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Asymptotic Expansion for the Density of States of the Magnetic Schrödinger Operator with a Random Potential

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Abstract: We study the asymptotics for the density of states of the magnetic Schrödinger operator with a random potential. By using the methods of effective Hamiltonian, complex dilation and complex translation, we obtain in the large magnetic field limit, the asymptotic expansion for the density of states measure considered as a distribution.

1. Introduction

We study the density of states of the magnetic Schrödinger operator with a random potential defined on $L^2(\mathbb{R}^2)$

$$P^{\omega}_{B,V} = (D_x + By)^2 + D_y^2 + V^{\omega}(x, y),$$

where $D_x = (1/i)\partial_x$, $D_y = (1/i)\partial_y$ and $B > 0$ is a constant. Let v be a C_0^∞ function, the potential V is defined as

$$V(\bar{x}) = \sum_{i \in \mathbb{Z}^2} \alpha_i v(\bar{x} - i) = \sum_{i \in \mathbb{Z}^2} \alpha_i v_i(\bar{x}), \tag{1.1}$$

where $\bar{x} = (x, y)$, $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}^2}$ form a random field, *i.e.* a family of random variables indexed by $\mathbb{Z}^2$ on a probability space (Ω, P). We denote by $\langle f \rangle$ the expectation value of the random variable f. One can always suppose that $\Omega = \mathbb{R}^{\mathbb{Z}^2}$. In this case,

$$\alpha_\omega(j) = \omega(j), \tag{1.2}$$

and the translation operators, $T_i (i \in \mathbb{Z}^2)$ in Ω are defined by

$$T_i\omega(j) = \omega(j - i). \tag{1.3}$$

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Hence for random potentials of the form (1.1), we have, by using (1.2) and (1.3),

$$V^{T_i\omega}(\bar{x}) = V^{\omega}(\bar{x} - i). \tag{1.4}$$

We take

$$dP = \prod_{i\in\mathbb{Z}^2} g(\alpha_i)d\alpha_i ,$$

where g is a C_0^∞ function. We say that the α_i are independent identically distributed (i.i.d.) random variables. The action of T_j is *ergodic* on (Ω, P) : if $A \subset \Omega$ satisfies

$$\forall j \in \mathbb{Z}^2, T_j^{-1}A = A ,$$

then $P(A) = 0$ or 1. The operator $P_{B,V}^{\omega}$ is an ergodic operator.

When $V = 0$, the Hamiltonian $P_{B,0}$ has eigenvalues $\lambda_n = (2n+1)B$, $n \in \mathbb{N}$ with infinite multiplicity. These are the so-called Landau levels.

We define the "magnetic translations" $\tau_j^B (j = (j_1, j_2) \in \mathbb{Z}^2)$ by

$$[\tau_j^B u](x, y) = e^{-iBj_2(x - \frac{1}{2}j_1)}u(x - j_1, y - j_2).$$

In particular

$$[\tau_{(j_1,0)}u](x, y) = u(x - j_1, y),$$

$$[\tau_{(0,j_2)}u](x, y) = e^{-iBj_2x}u(x, y - j_2).$$

Let $j \wedge k$ be the determinant of (j, k) in the canonical basis. We have the relations

$$[P_{B,0}, \tau_j^B] = 0,$$

$$\tau_j^B\tau_k^B = e^{\frac{1}{2}iB(j\wedge k)}\tau_{j+k}^B,$$

$$\tau_j^B\tau_k^B = e^{iB(j\wedge k)}\tau_k^B\tau_j^B.$$

When $V \neq 0$, we have

$$\tau_{-j}^B P_{B,V}^{\omega}\tau_j^B = P_{B,V}^{T_j\omega},$$

where we used (1.4). Hence by standard arguments concerning ergodic operators [CFKS, Pa], the spectrum of $P_{B,v}^{\omega}$, $\sigma(P_{B,v}^{\omega})$ is almost surely constant with respect to ω, *i.e.*

$$\sigma(P_{B,v}^{\omega}) = \sigma(P_{B,v}) \quad \text{a.s.},$$

where $\sigma(P_{B,v})$ is a non-random set. Without loss of generality we may assume range $v \subset [-1, 1]$. If moreover we take supp $g \subset (-p, p)$ with $p < B$, then $\sigma(P_{B,v}^{\omega})$ is contained in the union $\cup_n[\lambda_n - p, \lambda_n + p]$. We will show later that the density of states measure ρ^{ω} is also almost surely non-random:

$$\rho^{\omega} = \rho \quad \text{a.s.},$$

where ρ is a non-random measure.

The density of states of the magnetic Schrödinger operator with a random potential was first studied by the physicist Wegner [We] (see also [BGI, KP]). This is a physically measurable quantity. In the strong field limit $B \gg 1$, Wegner developed a heuristic argument. According to his argument, if one is only interested in the density of states for low energies, one only needs to take into account the lowest Landau level, and one can neglect the contributions of other Landau levels. This way Wegner was able to compute exactly the first term of the density of states for a Gaussian white noise potential.

In this paper, we justify the Wegner approximation in some weak sense, and obtain an asymptotic expansion for ρ considered as a distribution. We show that, there exist C^∞ functions $\rho_i(t)$, such that for all $N_0 \in \mathbb{N}$, $c > 0$, if $f \in C_0^\infty(((-B,-c) \cup (c,B)) + (2n+1)B)$ satisfies $|\partial^j f| = \mathbf{O}(B^{N_0 j})$ for $B \gg 1$, then for every $m \in \mathbb{N}$:

$$\begin{aligned}\int f(E)d\rho_B(E) = B\int f(t+(2n+1)B)\rho_0^{(n)}(t)\,dt \\ + \int f(t+(2n+1)B)\rho_1^{(n)}(t)\,dt \\ + B^{-1}\int f(t+(2n+1)B)\rho_2^{(n)}(t)\,dt + \cdots \\ + B^{-(m-1)}\int f(t+(2n+1)B)\rho_m^{(n)}(t)\,dt + \mathbf{O}(B^{-m})\,, \quad (1.5)\end{aligned}$$

where $t = E - (2n+1)B$ is the renormalized energy and $d\rho_B(E)$ is the "non-random" density of states measure that we mentioned earlier. In particular, if v has support contained in the unit square centered at the origin, we have

$$\rho_0^{(n)}(t) = \frac{1}{2\pi}\int F_0(\gamma)g\left(\frac{t}{\gamma}\right)\frac{d\gamma}{|\gamma|}\,,$$

$$\text{where} \quad F_0(\gamma) = \int_{v=\gamma}\frac{ds}{\|\nabla v\|}\,;$$

(note that $\rho_0^{(n)}(t)$ is independent of n) and

$$\rho_1^{(n)}(t) = \frac{2n+1}{8\pi}\int F_1(\gamma)g\left(\frac{t}{\gamma}\right)\frac{d\gamma}{|\gamma|}\,,$$

$$\text{where} \quad F_1(\gamma) = \frac{d}{d\gamma}\int_{v=\gamma}\frac{\Delta v}{\|\nabla v\|}ds\,.$$

If we further assume that $v \geqq 0$ and that the supports of v_i intersect so that

$$\sum_{i\in\mathbb{Z}^2} v_i \geqq s > 0 \quad (1.6)$$

for some postive constant s, then we show that the expansion (1.5) holds for all $f \in C_0^\infty((-B,B)+(2n+1)B)$ satisfying $|\partial^j f| = \mathbf{O}(B^{N_0 j})$.

We use the Grushin method to study this problem as Helffer and Sjöstrand [HS] did in the case where V is periodic. We take $h = B^{-1} \ll 1$ to be our semi-classical parameter. For all z' in $[\lambda_n - a, \lambda_n + a]$, the study of $P_{B,v}^\omega - z'$ can be reduced to the study of an effective one-dimensional operator denoted by E_{-+}/h: $z' \in \sigma(P_{B,v}^\omega)$ if and only if $0 \in \sigma(E_{-+}/h)$. The symbol of E_{-+} has an asymptotic expansion in h, the principal symbol being $h(V(x,\xi) - z)$, where $z = z' - \lambda_n$.

The effective Hamiltonian is the starting point of our analysis. The additional parameters α_i and the associated probability density $g(\alpha_i)$ enable us to use scaling arguments (see Sect. IV). To order $\mathbf{O}(h^\infty)$, $E_{-+}^\omega(x,\xi)$ only depends on the potential and its derivatives at (x,ξ). For $|\mathrm{Im}\, z'| \geqq h^{\frac{1}{2}-\varepsilon}$ $(\varepsilon > 0)$, $(E_{-+}^\omega)^{-1}$ admits a parametrix, *i.e.* an asymptotic expansion in powers of h, the first term being

$$\frac{1}{h(V^\omega - z)}\,.$$

In order to study the density of states, we need to have control over the non-elliptic regions where $|\alpha_j v_j - z| \ll 1$. By the ergodic theorem, the density of states ρ^{ω} exists almost surely and is equal to the averaged density of states ρ (see Sect. III). Once we take the average, we have

$$\langle E_{-+}^{-1}(x,\xi)\rangle = \langle E_{-+}^{-1}(x+i_1,\xi+i_2)\rangle$$

($(i_1,i_2)\in\mathbb{Z}^2$). Therefore we only need to have bounds on $\langle E_{-+}^{-1}(x,\xi)\rangle$ for $(x,\xi)\in\mathbb{E}$ with $\mathbb{E}$ the unit square centered at the origin.

Due to the "local" nature of the operator $E_{-+}(x,\xi)$; the influence of the non-elliptic regions on $E_{-+}^{-1}(x,\xi)$ decreases as j increases. To exhibit this, we complex dilate in α_j: $\alpha_j \to \alpha_j e^{i\theta_j}$, where $\theta_j \cong 1/|j|$.

This is our scaling argument in dimension 2. (The precise form of the scaling relation depends crucially on the dimension.) We can in this way control the non-elliptic regions up to the scale $L = \mathbf{O}(h^{-N})$ for all $N\in\mathbb{N}$. Hence for all z such that $\operatorname{Re} z \neq 0$, the asymptotic expansion for $\langle E_{-+}^{-1}\rangle$ is valid for $|\operatorname{Im} z|$ not too small ($|\operatorname{Im} z|^{-1} = \mathbf{O}(h^{-N})$). For all fixed $N_0\in\mathbb{N}$, we study the integral $\int f(t)d\rho(t)$ for all f such that $|\partial^j f| = \mathbf{O}(h^{-N_0 j})$ and with support away from $\{(2n+1)B, n\in\mathbb{N}\}$. We show that the ρ_i are C^∞ functions away from 0.

If we make the further assumption (1.6), then we complex translate in α_j: $\alpha_j \mapsto \alpha_j + i\delta_j$, where $\delta_j = 1/|j|$, to control the non-elliptic regions. We show that (1.5) holds for all f such that $|\partial^j f| = \mathbf{O}(h^{-N_0 j})$ and that the ρ_i are C^∞ functions.

