr{V}(\varphi, z)] = \left(z\frac{d}{dz} + \Delta_\varphi\right)\mathscr{V}(\varphi, z) \quad \text{if } \varphi \in \mathscr{F}_{(\Delta_\varphi)}. \qquad (2.43)$$

Using the well-known formula $e^A B e^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{ad}_A)^n B \equiv \sum_{n=0}^{\infty}\frac{1}{n!}[A, [A, \ldots [A, B]]\ldots]$ and Eqs. (2.19) and (2.20) above equations give, respectively,

(Translation property)

$$e^{yL_{(-1)}}\mathscr{V}(\varphi, z)e^{-yL_{(-1)}} = \mathscr{V}(\varphi, z + y), \qquad (2.44)$$

(Scaling property)

$$e^{yL_{(0)}} \mathscr{V}(\varphi, z) e^{-yL_{(0)}} = e^{y\Delta_\varphi} \mathscr{V}(\varphi, e^y z) \quad \text{if } \varphi \in \mathscr{F}_{(\Delta_\varphi)}, \tag{2.45}$$

for every $y \in z_0 \mathbb{C}[[z_0]]$. Thus $L_{(0)}$ generates scale transformations. Note that (2.43) also implies

$$\varphi_n \mathscr{F}_{(m)} \subset \mathscr{F}_{(\Delta_\varphi + m - n - 1)} \quad \text{if } \varphi \in \mathscr{F}_{(\Delta_\varphi)}, \tag{2.46}$$

which means that the operator φ_n shifts the grading by $\Delta_\varphi - n - 1$, i.e. it can be assigned "degree" $\Delta_\varphi - n - 1$. In view of this relation the reader might wonder again why we use subscripts in round brackets for the grading of $\mathscr{F}$ and for the Virasoro generators in contrast to the naked subscripts occurring in the mode expansion (2.31) of a vertex operator. This possibly causes some confusion but stems from the fact that we employ two different mode expansions. In conformal field theory we are familiar with the expansion

$$\psi(z) \equiv \mathscr{V}(\psi, z) = \sum_{n \in \mathbb{Z}} \psi_{(n)} z^{-n-\Delta_\psi}, \tag{2.47}$$

which depends on the conformal weight of the field $\psi(z)$. To exhibit explicitly the Virasoro algebra in the definition of a vertex algebra we used this expansion for the vertex operator associated with the conformal vector (stress-energy tensor!) in (2.26). It is quite easy to convert results obtained in one expansion into the other formalism, namely, simply by shifting the grading:

$$\psi_n \equiv \psi_{(n+1-\Delta_\psi)}, \qquad \psi_{(n)} \equiv \psi_{n-1+\Delta_\psi}, \tag{2.48}$$

for any homogeneous element $\psi \in \mathscr{F}$. For example we can rewrite (2.46) as

$$\varphi_{(n)} \mathscr{F}_{(m)} \subset \mathscr{F}_{(m-n)}, \tag{2.49}$$

so that $\varphi_{(n)}$ always has "degree" $-n$ irrespective of φ. The mode expansion (2.47) is therefore the more natural one because it respects the grading of $\mathscr{F}$. On the other hand for formal calculus it is more useful to stick to an expansion which does not refer to the conformal weight of a state. Hence we shall almost everywhere in the formulas assume the mode expansion (2.31).

Note that the Jacobi identity is obviously invariant under $(\psi, z_1, z_0) \leftrightarrow (\varphi, z_2, -z_0)$. This symmetry property together with (2.7), (2.13), (2.39) and (2.25) finally yields

(Skew-symmetry)

$$\mathscr{V}(\psi, z_0)\varphi = e^{z_0 L_{(-1)}} \mathscr{V}(\varphi, -z_0)\psi, \tag{2.50}$$

or, in components,

$$\psi_n \varphi = -(-1)^n \varphi_n \psi + \sum_{i \geq 1} \frac{1}{i!} (-1)^{i+n+1} L_{(-1)}^i (\varphi_{n+i} \psi). \tag{2.51}$$

In particular, we observe that the vertex operator $\mathscr{V}(\psi, z)$ "creates" the state $\psi \in \mathscr{F}$ when applied to the vacuum:

$$\mathscr{V}(\psi, z)\mathbf{1} = e^{zL_{(-1)}}\psi, \tag{2.52}$$

by (2.24). In components,

$$\psi_n \mathbf{1} = \begin{cases} 0 & \text{for } n \geq 0 \\ \psi & \text{for } n = -1 \\ \frac{1}{(-n-1)!} L_{(-1)}^{-n-1} \psi & \text{for } n \leq -2. \end{cases} \tag{2.53}$$

Hence the vacuum satisfies

$$L_{(n)} \mathbf{1} = 0 \qquad \forall n \geq -1. \tag{2.54}$$

We shall denote by $\mathscr{R}_{\Delta}$ the space of **(conformal) highest weight vectors** or **primary states** of weight Δ satisfying

$$(L_{(n)} - \delta_{n0}\Delta)\psi = 0 \qquad \forall n \geq 0. \tag{2.55}$$

Thus in any vertex algebra the vacuum is a primary state of weight zero. We immediately infer from (2.41) that, for $\psi \in \mathscr{R}_{\Delta}$,

$$[L_{(n)}, \mathscr{V}(\psi, z)] = z^n \left\{ z \frac{d}{dz} + (n+1)\Delta \right\} \mathscr{V}(\psi, z) \qquad \forall n \in \mathbb{Z}, \tag{2.56}$$

or

$$[L_{(n)}, \psi_m] = \{(\Delta - 1)(n+1) - m\} \psi_{m+n} \qquad \forall m, n \in \mathbb{Z}, \tag{2.57}$$

i.e. $\mathscr{V}(\psi, z)$ is a so called **(conformal) primary field** of weight Δ. We can rewrite (2.56) as

$$[L_{(n)}, z^{\Delta(n+1)} \mathscr{V}(\psi, z)] = z^{n+1} \frac{d}{dz} \left\{ z^{\Delta(n+1)} \mathscr{V}(\psi, z) \right\}, \tag{2.58}$$

so that, by (2.21),

$$e^{yL_{(n)}} \mathscr{V}(\psi, z) e^{-yL_{(n)}} = \left(\frac{\partial z_1}{\partial z} \right)^{\Delta} \mathscr{V}(\psi, z_1) \qquad \forall n \neq 0, \tag{2.59}$$

for every $y \in z_0 \mathbb{C}[\![z_0]\!]$ where $z_1 = (z^{-n} - ny)^{-1/n} = z(1 - nyz^n)^{-1/n}$.

We shall provide a certain subspace of the Fock space $\mathscr{F}$ with the structure of a Lie algebra (cf. [6, 5, 17]). We define a bilinear product on $\mathscr{F}$ by

$$[\psi, \varphi] := \psi_0 \varphi, \tag{2.60}$$

which turns out to be antisymmetric on the quotient space $\mathscr{F}/L_{(-1)}\mathscr{F}$ due to skew-symmetry (2.51). Putting $l = m = n = 0$ in the Jacobi identity (2.36) we get $\psi_0(\varphi_0\xi) - \varphi_0(\psi_0\xi) = (\psi_0\varphi)_0\xi$. But this equation translates precisely into the classical Jacobi identity for Lie algebras,

$$[[\psi, \varphi], \xi] + [[\varphi, \xi], \psi] + [[\xi, \psi], \varphi] = 0, \tag{2.61}$$

on $\mathscr{F}/L_{(-1)}\mathscr{F}$. Note that we may identify the Lie algebra $\mathscr{F}/L_{(-1)}\mathscr{F}$ with the Lie algebra of commutators of operators ψ_0, $\psi \in \mathscr{F}$. Indeed, the commutator formula (2.38) for $m = n = 0$,

$$[\psi_0, \varphi_0] = ([\psi, \varphi])_0, \tag{2.62}$$

together with definition (2.60) tells us that in the adjoint representation ψ acts on $\mathscr{F}/L_{(-1)}\mathscr{F}$ as the operator ψ_0. Moreover, if $\psi = L_{(-1)}\varphi \in L_{(-1)}\mathscr{F}_{(0)}$ for some

$\varphi \in \mathscr{F}_{(0)}$ then $\psi_0 = \text{Res}_z \left[\mathscr{V}(\text{L}_{(-1)}\varphi, z) \right] = \text{Res}_z \left[\frac{d}{dz} \mathscr{V}(\varphi, z) \right] = 0$ by (2.28) and (2.18). In other words, dividing out the subspace $\text{L}_{(-1)}\mathscr{F}$ reflects the fact that the zero mode ψ_0 of a vertex operator $\mathscr{V}(\psi, z)$ remains unchanged when a total derivative is added to $\mathscr{V}(\psi, z)$.

The Lie algebra $\mathscr{F}/\text{L}_{(-1)}\mathscr{F}$ is too large for further investigations. In physical applications such as string theory a distinguished role is played by the primary states of weight $\Delta = 1$ which we shall call **physical states** from now on. In fact, we learn from Eq. (2.57) that for a physical state ψ the corresponding zero mode operator ψ_0 commutes with the Virasoro algebra thereby preserving all subspaces $\mathscr{P}_{(n)}$ of primary states of weight n. In particular, it maps physical states into physical states, i.e. $[\mathscr{P}_{(1)}, \mathscr{P}_{(1)}] \subset \mathscr{P}_{(1)} \bmod \text{L}_{(-1)}\mathscr{P}_{(0)}$. Hence it is natural to look in detail at the **Lie algebra of physical states**,

$$\mathfrak{g}_{\mathscr{F}} := \mathscr{P}_{(1)}/\text{L}_{(-1)}\mathscr{P}_{(0)}, \tag{2.63}$$

where we used the fact that

$$\text{L}_{(-1)}\mathscr{F}_{(0)} \cap \mathscr{P}_{(1)} = \text{L}_{(-1)}\mathscr{P}_{(0)} \tag{2.64}$$

in any vertex algebra. To see this we start from the following identity:

$$\text{L}_{(n)}\text{L}_{(-1)}\psi = (n+1)\text{L}_{(n-1)}\psi + \text{L}_{(-1)}\text{L}_{(n)}\psi \qquad \forall \psi \in \mathscr{F}, n \in \mathbb{Z}. \tag{2.65}$$

Then the inclusion "$\supseteq$" in (2.64) obviously holds. On the other hand, let $\psi \in \mathscr{F}_{(0)}$ and demand $\text{L}_{(n)}\text{L}_{(-1)}\psi \overset{!}{=} 0 \ \forall n \geq 1$. Hence

$$\text{L}_{(n-1)}\psi = -\frac{1}{n+1}\text{L}_{(-1)}\text{L}_{(n)}\psi \qquad \forall n \geq 1, \tag{2.66}$$

which by induction yields the inclusion "$\subseteq$" in (2.64) when the regularity axiom (2.32) is applied to the right-hand side.

When defining the Lie algebra $\mathscr{F}/\text{L}_{(-1)}\mathscr{F}$ we had to divide out the space $\text{L}_{(-1)}\mathscr{F}$ for mathematical reasons. Surprisingly, this reasoning is motivated by physical considerations (cf. [27]). Suppose that $\mathscr{F}$ is equipped with an inner product $(\_,\_)$ such that the operator $\text{L}_{(-n)}$ is the adjoint of $\text{L}_{(n)}$. Then $(\text{L}_{(-1)}\varphi, \psi) = (\varphi, \text{L}_{(1)}\psi) = 0 \ \forall \varphi \in \mathscr{F}, \psi \in \mathscr{P}_{(n)}, n \in \mathbb{Z}$, i.e. the space $\text{L}_{(-1)}\mathscr{F}$ is orthogonal to all primary states. In particular, $\text{L}_{(-1)}\mathscr{P}_{(0)}$ consists of **null physical states**, physical states orthogonal to all physical states including themselves. Hence the Lie algebra $\mathfrak{g}_{\mathscr{F}}$ is obtained from $\mathscr{P}_{(1)}$ by dividing out (unwanted) null physical states.

It is well known that there are additional null physical states in $\mathscr{P}_{(1)}$ if and only if the central charge takes the critical value $c = 26$, namely the space $(\text{L}_{(-2)} + \frac{3}{2}\text{L}_{(-1)}^2)\mathscr{P}_{(-1)}$ (see [27] for the calculations). The existence of these additional null physical states is used in the proof of the no-ghost-theorem [26].

2.2. Toroidal compactification of the bosonic string. It is by no means obvious that nontrivial examples of vertex (operator) algebras exist. However, a class of vertex algebras is provided by the following result (see [6, 17]):
Associated with each nondegenerate even lattice Λ is a vertex algebra $(\mathscr{F}, \mathscr{V}, \mathbf{1}, \omega)$. If in addition Λ is positive definite then $(\mathscr{F}, \mathscr{V}, \mathbf{1}, \omega)$ has the structure of a vertex operator algebra.

The rest of this section will be concerned with the explicit construction of the vertex algebra stated above. For physical motivations of the construction below the

reader may consult the articles [24, 22, 25] or the comprehensive review [32]. A precursor of the construction is given in [28].

Let Λ be an even lattice of rank $d < \infty$ with a symmetric nondegenerate $\mathbb{Z}$-valued $\mathbb{Z}$-bilinear form $\_\cdot\_$ and corresponding metric tensor $\eta^{\mu\nu}$, $1 \leq \mu, \nu \leq d$ (Λ even means that $\mathbf{r}^2 \in 2\mathbb{Z}$ for all $\mathbf{r} \in \Lambda$). The vertex algebra $(\mathscr{F}, \mathscr{V}, \mathbf{1}, \boldsymbol{\omega})$ which we shall construct can be thought of as a bosonic string theory with d spacetime dimensions compactified on a torus. Thus Λ represents the allowed momentum vectors of the theory. We take $\Lambda_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ to be the real vector space in which Λ is embedded.

Introduce "zero mode states" $\Psi_{\mathbf{r}}$, $\mathbf{r} \in \Lambda$, which are by definition orthonormal,

$$(\Psi_{\mathbf{r}}, \Psi_{\mathbf{s}}) = \delta_{\mathbf{r},\mathbf{s}}, \tag{2.67}$$

and oscillators α_m^μ, $m \in \mathbb{Z}$, $1 \leq \mu \leq d$, satisfying the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{2.68}$$

and the hermiticity conditions

$$(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu, \tag{2.69}$$

and acting on zero mode states by

$$\alpha_m^\mu \Psi_{\mathbf{r}} = 0 \quad \text{if } m > 0, \tag{2.70}$$

$$p^\mu \Psi_{\mathbf{r}} = r^\mu \Psi_{\mathbf{r}}, \tag{2.71}$$

where $p^\mu \equiv \alpha_0^\mu$ and r^μ are the components of $\mathbf{r} \in \Lambda$. While the operators α_m^μ for $m > 0$ by definition act as annihilation operators, the creation operators α_m^μ, $m < 0$, generate the Fock space from the zero mode states. For convenience let us define

$$\mathbf{r}(m) := \sum_{\mu=1}^{d} r_\mu \alpha_m^\mu \equiv \mathbf{r}\cdot\boldsymbol{\alpha}_m \tag{2.72}$$

for $\mathbf{r} \in \Lambda_{\mathbb{R}}$, $m \in \mathbb{Z}$, such that

$$[\mathbf{r}(m), \mathbf{s}(n)] = m(\mathbf{r}\cdot\mathbf{s})\delta_{m+n,0}. \tag{2.73}$$

We denote the d-fold Heisenberg algebra spanned by the oscillators by

$$\hat{\mathbf{h}} := \{\mathbf{r}(m) \,|\, \mathbf{r} \in \Lambda_{\mathbb{R}}, m \in \mathbb{Z}\}, \tag{2.74}$$

and for the vector space of finite products of creation operators ($\equiv$ algebra of polynomials on the negative oscillator modes) we write

$$S(\hat{\mathbf{h}}^-) := \bigoplus_{N \in \mathbb{N}} \left\{ \prod_{i=1}^{N} \mathbf{r}_i(-m_i) \,\Big|\, \mathbf{r}_i \in \Lambda_{\mathbb{R}}, m_i > 0 \text{ for } 1 \leq i \leq N \right\}, \tag{2.75}$$

where "S" stands for "symmetric" because of the fact that the creation operators commute with each other.

If we introduce formally position operators q^μ, $1 \leq \mu \leq d$, commuting with α_m^μ for $m \neq 0$ and satisfying

$$[q^\nu, p^\mu] = i\eta^{\mu\nu}, \tag{2.76}$$

then we find that

$$e^{i\mathbf{r}\cdot\mathbf{q}}\Psi_{\mathbf{s}} = \Psi_{\mathbf{r+s}}, \tag{2.77}$$

i.e. the zero mode states can be generated from the vacuum $\Psi_{\mathbf{0}}$:

$$\Psi_{\mathbf{r}} = e^{i\mathbf{r}\cdot\mathbf{q}}\Psi_{\mathbf{0}}. \tag{2.78}$$

Thus the operators $e^{i\mathbf{r}\cdot\mathbf{q}}$, $\mathbf{r} \in \Lambda$, may be identified with the zero mode states and form an abelian group which is called the group algebra of the lattice Λ and is denoted by $\mathbb{R}[\Lambda]$. One might expect the full Fock space $\mathscr{F}$ of the vertex algebra to be $S(\hat{\mathbf{h}}^-) \otimes \mathbb{R}[\Lambda]$. However, it is well known that we must replace the group algebra $\mathbb{R}[\Lambda]$ by something more delicate in order to adjust the signs in the Jacobi identity for the vertex algebra. We will multiply $e^{i\mathbf{r}\cdot\mathbf{q}}$ by a so-called cocycle factor $c_{\mathbf{r}}$ which is a function of momentum $\mathbf{p}$. This means that it commutes with all oscillators α_m^μ and satisfies the eigenvalue equations

$$c_{\mathbf{r}}\Psi_{\mathbf{s}} = \epsilon(\mathbf{r},\mathbf{s})\Psi_{\mathbf{s}}. \tag{2.79}$$

More specifically, we define operators $e^{\mathbf{r}} := e^{i\mathbf{r}\cdot\mathbf{q}}c_{\mathbf{r}}$ and impose the conditions

$$
\begin{align}
e^{\mathbf{r}}e^{\mathbf{s}} &= \epsilon(\mathbf{r},\mathbf{s})e^{\mathbf{r+s}}, \tag{2.80}\\
e^{\mathbf{r}}e^{\mathbf{s}} &= (-1)^{\mathbf{r}\cdot\mathbf{s}}e^{\mathbf{s}}e^{\mathbf{r}}, \tag{2.81}\\
e^{\mathbf{r}}e^{-\mathbf{r}} &= (-1)^{\frac{1}{2}\mathbf{r}^2}, \tag{2.82}\\
e^{\mathbf{0}} &= 1, \tag{2.83}
\end{align}
$$

which are equivalent to requiring, respectively,

$$
\begin{align}
\epsilon(\mathbf{r},\mathbf{s})\epsilon(\mathbf{r+s},\mathbf{t}) &= \epsilon(\mathbf{r},\mathbf{s+t})\epsilon(\mathbf{s},\mathbf{t}), \tag{2.84}\\
\epsilon(\mathbf{r},\mathbf{s}) &= (-1)^{\mathbf{r}\cdot\mathbf{s}}\epsilon(\mathbf{s},\mathbf{r}), \tag{2.85}\\
\epsilon(\mathbf{r},-\mathbf{r}) &= (-1)^{\frac{1}{2}\mathbf{r}^2}, \tag{2.86}\\
\epsilon(\mathbf{0},\mathbf{0}) &= 1. \tag{2.87}
\end{align}
$$

For example, associativity of the product $e^{\mathbf{r}}e^{\mathbf{s}}e^{\mathbf{t}}$ and (2.80) yield (2.84). Note that the cocycle condition (2.84) implies $\epsilon(\mathbf{0},\mathbf{0}) = \epsilon(\mathbf{0},\mathbf{r}) = \epsilon(\mathbf{r},\mathbf{0})$ $\forall\mathbf{r}$. It is not difficult to show that it is always possible to construct cocycles with these properties (see [25], e.g.).[3] Also note that every 2-cocycle $\epsilon : \Lambda \times \Lambda \to \{\pm 1\}$ corresponds to a central extension $\hat{\Lambda}$ of Λ by $\{\pm 1\}$:

$$1 \to \{\pm 1\} \to \hat{\Lambda} \to \Lambda \to 1, \tag{2.88}$$

where we put $\hat{\Lambda} = \{\pm 1\} \times \Lambda$ as a set and define a multiplication in $\hat{\Lambda}$ by

$$(\rho,\mathbf{r}) * (\sigma,\mathbf{s}) := (\epsilon(\mathbf{r},\mathbf{s})\rho\sigma, \mathbf{r+s}) \quad \text{for } \rho,\sigma \in \{\pm 1\}, \ \mathbf{r},\mathbf{s} \in \Lambda. \tag{2.89}$$

We will take the twisted group algebra $\mathbb{R}\{\Lambda\}$ consisting of the operators $e^{\mathbf{r}}$, $\mathbf{r} \in \Lambda$, instead of $\mathbb{R}[\Lambda]$. This means nothing but working with the section in the double cover $\hat{\Lambda}$ of the lattice Λ.

---

[3] Without loss of generality we can assume that the function ϵ is bimultiplicative, i.e. $\epsilon(\mathbf{r+s},\mathbf{t}) = \epsilon(\mathbf{r},\mathbf{t})\epsilon(\mathbf{s},\mathbf{t})$ and $\epsilon(\mathbf{r},\mathbf{s+t}) = \epsilon(\mathbf{r},\mathbf{s})\epsilon(\mathbf{r},\mathbf{t})$, $\forall\mathbf{r},\mathbf{s},\mathbf{t}$. Together with (2.86) and the normalization condition (2.87) this then implies that $\epsilon(m\mathbf{r},n\mathbf{r}) = [\epsilon(\mathbf{r},\mathbf{r})]^{mn} = (-1)^{\frac{1}{2}mn\mathbf{r}^2}$, $\forall\mathbf{r}, \ m,n \in \mathbb{Z}$.

To summarize: The Fock space associated with the lattice Λ is defined to be

$$\mathscr{F} := S(\hat{\mathbf{h}}^-) \otimes \mathbb{R}\{\Lambda\}. \tag{2.90}$$

Note that the oscillators $\mathbf{r}(m)$, $m \neq 0$, act only on the first tensor factor, namely, creation operators as multiplication operators and annihilation operators via the adjoint representation, i.e. by (2.73). The zero mode operators α_0^μ, however, are only sensible for the twisted group algebra,

$$\mathbf{r}(0)e^{\mathbf{s}} = (\mathbf{r} \cdot \alpha_0)e^{\mathbf{s}} = (\mathbf{r} \cdot \mathbf{s})e^{\mathbf{s}} \qquad \forall \mathbf{r} \in \Lambda_{\mathbb{R}}, \mathbf{s} \in \Lambda, \tag{2.91}$$

while the action of $e^{\mathbf{r}}$ on $\mathbb{R}\{\Lambda\}$ is given by (2.80).

