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Homoclinic Orbits on Compact Hypersurfaces in $\mathbb{R}^{2N}$, of Restricted Contact Type

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Abstract: Consider a smooth Hamiltonian system in $\mathbb{R}^{2N}$, $\dot{x} = JH'(x)$, the energy surface $\Sigma = \{x/H(x) = H(0)\}$ being compact, and 0 being a hyperbolic equilibrium. We assume, moreover, that $\Sigma \backslash \{0\}$ is of restricted contact type. These conditions are symplectically invariant. By a variational method, we prove the existence of an orbit homoclinic, i.e. non-constant and doubly asymptotic, to 0.

I. Introduction

The goal of this work is to give a partial answer to a conjecture of Helmut Hofer, about homoclinic orbits in Hamiltonian systems (personal communication). Suppose that Σ is the zero energy surface of an autonomous Hamiltonian H in $\mathbb{R}^{2N}$ having $x_0 \in \Sigma$ as a hyperbolic equilibrium and no other equilibrium on Σ, and that $\Sigma \backslash \{x_0\}$ is of contact type. These conditions are symplectically invariant. The conjecture is that the Hamiltonian system

$$\dot{x} = X_H \,, \quad X_H = JH'(x) \,, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \tag{1.1}$$

has at least one solution $x(t)$ homoclinic to x_0, i.e. such that $x \not\equiv x_0$, and $\lim_{|t| \to \infty} x(t) = x_0$. It may be seen as an analogue for homoclinic orbits of the Weinstein conjecture in $\mathbb{R}^{2N}$, which was solved by Viterbo in 1987 (see [W, V, H-Z]). In the present paper, we replace the contact condition by a restricted contact condition, less general but also symplectically invariant. We find a homoclinic orbit, as the critical point of the action functional associated to a suitably chosen Hamiltonian.

More precisely, we consider the following set of hypotheses on Σ:

($\mathscr{H}1$): Σ is a compact set. It may be defined as $\Sigma = \{x/H(x) = 0\}$, where H is a smooth Hamiltonian defined on $\mathbb{R}^{2N}$, whose differential H' does not vanish on Σ, except at one point x_0 that we identify with 0 after translation. Moreover, $A = H''(0)$ is non-degenerate.

($\mathscr{H}2$): JA is hyperbolic, i.e. $sp(JA) \cap i\mathbb{R} = \emptyset$.

($\mathscr{H}3$): There is a C^1-field of vectors (η_x, $x \in \mathbb{R}^{2N}$), transverse to $\Sigma\backslash\{0\}$, and such that $\mathscr{L}_\eta\omega = \omega$ everywhere in $\mathbb{R}^{2N}$. Here, $\omega(X, Y) = \langle JX, Y\rangle$ is the usual symplectic form in $\mathbb{R}^{2N}$.

Remark 1. If Σ is fixed, the conditions ($\mathscr{H}2$, 3) do not depend on the choice of H satisfying ($\mathscr{H}1$). The existence of a solution of (1.1) homoclinic to x_0 is also independent of this choice. Moreover, ($\mathscr{H}1$, 2, 3) and the existence of a homoclinic orbit are invariant by symplectic diffeomorphism.

Remark 2. Denoting $\Omega_x = i_\eta\omega = \omega(\eta, \cdot)$, condition ($\mathscr{H}3$) implies that $\Omega_x - \frac{1}{2}\langle Jx, \cdot\rangle$ is closed (i.e. exact, in $\mathbb{R}^{2N}$), and that $\Omega_x(X_H) \neq 0$ for any $x \in \Sigma\backslash\{0\}$.

Remark 3. ($\mathscr{H}3$) is the restricted contact condition. "Restricted contact" means that η is defined on the whole space $\mathbb{R}^{2N}$. In the real "contact" condition, η would just be defined in a neighborhood of Σ. When $H^1(\Sigma) = 0$, there is no difference between "contact" and "restricted contact."

Our main result will be the following:

Theorem 1.1. *Assume that* ($\mathscr{H}1$, 2, 3) *are true. Then the system* (1.1) *has at least one solution homoclinic, i.e. doubly asymptotic, to* 0.

Our method of proof will be variational.

To our knowledge, the first result on homoclinic orbits obtained variationally is due to Bolotin [B]. It concerns Lagrangian systems in Riemannian manifolds, having an unstable equilibrium. In the last few years, this type of system has been extensively studied (see [K, B-G, R, R-T, A-B] and the references in these papers).