By a standard procedure, (1.5) can be extended to include Hölder continuous functions (B independent) $f\in C^\alpha, 0<\alpha<1$ with an error which is of order $\mathbf{O}(h^\infty)$.

This result should be compared with the periodic case:

$$V(\bar{x}) = \sum_{i\in\mathbb{Z}^2} v(\bar{x}-i),$$

i.e. $\alpha_i = 1$ for all $i\in\mathbb{Z}^2$. There one needs much stronger conditions on f, namely $|\partial^j f| = \mathbf{O}(h^{-j/(2+\varepsilon)})(\varepsilon>0)$. Moreover, in the periodic case, ρ_i has singularities. Of course this is due to the fact that our estimates on the parametrix for E_{-+}^{-1} is valid only for $|\operatorname{Im} z| \geqq h^{\frac{1}{2}-\varepsilon}$.

The nature of the spectrum of $P_{B,V}^{\omega}$ is presently not known. The operator $P_{B,V}^{\omega}$ is conjectured to have pure point spectrum with localized eigenfunctions (commonly known as *localization*) for certain ranges of energies. This conjecture has been used in building theories of the quantum Hall effect [Bel]. It is our eventual aim to prove this conjecture and this paper constitutes a first step toward research in that direction.

II. The Reduction and the Associated Grushin Problem

Classes of Symbols. We say that a symbol $Q(x,\xi,h)$ is in the class $S^0(\mathbb{R}^2)$ if it verifies

$$\exists h_0,\ \forall\alpha\in\mathbb{N}^2,\ \exists C_\alpha,\ \forall(x,\xi)\in\mathbb{R}^2,\ \forall h\in]0,h_0],|D_x^{\alpha_1}D_\xi^{\alpha_2}Q(x,\xi,h)| \leqq C_\alpha\,.$$

One can associate with it a pseudo-differential operator (p.d.o.) (this process is called the h-quantization of Weyl)

$$(Q^W(x, hD_x, h)u)(x) = (2\pi h)^{-1} \int e^{(i/h)\langle x-x', \xi\rangle} Q\left(\frac{x+x'}{2}, \xi, h\right) u(x')dx'd\xi .$$

Similarly, one can define the classes of operator valued symbols:

$$S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B^k_y, B^{k'}_y)) ,$$
$$S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B^k_y, \mathbb{C})) ,$$
$$S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(\mathbb{C}, B^k_y)) ,$$

where the $B^k_y (k \in \mathbb{Z})$ are defined by:

$$B^0_y = L^2(\mathbb{R}_y) ,$$
$$B^k_y = (1 + D^2_y + y^2)^{-k} B^0_y .$$

B^{-k}_y is the dual of B^k_y, and $\mathscr{L}(E, F)$ designates the space of bounded operators from E to F. Unless specified otherwise, we will in general denote the symbol and its corresponding operator by the same letter. We will use $\sharp(\sharp_h)$ to denote the composition of 1-quantized (h-quantized) Weyl symbols.

In this paper, we will also frequently encounter Q, symbols in $S^0(\mathbb{R}^4_{x,\xi,y,\eta})$. We associate with Q the following p.d.o. (the Weyl (h, 1)-quantization):

$$\begin{aligned}(Q^W(x, hD_x, y, D_y, h)u)(x, y) \\ = (4\pi^2 h)^{-1} \int e^{(i/h)\langle x-x', \xi\rangle} e^{i\langle y-y', \eta\rangle} Q\left(\frac{x+x'}{2}, \frac{y+y'}{2}, \xi, \eta, h\right) u(x, y')dx'dy'd\xi d\eta .\end{aligned}$$

We identify Q with operator valued symbols in $S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(L^2(\mathbb{R}_y), L^2(\mathbb{R}_y)))$. We use $\sharp$ to denote the composition of such operator valued symbols when the (x, ξ) variables are held fixed and the composition is only in the (y, η) variables.

We now consider the operator $P^\omega_{B,v}$ in the introduction. By a standard argument, there exists a unitary operator U such that $P^\omega_{B,V}$ is unitarily equivalent to

$$\tilde{P}^\omega_{B,V} = UP^\omega_{B,V}U^{-1} = B(D^2_y + y^2) + V^W(x + B^{-1/2}y, B^{-1}D_x - B^{1/2}D_y) .$$

(See Proposition 1.10 of [HS]). Renormalizing, we get (for simplicity, we now generally drop the superscript ω)

$$H^{(n)}_{B,V} = D^2_y + y^2 - (2n+1) + B^{-1}V^W(x + B^{-1/2}y, B^{-1}D_x - B^{-1/2}D_y) ,$$

where n is the Landau level that we are interested in, $B^{-1} = h$ is our semi-classical parameter as mentioned earlier. (Note that the symbol of V^W is $V(x + h^{1/2}y, \xi - h^{1/2}\eta)$.)

Let z be a complex parameter such that hz is in the open disc $B(0, 1)$ centered at 0 with radius 1. Define $R^{(n)}_-$ to be the operator from $L^2(\mathbb{R}_x)$ to $L^2(\mathbb{R}^2_{x,y})$ such that

$$(R^{(n)}_- v_-)(x, y) = h_n(y)v_-(x) ,$$

where h_n is the n^{th} eigenfunction of the harmonic oscillator, defined by

$$h_n(y) = \beta_n(\partial_y - y)^n e^{-y^2/2}$$

with the constant $\beta_n > 0$ chosen such that $\|h_n\| = 1$. The operator $R_+^{(n)}$ from $L^2(\mathbb{R}^2_{x,y})$ to $L^2(\mathbb{R}_x)$ is defined by

$$u \mapsto (R_+^{(n)}u)(x) = \int h_n(y)u(x,y)\,dy\,.$$

Note that $R_+^{(n)}$ is the adjoint of $R_-^{(n)}$. $R_-^{(n)}$ is in $S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(\mathbb{C}, B_y^k))$ and $R_+^{(n)}$ in $S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B_y^{k+2}, \mathbb{C}))$. We study the following associated Grushin operator:

$$\mathscr{P}_{B,V}(x,\xi) = \begin{pmatrix} hz - H_{B,V}^{(n)} & R_-^{(n)} \\ R_+^{(n)} & 0 \end{pmatrix},$$

where

$$H_{B,V}^{(n)} = D_y^2 + y^2 - (2n+1) + hV^W(x + h^{1/2}y, \xi - h^{1/2}D_y)$$

is in $S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B_y^{k+1}, B_y^k))$. The operator $\mathscr{P}$ is in the class $S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B_y^{k+1} \times \mathbb{C}, B_y^k \times \mathbb{C}))$ for all non-negative integers k.

1. The Case $V = 0$. We have

$$\mathscr{P}_{B,0}^{(n)}(z) = \begin{pmatrix} (2n+1) + hz - D_y^2 - y^2 & R_-^{(n)} \\ R_+^{(n)} & 0 \end{pmatrix}.$$

For $|hz| < 1$, $\mathscr{P}_{B,0}^{(n)}(z)$ is invertible [HS], and its inverse, denoted by $\mathscr{E}_0^{(n)}(z)$ can be written as

$$\mathscr{P}_{B,0}^{(n)}(z)^{-1} = \mathscr{E}_0^{(n)}(z) = \begin{pmatrix} E_0^{(n)}(z) & E_{+,0}^{(n)} \\ E_{-,0}^{(n)} & E_{-+,0}^{(n)}(z) \end{pmatrix}.$$

We decompose $L^2(\mathbb{R}^2_{x,y})$ as follows:

$$L^2(\mathbb{R}^2_{x,y}) = \sum_l \mathbb{C}h_l(y) \otimes L^2(\mathbb{R}_x) = \bigoplus_l E_l\,.$$

It is well known [HS] that $E_0^{(n)}(z)$ is a p.d.o. whose Weyl symbol $e_0(x,y,\xi,\eta;z)$ satisfies

$$e_0(x,y,\xi,\eta;z) = e_0(y,\eta;z)\,.$$

e_0 is holomorphic in z in $\mathbf{B}(0,h^{-1})$ and is in class S^0. In the decomposition $\bigoplus_l E_l$, we have:

$$(E_0^{(n)}(z))_{\mathscr{L}(E_l,E_k)} = (1 - \delta_{n,l})\delta_{l,k}/(2(l-n) - z)$$

(where one has identified E_l with $L^2(\mathbb{R}_x)$). One also has

$$E_{0,+}^{(n)} = R_-^{(n)}; \qquad E_{0,-}^{(n)} = R_+^{(n)}; \qquad E_{-+,0}^{(n)}(z) = -hz\,.$$

2. The Case $V \neq 0$. By standard perturbation theory, it can be shown [HS] that there exists an h_0 such that for all $h \in]0, h_0]$, $\mathscr{P}_{B,V}^{(n)W}$ is invertible with the inverse $\mathscr{E}^W$

whose (operator valued) symbol

$$\mathscr{E}(x,\xi;z) = \mathscr{P}_{B,v}^{(n)^{-1}} = \begin{pmatrix} E(x,\xi;z) & E_+(x,\xi;z) \\ E_-(x,\xi;z) & E_{-+}(x,\xi;z) \end{pmatrix},$$

where

$$h\mathscr{E}(x,\xi) \in S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B^k_y \times \mathbb{C}, B^{k+1}_y \times \mathbb{C})),$$

$$E_+(x,\xi) \in S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(\mathbb{C}, B^{k+1}_y)),$$

$$E_-(x,\xi) \in S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(B^k_y, \mathbb{C})),$$

$$E_{-+}(x,\xi)/h \in S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(\mathbb{C},\mathbb{C})) = S^0(\mathbb{R}^2_{x,\xi}).$$

E_{-+} is called the effective hamiltonian and one has

$$hz \in \sigma(H_{B,V}^{(n)}) \Longleftrightarrow 0 \in \sigma(E_{-+}).$$

This is the operator that we shall be studying for the rest of the paper.

If $hz \notin \sigma(H_{B,V}^{(n)})$, then

$$(hz - H_{B,V}^{(n)})^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

Moreover, it can be shown [HS] that there exists $B_0 = h_0^{-1}$ such that for all B with $|B| \geqq B_0$, E_{-+} is a symbol analytic in z and $\alpha_i (i \in \mathbb{Z}^2)$ for $z \in \mathbf{B}(0,h^{-1})$ and $\alpha_i \in \mathbf{B}(0,a)$, and is real for z real. E_{-+} has an asymptotic expansion in h:

$$E_{-+} = \sum_{n\geqq 0} h^{n+1} Q_n + \mathbf{O}(h^\infty).$$

The principal symbol is

$$Q_0(x,\xi;z) = (V(x,\xi) - z).$$

The next term is

$$Q_1(x,\xi;z) = \frac{2n+1}{4} \text{ Tr Hess } V(x,\xi),$$

where Hess denotes the hessian.