We shall define next the (untwisted) vertex operators $\mathscr{V}(\psi, z)$ for $\psi \in \mathscr{F}$. For $\mathbf{r} \in \Lambda_{\mathbb{R}}$ we introduce the formal sum

$$\mathbf{r}(z) := \sum_{m \in \mathbb{Z}} \mathbf{r}(m)z^{-m-1}, \tag{2.92}$$

which is an element of $\hat{\mathbf{h}}[[z, z^{-1}]]$ and may be regarded as a generating function for the operators $\mathbf{r}(m)$, $m \in \mathbb{Z}$, or as a "current" in contrast to the "states" in $\mathscr{F}$. It is convenient to split the current $\mathbf{r}(z)$ into three parts:

$$\mathbf{r}(z) = \mathbf{r}_<(z) + \mathbf{r}(0) + \mathbf{r}_>(z), \tag{2.93}$$

where

$$\mathbf{r}_<(z) := \sum_{m>0} \mathbf{r}(-m)z^{m-1}, \qquad \mathbf{r}_>(z) := \sum_{m>0} \mathbf{r}(m)z^{-m-1}. \tag{2.94}$$

We will employ the usual normal ordering procedure, i.e. colons indicate that in the enclosed expressions, q^ν is written to the left of p^μ, as well as the creation operators are to be placed to the left of the annihilation operators.

For $e^{\mathbf{r}} \in \mathbb{R}\{\Lambda\}$, we set

$$\mathscr{V}(e^{\mathbf{r}}, z) := e^{\int \mathbf{r}_<(z)dz} e^{\mathbf{r}} z^{\mathbf{r}(0)} e^{\int \mathbf{r}_>(z)dz}, \tag{2.95}$$

using an obvious formal integration notation, i.e.

$$\int \mathbf{r}_<(z)dz := \sum_{m>0} \frac{1}{m}\mathbf{r}(-m)z^m, \qquad \int \mathbf{r}_>(z)dz := -\sum_{m>0} \frac{1}{m}\mathbf{r}(m)z^{-m}. \tag{2.96}$$

This can be written in a way more familiar to physicists by introducing the Fubini-Veneziano coordinate field,

$$Q^\mu(z) \equiv q^\mu - ip^\mu \ln z + i \sum_{m \in \mathbb{Z}} \frac{1}{m}\alpha_m^\mu z^{-m}, \tag{2.97}$$

which really only has a meaning when exponentiated. We find that the vertex operator in (2.95) takes the familiar form

$$\mathscr{V}(e^{\mathbf{r}}, z) = {:}e^{i\mathbf{r} \cdot Q(z)}{:}c_{\mathbf{r}}, \tag{2.98}$$

and the current $\mathbf{r}(z)$ becomes

$$\mathbf{r}(z) = \mathrm{i}\frac{d}{dz}[\mathbf{r}\cdot\mathbf{Q}(z)]. \tag{2.99}$$

This shows that the vertex operators in (2.95) are indeed already normal ordered and carry a cocycle factor hidden in the elements of the twisted group algebra $\mathbb{R}\{\Lambda\}$.

Let $\psi = \left[\prod_{j=1}^{N} \mathbf{s}_j(-n_j)\right] \otimes \mathrm{e}^{\mathbf{r}}$ be a typical homogeneous element of $\mathscr{F}$ and define

$$\mathscr{V}(\psi, z) \;:=\; \;:\mathscr{V}(\mathrm{e}^{\mathbf{r}}, z)\prod_{j=1}^{N}\frac{1}{(n_j - 1)!}\left(\frac{d}{dz}\right)^{n_j - 1}\mathbf{s}_j(z): \tag{2.100}$$

$$\equiv \quad \mathrm{i}:\mathrm{e}^{\mathrm{i}\mathbf{r}\cdot\mathbf{Q}(z)}\prod_{j=1}^{N}\frac{1}{(n_j - 1)!)}\left(\frac{d}{dz}\right)^{n_j}(\mathbf{s}_j\cdot\mathbf{Q}(z)): c_{\mathbf{r}}.$$

Extending this definition by linearity we finally obtain a well-defined map

$$\mathscr{V}:\mathscr{F} \to (\mathrm{End}\,\mathscr{F})[\![z, z^{-1}]\!], \qquad \psi \mapsto \sum_{n\in\mathbb{Z}}\psi_n z^{-n-1}. \tag{2.101}$$

We shall prove the first four axioms in the definition of a vertex algebra.

1. **(Regularity)** Note that $\mathscr{F}$ contains only states with a finite occupation number of creation operators and the vertex operators are normal ordered expressions. Having this in mind it is clear that $\psi_n\varphi = 0$ for n large enough (depending on $\psi, \varphi \in \mathscr{F}$) since annihilation operators are always attached to negative powers of the formal variables.

2. **(Vacuum)** We choose the vacuum to be the zero mode state with no momentum and without any creation operators, i.e.

$$\mathbf{1} := 1 \otimes \mathrm{e}^{\mathbf{0}}, \tag{2.102}$$

   so that

$$\mathscr{V}(\mathbf{1}, z) = :\mathrm{e}^{\mathrm{i}\mathbf{0}\cdot\mathbf{Q}(z)}:c_{\mathbf{0}} = \mathrm{id}_{\mathscr{F}} \tag{2.103}$$

   by the normalization condition (2.83).

3. **(Injectivity)** Observe that, when acting on the vacuum, only terms involving creation operators survive in the expression for a vertex operator. Then it is obvious that

$$\psi_{-1}\mathbf{1} = \mathrm{Res}_z\left[z^{-1}\mathscr{V}(\psi, z)\mathbf{1}\right] = \psi \qquad \forall\psi \in \mathscr{F}. \tag{2.104}$$

   In particular, $\mathscr{V}(\psi, z) = 0$ implies $\psi = 0$.

4. **(Conformal vector)** We claim that the element

$$\boldsymbol{\omega} := \tfrac{1}{2}\sum_{\mu=1}^{d}\mathbf{e}^{(\mu)}(-1)\mathbf{e}_{(\mu)}(-1)(\otimes\mathrm{e}^{\mathbf{0}}) \tag{2.105}$$

   provides a conformal vector of dimension d which is independent of the choice of the basis $\{\mathbf{e}_{(\mu)}\}$ of $\Lambda_{\mathbb{R}}$ with dual basis $\{\mathbf{e}^{(\mu)}\}$ (w.r.t. $\eta^{\mu\nu}$). By (2.100) and (2.92), we have

$$\mathscr{V}(\omega, z) = \tfrac{1}{2} \sum_{\mu=1}^{d} {:}\mathbf{e}^{(\mu)}(z)\mathbf{e}_{(\mu)}(z){:}$$

$$= \tfrac{1}{2} \sum_{m,n\in\mathbb{Z}} {:}\boldsymbol{\alpha}_m \cdot \boldsymbol{\alpha}_n{:}\, z^{-m-n-2}. \tag{2.106}$$

(Note that in the last step we had to rely on nondegeneracy of the lattice, i.e. we used the completeness relation $\sum_{\mu=1}^{d} (\mathbf{e}^{(\mu)})_\rho (\mathbf{e}_{(\mu)})_\sigma = \eta_{\rho\sigma}$.) Thus

$$L_{(n)} \equiv \omega_{n+1} = \tfrac{1}{2} \sum_{m\in\mathbb{Z}} {:}\boldsymbol{\alpha}_m \cdot \boldsymbol{\alpha}_{n-m}{:}, \tag{2.107}$$

in agreement with the well-known expression from string theory. Using the oscillator commutation relations one indeed finds that the $L_{(n)}$'s obey (2.27) with central charge $c = d$ (see [27] for the calculation). To establish the translation property of $L_{(-1)}$ we find that

$$L_{(-1)}\mathbf{e}^{\mathbf{r}} = \mathbf{r}(-1)\mathbf{e}^{\mathbf{r}}, \tag{2.108}$$
$$L_{(-1)}\mathbf{r}(-m) = m\mathbf{r}(-m-1) \quad \text{for } m > 0, \tag{2.109}$$

by (2.91) and (2.73); but, on the other hand,

$$\frac{d}{dz}\mathscr{V}(\mathbf{e}^{\mathbf{r}}, z) = {:}\mathbf{r}(z)\mathscr{V}(\mathbf{e}^{\mathbf{r}}, z){:} = \mathscr{V}(\mathbf{r}(-1)\mathbf{e}^{\mathbf{r}}, z), \tag{2.110}$$

$$\frac{d}{dz}\mathscr{V}(\mathbf{r}(-m), z) = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^m \mathbf{r}(z) = \mathscr{V}(m\mathbf{r}(-m-1), z) \tag{2.111}$$

by (2.95), (2.100) and (2.92). Together with the derivation property of $L_{(-1)}$ and $\frac{d}{dz}$ this proves (2.28). Finally, let $\psi = \left[\prod_{j=1}^{N} \mathbf{s}_j(-n_j)\right] \otimes \mathbf{e}^{\mathbf{r}}$ be a typical homogeneous element of $\mathscr{F}$. Then

$$L_{(0)}\psi = \left(\tfrac{1}{2}\mathbf{r}^2 + \sum_{j=1}^{N} n_j\right)\psi \tag{2.112}$$

yields the desired grading of $\mathscr{F}$. Furthermore we observe that the spectrum of $L_{(0)}$ is nonnegative and the eigenspaces of $L_{(0)}$ are finite-dimensional provided that Λ is a positive definite lattice; while if Λ is Lorentzian then $\mathbf{r}^2$ can be arbitrarily negative so that the spectrum of $L_{(0)}$ is unbounded from above as well as from below.

It is not surprising that by far the hardest axiom to prove is the Jacobi identity because it contains most information about a vertex algebra. We will skip the proof and refer the interested reader to Ref. [17].

We turn now to the analysis of the Lie algebra of physical states, $\mathfrak{g}_\Lambda$, and work out some of its commutators. A closed formula for the commutator of zero mode operators associated to general weight one states and a related investigation of the Lie algebra of quasiprimary states of weight one can be found in [34] and [35].

Let us first list the simplest physical states:

1. **Tachyonic states:**

$$\mathfrak{g}_{\Lambda}^{[0]} := \{\mathrm{e}^{\mathbf{r}} \mid \mathbf{r} \in \Lambda_2\};$$ (2.113)

2. **Photonic states:**

$$\mathfrak{g}_{\Lambda}^{[1]} := \{\mathbf{s}(-1) \otimes \mathrm{e}^{\mathbf{r}} \mid \mathbf{r}\cdot\mathbf{s} = 0, \mathbf{s} \in \Lambda_{\mathbb{R}}, \mathbf{r} \in \Lambda_0\};$$ (2.114)

3. **Massive spin 2 states:**

$$\mathfrak{g}_{\Lambda}^{[2]} := \Big\{ \big[(\mathbf{s}\cdot\mathbf{r})\mathbf{t}(-2) + (\mathbf{t}\cdot\mathbf{r})\mathbf{s}(-2) - 2\mathbf{s}(-1)\mathbf{t}(-1)\big] \otimes \mathrm{e}^{\mathbf{r}} \Big| $$

$$\mathbf{s}\cdot\mathbf{t} = 2(\mathbf{s}\cdot\mathbf{r})(\mathbf{t}\cdot\mathbf{r}); \mathbf{s}, \mathbf{t} \in \Lambda_{\mathbb{R}}, \mathbf{r} \in \Lambda_{-2} \Big\};$$ (2.115)

where $\Lambda_n := \{\mathbf{r} \in \Lambda \mid \mathbf{r}^2 = n \ (\in 2\mathbb{Z})\}$ denotes the set of lattice vectors of squared length n and the superscript of $\mathfrak{g}_{\Lambda}$ counts the oscillator excitations. The relevant physical state conditions for above polarization vectors $\mathbf{s}, \mathbf{t} \in \Lambda_{\mathbb{R}}$ follow immediately from (2.112) and

$$\begin{aligned}
\mathrm{L}_{(m)}\psi \ = \ & \sum_{\substack{k=1 \\ n_k > m}}^{N} n_k \Bigg[\mathbf{s}_k(m - n_k) \prod_{\substack{j=1 \\ j \neq k}}^{N} \mathbf{s}_j(-n_j)\Bigg] \otimes \mathrm{e}^{\mathbf{r}} \\
& + m \sum_{k=1}^{N} \delta_{m,n_k}(\mathbf{s}_k\cdot\mathbf{r}) \Bigg[\prod_{\substack{j=1 \\ j \neq k}}^{N} \mathbf{s}_j(-n_j)\Bigg] \otimes \mathrm{e}^{\mathbf{r}} \\
& + \sum_{k<k'}^{N} n_k n_{k'} \delta_{m,n_k+n_{k'}}(\mathbf{s}_k\cdot\mathbf{s}_{k'}) \Bigg[\prod_{\substack{j=1 \\ j \neq k,k'}}^{N} \mathbf{s}_j(-n_j)\Bigg] \otimes \mathrm{e}^{\mathbf{r}} \quad (2.116)
\end{aligned}$$

for $\psi = \big[\prod_{j=1}^{N} \mathbf{s}_j(-n_j)\big] \otimes \mathrm{e}^{\mathbf{r}} \in \mathscr{F}$. The above formula also exhibits an explicit example for the regularity axiom (2.32), namely, that $\mathrm{L}_{(m)}\psi = 0$ for $m > \max_{j \neq k}(n_j + n_k)$. We want to stress again that the physical states in $\mathfrak{g}_{\Lambda}$ are only defined modulo $\mathrm{L}_{(-1)}\mathscr{F}_{(0)}$ which means for example that $\mathbf{r}(-1) \otimes \mathrm{e}^{\mathbf{r}} = \mathrm{L}_{(-1)}(\mathrm{e}^{\mathbf{r}}) \equiv 0$ in $\mathfrak{g}_{\Lambda}$ for $\mathbf{r} \in \Lambda_0$.

For the antisymmetric product (2.60) on $\mathscr{F}_{(1)}/\mathrm{L}_{(-1)}\mathscr{F}_{(0)}$ we obtain

$$\begin{aligned}
[\mathrm{e}^{\mathbf{r}}, \mathrm{e}^{\mathbf{s}}] \ := \ & \mathrm{Res}_z \Big[\mathrm{e}^{\int \mathbf{r}_<(z)dz} \mathrm{e}^{\mathbf{r}} z^{\mathbf{r}(0)} \mathrm{e}^{\int \mathbf{r}_>(z)dz} (1 \otimes \mathrm{e}^{\mathbf{s}})\Big] \\
= \ & \mathrm{Res}_z \Bigg[\sum_{m \geq 0} \mathrm{S}_m(\mathbf{r}) z^{m+\mathbf{r}\cdot\mathbf{s}} \mathrm{e}^{\mathbf{r}} \mathrm{e}^{\mathbf{s}}\Bigg] \\
= \ & \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{s} \geq 0, \\ \epsilon(\mathbf{r},\mathbf{s})\mathrm{S}_{-1-\mathbf{r}\cdot\mathbf{s}}(\mathbf{r}) \otimes \mathrm{e}^{\mathbf{r}+\mathbf{s}} & \text{if } \mathbf{r}\cdot\mathbf{s} < 0, \end{cases} \quad (2.117)
\end{aligned}$$

where we used the Schur polynomials $\mathrm{S}_m(\mathbf{r}) \equiv \mathrm{S}_m(\mathbf{r}(-1), \mathbf{r}(-2), \ldots, \mathbf{r}(-m))$ which are defined via the generating function

$$\mathrm{e}^{\sum_{n>0} \frac{1}{n}\mathbf{r}(-n)z^n} = \sum_{m \geq 0} \mathrm{S}_m(\mathbf{r}(-1), \mathbf{r}(-2), \ldots, \mathbf{r}(-m))z^m,$$ (2.118)

so that, e.g., $S_0(\mathbf{r}) = 1$, $S_1(\mathbf{r}) = \mathbf{r}(-1)$, $S_2(\mathbf{r}) = \frac{1}{2!}\big(\mathbf{r}(-1)^2 + \mathbf{r}(-2)\big)$. For notational convenience we put $S_m(\mathbf{r}) := 0 \; \forall m < 0, \mathbf{r} \in \Lambda$. We also find that

$$
\begin{aligned}
&\big[\mathbf{s}(-1) \otimes e^{\mathbf{r}}, e^{\mathbf{t}}\big] \\[2mm]
&= \operatorname{Res}_z \left[{:} e^{\int \mathbf{r}_<(z)dz} e^{\mathbf{r}} z^{\mathbf{r}(0)} e^{\int \mathbf{r}_>(z)dz} \mathbf{s}(z) {:} (1 \otimes e^{\mathbf{t}}) \right] \\[2mm]
&= \operatorname{Res}_z \left[\sum_{m \geq 0} S_m(\mathbf{r}) z^{m+\mathbf{r}\cdot\mathbf{t}} \left((\mathbf{s}\cdot\mathbf{t})z^{-1} + \sum_{n>0} \mathbf{s}(-n)z^{n-1} \right) \otimes e^{\mathbf{r}}e^{\mathbf{t}} \right] \\[2mm]
&= \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{t} \geq 1, \\[2mm] \epsilon(\mathbf{r},\mathbf{t})\left[(\mathbf{s}\cdot\mathbf{t})S_{-\mathbf{r}\cdot\mathbf{t}}(\mathbf{r}) + \displaystyle\sum_{m=0}^{-1-\mathbf{r}\cdot\mathbf{t}} S_m(\mathbf{r})\mathbf{s}(m+\mathbf{r}\cdot\mathbf{t}) \right] \otimes e^{\mathbf{r}+\mathbf{t}} & \text{if } \mathbf{r}\cdot\mathbf{t} \leq 0; \end{cases}
\end{aligned}
\tag{2.119}
$$

$$
\begin{aligned}
&\big[\mathbf{s}(-1) \otimes e^{\mathbf{r}}, \mathbf{u}(-1) \otimes e^{\mathbf{t}}\big] \\[2mm]
&= \operatorname{Res}_z \left[{:} e^{\int \mathbf{r}_<(z)dz} e^{\mathbf{r}} z^{\mathbf{r}(0)} e^{\int \mathbf{r}_>(z)dz} \mathbf{s}(z) {:} (\mathbf{u}(-1) \otimes e^{\mathbf{t}}) \right] \\[2mm]
&= \operatorname{Res}_z \left[\sum_{m \geq 0} S_m(\mathbf{r}) z^{m+\mathbf{r}\cdot\mathbf{t}} \Big[\mathbf{s}\cdot\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})(\mathbf{s}\cdot\mathbf{t}) \Big] z^{-2} \right. \\[2mm]
&\qquad\qquad + \big((\mathbf{s}\cdot\mathbf{t})\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})\mathbf{s}\big)(-1)z^{-1} \\[2mm]
&\qquad\qquad \left. + \sum_{n>0} \big(\mathbf{s}(-n)\mathbf{u}(-1) - (\mathbf{r}\cdot\mathbf{u})\mathbf{s}(-n-1)\big)z^{n-1} \right] \otimes e^{\mathbf{r}}e^{\mathbf{t}} \Bigg] \\[2mm]
&= \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{t} \geq 2, \\[2mm] \epsilon(\mathbf{r},\mathbf{t})\Big[[\mathbf{s}\cdot\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})(\mathbf{s}\cdot\mathbf{t})]S_{1-\mathbf{r}\cdot\mathbf{t}}(\mathbf{r}) & \\[2mm] \qquad + \big((\mathbf{s}\cdot\mathbf{t})\mathbf{u} - (\mathbf{r}\cdot\mathbf{u})\mathbf{s}\big)(-1)S_{-\mathbf{r}\cdot\mathbf{t}}(\mathbf{r}) & \\[2mm] \qquad + \displaystyle\sum_{m=0}^{-1-\mathbf{r}\cdot\mathbf{t}} \big(\mathbf{s}(-m-\mathbf{r}\cdot\mathbf{t})\mathbf{u}(-1) & \\[2mm] \qquad - (\mathbf{r}\cdot\mathbf{u})\mathbf{s}(-m-1-\mathbf{r}\cdot\mathbf{t})\big)S_m(\mathbf{r})\Big] \otimes e^{\mathbf{r}+\mathbf{t}} & \text{if } \mathbf{r}\cdot\mathbf{t} \leq 1. \end{cases}
\end{aligned}
\tag{2.120}
$$

These formulas simplify drastically in the special case where Λ is a positive definite even lattice. Obviously, $\mathscr{F}_{(0)} = \mathbb{R}\mathbf{1}$ and the spectrum of $L_{(0)}$ is nonnegative so that $\mathfrak{g}_\Lambda = \mathscr{P}_{(1)} = \mathscr{F}_{(1)}$. Its elements are easy to describe,

$$
\mathfrak{g}_\Lambda = \{\mathbb{R}e^{\mathbf{r}} \mid \mathbf{r} \in \Lambda_2\} \oplus \{\mathbf{s}(-1) \mid \mathbf{s} \in \Lambda_{\mathbb{R}}\}.
\tag{2.121}
$$

The commutators become

$$
\big[\mathbf{r}(-1), \mathbf{s}(-1)\big] = 0,
\tag{2.122}
$$

$$
\big[\mathbf{r}(-1), e^{\mathbf{s}}\big] = (\mathbf{r}\cdot\mathbf{s})e^{\mathbf{s}},
\tag{2.123}
$$

$$
\big[e^{\mathbf{r}}, e^{\mathbf{s}}\big] = \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{s} \geq 0, \\ \epsilon(\mathbf{r},\mathbf{s})e^{\mathbf{r}+\mathbf{s}} & \text{if } \mathbf{r}\cdot\mathbf{s} = -1, \\ -\mathbf{r}(-1) & \text{if } \mathbf{r}\cdot\mathbf{s} = -2. \end{cases}
\tag{2.124}
$$

Note that in this special case the Schwarz inequality yields $|\mathbf{r} \cdot \mathbf{s}| \leq 2$. Moreover, $\mathbf{r} \cdot \mathbf{s} = -1 \iff \mathbf{r} + \mathbf{s} \in \Lambda_2$ and $\mathbf{r} \cdot \mathbf{s} = -2 \iff \mathbf{r} + \mathbf{s} = 0$ for $\mathbf{r}, \mathbf{s} \in \Lambda_2$.

We are not interested here in this well-understood case of positive definite Λ which leads to a finite-dimensional Lie algebra $\mathfrak{g}_\Lambda$, but rather the case of Lorentzian lattices, which is of course far more complicated.

