

Bispectral KP Solutions and Linearization of Calogero–Moser Particle Systems

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Abstract: Rational and soliton solutions of the KP hierarchy in the subgrassmannian Gr_1 are studied within the context of finite dimensional dual grassmannians. In the rational case, properties of the tau function, τ , which are equivalent to bispectrality of the associated wave function, ψ , are identified. In particular, it is shown that there exists a bound on the degree of all time variables in τ if and only if ψ is a rank one bispectral wave function. The action of the bispectral involution, β , in the generic rational case is determined explicitly in terms of dual grassmannian parameters. Using the correspondence between rational solutions and particle systems, it is demonstrated that β is a linearizing map of the Calogero-Moser particle system and is essentially the map σ introduced by Airault, McKean and Moser in 1977 [2].

1. Introduction

Among the surprises in the history of rational solutions of the KP hierarchy (and the PDE's which make it up) are the existence of rational initial conditions to a non-linear evolution equation which remain rational for all time [1, 2], that these solutions are related to completely integrable systems of particles [2, 6, 7], and that a large class of wave functions which have been found to have the bispectral property turn out to be associated with potentials that are rational KP solutions [3, 16, 17]. Within the grassmannian which is used to study the KP hierarchy, the rational solutions, along with the N -soliton solutions, reside in the subgrassmannian Gr_1 [13]. This paper develops a general framework of finite dimensional grassmannians for studying the KP solutions in Gr_1 and then applies this to the bispectral rational solutions. New results include information about the geometry of KP orbits in Gr_1 and identification of properties equivalent to bispectrality. In addition, an explicit description of the bispectral involution in terms of dual grassmannian coordinates leads to the conclusion that it is, in fact, essentially the linearizing map σ [2].

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Outline. A brief review of previous results which will be necessary to understand the results of this paper follows in Sect. 1.1. For details, please refer to [8, 12] for a review of the theory of spectral curves, [10] for the algebraic theory of the KP hierarchy and to [13] for the analytic theory of the grassmannian which will be used in this paper.

Section 2 develops the technique of dual grassmannians for studying the Gr_1 solutions of the KP hierarchy. This construction takes advantage of the fact that the orbits of subspaces in Gr_1 are infinite dimensional subspaces in which each point has a finite dimensional complement. Consequently, the dual grassmannians are finite dimensional and simpler to work with. The decomposition of the dual grassmannians into the disjoint union of generalized jacobians and the representation of their Plücker coordinates as a τ function are discussed.

In Sect. 3, the technique of the dual grassmannian is utilized to study bispectrality of the wave function. Using results of Wilson [16] which completely identified the bispectral wave functions in Gr_1 , a set of conditions on W or τ_W which are equivalent to bispectrality of ψ_W are listed (Theorem 3.3). In particular, it is shown that for $W \in Gr_1$, the wave function ψ_W is bispectral if and only if τ_W is of *bounded degree* in all time variables.

The bispectral involution introduced by Wilson [16] is studied in Sect. 4. It is noted that the composition of the bispectral involution with the KP flows results in a non-isospectral flow under which the singular points of the spectral curve move as a Calogero–Moser particle system. The action of the involution is determined explicitly (Theorem 4.2) in terms of dual grassmannian parameters in the generic case that the spectral curve of the image has only simple cusps. Using the correspondence between rational solutions and particle systems, the bispectral involution is determined as an involution on Calogero–Moser particle systems (Theorem 4.1), and it is shown (Theorem 4.3) that it acts as a linearizing map. The bispectral involution acting on particle systems, after simple rescaling, is seen (Theorem 4.4) to restrict to the linearizing map σ introduced in [2] and thus explains the surprising involutive nature of σ in terms of bispectrality. The paper concludes with two examples. The first example demonstrates how the bispectral involution can be used to determine the τ -function of a rational KP solution. Finally, the second example demonstrates Theorem 4.2 by determining the condition space of the image under the bispectral involution of a particular rational solution of the form given by Wilson [16] in Example 10.3.

1.1. Review

Rational Solutions to the KP Equation. For the purposes of this paper, the term “rational solution of the KP Equation” will mean a function $u(x, y, t)$ which is rational in the variable x , satisfies the KP equation

$$\frac{3}{4}u_{yy} = \left(u_t - \frac{1}{4}(6uu_x + u_{xxx}) \right)_x$$

and has the property that $\lim_{|x| \rightarrow \infty} u = 0$. Although there do exist rational solutions which do not have the latter property [4, 15], these will only be referred to explicitly as “non-vanishing rational solutions.”

Along with the ideas of the algebro-geometric construction [8], the correspondence between poles of rational solutions and the Calogero–Moser system [2] is utilized by Krichever [6, 7] to find rational solutions to the KP equation. The motion of the poles is shown to be equivalent to the motion of Calogero–Moser particles where the y -flow corresponds to the second Hamiltonian and the t -flow corresponds to the third. Furthermore, this paper completely identified the rational solutions of the KP equation.

The following results from [7] will be important in the present paper.

Theorem 1.1. (Krichever). *A function $u(x, y, t)$ which is rational in x and decreases for $|x| \rightarrow \infty$ is a solution of the KP equation if and only if*

$$u(x, y, t) = \sum_{j=1}^N \frac{-2}{(x - x_j(y, t))^2}$$

and there exists a function

$$(B) \quad \psi(x, y, t, z) = \left(1 + \sum_{j=1}^N \frac{P_j(y, t, z)}{(x - x_j(y, t))} \right) e^{xz + yz^2 + tz^3}$$

such that

$$\begin{aligned} L_1\psi &= \frac{\partial}{\partial y}\psi, & L_2\psi &= \frac{\partial}{\partial t}\psi, \\ L_1 &= \frac{\partial^2}{\partial x^2} + u(x, y, t), & L_2 &= \frac{\partial^3}{\partial x^3} + \frac{3}{2}u \frac{\partial}{\partial x} + w(x, y, t), \\ w(x, y, t) &= \sum_{j=1}^N \left[3(x - x_j)^{-3} + \frac{3}{2}(x - x_j)^{-2} \frac{\partial}{\partial y} x_j \right]. \end{aligned}$$

In general, a wave function ψ corresponding to the solution u is any function of the form

$$(1 + \alpha_1(x, y, t)z^{-1} + \alpha_2(x, y, t)z^{-2} + \dots) e^{xz + yz^2 + tz^3}$$

which satisfies $L_1\psi = \frac{\partial}{\partial y}\psi$ and $L_2\psi = \frac{\partial}{\partial t}\psi$ for L_1, L_2 and u as in Theorem 1.1. Multiplication by any series of the form $1 + c_1z^{-1} + c_2z^{-2} + \dots$ (with $c_i \in \mathbb{C}$) will take one wave function for u to another wave function for the same u . Since a series of this form can be seen to alter ψ but not the associated solution u , they will be referred to as *gauge transformations* of the wave function. As will be shown below by Corollary 3.1, there is a unique wave function of the form (B) corresponding to each rational solution. This form of the wave function is the *bispectral gauge* and can be identified by the fact that $\lim_{x \rightarrow \infty} \exp(-xz - yz^2 - tz^3)\psi = 1$.

A method for generating almost every rational solution of the KP equation is developed (also in [7]) utilizing *another* gauge of the wave function, which is identified by the fact that $\exp(-xz - yz^2 - tz^3)\psi|_{x=0} = 1$. In this gauge, denoted (Kr) , there exist N distinct numbers λ_i , such that $\frac{\partial}{\partial z}\psi|_{z=\lambda_i} = 0$. Consequently, it is clear that the spectral curve in this case is a rational curve with singularities only in the form of simple cusps at the points $z = \lambda_i$. In the standard algebro-geometric construction of solutions to integrable equations with a *non-singular* spectral curve, the wave function ψ is chosen to be an *Akhiezer function*. That is, it is chosen so as to be the unique wave function which is holomorphic off of a given non-special

divisor and specified point. The wave function in the gauge (Kr) is the limit of the Akhiezer functions in the cuspidal case. It is for its relevance to the algebro-geometric construction that this gauge is introduced here, although it is the bispectral gauge that will be utilized in the proofs to follow.