In the case of autonomous Lagrangian systems in $\mathbb{R}^N$, $\ddot{q} + V'(q) = 0$, the existence of a homoclinic orbit was proved by two different methods in [A-B, R-T], under very natural assumptions on the potential V : 0 is a point of non-degenerate local maximum for V, with $V(0) = 0$, and the set $\{q/V(q) \leqq 0\}$ is compact, its frontier being a regular hypersurface on which V' never vanishes. Independently of these two works, Scholze [Sc] proved the existence of a homoclinic orbit for a Hamiltonian system in $\mathbb{R}^{2N}$, assuming that ($\mathscr{H}1$, 2) are true, and that $\partial_p H(p,q) \cdot p > 0$ for any (p,q) near Σ with $p \neq 0$. It is clear that, if $H = \frac{p^2}{2} + V(q)$, with V as in [A-B, R-T], then the assumptions of [Sc] are satisfied.

In the field of variational methods for homoclinic orbits, another direction of research is the study of Hamiltonians in $\mathbb{R}^{2N}$ which have the form $H(x, t) = \frac{1}{2}\langle x, Ax\rangle + R(x, t)$, with JA hyperbolic, R smooth, one-periodic in time, and superquadratic[1] in x (see [CZ-E-S, H-W, T, Se1, Se2]).

In [CZ-E-S], under the additional assumption that R is strictly convex in x, the existence of a homoclinic orbit was proved thanks to a dual action principle (a multiplicity result for non-autonomous systems was also given). This existence result was improved by [H-W] and [T], who relaxed the convexity assumption by two different methods.

The following proposition shows that the results in [A-B, R-T, Sc] can easily be obtained as a consequence of Theorem 1.1:

Proposition 1.2. *Assume that Σ satisfies* ($\mathscr{H}1$, 2), *and suppose that* $\partial_p H(p, q) \cdot p > 0$ *for any* (p, q) *in a neighborhood* $\mathscr{U}$ *of* Σ *with* $p \neq 0$.

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[1] I.e., such that for some $c_1, c_2 > 0$ and $\alpha > 2$, and any $(x, t) \in \mathbb{R}^{2N+1}$, $c_1|x|^\alpha \leqq R(x) \leqq c_2|x|^\alpha$ and $R'(x, t) \cdot x \geqq \alpha R(x, t)$.

Then Σ also satisfies $(\mathscr{H}3)$.

Proof. Geometrically, the assumption "$\partial_p H(p, q) \cdot p > 0$ for any $(p, q) \in \mathscr{U}$ with $p \neq 0$" implies that Σ is starshaped with respect to the Lagrangian plane $L_1 = \{(q, 0), q \in \mathbb{R}^N\}$. So a natural idea is to take $\eta(q, p) = (0, p)$. We find that $\mathscr{L}_\eta \omega = \omega$, and η is transverse to $\Sigma \backslash L_1$, but vanishes on L_1. To avoid this problem, we remark that the Lagrangian plane $L_2 = \{(0, p), p \in \mathbb{R}^N\}$ is contained in the tangent space to Σ at $(q, 0)$ for any non-zero $(q, 0) \in \Sigma \cap L_1$. This comes from the inequality $\partial_p H(p, q) \cdot p \geqq 0$, true in a neighborhood of $(q, 0)$. As a consequence, the set $\mathscr{S} = \{q \neq 0 / (q, 0) \in \Sigma\}$ is a regular hypersurface of $\mathbb{R}^N$. Let v_0 be a smooth vector field in $\mathbb{R}^N$, null in a neighborhood of 0, and transverse to $\mathscr{S}$, pointing outward. We choose $\eta_\varepsilon(q, p) = (0, p) + \varepsilon v(q, p)$, where $\varepsilon > 0$ and $Jv(q, p)$ is the gradient of the real-valued function $f(q, p) = \langle p, v_0(q)\rangle_{\mathbb{R}^N}$.