We have seen that a special role is played by the norm 2 vectors of Λ which we call **real roots** of the lattice. The **reflection** $\mathfrak{w}_\mathbf{r}$ associated with a real root $\mathbf{r}$ is defined as $\mathfrak{w}_\mathbf{r}(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{r})\mathbf{r}$ for $\mathbf{x} \in \Lambda_\mathbb{R}$. It is easy to see that a reflection in a real root is an automorphism of the lattice. The hyperplanes perpendicular to these real roots divide the vector space $\Lambda_\mathbb{R}$ into regions called **Weyl chambers**. The reflections in the real roots of Λ generate a group called the **Weyl group** $\mathfrak{W}$ of Λ, which acts simply transitively on the Weyl chambers of Λ. This means that if we fix one Weyl chamber $\mathscr{C}$ then any real root from the interior of another Weyl chamber can be transported via Weyl reflection to a unique real root in $\mathscr{C}$. The real roots $\mathbf{r}_i$ that are perpendicular to the faces of $\mathscr{C}$ and have inner product at most 0 with the elements of $\mathscr{C}$ are called the **simple roots** of $\mathscr{C}$. The **Coxeter-Dynkin diagram** $\mathscr{G}$ of $\mathscr{C}$ is the set of simple roots of $\mathscr{C}$, drawn as a graph with one vertex for each simple root of $\mathscr{C}$ and two vertices corresponding to the distinct roots $\mathbf{r}_i, \mathbf{r}_j$ are joined by $-\mathbf{r}_i \cdot \mathbf{r}_j$ lines.

Let us denote the group of graph automorphisms of the Coxeter-Dynkin diagram by $\mathrm{Aut}(\mathscr{G})$. Note that an automorphism $\sigma \in \mathrm{Aut}(\mathscr{G})$ induces an automorphism of Λ by $\sigma(\mathbf{r}_i) := \mathbf{r}_{\sigma(i)}$ $\forall i$. Hence $\mathrm{Aut}(\mathscr{G})$ may be identified with the group of automorphisms of Λ fixing $\mathscr{C}$. Furthermore, one can show that $\sigma \mathfrak{w}_{\mathbf{r}_i} \sigma^{-1} = \mathfrak{w}_{\mathbf{r}_{\sigma(i)}}$ and that $\mathfrak{W} \cap \mathrm{Aut}(\mathscr{G}) = 1$. Then the group of all autochronous automorphisms of the lattice Λ is a split extension of its Weyl group by $\mathrm{Aut}(\mathscr{G})$,

$$0 \longrightarrow \mathfrak{W} \overset{\iota}{\longrightarrow} \mathrm{Aut}(\Lambda)^+ \overset{\pi}{\longrightarrow} \mathrm{Aut}(\mathscr{G}) \longrightarrow 0, \qquad \mathrm{im}\, \iota = \ker \pi,$$

i.e. it is equivalent to a semidirect product of the Weyl group and the group of graph automorphisms:

$$\mathrm{Aut}(\Lambda)^+ = \mathfrak{W} \rtimes \mathrm{Aut}(\mathscr{G}).$$

The full automorphism group of Λ is just the autochronous subgroup extended by the negative of the identity operation (which interchanges the forward and backward light cones).

Returning to the vertex algebra $(\mathscr{F}, \mathscr{V}, \mathbf{1}, \omega)$ associated with the even Lorentzian lattice Λ, we immediately infer from (2.113) and (2.114) that, for any simple root $\mathbf{r}_i$, the elements $e^{\mathbf{r}_i}$, $e^{-\mathbf{r}_i}$, and $\mathbf{r}_i(-1)$ describe physical states, i.e. they lie in $\mathscr{P}_{(1)}$. Define generators for a Lie algebra $\mathfrak{g}(A)$ by

$$e_i \mapsto e^{\mathbf{r}_i}, \tag{2.125}$$

$$f_i \mapsto -e^{-\mathbf{r}_i}, \tag{2.126}$$

$$h_i \mapsto \mathbf{r}_i(-1). \tag{2.127}$$

Then, by (2.122) – (2.124), we find the following relations to hold:

$$[h_i, h_j] = 0, \tag{2.128}$$

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \tag{2.129}$$

$$[e_i, f_j] = \delta_{ij} h_i, \tag{2.130}$$

where we defined the **Cartan matrix** $A = (a_{ij})$ associated with $\mathscr{C}$ by $a_{ij} := \mathbf{r}_i \cdot \mathbf{r}_j$. The elements h_i obviously form a basis for an abelian subalgebra of $\mathfrak{g}(A)$ called the **Cartan subalgebra** $\mathfrak{h}(A)$. In technical terms, from the above commutators we learn that the elements $\{e_i, f_i, h_i \mid i\}$ generate the so-called free Lie algebra associated with A. But even more is true; for we can show that the **Serre relations**

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0, \tag{2.131}$$

are fulfilled for all i, j. To see this we recall that $\mathscr{F}$ is Λ-graded by construction,

$$\mathscr{F} = \bigoplus_{\mathbf{x} \in \Lambda} S(\hat{\mathbf{h}}^-) \otimes e^{\mathbf{x}} \equiv \bigoplus_{\mathbf{x} \in \Lambda} \mathscr{F}^{(\mathbf{x})}. \tag{2.132}$$

Then the Lie algebra of physical states inherits a natural Λ-gradation from $\mathscr{F}$ by defining

$$\mathfrak{g}_\Lambda^{(\mathbf{x})} := \mathfrak{g}_\Lambda \cap \left[S(\hat{\mathbf{h}}^-) \otimes e^{\mathbf{x}} \right], \tag{2.133}$$

so that

$$[\mathfrak{g}_\Lambda^{(\mathbf{x})}, \mathfrak{g}_\Lambda^{(\mathbf{y})}] \subset \mathfrak{g}_\Lambda^{(\mathbf{x}+\mathbf{y})} \tag{2.134}$$

for $\mathbf{x}, \mathbf{y} \in \Lambda$.[4] In particular,

$$(\operatorname{ad} e^{\mathbf{r}})^j e^{\mathbf{s}} \in \mathfrak{g}_\Lambda^{(j\mathbf{r}+\mathbf{s})} \quad \forall j \geq 0, \ \mathbf{r}, \mathbf{s} \in \Lambda_2. \tag{2.135}$$

From (2.112) we infer that the element $(\operatorname{ad} e^{\mathbf{r}})^j e^{\mathbf{s}}$ has an $L_{(0)}$ eigenvalue of at least $\frac{1}{2}(j\mathbf{r}+\mathbf{s})^2 = 1 + j(j + \mathbf{r}\cdot\mathbf{s})$. Comparing this with the physical state condition $L_{(0)}\psi = \psi$ we conclude that

$$(\operatorname{ad} e^{\mathbf{r}})^j e^{\mathbf{s}} = 0 \quad \text{for } j \geq 1 - \mathbf{r}\cdot\mathbf{s}. \tag{2.136}$$

Having established the Serre relations, the Gabber-Kac theorem [29, Thm.9.11] tells us that the Lie algebra $\mathfrak{g}(A)$ generated by the elements $\{e_i, f_i, h_i \mid i\}$ is just the **Kac Moody algebra** associated with the Cartan matrix A. Namely, the latter is defined as the above free Lie algebra divided by the maximal ideal intersecting $\mathfrak{h}(A)$ trivially, and the theorem states that this maximal ideal is generated by the elements $\{(\operatorname{ad} e_i)^{1-a_{ij}} e_j, (\operatorname{ad} f_i)^{1-a_{ij}} f_j \mid i \neq j\}$.

   We would like to emphasize the remarkable fact that the physical state condition $L_{(0)}\psi = \psi$ accounts for all Serre relations which are usually very difficult to deal with in the theory of Kac Moody algebras; or, in string theory language, the absence of particles with squared mass below the tachyon reflects the validity of the Serre relations for the Lie algebra $\mathfrak{g}(A)$.

   To summarize (cf. [6]): The physical states $\{e^{\mathbf{r}_i}, e^{-\mathbf{r}_i}, \mathbf{r}_i(-1) \mid i\}$ generate via multiple commutators the Kac Moody algebra $\mathfrak{g}(A)$ associated with the Cartan matrix $A = (\mathbf{r}_i \cdot \mathbf{r}_j)$ which is a subalgebra of the Lie algebra of physical states, $\mathfrak{g}_\Lambda$.

   Only in the Euclidean case these two Lie algebras coincide. In general, we have a *proper* inclusion

$$\mathfrak{g}(A) \subset \mathfrak{g}_\Lambda, \tag{2.137}$$

---

[4] Some of the subspaces $\mathfrak{g}_\Lambda^{(\mathbf{x})}$ may be empty, e.g. for $\mathbf{x}^2 > 2$.

and the characterization of the elements of $\mathfrak{g}_\Lambda$ not contained in the Lie algebra $\mathfrak{g}(A)$, is the key problem for the vertex operator construction of hyperbolic Kac Moody algebras. The special feature of (2.137) is that the root system of the Kac Moody algebra $\mathfrak{g}(A)$ is well understood though its root multiplicities are not completely known for a single example; whereas the root system of $\mathfrak{g}_\Lambda$ is certainly not compatible with that of a Kac Moody algebra although the root multiplicities are always known. Thus a complete understanding of (2.137) requires a "mechanism" which tells us how $\mathfrak{g}(A)$ has to be filled up with physical states to reach the complete Lie algebra of physical states. For the special case of the unique self-dual Lorentzian lattice $II_{25,1}$, this was accomplished in [4] by the addition of imaginary simple roots, or, equivalently, by adjoining new generators to the Kac Moody algebra $L_\infty(=\mathfrak{g}(A))$, where the infinite matrix A corresponds to the Coxeter-Dynkin diagram built up from the Leech roots [7]), thereby furnishing the transition to the "fake monster" Lie algebra $\mathfrak{g}_{II_{25,1}}$ [4]. See also [39] for an attempt to determine the structure constants of this algebra.

3. Discrete DDF Construction

As can be seen from Eqs. (2.113) – (2.115) and (2.116) the Virasoro conditions $(L_{(n)} - \delta_{n0})\psi = 0$, $n \geq 0$, which should be obeyed by physical states ψ, become increasingly complicated at higher excitations. In fact, we cannot hope to arrive at a general description of the physical states by this method of calculating polarization vectors. However, there is an elegant resolution of this problem by Del Giudice, Di Vecchia and Fubini [11] which allows an explicit construction of all the physical excited states. The idea is to find a set of operators that commute with the Virasoro operators, and which when applied successively to the tachyonic ground states give all possible physical states. These operators form a closed algebra called the **spectrum generating algebra**. It turns out that the latter consists of transversal DDF operators A_n^i, $1 \leq i \leq d - 2$, $n \in \mathbb{Z}$, describing the transversal modes of the string, and of longitudinal DDF operators $\mathfrak{L}_n$, $n \in \mathbb{Z}$ for the longitudinal excitations. We shall now introduce the discrete version of these operators taking into account that the momenta lie on the even lattice Λ so that we are not allowed to use Lorentz transformations to rotate them into convenient frames. Apparently, the longitudinal DDF operators have so far not been considered in this discrete context.

3.1. DDF vertex operators. Let $\mathbf{k}$ be a primitive lightlike lattice vector, i.e., $\mathbf{k} \in \Lambda_0$ and $\frac{1}{n}\mathbf{k} \notin \Lambda_0 \, \forall n > 1$. Using (2.120) we can immediately write down the commutator of physical states $\boldsymbol{\xi}(-1)e^{m\mathbf{k}}$ and $\boldsymbol{\eta}(-1)e^{n\mathbf{k}}$, $m, n \in \mathbb{Z}$:

$$\left[\boldsymbol{\xi}(-1)e^{m\mathbf{k}}, \boldsymbol{\eta}(-1)e^{n\mathbf{k}}\right] = \epsilon(m\mathbf{k}, n\mathbf{k})(\boldsymbol{\xi}\cdot\boldsymbol{\eta})m\mathbf{k}(-1)e^{(m+n)\mathbf{k}}$$
$$= m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}\mathbf{k}(-1), \tag{3.1}$$

since $\boldsymbol{\xi}\cdot\mathbf{k} = \boldsymbol{\eta}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{k} = 0$ and $\mathbf{k}(-1)e^{n\mathbf{k}} = \frac{1}{n}L_{(-1)}(e^{n\mathbf{k}}) \equiv 0$ for $n \neq 0$. Recall that we assumed the cocycle ϵ to be bimultiplicative so that $\epsilon(m\mathbf{k}, n\mathbf{k}) = (-1)^{\frac{1}{2}mn\mathbf{k}^2} = 1$. We define the **transversal DDF operator** $A_m^{\boldsymbol{\xi}} = A_m(\boldsymbol{\xi}, \mathbf{k})$ as the zero mode operator corresponding to the physical state $\boldsymbol{\xi}(-1)e^{m\mathbf{k}}$,

$$A_m^{\boldsymbol{\xi}} := \left(\boldsymbol{\xi}(-1)e^{m\mathbf{k}}\right)_0 \tag{3.2}$$

$$= \text{Res}_z\left[\mathscr{V}(\boldsymbol{\xi}(-1)e^{m\mathbf{k}}, z)\right]$$

$$= \text{Res}_z\left[\boldsymbol{\xi}(z)\mathscr{V}(e^{m\mathbf{k}}, z)\right], \tag{3.3}$$

where normal ordering in the last line is unnecessary due to $\boldsymbol{\xi}\cdot\mathbf{k} = 0$. According to (2.62) the above commutator then translates into

$$
\begin{aligned}
\left[A_m^{\xi}, A_n^{\eta}\right] &= m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}(\mathbf{k}(-1))_0 \\
&= m(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\delta_{m+n,0}\mathbf{k}(0).
\end{aligned}
\tag{3.4}
$$

We observe that apart from the operator $\mathbf{k}(0) = \mathbf{k}\cdot\boldsymbol{\alpha}_0$, this is just an oscillator commutation relation like (2.73) but now for $d-2$ oscillators since the space $\{\boldsymbol{\xi} \in \Lambda_{\mathbb{R}} \mid \boldsymbol{\xi}\cdot\mathbf{k} = 0, \ \boldsymbol{\xi} \equiv \boldsymbol{\xi} \bmod \mathbb{R}\mathbf{k}\}$ has indeed dimension $d-2$. Moreover, it is clear from (2.57) that these operators commute with the Virasoro algebra,

$$
\left[\mathbf{L}_{(n)}, A_m^{\xi}\right] = 0 \qquad \forall n, m \in \mathbb{Z}.
\tag{3.5}
$$

Since we shall encounter the DDF operators only when acting on physical states with certain momentum $\mathbf{r}$, say, the operator $\mathbf{k}(0)$ can be thought of as an integer $\mathbf{k}\cdot\mathbf{r}$. The crucial feature of the DDF construction is then that for given momentum $\mathbf{r}$, one has to find a lightlike vector $\mathbf{k} = \mathbf{k}(\mathbf{r})$ such that $\mathbf{k}\cdot\mathbf{r} = 1$. In this case the transversal DDF operators $A_m^{\xi}(\mathbf{k})$ realize precisely the algebra of $d-2$ transversal oscillators on the ground state $e^{\mathbf{r}}$. Indeed, we learn from (2.119) that the DDF operators $A_m^{\xi}(\mathbf{k})$ for positive m annihilate the tachyonic ground state $e^{\mathbf{r}}$,

$$
A_m^{\xi}(\mathbf{k})|\mathbf{r}\rangle = 0 \qquad \forall m > 0,
\tag{3.6}
$$

the operator $A_0^{\xi}(\mathbf{k}) = \boldsymbol{\xi}(0)$ acts diagonally with eigenvalue $\boldsymbol{\xi}\cdot\mathbf{r}$, while the operators $A_m^{\xi}(\mathbf{k})$ for negative m when applied to the ground state generate new physical states called **transversal DDF states**,

$$
A_{-m_1}^{\xi_1} \ldots A_{-m_N}^{\xi_N}|\mathbf{r}\rangle \equiv \left[\boldsymbol{\xi}_1(-1)e^{-m_1\mathbf{k}}, \left[\ldots, \left[\boldsymbol{\xi}_N(-1)e^{-m_N\mathbf{k}}, e^{\mathbf{r}}\right]\ldots\right]\right],
\tag{3.7}
$$

where we wrote $e^{\mathbf{r}} \equiv |\mathbf{r}\rangle$ to make contact with the standard physics notation. For later purposes we denote the $d-2$-fold Heisenberg algebra spanned by the transversal DDF operators by

$$
\hat{\mathbf{t}} := \{A_m^{\xi} \mid \boldsymbol{\xi} \in \Lambda_{\mathbb{R}}, \ \boldsymbol{\xi}\cdot\mathbf{k} = \boldsymbol{\xi}\cdot\mathbf{r} = 0, \ m \in \mathbb{Z}\},
\tag{3.8}
$$

and the vector space of finite products of creation operators ($\equiv$ algebra of polynomials on the transversal oscillators) is written as

$$
S(\hat{\mathbf{t}}^-) := \bigoplus_{N\in\mathbb{N}} \left\{ \prod_{i=1}^{N} A_{-m_i}^{\xi_i} \mid \boldsymbol{\xi}_i \in \Lambda_{\mathbb{R}}, \ \boldsymbol{\xi}_i\cdot\mathbf{k} = \boldsymbol{\xi}_i\cdot\mathbf{r} = 0, \ m_i > 0 \ \forall i \right\},
\tag{3.9}
$$

where "S" stands for "symmetric" because of the fact that the creation operators commute with each other.

The above identification of DDF physical states with multiple commutators in the Lie algebra $\mathfrak{g}_A$ will be our main guide in the analysis of hyperbolic Lie algebras; for the DDF construction allows us to write down elements of the Kac Moody algebra $\mathfrak{g}(A)$ *explicitly* and to introduce the notion of *polarization* into the framework of these algebras.

Recall that the photonic physical states in (2.114) deserve the attribute "transversal" in the sense that the polarization vector $\mathbf{s}$ in $\mathbf{s}(-1)e^{\mathbf{r}}$ has to be orthogonal to the momentum vector $\mathbf{r}$. Thus, we cannot expect to obtain "longitudinal" physical states

in a straightforward way. Nevertheless, there is a "dirty trick" [8]. Let $\mathbf{r} \in \Lambda$, $\mathbf{k} \in \Lambda_0$ and suppose that $\mathbf{k} \cdot \mathbf{r} \neq 0$. Then Eq. (2.41) yields

$$\left[\mathrm{L}_{(n)}, \mathscr{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}}, z) \right] = z^n \left\{ z \frac{d}{dz} + n + 1 \right\} \mathscr{V}(\mathbf{r}(-1)\mathrm{e}^{\mathbf{k}}, z)$$
$$+ \tfrac{1}{2} n(n+1)(\mathbf{k} \cdot \mathbf{r}) \mathscr{V}(\mathrm{e}^{\mathbf{k}}, z) z^{n-1}. \qquad (3.10)$$

The unwanted term on the right-hand side which destroys the conformal transformation properties (2.59) can be removed by the following trick: introduce the formal series

$$\mathbf{k}^{\times}(z) := z \frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} - 1 = \frac{1}{\mathbf{r} \cdot \mathbf{k}} \sum_{n \neq 0} \mathbf{k}(n) z^{-n} + \left(\frac{\mathbf{k}(0)}{\mathbf{r} \cdot \mathbf{k}} - 1 \right), \qquad (3.11)$$

and define

$$\log \left(z \frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} \right) = \log[1 + \mathbf{k}^{\times}(z)] := \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} (\mathbf{k}^{\times}(z))^i, \qquad (3.12)$$

which is only defined on states with momentum $\mathbf{s}$ *such that* $\mathbf{s} \cdot \mathbf{k} = \mathbf{r} \cdot \mathbf{k}$: if the second term on the right-hand side of (3.11) does not vanish on a given state, an infinite number of terms will contribute when (3.12) is applied to it. This means that the above series is not (algebraically) summable on the whole space $\mathscr{F}$. In particular, it is not summable on the vacuum state $\mathbf{1} \equiv |\mathbf{0}\rangle$ which, in view of (2.53), makes it impossible to recover *the* state corresponding to the log series: there does not exist a universal state whose vertex operator is given by $\log(1 + \mathbf{k}^{\times}(z))$. Luckily, however, we shall only need the action of this log series on states with momentum $\mathbf{r} - n\mathbf{k}$ (with $n \in \mathbb{N}$), so that the resulting series will be well-defined. And if this is the case then we may indeed find a state whose vertex operator has the same action as the log series. Thus, the log series should be interpreted as some sort of generating series for a class of genuine vertex operators which can be revealed by acting on states. Keeping in mind this subtlety let us perform some calculations in connection with the log series.

$$\left[\mathrm{L}_{(n)}, \mathbf{k}^{\times}(z) \right] = z^n \left\{ z \frac{d}{dz} + n \right\} \mathbf{k}^{\times}(z) + n z^n, \qquad (3.13)$$

since the current $\mathbf{k}(z)$ is a primary field of weight 1. For the formal series $\log[1 + \mathbf{k}^{\times}(z)]$ we therefore obtain

$$\left[\mathrm{L}_{(n)}, \log[1 + \mathbf{k}^{\times}(z)] \right] = \sum_{i \geq 1} (-1)^{i+1} (\mathbf{k}^{\times}(z))^{i-1} \left[\mathrm{L}_{(n)}, \mathbf{k}^{\times}(z) \right] \qquad \text{since } \mathbf{k} \in \Lambda_0$$

$$= z^{n+1} \frac{d}{dz} \log[1 + \mathbf{k}^{\times}(z)] + n z^n, \qquad (3.14)$$

so that

$$\left[\mathrm{L}_{(n)}, \frac{d}{dz} \log[1 + \mathbf{k}^{\times}(z)] \right] = z^n \left\{ z \frac{d}{dz} + n + 1 \right\} \frac{d}{dz} \log[1 + \mathbf{k}^{\times}(z)] + n^2 z^{n-1}. \quad (3.15)$$

Using this formula and the fact that

$$\frac{d}{dz} \log \frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} = \frac{d}{dz} \log[1 + \mathbf{k}^{\times}(z)] - z^{-1}, \tag{3.16}$$

we deduce that

$$\left[L_{(n)}, \frac{d}{dz} \log \frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} \right] = z^n \left\{ z\frac{d}{dz} + n + 1 \right\} \left(\frac{d}{dz} \log \frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} \right) + n(n+1)z^{n-1}. \tag{3.17}$$

Putting everything together we conclude that the **DDF vertex operators**

$$\mathscr{Y}_{\mathbf{k}}(\mathbf{r}, z) := \mathscr{V}(\mathbf{r}(-1)e^{\mathbf{k}}, z) - \tfrac{1}{2}(\mathbf{r} \cdot \mathbf{k})\frac{d}{dz} \log \left(\frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} \right) \mathscr{V}(e^{\mathbf{k}}, z), \tag{3.18}$$

enjoy the correct conformal transformation properties for primary fields of weight 1:[5]

$$\left[L_{(n)}, \mathscr{Y}_{\mathbf{k}}(\mathbf{r}, z) \right] = z^n \left\{ z\frac{d}{dz} + n + 1 \right\} \mathscr{Y}_{\mathbf{k}}(\mathbf{r}, z) \qquad \forall n \in \mathbb{Z}. \tag{3.19}$$

For $\mathbf{r}\mathbf{k} \neq 0$ we call $\mathscr{Y}_{\mathbf{k}}(\mathbf{r}, z)$ **longitudinal vertex operator** since otherwise we recover the transversal vertex operator $\mathscr{V}(\mathbf{r}(-1)e^{\mathbf{k}}, z)$.[6] Also note that the log term in (3.18) does not require normal ordering because of $\mathbf{k} \in \Lambda_0$.