III. The Density of States Measure

Let $f \in C_0(\mathbb{R})$; it can be shown [HS] by using the various Sobolev spaces naturally associated to $P_{B,V}$ that the distribution kernel $K^\omega_{f,B}$ of $f(P^\omega_{B,V}) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. Since

$$(\tau^B_j)^{-1} P^\omega_{B,V} \tau^B_j = P^{T_j\omega}_{B,V},$$

we have

$$(\tau^B_j)^{-1} f(P^\omega_{B,V}) \tau^B_j = f(P^{T_j\omega}_{B,V}).$$

Let $K^\omega_{f,B}(x,y)$ be the kernel of $f(P^\omega_{B,V})$; we have

$$K^\omega_{f,B}(x+j, x+j) = K^{T_j\omega}_{f,B}(x,x)$$

for all $j \in \mathbb{Z}^2$. By the ergodic theorem, we have

$$\lim_{L\to\infty} \frac{1}{(2L+1)^2} \sum_{|j|\leqq L} \int_{x\in \mathbb{E}} K_{f,B}^{T_j\omega}(x,x)\,dx = \int \int_{x\in\mathbb{E}} K_{f,B}^{\omega}(x,x)dx P(d\omega) \quad \text{a.s.}$$
$$= \left\langle \int_E K_{f,B}(x,x)\,dx \right\rangle , \tag{3.1}$$

where $L \in \mathbb{N}$, $|\ \ |$ denotes the sup-norm, $\mathbb{E}$ is the unit square centered at the origin and $\langle\ \rangle$ is, like before, the expectation with respect to ω. Define

$$\tilde{\operatorname{Tr}} f(P_{B,V}^{\omega}) = \lim_{R\to\infty} \frac{1}{4R^2} \operatorname{Tr}(\chi_R' f(P_{B,V}^{\tilde{\omega}})\chi_R') , \tag{3.2}$$

where χ_R' is a C_0^∞ function on $\mathbb{R}^2$ with support in the square of center 0 and length $2R$ and equal to 1 in the square of center 0 and length $2(R-1)$ with derivatives bounded independently of R.

We now show that almost surely $\tilde{\operatorname{Tr}} f(P_{B,V}^{\omega})$ exists and is finite. Since for any $R \in (L-1/2, L+1/2)$ $(L > 1)$

$$(\operatorname{Tr} \chi_R' f(P_{B,V}^{\omega})\chi_R' - \operatorname{Tr} \chi_{L+1/2}' f(P_{B,V}^{\omega})\chi_{L+1/2}')/4R^2 = \mathbf{O}(1/R) ,$$

we only need to show that the limit in the R.H.S. of (3.2) exists along half integers $(R = L+1/2)$. Let

$$D = \int_{|X|\leqq L+1/2} K^{\omega}(x,x)dx - \int_{|X|\leqq L+1/2} \chi_R'(x)K^{\omega}(x,x)\chi_R'(x)\,dx .$$

Note that the difference in the integrands has support only for $L-1/2 \leqq |x| \leqq L+1/2$. We conclude that $|D| \leqq CL$. Therefore the limit in the R.H.S. of (3.2) exists almost surely and we have

$$\tilde{\operatorname{Tr}} f(P_{B,V}^{\omega}) = (\operatorname{vol} \mathbb{E})^{-1} \int_{\mathbb{E}} K_{f,B}^{\omega}(x,x)P(d\omega)$$
$$= (\operatorname{vol} \mathbb{E})^{-1} \int_{\mathbb{E}} \langle K_{f,B}(x,x)\rangle dx \quad \text{a.s.}$$
$$= \langle \tilde{\operatorname{Tr}} f(P_{B,V}^{\omega})\rangle .$$

We can associate to $\tilde{\operatorname{Tr}}$ a positive measure $\rho_{B,V}$. It is the *density of states measure*. We have

$$\langle \tilde{\operatorname{Tr}} f(P_{B,V}^{\omega})\rangle = \int f(E)\rho_{B,V}(dE)$$

for all $f \in C_0^\infty(\mathbb{R})$.

If $f \in C_0^\infty(\mathbb{R})$, we can use the usual functional calculus [HS] (see also [S]) to obtain for A self-adjoint

$$f(A) - \frac{i}{2\pi} \int \partial_{\bar{z}} \tilde{f}(z)(z-A)^{-1} d\bar{z} \wedge dz ,$$

where $\tilde{f} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of f, i.e. $\tilde{f} = f$ on $\mathbb{R}$ and $\partial_{\bar{z}}\tilde{f}$ vanishes on $\mathbb{R}$ to infinite order. In order for $f(A)$ to be a pseudo-differential operator, we shall choose f in a restricted class of h-dependent functions in C_0^∞ that will be made more precise later.

Using the same unitary transformation U as the one we used for $P^{\omega}_{B,V}$, we obtain

$$\operatorname{Tr}\chi'_R f(P^{\omega}_{B,V})\chi'_R = \operatorname{Tr} U\chi'_R U^{-1} U\, f(P^{\omega}_{B,V}) U^{-1} U\chi'_R U^{-1} .$$

Writing $\chi_R = U\chi'_R U^{-1}$, the symbol of χ_R is $\chi_R(x,y,\xi,\eta) = \chi'_R(x + h^{1/2}y, \xi - h^{1/2}\eta)$. We have

$$\chi^W_R(x + h^{1/2}y, \xi - h^{1/2}D_y) \in S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(L^2(\mathbb{R}_y), B^{-k}_y))$$

with $k \geqq 0$. For all z such that $|\operatorname{Im} z| \neq 0$, we have

$$((2n+1)B + z - \tilde{P}_{B,V})^{-1} = h(hz - H_{B,V})^{-1} = h[E(z) - E_+(z)E^{-1}_{-+}(z)E_-(z)] .$$

Since $E(z)$ is holomorphic in z, we have

$$U f(P^{\omega}_{B,V}) U^{-1} = f(\tilde{P}^{\omega}_{B,V}) = -\frac{ih}{2\pi}\int \partial_{\bar z}\tilde{f} E_+(z)E^{-1}_{-+}(z)E_-(z) dz \wedge d\bar z$$

as before. Therefore

$$\operatorname{Tr}\chi'_R f(P^{\omega}_{B,V})\chi'_R = -\frac{ih}{2\pi}\operatorname{Tr}\left\{\chi_R\left(\int \partial_{\bar z}\tilde{f} E_+(z)E^{-1}_{-+}(z)E_-(z) d\bar z \wedge dz\right)\chi_R\right\} .$$

We now show that $\chi_R E_+ E^{-1}_{-+} E_- \chi_R$ is trace class for $|\operatorname{Im} z| \neq 0$. Hence we will have

$$\operatorname{Tr}\chi'_R f(P^{\omega}_{B,V})\chi'_R = -\frac{ih}{2\pi}\int \partial_{\bar z}\tilde{f}\operatorname{Tr}(\chi_R E_+ E^{-1}_{-+} E_- \chi_R) d\bar z \wedge dz . \tag{3.3}$$

Lemma 1. *The operator $E_-\chi_R$ is Hilbert–Schmidt from $L^2(\mathbb{R}_x, L^2(\mathbb{R}_y))$ to $L^2(\mathbb{R}_x, \mathbb{C}) = L^2(\mathbb{R}_x)$. Moreover $\|E_-\chi_R\|_{HS} \leqq Ch^{-1/2}R$, where C is a constant independent of R, h and ω.*

To prove the above lemma, we need the following well known fact:

Proposition 1. *The operator $P^W(x, hD_x)$ from $L^2(\mathbb{R}_x; \mathscr{H}_1)$ to $L^2(\mathbb{R}_x; \mathscr{H}_2)$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are arbitrary Hilbert spaces, is Hilbert–Schmidt if and only if*

$$\int \|P^W(x,\xi)\|^2_{HS(\mathscr{H}_1,\mathscr{H}_2)} dx\, d\xi < \infty .$$

Moreover, when $P^W(x, hD_x)$ is Hilbert–Schmidt, we have

$$\|P^W(x, hD_x)\|^2_{HS} = \frac{1}{2\pi h}\int \|P^W(x,\xi)\|^2_{HS(\mathscr{H}_1,\mathscr{H}_2)} dx\, d\xi .$$

Proof of Lemma 1. We consider $E_- \# \chi_R(x,\xi)$ as the composition:

$$\mathbb{C} \xleftarrow{E_-} B^{-k'}_y \xleftarrow{I} B^{-k}_y \xleftarrow{\chi_R} L^2 \quad k' > k > 0 .$$

Let Λ_{-k} be the symbol for $[1 + y^2 + D^2_y]^{-k}$. It verifies

$$|\partial^{\alpha}_y \partial^{\beta}_{\eta} \Lambda_{-k}(y,\eta)| \leqq C_{\alpha,\beta}(1 + \|(y,\eta)\|)^{-2k-\alpha-\beta} ,$$

for some $C_{\alpha,\beta}$ and where $\|(y,\eta)\| = (y^2 + \eta^2)^{1/2}$. Let

$$K_{x,\xi}(y,\eta) = (\Lambda_{-k} \# \chi_R)(y,\eta) .$$

Using the composition formula for two Weyl 1-quantization symbols a, b:

$$a^W \sharp b^W(y,\eta) = \left(\frac{1}{2\pi}\right)^2 \int\int e^{i(y_1\eta_2 - y_2\eta_1)} a(y - y_1, \eta - \eta_1)$$

$$\times\, b(y - y_2, \eta - \eta_2) dy_1 dy_2 d\eta_1 d\eta_2 \,,$$

we have

$$K_{x,\xi}(y,\eta) = \frac{1}{4\pi^2}\int\int e^{i(y_1\eta_2 - y_2\eta_1)} \Lambda_{-k}(y - y_1, \eta - \eta_1)$$

$$-\, \chi_R(x + h^{1/2}(y - y_2), \xi - h^{1/2}(\eta - \eta_2)) dy_1 dy_2 d\eta_1 d\eta_2 \,.$$

Since χ_R has support in a square of length $2R$, we have

$$|(y - y_2, \eta - \eta_2)| \geqq \frac{1}{Ch^{1/2}}[|(x,\xi)| - R] \tag{3.4}$$

for some $C > 0$. Let $|(x,\xi)| \geqq R + 1$. Define:

$$Y = (y_1, \eta_1, y_2, \eta_2)$$

$$Y_i = \text{components of } Y$$

$$Q(Y) = y_1\eta_2 - \eta_1 y_2$$

$$L^t(Y, D_Y) = (\gamma_Y + (1 - \gamma_Y)\|\nabla Q(Y)\|^2)^{-1}(\gamma_Y + (1 - \gamma_Y)\nabla Q(Y) D_Y)\,, \tag{3.5}$$

where γ_Y is a C_0^∞ function: $\gamma_Y = 1$ for $\|Y\| \leqq 1 - \beta, \gamma_Y = 0$ for $\|Y\| \geqq 1$, $(0 < \beta < 1)$ then

$$L^t e^{iQ(Y)} = e^{iQ(Y)} \,.$$

By a straightforward computation, we have

$$L(Y, D_Y) = a_0(Y) + \sum_{i=1}^{4} a_i(Y) D_{Y_i}$$

where

$$|\partial^\alpha a_0| \leqq C(\|Y\| + 1)^{-2-|\alpha|} \,,$$

$$|\partial^\alpha a_j| \leqq C(\|Y\| + 1)^{-1-|\alpha|} \,. \tag{3.6}$$

Integration by parts $N + 5$ times, with $N \geqq k$, we have

$$K_{x,\xi}(y,\eta) = \frac{1}{4\pi^2}\int\int e^{iQ(Y)} \Gamma^{(x,\xi)}_{N,-k,R}(y,\eta;Y) dY \,,$$

where

$$\Gamma^{(x,\xi)}_{N,-k,R}(y,\eta;Y) = L^{N+5}[\Lambda_{-k}(y - y_1, \eta - \eta_1)\chi_R(x + h^{1/2}(y - y_2), \xi - h^{1/2}(\eta - \eta_2))] \,.$$