The form $i_v \omega$ is exact, so from Remark 2, $\mathscr{L}_{\eta_\varepsilon} \omega = \omega$. We easily check that for ε small enough, $\langle H'(x), \eta_\varepsilon \rangle$ is positive on $\Sigma \backslash \{0\} \cdot (\mathscr{H}3)$ is thus satisfied. □

Similarly, in the autonomous case (i.e. when R does not depend on t), the existence results of [CZ-E-S, H-W and T] can be deduced from Theorem 1.1:

Proposition 1.3. *Assume that Σ is the zero energy surface of a Hamiltonian $H(x) = \frac{1}{2}\langle x, Ax\rangle + R(x)$, with JA hyperbolic, R smooth, $c_1|x|^\alpha \leqq R(x) \leqq c_2|x|^\alpha$ and $\langle R'(x), x\rangle \geqq \alpha R(x)$ for some $c_1, c_2 > 0$, $\alpha > 2$, and any $x \in \mathbb{R}^{2N}$. Then Σ satisfies the assumptions $(\mathscr{H}1, 2, 3)$ of Theorem* 1.1.

Proof. By assumption, $(\mathscr{H}2)$ is satisfied. Since $c_1|x|^\alpha \leqq R(x)$, Σ is compact. We choose $\eta = \frac{x}{2}$. Clearly, $\mathscr{L}_\eta \omega = \omega$. Moreover,

$$\langle H'(x), \eta\rangle = \frac{1}{2}(\langle x, Ax\rangle + \langle R'(x), x\rangle) \geqq H(x) + \frac{\alpha - 2}{2} R(x) .$$

So for any $x \in \Sigma \backslash \{0\}$, $\langle H'(x), \eta\rangle$ is positive (geometrically, this implies that Σ is regular outside 0 and starshaped with respect to 0). So $(\mathscr{H}1, 3)$ are satisfied, and Theorem 1.1 can be applied to Σ. □

In the proof of Theorem 1.1, we shall write $H(x) = \frac{1}{2}(x, Ax) + R(x)$, with JA hyperbolic, R smooth, $R(0) = 0$, $R'(0) = 0$, $R''(0) = 0$, as in [CZ-E-S, H-W, T]. In our situation, R is not necessarily superquadratic, moreover it may be negative on some parts of Σ. Our functional will be similar to the one in [H-W],

$$\begin{aligned} I(x) &= \frac{1}{2}\int_{\mathbb{R}} \langle -J\dot{x} - Ax, x\rangle - \int_{\mathbb{R}} R(x) \\ &= \frac{1}{2}\int_{\mathbb{R}} \langle Jx, \dot{x}\rangle - \int_{\mathbb{R}} H(x), \quad x \in H^{1/2}(\mathbb{R}, \mathbb{R}^{2N}) . \end{aligned} \tag{1.2}$$

Hofer and Wysocki [H-W] used the theory of first order elliptic systems of the type $\partial_s u + J\partial_t u + H'(u) = 0$, to find a Palais–Smale sequence for I. In the present work, we won't use this "elliptic" approach.

Our assumptions being much weaker than those of [H-W, T], we will have to solve new problems:

- Since the superquadraticity of R is relaxed, it is more difficult to prove that the Palais–Smale sequences are bounded in $L^2(\mathbb{R}, \mathbb{R}^{2N})$. Assumption $(\mathscr{H}3)$ will be crucial here.

• Since the sign of R near Σ is unknown, it is more difficult to find a topological argument for the existence of a Palais–Smale sequence. This second problem is the most important one.

II. Main Notations and Sketch of the Proof of Theorem 1.1

Throughout this paper, we shall use the standard notations $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_+^* = \mathbb{R}_+ \backslash \{0\}$, $\mathbb{N} = \mathbb{Z} \cap \mathbb{R}_+$, and $\mathbb{N}^* = \mathbb{N} \backslash \{0\}$.

If $N = 1$, then the Hamiltonian system is integrable, and Theorem 1.1 is trivial. So we only have to treat the case $N \geqq 2$. Consider Σ satisfying ($\mathscr{H}$1 to 3). We can replace Σ by the connected component of 0 in Σ, ($\mathscr{H}$1 to 3) remaining true, and the homoclinic problem being unchanged. So, using the freedom in the choice of H, η, we may impose the following additional condition, without restriction of generality:

($\mathscr{H}$4): $N \geqq 2$, Σ is connected, and, outside some bounded neighborhood of Σ:

$$\begin{cases} H \equiv \frac{1}{2}\langle x, Ax\rangle + a|x|^2 & \text{with } a > \dfrac{\|A\|}{2}\,, \\ \Omega_x \equiv \frac{1}{2}\langle Jx, \cdot\rangle, & \text{with } \Omega = i_\eta \omega \text{ as in Remark 2.} \end{cases}$$

JA being hyperbolic, we have signature $(A) = \text{signature}(iJ) = (N, N)$. So, for $N \geqq 2$, $C^* = \{x \neq 0 / \langle x, Ax\rangle = 0\}$ is connected, and, if Σ is connected, then $\Sigma\backslash 0$ is also connected. We will use this in Lemma 3.1.