3.2. Longitudinal Virasoro operators.

We define the **longitudinal Virasoro operator** $\mathfrak{L}_m$ as the zero mode operator of the longitudinal vertex operator $\mathscr{Y}_{m\mathbf{k}}(\mathbf{r}, z)$,

$$
\begin{aligned}
\mathfrak{L}_m &:= -\text{Res}_z \left[\mathscr{Y}_{m\mathbf{k}}(\mathbf{r}, z) \right] \\
&\equiv \text{Res}_z \left[-\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z) + \frac{m}{2}(\mathbf{r} \cdot \mathbf{k})\frac{d}{dz} \log \left(\frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} \right) \mathscr{V}(e^{m\mathbf{k}}, z) \right]
\end{aligned}
\tag{3.20}
$$

These operators satisfy the commutation relations of a Virasoro algebra with central charge $c = 24$. To see this, we first note that

$$
\begin{aligned}
&\left[\mathbf{r}(-1)e^{m\mathbf{k}}, \mathbf{r}(-1)e^{n\mathbf{k}} \right] \\
&= \epsilon(m\mathbf{k}, n\mathbf{k}) \left[m[\mathbf{r}^2 - mn(\mathbf{r} \cdot \mathbf{k})]\mathbf{k}(-1) + (n-m)(\mathbf{r} \cdot \mathbf{k})\mathbf{r}(-1) \right] e^{(m+n)\mathbf{k}} \\
&= (n-m)(\mathbf{r} \cdot \mathbf{k})\mathbf{r}(-1)e^{(m+n)\mathbf{k}} + m(\mathbf{r}^2 + m^2(\mathbf{r} \cdot \mathbf{k}))\delta_{m+n,0}\mathbf{k}(-1) \tag{3.21}
\end{aligned}
$$

by (2.120) so that

---

[5] This is in perfect agreement with [8] since we employ a different normal ordering prescription for p^μ and q^ν; we use $:q^\nu p^\mu: = q^\nu p^\mu$ in contrast to the "standard" symmetric normal ordering $_\times^\times q^\nu p^\mu _\times^\times = \tfrac{1}{2}(q^\nu p^\mu + p^\mu q^\nu) = :q^\nu p^\mu: - \tfrac{i}{2}\eta^{\mu\nu}$ which leads to

$$\mathscr{V}_{\text{symm.}}(\mathbf{r}(-1)e^{\mathbf{k}}, z) = \mathscr{V}(\mathbf{r}(-1)e^{\mathbf{k}}, z) + \tfrac{1}{2}(\mathbf{k} \cdot \mathbf{r})\mathscr{V}(e^{\mathbf{k}}, z)z^{-1},$$

so that indeed

$$
\begin{aligned}
&\mathscr{V}_{\text{symm.}}(\mathbf{r}(-1)e^{\mathbf{k}}, z) - \tfrac{1}{2}(\mathbf{k} \cdot \mathbf{r})\frac{d}{dz} \log[1 + \mathbf{k}^{\times}(z)]\mathscr{V}(e^{\mathbf{k}}, z) \\
&\qquad = \mathscr{V}(\mathbf{r}(-1)e^{\mathbf{k}}, z) - \tfrac{1}{2}(\mathbf{k} \cdot \mathbf{r})\frac{d}{dz} \log \left(\frac{\mathbf{k}(z)}{\mathbf{r} \cdot \mathbf{k}} \right) \mathscr{V}(e^{\mathbf{k}}, z).
\end{aligned}
$$

[6] Apparently, the essential log term was missed in [14].

$$\Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \mathrm{Res}_{z_2}\big[\mathscr{V}(\mathbf{r}(-1)e^{n\mathbf{k}}, z_2)\big]\Big]$$

$$\begin{aligned}
= \quad & (m-n)(\mathbf{r}\cdot\mathbf{k})\mathrm{Res}_z\big[-\mathscr{V}(\mathbf{r}(-1)e^{(m+n)\mathbf{k}}, z)\big] \\
& + m[\mathbf{r}^2 + m^2(\mathbf{r}\cdot\mathbf{k})]\delta_{m+n,0}\mathbf{k}(0).
\end{aligned} \tag{3.22}$$

It is also clear that the commutator of two log terms vanishes due to lightlikeness of $\mathbf{k}$. Finally, we have to calculate the cross commutator:

$$\Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \mathrm{Res}_{z_2}\Big[\frac{d}{dz_2}\log\Big(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\Big)\mathscr{V}(e^{n\mathbf{k}}, z_2)\Big]\Big]$$

$$\begin{aligned}
= \quad & \mathrm{Res}_{z_2}\Big\{\frac{d}{dz_2}\log\Big(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\Big)\Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \mathscr{V}(e^{n\mathbf{k}}, z_2)\Big] + \\
& + \Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \frac{d}{dz_2}\log\Big(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\Big)\Big]\mathscr{V}(e^{n\mathbf{k}}, z_2)\Big\}
\end{aligned} \tag{3.23}$$

To calculate these two commutators we first recall the following version of the commutator formula (2.38):

$$\begin{aligned}
\big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\psi, z_1)\big], \mathscr{V}(\varphi, z_2)\big] \quad &= \quad \mathscr{V}(\psi_0\varphi, z_2) \\
&\equiv \quad \mathscr{V}([\psi, \varphi], z_2).
\end{aligned} \tag{3.24}$$

From (2.119) and (2.120) we therefore deduce that

$$\Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \mathscr{V}(e^{n\mathbf{k}}, z_2)\Big] = n(\mathbf{r}\cdot\mathbf{k})\mathscr{V}(e^{(m+n)\mathbf{k}}, z_2), \tag{3.25}$$

and

$$\Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \mathscr{V}(\mathbf{k}(-1), z_2)\Big] = m(\mathbf{r}\cdot\mathbf{k})\mathscr{V}(\mathbf{k}(-1)e^{m\mathbf{k}}, z_2), \tag{3.26}$$

respectively. The last formula then yields

$$\Big[\mathrm{Res}_{z_1}\big[\mathscr{V}(\mathbf{r}(-1)e^{m\mathbf{k}}, z_1)\big], \frac{d}{dz_2}\log\Big(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}}\Big)\Big]$$

$$\begin{aligned}
= \quad & \frac{d}{dz_2}\Big[\sum_{i\geq 1}(-1)^{i+1}\Big(\frac{\mathbf{k}(z_2)}{\mathbf{r}\cdot\mathbf{k}} - 1\Big)^{i-1} m(\mathbf{r}\cdot\mathbf{k})\mathscr{V}(\mathbf{k}(-1)e^{m\mathbf{k}}, z_2)\Big] \\
= \quad & \frac{d}{dz_2}\big[m(\mathbf{r}\cdot\mathbf{k})\mathscr{V}(e^{m\mathbf{k}}, z_2)\big] \\
= \quad & m^2(\mathbf{r}\cdot\mathbf{k})\mathscr{V}(\mathbf{k}(-1)e^{m\mathbf{k}}, z_2).
\end{aligned} \tag{3.27}$$

Collecting the above commutators we finally get

$$\begin{aligned}
\big[\mathfrak{L}_m, \mathfrak{L}_n\big] \quad = \quad & (m-n)(\mathbf{r}\cdot\mathbf{k})\mathrm{Res}_z\big[-\mathscr{V}(\mathbf{r}(-1)e^{(m+n)\mathbf{k}}, z)\big] \\
& + m(\mathbf{r}^2 + m^2(\mathbf{r}\cdot\mathbf{k}))\delta_{m+n,0}\mathbf{k}(0) \\
& - \frac{n^2}{2}(\mathbf{r}\cdot\mathbf{k})^2\mathrm{Res}_z\Big[\frac{d}{dz}\log\Big(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}}\Big)\mathscr{V}(e^{(m+n)\mathbf{k}}, z)\Big] \\
& - \frac{nm^2}{2}(\mathbf{r}\cdot\mathbf{k})^2\mathrm{Res}_z\big[\mathscr{V}(\mathbf{k}(-1)e^{(m+n)\mathbf{k}}, z)\big]
\end{aligned}$$

$$+ \frac{m^2}{2}(\mathbf{r}\cdot\mathbf{k})^2 \mathrm{Res}_z \left[\frac{d}{dz} \log \left(\frac{\mathbf{k}(z)}{\mathbf{r}\cdot\mathbf{k}} \right) \mathscr{V}(\mathrm{e}^{(m+n)\mathbf{k}}, z) \right]$$

$$+ \frac{mn^2}{2}(\mathbf{r}\cdot\mathbf{k})^2 \mathrm{Res}_z \left[\mathscr{V}(\mathbf{k}(-1)\mathrm{e}^{(m+n)\mathbf{k}}, z) \right]$$

$$= (m-n)\mathfrak{L}_{m+n} + [(\mathbf{r}\cdot\mathbf{k})^2 + (\mathbf{r}\cdot\mathbf{k})]m^3 + \mathbf{r}^2 m \delta_{m+n,0}\mathbf{k}(0). \quad (3.28)$$

As for the central term, we shall assume from now on that $\mathbf{r}\cdot\mathbf{k} = 1$ so that the factor in the central term reads $2m^3 + \mathbf{r}^2 m$. The standard form $\frac{c}{12}(m^3 - m)$ can be obtained by redefining $\mathfrak{L}_0 \to \mathfrak{L}_0 + (1 + \frac{1}{2}\mathbf{r}^2)\mathbf{k}(0)$ so that we end up with

$$[\mathfrak{L}_m, \mathfrak{L}_n] = (m-n)\mathfrak{L}_{m+n} + 2(m^3 - m)\delta_{m+n,0}\mathbf{k}(0). \quad (3.29)$$

We conclude that the longitudinal Virasoro operators $\mathfrak{L}_m$, when applied to physical states with momentum $\mathbf{r}$, realize a Virasoro algebra, $\mathrm{Vir}_{\mathfrak{L}}$, with central charge $c_{\mathfrak{L}} = 24$. Remarkably, this Virasoro algebra is universal in the sense that its central charge does *not* depend on the dimension of the lattice.

Let us proceed by determining the commutator of the transversal DDF operators and the longitudinal Virasoro operators,

$$
\begin{aligned}
[\mathfrak{L}_m, A_n^\xi] &= \left[\mathrm{Res}_{z_1}\left[\mathscr{V}(\boldsymbol{\xi}(-1)\mathrm{e}^{n\mathbf{k}}, z_1) \right], \mathrm{Res}_{z_2}\left[\mathscr{V}(\mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}}, z_2) \right] \right] \\
&= \mathrm{Res}_z \left[\mathscr{V}([\boldsymbol{\xi}(-1)\mathrm{e}^{n\mathbf{k}}, \mathbf{r}(-1)\mathrm{e}^{m\mathbf{k}}], z) \right] \\
&= \mathrm{Res}_z \left[n(\boldsymbol{\xi}\cdot\mathbf{r})\mathscr{V}(\mathbf{k}(-1)\mathrm{e}^{(m+n)\mathbf{k}}, z) - n(\mathbf{r}\cdot\mathbf{k})\mathscr{V}(\boldsymbol{\xi}(-1)\mathrm{e}^{(m+n)\mathbf{k}}, z) \right] \\
&= -n(\mathbf{r}\cdot\mathbf{k})A_{n+m}^\xi + n(\mathbf{r}\cdot\boldsymbol{\xi})\delta_{m+n,0}\mathbf{k}(0) \quad\quad (3.30)
\end{aligned}
$$

by (2.120). Obviously, we can remove the second term by choosing $\boldsymbol{\xi}$ orthogonal to $\mathbf{r}$; and if we make our standard assumption that $\mathbf{r}\cdot\mathbf{k} = 1$ we arrive at the important formula

$$[\mathfrak{L}_m, A_n^\xi] = -n A_{n+m}^\xi. \quad (3.31)$$

We claim that the tachyonic ground state $\mathrm{e}^\mathbf{r}$ is annihilated by the longitudinal Virasoro operators $\mathfrak{L}_m$ for nonnegative m,

$$\mathfrak{L}_m|\mathbf{r}\rangle = 0 \qquad \forall m \geq 0. \quad (3.32)$$

First note that the operator $\mathfrak{L}_0 = -\mathbf{r}(0) + (1 + \frac{1}{2}\mathbf{r}^2)\mathbf{k}(0)$ acts diagonally with eigenvalue $(1 - \frac{1}{2}\mathbf{r}^2)$ which indeed vanishes because $\mathbf{r} \in \Lambda_2$. Next, using the Λ-gradation (2.134) of $\mathfrak{g}_\Lambda$ we observe that the state $\mathfrak{L}_m|\mathbf{r}\rangle$ carries momentum $\mathbf{r}+m\mathbf{k}$. But $\frac{1}{2}(\mathbf{r}+m\mathbf{k})^2 = 1+m$ contradicts the physical state condition $\mathrm{L}_{(0)}\psi = \psi$ for positive m in view of (2.112) unless the state itself vanishes. We conclude that only the operators $\mathfrak{L}_m$ for negative m generate new physical states when applied to the ground state

$$\mathfrak{L}_{-n_1}\ldots\mathfrak{L}_{-n_P}|\mathbf{r}\rangle \in \mathscr{P}_{(1)} \quad (3.33)$$

for $n_1, \ldots, n_P \geq 1$. Further, we can verify that the state $\mathfrak{L}_{-1}|\mathbf{r}\rangle$ is a null physical state, i.e. the action of the operator $\mathfrak{L}_{-1}$ is essentially the same as the action of $\mathrm{L}_{(-1)}$:

$$\mathfrak{L}_{-1}|\mathbf{r}\rangle = \epsilon(\mathbf{r}, \mathbf{k})\mathrm{L}_{(-1)}|\mathbf{r} - \mathbf{k}\rangle, \quad (3.34)$$

which vanishes as an element of $\mathfrak{g}_\Lambda$! To prove this equation we first deduce from (2.119) that

$$
\begin{aligned}
\mathrm{Res}_z \left[-\mathscr{V}(\mathbf{r}(-1)e^{-\mathbf{k}}, z) \right] (e^{\mathbf{r}}) &= -\left[\mathbf{r}(-1)e^{-\mathbf{k}}, e^{\mathbf{r}} \right] \\
&= \epsilon(\mathbf{r}, \mathbf{k})(\mathbf{r} - 2\mathbf{k})(-1) \otimes e^{\mathbf{r}-\mathbf{k}}. \quad (3.35)
\end{aligned}
$$

The calculations for the log term have to be performed explicitly:

$$
\begin{aligned}
\mathrm{Res}_z & \left[-\tfrac{1}{2} \frac{d}{dz} \log \mathbf{k}(z) \mathscr{V}(e^{-\mathbf{k}}, z) \right] (e^{\mathbf{r}}) \\
&= \mathrm{Res}_z \left[-\tfrac{1}{2} \frac{d}{dz} \log \mathbf{k}(z) \mathscr{V}(e^{-\mathbf{k}}, z) \right] (e^{\mathbf{r}}) \\
&= -\tfrac{1}{2} \mathrm{Res}_z \left[\left[\frac{d}{dz} \log[1 + \mathbf{k}^\times(z)] - z^{-1} \right] \mathscr{V}(e^{-\mathbf{k}}, z) \right] (e^{\mathbf{r}}) \\
&= -\tfrac{1}{2} \mathrm{Res}_z \left[e^{-\mathbf{k}} \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} (\mathbf{k}^\times(z))^i \mathbf{k}(z) \sum_{m \geq 0} S_m(-\mathbf{k}) z^{m-1} \right] (e^{\mathbf{r}}) \\
&\quad + \tfrac{1}{2} \mathrm{Res}_z \left[z^{-1} \sum_{m \geq 0} S_m(-\mathbf{k}) z^{m-1} e^{-\mathbf{k}} e^{\mathbf{r}} \right] \\
&= -\tfrac{1}{2} \mathrm{Res}_z \left[\sum_{i \geq 1} \frac{(-1)^{i+1}}{i} \left(\sum_{n > 0} \mathbf{k}(-n) z^n \right)^i \left(z^{-1} + \sum_{n > 0} \mathbf{k}(-n) z^{n-1} \right) \times \right. \\
&\quad \left. \times \sum_{m \geq 0} S_m(-\mathbf{k}) z^{m-1} e^{-\mathbf{k}} e^{\mathbf{r}} \right] - \tfrac{1}{2} \epsilon(-\mathbf{k}, \mathbf{r}) \mathbf{k}(-1) e^{\mathbf{r}-\mathbf{k}} \\
&= -\epsilon(-\mathbf{k}, \mathbf{r}) \mathbf{k}(-1) e^{\mathbf{r}-\mathbf{k}}. \quad (3.36)
\end{aligned}
$$

Putting together the above two results we obtain $\epsilon(\mathbf{r}, \mathbf{k})(\mathbf{r} - \mathbf{k})(-1)|\mathbf{r} - \mathbf{k}\rangle$ as desired. Thus, using the commutation relations and (3.34), we can rewrite any state of the form (3.33) as a linear combination of states not containing $\mathfrak{L}_{-1}$. As a basis for states of the form (3.33) in $\mathfrak{g}_\Lambda$ we may therefore choose

$$
\mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_P} |\mathbf{r}\rangle, \quad (3.37)
$$

with fixed ordering $n_1 \geq \ldots \geq n_P \geq 2$.

We turn now to the no-ghost theorem applied to our discrete construction. We fix a tachyonic groundstate $e^{\mathbf{r}} \equiv |\mathbf{r}\rangle$, $\mathbf{r} \in \Lambda_2$, and suppose that there exists a lightlike vector $\mathbf{k} = \mathbf{k}(\mathbf{r}) \in \Lambda_0$ such that $\mathbf{r} \cdot \mathbf{k} = 1$. Then we can always find $d - 2$ orthonormal vectors $\boldsymbol{\xi}_i \in \Lambda_{\mathbb{R}}$, $1 \leq i \leq d - 2$, orthogonal to both $\mathbf{r}$ and $\mathbf{k}$. If we put $A^i_m \equiv A^{\boldsymbol{\xi}_i}_m$ then the no-ghost theorem [26] tells us that the states

$$
A^{i_1}_{-m_1} \ldots A^{i_N}_{-m_N} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_P} |\mathbf{r}\rangle \quad (3.38)
$$

for $i_1, \ldots, i_N \in \{1, \ldots, d-2\}$, $m_1, \ldots, m_N \geq 1$ and $n_1 \geq \ldots \geq n_P \geq 1$, account for *all* physical states (including null physical states!) with momentum

$$
\mathbf{r} - \left(\sum_{a=1}^N m_a + \sum_{b=1}^P n_b \right) \mathbf{k}. \quad (3.39)
$$

Reformulated in the language of the Lie algebra $\mathfrak{g}_\Lambda$, the subspace

$$\mathfrak{g}_\Lambda(\mathbf{r}) := \bigoplus_{n \in \mathbb{N}} \mathfrak{g}_\Lambda^{(\mathbf{r} - n\mathbf{k})}, \qquad \mathbf{r} \in \Lambda_2, \tag{3.40}$$

is spanned by elements of the form

$$A^{i_1}_{-m_1} \dots A^{i_N}_{-m_N} \mathfrak{L}_{-n_1} \dots \mathfrak{L}_{-n_P} |\mathbf{r}\rangle, \tag{3.41}$$

where $i_1, \dots, i_N \in \{1, \dots, d-2\}$, $m_1, \dots, m_N \geq 1$ and $n_1 \geq \dots \geq n_P \geq 2$.

3.3. Spectrum generating algebra. Note that due to (3.31) we had to fix some ordering of the operators in (3.38). Historically, this was the reason for replacing the longitudinal Virasoro operators $\mathfrak{L}_m$ by **longitudinal DDF operators** A^-_m which commute with the transversal DDF operators. To see this, we introduce the standard normal ordering of the transversal DDF operators by placing DDF operators with positive modes to the left of the ones with negative modes, and we define

$$\mathscr{L}_n := \frac{1}{2} \sum_{i=1}^{d-2} \sum_{m \in \mathbb{Z}} :A^i_m A^i_{n-m}:. \tag{3.42}$$

Comparing this with (2.107) we immediately infer that the $\mathscr{L}_n$'s obey a Virasoro algebra, $\mathrm{Vir}_\mathscr{L}$, with central charge $c_\mathscr{L} = d - 2$. Furthermore, it is straightforward to show that

$$\left[\mathscr{L}_m, A^i_n\right] = -n A^i_{n+m}. \tag{3.43}$$

Hence, if we define

$$A^-_n := \mathfrak{L}_n - \mathscr{L}_n = \mathfrak{L}_n - \frac{1}{2} \sum_{i=1}^{d-2} \sum_{m \in \mathbb{Z}} :A^i_m A^i_{n-m}:, \tag{3.44}$$

we get

$$\left[A^i_m, A^-_n\right] = 0 \qquad \forall m, n \in \mathbb{Z}, \ 1 \leq i \leq d - 2. \tag{3.45}$$

The last equation can be used to show that longitudinal DDF operators form a "coset" Virasoro algebra, Vir_{A^-}, with central charge $c_{A^-} = c_\mathfrak{L} - c_\mathscr{L} = 26 - d$:

$$\left[A^-_m, A^-_n\right] = (m-n) A^-_{m+n} + \frac{26-d}{12}(m^3 - m)\delta_{m+n,0}. \tag{3.46}$$

Thus we may rewrite the basis of all physical states (including null states!) as

$$A^{i_1}_{-m_1} \dots A^{i_N}_{-m_N} A^-_{-n_1} \dots A^-_{-n_P} |\mathbf{r}\rangle \tag{3.47}$$

where $i_1, \dots, i_N \in \{1, \dots, d-2\}$, $m_1, \dots, m_N \geq 1$ and $n_1 \geq \dots \geq n_P \geq 1$, which exhibits explicitly how the space of physical states with momentum $\mathbf{r} - n\mathbf{k}$, $n \geq 0$, splits into a tensor product of the algebra of polynomials in the transversal oscillators with a Virasoro Verma module:

$$\mathscr{P}_{(1)}(\mathbf{r}) := \bigoplus_{n \geq 0} \mathscr{P}_{(1)}^{(\mathbf{r} - n\mathbf{k})} = S(\hat{\mathbf{t}}^-) \otimes V(26 - d, 0), \tag{3.48}$$

where $V(c,h)$ denotes the irreducible highest weight Vir-module. In other words, we may regard the associative algebra

$$S(\hat{\mathfrak{t}}^-) \otimes \text{Vir}_{A^-}, \tag{3.49}$$

as the **spectrum generating algebra** associated with $\mathbf{r}$, since it generates all physical states with momentum $\mathbf{r} - n\mathbf{k}$, $n \in \mathbb{N}$, by acting on the fixed tachyonic groundstate $|\mathbf{r}\rangle$. In particular, we observe how the critical dimension $d = 26$ arises: in 26 dimensions the longitudinal and the transversal modes decouple because the coset Virasoro module $V(26 - d, 0)$ becomes trivial. Moreover, (3.47) enables us to write down a formula for the dimension of the physical subspaces with momentum $\mathbf{r} - n\mathbf{k}$, $\mathbf{r} \in \Lambda_2$:

$$\dim \mathfrak{g}_A^{(\mathbf{r}-n\mathbf{k})} = p_{d-1}(n) - p_{d-1}(n - 1), \tag{3.50}$$

where $p_{d-1}(n)$ counts the partitions of $n \in \mathbb{N}$ into "parts" of $d - 1$ "colours", i.e.

$$
\begin{aligned}
\phi(q)^{1-d} \quad &:= \quad \prod_{l=1}^{\infty}(1 - q^l)^{1-d} \\
&= \quad \sum_{n\in\mathbb{N}} p_{d-1}(n)q^n \\
&= \quad 1 + (d - 1)q + \tfrac{1}{2}(d - 1)(d + 2)q^2 + \tfrac{1}{6}(d - 1)d(d + 7)q^3 + \dots, \quad (3.51)
\end{aligned}
$$

in terms of the generating Euler function $\phi(q)$. Hence

$$\sum_{n=0}^{\infty} \dim \mathfrak{g}_A^{(\mathbf{r}-n\mathbf{k})}q^n = 1 + (d - 2)q + \tfrac{1}{2}(d - 1)dq^2 + \tfrac{1}{6}(d - 1)(d^2 + 4d - 6)q^3 + \dots \ .$$

Note that the second term in (3.50) is due to the null physical states.