From (3.5) and the bounds on Λ_{-k}, χ_R and their derivatives, we have

$$|\Gamma^{(x,\xi)}_{N,-k,R}(y,\eta;Y)| \leqq C_N(1 + y_1^2 + \eta_1^2 + y_2^2 + \eta_2^2)^{-(N+5)/2}(1 + \|(y - y_1, \eta - \eta_1)\|)^{-2k}$$

$$F(|(y - y_2, \eta - \eta_2)| - \frac{|(x,\xi)| - R}{Ch^{1/2}}) \,,$$

where F is such that

$$F(u) = 1 \quad \text{if } u \geqq 0\,,$$

$$F(u) = 0 \quad \text{otherwise}\,.$$

Case 1.

$$|(y,\eta)| \leqq \frac{|(x,\xi)| - R}{2Ch^{1/2}}\,.$$

From (3.4) we have:

$$|(y_2,\eta_2)| \geqq \frac{1}{2Ch^{1/2}}(|(x,\xi)| - R)\,.$$

Therefore

$$|\Gamma_N^{(x,\xi)}(y,\eta)| \leqq \frac{C_N}{(1+\|Y\|^2)^{5/2}}\frac{1}{[(|(x,\xi)|-R)_+ + 1]^N}$$

for all (x,ξ).

Case 2.

$$|(y,\eta)| \geqq \frac{|(x,\xi)| - R}{2Ch^{1/2}}\,, \tag{3.7}$$

then

$$|\Gamma_N^{(x,\xi)}(y,\eta;Y)| \leqq \frac{C_N}{[1+\|Y\|^2]^{5/2}}[1+|(y-y_1,\eta-\eta_1)|]^{-2k}[1+|(y_1,\eta_1)|]^{-N}\,.$$

Using (3.7), we have

$$|\Gamma_N^{(x,\xi)}(y,\eta;Y)| \leqq \frac{C_N}{[1+\|Y\|^2]^{5/2}}\frac{1}{[(|(x,\xi)|-R)_+ + 1]^{\min(N,2k)}}$$

for all (x,ξ).

Hence, taking $N = k$, we have

$$|K_{x,\xi}(y,\eta)| \leqq \frac{1}{4\pi^2}\int \Gamma_N^{(x,\xi)}(y,\eta;Y)\,dY \leqq C_k[1+(|(x,\xi)|-R)_+]^{-k}$$

for all (x,ξ). Similarly, we have

$$|\partial_y^\alpha \partial_\eta^\beta K_{x,\xi}(y,\eta)| \leqq C_{k,\alpha,\beta}[1+(|(x,\xi)|-R)_+]^{-k}$$

for all α,β and all (x,ξ). Therefore

$$\begin{aligned}
&\|\chi_R^W(x+h^{1/2}y,\xi-h^{1/2}D_y)\|_{\mathscr{L}(L^2(\mathbb{R}_y),B_y^{-k})} \\
&= \|(1+y^2+D_y^2)^{-k}\chi_R^W(x+h^{1/2}y,\xi-h^{1/2}D_y)\|_{\mathscr{L}(L^2(\mathbb{R}_y),L^2(\mathbb{R}_y))} \quad (k \geqq 0) \\
&\leqq C_k[1+(|(x,\xi)|-R)_+]^{-k}\,.
\end{aligned} \tag{3.8}$$

Since the embedding of B^m in B^n for $m > n+1$ is Hilbert–Schmidt, we have

$$\begin{aligned}
\|\tilde{\chi}_R(x,\xi)\|_{HS(L^2,B^{-k'})} &\leqq \|I\|_{HS(B^{-k},B^{-k'})}\|\tilde{\chi}_R(x,\xi)\|_{\mathscr{L}(L^2,B^{-k})} \\
&\leqq C_{k,k'}[1+(|(x,\xi)|-R)_+]^{-k}
\end{aligned}$$

for all $k' > k+1$ and all (x,ξ), where $\tilde{\chi}_R$ is χ_R viewed as an operator valued symbol. Similarly, we have

$$\|\partial_x^\alpha \partial_\xi^\beta \tilde{\chi}_R(x,\xi)\|_{HS(L^2,B^{-k'})} \leqq C_{\alpha,\beta,k,k'}[1+(|(x,\xi)|-R)_+]^{-k}$$

for all $k' > k+1$. Since χ_R is chosen such that the bounds on χ_R and its derivatives are independent of R, the composition of the symbols:

$$E_- \sharp \chi_R \in S^0(\mathbb{R}^2_{x,\xi}; \mathscr{L}(L^2(\mathbb{R}_y), \mathbb{C}))$$

uniformly in R. Hence

$$\|E_- \sharp \tilde{\chi}_R(x,\xi)\|_{HS(L^2_y,\mathbb{C})} \leqq C_k[1+(|(x,\xi)|-R)_+]^{-k} \tag{3.9}$$

for all $k > 0$. By Proposition 1, we have

$$\|E_-\chi_R\|_{HS} \leqq Ch^{-1/2}R\,.$$

Proposition 2. *The operator* $\chi_R E_+ E_{-+}^{-1} E_- \chi_R$ *is trace class on* $L^2(\mathbb{R}_x, L^2(\mathbb{R}_y))$ *for* $|\mathrm{Im}\, z| \neq 0$.

Proof.

$$\|\chi_R E_+ E_{-+}^{-1} E_- \chi_R\|_{\mathrm{T}_1} \leqq \|\chi_R E_+\|_{HS} \|E_{-+}^{-1}\| \|E_-\chi_R\|_{HS} \leqq (C^2 h^{-1}/|\mathrm{Im}\, z|)R^2\,,$$

where we have used the fact that $\chi_R E_+$ is the adjoint of $E_-\chi_R$, and is therefore Hilbert–Schmidt.

Using the property of cyclic invariance of the trace, we have

$$\begin{aligned}
\mathrm{Tr}\,(\chi_R E_+ E_{-+}^{-1} E_- \chi_R) &= \mathrm{Tr}\,(E_- \chi_R^2 E_+ E_{-+}^{-1}) \\
&= \mathrm{Tr}\,((\chi_R^{(1)})^2 E_- E_+ E_{-+}^{-1}) + \mathrm{Tr}\,([E_-,\chi_R]_0(\chi_R E_+)E_{-+}^{-1}) + \mathrm{Tr}\,(\chi_R^{(1)}[E_-,\chi_R]_0 E_+ E_{-+}^{-1}) \\
&= \mathrm{Tr}\, A_1 + \mathrm{Tr}\, A_2 + \mathrm{Tr}\, A_3\,,
\end{aligned} \tag{3.10}$$

where $\chi_R^{(1)}(x,\xi) = \chi_R(x,\xi)$ and we have written

$$[E_-,\chi_R]_0 = E_-\chi_R - \chi_R^{(1)} E_-\,.$$

By the classical equality:

$$\|\chi_R^{(1)}\|_{HS}^2 = \frac{1}{2\pi h}\int |\chi_R(x,\xi)|^2 dx\, d\xi\,,$$

we obviously have:

Lemma 2. $\chi_R^{(1)}$ *is Hilbert–Schmidt, and*

$$\|\chi_R^{(1)}\|_{HS} \leqq Ch^{-1/2}R\,.$$

Corollary. $A_1 = (\chi_R^{(1)})^2 E_- E_+ E_{-+}^{-1}$ *is trace-class for* $|\mathrm{Im}\, z| \neq 0$:

$$\mathrm{Tr}\,((\chi_R^{(1)})^2 E_- E_+ E_{-+}^{-1}) = \mathrm{Tr}\,(\chi_R^{(1)} E_- E_+ E_{-+}^{-1} \chi_R^{(1)})\,.$$

Furthermore, $\mathrm{Tr}\, A_1 \leqq Ch^{-1}|\mathrm{Im}\, z|^{-1} R^2$, *with* C *independent of* R, h *and* ω.

Lemma 3. *A_2 and A_3 are trace-class and*

$$\operatorname{Tr} A_2 \leqq Ch^{-1}|\operatorname{Im} z|^{-1}R^{3/2} \quad \operatorname{Tr} A_3 \leqq Ch^{-1}|\operatorname{Im} z|^{-1}R^{3/2} ,$$

where C is independent of R, h and ω.

Proof. We already know that

$$\|[E_-, \chi_R]_0\|_{HS} \leqq Ch^{-1/2}R .$$

We now show that, in fact

$$\|[E_-, \chi_R]_0\|_{HS} \leqq Ch^{-1/2}R^{1/2} .$$

Let $\chi_R^c = 1 - \chi_R$. Replacing $\tilde{\chi}_R$ by $\tilde{\chi}_R^c$ in (3.8), we have

$$|E_- \sharp \tilde{\chi}_R^c(x,\xi)\|_{HS(L_y^2, \mathbb{C})} \leqq C_k[1 + (R - |(x,\xi)|)_+]^{-k} \quad \text{for all } k > 0 . \tag{3.11}$$

Let $\chi_R^{(1)c} = 1 - \chi_R^{(1)}$. Viewing $\chi_R^{(1)c} \sharp E_-(x,\xi)$ as the composition of operators

$$\mathbb{C} \xleftarrow{\chi_R^{(1)c}} \mathbb{C} \xleftarrow{E_-} B^{-k'} \xleftarrow{I} L^2 \quad k' > 0 ,$$

and using the fact that the embedding of L^2 in $B^{-k'}$ is Hilbert–Schmidt for $k' > 1$, we have

$$\begin{aligned} \|\chi_R^{(1)c} \sharp E_-(x,\xi)\|_{HS(L_y^2, \mathbb{C})} &\leqq C_{k'} \|\chi_R^{(1)c} \sharp E_-(x,\xi)\|_{\mathscr{L}(B^{-k'}, \mathbb{C})} \|I\|_{HS(L^2, B^{-k'})} \\ &\leqq C_k[1 + R - |(x,\xi)|)_+]^{-k} \end{aligned} \tag{3.12}$$

for all $k > 0$, where we used the composition formula for two symbols. Hence from (3.11) and (3.12),

$$\|[E_-, \chi_R^c]_0(x,\xi)\|_{HS} \leqq C_k[1 + (R - |(x,\xi)|)_+]^{-k} \quad k > 0 .$$

Since

$$[E_-, \chi_R]_0 = -[E_-, \chi_R^c]_0$$

and

$$\|[E_-, \chi_R]_0(x,\xi)\|_{HS} \leqq C_k[1 + (|(x,\xi)| - R)_+]^{-k} \quad k > 0 ,$$

we have

$$\|[E_-, \chi_R]_0(x,\xi)\|_{HS(L_y^2, \mathbb{C})} \leqq C_k[1 + ||(x,\xi)| - R|]^{-k} \quad k > 0 .$$

Therefore

$$\|[E_-, \chi_R]_0\|_{HS(L^2(\mathbb{R}_x; L_y^2), L^2(\mathbb{R}_x))} \leqq Ch^{-1/2}R^{1/2} .$$

This yields the results for $\operatorname{Tr} A_2$ and $\operatorname{Tr} A_3$.