If ($\mathscr{H}$1 to 4) are satisfied, then the functional I given by formula (1.2) is well-defined and smooth on the Hilbert space $E = H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$.

Indeed, these conditions imply that

$$R = O(|x|^3)\,,\ R' = O(|x|^2)\,, \qquad R'' = O(|x|) \quad \text{for } x \text{ near } 0\,, \tag{2.1}$$

$$R = O(|x|^2),\ R' = O(|x|), \qquad R'' = O(1) \quad \text{for } |x| \text{ large}\,, \tag{2.2}$$

and that the higher derivatives $R^{(n)}$, $n \geqq 3$, are bounded and compactly supported.

Now, denote

$$\begin{aligned} D : E &\to E' = H^{-1/2}(\mathbb{R}, \mathbb{R}^{2N}) \\ x &\mapsto \left(-J\frac{d}{dt} - A\right) x\,, \\ L &= D^{-1}\,. \end{aligned} \tag{2.3}$$

In the Fourier representation, we have $\widehat{Dx}(\xi) = \hat{D}(\xi)\hat{x}(\xi)$, where

$$\hat{D}(\xi) = -iJ\xi - A \tag{2.4}$$

is a hermitian matrix, always non-degenerate because JA is hyperbolic.

We denote

$$|\hat{D}(\xi)| = \sqrt{\hat{D}^2(\xi)},\ \hat{D}_\pm(\xi) = \frac{1}{2}(|\hat{D}(\xi)| \pm \hat{D}(\xi))\,. \tag{2.5}$$

We have $\hat{D}(\xi) = \hat{D}_+(\xi) - \hat{D}_-(\xi)$, $|\hat{D}(\xi)| = \hat{D}_+(\xi) + \hat{D}_-(\xi)$.

Let us define

$$\pi_\pm(\xi) = |\hat{D}(\xi)|^{-1}\hat{D}_\pm(\xi)\,. \tag{2.6}$$

We have $\pi_+ + \pi_- = 1_{\mathbb{C}^{2N}}$, $\pi_+ \circ \pi_- = \pi_- \circ \pi_+ = 0$, and $\pi_+(\xi)$, $\pi_-(\xi)$ are two orthogonal projectors on $\mathbb{C}^{2N}$ for each of the hermitian products $\langle X, Y\rangle_{\mathbb{C}^{2N}}$ and $\langle X, |\hat{D}(\xi)|Y\rangle_{\mathbb{C}^{2N}}$.

Now, if we consider the bilinear form on E

$$(x, y)_E = \int_{\mathbb{R}} d\xi \langle \hat{x}(\xi), |\hat{D}(\xi)|\hat{y}(\xi)\rangle\,, \tag{2.7}$$

$(\cdot, \cdot)_E$ is a scalar product whose associated norm $\|\cdot\|_E$ is equivalent to the classical $H^{1/2}$-norm. Moreover, denoting

$$\widehat{P_\pm x}(\xi) = \pi_\pm(\xi)\hat{x}(\xi), \qquad P_\pm E = E_\pm, \tag{2.8}$$

we see that $E = E_+ \oplus E_-$, the sum being orthogonal for the classical L^2 scalar product and for $(\cdot, \cdot)_E$. Thus, for any $x \in E$,

$$\|x\|_E^2 = \|P_+x\|_E^2 + \|P_-x\|_E^2\,, \qquad (x, Dx)_{E\times E'} = \|P_+x\|_E^2 - \|P_-x\|_E^2\,.$$

So we may write

$$I(x) = \frac{1}{2}(\|P_+x\|_E^2 - \|P_-x\|_E^2) - \int_{\mathbb{R}} R(x)\,. \tag{2.9}$$

The differential of I at $x \in E$ is

$$I'(x) = Dx - R'(x) \in E'\,, \tag{2.10}$$

and denoting

$$\widehat{|L|x}(\xi) = |\hat{D}(\xi)|^{-1}\hat{x}(\xi)\,, \tag{2.11}$$

we can define the gradient of I for the scalar product $(\cdot, \cdot)_E$,

$$\nabla I(x) = P_+x - P_-x - |L|R'(x) \in E\,. \tag{2.12}$$

In Sect. III, we will prove the following compactness result.