Since we will mainly focus on a deeper understanding of the Kac Moody algebra $\mathfrak{g}(A)$ the question arises how to make contact between the elegant DDF formulation of $\mathfrak{g}_A$ and the construction of $\mathfrak{g}(A)$ in terms of generators and relations. In other words, we have to face the problem how to separate the DDF states contained in $\mathfrak{g}(A)$ from those which cannot be generated by the set $\{e_i, f_i, h_i \,|\, i\}$. Note that a special case of (2.117) gives us a recipe for writing physical states $\boldsymbol{\xi}(-1)\mathrm{e}^{\mathbf{r}}$ as Lie algebra commutators:

$$
\begin{aligned}
\left[\mathrm{e}^{\mathbf{s}}, \mathrm{e}^{\mathbf{t}}\right] \quad &= \quad \epsilon(\mathbf{s}, \mathbf{t})\mathbf{s}(-1)\mathrm{e}^{\mathbf{s}+\mathbf{t}} \\
&= \quad \tfrac{1}{2}\epsilon(\mathbf{s}, \mathbf{t})(\mathbf{s} - \mathbf{t})(-1) \otimes \mathrm{e}^{\mathbf{s}+\mathbf{t}} \quad (3.52)
\end{aligned}
$$

for $\mathbf{s}, \mathbf{t} \in \Lambda_2$ such that $\mathbf{s} \cdot \mathbf{t} = -2$. The last equality is obtained by adding the null physical state ("total derivative") $\tfrac{1}{2}\epsilon(\mathbf{s}, \mathbf{t})\mathrm{L}_{(-1)}\mathrm{e}^{\mathbf{s}+\mathbf{t}}$. Hence we may put $\boldsymbol{\xi} = \mathbf{s} - \mathbf{t}$ and $\mathbf{r} = \mathbf{s} + \mathbf{t}$. This observation will be useful later.

We conclude with a comment that will be crucial for the discrete DDF construction of E_{10}. So far, we have tacitly assumed the DDF vectors $\mathbf{k}$ and $\mathbf{r}$ to be elements of the root lattice. However, inspection of the computations presented above shows that all arguments remain valid if only $\mathbf{k}^2 = 0, \mathbf{r}^2 = 2, \mathbf{r} \cdot \mathbf{k} = 1$ and $\boldsymbol{\xi} \cdot \mathbf{r} = \boldsymbol{\xi} \cdot \mathbf{k} = 0$. Thus, there is actually no need to assume the vectors $\mathbf{k}$ and $\mathbf{r}$ to be on the root lattice as long as these conditions are satisfied. In particular, under these circumstances we may choose $\mathbf{k}$ and $\mathbf{r}$ on the rational extension $\Lambda_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$, and the discrete DDF construction still works. All our formulas will continue to make sense, whereas the interpretation

of physical states and the identification of Lie algebra elements need some care. This subtlety arises because, rigorously speaking, we are dealing with a **generalized vertex algebra** associated with $\Lambda_\mathbb{Q}$, into which the original vertex algebra (associated with Λ) can be embedded. The generalized vertex operators are then defined as in (2.95) and (2.100), but are no longer elements of $(\text{End}\,\mathscr{F})[\![z, z^{-1}]\!]$; rather, the generalized vertex operator associated with a typical homogeneous element $\psi = \left[\prod_{j=1}^{N} \mathbf{s}_j(-n_j)\right] \otimes e^{\mathbf{r}}$ (where now $\mathbf{r} \in \Lambda_\mathbb{Q}$) is an element of $(\text{End}\,\mathscr{D}_{\mathbf{r}})[\![z, z^{-1}]\!]$ with

$$\mathscr{D}_{\mathbf{r}} := \bigoplus_{\substack{\mathbf{s} \in \Lambda_\mathbb{Q} \\ \mathbf{r} \cdot \mathbf{s} \in \mathbb{Z}}} \mathscr{F}^{(\mathbf{s})}. \tag{3.53}$$

This means that the modes of the generalized vertex operators are *not* well defined operators on the whole Fock space $\mathscr{F}$, but only on certain of its subspaces.

4. The Hyperbolic Algebra E_{10} and the DDF Construction

We now want to apply the concepts developed in the foregoing chapters to the study of Kac Moody algebras $\mathfrak{g}(A)$ whose Cartan matrix A is of hyperbolic type, choosing the hyperbolic Kac Moody algebra E_{10} as our example. We remind the reader that hyperbolic algebras are distinguished from the more general algebras based on arbitrary indefinite Cartan matrices by the additional requirement that the removal of any point from the Dynkin diagram leaves a Kac Moody algebra which is either of affine or finite type (for a review of hyperbolic root systems, see [36]). As shown in [29], the rank can then be 10 at most, and the root lattice must be Minkowskian, i.e. with metric signature $(+\ldots+|-)$. There are altogether three hyperbolic algebras of maximal rank. Of these, E_{10} is not only the most interesting, containing E_8 and its affine extension E_9 as subalgebras, but also distinguished by the fact that it has only *one* affine subalgebra that can be obtained by removing a point from the E_{10} Dynkin diagram, while the other two rank 10 algebras contain at least two regular affine subalgebras, (see e.g. [40]). Furthermore, the root lattice $Q(E_{10})$ coincides with the (unique) 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$ [10], whereas the root lattices of the other two maximal rank hyperbolic algebras are not self-dual.

Overall, our knowledge about Kac Moody algebras of hyperbolic type is rather limited. As already explained in Sect. 2.2, they are generally defined in terms of multiple commutators of the basic generators e_i, f_i, h_i and the multilinear Serre relations (2.131). In contradistinction to the finite and affine cases, a manageable representation of all the Lie algebra elements obtained in this way has not yet been found. In principle, the string vertex operator construction provides a more concrete realization with the additional advantage that the Serre relations (2.131) are built in from the outset (see the discussion at the end of Sect. 2.2), but the problem of characterizing the missing elements belonging to $\mathfrak{g}_\Lambda$ and not to $\mathfrak{g}(A)$ in (2.137) remains. We emphasize that we face essentially the same problem if instead we want to define a Borcherds-type algebra [5] based on $II_{9,1}$, because we then would have to supply the missing generators "by hand" by adding extra imaginary simple roots, which again presupposes knowledge of what the missing Lie algebra elements are (not to mention the potential arbitrariness as to the number of ways in which this can be consistently done).

As already mentioned, our analysis makes use of a discretized version of the DDF construction and relies in a crucial way on the identification of Lie algebra elements

with physical Fock space states. In the previous section we have seen that a central role is played by the tachyon momentum $\mathbf{a}$ of the ground state (so $\mathbf{a}^2 = 2$) and the null vector $\mathbf{k}$, subject to the condition $\mathbf{k}\cdot\mathbf{a} = 1$. For continuous momenta $\mathbf{a}$, we can always find suitable $\mathbf{k} = \mathbf{k}(\mathbf{a})$; moreover, we can rotate these vectors into a convenient frame by means of a Lorentz transformation [41]. On the lattice, however, the full Lorentz invariance is broken to a discrete subgroup (containing the Weyl group generated by the fundamental Weyl reflections), and for generic roots Λ, the associated DDF vectors $\mathbf{a}$ and $\mathbf{k}$ will *not* be elements of the root lattice $II_{9,1}$ in general[7]. Nevertheless, we employ these vectors in our analysis because we can still use the associated (transversal and longitudinal) DDF operators to construct a complete basis for any root space of the Lie algebra of physical states $\mathfrak{g}_{II_{9,1}}$. The corresponding root space of the Kac Moody algebra $\mathfrak{g}(A)$ is then a proper subspace thereof. As we will see, longitudinal states are absent only for level zero and level one; this accounts for the comparative simplicity of the corresponding multiplicity formulas.

Although it is possible in principle (with some effort) to extend our discussion to other hyperbolic Kac Moody algebras, the following points must be kept in mind. Our method may not apply to strictly hyperbolic algebras, which by definition have no affine, but only finite subalgebras, because their associated Weyl chambers contain no null vectors (i.e. they lie entirely within the light-cone), so the DDF operators cannot be defined. On the other hand, the Weyl chambers of arbitrary Kac Moody algebras of indefinite type generically contain several linearly independent null directions, a feature that will greatly complicate (if not vitiate) the application of our method, because one must then deal with at least two different sets of photon momenta for the DDF operators. Moreover, if the algebra contains more than one regular affine subalgebra, the level of a root is no longer uniquely defined; for indefinite algebras, which are not hyperbolic (such as the fake monster), it is not even clear whether this notion can be sensibly defined at all. We thus begin to understand the possible significance of the fact that the fundamental Weyl chamber of E_{10} touches the light-cone at precisely one edge. In view of the limitations of the method, we will make no attempt to state the results in the most general way.

In Sect. 4.1 we will summarize the pertinent results about E_{10}. In Sect. 4.2, we apply the discrete DDF construction to level-zero and level-one elements of $\mathfrak{g}(A)$, thereby recovering some known results. In Sect. 4.4, we turn to the level-two states, analyzing one example in complete detail.

4.1. Basic results about E_{10}. The hyperbolic Kac Moody algebra E_{10} is defined via its Coxeter-Dynkin diagram and the Serre relations following from it. As already mentioned, the root lattice $Q(E_{10})$ coincides with the unique 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$. The latter can be defined as the lattice of all points $\mathbf{x} = (x_1, \ldots, x_9 | x_0)$ for which the x_μ's are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\mathbf{l} = (\frac{1}{2}, \ldots, \frac{1}{2} | \frac{1}{2})$, all norms and inner products being evaluated in the Minkowskian metric $\mathbf{x}^2 = x_1^2 + \ldots + x_9^2 - x_0^2$ (cf. [42]). In more physical parlance, we are dealing with a subcritical open bosonic string moving in 10-dimensional space-time fully compactified on a torus (hence "finite in all directions" [37]), so that the momenta lie on $II_{9,1}$. According to [10], a set of positive norm simple roots for $II_{9,1}$ is given by the ten vectors $\mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_8$ in

---

[7] To make this explicit in the notation, we designate the tachyon momentum by $\mathbf{a}$ rather than $\mathbf{r}$ as in the previous sections.

$II_{9,1}$ for which $r_i^2 = 2$ and $\mathbf{r}_i \cdot \rho = -1$ where the **Weyl vector** is $\rho = (0, 1, 2, \ldots, 8 | 38)$ with $\rho^2 = -1240$. Explicitly,

$$
\begin{aligned}
\mathbf{r}_{-1} &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -1\, |\, 0), \\
\mathbf{r}_0 &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -1,\ 0\, |\, 0), \\
\mathbf{r}_1 &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1, -1,\ 0,\ 0\, |\, 0), \\
\mathbf{r}_2 &= (\ 0,\ 0,\ 0,\ 0,\ 1, -1,\ 0,\ 0,\ 0\, |\, 0), \\
\mathbf{r}_3 &= (\ 0,\ 0,\ 0,\ 1, -1,\ 0,\ 0,\ 0,\ 0\, |\, 0), \\
\mathbf{r}_4 &= (\ 0,\ 0,\ 1, -1,\ 0,\ 0,\ 0,\ 0,\ 0\, |\, 0), \\
\mathbf{r}_5 &= (\ 0,\ 1, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\, |\, 0), \\
\mathbf{r}_6 &= (-1, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\, |\, 0), \\
\mathbf{r}_7 &= (\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\, |\, \tfrac{1}{2}), \\
\mathbf{r}_8 &= (\ 1, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\, |\, 0).
\end{aligned}
$$

These simple roots indeed generate the reflection group of $II_{9,1}$. The corresponding Coxeter-Dynkin diagram looks as follows

$$(4.1)$$
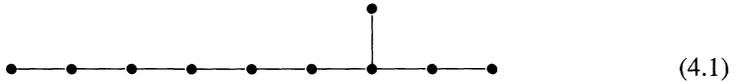
and is associated with the Cartan matrix:

$$
A \equiv (a_{ij}) = (\mathbf{r}_i \cdot \mathbf{r}_j) =
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
$$

whose inverse

$$
A^{-1} = -
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\
1 & 2 & 4 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\
2 & 4 & 6 & 9 & 12 & 15 & 18 & 12 & 6 & 9 \\
3 & 6 & 9 & 12 & 16 & 20 & 24 & 16 & 8 & 12 \\
4 & 8 & 12 & 16 & 20 & 25 & 30 & 20 & 10 & 15 \\
5 & 10 & 15 & 20 & 25 & 30 & 36 & 24 & 12 & 18 \\
6 & 12 & 18 & 24 & 30 & 36 & 42 & 28 & 14 & 21 \\
4 & 8 & 12 & 16 & 20 & 24 & 28 & 18 & 9 & 14 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 & 7 \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 14 & 7 & 10
\end{pmatrix}
$$

we shall need below. The E_9 **null root** is

$$\delta = \sum_{i=0}^{8} n_i \mathbf{r}_i = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\,|\,1), \qquad (4.2)$$

where the coefficients n_i (called **marks** of E_9) can be read off from

$$\begin{bmatrix} & & & & & & 3 & & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{bmatrix}. \qquad (4.3)$$

The fundamental Weyl chamber $\mathscr{C}$ of E_{10} is the convex cone generated by the **fundamental weights** Λ_i [8],

$$\Lambda_i = -\sum_{j=-1}^{8} (A^{-1})_{ij} \mathbf{r}_j \quad \text{for } i = -1, 0, 1, \ldots 8 \qquad (4.4)$$

with the inverse Cartan matrix from above. Thus,

$$\Lambda \in \mathscr{C} \iff \Lambda = \sum_{i=-1}^{8} k_i \Lambda_i \qquad (4.5)$$

for $k_i \in \mathbb{Z}_+$. A special feature of E_{10} is that we need not distinguish between root and weight lattice, since these are the same for self-dual root lattices[9]. Since Weyl transformations preserve multiplicities and since every root can be brought into $\mathscr{C}$ by means of a Weyl transformation, the structure of the algebra is completely understood once the root spaces for roots belonging to $\mathscr{C}$ are under control. Note also that the null root plays a special role: the first fundamental weight is just $\Lambda_{-1} = \delta$, and all null-vectors in $\mathscr{C}$ must be multiples of Λ_{-1} since $\Lambda_i^2 < 0$ for all other fundamental weights.

As described in Sect. 2.2, the algebra $\mathfrak{g}(A) = E_{10}$ consists of all multiple commutators of the Chevalley-Serre generators e_i, f_i, h_i with $i = -1, 0, 1, \ldots, 8$. It is a standard result [29] that this algebra can be written as a direct sum

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \qquad (4.6)$$

where the subalgebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are defined to consist of all linear combinations of multiple commutators of the form $[f_{i_1}, [f_{i_2}, \ldots [f_{i_{n-1}}, f_{i_n}] \ldots]]$ and $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]]$, respectively, modulo the multilinear Serre relations (2.131). Since $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are conjugate and thus enjoy analogous properties, it is enough in practice to consider only multiple commutators made out of e_i generators (corresponding to positive roots). To classify such commutators one introduces the **level** $\ell \in \mathbb{Z}$ of a root, such that positive ℓ counts the number of e_{-1} generators in $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]]$ (similarly, if ℓ is negative, $-\ell$ counts the number of f_{-1} generators in $[f_{i_1}, [f_{i_2}, \ldots [f_{i_{n-1}}, f_{i_n}] \ldots]]$). In terms of the corresponding root $\Lambda = \mathbf{r}_{i_1} + \ldots + \mathbf{r}_{i_n}$, ℓ is defined by

$$\ell := -\Lambda \cdot \delta. \qquad (4.7)$$

---

[8] Notice that our convention is opposite to the one adopted in [30]. The fundamental weights here are *positive* and satisfy $\Lambda_i \cdot \mathbf{r}_j = -\delta_{ij}$ Thus, we will be dealing with "lowest weight" rather than "highest weight" representations in accordance with physics usage.

[9] In the remainder, we will consequently denote arbitrary roots by Λ and reserve the letter $\mathbf{r}$ for real roots (i.e. $\mathbf{r}^2 = 2$).

Observe that ℓ is not preserved under arbitrary E_{10} Weyl transformations, but only under the subgroup $\mathfrak{W}(E_9)$ corresponding to the E_9 subalgebra. Therefore, we can freely use this notion also for roots Λ which are not in $\mathscr{C}$, but can be brought into $\mathscr{C}$ by an E_9 Weyl transformation.

The level derives its importance from the fact that it grades the algebra E_{10} with respect to its affine subalgebra E_9 [12]. The subspaces belonging to a fixed level can be decomposed into irreducible representations of E_9, the level being equal to the eigenvalue of the central term of the E_9 algebra on this representation (the full E_{10} algebra contains E_9 representations of *all* integer levels!). Let us emphasize that for general hyperbolic algebras there would be a separate grading associated with every regular affine subalgebra, and therefore the graded structure would no longer be unique. An important result is the following [12] (see also [13]).

Theorem 1. *Suppose that x is an element of E_{10} at level n. Then it can be represented as a linear combination of n-fold commutators of level-one elements, viz.*

$$x = [x_1, [x_2, \ldots [x_{n-1}, x_n] \ldots]] \tag{4.8}$$

where each x_i contains exactly one generator e_{-1} in the right-most position[10], i.e.

$$x_1 = [e_{i_1}, [e_{i_2}, \ldots [e_{i_k}, e_{-1}] \ldots]] \tag{4.9}$$

with $i_\nu \in \{0, 1, \ldots, 8\}$, and similarly for the other x_i.

We are going to make use of this result in the next section in order to effectively construct higher level elements.

As already mentioned, little is known about the general structure of this algebra. Partial progress has been made in determining the multiplicity of certain roots, i.e. the number of linearly independent Lie algebra elements in the corresponding root space. Although the general form of the multiplicity formulas for arbitrary levels appears to be beyond reach for the moment, the following results for levels $\ell \leq 2$ have been established. For $\ell = 0$ and $\ell = 1$, we have [29]

$$\text{mult}(\Lambda) = p_8(1 - \tfrac{1}{2}\Lambda^2) \tag{4.10}$$

i.e. the multiplicities are just given by the number of transversal states; we will see in the next section that the corresponding states are indeed transversal. For $\ell = 2$, it was shown in [30] that

$$\text{mult}(\Lambda) = \xi(3 - \tfrac{1}{2}\Lambda^2) \tag{4.11}$$

where

$$\sum_{n \geq 0} \xi(n) q^n = \frac{1}{\phi(q)^8} \left[1 - \frac{\phi(q^2)}{\phi(q^4)} \right] \tag{4.12}$$

and the Euler function $\phi(q)$ is defined in (3.51). For sufficiently large (negative) Λ^2, one can check from this formula that there are roots Λ such that $\text{mult}(\Lambda) > p_8(1 - \tfrac{1}{2}\Lambda^2)$; this clearly implies the presence of longitudinal states. Beyond $\ell = 2$, no general formula seems to be known although the multiplicities can be determined recursively from the Peterson formula (see e.g. [31]).

---

[10] All level-one elements can be cast into this form by use of the Jacobi identity and by taking appropriate linear combinations.

The derivation of the above result in [30] is based on the E_9 decomposition

$$L(\Lambda_0) \wedge L(\Lambda_0) \cong L(\Lambda_1) \otimes V\left(\tfrac{1}{2}, \tfrac{1}{16}\right)$$

where $L(\Lambda_i)$ denotes the irreducible E_9 module with lowest weight Λ_i and $V\left(\tfrac{1}{2}, \tfrac{1}{16}\right)$ the irreducible Virasoro module with $c = \tfrac{1}{2}$ and $h = \tfrac{1}{16}$ (by abuse of notation, we use the same labels for the E_9 weights as for the E_{10} weights). Observe that the module $L(\Lambda_1)$ precisely corresponds to the ideal generated by the double commutator $[[e_0, e_{-1}], e_{-1}]$. For higher levels, analogous decompositions contain more than one term on the right-hand side, and it seems no longer possible to divide out the Serre relations by this method.