We now put $\tilde{\operatorname{Tr}} f(P_{B,V}^\omega)$ in a convenient form, which we will use later to compute the asymptotics.

Proposition 3.

$$\begin{aligned} \tilde{\operatorname{Tr}} f(P_{B,V}^\omega) &= \langle \tilde{\operatorname{Tr}} f(P_{B,V}^\omega) \rangle \\ &= -\frac{1}{4\pi^2 ih} \int_{\mathbb{E}} \partial_{\bar{z}} \tilde{f} \langle [\partial_{\bar{z}} E_{-+} \sharp_h E_{-+}^{-1}](x,\xi,z) \rangle (d\bar{z} \wedge dz) dx\, d\xi \quad \text{a.s.} , \end{aligned}$$

where $\mathbb{E}$ is a unit square.

Since E_{-+} is holomorphic in z, it is now enough to control E_{-+}^{-1} when $|\mathrm{Im}\, z| \to 0$.

Proof of Proposition 3. Combining (3.2), (3.3), (3.10), the corollary to Lemma 2 and Lemma 3, we have

$$\begin{aligned}\tilde{\mathrm{Tr}}\, f(P^{\omega}_{B,V}) &= \langle \tilde{\mathrm{Tr}}\, f(P^{\omega}_{B,V})\rangle \\ &= \lim_{R\to\infty} \frac{-1}{16 i\pi^2 R^2 h} \int \partial_{\bar z}\tilde f \langle [\chi_R^{(1)} \sharp_h \partial_z E_{-+} \sharp_h E_{-+}^{-1} \sharp_h \chi_R^{(1)}](x,\xi,z)\rangle \\ &\quad \times (d\bar z \wedge dz)dx\, d\xi \quad \text{a.s.},\end{aligned}$$

where we used the identity (see [HS])

$$h^{-1}\partial_z E_{-+} = E_{-}\sharp E_{+}\,.$$

Let $T = \partial_z E_{-+}\sharp_h E_{-+}^{-1}$, $\mathscr{T} = \chi_R^{(1)}\sharp_h T \sharp_h \chi_R^{(1)}$. For convenience, we view $\mathscr{T}$ as $\mathscr{T} I$, where I is the identity operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, with symbol the function $\mathbf{1}$. Using the composition formula for four Weyl h-quantization symbols, we have (writing $X = (x,\xi)$)

$$\mathscr{T}(X) = \frac{1}{(2\pi h)^4}\int e^{iQ_4(Y)/h}\chi_R^{(1)}(X-Y_1)T(X-Y_2)\chi_R^{(1)}(X-Y_3)\mathbf{1}(X-Y_4)\,dY\,,$$

where $Y = (Y_1,Y_2,Y_3,Y_4)$, $Y_i = (y_i,\eta_i)$, $dY = \prod_i dy_i d\eta_i$ and

$$Q_4(Y) = \sigma(Y_1,Y_2)+\sigma(Y_2,Y_3)+\sigma(Y_3,Y_4)+\sigma(Y_1,Y_4)+\sigma(Y_4,Y_2)+\sigma(Y_3,Y_1)\,;$$

σ is the canonical symplectic form:

$$\sigma(Y_i,Y_j) = y_j\eta_i - y_i\eta_j\,.$$

We first treat the case $|X| > R+1$. Noting that $Q_4(Y)/h = Q_4(h^{-1/2}Y)$, we see that

$$\frac{L^t}{i}(h^{-1/2}Y, h^{1/2}D_Y)e^{iQ_4/h} = e^{iQ_4/h}\,,$$

where L^t is as defined in (3.5) with obvious modifications. By straightforward computation, we have that

$$\|\nabla Q_4(Y)\|^2 = \|Y'\|^2\,,$$

where

$$Y' = \begin{pmatrix} A & O \\ 0 & -A\end{pmatrix} Y\,,$$

$$Y = \begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\\ \eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix},$$

and similarly for Y', and

$$A = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix} .$$

A is clearly invertible, hence $\|Y\|^2/C \leqq \|Y'\|^2 \leqq C\|Y\|^2$ for some constant $C > 0$. Therefore

$$L = b_0(h^{-1/2}Y) + \sum_{i=1}^{8} b_i(h^{-1/2}Y)h^{1/2}D_{Y_i}$$

with $b_0, b_i(i \neq 0)$ satisfy the same estimate as $a_0, a_i(i \neq 0)$ in (3.6). Using the estimates on $b_0, b_i(i \neq 0)$, we obtain that

$$L^N = \sum_{|\beta| \leqq N} b_\beta^N(h^{-1/2}Y)(h^{1/2}D_Y)^\beta \tag{3.13}$$

with

$$|\partial^\alpha b_\beta^N(Y)| \leqq C(\|Y\| + 1)^{-2N+|\beta|-|\alpha|} . \tag{3.14}$$

We therefore have, for all $N \in \mathbb{N}$,

$$\mathscr{T}(X) = \frac{1}{(2\pi h)^4} \int e^{iQ_4(Y)/h} L^N [\chi_R^{(1)}(X - Y_1)T(X - Y_2)\chi_R^{(1)}(X - Y_3)I(X - Y_4)]\, dY .$$

Since $|X| > R + 1$, we have $|Y_1| > (|X| - R)/C, |Y_3| > (|X| - R)/C$. Since

$$\|T\| \leqq \frac{C}{|\operatorname{Im} z|} ,$$

with C independent of ω, we have by Beals' Lemma [Bea] as stated in [S],

$$|\partial_x^\alpha \partial_\xi^\beta T| \leqq C_{\alpha,\beta} \max\left(1, \left(\frac{h}{|\operatorname{Im} z|^2}\right)^3\right) |\operatorname{Im} z|^{-1-\alpha-\beta} ,$$

where $C_{\alpha,\beta}$ is independent of ω. Hence using (3.13), (3.14) and taking $N > 8$, we have

$$|\mathscr{T}(X)| \leqq \frac{C_N}{(|X| - R)^{N-8} |\operatorname{Im} z|^{7+N}} .$$

(We do not keep track of powers of h, since it is not important here.) For $|X| < R - 1$, we write

$$\begin{aligned}
\mathscr{T}(X) = {} & \frac{1}{(2\pi h)^4} \int e^{iQ_4(Y)/h} \mathbf{1}(X - Y_1)T(X - Y_2)\mathbf{1}(X - Y_3)\mathbf{1}(X - Y_4)]\, dY \\
& - \frac{1}{(2\pi h)^4} \int e^{iQ_4(Y)/h} (\mathbf{1} - \chi_R^{(1)})(X - Y_1)T(X - Y_2)\mathbf{1}(X - Y_3)\mathbf{1}(X - Y_4)]\, dY \\
& - \frac{1}{(2\pi h)^4} \int e^{iQ_4(Y)/h} \mathbf{1}(X - Y_1)T(X - Y_2)(\mathbf{1} - \chi_R^{(1)})(X - Y_3)\mathbf{1}(X - Y_4)]\, dY \\
& + \frac{1}{(2\pi h)^4} \int e^{iQ_4(Y)/h} (\mathbf{1} - \chi_R^{(1)})(X - Y_1)T(X - Y_2)(\mathbf{1} - \chi_R^{(1)})(X - Y_3) \\
& \times \mathbf{1}(X - Y_4)]\, dY .
\end{aligned}$$

The last three terms in the sum are bounded by

$$\frac{C_N}{(R-|X|)^{N-8}|\mathrm{Im}\, z|^{7+N}}$$

with C_N independent of ω, by using the same method as before. The first term is precisely

$$T(X) = \partial_z E_{-+} \sharp_h E_{-+}^{-1}(X) .$$

For $R-1 < |X| < R+1, \mathcal{T}(X)$ is bounded for $|\mathrm{Im}\, z| \neq 0$. Hence upon taking the limit $R \to \infty$ and taking into account the periodicity of $\langle \partial_z E_{-+} \sharp_h E_{-+}^{-1}(x,\xi)\rangle$, we obtain the proposition.

IV. Study of the Operator $f(P_{B,V}^{\omega})$

Let S_δ^k denote the class of symbols a such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leqq C_{\alpha,\beta} h^{-k-\delta(\alpha+\beta)} .$$

For $|\mathrm{Im}\, z| \geqq h^\delta (0<\delta<1/2)$, hE_{-+}^{-1} is a p.d.o. whose symbol admits the asymptotic expansion:

$$hE_{-+}^{-1} \sim \sum_{j\geqq 0} F_j ,$$

where

$$F_j = \frac{h^j \tilde{Q}_j}{(V-z)^{2j}} \in S_\delta^{-j(1-2\delta)} = S_\delta^{k_j} \quad (j>0) ,$$

$$F_0 = \frac{1}{V-z} \in S_\delta^\delta = S_\delta^{k_0} ,$$

$$hE_{-+}^{-1} - \sum_{j=0}^{N} F_j \in S_\delta^{k_{N+1}} \quad \text{for all } N \in \mathbb{N} .$$

$\tilde{Q}_j$ are functions of the $Q_i (i \leqq j)$ and their derivatives; in particular, $\tilde{Q}_1 = -Q_1$. (Recall that $E_{-+} = \sum_{n\geqq 0} h^{n+1} Q_n + \mathbf{O}(h^\infty)$.) Since $\partial_z E_{-+}$ is also in class S^0 and has the asymptotic expansion:

$$\partial_z E_{-+}^\omega = h(1 + hS_1^\omega + h^2 S_2^\omega + \cdots)$$

with $S_i \in S^0$ for all $\omega, \partial_z E_{-+}^\omega \sharp_h (E_{-+}^\omega)^{-1}$ admits the following asymptotic expansion:

$$\partial_z E_{-+}^\omega \sharp_h E_{-+}^{\omega\ -1} = \frac{1}{V^\omega - z} + h\left(\frac{S_1^\omega}{V^\omega - z} + \frac{\tilde{Q}_1^\omega}{(V^\omega - z)^2}\right) + \cdots$$

for $|\mathrm{Im}\, z| \geqq h^\delta (0<\delta<1/2)$.