The wave function of a rational KP solution can be written in the form

$$\psi(x, y, t, z) = \left(1 + \frac{p(x, y, t, z)}{q(z)} \right) e^{xz + yz^2 + tz^3},$$

where q is a monic polynomial of degree N and p is a polynomial of degree at most $N - 1$ in z . There is a unique such function in the gauge (Kr) for any choice of the λ_i and the coefficients of q and thus these parameters determine a KP solution u . After specifying λ_i and q , the problem of finding ψ reduces to a problem of linear algebra. The matrix, Θ , which arises in this problem leads to a solution to the KP equation as shown in this theorem.

Theorem 1.2. (Krichever) *For almost all solutions of the KP equation, depending rationally on x and decreasing for $|x| \rightarrow \infty$, we have the formula*

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \det \Theta,$$

where the matrix elements Θ_{ij} are given by²

$$\Theta_{ij} = \frac{\partial}{\partial z} \frac{z^{j-1} e^{xz + yz^2 + tz^3}}{q(z)} \Bigg|_{z=\lambda_i} e^{-(x\lambda_i + y\lambda_i^2 + t\lambda_i^3)} \quad 1 \leq i, j \leq N.$$

Since the elements of Θ are all linear in x, y and t , letting $\vartheta = \det \Theta$ we have

Corollary 1.3. *Almost every rational solution can be expressed as $2 \frac{\partial^2}{\partial x^2} \log \vartheta$ where ϑ is a polynomial in x, y and t .*

Note. Those solutions which can be determined explicitly from Theorem 1.2 are those for which all of the cusps are *simple*. Rational solutions whose spectral curves have higher cusps can be determined as limits of these solutions.

The KP Hierarchy and Sato's Grassmannian. The KP hierarchy is defined in terms of pseudo-differential operators in the variable x . Given a pseudo-differential operator $L = \sum_{l=-\infty}^1 a_l(x) \partial^l$, where $\partial = \frac{\partial}{\partial x}$ and $a_1(x) = 1$, one may define the flows of the KP hierarchy as the compatibility of the conditions

$$(KP) \quad L\psi = z\psi \quad \frac{\partial}{\partial t_i} \psi = (L^i)_+ \psi$$

with $(K)_+$ denoting the projection of K onto ordinary differential operators. Here and throughout the paper, I will use the notation $x = t_1, y = t_2$ and $t = t_3$.

The geometry of an infinite dimensional grassmannian, Gr , is utilized to study the dynamical system of pseudo-differential operators given above. This is a grassmannian of the Hilbert space H which is spanned by all integer powers of the

²Whereas the matrix Θ in [7] was indexed from 0 to $N - 1$, I have altered the notation to be compatible with the matrix \mathcal{M} below. In addition, the substitution $t \rightarrow -t$ is understood when comparing the present paper to [7].

variable z . Thus, a point $W \in Gr$ is a subspace of H (satisfying certain conditions which will not be emphasized here). As will be seen below, the action of a certain multiplicative group on Gr results in the KP flow of the associated operators.

The association between a point W and its corresponding pseudo-differential operator, L_W , is described most easily in terms of certain intermediate objects. Associated to a point $W \in Gr$ is the stationary wave function $\psi_W(x, z)$ which is the unique function of the form

$$\psi_W(x, z) = (1 + \alpha_1(x)z^{-1} + \alpha_2(x)z^{-2} \dots) e^{xz}$$

that is contained in W for each fixed value x in its domain. This function can also be written as $K_W e^{xz}$ for some monic, zero order, pseudo-differential operator K_W . Finally, we have the associated pseudo-differential operator $L_W = K_W \hat{\partial} (K_W)^{-1}$. Also frequently associated to W are the time dependent version ψ_W given by the equations (KP), the tau function τ_W (a “bosonic” representation of the Plücker coordinates of W), and the function $u_W(t_1, t_2, \dots)$ which solves the original KP equation. Depending on the context, any of these associated objects could be considered a “solution” of the KP hierarchy. Again, for the purposes of this paper, a rational solution of the KP hierarchy is a solution such that the function u_W is a rational solution of the KP equation in the sense discussed earlier. Although there is a one to one correspondence between points $W \in Gr$ and the objects τ_W, K_W or ψ_W , the map to L_W and u_W is many to one.

The association of a τ -function to a point of the grassmannian comes through the representation of Gr on $\mathbb{P}(\mathbb{C}[[t_i]])$. It is well known that the set of Schur polynomials form a basis over \mathbb{C} of $\mathbb{C}[[t_i]]$. In the representation of Gr , a Schur polynomial is associated in the standard way to each of the points of Gr where all but one of the Plücker coordinates are zero. Let W be an arbitrary point of Gr , and let τ_W be the infinite series written as a sum of the Schur functions with the corresponding Plücker coordinates as the coefficients. That is, if $\pi_\nu \in \mathbb{C}$ are the Plücker coordinates of a point $W \in Gr$, then

$$\tau_W = \sum_\nu \pi_\nu S_\nu(t_i),$$

where S_ν is the Schur polynomial associated to π_ν . The function $u(t_i) = 2 \frac{\partial^2}{\partial x^2} \log \tau_W$ is then a solution to the KP equation. Since it is exactly those coefficients which are valid Plücker coordinates that yield solutions, the KP hierarchy can be viewed as the Plücker relations of the grassmannian.

Two groups of interest to this paper act on Gr via multiplication. The group Γ_+ consists of all real-analytic functions $f : S^1 \rightarrow \mathbb{C}^\times$ which extend to holomorphic functions $f : D_0 \rightarrow \mathbb{C}^\times$ in the disc $D_0 = \{z \in \mathbb{C} : |z| \leq 1\}$ satisfying $f(0) = 1$. The other group, Γ_- consists of functions f which extend to non-vanishing holomorphic functions in $D_\infty = \{z \in \mathbb{C} \cup \infty : |z| \geq 1\}$ satisfying $f(\infty) = 1$. If $g = e^{-cz'} \in \Gamma_+$ and $W \in Gr$, then

$$\tau_{gW}(t_1, t_2, \dots, t_i, \dots) = \tau_W(t_1, t_2, \dots, t_i + c, \dots).$$

Thus, the KP flows are given by the action of the group Γ_+ . However, the group Γ_- , since it contains only elements of the form $1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} \dots$, is again a group of gauge transformations which alter the point, W , but not the associated u_W or L_W .

The solution u_W corresponding to a point which only has a *finite* number of non-zero Plücker coordinates is going to be a rational solution since τ_W (now a *finite* sum of Schur polynomials) will be a polynomial. The subset of Gr for which this is true is called Gr_0 , and it corresponds to rational solutions whose spectral curve is a rational curve with only one cusp, at the point $z = 0$. It therefore generates some of Krichever’s rational solutions, and all of the rational KdV solutions. This is contained in a larger subset, Gr_1 , which is characterized by the fact that if $W \in Gr_1$, then there are polynomials $p(z)$ and $q(z)$ such that $pH_+ \subset W \subset q^{-1}H_+$. (Here, H_+ denotes the subspace of H spanned by non-negative powers of z .) The points of Gr_1 correspond to rank one solutions with rational spectral curves and the converse is true if the data are suitably normalized [13 (Proposition 7.1)]. Thus, all of Krichever’s rational solutions can be derived from this sub-grassmannian. However, not every solution from Gr_1 is rational since it also includes the nodal rational curves which are associated with solitons³. Only solutions in Gr_1 will be considered for the remainder of the paper.

It was shown in [9] that the orbit of a pseudo-differential operator L which is a solution of the KP hierarchy and can be determined by an isospectral flow of line bundles is isomorphic to the jacobian variety of the spectral curve. Thus, for a point W which is determined by a line bundle over a curve, the orbit under Γ_+ modulo the action of Γ_- is isomorphic to the jacobian of the curve. In general, there is no reason to expect that the orbit in Gr will be a jacobian prior to taking the quotient by Γ_- . The next section will study the Γ_+ orbits of points in Gr_1 and their relationship to the geometry of the grassmannian.

2. Dual Grassmannians and Rational KP Solutions

2.1. Differential Conditions and Gr_1 . Every subspace $W \in Gr_1$ can be derived from a line bundle over a singular rational curve [13]. As will be explained below, it can also be written as the closure in H of the set of polynomials in z satisfying a finite number of differential conditions at a finite number of points divided by a polynomial q [16]. By forming the grassmannian of the linear space of such conditions, one is able to construct finite dimensional grassmannians dual to those in Gr_1 .