In the physical interpretation, the multiplicity of a root Λ is nothing but the number of linearly independent polarization states of the associated vertex operator of momentum Λ, which can be generated by multiple commutators (recall that not all physical states can be obtained in this way, cf. (2.137)). Given a root $\Lambda \in \mathscr{C}$, we call a polarization vector $\boldsymbol{\xi}$ **transversal** if $\boldsymbol{\xi}\cdot\Lambda = \boldsymbol{\xi}\cdot\boldsymbol{\delta} = 0$, and **longitudinal** otherwise. This terminology is, of course, physically motivated. We also define the **little group** $\mathfrak{W}(\Lambda, \boldsymbol{\delta})$ to be that subgroup of the full Weyl group of E_{10} which leaves the vectors Λ and $\boldsymbol{\delta}$ invariant. Unless Λ is collinear with $\boldsymbol{\delta}$ (corresponding to $\ell = 0$), $\mathfrak{W}(\Lambda, \boldsymbol{\delta})$ is a *finite* subgroup of $\mathfrak{W}(E_{10})$, as well as a discrete subgroup of $SO(8)$. As an example consider $\ell = 1$; then $\Lambda = \Lambda_0 = \mathbf{r}_{-1} + 2\boldsymbol{\delta}$ and $\mathfrak{W}(\Lambda, \boldsymbol{\delta})$ is isomorphic to the Weyl group of E_8. In fact, for $\Lambda \in \mathscr{C}$, it is known (cf. [29, Prop. 3.12]) that $\mathfrak{W}(\Lambda, \boldsymbol{\delta})$ is generated by the reflections $\mathfrak{w}_i$ corresponding to those simple roots $\mathbf{r}_i$ for which $\Lambda\cdot\mathbf{r}_i = \boldsymbol{\delta}\cdot\mathbf{r}_i = 0$. This indicates that the little group will not be quite as useful in this context as it is in conventional quantum field theory, because it becomes trivial for sufficiently high levels. However, at low levels, this problem does not yet arise, and the polarization states and hence the elements belonging to the root space $\mathfrak{g}^{(\Lambda)}_{II_{9,1}}$ can be classified as representations of $\mathfrak{W}(\Lambda, \boldsymbol{\delta})$.

Any root $\Lambda \in \mathscr{C}$ can be represented in the form

$$\Lambda = \ell\mathbf{r}_{-1} + M\boldsymbol{\delta} + \mathbf{b} \tag{4.13}$$

where ℓ is the level of Λ and $\mathbf{b}$ an element of the E_8-root lattice $Q(E_8)$ ($\mathbf{b}$ need not be positive by itself as only $M\boldsymbol{\delta} + \mathbf{b}$ must be positive). We now define the **DDF decomposition** of Λ by

$$\Lambda = \mathbf{a} - n\mathbf{k}(\mathbf{a}), \tag{4.14}$$

where

$$\mathbf{k}(\mathbf{a}) := -\tfrac{1}{\ell}\boldsymbol{\delta} \tag{4.15}$$

and

$$n = 1 - \tfrac{1}{2}\Lambda^2 = 1 + (M - \ell)\ell - \tfrac{1}{2}\mathbf{b}^2. \tag{4.16}$$

By construction, $\mathbf{a}$ obeys $\mathbf{a}^2 = 2$ and is therefore associated with a tachyon state, and n is the number of steps required to reach the root Λ by starting from $\mathbf{a}$ and decreasing the momentum by $\mathbf{k}$ at each step (n is non-negative because $\Lambda^2 \leq 2$; note also that $\mathbf{k}$ is always a *negative* root, so Λ is positive for all n). Obviously, for $\ell > 1$, neither $\mathbf{k}$ nor $\mathbf{a}$ belong to the lattice in general. As a consequence, the intermediate DDF states

associated with momenta $\mathbf{a} - m\mathbf{k}$ not on the lattice will not correspond to elements of the algebra. On the other hand, states associated with the root Λ do belong to the algebra of physical states, and the DDF decomposition enables us to write down all possible polarization states associated with the root $\Lambda \in \mathscr{C}$ in terms of transversal and longitudinal DDF states; the totality of these states constitutes the complete set of elements in the root space $\mathfrak{g}_{H_{9,1}}^{(\Lambda)}$.

Of course, we could also try to apply the DDF decomposition to roots Λ not $\mathfrak{W}(E_9)$-equivalent to roots in $\mathscr{C}$. Whenever we succeed in finding a suitable null vector $\mathbf{k}$ on the lattice obeying $\Lambda \cdot \mathbf{k} = 1$, we can also find a Weyl transformation $\mathfrak{w} \in \mathfrak{W}(E_{10})$ such that $\mathfrak{w}(\mathbf{k}) = -\delta$ because δ is the only primitive null vector in $\mathscr{C}$. Since $\Lambda \cdot \mathbf{k} = -\mathfrak{w}(\Lambda) \cdot \delta$ is just the level, it follows that $\mathfrak{w}(\Lambda) = \mathbf{a} + n\delta$ is a level-one root with tachyon momentum

$$\mathbf{a} = \mathbf{r}_{-1} + \left(\tfrac{1}{2}\mathbf{b}^2\right)\delta + \mathbf{b} \tag{4.17}$$

for some $\mathbf{b} \in Q(E_8)$. Therefore, nothing is gained by searching for DDF vectors outside the $\mathfrak{W}(E_9)$ transforms of the fundamental Weyl chamber. Note that the elements of the form (4.17) constitute the E_9 Weyl orbit of $\mathbf{r}_{-1}$ [29].

4.2. The DDF states at levels zero and one. Although the multiplicity formulas for levels $\ell = 0$ and $\ell = 1$ are understood [29], we here derive them once more, because our explicit DDF representation of the level-one elements has apparently not been exhibited in the literature so far. The level-zero elements make up the E_9 subalgebra of E_{10}. The allowed (positive and negative) roots are all $\mathbf{r} \in H_{9,1}$ obeying $\mathbf{r}^2 = 2$ and $\mathbf{r} \cdot \delta = 0$ (hence having no $\mathbf{r}_{-1}$ component), and $m\delta$ for $m \in \mathbb{Z}^\times$. These correspond to the tachyonic and photonic states with multiplicities 1 and 8, respectively:

$$|\mathbf{r}\rangle \equiv e^{\mathbf{r}} \quad \text{for } \mathbf{r}^2 = 2, \tag{4.18}$$

$$\boldsymbol{\xi}_i(-1)|m\delta\rangle \equiv \boldsymbol{\xi}_i(-1)e^{m\delta}, \tag{4.19}$$

where $\boldsymbol{\xi}_i \cdot \delta = 0$ and $\boldsymbol{\xi}_i$ has no component along δ (i.e. $\mathbf{r}_{-1} \cdot \boldsymbol{\xi}_i = 0$). The Cartan subalgebra of E_9 is spanned by the states

$$\delta(-1)|\mathbf{0}\rangle =: K, \tag{4.20}$$

$$(\mathbf{r}_{-1} + \delta)(-1)|\mathbf{0}\rangle =: d, \tag{4.21}$$

$$\boldsymbol{\xi}_i(-1)|\mathbf{0}\rangle \quad \text{for } i = 1, \ldots, 8, \tag{4.22}$$

where K represents the central element, d is the derivation of E_9, and $\{\boldsymbol{\xi}_i(-1)|\mathbf{0}\rangle \mid i = 1, \ldots, 8\}$ span the Cartan subalgebra of E_8. This is the standard "light-cone" basis of $\mathfrak{h}(E_9)$ in the sense that K and d are lightlike. As for the commutators we rewrite (2.120) and (2.122) – (2.124) as

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, \boldsymbol{\zeta}(-1)|\mathbf{0}\rangle\right] = 0, \tag{4.23}$$

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, \boldsymbol{\xi}_i(-1)|m\delta\rangle\right] = m(\boldsymbol{\eta}\cdot\delta)\boldsymbol{\xi}_i(-1)|m\delta\rangle, \tag{4.24}$$

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, |\mathbf{r}\rangle\right] = (\boldsymbol{\eta}\cdot\mathbf{r})|\mathbf{r}\rangle, \tag{4.25}$$

$$\left[\boldsymbol{\xi}_i(-1)|m\delta\rangle, \boldsymbol{\xi}_j(-1)|n\delta\rangle\right] = m\delta_{m+n,0}(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\delta(-1)|\mathbf{0}\rangle, \tag{4.26}$$

$$\left[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, |\mathbf{r}\rangle\right] \;=\; (\boldsymbol{\xi}_i \cdot \mathbf{r})|\mathbf{r} + m\boldsymbol{\delta}\rangle, \tag{4.27}$$

$$\left[|\mathbf{r}\rangle, |\mathbf{s}\rangle\right] \;=\; \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{s} \geq 0, \\ \epsilon(\mathbf{r},\mathbf{s})|\mathbf{r} + \mathbf{s}\rangle & \text{if } \mathbf{r}\cdot\mathbf{s} = -1, \\ -\mathbf{r}(-1)|m\boldsymbol{\delta}\rangle & \text{if } \mathbf{r} + \mathbf{s} = m\boldsymbol{\delta}, \end{cases} \tag{4.28}$$

for $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathfrak{h}(E_9)$ and E_9 roots $\mathbf{r}, \mathbf{s}$. To see that photonic states with all required transversal polarizations can be generated by commuting tachyonic states, we recall (3.52) (a special case of (2.117)): choosing $\mathbf{s} = \mathbf{r}_i$ and $\mathbf{t} = m\boldsymbol{\delta} - \mathbf{r}_i$ (where $\mathbf{r}_i$ is any simple root of E_9), we obtain all transversal polarizations. There is obviously no way to generate longitudinal states, because the polarization vectors $\boldsymbol{\xi}_i$ would then have to have components along $\mathbf{r}_{-1}$, which we cannot generate by commuting tachyonic states belonging to E_9 roots only. Since we can ignore null physical states (for which $\boldsymbol{\xi} \propto \boldsymbol{\delta}$), we can in addition impose the requirement $\boldsymbol{\xi}\cdot\mathbf{r}_{-1} = 0$, so $\boldsymbol{\xi} \in \mathrm{span}_{\mathbb{R}}\{\mathbf{r}_1, \ldots, \mathbf{r}_8\}$, so that by taking appropriate linear combinations we can arrange that $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j = \delta_{ij}$ with $\boldsymbol{\xi}_i\cdot\boldsymbol{\delta} = \boldsymbol{\xi}_i\cdot\mathbf{r}_{-1} = 0$ for $i,j = 1,\ldots,8$. It is clear that an infinity of conjugate E_9 subalgebras in E_{10} can be obtained by Weyl conjugation of these states with elements of $\mathfrak{W}(E_{10})$ not in $\mathfrak{W}(E_9)$.

Let us now turn to the level-one roots. Inspection of the inverse Cartan matrix shows that the only such roots in $\mathscr{C}$ are of the form

$$\boldsymbol{\Lambda} = k_{-1}\boldsymbol{\Lambda}_{-1} + \boldsymbol{\Lambda}_0 = \mathbf{r}_{-1} + (2 + k_{-1})\boldsymbol{\delta} \tag{4.29}$$

corresponding to the DDF decomposition (4.14) with $\mathbf{a} = \mathbf{r}_{-1}$, $\mathbf{k} = -\boldsymbol{\delta}$ and $n = 2 + k_{-1}$. Since all these vectors are elements of the lattice, we can straightforwardly apply the DDF construction to obtain the physical states

$$A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle, \tag{4.30}$$

where $m_1 + \ldots + m_N = 2 + k_{-1}$ and with the polarization vectors chosen as above. Recall that $A^i_{-m} \equiv \left(\boldsymbol{\xi}_i(-1)\mathrm{e}^{m\boldsymbol{\delta}}\right)_0$. These states correspond to the multiple commutators

$$\left[\boldsymbol{\xi}_{i_1}(-1)|m_1\boldsymbol{\delta}\rangle, \left[\ldots, \left[\boldsymbol{\xi}_{i_N}(-1)|m_N\boldsymbol{\delta}\rangle, |\mathbf{r}_{-1}\rangle\right]\ldots\right]\right], \tag{4.31}$$

as we have already shown. Moreover, we can explicitly verify that they form part of the basic representation of E_9 with lowest weight vector $|\mathbf{r}_{-1}\rangle$. To see this we have to work out the commutators of the E_9 elements (4.18) – (4.22) with the level-one states (4.30):

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right]$$
$$= \;\left[(m_{i_1} + \ldots + m_{i_N})\boldsymbol{\delta}\cdot\boldsymbol{\eta} + \mathbf{r}_{-1}\cdot\boldsymbol{\eta}\right]A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle, \tag{4.32}$$

$$\left[\boldsymbol{\xi}_j(-1)|n\boldsymbol{\delta}\rangle, A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right]$$
$$= \;A^j_{-n}\left(A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right)$$
$$= \;\begin{cases} -\displaystyle\sum_{k=1}^{N} n\delta_{j,i_k}\delta_{n,m_k}\prod_{l\neq k} A^{i_l}_{-m_l}|\mathbf{r}_{-1}\rangle & \text{if } n < 0, \\[4mm] A^j_{-n}A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle & \text{if } n > 0, \end{cases} \tag{4.33}$$

$$\left[|\mathbf{s}\rangle, A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right]$$

$$= -\sum_{k=1}^{N}(\mathbf{s}\cdot\boldsymbol{\xi}_{i_k})A^{i_1}_{-m_1}\cdots A^{i_{k-1}}_{-m_{k-1}}\left[|\mathbf{s}+m_k\boldsymbol{\delta}\rangle, A^{i_{k+1}}_{-m_{k+1}}\cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle\right]$$

$$+ A^{i_1}_{-m_1}\cdots A^{i_N}_{-m_N}\left[|\mathbf{s}\rangle, |\mathbf{r}_{-1}\rangle\right]. \tag{4.34}$$

The first commutator tells us that the Cartan subalgebra of E_9 acts diagonally on the DDF states, giving the components of the lowest weight $\boldsymbol{\eta}\cdot\boldsymbol{\Lambda}$ of the representation. The second commutator which directly follows from the definitions (2.60) and (3.2), reveals that the E_9 elements corresponding to multiples of the null root $\boldsymbol{\delta}$ act by multiplication with a DDF operator. The last commutator is obtained by rewriting the DDF states (4.30) as multiple commutators and repeated application of the following version of the Jacobi identity:

$$\begin{aligned}\left[|\mathbf{s}\rangle, A^{i}_{-m}\psi\right] &\equiv \left[|\mathbf{s}\rangle, \left[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, \psi\right]\right]\\ &= \left[\left[|\mathbf{s}\rangle, \boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle\right], \psi\right] + \left[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, \left[|\mathbf{s}\rangle, \psi\right]\right]\\ &= -m(\mathbf{s}\cdot\boldsymbol{\xi}_i)\left[|\mathbf{s}+m\boldsymbol{\delta}\rangle, \psi\right] + A^i_{-m}\left[|\mathbf{s}\rangle, \psi\right] \end{aligned} \tag{4.35}$$

for any state ψ. Note that the commutator $\left[|\mathbf{s}\rangle, |\mathbf{r}_{-1}\rangle\right]$ above can be evaluated using (2.117). For example it vanishes whenever $\mathbf{s}$ is a negative root of E_9 as it should be since $\mathbf{r}_{-1}$ is a lowest weight vector; furthermore, we always get a level-one state since $\mathbf{s}$ is a E_9 root. Weyl-equivalent level-one states can be generated by Weyl conjugation with elements $\mathfrak{w} \in \mathfrak{W}$ leaving the level fixed, i.e. $\mathfrak{w} \in \mathfrak{W}(E_9)$. The tachyonic momentum $\mathbf{r}_{-1}$ is then mapped to a vector of the form (4.17) with $\mathbf{a} = \mathfrak{w}(\mathbf{r}_{-1})$, while $\boldsymbol{\delta}$ is left invariant. The polarizations used above must be replaced by the rotated polarization vectors $\boldsymbol{\xi}_{\mathfrak{w}(i)} := \mathfrak{w}(\boldsymbol{\xi}_i)$ with corresponding changes in the DDF vectors. Denoting the rotated DDF operators by $A^{\mathfrak{w}(i)}_{-m} \equiv A^{\mathfrak{w}(\boldsymbol{\xi}_i)}_{-m}$, we obtain the new states

$$A^{\mathfrak{w}(i_1)}_{-m_1} \cdots A^{\mathfrak{w}(i_N)}_{-m_N}|\mathfrak{w}(\mathbf{r}_{-1})\rangle. \tag{4.36}$$

The so-called basic representation is spanned by all elements of the form (4.36). Notice that although we are using transversal indices these now transform under different little groups (which are all conjugate to $\mathfrak{W}(E_8)$). The multiplicity formula for the level-zero and level-one roots [29]

$$\text{mult}(\Lambda) = p_8(n) = p_8(1 - \tfrac{1}{2}\Lambda^2) \tag{4.37}$$

can be read off immediately from (4.18), (4.19) and (4.30). This multiplicity formula holds likewise for roots related by an arbitrary Weyl rotation to a level-one root.

As already mentioned before, the states (4.30) transform covariantly under the corresponding little group $\mathfrak{W}(\mathbf{r}_{-1}, \boldsymbol{\delta})$, which is just the Weyl group of E_8. Now it is known that $\mathfrak{W}(E_8) = D_4(2) \times (\mathbb{Z}_2)^2$, where $D_4(2)$ is the Chevalley group of order $2^{12}3^55^27$, or, equivalently, the set of $SO(8)$ matrices with entries in the field $\mathbb{Z}_2$ (see e.g. [9]). Since it is the maximal discrete subgroup of $SO(8)$ of this type in the sense that the little groups of all higher level roots will be much smaller, this also explains why the states (4.30) look "$SO(8)$ covariant" (although the polarization indices $i, j, \ldots$ should by no means be regarded as $SO(8)$ indices). The higher level root spaces will exhibit much less symmetry.

4.3. Higher level: Generalities. Before turning to the discussion of an explicit exam-
ple of a level-two root space, we would like to explain the general ideas underlying
the description of higher level elements in terms of the DDF construction. As we
have already mentioned, the DDF states constitute a complete basis of physical states
for any allowed momentum on the root lattice. Consequently, the root space $E_{10}^{(\Lambda)}$ is a
(proper for $\ell > 1$) subspace of $\mathfrak{g}_{II_{9,1}}^{(\Lambda)}$ for any root Λ (this inclusion is a special case of
(2.137)). The physical states are explicitly given by (3.47) or, equivalently, by (3.38).
Anticipating that the final results are somewhat simpler in terms of (3.38), we will
use the basis

$$A^{i_1}_{-m_1}(\mathbf{a}) \ldots A^{i_M}_{-m_M}(\mathbf{a}) \mathcal{L}_{-n_1}(\mathbf{a}) \ldots \mathcal{L}_{-n_N}(\mathbf{a}) |\mathbf{a}\rangle, \tag{4.38}$$

explicitly indicating the dependence of the DDF operators and their polarizations
on the tachyon momentum $\mathbf{a}$ and the associated lightlike vector $\mathbf{k}(\mathbf{a}) = -\frac{1}{\ell}\boldsymbol{\delta}$, and
assuming $n_i \geq 2$ from now on to exclude null states. Since $\ell \neq 1$, we have

$$A^i_{-m}(\mathbf{a}) \equiv \left((\boldsymbol{\xi}_i(\mathbf{a}))(-1) e^{\frac{m}{\ell}\boldsymbol{\delta}} \right)_0 \tag{4.39}$$

with an extra factor of $\frac{1}{\ell}$ in the exponent, as appropriate for level ℓ by (4.15). In
accordance with the DDF decomposition $\Lambda = \mathbf{a} - n\mathbf{k}(\mathbf{a})$, the indices obey the sum
rule $m_1 + \ldots + m_M + n_1 + \ldots + n_N = n$. We emphasize once more that neither $\mathbf{a}$ nor $\mathbf{k}(\mathbf{a})$
need be on the root lattice for $\ell > 1$ any more. The problem of characterizing the root
spaces of the hyperbolic Kac Moody algebra is now no longer one of dividing out the
Serre relations (2.131) (these are automatically taken care of by the vertex operator
formalism as we pointed out already), but rather one of identifying the missing states
which cannot be generated by multiple commutators of the Chevalley generators e_i
or f_i. The above representation immediately yields the following upper bound on the
root multiplicities [6]

$$\text{mult}\,(\Lambda) \leq p_9(1 - \tfrac{1}{2}\Lambda^2) - p_9(-\tfrac{1}{2}\Lambda^2). \tag{4.40}$$

To effectively construct higher level elements we invoke Theorem 1 of Sect. 4.1.
For instance, given a level-two root Λ in the fundamental Weyl chamber $\mathscr{C}$, we write

$$\Lambda = \mathbf{r} + \mathbf{s} + m\boldsymbol{\delta}, \tag{4.41}$$

where $\mathbf{r}$ and $\mathbf{s}$ are real positive level-one roots (i.e. containing the simple root $\mathbf{r}_{-1}$
exactly once and obeying $\mathbf{r}^2 = \mathbf{s}^2 = 2$). In general, there will be many different ways
to split Λ in this manner, as well as different integers m. For fixed value of m, these
decompositions are related by the little group, which leaves Λ and $\boldsymbol{\delta}$ fixed, but varies
$\mathbf{r}$ and $\mathbf{s}$. Thus, we work with a fixed decomposition and then act on the resulting
commutator states with the little group so as to obtain all possible states with the
same value of m. The commutator to be computed is

$$\left[A^{i_1}_{-m_1}(\mathbf{s}) \ldots A^{i_M}_{-m_M}(\mathbf{s}) |\mathbf{s}\rangle, \ A^{j_1}_{-n_1}(\mathbf{r}) \ldots A^{j_N}_{-n_N}(\mathbf{r}) |\mathbf{r}\rangle \right] \tag{4.42}$$

where $m_1 + \ldots m_M + n_1 + \ldots + n_N = m$. For the special example to be discussed
below, this expression can be evaluated with the help of the formulas given in the
appendix. Expanding it in terms of the basis (4.38), we arrive at

$$(4.42) = \sum_{\substack{p_1 + \ldots + q_Q = n \\ k_1, \ldots, k_P}} c^{i_1 \ldots i_M j_1 \ldots j_N}_{k_1 \ldots k_P} A^{k_1}_{-p_1}(\mathbf{a}) \ldots A^{k_P}_{-p_P}(\mathbf{a}) \mathcal{L}_{-q_1}(\mathbf{a}) \ldots \mathcal{L}_{-q_Q}(\mathbf{a}) |\mathbf{a}\rangle \tag{4.43}$$

with the "Clebsch Gordan coefficients" $c^{i_1 \ldots i_M j_1 \ldots j_N}_{k_1 \ldots k_P}$, into which all the information about the missing states is encoded. For the Fock space states, this equality holds of course only modulo terms $L_{(-1)}(\ldots)$, which can however be ignored for the Lie algebra, as they are factored out by (2.63). (4.43) is the crucial formula, containing both transversal and longitudinal excitations[11]. For the calculations, we note that the polarization vectors $\boldsymbol{\xi}_i(\mathbf{r})$ and $\boldsymbol{\xi}_i(\mathbf{s})$ can always be chosen orthonormal and such that they agree for $i = 1, \ldots, 7$; from (4.41) we then see that $\boldsymbol{\xi}_i(\mathbf{a}) = \boldsymbol{\xi}_i(\mathbf{r})$ as well for these values of the indices. As for the remaining components $\boldsymbol{\xi}_8(\mathbf{r})$, $\boldsymbol{\xi}_8(\mathbf{s})$ and $\boldsymbol{\xi}_8(\mathbf{a})$, one can convince oneself that their differences are proportional to the null vector $\boldsymbol{\delta}$. Since such contributions drop out in the non-zero mode parts of the DDF operators (cf. the discussion after (2.62)), the respective DDF operators are really the same except for their zero mode parts and the crucial fact that their photon momenta depend on the level. We stress that this would not be true if the Weyl chamber contained more than one null direction.