Using the random character of the parameters α_i, we have:

Proposition 4. *For all $N \in \mathbb{N}$, $C>0$ there exists h_0, such that for all $h \in]0,h_0]$, all z satisfying $|z| < h^{-1}$ with $|\mathrm{Re}\, z| \geqq 1/C$ and $|\mathrm{Im}\, z| \geqq h^N/C, \langle \partial_z E_{-+} \sharp_h E_{-+}^{-1}\rangle$ is* *in class* S^0, *i.e.* $|\partial_x^\alpha \partial_y^\beta \langle (\partial_z E_{-+} \sharp_h E_{-+}^{-1})(x,\xi,z)\rangle| \leqq C_{N,\alpha,\beta}$. *Moreover,* $\langle \partial_z E_{-+} \sharp_h E_{-+}^{-1}\rangle$ *is a classical symbol in* S_0, *i.e. there exist* $A_j \in S_0$, *such that*

$$\langle \partial_z E_{-+} \sharp_h E_{-+}^{-1}\rangle = \sum_{j\geqq 0} h^j A_j + \mathbf{O}(h^\infty)\,,$$

where $A_j \in S^0$.

We shall prove the above proposition by scaling: we gradually take into account the potentials at larger and larger length scales. Since $\langle \partial_z E_{-+} \sharp_h E_{-+}^{-1}\rangle$ is $\mathbb{Z}^2$ translationally invariant, we only need to prove Proposition 4 for $(x,\xi) \in \mathbb{E}$, where $\mathbb{E}$ is the unit square centered at (0,0).

Without loss of generality, we may assume v is a C_0^∞ function with support contained in the square centered at $(0,0)$ and of length b, with $b \in \mathbb{N}$. Let Λ_n be an increasing sequence of squares centered at $(0,0)$ of length $l_n = 2^n b + 2$. Let Λ_n', Λ_n'' be two corresponding sequences of squares centered at $(0,0)$ of lengths $l_n' = l_n - b$ and $l_n'' = l_n + b$ respectively. Clearly for all $(x,\xi) \in \mathbb{E}$, $\mathrm{dist}((x,\xi), \partial\Lambda_n') \geqq (l_n' - 1)/2$, where $\partial\Lambda_n'$ is the boundary of Λ_n'. Let H_n be the reduced operator when $V_n(\bar{x}) = \sum_{i\in\Lambda_n} \alpha_i v(\bar{x} - i)(E_{-+}$ is the reduced operator when $V(\bar{x}) = \sum_{i\in\mathbb{Z}^2} \alpha_i v(\bar{x} - i))$. The principal symbol for H_n is clearly $h(V_n(x,\xi) - z)$. We write

$$\begin{aligned} E_{-+}^{-1} &= H_0^{-1} + H_1^{-1} - H_0^{-1} + H_2^{-1} - H_1^{-1} + \cdots + H_{N_0}^{-1} - H_{N_0-1}^{-1} + E_{-+}^{-1} - H_{N_0}^{-1} \\ &= H_0^{-1} + H_0^{-1}\Delta_1 H_1^{-1} + H_1^{-1}\Delta_2 H_2^{-1} + \cdots + H_{N_0}^{-1}\Delta H_{N_0-1}^{-1} + H_{N_0}^{-1}\tilde{\Delta}_{N_0+1}E_{-+}^{-1}\,, \end{aligned}$$

where $\Delta_n = H_{n-1} - H_n$, $\tilde{\Delta}_{N_0+1} = H_{N_0} - E_{-+}$ and we used the resolvent equation: $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$; N_0 is yet to be determined. Hence:

$$\begin{aligned} \partial_z E_{-+} E_{-+}^{-1} =& (\partial_z E_{-+})H_0^{-1} + (\partial_z E_{-+})H_0^{-1}\Delta_1 H_1^{-1} + (\partial_z E_{-+})H_1^{-1}\Delta_2 H_2^{-1} \\ &+ \cdots + (\partial_z E_{-+})H_{N_0}^{-1}\tilde{\Delta}_{N_0+1}E_{-+}^{-1}\,. \end{aligned} \tag{4.1}$$

To prove the above proposition, we need to complex dilate in α. The Grushin problem remains well posed and we have the following:

Lemma 4. *Let* $h \in]0, h_0]$. *Suppose* z *is such that* $|z| < h^{-1}$, *and* α *is such that* $|\alpha|_\infty \leqq 2R_0$ *for some* $R_0 > 0$, *then for* t *such that* $|t|_\infty \leqq R_0$, *we have*

$$|\langle \nabla_\alpha H_n(\alpha), t\rangle| \leqq \sup_{|w|=1} |H_n(\alpha + wt)|\,,$$

$$|\langle \nabla_\alpha E_{-+}(\alpha), t\rangle| \leqq \sup_{|w|=1} |E_{-+}(\alpha + wt)|\,.$$

Proof. We prove the second assertion, the first can be proved in the same way. We have

$$\langle \nabla_\alpha E_{-+}(\alpha), t\rangle = \frac{\partial}{\partial w} E_{-+}(\alpha + wt)|_{w=0}\,.$$

Hence

$$|\langle \nabla_\alpha E_{-+}(\alpha), t\rangle| \leqq \sup_{|w|=1} |E_{-+}(\alpha + wt)|$$

by the Cauchy inequality.

For z such that $\operatorname{Re} z \operatorname{Im} z < 0 (\operatorname{Re} z \operatorname{Im} z > 0)$, we define $V_\theta = e^{i\theta} V (V_\theta = e^{-i\theta} V)$ $(\theta > 0)$. Let $E_{-+}(\theta)$, $H_n(\theta)$ be the corresponding reduced operator.

Corollary. *$h^{-1}\partial E_{-+}(\theta)/\partial\theta$ and $h^{-1}\partial H_n(\theta)/\partial\theta$ are bounded symbols in S_0.*

From [HS] we have the following asymptotic expansion for H_n:

$$H_n(x,\xi,h,z) = \sum_{j\geqq 0} \bar{Q}_j^{(n)}(x,\xi,g,z,h)h^{j+1} - hz ,$$

with

$$g = h^{1/2} ,$$

$$\bar{Q}_j^{(n)}(x,\xi,g,z) = (-1)^j \int h_m(y)(u_j^{(n)}(y,x,D_y,\xi,g)h_m)(y)\,dy ,$$

$$u_j^{(n)}(y,x,D_y,\xi) = V_n^W[x+gy,\xi-gD_y]\sharp_h(e_0(y,D_y,z)V_n^W)^j ,$$

and $e_0(y,D_y,z)$ is the symbol of the operator $E_0^{(n)}$ as in Sect. II.1. E_{-+} corresponds to taking the potential to be V. By symmetry arguments, it is easy to see that

$$\bar{Q}_j^{(n)}(x,\xi,g,z) = \bar{Q}_j^{(n)}(x,\xi,-g,z) .$$

So $H_n(x,\xi,g,z)$ has an asymptotic expansion in $h(=g^2)$,

$$H_n(x,\xi,h,z) = \sum_{j\geqq 0} Q_j^{(n)}(x,\xi,z,h)h^{j+1} - hz .$$

It is easy to see that $Q_j^{(n)}(x,\xi)$ only depends on the potential and its derivatives at (x,ξ). We deduce that:

$$|\partial_x^\alpha \partial_\xi^\beta \Delta_n(x,\xi)| \leqq C_{\alpha,\beta}^{(N)} h^N \tag{4.2}$$

for all N, (x,ξ) in Λ'_{n-1} or $\mathbb{R}^2 \backslash \Lambda''_n$ and $C_{\alpha,\beta}^{(N)}$ is independent of ω. (Recall that $\Delta_n = H_{n-1} - H_n$.)

Proof of Proposition 4. We use the expansion (4.1). We assume $\operatorname{Re} z \operatorname{Im} z < 0$, the other case can clearly be treated in a parallel manner. We first estimate the term $\langle \partial_z E_{-+} H_{n-1}^{-1} \Delta_n H_n^{-1} \rangle$. We complex dilate: $\alpha_i \to \alpha_i e^{i\theta_n}$ with $\theta_n = 1/l_n$ for all $i \in \Lambda_n$. We see that for n such that $1/l_n \leqq |\operatorname{Im} z|$, complex dilation will not give us a better estimate on H_n^{-1}. Hence for a given N, N_0 is chosen to be the largest integer such that $1/l_{N_0} \geqq |\operatorname{Im} z| \geqq h^N/C$; N_0 is therefore at most of order $N \log(1/h)$. Let

$$\langle \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \rangle_c \stackrel{\text{def}}{=} \int_{\Gamma(0)} \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \mathcal{G}(\alpha) d\alpha ,$$

where $\alpha = \{\alpha_i\}_{i\in\Lambda_n}$, $d\alpha = \prod_{i\in\Lambda_n} d\alpha_i$, $\mathcal{G}(\alpha) = \prod_{i\in\Lambda_n} g(\alpha_i)$, i.e. the LHS is the conditional expectation value conditioned upon $\alpha_i \notin \Lambda_n$. Then

$$|\langle \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \rangle| \leqq |\langle \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \rangle_c|_\infty ,$$

where $|\ |_\infty$ is the L^∞-norm with respect to α_i, $i \notin \Lambda_n$. Hence it is enough to estimate $|\langle \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \rangle_c|$. By the Stokes formula, we have

$$\begin{aligned}\langle \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \rangle_c &= \int_{\Gamma(0)} \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \mathscr{G}(\alpha)\, d\alpha \\ &= \int_{\Gamma(\theta_n)} \partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \tilde{\mathscr{G}}(a)\, da \\ &\quad + \int_{\Omega(\theta_n)} \sum_{j=1}^{|\Lambda_n|} \frac{\partial \tilde{\mathscr{G}}}{\partial \bar{a}_j} [\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}]\, d\bar{a}_j \wedge da\,,\end{aligned}$$

where $a = \{\alpha_i e^{i\theta}\}_{i \in \Lambda_n} = \{a_i\}_{i \in \Lambda_n}$, $\tilde{\mathscr{G}}$ is an almost analytic extension of $\mathscr{G}$, such that $\partial \tilde{\mathscr{G}} / \partial \bar{a}|_{\mathbb{R}^{\Lambda_n}} = 0$ to infinite order and $\Omega(\theta_n)$ is the volume enclosed by $\Gamma(0)$, $\Gamma(\theta_n)$ and $|\alpha_i| = p$ for all $i \in \Lambda_n$. (Recall that $\operatorname{supp} g \subset (-p, p)$.) We now choose

$$\tilde{\mathscr{G}}(a) = \prod_{i \in \Lambda_n} \tilde{g}(a_i)\,,$$

where $\tilde{g}(a_i)$ is an almost analytic extension of $g(\alpha_i)$ such that $\tilde{g}(a_i) \in C_0^\infty(\mathbb{C})$ and $\partial \tilde{g} / \partial \bar{a}_i|_{\mathbb{R}} = 0$ to infinite order.