Definitions. Let $d(l, \lambda)$ denote the linear functional on the space $\mathbb{C}[z]$ which takes $f(z)$ to $f^{(l)}(\lambda)$, the l^{th} derivative evaluated at λ . Then let \mathcal{C} be the infinite dimensional vector space over \mathbb{C} generated by $d(l, \lambda)$ for all $l \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$. Let $\mathcal{C}(\lambda) \subset \mathcal{C}$ for $\lambda \in \mathbb{C}$ be the subspace spanned by $d(l, \lambda)$ for all $l \in \mathbb{N}$. Stratify these subsets into $\mathcal{C}(l, \lambda) \subset \mathcal{C}(\lambda)$ which is the subspace spanned by $d(\alpha, \lambda)$ for $0 \leq \alpha \leq l - 1$. (For convenience, we define $\mathcal{C}(0, \lambda)$ to be the empty set.)

Let C be any M dimensional subspace of \mathcal{C} . Here I will recall the mapping which associates a point of Gr_1 to C [16]. Let V_C be the vector space of polynomials in z which satisfy the condition $c(f) = 0$ for each $c \in C$. Finally, picking a monic polynomial, $q(z)$, of degree M the Hilbert closure of the set $q^{-1}V_C$ in H is a point $W \in Gr_1$.

³Although the rational solutions are technically soliton solutions as well, within this paper I will refer only to non-rational solutions as solitons.

This suggests the following definition. Let $Gr(\mathcal{C})$ be the set of finite dimensional linear subspaces of \mathcal{C} . Then letting

$$Gr_1^* = \left\{ (C, q) \in Gr(\mathcal{C}) \times \mathbb{C}[z] \mid \dim C = M \text{ and } q = z^M + \sum_{i=0}^{M-1} c_i z^i \right\}$$

allows us to associate a point $W \in Gr_1$ to each $W^* = (C, q) \in Gr_1^*$ by the *dual mapping* described above.⁴ Although this mapping is onto (since every element of Gr_1 can be expressed in this way) it is not injective. Note, for example, that $(\{d(0, 0)\}, z)$ and $(\{d(0, 0), d(1, 0)\}, z^2)$ both get sent to the vacuum solution $H_+ \in Gr_1$.

The following lemma demonstrates that the choice of a polynomial q only alters the gauge of the associated KP solution. Consequently, for most applications it will be sufficient to consider only the case $q = z^M$, where M is the dimension of C . Only later when bispectrality is being considered will gauge become significant.

Lemma 2.1. *Varying the choice of q merely affects the associated solution by a gauge transformation.*

Proof. Let $W_1^* = (C, q_1)$ and $W_2^* = (C, q_2)$ be two points of Gr_1^* with the same condition space C and let $W_i \in Gr_1$ be the image of W_i^* under the dual mapping. It is clear from the definition of the mapping that the gauge transformation $q_1/q_2 \in \Gamma_-$ will take W_1 to W_2 . \square

For any point of Gr_1^* , given by the condition space $C = \{c_i\}$ with basis $\{c_1, c_2, \dots, c_M\}$ and $q = z^M$, it is simple to calculate the corresponding τ . The definition of τ in [13] is equivalent to $\tau = \det \mathcal{M}$ [16], where \mathcal{M} is the $M \times M$ matrix

$$\mathcal{M}_{ij} = c_i(z^{j-1} e^{\sum_{t=1}^j z^t}) \quad 1 \leq i, j \leq M.$$

(This determinant can be viewed as the Wronskian of the M one condition solutions given by the individual c_i 's.) Notice that another choice of basis for C only affects τ by a constant multiple.

2.2. The Finite Grassmannians. Recall that the grassmannian $Gr(M, N)$, which is made up of M dimensional subspaces of an N dimensional vector space V , is isomorphic to the grassmannian $Gr(N - M, N)$. This isomorphism follows from the Principle of Duality [5], since such an isomorphism is given by sending $W \in Gr(M, N)$ to the $N - M$ dimensional subspace $W^* \subset V^*$ such that $w^*(w) = 0$ for all $w \in W$ and $w^* \in W^*$. Consequently, $Gr(N - M, N)$ is referred to as the dual grassmannian of $Gr(M, N)$. This section will demonstrate that the Γ_+ orbit of any point W^* is contained in a finite dimensional sub-grassmannian of Gr_1^* . These finite dimensional grassmannians are actually the dual grassmannians of the sub-grassmannians of Gr_1 which are their image under the dual mapping (thereby

⁴Technically, the image of certain points in Gr_1^* are not contained in Gr as formulated by Segal and Wilson. In particular, if $q(\zeta) = 0$ for some $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$, then the corresponding subspace may not be contained in $L^2(S^1, z)$ unless it undergoes a scaling transformation. Without altering the results of the present paper, one may resolve this problem either by restricting the definition of Gr_1^* to contain only points whose image is in Gr or by extending the definition of Gr_1 to contain subspaces which would be in Gr after rescaling.

justifying the terminology). In addition, it will be seen that they decompose into a disjoint union of Γ_+ orbits.

Let $\mu : \mathbb{C} \rightarrow \mathbb{N}$ be called a *singularity bounding function* (or simply bounding function) if the set $\mathbb{C} - \mu^{-1}(0)$ is a finite set of points. This finite set on which μ is non-zero is called the support of μ ($\text{supp } \mu$). Define the dimension of μ by $\dim \mu = \sum_{\lambda \in \mathbb{C}} \mu(\lambda)$.

Given a bounding function, μ , let

$$\mathcal{C}(\mu) = \bigoplus_{\lambda \in \mathbb{C}} \mathcal{C}(\mu(\lambda), \lambda).$$

Then $\mathcal{C}(\mu)$ is a $\dim \mu$ dimensional vector space. For any $M < \dim \mu$, the set of M dimensional subspaces of $\mathcal{C}(\mu)$ is then a well defined grassmannian: $Gr(M, \mathcal{C}(\mu))$. In order to consider the images under the dual mapping of points in this grassmannian we must specify a choice of q . We will denote this by $Gr_q(M, \mathcal{C}(\mu))$ and assume that $q = z^M$ when none is specified. The dual mapping from $Gr_q(M, \mathcal{C}(\mu))$ to Gr_1 is injective. (In fact, once the choice of M and q is fixed, the dual mapping is injective.)

Given a point $W^* = (C, z^M) \in Gr(M, \mathcal{C}(\mu))$, consider the form of the associated point $W \in Gr_1$. Note first that the polynomial $p(n, z) = z^n \prod (z - \lambda)^{\mu(\lambda)}$ will be in V_C for any $n \in \mathbb{N}$. Thus, $p(0, z)H_+ \subset W \subset z^{-M}H_+$ demonstrates that $W \in Gr_1$ according to the definition in [13]. Then W will have this form:

$$W = z^{-M} \{ \omega_1, \omega_2, \dots, \omega_N, p(0, z), p(1, z), \dots \},$$

in which everything is fixed apart from the choice of ω_i . The ω_i can be any basis of the N dimensional space of polynomials of degree less than $\dim \mu$ which satisfy $c(\omega_i) = 0$ for all $c \in C$. Since M independent conditions will have been applied to get this set, we know that $N = \dim \mu - M$. Then, the dual mapping is an isomorphism from $Gr(M, \mathcal{C}(\mu))$ to the $Gr(\dim \mu - M, \dim \mu)$ which is its image in Gr_1 . In particular, the two finite dimensional grassmannians are dual to each other and W is the space “perpendicular” to W^* under the inner-product defined by application of the condition to the function. The grassmannians corresponding to different choices of q are trivially isomorphic and all statements above apply to them as well.

Note. One may coordinatize the image in Gr_1 with the Plücker coordinates of a $Gr(M, \dim \mu)$ given by the subspace spanned by $\{\omega_i\}$ in the vector space spanned by $\{z^i | 0 \leq i \leq \dim \mu - 1\}$. It is then possible to choose a set of functions for which these finite dimensional Plücker coordinates are represented as the coefficients in a series summing to τ . Denote by S_μ^q the finite set of τ -functions which correspond to the points in the image with only one non-zero Plücker coordinate. In a method analogous to the infinite dimensional case, the τ -function of an arbitrary point of $Gr_q(M, \mathcal{C}(\mu))$ can then be written as the finite sum of these functions with the corresponding Plücker coordinates as coefficients. In particular, all τ -functions from this grassmannian are contained in the finite dimensional space spanned by S_μ^q . If we write τ as an arbitrary sum of elements from S_μ^q , then the equations of the KP hierarchy will act as Plücker relations for the coefficients.