Just as for the level-one states, one can determine how the states (4.43) transform under E_9. Suppressing the label $(\mathbf{a})$ on the DDF operators to make the formulas less cumbersome, this calculation requires the commutators (for $\ell = 2$)

$$\left[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} |\mathbf{a}\rangle \right]$$
$$= \left(\tfrac{1}{\ell}(m_1 + \ldots + n_N)\boldsymbol{\delta} \cdot \boldsymbol{\eta} + \mathbf{a} \cdot \boldsymbol{\eta} \right) A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} |\mathbf{a}\rangle. \quad (4.44)$$

The scalar product in parentheses is easily seen to reduce to $\boldsymbol{\eta} \cdot \Lambda$, giving the components of the lowest weight of the representation. Furthermore,

$$\left[\boldsymbol{\xi}_j(-1)|n\boldsymbol{\delta}\rangle, A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} |\mathbf{a}\rangle \right]$$
$$= A^j_{-\ell n} \left(A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} |\mathbf{a}\rangle \right), \quad (4.45)$$

(notice that the index on the first operator is $(-\ell n)$ rather than $(-n)$!) and

$$\left[|\mathbf{s}\rangle, A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} |\mathbf{a}\rangle \right]$$
$$= -\sum_{k=1}^{M} (\mathbf{s} \cdot \boldsymbol{\xi}_{i_k}) A^{i_1}_{-m_1} \ldots A^{i_{k-1}}_{m_{k-1}} \left[|\mathbf{s} + \tfrac{1}{\ell}m_k\boldsymbol{\delta}\rangle, A^{i_{k+1}}_{m_{k+1}} \ldots A^{i_M}_{m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} |\mathbf{a}\rangle \right]$$
$$+ \sum_{l=1}^{N} (\mathbf{s} \cdot \mathbf{a}) A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{n_{l-1}} \left[|\mathbf{s} + \tfrac{1}{\ell}n_l\boldsymbol{\delta}\rangle, \mathfrak{L}_{n_{l+1}} \ldots \mathfrak{L}_{n_N} |\mathbf{a}\rangle \right]$$
$$+ A^{i_1}_{-m_1} \ldots A^{i_M}_{-m_M} \mathfrak{L}_{-n_1} \ldots \mathfrak{L}_{-n_N} \left[|\mathbf{s}\rangle, |\mathbf{a}\rangle \right]. \quad (4.46)$$

Observe that there are no contributions from the logarithmic terms in $\mathfrak{L}_{-m}$ to the last commutator because $\mathbf{s} \cdot \boldsymbol{\delta} = 0$ for any E_9 root $\mathbf{s}$. The proof of these formulas is analogous to the proof of the corresponding formulas for the level-one states in the previous section, save for the following important caveat. When building up the states from the tachyonic groundstate $|\mathbf{a}\rangle$ by successive application of the DDF operators, the intermediate states, whose momenta are not on the root lattice, do *not* belong

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[11] This formula also shows why the fake monster Lie algebra of [6] is, in a certain sense, much simpler (though bigger) than E_{10}. The longitudinal components generated by commuting two transversal DDF states decouple in 26 dimensions, and therefore only the terms *without* longitudinal states survive in the expansion (4.43). To be sure, one must still prove that indeed *all* transversal states can be generated in this way if one takes into account the imaginary simple roots.

to the Kac Moody algebra, because the Lie bracket with arbitrary elements is in general not defined due to branch cuts in the relevant operator product expansions[12]. Therefore, the "commutators" in (4.46) are neither commutators in E_{10} nor even in the Lie algebra of physical states $\mathfrak{g}_\Lambda$; nevertheless, the above calculation does make sense because all relevant products of momenta are integer, and therefore the generic branch cuts are absent. So we must keep in mind that only the final result including summation according to (4.43) is an element of E_{10} again. The fact that the direct construction of the DDF states has no Lie algebra analog beyond level one explains the emergence of longitudinal states as well as the disappearance of certain transversal states.

In the next section, we will work out one non-trivial level-two root, arriving at a complete description of its root space in terms of DDF states, which decompose into irreducible representations of the little group $\mathfrak{W}(\Lambda, \delta)$; as a by-product, we verify the multiplicity formula of [30] for a concrete example. The comparative simplicity of the representation obtained in this manner is perhaps best appreciated by noting that the number of Lie brackets needed to represent any of its elements in terms of Chevalley generators is equal to $(-\rho \cdot \Lambda - 1)$, where ρ is the Weyl vector.

4.4. A level-two example: $\Lambda = \Lambda_7$. Any level-two root in $\mathscr{C}$ must be of the form $\Lambda_1 + n\delta$ or $\Lambda_7 + n\delta$ or $2\Lambda_0 + n\delta$ for some $n \in \mathbb{N}$. We will here only discuss the root $\Lambda = \Lambda_7$, dual to the simple root $\mathbf{r}_7$. Explicitly, Λ_7 is given by

$$\Lambda_7 = \begin{bmatrix} & & & & & & 7 & & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,2), \quad (4.47)$$

so $\Lambda_7^2 = -4$. Its decomposition into two level-one tachyonic roots is $\Lambda_7 = \mathbf{r} + \mathbf{s} + 2\delta$, where

$$\mathbf{r} := \mathbf{r}_{-1} \;\; = \;\; \begin{bmatrix} & & & & & & & 0 & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ -1\,|\,0),$$

$$\mathbf{s} \;\; := \;\; \begin{bmatrix} & & & & & & 1 & & \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \end{bmatrix} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -1,\ -1\,|\,0).$$

Since $n = 1 - \frac{1}{2}\Lambda_7^2 = 3$ we have the DDF decomposition $\Lambda_7 = \mathbf{a} - 3\mathbf{k}$ where $\mathbf{k} := -\frac{1}{2}\delta$ and

$$\mathbf{a} := \mathbf{r} + \mathbf{s} - \mathbf{k} = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -\tfrac{3}{2}\,|\,\tfrac{1}{2}).$$

As expected, neither $\mathbf{k}$ nor $\mathbf{a}$ are elements of $\Pi_{9,1}$. Nevertheless, since $\mathbf{a} \cdot \mathbf{k} = 1$, the action of the DDF operators $A^i_{-n}(\mathbf{k})$ on the tachyonic ground-state $|\mathbf{a}\rangle$ is perfectly well-defined as we already pointed out. As for the three sets of polarization vectors associated with the tachyon momenta $|\mathbf{r}\rangle$, $|\mathbf{s}\rangle$ and $|\mathbf{a}\rangle$, respectively, a convenient choice is

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[12] We note that the cocycle conditions (2.80)–(2.87) can be solved on a rational extension of the root lattice [17].

$$\boldsymbol{\xi}_\alpha \quad \equiv \quad \boldsymbol{\xi}_\alpha(\mathbf{r}) = \boldsymbol{\xi}_\alpha(\mathbf{s}) = \boldsymbol{\xi}_\alpha(\mathbf{a}) \quad \text{for} \quad \alpha = 1, \ldots, 7 \ ,$$
$$\boldsymbol{\xi}_1 \quad := \quad (1, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0 \,|\, 0),$$

$$\vdots$$

$$\boldsymbol{\xi}_7 \quad := \quad (0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 1, \ 0, \ 0 \,|\, 0);$$

$$\boldsymbol{\xi}_8(\mathbf{r}) \quad := \quad (0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 1, \ 1 \,|\, 1),$$
$$\boldsymbol{\xi}_8(\mathbf{s}) \quad := \quad (0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, -1, \ 1 \,|\, 1),$$
$$\boldsymbol{\xi}_8 \equiv \boldsymbol{\xi}_8(\mathbf{a}) \quad := \quad (0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0, \ 1, \ 0 \,|\, 0). \tag{4.48}$$

The little group is $\mathfrak{W}(\Lambda_7, \delta) = \mathfrak{W}(D_8) = S_8 \ltimes (\mathbb{Z}_2)^7$ of order $2^{14} 3^1 5^1 7^1$. This group is generated by the fundamental reflections $\{\mathfrak{w}_0, \mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3, \mathfrak{w}_4, \mathfrak{w}_5, \mathfrak{w}_6, \mathfrak{w}_8\}$. On the polarization vectors $\boldsymbol{\xi}_i(\mathbf{a})$ it acts as follows:

$$\begin{aligned}
\mathfrak{w}_0(\boldsymbol{\xi}_7) &= \boldsymbol{\xi}_8 \ , \quad \mathfrak{w}_0(\boldsymbol{\xi}_8) = \boldsymbol{\xi}_7, \\
\mathfrak{w}_1(\boldsymbol{\xi}_6) &= \boldsymbol{\xi}_7 \ , \quad \mathfrak{w}_1(\boldsymbol{\xi}_7) = \boldsymbol{\xi}_6, \\
\mathfrak{w}_2(\boldsymbol{\xi}_5) &= \boldsymbol{\xi}_6 \ , \quad \mathfrak{w}_2(\boldsymbol{\xi}_6) = \boldsymbol{\xi}_5, \\
\mathfrak{w}_3(\boldsymbol{\xi}_4) &= \boldsymbol{\xi}_5 \ , \quad \mathfrak{w}_3(\boldsymbol{\xi}_5) = \boldsymbol{\xi}_4, \\
\mathfrak{w}_4(\boldsymbol{\xi}_3) &= \boldsymbol{\xi}_4 \ , \quad \mathfrak{w}_4(\boldsymbol{\xi}_4) = \boldsymbol{\xi}_3, \\
\mathfrak{w}_5(\boldsymbol{\xi}_2) &= \boldsymbol{\xi}_3 \ , \quad \mathfrak{w}_5(\boldsymbol{\xi}_3) = \boldsymbol{\xi}_2, \\
\mathfrak{w}_6(\boldsymbol{\xi}_1) &= -\boldsymbol{\xi}_2 \ , \quad \mathfrak{w}_6(\boldsymbol{\xi}_2) = -\boldsymbol{\xi}_1, \\
\mathfrak{w}_8(\boldsymbol{\xi}_1) &= \boldsymbol{\xi}_2 \ , \quad \mathfrak{w}_8(\boldsymbol{\xi}_2) = \boldsymbol{\xi}_1, \tag{4.49}
\end{aligned}$$

and as the identity on all those that have not been listed. Furthermore, $\{\mathfrak{w}_1, \ldots, \mathfrak{w}_6, \mathfrak{w}_8\}$ leave $\mathbf{r}$ and $\mathbf{s}$ invariant, whereas

$$\mathfrak{w}_0(\mathbf{r}) = \mathbf{r} + \mathbf{r}_0 \qquad , \qquad \mathfrak{w}_0(\mathbf{s}) = \mathbf{s} - \mathbf{r}_0 \ .$$

The Weyl group element $\mathfrak{w} = \mathfrak{w}_0 \mathfrak{w}_1 \mathfrak{w}_2 \mathfrak{w}_3 \mathfrak{w}_4 \mathfrak{w}_5 \mathfrak{w}_8 \mathfrak{w}_6 \mathfrak{w}_5 \mathfrak{w}_4 \mathfrak{w}_3 \mathfrak{w}_2 \mathfrak{w}_1 \mathfrak{w}_0$ interchanges $\mathbf{r}$ and $\mathbf{s}$.

There are three sets of DDF operators acting on different tachyonic ground states $|\mathbf{r}\rangle$, $|\mathbf{s}\rangle$, and $|\mathbf{a}\rangle$, respectively. Now, since $\mathfrak{g}_{II_{9,1}}^{(\Lambda_7)}$ is spanned by the 192 transversal and the 9 longitudinal DDF states

$$A_{-1}^i(\mathbf{a}) A_{-1}^j(\mathbf{a}) A_{-1}^k(\mathbf{a}) |\mathbf{a}\rangle \quad ,$$
$$A_{-2}^i(\mathbf{a}) A_{-1}^j(\mathbf{a}) |\mathbf{a}\rangle \quad ,$$
$$A_{-3}^i(\mathbf{a}) |\mathbf{a}\rangle \quad ,$$
$$A_{-1}^i(\mathbf{a}) \mathcal{L}_{-2}(\mathbf{a}) |\mathbf{a}\rangle \quad ,$$
$$\mathcal{L}_{-3}(\mathbf{a}) |\mathbf{a}\rangle \quad ,$$

we can express any element of the root space $E_{10}^{(\Lambda_7)}$ as a linear combination of the above elements modulo $L_{(-1)}$ terms. This is done by using the formulas from the appendix and solving the resulting (overdetermined!) systems of linear equations for the coefficients. We will suppress in our notation the dependence of the DDF operators on the tachyon momenta. *In the following we adopt the convention that DDF operators are always understood to be associated with the tachyons on which they act.* Hence the DDF operators occurring on the left hand side and on the right-hand side of the

formulas below are *not* the same. In Eq. (4.50), for example, we have $A^\alpha_{-1} \equiv A^\alpha_{-1}(\mathbf{r})$ on the left hand side but $A^\alpha_{-1} \equiv A^\alpha_{-1}(\mathbf{a})$ on the right-hand side. Here are our results:

$$\left[|s\rangle, A^\alpha_{-1} A^\beta_{-1} |\mathbf{r}\rangle\right] = \epsilon\left\{-\tfrac{1}{2}A^\alpha_{-2}A^\beta_{-1} - \tfrac{1}{2}A^\beta_{-2}A^\alpha_{-1} - A^\alpha_{-1}A^\beta_{-1}A^8_{-1}\right.$$
$$\left. +\tfrac{1}{24}\delta^{\alpha\beta}\left[A^8_{-3} + 3A^8_{-1}\mathfrak{L}_{-2} - 4A^8_{-1}A^8_{-1}A^8_{-1}\right]\right\} |\mathbf{a}\rangle, \quad (4.50)$$

$$\left[|s\rangle, A^\alpha_{-1} A^8_{-1} |\mathbf{r}\rangle\right] = \epsilon\left\{\tfrac{1}{4}A^\alpha_{-3} + \tfrac{1}{2}A^\alpha_{-2}A^8_{-1} - \tfrac{1}{2}A^8_{-2}A^\alpha_{-1} - \tfrac{1}{4}A^\alpha_{-1}\mathfrak{L}_{-2}\right\} |\mathbf{a}\rangle, \quad (4.51)$$

$$\left[|s\rangle, A^8_{-1} A^8_{-1} |\mathbf{r}\rangle\right] = \epsilon\left\{\tfrac{17}{24}A^8_{-3} + A^8_{-2}A^8_{-1} + \tfrac{1}{8}A^8_{-1}\mathfrak{L}_{-2} + \tfrac{1}{6}A^8_{-1}A^8_{-1}A^8_{-1}\right\} |\mathbf{a}\rangle, (4.52)$$

$$\left[|s\rangle, A^\alpha_{-2} |\mathbf{r}\rangle\right] = \epsilon\left\{-\tfrac{3}{4}A^\alpha_{-3} - \tfrac{1}{4}A^\alpha_{-1}\mathfrak{L}_{-2} + A^\alpha_{-1}A^8_{-1}A^8_{-1}\right\} |\mathbf{a}\rangle, \quad (4.53)$$

$$\left[|s\rangle, A^8_{-2} |\mathbf{r}\rangle\right] = \epsilon\left\{-\tfrac{1}{2}A^8_{-3} + \tfrac{1}{2}A^8_{-1}\mathfrak{L}_{-2}\right\} |\mathbf{a}\rangle, \quad (4.54)$$

$$\left[A^\alpha_{-1}|s\rangle, A^\beta_{-1} |\mathbf{r}\rangle\right] = \epsilon\left\{-\tfrac{1}{2}A^\alpha_{-2}A^\beta_{-1} + \tfrac{1}{2}A^\beta_{-2}A^\alpha_{-1} + A^\alpha_{-1}A^\beta_{-1}A^8_{-1}\right.$$
$$\left. -\tfrac{1}{24}\delta^{\alpha\beta}\left[A^8_{-3} + 3A^8_{-1}\mathfrak{L}_{-2} - 4A^8_{-1}A^8_{-1}A^8_{-1}\right]\right\} |\mathbf{a}\rangle, \quad (4.55)$$

$$\left[A^\alpha_{-1}|s\rangle, A^8_{-1} |\mathbf{r}\rangle\right] = \epsilon\left\{-\tfrac{1}{4}A^\alpha_{-3} + \tfrac{1}{2}A^\alpha_{-2}A^8_{-1} + \tfrac{1}{2}A^8_{-2}A^\alpha_{-1} + \tfrac{1}{4}A^\alpha_{-1}\mathfrak{L}_{-2}\right\} |\mathbf{a}\rangle, \quad (4.56)$$

$$\left[A^8_{-1}|s\rangle, A^8_{-1} |\mathbf{r}\rangle\right] = \epsilon\left\{\tfrac{17}{24}A^8_{-3} + \tfrac{1}{8}A^8_{-1}\mathfrak{L}_{-2} + \tfrac{1}{6}A^8_{-1}A^8_{-1}A^8_{-1}\right\} |\mathbf{a}\rangle, \quad (4.57)$$

for $\alpha, \beta = 1, \ldots, 7$ and with $\epsilon \equiv \epsilon(\mathbf{s},\mathbf{r})\epsilon(\mathbf{k},\mathbf{a})$; contributions involving $L_{(-1)}(\ldots)$ have been discarded in accordance with (2.63). Since these terms are, however, indispensable for the actual calculation, we have listed them in the appendix. An immediate consequence is the following simple formula:

$$-(-1)^{\delta_{j8}+\delta_{i8}}\left[A^i_{-1}|s\rangle, A^j_{-1}|\mathbf{r}\rangle\right] - (-1)^{\delta_{i8}}\left[|s\rangle, A^i_{-1}A^j_{-1}|\mathbf{r}\rangle\right] = A^i_{-2}A^j_{-1}|\mathbf{a}\rangle. \quad (4.58)$$

Further careful analysis of the above results and use of the little Weyl group action (4.49) finally reveals that the following states form a complete basis of the root space $E_{10}^{(\Lambda_7)}$ (no summation convention!)

$$A^i_{-2}A^j_{-1}|\mathbf{a}\rangle \qquad \text{for } i,j \text{ arbitrary,}$$
$$A^i_{-1}A^j_{-1}A^k_{-1}|\mathbf{a}\rangle \qquad \text{for } i \neq j \neq k \neq i,$$
$$\{A^i_{-3} - A^i_{-1}A^j_{-1}A^j_{-1}\}|\mathbf{a}\rangle \qquad \text{for } i \neq j,$$
$$\{5A^i_{-3} + A^i_{-1}A^i_{-1}A^i_{-1}\}|\mathbf{a}\rangle \qquad \text{for } i \text{ arbitrary,}$$
$$\{A^i_{-3} - A^i_{-1}\mathfrak{L}_{-2}\}|\mathbf{a}\rangle \qquad \text{for } i \text{ arbitrary.} \quad (4.59)$$

Remarkably, this choice is consistent with the above eight commutator equations and their Weyl-rotated analogs thereby proving the viability of our method. Altogether, we get $64 + 2\cdot56 + 2\cdot8 = 192$ states in agreement with the formula (4.11) predicting $\xi(3) = 192$ [30]. Despite the fact that this number coincides with the number of transversal states, our result explicitly shows the appearance of longitudinal as well

as the disappearance of some transversal states. The above states form irreducible representations of the little group, whose action on the polarizations can be determined from (4.49) in a straightforward fashion; in particular, the longitudinal DDF operator is inert under the little Weyl group. We note that the states (4.59) do not even look "$SO(8)$ covariant" any more, unlike the level-one states (4.30).

Having a complete description of the root space $E_{10}^{(\Lambda_7)}$, we can now in principle explore root spaces associated with other level-two roots of the form $\Lambda = \Lambda_7 + n\delta$ (i.e. the **root string** associated with Λ_7) by commuting the states (4.59) with the E_9 elements (4.19). From (4.45) it is evident that all states obtained by acting with a product $A^{i_1}_{-2m_1} \ldots A^{i_M}_{-2m_M}$ on any of the states (4.59) belong to the root space of $\Lambda = \Lambda_7 + (m_1 + \ldots m_M)\delta$ (note that each operator A^i_{-2m} shifts the momentum by $m\delta$!). However, it is also clear that we cannot obtain all root space elements in this way. For this, it is necessary to calculate DDF commutators of the form (4.42). An alternative, more elucidating way might be to consider the action of the Sugawara generators defined by

$$\mathscr{L}_m^{\text{Sug.}} := \frac{1}{2(\ell + h^\vee)} \left\{ \sum_{n \in \mathbb{Z}} {:}A^i_n A^i_{m-n}{:} + \sum_{\mathbf{s} \in \Delta^{\text{real}}(E_9)} {:}\text{ad}_{|\mathbf{s}\rangle}\, \text{ad}_{|-\mathbf{s}-m\delta\rangle}{:} \right\} \tag{4.60}$$

on the states (4.59) (with $A^i_n \equiv A^i_n(\mathbf{r_{-1}})$); here, $h^\vee = 30$ is the dual Coxeter number of E_8, the level is $\ell = 2$, and the normal ordering of the operators $\text{ad}_{|\mathbf{r}\rangle} \equiv (e^{\mathbf{r}})_0$ is chosen as

$$
{:}\text{ad}_{|\mathbf{s}+m\delta\rangle}\, \text{ad}_{|\mathbf{t}+n\delta\rangle}{:} := \begin{cases} \text{ad}_{|\mathbf{s}+m\delta\rangle}\, \text{ad}_{|\mathbf{t}+n\delta\rangle} & \text{if } m \geq n, \\ \text{ad}_{|\mathbf{t}+n\delta\rangle}\, \text{ad}_{|\mathbf{s}+m\delta\rangle} & \text{if } m < n, \end{cases} \tag{4.61}
$$

for E_8 roots $\mathbf{s}, \mathbf{t}$ and $m, n \in \mathbb{Z}$. It is now not difficult to check that

$$\mathscr{L}_m^{\text{Sug.}} |\mathbf{a}\rangle = 0 \tag{4.62}$$

for $m \geq 1$. Furthermore, when evaluating $\mathscr{L}_0^{\text{Sug.}}$ on the ground state $|\mathbf{a}\rangle$, only the term with $A^8_0 A^8_0$ contributes in the sum with our choice of polarization vectors. With $A^8_0 |\mathbf{a}\rangle = -2|\mathbf{a}\rangle$, we thus obtain

$$\mathscr{L}_0^{\text{Sug.}} |\mathbf{a}\rangle = \tfrac{1}{16} |\mathbf{a}\rangle, \tag{4.63}$$

showing that the state $|\mathbf{a}\rangle$ is a highest weight vector of weight $h = \frac{1}{16}$ for the level-two Sugawara generators. In accordance with the remark after Eq. (4.12), we therefore expect these states to belong to the irreducible Virasoro module with $c = \frac{1}{2}$ and $h = \frac{1}{16}$. The problem that remains is to relate the Sugawara generators to the longitudinal DDF operators. If this can be done, a completely explicit description of *all* level-two root spaces is within reach.