Theorem 2.1. *Suppose that* $(\mathscr{H}1)$ *to* $(\mathscr{H}4)$ *are true. Let* $(x_n)_{n\geqq 0}$ *be a Cerami sequence for* I, *i.e. a sequence in* E *such that* $I(x_n) \to c \in \mathbb{R}$, *and* $(1 + \|x_n\|_E)\|I'(x_n)\|_{E'} \to 0$ *as* $n \to +\infty$.

Then there is a finite integer $p \geqq 0$, *and homoclinic orbits* $Z^1, \dots, Z^p$ *such that, after extraction, we have*

$$\left\| x_n(t) - \sum_{i=1}^{p} Z^i(t - t_n^i)\right\|_E \to 0 \quad \text{as } n \to +\infty\,,$$

where $t_n^i \in \mathbb{R}$, $|t_n^j - t_n^i| \to +\infty$ *as* $n \to +\infty$, *if* $i \neq j$.

Moreover, for any i, $I(Z^i) > 0$. *As a consequence,* $c = \sum_{i=1}^{p} I(Z^i)$, *and* $\|x_n\|_E \to \sum_{i=1}^{p} \|Z^i\|_E$. *So* $c \geqq 0$, *and if* $c = 0$, *then* $\|x_n\|_E \to 0$ *as* $n \to +\infty$ *(case* $p = 0$*).*

This type of compactness result is well-known for functionals invariant under the action of a non-compact group (see [St, L]). A general method of proof, based on the Concentration-Compactness principle of P.L. Lions, is described in

[L]. To apply this method, it is necessary to have an "a priori" estimate on the norm in E of the critical sequence (x_n). To obtain this bound, we use the assumption $(1 + \|x_n\|)I'(x_n) \to 0$, $I(x_n) \to c)$, introduced by Cerami for general variational problems, (see [C, E]), instead of the more classical Palais–Smale assumption $(I'(x_n) \to 0$, $I(x_n) \to c)$, which seems too weak here (see the proof of Lemma 3.3).

The second step in the variational proof of Theorem 1.1 is to find a topological reason for the existence of a critical sequence. In [H-W, T], a linking argument is used, based on the fact that for some $e \in E_+$, I is negative on $E_-\backslash\{0\}$ and outside a bounded subset of $M_- = E_- \oplus \mathbb{R}_+ e$. Unfortunately, we cannot use this standard approach: since our conditions $(\mathscr{H}1)$ to $(\mathscr{H}4)$ do not imply that $\Sigma \subset \{x/\langle x, Ax\rangle \leqq 0\}$, the sign of R may be negative near some parts of Σ. So it is not always true that $I(E_-) \leqq 0$.

Our solution will be to replace E_- by the unstable manifold $\mathscr{W}_-$ for the flow $(\phi_\tau, \tau \in \mathbb{R})$ of $-\nabla I$, issuing from 0 (it is obvious that I is negative on $\mathscr{W}_-\backslash\{0\}$). We will have to choose $e \in E_+$ very carefully, and to introduce an auxiliary functional $\tilde{I} \leqq I$, with $(I'(x) = 0 \Leftrightarrow \tilde{I}'(x) = 0)$. Then we will prove that $\tilde{I}$ is negative outside a bounded part of $\mathscr{W}_- \oplus \mathbb{R}_+ e$. We will conclude by arguments inspired of [H-W] (see Lemma 4.4).

To start this program, we define

$$B(\varepsilon) = \{x \in E/\|x\|_E \leqq \varepsilon\}, \qquad S(\varepsilon) = \{x \in E/\|x\|_E = \varepsilon\}\,,$$
$$B_\pm(\varepsilon) = B(\varepsilon) \cap E_\pm, \qquad S_\pm(\varepsilon) = S_\pm(\varepsilon) \cap E_\pm\,. \tag{2.13}$$

The flow $(\phi_\tau, \tau \in \mathbb{R})$ is defined by

$$\begin{cases} \phi_0(x) = x, & \forall x \in E, \\ \partial_\tau \phi_\tau(x) = -\nabla I \circ \phi_\tau(x), & \forall (\tau, x) \in \mathbb{R} \times E\,. \end{cases} \tag{2.14}$$