Example. Consider the grassmannian $Gr(2, \mathcal{C}(\mu))$, where $\mu(n) = 3\delta_{0n} + \delta_{1n}$ and $q = z^2$. The Plücker coordinates of the grassmannian will be a set of six coordinates π_{ij} with $0 \leq i < j \leq 3$. Denote by τ_{ij} the τ -function corresponding to the point where only π_{ij} is non-zero. These six τ -functions are as follows:

$$\begin{aligned} \tau_{01} &= x^2 + 2x - 2y + (x^2 + 2y - 2x)e^{\sum t_i}, & \tau_{03} &= x^2 - 2y, & \tau_{13} &= x, \\ \tau_{02} &= -2x^2 + 4y + 2 - 2(1 - x)e^{\sum t_i}, & \tau_{12} &= -2x - 1 + e^{\sum t_i}, & \tau_{23} &= 1. \end{aligned}$$

Notice that three of these functions are *not* the Schur polynomials which arise in the standard representation. Every τ -function from $Gr(2, \mathcal{C}(\mu))$ is contained in the vector space spanned by the τ_{ij} , and can be written as

$$\tau = \sum_{0 \leq i < j \leq 3} \pi_{ij} \tau_{ij},$$

although an arbitrary sum of this form is not necessarily a τ -function. Inserting this arbitrary sum into the Hirota bilinear form of the KP equation yields

$$48e^{\sum t_i} (\pi_{03}\pi_{12} - \pi_{02}\pi_{13} + \pi_{12}\pi_{13}) = 0$$

which, apart from the exponential coefficient, is the Plücker relation for the coordinates of $Gr(2, 4)$.

Definition 2.2. Define the action of $g(z) = e^{cz^n} \in \Gamma_+$ on $d(l, \lambda) \in \mathcal{C}(\mu)$ by the formula

$$\begin{aligned} g(d(0, \lambda)) &= e^{-cz^n} d(0, \lambda) \\ g(d(l, \lambda)) &= e^{-cz^n} \left(d(l, \lambda) - \sum_{i=1}^l k_{l-i} d(l-i, \lambda) \right), \quad \text{for } l \geq 1 \end{aligned}$$

with the k_{l-i} for $0 \leq i \leq l$ defined recursively as

$$k_{l-1} = l \frac{\partial}{\partial z} \log g(z) \Big|_{z=i}, \quad k_{l-i} = \frac{1}{g} \left(\binom{l}{i} g^{(i)} - \sum_{j=1}^{i-1} k_{l-j} \binom{l-i}{i-j} g^{(i-l)} \right) \Big|_{z=i}.$$

Extend this linearly to define the action of the group Γ_+ on $\mathcal{C}(\mu)$.

Since $g(c)$ has the same support and bound as the condition c , Γ_+ acts on the linear space $\mathcal{C}(\mu)$. Furthermore, since Γ_+ acts linearly and has no kernel, it takes one M dimensional subspace to another. Consequently, Γ_+ acts on the finite dimensional grassmannian $Gr_q(M, \mathcal{C}(\mu))$ under Definition 2.2.

Definition 2.2 was chosen so that for any $c \in \mathcal{C}(\mu)$, $g \in \Gamma_+$ and $f \in \mathbb{C}[[z]]$,

$$c(f) = 0 \quad \text{iff } g(c)(g \cdot f) = 0.$$

Therefore, this definition of the action of Γ_+ on Gr_1^* coincides with the multiplicative action on Gr_1 via the dual mapping.

Grassmannians and Generalized Jacobians. The Γ_+ orbit of any point W in Gr which is “rank one algebro-geometric,” modulo the action of Γ_- , will be the generalized jacobian of the spectral curve. In fact, in the case of Gr_1 , taking the quotient by Γ_- is unnecessary and we have the following result:

Proposition 2.3. *The Γ_+ orbit of a point $W \in Gr_1$ is isomorphic to the (generalized) jacobian of the corresponding spectral curve.*

Proposition 2.3 follows immediately from Lemma 2.4.

Lemma 2.4. *If $g \neq 1 \in \Gamma_-$ is a non-trivial gauge transformation and $W \in Gr_1$, then gW is not in the Γ_+ orbit of W .*

Proof. By the formula for τ as the determinant of the Wronskian matrix \mathcal{M} , one can write τ_W as

$$\tau_W = \sum_{j=1}^N p_j(\mathbf{t}) e^{\sum a_{ij} t_i},$$

where the p_j are of bounded degree in all the t_i . Then, define $\omega(\tau_W, i)$ to be the unordered N -tuple $\{a_{ij} | 0 \leq j \leq N\}$, which is well defined despite the projective ambiguity in τ . Note that the $\omega(\tau_W, i)$ are preserved under Γ_+ translation of W . Now suppose that $g \in \Gamma_-$ is a non-trivial gauge transformation. Then, $\tau_{gW} = \hat{g}\tau_W$, where

$$\hat{g} = e^{\sum \alpha_i t_i}$$

with not all $\alpha_i = 0$. Let i_0 be such that $\alpha_{i_0} \neq 0$ and note that $\omega(\tau_{gW}, i_0) \neq \omega(\tau_W, i_0)$ since every element has undergone an additive shift of α_{i_0} . Therefore, $gW \notin \Gamma_+W$. \square

Therefore, the Γ_+ orbit of any point $W^* \in Gr_q(M, \mathcal{C}(\mu))$ will be the (generalized) jacobian of a (singular) rational curve. Furthermore, it is clear that these orbits are disjoint sets. Thus, one may conclude that any $Gr(M, \dim \mu)$ can be written as the disjoint union of the jacobians of all rational curves which can be described by M differential conditions bounded by μ . Since any $N \in \mathbb{N}$ is realizable as $\dim \mu$ for many different choices of μ (essentially determined by a partition of N), one can actually say:

Proposition 2.5. *For any $M \leq N \in \mathbb{N}$, the grassmannian $Gr(M, N)$, can be decomposed into the disjoint union of the generalized jacobians of all curves represented by M conditions bounded by μ for any μ of dimension N .*

The two decompositions of $Gr(1, 2)$, corresponding to the two partitions of the number 2 serve as an enlightening example. Consider first the bounding function $\mu_a(n) = 2\delta_{an}$ and $M = 1$. A point of the corresponding grassmannian is determined by a condition $c_1 f'(a) + c_2 f(a) = 0$ for $(c_1, c_2) \in \mathbb{P}^1 \mathbb{C}$. There are two distinct Γ_+ orbits which form this set. In the case $c_1 = 0$, we have the trivial solution $u = 0$ which is fixed by all Γ_+ action. Alternatively, if $c_1 \neq 0$, let $c = c_2/c_1$ and the condition can be written simply as $f'(a) + cf(a) = 0$. One may check that the subgroup of Γ_+ generated by elements of the form $e^{z - id^{-1}z}$ is the stabilizer of this condition (for any value of c). The action of Γ_+ can thus be understood by taking the quotient by this stabilizer, any element of which has a representative of the form $e^{\theta z}$. Note that this element takes the condition above to $f'(a) + (c - \theta)f(a) = 0$. In particular, the entire orbit is isomorphic to \mathbb{C} , which is indeed the generalized jacobian of a rational curve with a simple cusp at a .

Alternatively, consider the grassmannian corresponding to $\mu_{ab}(n) = \delta_{an} + \delta_{bn}$ and $M = 1$. A point of this grassmannian is specified by a condition $c_1 f(a) + c_2 f(b) = 0$ for $(c_1, c_2) \in \mathbb{P}^1 \mathbb{C}$ as before. Note, however, that the case in which either c_i is

zero corresponds to the trivial solution, and thus to two distinct Γ_+ orbits. We are then left with the case $c_1, c_2 \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. The condition $f(a) + cf(b) = 0$ (where $c = c_2/c_1 \neq 0$) is stabilized by any element of the form

$$\exp\left(z^i - \frac{b' - a'}{b - a} z\right).$$

Once again, we may now consider any element of Γ_+ to be represented by an element of the form $e^{\theta z}$. (In this case, we have not taken the quotient by the entire stabilizer since there remains a periodicity in θ .) The action of this element on the condition gives $f(a) + e^{\theta(b-a)}cf(b) = 0$. In particular, we can translate c to any value in \mathbb{C}^\times , which is the generalized jacobian of a rational curve with a node given by identifying the points a and b .