A. DDF States and Commutators

We here list the transversal and longitudinal DDF states, required in Sect. 4.4. For the special example discussed there, we must only evaluate them for the following scalar products: $\mathbf{r}^2 = \mathbf{s}^2 = 2$, $\mathbf{k}^2 = 0$, $\mathbf{r}\cdot\mathbf{k} = \mathbf{s}\cdot\mathbf{k} = 1$, $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j = \delta_{ij}$, $\boldsymbol{\eta}_{i'}\cdot\boldsymbol{\eta}_{j'} = \delta_{i'j'}$, $\boldsymbol{\xi}_i\cdot\mathbf{r} = \boldsymbol{\xi}_i\cdot\mathbf{k} = \boldsymbol{\eta}_{i'}\cdot\mathbf{s} = \boldsymbol{\eta}_{i'}\cdot\mathbf{k} = 0$, $\boldsymbol{\xi}_i\cdot\mathbf{s} =: \delta_{is}$, $\boldsymbol{\eta}_{i'}\cdot\mathbf{r} =: \delta_{i'r}$, $\boldsymbol{\eta}_{i'}\cdot\boldsymbol{\xi}_j =: g_{i'j}$. Also put $\epsilon \equiv \epsilon(\mathbf{k}, \mathbf{r})$, $\epsilon' \equiv \epsilon(\mathbf{s}, \mathbf{r})$.

The transversal states are:

$$A^i_{-1}|\mathbf{r}\rangle = \epsilon\boldsymbol{\xi}_i(-1)|\mathbf{r} - \mathbf{k}\rangle, \tag{A.1}$$

$$A^i_{-1}A^j_{-1}|\mathbf{r}\rangle = \left\{\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1) + \tfrac{1}{2}\delta_{ij}\left[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\right]\right\}|\mathbf{r} - 2\mathbf{k}\rangle, \tag{A.2}$$

$$A^i_{-2}|\mathbf{r}\rangle = \left\{\boldsymbol{\xi}_i(-2) - 2\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)\right\}|\mathbf{r} - 2\mathbf{k}\rangle, \tag{A.3}$$

$$A^i_{-1}A^j_{-1}A^k_{-1}|\mathbf{r}\rangle = \epsilon\Big\{\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\boldsymbol{\xi}_k(-1) + \tfrac{1}{2}\big[\delta_{ij}\boldsymbol{\xi}_k(-1) + \delta_{jk}\boldsymbol{\xi}_i(-1)$$
$$+ \delta_{ki}\boldsymbol{\xi}_j(-1)\big]\big[\mathbf{k}(-1)^2 - \mathbf{k}(-2)\big]\Big\}|\mathbf{r} - 3\mathbf{k}\rangle, \tag{A.4}$$

$$A^i_{-2}A^j_{-1}|\mathbf{r}\rangle = \epsilon\Big\{\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_j(-1) - 2\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathbf{k}(-1)$$
$$- \tfrac{2}{3}\delta_{ij}\big[2\mathbf{k}(-1)^3 - 3\mathbf{k}(-2)\mathbf{k}(-1) + \mathbf{k}(-3)\big]\Big\}|\mathbf{r} - 3\mathbf{k}\rangle, \tag{A.5}$$

$$A^i_{-3}|\mathbf{r}\rangle = \epsilon\Big\{\boldsymbol{\xi}_i(-3) - 3\boldsymbol{\xi}_i(-2)\mathbf{k}(-1)$$
$$+ \tfrac{3}{2}\boldsymbol{\xi}_i(-1)\big[3\mathbf{k}(-1)^2 - \mathbf{k}(-2)\big]\Big\}|\mathbf{r} - 3\mathbf{k}\rangle. \tag{A.6}$$

The longitudinal states are:

$$A^-_{-1}|\mathbf{r}\rangle = \epsilon\{-\mathbf{r}(-1) + \mathbf{k}(-1)\}|\mathbf{r} - \mathbf{k}\rangle = -\epsilon L_{(-1)}|\mathbf{r} - \mathbf{k}\rangle, \tag{A.7}$$

$$A^-_{-2}|\mathbf{r}\rangle = \Big\{-\mathbf{r}(-2) + \tfrac{d-6}{4}\mathbf{k}(-2) + 2\mathbf{r}(-1)\mathbf{k}(-1)$$
$$- \tfrac{1}{2}\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-1)^2 + \tfrac{6-d}{4}\mathbf{k}(-1)^2\Big\}|\mathbf{r} - 2\mathbf{k}\rangle, \tag{A.8}$$

$$A^i_{-1}A^-_{-2}|\mathbf{r}\rangle = \epsilon\Big\{-\boldsymbol{\xi}_i(-3) - \mathbf{r}(-2)\boldsymbol{\xi}_i(-1) + 3\boldsymbol{\xi}_i(-2)\mathbf{k}(-1) + \tfrac{d-2}{4}\mathbf{k}(-2)\boldsymbol{\xi}_i(-1)$$
$$+ 2\mathbf{r}(-1)\boldsymbol{\xi}_i(-1)\mathbf{k}(-1) - \tfrac{1}{2}\sum_{j=1}^{d-2}\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)^2$$
$$- \tfrac{d+6}{4}\boldsymbol{\xi}_i(-1)\mathbf{k}(-1)^2\Big\}|\mathbf{r} - 3\mathbf{k}\rangle, \tag{A.9}$$

$$A_{-3}^-|\mathbf{r}\rangle = \epsilon\left\{-\mathbf{r}(-3) + \tfrac{2d-16}{3}\mathbf{k}(-3) + 3\mathbf{r}(-2)\mathbf{k}(-1) - \sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-2)\boldsymbol{\xi}_i(-1)\right.$$

$$+ \tfrac{3}{2}\mathbf{k}(-2)\mathbf{r}(-1) + \tfrac{35-4d}{2}\mathbf{k}(-2)\mathbf{k}(-1) - \tfrac{9}{2}\mathbf{r}(-1)\mathbf{k}(-1)^2$$

$$\left. + 2\sum_{i=1}^{d-2}\boldsymbol{\xi}_i(-1)^2\mathbf{k}(-1) + \tfrac{8d-79}{6}\mathbf{k}(-1)^3\right\}|\mathbf{r}-3\mathbf{k}\rangle. \qquad (A.10)$$

Some commutators for $\mathbf{r}\cdot\mathbf{s} = 0$:

$$\left[|\mathbf{s}\rangle, A_{-1}^i A_{-1}^j|\mathbf{r}\rangle\right] = \epsilon'\left\{\boldsymbol{\xi}_i(-1)\boldsymbol{\xi}_j(-1)\mathbf{s}(-1)\right.$$

$$-\tfrac{1}{2}\left[\delta_{is}\boldsymbol{\xi}_j(-1) + \delta_{js}\boldsymbol{\xi}_i(-1)\right]\left[\mathbf{s}(-1)^2 + \mathbf{s}(-2)\right]$$

$$+\tfrac{1}{6}\delta_{is}\delta_{js}\left[\mathbf{s}(-1)^3 + 3\mathbf{s}(-1)\mathbf{s}(-2) + 2\mathbf{s}(-3)\right]$$

$$+\tfrac{1}{6}\delta_{ij}\left[\mathbf{s}(-1)^3 - 3\mathbf{s}(-1)^2\mathbf{k}(-1) + 3\mathbf{s}(-1)\mathbf{k}(-1)^2\right.$$

$$-3\mathbf{s}(-1)\mathbf{k}(-2) + 3\mathbf{s}(-1)\mathbf{s}(-2)$$

$$\left.\left. -3\mathbf{s}(-2)\mathbf{k}(-1) + 2\mathbf{s}(-3)\right]\right\}|\mathbf{r}-2\mathbf{k}+\mathbf{s}\rangle, \qquad (A.11)$$

$$\left[|\mathbf{s}\rangle, A_{-2}^i|\mathbf{r}\rangle\right] = \epsilon'\left\{\boldsymbol{\xi}_i(-2)\mathbf{s}(-1) + \boldsymbol{\xi}_i(-1)\left[\mathbf{s}(-1)^2 - 2\mathbf{s}(-1)\mathbf{k}(-1) + \mathbf{s}(-2)\right]\right.$$

$$-\tfrac{1}{2}\delta_{is}\left[\mathbf{s}(-1)^3 - 2\mathbf{s}(-1)^2\mathbf{k}(-1) + 3\mathbf{s}(-1)\mathbf{s}(-2)\right.$$

$$\left.\left. -2\mathbf{k}(-1)\mathbf{s}(-2) + 2\mathbf{s}(-3)\right]\right\}|\mathbf{r}-2\mathbf{k}+\mathbf{s}\rangle, \qquad (A.12)$$

$$\left[A_{-1}^{i'}|\mathbf{s}\rangle, A_{-1}^j|\mathbf{r}\rangle\right] = \epsilon'\left\{\delta_{js}\boldsymbol{\eta}_{i'}(-3) - \boldsymbol{\eta}_{i'}(-2)\boldsymbol{\xi}_j(-1) + \delta_{js}\boldsymbol{\eta}_{i'}(-2)\left[\mathbf{s}(-1) - \mathbf{k}(-1)\right]\right.$$

$$-\boldsymbol{\eta}_{i'}(-1)\boldsymbol{\xi}_j(-1)\left[\mathbf{s}(-1) - \mathbf{k}(-1)\right]$$

$$-\tfrac{1}{2}\left[\delta_{i'\mathbf{r}}\boldsymbol{\xi}_j(-1) - \delta_{js}\boldsymbol{\eta}_{i'}(-1)\right]\left[\mathbf{s}(-1)^2 - 2\mathbf{s}(-1)\mathbf{k}(-1)\right.$$

$$\left. + \mathbf{k}(-1)^2 + \mathbf{s}(-2) - \mathbf{k}(-2)\right]$$

$$+\tfrac{1}{6}\left[\delta_{i'\mathbf{r}}\delta_{js} - g_{i'j}\right]\left[\mathbf{s}(-1)^3 - 3\mathbf{s}(-1)^2\mathbf{k}(-1)\right.$$

$$+ 3\mathbf{s}(-1)\mathbf{k}(-1)^2 - \mathbf{k}(-1)^3 + 3\mathbf{s}(-2)\mathbf{s}(-1)$$

$$-3\mathbf{s}(-2)\mathbf{k}(-1) - 3\mathbf{k}(-2)\mathbf{s}(-1) + 3\mathbf{k}(-2)\mathbf{k}(-1)$$

$$\left.\left. + 2\mathbf{s}(-3) - 2\mathbf{k}(-3)\right]\right\}|\mathbf{r}-2\mathbf{k}+\mathbf{s}\rangle. \qquad (A.13)$$

The following commutators are written in terms of the basis (3.47) rather than (3.38) as in the main text. Because the contributions $L_{(-1)}(\ldots)$ are needed to find the correct results, we list them here. We put $\epsilon \equiv \epsilon(\mathbf{s},\mathbf{r})\epsilon(\mathbf{k},\mathbf{a})$ and stress again that we are

dealing with different sets of DDF operators depending on the tachyon momenta they are acting on (see remarks before Eq. (4.50)).

$$
\left[|\mathbf{s}\rangle, A^{\alpha}_{-1} A^{\beta}_{-1} |\mathbf{r}\rangle\right] =
$$
$$
\epsilon\Bigg\{ -\tfrac{1}{2} A^{\alpha}_{-2} A^{\beta}_{-1} - \tfrac{1}{2} A^{\beta}_{-2} A^{\alpha}_{-1} - A^{\alpha}_{-1} A^{\beta}_{-1} A^{8}_{-1}
$$
$$
+ \delta^{\alpha\beta} \Big[\tfrac{1}{24} A^{8}_{-3} + \tfrac{1}{8} A^{8}_{-1} A^{-}_{-2} + \tfrac{1}{16} \sum_{\gamma=1}^{7} A^{8}_{-1} A^{\gamma}_{-1} A^{\gamma}_{-1} - \tfrac{5}{48} A^{8}_{-1} A^{8}_{-1} A^{8}_{-1} \Big] \Bigg\} |\mathbf{a}\rangle
$$
$$
+ \mathrm{L}_{(-1)} \Big\{ \tfrac{1}{2} \boldsymbol{\xi}_{\alpha}(-1) \boldsymbol{\xi}_{\beta}(-1) + \delta^{\alpha\beta} \big[-\tfrac{1}{8} \boldsymbol{\xi}_8(-2) + \tfrac{1}{12} \boldsymbol{\Lambda}(-2) + \tfrac{1}{4} \boldsymbol{\xi}_8(-1)^2
$$
$$
+ \tfrac{1}{48} \boldsymbol{\Lambda}(-1)^2 - \tfrac{1}{8} \boldsymbol{\xi}_8(-1)[\boldsymbol{\Lambda}(-1) - \boldsymbol{\delta}(-1)] \big] \Big\} |\boldsymbol{\Lambda}_7\rangle, \qquad (A.14)
$$

$$
\left[|\mathbf{s}\rangle, A^{\alpha}_{-1} A^{8}_{-1} |\mathbf{r}\rangle\right] =
$$
$$
\epsilon\Bigg\{ \tfrac{1}{4} A^{\alpha}_{-3} + \tfrac{1}{2} A^{\alpha}_{-2} A^{8}_{-1} - \tfrac{1}{2} A^{8}_{-2} A^{\alpha}_{-1} - \tfrac{1}{4} A^{\alpha}_{-1} A^{-}_{-2}
$$
$$
- \tfrac{1}{8} \sum_{\gamma=1}^{7} A^{\alpha}_{-1} A^{\gamma}_{-1} A^{\gamma}_{-1} - \tfrac{1}{8} A^{\alpha}_{-1} A^{8}_{-1} A^{8}_{-1} \Bigg\} |\mathbf{a}\rangle
$$
$$
+ \mathrm{L}_{(-1)} \Big\{ -\tfrac{1}{4} \boldsymbol{\xi}_{\alpha}(-2) - \tfrac{1}{4} \boldsymbol{\xi}_{\alpha}(-1)[2\boldsymbol{\xi}_8(-1) - \boldsymbol{\Lambda}(-1) + 3\boldsymbol{\delta}(-1)] \Big\} |\boldsymbol{\Lambda}_7\rangle, \qquad (A.15)
$$

$$
\left[|\mathbf{s}\rangle, A^{8}_{-1} A^{8}_{-1} |\mathbf{r}\rangle\right] =
$$
$$
\epsilon\Bigg\{ \tfrac{17}{24} A^{8}_{-3} + A^{8}_{-2} A^{8}_{-1} + \tfrac{1}{8} A^{8}_{-1} A^{-}_{-2} + \tfrac{1}{16} \sum_{\gamma=1}^{7} A^{8}_{-1} A^{\gamma}_{-1} A^{\gamma}_{-1} + \tfrac{11}{48} A^{8}_{-1} A^{8}_{-1} A^{8}_{-1} \Bigg\} |\mathbf{a}\rangle
$$
$$
+ \mathrm{L}_{(-1)} \Big\{ -\tfrac{9}{8} \boldsymbol{\xi}_8(-2) + \tfrac{5}{12} \boldsymbol{\Lambda}(-2) - \boldsymbol{\delta}(-2) - \tfrac{1}{4} \boldsymbol{\xi}_8(-1)^2 + \tfrac{5}{48} \boldsymbol{\Lambda}(-1)^2
$$
$$
- \tfrac{1}{2} \boldsymbol{\delta}(-1)^2 - \tfrac{1}{8} \boldsymbol{\xi}_8(-1)[\boldsymbol{\Lambda}(-1) + 7\boldsymbol{\delta}(-1)] \Big\} |\boldsymbol{\Lambda}_7\rangle, \qquad (A.16)
$$

$$
\left[|\mathbf{s}\rangle, A^{\alpha}_{-2} |\mathbf{r}\rangle\right] =
$$
$$
\epsilon\Bigg\{ -\tfrac{3}{4} A^{\alpha}_{-3} - \tfrac{1}{4} A^{\alpha}_{-1} A^{-}_{-2} - \tfrac{1}{8} \sum_{\gamma=1}^{7} A^{\alpha}_{-1} A^{\gamma}_{-1} A^{\gamma}_{-1} + \tfrac{7}{8} A^{\alpha}_{-1} A^{8}_{-1} A^{8}_{-1} \Bigg\} |\mathbf{a}\rangle
$$
$$
+ \mathrm{L}_{(-1)} \Big\{ \tfrac{1}{4} \boldsymbol{\xi}_{\alpha}(-2) - \tfrac{1}{4} \boldsymbol{\xi}_{\alpha}(-1)[4\boldsymbol{\xi}_8(-1) - \boldsymbol{\Lambda}(-1) + \boldsymbol{\delta}(-1)] \Big\} |\boldsymbol{\Lambda}_7\rangle, \qquad (A.17)
$$

$$
\left[|\mathbf{s}\rangle, A^{8}_{-2} |\mathbf{r}\rangle\right] =
$$
$$
\epsilon\Bigg\{ -\tfrac{1}{2} A^{8}_{-3} + \tfrac{1}{2} A^{8}_{-1} A^{-}_{-2} + \tfrac{1}{4} \sum_{\gamma=1}^{7} A^{8}_{-1} A^{\gamma}_{-1} A^{\gamma}_{-1} + \tfrac{1}{4} A^{8}_{-1} A^{8}_{-1} A^{8}_{-1} \Bigg\} |\mathbf{a}\rangle
$$
$$
+ \mathrm{L}_{(-1)} \Big\{ -\tfrac{1}{2} \boldsymbol{\xi}_8(-2) + \tfrac{1}{2} \boldsymbol{\Lambda}(-2) - \boldsymbol{\delta}(-2) + \tfrac{1}{2} \boldsymbol{\xi}_8(-1)^2 + \tfrac{1}{8} \boldsymbol{\Lambda}(-1)^2
$$
$$
- \tfrac{1}{2} \boldsymbol{\delta}(-1)^2 - \tfrac{1}{2} \boldsymbol{\xi}_8(-1)[\boldsymbol{\Lambda}(-1) - \boldsymbol{\delta}(-1)] \Big\} |\boldsymbol{\Lambda}_7\rangle, \qquad (A.18)
$$

$$\left[A^\alpha_{-1}|\mathbf{s}\rangle, A^\beta_{-1}|\mathbf{r}\rangle\right] =$$

$$\epsilon\Bigg\{ -\tfrac{1}{2}A^\alpha_{-2}A^\beta_{-1} + \tfrac{1}{2}A^\beta_{-2}A^\alpha_{-1} + A^\alpha_{-1}A^\beta_{-1}A^8_{-1}$$

$$-\delta^{\alpha\beta}\bigg[\tfrac{1}{24}A^8_{-3} + \tfrac{1}{8}A^8_{-1}A^-_{-2} + \tfrac{1}{16}\sum_{\gamma=1}^{7}A^8_{-1}A^\gamma_{-1}A^\gamma_{-1} - \tfrac{5}{48}A^8_{-1}A^8_{-1}A^8_{-1}\bigg]\Bigg\}|\mathbf{a}\rangle$$

$$+\,L_{(-1)}\Bigg\{ -\tfrac{1}{2}\xi_\alpha(-1)\xi_\beta(-1) - \delta^{\alpha\beta}\bigg[-\tfrac{1}{8}\xi_8(-2) + \tfrac{1}{12}\Lambda(-2) + \tfrac{1}{4}\xi_8(-1)^2$$

$$+\tfrac{1}{48}\Lambda(-1)^2 - \tfrac{1}{8}\xi_8(-1)[\Lambda(-1) - \delta(-1)]\bigg]\Bigg\}|\Lambda_7\rangle, \qquad (A.19)$$

$$\left[A^\alpha_{-1}|\mathbf{s}\rangle, A^8_{-1}|\mathbf{r}\rangle\right] =$$

$$\epsilon\Bigg\{ -\tfrac{1}{4}A^\alpha_{-3} + \tfrac{1}{2}A^\alpha_{-2}A^8_{-1} + \tfrac{1}{2}A^8_{-2}A^\alpha_{-1} + \tfrac{1}{4}A^\alpha_{-1}A^-_{-2}$$

$$+\tfrac{1}{8}\sum_{\gamma=1}^{7}A^\alpha_{-1}A^\gamma_{-1}A^\gamma_{-1} + \tfrac{1}{8}A^\alpha_{-1}A^8_{-1}A^8_{-1}\Bigg\}|\mathbf{a}\rangle$$

$$+\,L_{(-1)}\Bigg\{ -\tfrac{3}{4}\xi_\alpha(-2) - \tfrac{1}{4}\xi_\alpha(-1)[-2\xi_8(-1) + \Lambda(-1) + \delta(-1)]\Bigg\}|\Lambda_7\rangle, \quad (A.20)$$

$$\left[A^8_{-1}|\mathbf{s}\rangle, A^8_{-1}|\mathbf{r}\rangle\right] =$$

$$\epsilon\Bigg\{ \tfrac{17}{24}A^8_{-3} + \tfrac{1}{8}A^8_{-1}A^-_{-2} + \tfrac{1}{16}\sum_{\gamma=1}^{7}A^8_{-1}A^\gamma_{-1}A^\gamma_{-1} + \tfrac{11}{48}A^8_{-1}A^8_{-1}A^8_{-1}\Bigg\}|\mathbf{a}\rangle$$

$$+\,L_{(-1)}\Bigg\{ -\tfrac{9}{8}\xi_8(-2) + \tfrac{5}{12}\Lambda(-2) - \delta(-2) - \tfrac{1}{4}\xi_8(-1)^2 + \tfrac{5}{48}\Lambda(-1)^2$$

$$-\tfrac{1}{2}\delta(-1)^2 - \tfrac{1}{8}\xi_8(-1)[\Lambda(-1) - \delta(-1)]\Bigg\}|\Lambda_7\rangle. \qquad (A.21)$$

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