It is easy to see that the formula

$$\Phi(x) = \lim_{\tau \to +\infty} \phi_\tau(e^{-\tau} x), \quad x \in E_- \tag{2.15}$$

defines a smooth mapping $\Phi : E_- \to \mathscr{W}_-$. From (2.12) we find, by the method of variation of the constant, that

$$\Phi(x) = x + |L| k(x)\,, \tag{2.16}$$

where k is solution of

$$k(x) = \int_0^\infty e^{(P_- - P_+)\tau} R'(e^{-\tau}x + |L|k(e^{-\tau}x))\, d\tau\,. \tag{2.17}$$

The following consequence of Theorem 2.1 will be very useful:

Corollary 2.2. *Suppose that* $(\mathscr{H}1)$ *to* $(\mathscr{H}4)$ *are true. Consider the function*

$$l(\sigma) = \sup I \circ \Phi(S_-(\sigma))\,. \tag{2.18}$$

We have $\lim_{\sigma \to +\infty} l(\sigma) = -\infty$.

Proof. From the relation $\Phi(e^\tau x) = \phi_\tau \circ \Phi(x)$, it is clear that l is strictly decreasing. Let $l_\infty < 0$ be its limit as σ goes to infinity. Assume by contradiction that l_∞

is finite. Then there is a sequence $(X_n)_{n \geqq 0}$ in $\Phi(B(0,1))$ such that $I \circ \phi_{2^n}(X_n) \to l_\infty$, $I \circ \phi_{2^{n+1}}(X_n) \to l_\infty$.

Since $\int_{2^n}^{2^{n+1}} \|I' \circ \phi_s(X_n)\|_{E'}^2 \, d\tau = I \circ \phi_{2^n}(X_n) - I \circ \phi_{2^{n+1}}(X_n) = o(1)$, there is a sequence $(\tau_n)_{n \geqq 0}$, $2^n \leqq \tau_n \leqq 2^{n+1}$, such that

$$\|I' \circ \phi_{\tau_n}(X_n)\|_{E'} = o\left(\frac{1}{2^{n/2}}\right) .$$

But we also have $\int_0^{2^{n+1}} \|I' \circ \phi_\tau(X_n)\|_{E'}^2 d\tau \leqq -l_\infty$, thus

$$\|\phi_{\tau_n}(X_n)\|_E \leqq M + |l_\infty|^{\frac{1}{2}} 2^{\frac{n+1}{2}} ,$$

where $M = \sup\{\|\Phi(x)\|_E / x \in B(0,1)\}$.

As a consequence, $(1 + \|\phi_{\tau_n}(X_n)\|_E)\|I' \circ \phi_{\tau_n}(X_n)\|_{E'} = o(1)$.

Moreover, $l_\infty \leqq I \circ \phi_{\tau_n}(X_n) \leqq I \circ \phi_{2^n}(X_n) = l_\infty + o(1)$.

So $\phi_{\tau_n}(X_n)$ is a Cerami sequence at the negative level l_∞. But from Theorem 2.1, this is impossible, and Corollary 2.2 is proved by contradiction. □

Given $\sigma > 0$, $e \in E_+$, we denote

$$\mathscr{M}_-(\sigma, e) = \Phi(B_-(\sigma)) + [0,\sigma]e , \tag{2.19}$$

$$\partial\mathscr{M}_-(\sigma, e) = (\Phi(S_-(\sigma)) + [0,\sigma]e) \cup (\Phi(B_-(\sigma)) + \{0,\sigma\}e) . \tag{2.20}$$

Finally, $\tilde{I}$ is called I-admissible if

$$\begin{cases} \tilde{I}(x) = I(x) - \int \tilde{R}(x) , \\ \text{with } \tilde{R} \geqq 0,\ \tilde{R} \text{ smooth, } \tilde{R} \equiv 0 \text{ in a neighborhood of } \Sigma , \\ \tilde{R} \equiv \tilde{a}|x|^2,\ \tilde{a} \geqq 0, \text{ outside another neighborhood of } \Sigma . \end{cases} \tag{2.21}$$

Let $(\tilde{\phi}_\tau, \tau \in \mathbb{R})$ be the flow of $-\nabla\tilde{I}$, defined by

$$\begin{cases} \tilde{\phi}_0(x) = x, \ \forall x \in E , \\ \partial_\tau \tilde{\phi}_\tau(x) = -\nabla\tilde{I} \circ \tilde{\phi}_\tau(x), \ \forall (\tau, x) \in \mathbb{R} \times E . \end{cases} \tag{2.22}$$