Note. A clever argument of G. Segal (related to me by E. Previato) demonstrates that the result of Lemma 2.4 is not necessarily true outside of Gr_1 . More precisely, if any non-trivial gauge transformation takes W outside its Γ_+ orbit, then $A_W \subset \mathbb{C}[z]$. Thus, in general, we should not expect that the KP orbit of the operator L_W is isomorphic to the Γ_+ orbit of W .

2.3. Rational Solutions. Given a point $(C, q) \in Gr_1^*$, the corresponding solution will be rational if and only if C is spanned by $\{c_1, c_2, \dots, c_M\}$, where $c_i \in \mathcal{C}(\lambda_i)$ for some $\lambda_i \in \mathbb{C}$. (That is, each condition must involve only one point. This limits the spectral curves to rational curves with cusps.) This is proved in [16], but the basic idea can be understood by studying the matrix \mathcal{M} which corresponds to the gauge $q = z^M$. Note that it is only when C has this property that the i^{th} column is made up of polynomials multiplied by $e^{\sum \lambda_i' t_i}$. Then, using elementary properties of the determinant, one can factor out these exponentials and find that τ is the determinant of a matrix of polynomials multiplied by the exponential of a linear function of x . Consequently, the general form of such a τ is easily seen to be:

$$(*) \quad \tau = p(\mathbf{t}) \prod_{i=1}^M \left(\exp\left(\sum_{j=1}^{\infty} \lambda_i' t_j\right) \right),$$

where $p(\mathbf{t})$ is a polynomial of degree N in x and a polynomial of degree less than or equal to N in each of the other time variables, t_i , and λ_i are the singular points. This clearly makes $u = 2 \frac{\partial^2}{\partial x^2} \log \tau$ into a rational function.

2.4. Translating Krichever’s Parameters. The function q plays a fundamentally different role in the two methods discussed for generating rational solutions. In the dual grassmannian, the choice of q merely determines the gauge. In Krichever’s method, however, the gauge is fixed by the form (Kr) of the wave function. Thus, once the λ_i have been fixed, the choice of q is the only way of determining the solution. The difference arises from the fact that Krichever’s condition is applied to the function ψ , which is an element of $q^{-1}\overline{V}_C$ rather than an element of \overline{V}_C itself. Then the “chain rule” gives us the ability to relate the two sets of parameters.

Claim 2.6. *If we choose the polynomial q and $c_i(f(z)) = f'(z) - \frac{q'(\lambda_i)}{q(\lambda_i)}f(z)|_{z=\lambda_i}$ as our conditions in the dual grassmannian, this will lead to the same solution as choosing q and $\{\lambda_i\}$ in Krichever’s method.*

Proof.

$$\begin{aligned} \mathcal{M}_{ij} &= c_i(z^{j-1}e^{\sum t_n z^n}) = \left(z^{j-1}e^{\sum t_n z^n}\right)' - \frac{q'(\lambda_i)}{q(\lambda_i)}e^{\sum t_n z^n} \Big|_{z=\lambda_i} \\ &= q(\lambda_i) \frac{\partial}{\partial z} z^{j-1} \frac{e^{\sum t_n z^n}}{q(z)} \Big|_{z=\lambda_i} = \Theta_{ij} q(\lambda_i) e^{\sum \lambda_i^n t_n} . \end{aligned}$$

The presence of the factor $q(\lambda_i)$ merely affects the determinant by a constant multiple. Furthermore, in [7], Krichever has removed the exponential part of each θ_{ij} . (As we shall see below, this latter difference corresponds to a gauge transformation.) To complete the proof it is sufficient to see that these differences will not alter the associated rational solution. \square

Note. It is known that the gauge of Krichever’s wave function is identified by the property that $1 \in W$. This can be determined, for example, from the results of [13]. This statement is also easily observed from Claim 2.6 since $q(z)$ clearly satisfies the conditions.

3. Bispectrality

Given a wave function $\psi(x, z)$, which is an eigenfunction in a generalized Schrödinger equation with spectral parameter z :

$$L \left(x, \frac{\partial}{\partial x} \right) \psi = \Theta(z)\psi ,$$

we say that ψ is *bispectral* if it also satisfies an analogous equation with the roles of x and z reversed:

$$Q \left(z, \frac{\partial}{\partial z} \right) \psi = \Phi(x)\psi .$$

This property was first discussed in [3] in connection with some questions arising in medical imaging. Surprisingly, it was found that the wave functions corresponding to rational KdV solutions were bispectral. Zubelli [17] extended this result to all the rational KP solutions from Gr_0 by explicitly constructing operators in the spectral parameter which demonstrate the bispectrality of the corresponding wave functions.

Wilson [16] completely classified the bispectral wave functions of the KP hierarchy which correspond to a rank one ring of commuting differential operators. (That is, [16] determined all bispectral wave functions which can be constructed from an isospectral flow of *line* bundles.) He first demonstrated that such wave functions must correspond to (vanishing or non-vanishing) rational solutions. Moreover, he concluded that a *unique* point in the Γ_- orbit of each rational solution of Gr_1 corresponds to a bispectral wave function. In particular, it was shown that given a condition space $C = \{c_i\}$ such that c_i is a differential condition at the point λ_i , the unique bispectral point corresponding to this condition space is $q^{-1}\overline{V_C}$ for $q(z) = \prod (z - \lambda_i)$. I will refer to the point corresponding to this choice of q as the *bispectral gauge*.

Furthermore, it was shown that the stationary wave function corresponding to the bispectral gauge, $\psi(x, z)$, remains a wave function of Gr in bispectral gauge

when x and z are interchanged⁵. It is in this context that the *bispectral involution* arises naturally. Consider the involution, β , acting on functions of x and z which switches the variables (i.e., $\beta(f(x, z)) = f(z, x)$). Then, as a result of [16], β is an involution on bispectral wave functions corresponding to rational KP solutions.

If we write $\psi(x, z) = \hat{\psi}e^{xz}$, then $\hat{\psi} = 1 + \alpha_1(x)z^{-1} + \alpha_2(x)z^{-2} + \dots$. It is therefore clear that $\lim_{z \rightarrow \infty} \hat{\psi} = 1$. Note also, if $g \neq 1 \in \Gamma_-$ and $\lim_{x \rightarrow \infty} \hat{\psi} = 1$ then

$$\lim_{x \rightarrow \infty} g\hat{\psi} = g \neq 1 .$$

From these facts it is simple to deduce the following corollary.

Corollary 3.1. *A stationary wave function $\psi = \hat{\psi}e^{xz}$ of a rational solution to the KP hierarchy satisfies*

$$\lim_{x \rightarrow \infty} \hat{\psi} = 1$$

if and only if ψ is in the bispectral gauge.

3.1. Tau Functions of Bounded Degree

Lemma 3.2. *The τ function of a rational solution in the bispectral gauge is of bounded degree in each time variable. (That is, there is one N such that τ is a polynomial of degree at most N in each time variable.)*

Proof. Recall from Sect. 2 that the τ corresponding to a rational solution in Gr_1^* with $q = z^M$ is in the form (*). It will be demonstrated that the element of Γ_- which will take this rational solution to its bispectral gauge results in a τ of bounded degree.

The action of the gauge group Γ_- on the representation τ can be expressed as follows. Let $g = e^{xz^{-1}} \in \Gamma_-$ and $\hat{g} = e^{-\sum t_i}$ and denote by gW the translation of $W \in Gr$ by g . Then we have:

$$\tau_{gW} = \hat{g}\tau_W .$$

It is then simple to see that multiplication by the gauge transformation

$$g(a) = 1 - az^{-1} = e^{\ln 1 - az^{-1}} = e^{-\sum (a^i / i) z^{-i}}$$

corresponds to multiplying τ by $e^{\sum a^i t_i}$.

To move a point with $q = z^M$ to its bispectral gauge, we would multiply by

$$\frac{z^M}{\prod (z - \lambda_i)} = \prod \frac{z}{z - \lambda_i} = \prod \frac{1}{1 - \lambda_i z^{-1}} = (\prod g(\lambda_i))^{-1} .$$

Note that the result of this transformation on a τ in the form (*) is the elimination of the exponential part, leaving $\tau = p(\mathbf{t})$. \square

Theorem 3.3. *The following are equivalent conditions on $W \in Gr$:*

1. *There exists an $N \in \mathbb{N}$ such that τ_W is a polynomial of degree at most N in each of the variables t_i .*

⁵It is clear that if $\psi(z, x)$ is also a wave function then $\psi(x, z)$ is bispectral. To see the more powerful fact that these are the only bispectral wave functions in Gr_1 , please refer to the proof in [16].