Let $hS_{\theta,z} = h^{-1} H_n(\theta) - e^{i\theta} V_n + z$, $(0 \leqq \theta \leqq \theta_n)$, (Note that $S_{\theta,z} \in S_0$.), then

$$h^{-1} \langle \phi, H_n(\theta)\phi \rangle = e^{i\theta} A_0 + h\langle \phi, S_{\theta,z}\phi \rangle - z\,,$$

where

$$A_0 = \frac{1}{2\pi h} \int \bar{\phi}(x) e^{i\frac{(x-y)\xi}{h}} \phi(y) V_n\left(\frac{x+y}{2}, \xi\right) dx\, dy\, d\xi$$

is real. Since $\partial_\theta S$ and $\partial_z S$ are in S^0, $\langle \phi, S_{\theta,z}\phi \rangle$ is real when $\theta = 0$, $\operatorname{Im} z = 0$ we have

$$\operatorname{Im} |\langle \phi, S_{\theta,z}\phi \rangle| \leqq C(\theta + |\operatorname{Im} z|)\|\phi\|\,,$$

where C is independent of ω. Hence there exists h sufficiently small, such that for z with $|\operatorname{Re} z| > c'$,

$$\operatorname{dist}(0, \mathscr{NR}_{H_n(\theta)}) \geqq ch(\theta + |\operatorname{Im} z|)\,,$$

where $\mathscr{NR}_{H_n(\theta)}$ denotes the numerical range of $H_n(\theta)$ and c is independent of ω. We therefore have

$$\|H_n^{-1}(\theta)\| \leqq \frac{1}{ch(\theta + |\operatorname{Im} z|)}\,.$$

Similarly,

$$\|H_{n-1}^{-1}(\theta)\| \leqq \frac{1}{ch(\theta + |\operatorname{Im} z|)}\,.$$

Using Beals' lemma, we have

$$h\left|\partial_x^\alpha \partial_\xi^\beta \begin{pmatrix} H_{n-1}^{-1} \\ H_n^{-1} \end{pmatrix}\right| \leqq C_{\alpha,\beta} \max\left(1, \left(\frac{h}{(\theta + |\operatorname{Im} z|)^2}\right)^3\right) (\theta + |\operatorname{Im} z|)^{-1-\alpha-\beta}\,, \quad (4.3)$$

where $C_{\alpha,\beta}$ is independent of ω. We now show that $(\partial_z E_{-+}) \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}(x, \xi)$ is small for $(x, \xi) \in \mathbb{E}$.

Let χ_n be a C_0^∞ function such that

$$\chi_n(X) = 1 \quad l'_{n-1} \leqq |X| \leqq l''_n$$
$$= 0 \quad |X| \leqq l'_{n-1} - 1 \quad \text{or} \quad |X| \geqq l''_n + 1$$

with derivatives bounded uniformly in n. We have by the composition formula,

$$[\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}](X)$$
$$= \left(\frac{1}{2\pi h}\right)^4 \int e^{iQ_4(Y)/h}[\partial_z E_{-+}(X - Y_1)H_{n-1}^{-1}(X - Y_2)$$
$$\times \{\chi_n \Delta_n\}(X - Y_3)H_n^{-1}(X - Y_4)]\, dY$$
$$+ \left(\frac{1}{2\pi h}\right)^4 \int e^{iQ_4(Y)/h}[\partial_z E_{-+}(X - Y_1)H_{n-1}^{-1}(X - Y_2)$$
$$\times \{(1 - \chi_n)\Delta_n\}(X - Y_3)H_n^{-1}(X - Y_4)]\, dY\, .$$

The second term of the R.H.S. is of order $\mathbf{O}(h^\infty)$ by using (4.2). To estimate the first term, we do integration by parts $m + 9$ times. We have

$$\int e^{iQ_4(Y)/h}[\partial_z E_{-+}(X - Y_1)H_{n-1}^{-1}(X - Y_2)\{\chi_n \Delta_n\}(X - Y_3)H_n^{-1}(X - Y_4)]\, dY$$
$$= \int e^{iQ_4(Y)/h} L^{m+9}[\partial_z E_{-+}(X - Y_1)H_{n-1}^{-1}(X - Y_2)$$
$$\times \{\chi_n \Delta_n\}(X - Y_3)H_n^{-1}(X - Y_4)]\, dY\, .$$

Using (3.13), (3.14), (4.3) and the fact that $|X| \leqq 1/2$, we have

$$|L^{m+9}[\partial_z E_{-+}(X - Y_1)H_{n-1}^{-1}(X - Y_2)\{\chi_n \Delta_n\}(X - Y_3)H_n^{-1}(X - Y_4)]|$$
$$\leqq C \sum_{|\beta| \leqq m+9} \left(\frac{1 + \|Y\|}{h^{1/2}}\right)^{-2(m+9)+|\beta|} h^{|\beta|/2} \max\left(1, \left(\frac{h}{(\theta + |\operatorname{Im} z|)^2}\right)^3\right)^2$$
$$\times (\theta + |\operatorname{Im} z|)^{-1-|\beta|}\, . \tag{4.4}$$

On $\Gamma(\theta_n)$, using the scaling relations: $\theta = \theta_n = 1/l_n$ and $|Y| \geqq |Y_3| \geqq l_n/c (c > 0)$, we have

$$\text{L.H.S.} \leqq C \frac{h^{m+9}}{(1 + \|Y\|)^9} \max(1, (hl_n^2)^3)^2 \sum_{|\beta| \leqq m+9} l_n^{-2m+2|\beta|-7}$$
$$\leqq C \frac{h^{m+9}}{(1 + \|Y\|)^9} \max(1, (hl_n^2)^3)^2 l_n^{11}\, . \tag{4.5}$$

Since $l_n \leqq Ch^{-N}$, we see that for $m > 23N - 9$, the R.H.S. is small. Using the estimate in (4.5), we have

$$|[\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}](X)| \leqq C_M h^M$$

on $\Gamma(\theta_n)$ for all M and $n \leqq N_0$.

Since the function $\theta \mapsto \int |\tilde{g}(\alpha_j e^{i\theta})| d\alpha_j$ is of class $C^{1,1}$, i.e. the function is differentiable and whose derivative is Lipshitz, we have

$$\int_{\Gamma(\theta_n)} |\tilde{g}(a)| da = \int \prod_{j \in \Lambda_n} |\tilde{g}(\alpha_j e^{i\theta_n})| d\alpha_j \leqq (1 + C'\theta_n^2)^{l_n^2} \leqq C ,$$

where we used the fact that $\theta_n = 1/l_n$. We have finally

$$\sum_{n=1}^{N_0} \int_{\Gamma(\theta_n)} |\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1} \tilde{g}(a)| \, ds \leqq C_M h^M$$

for all z satisfying $|z| \leqq h^{-1}$ with $|\mathrm{Re}\, z| > 1/C$ and $|\mathrm{Im}\, z| > h^N/C$ and we used the fact that $N_0 \leqq$ const $N \log h$.

We are now left with the task of estimating the volume integral $\int_{\Omega(\theta_n)}$. Expanding the wedge product, we have

$$\left| \int_{\Omega(\theta_n)} \sum_{j=1}^{|\Lambda_n|} \frac{\partial \tilde{\mathscr{G}}}{\partial \bar{a}_j} (\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}) d\bar{a}_j \wedge da \right|$$

$$\leqq C_{\bar{N}} \int_0^{\theta_n} \int_{\mathbb{R}^{|\Lambda_n|}} \sum_{j=1}^{|\Lambda_n|} \theta^{\bar{N}} \prod_{i \neq j} |\tilde{g}(\alpha_i e^{i\theta})| \, |(\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1})|$$

$$\times \, d\bar{a}_j \wedge da_i \wedge \cdots \wedge da_{|\Lambda_n|} ,$$

where $\bar{N}$ is an integer yet to be determined. We write

$$da_j = e^{i\theta} d\alpha_j + i e^{i\theta} \alpha_j d\theta ,$$

$$d\bar{a}_j = e^{-i\theta} d\alpha_j - i e^{-i\theta} \alpha_j d\theta .$$

From (4.4), we have

$$|\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}| \leqq C_m \frac{h^{m+9}}{l_n^m} \frac{1}{\theta^{23+m}}$$

for all m positive integers. Hence

$$\left| \int_{\Omega(\theta_n)} \sum_{j=1}^{\Lambda_n} \frac{\partial \tilde{\mathscr{G}}}{\partial \bar{a}_j} (\partial_z E_{-+} \sharp_h H_{n-1}^{-1} \sharp_h \Delta_n \sharp_h H_n^{-1}) d\bar{a}_j \wedge da \right|$$

$$\leqq C_m C_{\bar{N}} |\Lambda_n| \frac{h^{m+9}}{l_n^m} \int_0^{\theta_n} \theta^{\bar{N}-23-m} \int \left(\prod_{j \neq 1} \tilde{g}(\alpha_j e^{i\theta}) \right) \left(\prod_{j \in |\Lambda_n|} d\alpha_j \right) d\theta ,$$

where the factor $|\Lambda_n|$ comes from summing over j. Let $\bar{N} > 23 + m$, then

$$\text{R.H.S.} \leqq C_m C_{\bar{N}} \frac{\theta_n^{\bar{N}-23-m}}{l_n^{m-4}} h^{m+9} (1 + C\theta_n^2)^{l_n^2} \leqq C_N h^N$$

for all N.

Since $\|E_{-+}^{-1}(z)\| \leqq C/|\mathrm{Im}\, z|$, with C independent of ω, we have from Beals' lemma

$$|\partial_x^\alpha \partial_\xi^\beta E_{-+}^{-1}| \leqq C_{\alpha,\beta} |\mathrm{Im}\, z|^{-7+\alpha+\beta} \leqq C_{\alpha,\beta} h^{-N(7+\alpha+\beta)} ,$$

with $C_{\alpha,\beta}$ independent of ω. Hence by the same reasoning we obtain

$$|\langle \partial_z E_{-+} \sharp_h H_{N_0}^{-1} \sharp_h \tilde{\Delta}_{N_0+1} \sharp_h E_{-+}^{-1} \rangle_c| \leqq C_N h^N$$

for all N.

Similarly we obtain, after averaging, that the first (dominant) term in the R.H.S. of (4.1) is bounded:

$$|\langle \partial_z E_{-+} \sharp_h H_0^{-1} \rangle_c| \leqq C .$$

Hence we have

$$|\langle \partial_z E_{-+} \sharp_h E_{-+}^{-1} \rangle| \leqq C$$

by summing the series. Clearly by the same argument we have

$$|\partial_x^\alpha \partial_\xi^\beta \langle \partial_z E_{-+} \sharp_h E_{-+}^{-1} \rangle| \leqq C_{\alpha,\beta} .$$

This proves that $\langle \partial_z E_{-+} \sharp_h E_{-+}^{-1} \rangle$ is a classical symbol in class S^0.

Proposition 4 yields:

Corollary. *For all $N_0 \in \mathbb{N}$, $c > 0$; there exists h_0 such that for all $h \in]0, h_0]$, if f is a C_0^∞ function, with support contained in $(-B, -c) \cup (c, B)$ satisfying $|\partial^j f| \leqq C_j h^{-N_0 j}$, then*

$$A_f = \int \partial_{\bar{z}} \tilde{f} \langle \partial_z E_{-+}^{(n)} \sharp_h E_{-+}^{-1\ (n)} \rangle d\bar{z} \wedge dz$$

is a classical symbol in S_0.