In Sect. IV, we will prove the following result:

Theorem 2.3. *Suppose that* $(\mathscr{H}1)$ *to* $(\mathscr{H}4)$ *are true. Then there are* $\overline{\sigma}, \overline{\rho} > 0, \overline{e} \in E_+$ *and* $\tilde{I}$, *I-admissible, with*:

(i) $\tilde{I}(\partial\mathscr{M}_-(\overline{\sigma}, \overline{e})) < 0$,
(ii) $\inf \tilde{I}(S_+(\overline{\rho})) > 0$,
(iii) For any $\tau \geqq 0$, $\phi_\tau(\mathscr{M}_-(\overline{\sigma}, \overline{e})) \cap S_+(\overline{\rho})$ *is non-empty.*

Theorem 2.3 has the following consequence:

Corollary 2.4. *Suppose that* $(\mathscr{H}1)$ *to* $(\mathscr{H}4)$ *are true. Then there is a Cerami sequence for* $\tilde{I}$ *at the positive level*

$$c = \inf_{\tau \geqq 0} \sup \tilde{I} \circ \tilde{\phi}_\tau(\mathscr{M}_-(\overline{\sigma}, \overline{e})) > 0 .$$

The proof of Corollary 2.4 is standard (see [C]).

Clearly, Σ and $\tilde{H} = H + \tilde{R}$ satisfy the conditions $(\mathscr{H}1)$ to $(\mathscr{H}4)$ of Theorem 2, and $\tilde{H} \equiv H$ in a neighborhood of Σ. So Theorem 1.1 is an immediate consequence of Theorem 2.1 and Corollary 2.4.

III. Proof of Theorem 2.1

We consider a smooth function $\theta : \mathbb{R} \to [0, 1]$ such that

$$\begin{cases} \theta \equiv 0 \text{ on } \left(-\infty, -\frac{1}{2}\right] , \\ \theta \equiv 1 \text{ on } \left[\frac{1}{2}, +\infty\right) , \\ \theta(t) + \theta(-t) = 1, \ \forall t \in \mathbb{R} . \end{cases} \tag{3.1}$$

We define

$$\begin{cases} \Theta_{\mathbb{R}}(t) = 1 , \\ \Theta_{[a,+\infty)}(t) = \theta(t-a) , \\ \Theta_{(-\infty,b]}(t) = \theta(b-t) , \\ \Theta_{[a,b]}(t) = \Theta_{[a,+\infty)}(t) \cdot \Theta_{(-\infty,b]}(t) . \end{cases} \tag{3.2}$$

In what follows, the "action of x on the interval of time $\mathscr{J}$" is the integral

$$\mathscr{S}_{\mathscr{J}}(x) = \int_{\mathbb{R}} [\Omega_x(\dot{x}) - H(x)]\, \Theta_{\mathscr{J}}(t)\, dt . \tag{3.3}$$

$\mathscr{S}_{\mathscr{J}}$ is well-defined on E, and $\mathscr{S}_{\mathbb{R}} = I$, as we will see in Lemma 3.2. Moreover, if $a + 1 \leqq b \leqq c - 1$, then $S_{[a,c]} = S_{[a,b]} + S_{[b,c]}$.

Our arguments are organized as follows:

First of all, by the restricted contact assumption, the orbits of (1.1) on Σ must have a positive action on $\mathbb{R}$, and the ones having a finite action on $\mathbb{R}$ are homoclinic to 0. We make this precise in Lemma 3.1 (and later in Lemma 3.6, for the consequence on Cerami sequences).

Lemma 3.3 tells us that Cerami sequences must stay "near" Σ. The proof of Lemma 3.3 is the step where we need "$(1 + \|x_n\|)I'(x_n) \to 0$" instead of "$I'(x_n) \to 0$."

Thanks to Lemma 3.3, we prove that (x_n) is bounded in L^2_{loc}-norm (see Lemma 4), and that, for a given sequence $(t_n)_{n \geqq 0}$ of translations in time, we have, after extraction, $x_n(t - t_n) \to x_\infty(t)$ in $H^{1/2}_{\mathrm{loc}}$ norm, where x_∞ is an orbit of (1.1) on Σ (see the proof of Lemma 3.6). Lemma 3.5 means that, if x_n is small in local norm on a bounded or unbounded interval of time $\mathscr{J}$, then the action and L^2-norm of x_n on $\mathscr{J}$ are small. It is closely related to the "concentration-compactness lemma" in the "vanishing case" (see [L]).