2. $u = 2 \frac{\partial^2}{\partial x^2} \log \tau_W$, is a vanishing rational solution and $\psi_W(x, z) = \psi_{W'}(z, x)$ for some W' with the same properties.
3. $W = q^{-1} \overline{V}_C$, where V_C is the set of polynomials in z satisfying the conditions $\{c_1, \dots, c_m\}$, the condition c_i involves only derivatives evaluated at the point λ_i , and $q = \prod (z - \lambda_i)$.
4. τ_W , is a polynomial in x and the coefficient of the highest degree term of x is constant.⁶

Proof (1) implies (2). A τ of bounded degree is necessarily a polynomial in x , and thus $u = 2 \frac{\partial^2}{\partial x^2} \log \tau$ is a (vanishing) rational solution. Given that [7] finds all such solutions as resulting from cuspidal rational curves, one may conclude that W comes from Wilson’s construction for rational curves with cusps. Furthermore, it was shown in Lemma 3.2 that one only gets a τ of bounded degree in this construction with the bispectral choice of q . Consequently, we also have that $W \rightarrow W'$ under the bispectral involution.

(2) implies (3). That (2) and (3) are equivalent is a major result of [16].

(3) implies (1). This is what I have shown above.

Thus we have that (1), (2) and (3) are equivalent.

That (4) is equivalent to (1–3) is most simply seen by comparing it to (2). Since τ_W is a polynomial in x , the corresponding solution is clearly a vanishing rational solution. Furthermore, the fact that the coefficient of the highest degree term in x is constant is equivalent to the fact that $\lim_{x \rightarrow \infty} \hat{\psi} = 1$. (If τ is a polynomial in t_i and

$$\hat{\psi} = 1 + \frac{h(\mathbf{t}, z)}{\tau(\mathbf{t})q(z)}$$

then h is of lower degree than τ in t_i if and only if the coefficient of the highest power of t_i in τ is constant. This can be determined from the formula relating τ and ψ [13].) So, (4) is really only a restatement of (2). \square

Since every rational solution has a τ of bounded degree, this can be seen as an extension of Corollary 1.3 to the arbitrary case and all of the time variables. Furthermore, from the fact that (4) implies (1), we get the following corollary.

Corollary 3.4. *If τ_W is polynomial in x with constant coefficient on the highest degree term, then τ_W is polynomial in each t_i .*

4. The Bispectral Involution

The correspondence between rational KP solutions and Calogero–Moser particle systems is simple to describe in the case that the solution is in the bispectral gauge. In that case, the positions of the particles x_j are the roots of the polynomial τ_W in the variable x and the momenta y_j are the instantaneous velocity of the x_j ’s under the second time flow. By using this correspondence, it is possible to consider the bispectral involution as a map on particle systems. It can then be compared to the well known linearizing map σ [2] which is also an involution on Calogero–Moser particle systems. The results are summarized by the following theorems and proved below.

⁶T. Shiota [14] has recently completed a study of the τ -functions of the KP equation which are monic polynomials in x and their extension to the KP hierarchy.

In the case that a KP solution can be represented by conditions of the form $f'(\lambda_i) + \gamma_i f(\lambda_i) = 0$ for distinct λ_i , the parameters λ_i and γ_i will be referred to as the *dual grassmannian coordinates* of the solution. The bispectral involution demonstrates a symmetry between the dual grassmannian coordinates of a solution and the corresponding Calogero–Moser particle system. This can be applied to determine the action of β on Calogero–Moser particle systems in terms of dual grassmannian coordinates or the action on dual grassmannian coordinates in terms of Calogero–Moser particle systems as shown in the following two theorems:

Theorem 4.1. *If the Calogero–Moser particle system (\vec{x}, \vec{y}) is determined by the dual grassmannian parameters $(\vec{\lambda}, \vec{\gamma})$ ($\lambda_i \neq \lambda_j$ for $i \neq j$), then $\beta(\vec{x}, \vec{y}) = (\vec{\zeta}, \vec{\mu})$ is given by*

$$\zeta_i = \lambda_i$$

and

$$\mu_i = \gamma_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} .$$

Theorem 4.2. *Given a bispectral rational KP solution $W^* \in Gr_1^*$ that corresponds to a Calogero–Moser particle system (\vec{x}, \vec{y}) such that the x_j are distinct, the condition space with basis:*

$$c_i(f) = f(x_i) + \left(y_i - \sum_{j \neq i} \frac{1}{x_i - x_j} \right) f'(x_i) = 0$$

and the polynomial $q(z) = \prod (z - x_i)$ are the point $\beta(W^*) \in Gr_1^*$.

Then, by considering the composition of the bispectral involution with the motion of the particle systems, it is determined that:

Theorem 4.3. *β acting on Calogero–Moser particle systems with distinct particle positions is a linearizing map in the sense that it fixes the positions and linearizes the momenta under composition with time flows.*

Thus, β , which is an involution by construction, has the surprising property of being a linearizing map. Alternatively, the map σ which is a linearizing map by construction, is mysteriously an involution. Finally, it is noted that the two involutions are basically the same.

Theorem 4.4. *Given a Calogero–Moser particle system $(\vec{x}(t_2), \vec{y}(t_2))$,*

$$\sigma(\vec{x}(t_2), \vec{y}(t_2)) = (\vec{\zeta}, 2\vec{\zeta}t_2 + \vec{\eta})$$

and

$$\beta(\vec{x}(t_2), \vec{y}(t_2)) = (\vec{\zeta}, 2\vec{\zeta}t_2 + \vec{\mu})$$

for constant $\vec{\zeta}$, $\vec{\eta}$, $\vec{\zeta}$ and $\vec{\mu}$. Furthermore, they are related by the fact that $\vec{\zeta} = -\vec{\zeta}$.

4.1. *Bispectral Flow.* Let C be an N dimensional space of conditions with basis $\{c_1, \dots, c_N\}$ such that $c_i \in \mathcal{C}(\lambda_i)$ and let $q = \prod(z - \lambda_i)$. Denote by W^* the point $(C, q) \in Gr_1^*$ and its image in Gr_1 by W . The spectral curve corresponding to this KP solution is a rational curve with cusps at the points λ_i and its stationary wave function ψ_W is in the form (B) with poles in z at the cusps of the spectral curve and poles in x at the zeroes of the polynomial $\tau_W(x, 0, 0, \dots)$. These values of x are those for which the corresponding point of the t_1 orbit of W leaves the “big cell” of Gr [13].

The bispectrality of ψ_W implies that its image under the bispectral involution,

$$\beta(\psi_W(x, z)) = \psi_W(z, x),$$

is the wave function of another bispectral rational solution. Since $\beta(\psi_W)$ exchanges the roles of x and z , it is clear from the observations in the previous paragraph that the spectral curve of the KP solution with wave function $\beta(\psi_W)$ has cusps at those points z for which $\tau_W(z, 0, 0, \dots) = 0$.

Let $W(t_i)$ be the time dependence of the point $W \in Gr_1$ under the i^{th} KP flow and $\beta(W(t_i))$ its image under the bispectral involution. Recall that the zeroes of the polynomial $\tau_W(x, 0, 0, \dots)$ move as a Calogero–Moser system of particles under the KP flow. Then the composition of any KP flow with the bispectral involution, the *bispectral flow*, is seen to be a non-isospectral flow for which the cusps of the corresponding spectral curve behave as a Calogero–Moser system. In the case of the first KP flow given by $x \rightarrow x + c$, the induced bispectral flow would clearly be given by a linear deformation of the spectral parameter z , which is an example of a Virasoro flow [11]. As will be demonstrated below, it was essentially a bispectral flow which was utilized in [2] as a linearized flow of the Calogero–Moser particle system.