Proof. We construct $\tilde{f}$ such that $\bar{\partial}_{\bar{z}} \tilde{f} \leqq C_M h^{-N_0 M} |\mathrm{Im}\, z|^M$ for all $M \geqq 0$. If $|\mathrm{Im}\, z| \geqq h^{N'}/C$, for some $N' \geqq 0$, then we take $M = 0$ and apply Proposition 4. If $|\mathrm{Im}\, z| < h^{N'}/C$, then from Beals' lemma, we have

$$|\partial_x^\alpha \partial_\xi^\beta \langle \partial_z E_{-+} \sharp_h E_{-+}^{-1} \rangle| \leqq C_{\alpha,\beta} \frac{1}{|\mathrm{Im}\, z|^{7+\alpha+\beta}} .$$

Hence

$$\begin{aligned} |\partial_{\bar{z}} \tilde{f} \partial_x^\alpha \partial_\xi^\beta \langle \partial_z E_{-+} \sharp_h E_{-+}^{-1} \rangle| &\leqq C_{\alpha,\beta,M} h^{-N_0 M} |\mathrm{Im}\, z|^{M-\alpha-\beta-7} \\ &\leqq C_{\alpha,\beta,M} h^{-N_0 M} h^{(M-\alpha-\beta-7)N'} . \end{aligned}$$

By choosing $N' = 2N_0$, we have

$$|\partial_{\bar{z}} \tilde{f} \partial_x^\alpha \partial_\xi^\beta \langle \partial_z E_{-+} \sharp_h E_{-+}^{-1} \rangle| \leqq C_{\alpha,\beta,M} h^{N_0(M-2\alpha-2\beta-14)} .$$

Clearly, if $2(\alpha + \beta) \leqq M - 14$, we have a well defined p.d.o.. Hence with the condition $|\partial^j f| \leqq C_j h^{-N_0 j}$, A_f is a classical symbol in the class S_0.

V. Conclusion

We now state our main theorems. (Recall that $\operatorname{supp} g \subset (-p, p)$ and range $v \subset [-1, 1]$.)

Theorem 1. *There exist* $\rho_i^{(n)}(t)$ $(n \in \mathbb{N})$ *in* $C_0^\infty(\mathbb{R}\backslash\{0\})$ *with support contained in* $(-p, p)$ *such that for all* $N_0 \in \mathbb{N}$, $c > 0$, *there exists* B_0, *such that if* $f \in C_0^\infty((-B, -c) \cup (c, B) + (2n+1)B)$ *satisfies* $|\partial^j f| = \underline{O}(B^{N_0 j})$ *with* $B \geqq B_0 > 0$, *then for every* $m \in \mathbb{N}$:

$$
\begin{aligned}
\langle \tilde{\operatorname{Tr}}\, f(P^{\omega}_{B,V}) \rangle &= \int f(E)\, d\rho_B(E) \\
&= B \int f(t + (2n+1)B)\rho_0^{(n)}(t)\, dt \\
&\quad + \int f(t + (2n+1)B)\rho_1^{(n)}(t)\, dt \\
&\quad + B^{-1} \int f(t + (2n+1)B)\rho_2^{(n)}(t)\, dt + \cdots \\
&\quad + B^{-(m-1)} \int f(t + (2n+1)B)\rho_m^{(n)}(t)\, dt + \mathbf{O}(B^{-m}), \qquad (5.1)
\end{aligned}
$$

where $d\rho_B(E)$ *is the "non-random" density of states measure. One has for example*:

$$
\rho_0^{(n)}(t) = \frac{1}{2\pi} \int \prod_l g(\alpha_l)\, d\alpha_l \int_{\mathscr{V}_t} \frac{ds}{\|\nabla V\|},
$$

$$
\rho_1^{(n)}(t) = \frac{2n+1}{8\pi} \int \prod_l g(\alpha_l)\, d\alpha_l \frac{d}{dt}\left(\int_{\mathscr{V}_t} \frac{\Delta V}{\|\nabla V\|}\, ds \right),
$$

where $\mathscr{V}_t$ *is the intersection of the curve* $V(x, \xi) = t$ *with the unit square centered at* $(0, 0)$. *Note that* $\rho_0^{(n)}$ *is independent of* n.

In particular, if v *has its support contained in the unit square, then*

$$
\rho_0^{(n)} = \frac{1}{2\pi} \int F_0(\gamma)\, g\left(\frac{t}{\gamma}\right) \frac{d\gamma}{|\gamma|},
$$

where

$$
F_0(\gamma) = \int_{v=\gamma} \frac{ds}{\|\nabla v\|};
$$

and

$$
\rho_1^{(n)} = \frac{2n+1}{8\pi} \int F_1(\gamma)\, g\left(\frac{t}{\gamma}\right) \frac{d\gamma}{|\gamma|},
$$

where

$$
F_1(\gamma) = \frac{d}{d\gamma}\left(\int_{v=\gamma} \frac{\Delta v}{\|\nabla v\|}\, ds \right).
$$

Proof. Direct consequence of Proposition 4 and its corollary and straightforward computation by using the asymptotic expansion for E_{-+}^{-1} and $\partial_z E_{-+}$.

Remark. Because of the extra integration over the random variable α, when we compute $d\rho_m/dt$, we somehow are always computing $\partial g/\partial \alpha$. Since g is a C_0^∞ function, so is ρ_m.

If we further assume that $v \geqq 0$ and that the support of v_i intersect so that $\sum_{i\in\mathbb{Z}^2} v_i \geqq s > 0$, then we have:

Theorem 2. *There exist $\rho_i^{(n)}(t)$ $(n \in \mathbb{N})$ in $C_0^\infty(-p, p)$ such that for all $N_0 \in \mathbb{N}$, $c > 0$, there exists B_0, such that if $f \in C_0^\infty((-B,B) + (2n+1)B)$ satisfies $|\partial^j f| = \mathbf{O}(B^{N_0 j})$ with $B \geqq B_0 > 0$, then for every $m \in \mathbb{N}$ the expansion (5.1) holds.*

Proof. We only need to prove that the expansion (5.1) holds for $f \in C_0^\infty((-c,c) + (2n+1)B)$ for some $c > 0$ to be determined. We use a similar construction to the one used in the proof of Proposition 4 in Sect. IV. We take H_n to be the reduced operator corresponding to the potential

$$V_n(\bar{x}) = \sum_{i\in\Lambda_n} \alpha_i v(\bar{x} - i) + \sum_{i\in\mathbb{Z}^2\setminus\Lambda_n} v(\bar{x} - i)\,.$$

Instead of complex dilating in α as in the proof of Proposition 4, we complex translate in α:

$$\alpha_j \mapsto \alpha_j - i(\operatorname{sign}\operatorname{Im} z)\delta_n\,,$$

where $\delta_n = 1/l_n$. We assume that $\operatorname{Im} z < 0$ (the other case can be treated in the same way). Let $H_n(\delta)$ be the reduced operator corresponding to α_j replaced by $\alpha_j + i\delta(0 < \delta \leqq \delta_n)$. The principal symbol for $h^{-1}H_n(\delta)$ is

$$V_n^\delta(x,\xi) - z = \sum_{j\in\Lambda_n} \alpha_j v_j(x,\xi) + i\delta \sum_{j\in\Lambda_n} v_j(x,\xi) + \sum_{j\in\mathbb{Z}^2\setminus\Lambda_n} v_j(x,\xi) - z\,.$$

Let $\phi \in L^2(\mathbb{R})$; we have

$$\operatorname{Re}\langle\phi, (V_n^\delta - z)\phi\rangle = \langle\phi, \sum_{j\in\Lambda_n} \alpha_j v_j \phi\rangle + \langle\phi, \sum_{j\in\mathbb{Z}^2\setminus\Lambda_n} v_j\phi\rangle - \operatorname{Re} z\langle\phi,\phi\rangle\,,$$

$$\operatorname{Im}\langle\phi, (V_n^\delta - z)\phi\rangle = \delta\langle\phi, \sum_{j\in\Lambda_n} v_j\phi\rangle + |\operatorname{Im} z|\langle\phi,\phi\rangle$$

$$\geqq \delta\langle\phi, \sum_{j\in\Lambda_n} v_j\phi\rangle \geqq 0\,.$$

Since $\sum_{j\in\mathbb{Z}^2} v_j \geqq s > 0$, we certainly have, for h small enough

$$\langle\phi, \sum_{j\in\mathbb{Z}^2} v_j\phi\rangle > s/2\langle\phi,\phi\rangle\,. \tag{5.2}$$

Let $\tilde{\alpha} = \max(|\alpha|_\infty, 3)$. For ϕ such that $\langle\phi, \sum_{j\in\Lambda_n} v_j\phi\rangle \leqq s/(\tilde{\alpha}^2)\langle\phi,\phi\rangle$, we have

$$|\langle\phi, \sum_{j\in\Lambda_n} \alpha_j v_j\phi\rangle| \leqq \frac{s}{\tilde{\alpha}}\langle\phi,\phi\rangle\,.$$

We also have from (5.2)

$$\langle\phi, \sum_{j\in\mathbb{Z}^2\setminus\Lambda_n} v_j\phi\rangle \geqq s\left(\frac{1}{2} - \frac{1}{\tilde{\alpha}^2}\right)\langle\phi,\phi\rangle\,.$$

Hence

$$\operatorname{Re}\langle \phi, (V_n^\delta - z)\phi\rangle \geqq s\left(\frac{1}{2} - \frac{1}{\tilde{\alpha}^2}\right)\langle \phi, \phi\rangle - \operatorname{Re} z\langle \phi, \phi\rangle .$$

Taking $c = s/19$, we have $|\operatorname{Re} z| \leqq s/19$. Therefore

$$\max(|\operatorname{Re}\langle \phi, (V_n^\delta - z)\phi\rangle|,\ |\operatorname{Im}\langle \phi, (V_n^\delta - z)\phi\rangle|) \geqq \frac{\delta s}{\tilde{\alpha}^2}\langle \phi, \phi\rangle .$$

Let $hS_{\delta,z} = h^{-1}H_n(\delta) - V_n^\delta + z$. The symbols $\partial_\delta S$, $\partial_z S$ belong to S^0. Hence for h sufficiently small, we have

$$\operatorname{dist}(0, \mathcal{N}\mathcal{R}_{H_n(\delta)}) \geqq c'h(\delta + |\operatorname{Im} z|) ,$$

where $\mathcal{N}\mathcal{R}_{H_n(\delta)}$ is the numerical range of $H_n(\delta)$. We therefore have

$$\|H_n^{-1}(\delta)\| \leqq \frac{1}{c'h(\delta + |\operatorname{Im} z|)} .$$

The rest of the proof follows exactly that of Theorem 1, with δ replacing θ.

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