4.2. *Calogero–Moser Particle Systems.* The phase space of an n -particle Calogero–Moser particle system is given by $2n$ complex numbers, x_i and y_i , $1 \leq i \leq n$, $x_i \neq x_j$ for $i \neq j$, where $\vec{x} = (x_1, \dots, x_n)$ represent the positions of n particles and $\vec{y} = (y_1, \dots, y_n)$ represent their instantaneous velocity. Here we will be mainly interested in their motion under the second Hamiltonian: $H = \sum y_i^2 - \sum (x_i - x_j)^{-2}$. Airault, McKean and Moser [2] introduced the linearizing map $\sigma(\vec{x}, \vec{y}) \rightarrow (\vec{\xi}, \vec{\eta})$ in the case when all the phase space variables are real, where ξ_i is the asymptotic velocity of the i^{th} particle and η_i is an asymptotic relative position. It is clear, by physical consideration, that ξ_i is a constant of motion and that η_i is a linear function of the time variable. It was unexpected, however, that the map σ would be an involution. Yet, it was shown in [2], that $\sigma(\vec{\xi}, \vec{\eta}) = (\vec{x}, \vec{y})$. It is important to note that the matrix

$$A_{ij} = y_i \delta_{ij} + \frac{1 - \delta_{ij}}{x_i - x_j}$$

has $\{\xi_i\}$ as its eigenvalues and that the analogous matrix A^σ written in terms of ξ_i and η_i has $\{x_i\}$ as its eigenvalues. Therefore, one can view σ as a map which exchanges spacial and spectral values. This, superficially, indicates a relationship to the bispectral involution.

Note. Although vector notation is being used, particle systems within this paper are unordered sets of pairs (x_i, y_i) . Thus, two particle systems which dif-

fer only by a permutation of the index i are considered to be the same particle system.

4.3. Dual Grassmannian Coordinates of Krichever’s Solutions and the Associated Particle System. It is clear from Theorem 2.6 that the rational solutions that are constructed in Theorem 1.2 are also given by the dual grassmannian construction with a choice of $2n$ parameters: λ_i ($\lambda_i \neq \lambda_j$ if $i \neq j$) and γ_i for $1 \leq i \leq n$, where the point W^* has a condition space spanned by n differential conditions:

$$c_i(f) = f'(\lambda_i) + \gamma_i f(\lambda_i) = 0$$

and can be placed in the bispectral gauge by the choice of polynomial $q(z) = \prod(z - \lambda_i)$. In the case that a KP solution can be represented by a condition space of this form with distinct singular points λ_i , the parameters λ_i and γ_i will be referred to as the *dual grassmannian coordinates* of the solution. The dual grassmannian coordinates will be written using the same notation as a particle system: $(\vec{\lambda}, \vec{\gamma})$.

In this section, we will only be concerned with this solution and its motion under the second time flow. We therefore suppress all time variables other than x and t_2 . Then the time dependent wave function of the solution described above can be written in the form

$$\begin{aligned} \psi(x, t_2, z) &= \left(1 + \frac{p(x, t_2, z)}{\prod(x - x_i(t_2)) \prod(z - \lambda_i)} \right) e^{xz + t_2 z^2} \\ &= \left(1 + \frac{1}{\prod(z - \lambda_i)} \sum \frac{a_i(t_2, z)}{x - x_i(t_2)} \right) e^{xz + t_2 z^2} \\ &= \left(1 + \frac{1}{\prod(x - x_i(t_2))} \sum \frac{\alpha_i(x, t_2)}{z - \lambda_i} \right) e^{\lambda z + t_2 z^2}, \end{aligned}$$

where $p(x, t_2, z)$ is a polynomial of degree less than n each variable and the residues a_i and α_i are merely determined by a partial fractions expansion in x and z respectively. The stationary wave function $\psi(x, z)$ is merely $\psi(x, 0, z)$, i.e. the wave function at $t_2 = 0$.

Notice that the positions of the poles of ψ in the variable x , which are given by the functions x_i , move in time whereas the positions of the poles in z are fixed. As is often the case in the study of rational solutions to integrable equations, the motion of the poles in x are equivalent to an integrable Hamiltonian system of n particles. Let $y_i = \frac{1}{2} \frac{d}{dt_2} x_i|_{t_2=0}$ be the “instantaneous velocity” of the particle at position x_i . Thus we associate the particle system (\vec{x}, \vec{y}) to the wave function ψ .

Denote by \mathcal{CM}_0 the subset of the phase space of an n particle Calogero–Moser particle system for which the positions, x_i , are distinct. If the dual grassmannian parameters λ_i and γ_i were chosen such that $(\vec{x}, \vec{y}) \in \mathcal{CM}_0$, let A be the Moser matrix

$$A_{ij} = y_i \delta_{ij} + \frac{1 - \delta_{ij}}{x_i - x_j}.$$

Then the motion of the x_i under the t_2 flow is a Calogero–Moser particle system, given by the Hamiltonian $H = \text{tr } A^2$.

4.4. *The Particle System of the Bispectral Dual of the Wave Function.* Recall that in the bispectral gauge, the stationary wave function remains a stationary wave function of the KP hierarchy if the variables x and z are interchanged [16]. So, in particular, we may talk about the bispectral dual of the wave function

$$\begin{aligned} \psi^\beta(x, z) &= \psi(z, x) = \left(1 + \frac{1}{\prod(x - \lambda_i)} \sum \frac{a_i(x)}{z - x_i} \right) e^{xz} \\ &= \left(1 + \frac{1}{\prod(z - x_i)} \sum \frac{\alpha_i(z)}{x - \lambda_i} \right) e^{xz}. \end{aligned}$$

There are two reasonable ways to consider the time evolution of the stationary wave function $\psi^\beta(x, z)$.

- We can maintain the t_2 dependence of the functions x_i , a_i and α_i from the time dependent ψ . This composition of the KP flow with the bispectral involution is the bispectral flow which was introduced in Sect. 4.
- Alternatively, since ψ^β is a stationary wave function of the KP hierarchy, one may add dependence on time so as to make it a time dependent wave function. Under this flow, the functions λ_i , a_i and α_i would become time dependent while the x_i would remain constant.

To avoid confusing these two flows of the wave function or the two time dependencies of the functions a_i and α_i , the usual KP flow of the function ψ^β will be indicated by the variables T_n rather than the variables t_n . That is, define the function $\psi^\beta(x, t_2 | T_2, z)$ as follows: let the stationary wave function $\psi(x, z)$ follow the second flow of the KP hierarchy until time t_2 , exchange the variables x and z in this new stationary wave function, then let this function follow the second flow of the KP hierarchy until time T_2 (treating t_2 as a constant).

The motion of the poles in x of the function $\psi^\beta(x, t_2 | T_2, z)$, under the variable T_2 gives us another Calogero–Moser particle system: $(\vec{\zeta}(T_2), \vec{\mu}(T_2))$. In this way, we may view β as a map on the phase space of Calogero–Moser particle systems given by $\beta(\vec{x}, \vec{y}) = (\vec{\zeta}, \vec{\mu})$.

Since the positions of the particles associated to a wave function are given by the positions of the poles in x , it is clear that the positions of the particles associated to ψ^β are given by the poles in z of ψ . Therefore, we have

$$\zeta_i = \lambda_i.$$

By definition, we have $\mu_i = \frac{1}{2} \frac{d}{dT_2} \lambda_i |_{T_2=0}$. However, the next section will demonstrate that $\vec{\mu}$ is more easily computed algebraically in terms of the dual grassmannian coordinates λ_i and γ_i .

4.5. *β in terms of Dual Grassmannian Coordinates.* As demonstrated above, the first component of $(\vec{\zeta}, \vec{\mu}) = \beta(\vec{x}, \vec{y})$ is given by $\vec{\zeta} = \vec{\lambda}$, where λ_i are the singular points in the dual grassmannian coordinates of the solution associated to (\vec{x}, \vec{y}) . As a result of the two calculations described below, we will similarly be able to determine $\vec{\mu}$ in terms of the dual grassmannian coordinates.

Since the function $\psi(x, t_2, z)$ is a wave function for a solution to the KP equation, it satisfies the non-stationary Schrödinger equation

$$L\psi = \left(\frac{\partial}{\partial t_2} - \frac{\partial^2}{\partial x^2} + \sum \frac{2}{(x - x_j)^2} \right) \psi = 0.$$

Then for any $1 \leq i \leq n$ the function $(x - x_i)L\psi = 0$, but its residue at the point $x = x_i$ is given by

$$\frac{a_i \dot{x}_i}{q} + \frac{2za_i}{q} + 2 + \frac{2}{q} \sum_{j \neq i} \frac{a_j}{x_i - x_j},$$

where $\dot{x}_i = \frac{d}{dt_2} x_i$. Setting this equal to zero and solving for \dot{x}_i , we find that

$$y_i = \frac{1}{2} \dot{x}_i = - \left(z + \frac{\prod_{1 \leq j \leq n} (z - \lambda_j)}{a_i(z)} + \frac{1}{a_i(z)} \sum_{j \neq i} \frac{a_j(z)}{x_i - x_j} \right).$$

Then, considering ψ^β as a function of T_2 , it is clear that

$$\mu_i = - \left(x + \frac{\prod_{1 \leq j \leq n} (x - x_j)}{\alpha_i(x)} + \frac{1}{\alpha_i(x)} \sum_{j \neq i} \frac{\alpha_j(x)}{\lambda_i - \lambda_j} \right).$$

Alternatively, we can determine a similar formula for the dual grassmannian coordinate γ_i . Recall that the function $\phi(x, z) = q(z)\psi(x, z)$ satisfies the conditions c_i which determine the solution. In particular,

$$\left. \phi' + \gamma_i \phi \right|_{z=\lambda_i} = 0$$

for all x in the domain of ϕ , where differentiation is done with respect to the spectral parameter z . Consequently,

$$\left. \frac{\phi'}{\phi} \right|_{z=\lambda_i} = -\gamma_i.$$

Using the form of ψ which has been written in terms of the residues α_j in z , it is determined that

$$\gamma_i = - \left(x + \frac{\prod_{1 \leq j \leq n} (x - x_j)}{\alpha_i(x)} + \frac{1}{\alpha_i(x)} \sum_{j \neq i} \frac{\alpha_j(x)}{\lambda_i - \lambda_j} + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right).$$

Therefore,

$$\mu_i = \gamma_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}.$$

This provides us with information about the image of β as a map from $\mathcal{C.M}_0$. In addition, it gives us the ability to determine explicitly the dual grassmannian coordinates of a point under the bispectral involution on a large class of bispectral rational solutions. These results are summarized by Theorem 4.1 and Theorem 4.2.

4.6. Linearization of Particle Systems. In this section, we will consider the bispectral flow of $(\vec{\zeta}, \vec{\mu})$. That is, given a particle system $(\vec{x}(t_2), \vec{y}(t_2))$, we study

$(\vec{\zeta}(t_2), \vec{\mu}(t_2)) = \beta(\vec{x}(t_2), \vec{y}(t_2))$. Using Theorem 4.1, this flow can be determined simply by observing the action of the KP flow on the parameters λ_i and γ_i .

The action of the KP flows on a condition of the form

$$f'(\lambda) + \gamma f(\lambda)$$

is determined simply in terms of Definition 2.2. Since the KP flows are isospectral, the positions of the singular points, λ_i , are constant. As shown earlier, the positions of the particles, ζ_i , are given exactly by the coordinates λ_i . Consequently, the positions of the particles, ζ_i , are fixed under the bispectral flow. Furthermore, the result of the n^{th} flow for time t on the parameter γ is $\gamma \rightarrow n\lambda^{n-1}t + \gamma$. As shown earlier, we can determine $\vec{\mu}$ by the formula $\mu_i = \gamma_i + \sum \frac{1}{\lambda_i - \lambda_j}$. Since γ is a linear function of all time variables and the λ_i are constant, it is clear that the μ_i have been linearized by β . This proves Theorem 4.3.

The map σ was originally considered only on the subset of $\mathcal{C.M}_0$ for which the parameters x_i and y_i are real numbers. Recall⁷ that $\sigma(x_i, y_i) = (\xi_i, \eta_i)$ was defined so that $\xi_i = \lim_{t_2 \rightarrow \infty} y_i$ is the asymptotic velocity of the i^{th} particle. The map σ can then be related to β using the fact that the eigenvalues of the matrix A are $\{-\lambda_i\}$ [7] and, as always, are preserved by the t_2 flow. Then, since

$$\lim_{t_2 \rightarrow \infty} A_{ij} = \lim_{t_2 \rightarrow \infty} y_i \delta_{ij},$$

we determine $-\xi = \lambda = \zeta$.

Similarly, note that η is given as a linear function by the formula $\eta(t_2) = f'(\xi)t_2 + \eta(0)$ where the Hamiltonian is given by $\text{tr } f(A)$ [2] (Amplification 1). In this case $f(A) = A^2$, and so we get agreement with the first coefficient of μ above. That is,

$$-2\xi = -\frac{d}{dt_2}\mu = \frac{d}{dt_2}\eta = 2\xi.$$

This is sufficient to prove Theorem 4.4. Thus, β restricted to the domain of the map σ is given simply by $-\sigma + (0, \vec{c})$, where \vec{c} is a constant of evolution.

4.7. Examples

Solving the System with the Bispectral Involution. Consider the generic “2 particle” case given by

$$c_i(f) = f'(\lambda_i) + \gamma_i f(\lambda_i) = 0, \quad i = 1, 2,$$

and $q(z) = (z - \lambda_1)(z - \lambda_2)$. The parameters λ_i and γ_i can form a matrix whose determinant is τ in two ways. First, we can use the matrix \mathcal{M} which gives τ in the gauge given by $q = z^2$ and translate it to the bispectral gauge. Then

$$\begin{aligned} \tau(x, 0, \dots) &= (x - x_1)(x - x_2) \\ &= e^{-(\lambda_1 + \lambda_2)x} \det \mathcal{M} \\ &= e^{-(\lambda_1 + \lambda_2)x} \det(c_i(z^{j-1} e^{xz})) \\ &= \det M_1, \end{aligned}$$

⁷I have changed the notation slightly to agree with that used in this paper.

where

$$M_1 = \begin{pmatrix} x + \gamma_1 & x + \gamma_2 \\ \lambda_1 x + \lambda_1 \gamma_1 + 1 & \lambda_2 x + \lambda_2 \gamma_2 + 1 \end{pmatrix}.$$

On the other hand, we know that the eigenvalues of the Moser matrix A are simply $-\lambda_i$. Since, the bispectral involution exchanges λ_i and x_i , we can also determine the τ -function as

$$\begin{aligned} \tau(x) &= \det(xI + A^\beta) = \det \left((x + \mu_i) \delta_{ij} + \frac{1 - \delta_{ij}}{\zeta_i - \zeta_j} \right) \\ &= \det \left((x + \gamma_i + \sum \frac{1}{\lambda_i - \lambda_j}) \delta_{ij} + \frac{1 - \delta_{ij}}{\lambda_i - \lambda_j} \right) = \det M_2, \end{aligned}$$

where

$$M_2 = \begin{pmatrix} x + \gamma_1 + \frac{1}{\lambda_1 - \lambda_2} & \frac{1}{\lambda_1 - \lambda_2} \\ \frac{1}{\lambda_2 - \lambda_1} & x + \gamma_2 + \frac{1}{\lambda_2 - \lambda_1} \end{pmatrix}.$$

Note that the matrix M_2 is linear in time along the diagonal and constant off of the diagonal.

The matrices M_1 and M_2 , which arise in very different contexts and are constructed in ways which do not appear similar, can be made equal with only a few basic row and column operations.

Determining the Action of the Bispectral Involution. As an application of Theorem 4.2, the dual grassmannian coordinates of the image under the bispectral involution of a specific case of the previous example are determined below.

Let $\lambda_1 = \gamma_1 = 1$ and $\lambda_2 = \gamma_2 = 2$. Then one may determine that

$$\begin{aligned} \psi_1 &= \left(1 + \frac{7 + 3x - 3z - 2xz}{(1 + 3x + x^2)(2 - 3z + z^2)} \right) e^{xz}, \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \frac{-3 + \sqrt{5}}{2} \\ \frac{-3 - \sqrt{5}}{2} \end{pmatrix}, \end{aligned}$$

and

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{-6 + \frac{1}{\sqrt{5}} + \sqrt{5}}{4} \\ \frac{-6 - \frac{1}{\sqrt{5}} - \sqrt{5}}{4} \end{pmatrix}.$$

Then, the wave function of the solution given by the dual parameters $\lambda_1 = x_1$, $\lambda_2 = x_2$, $\gamma_1 = y_1 - \frac{1}{x_1 - x_2}$ and $\gamma_2 = y_2 - \frac{1}{x_2 - x_1}$ is

$$\psi_2 = \left(1 + \frac{7 + 3z - 3x - 2zx}{(1 + 3z + z^2)(2 - 3x + x^2)} \right) e^{xz} = \beta(\psi_1).$$

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