Proposition 3.7 is a key step in the proof of Theorem 2.1. It gives a global L^2 estimate on Cerami sequences. To obtain it, we cut the time domain $\mathbb{R}$ into a family of intervals $(\mathscr{J}^n_l)_{l \in A^n}$. On some of them, x_n is small in local norm. On the others, the action of x_n is greater than some $\gamma > 0$. Moreover, any interval of the first type is adjacent to at least one interval of the second type. This decomposition is based on Lemma 3.6, and gives the estimate of Proposition 3.7, because Card(A^n) and $\|x_n \chi_{\mathscr{J}^n_l}\|_{L^2}$ are bounded independently of $n \geqq 0$, $l \in A^n$.

To end the proof of Theorem 2.1, one can either apply the general concentration-compactness theory [L], or use once again the partition $(\mathcal{I}_l^n)_{l\in A^n}$.

To start our program, we prove

Lemma 3.1. *Assume, for $N \geqq 2$, that ($\mathcal{H}1$) to ($\mathcal{H}3$) are true, that Σ is connected, and that H is positive outside some bounded set. Then $\Omega_0 = 0$, and for any $x \in \Sigma\backslash\{0\}$, we have $\Omega_x(X_H) > 0$. Moreover, there is $\gamma > 0$ such that, for any $\zeta > 0$, exists a positive integer $T(\zeta)$, with the following property*:

For any orbit $x(t)$ lying on Σ, if $|S_{[0,1]}(x)| + \int_0^1 |x|^2 \geqq \zeta^2$, then

$$\mathcal{S}_{[-T(\zeta),T(\zeta)]}(x) = \int \Theta_{[-T(\zeta),T(\zeta)]} \cdot \Omega_x(X_H) > \gamma \,. \tag{3.4}$$

So a solution $x(t)$ of (1.1) *is homoclinic to* 0 *iff it lies on $\Sigma\backslash\{0\}$ and has a finite action on* $\mathbb{R}$, *and we then have $I(x) = \int_{\mathbb{R}} \Omega_x(\dot{x}) = \int_{\mathbb{R}} \Omega_x(X_H) > \gamma$.*

Proof. Since $N \geqq 2, \Sigma\backslash\{0\}$ is connected, so the sign of $\Omega_x(X_H)$ is constant on $\Sigma\backslash\{0\}$. As a first consequence, the sign of $\Omega_0(JAx)$ is constant on $C^* = \{x \neq 0/\langle x, Ax\rangle = 0\}$. So $\Omega_0 = 0$.

Σ is the frontier of the bounded volume $\mathcal{V} = \{x/H(x) \leqq 0\}$. For $x \in \Sigma\backslash\{0\}$, we have $\Omega_x(X_H) = \omega(\eta, X_H) = \langle \eta, H'(x)\rangle$. So $\Omega_x(X_H) > 0$ iff η points outward.

Denoting ψ_t the flow of η, if the volume of $\psi_t(\mathcal{V})$ is an increasing function of t, then η points outward. We write

$$\frac{d}{dt}\left[\int_{\psi_t(\mathcal{V})} \omega^n\right]\Big|_{t=0} = \int_{\mathcal{V}} \mathcal{L}_\eta(\omega^n) = n\int_{\mathcal{V}} \omega^n > 0\,.$$

So the first part of Lemma 3.1 is proved.

Now, the flow φ of X_H is hyperbolic at 0, and its restriction to a small neighborhood of 0 is C^0-conjugate to the flow of the linearized vector field JAx. So there is $\beta > 0$ such that, if $x(t)$ is an orbit of (1.1) on Σ satisfying $|x(0)| \geqq \rho > 0$, then either $|x(\mathcal{T}(\rho))| \geqq \beta$, or $|x(-\mathcal{T}(\rho))| \geqq \beta$, for some $\mathcal{T}(\rho) \in \mathbb{N}^*$, independent of x.

We choose $\gamma = \inf\{S_{[-1,1]}(y)/y(0) \in \Sigma,\ |y(0)| \geqq \beta,\ \dot{y} = X_H(y)\}$. Given $\zeta > 0$, we take $\rho(\zeta) > 0$ such that, if an orbit x of (1.1) on Σ satisfies $|x(0)| < \rho(\zeta)$, then $|S_{[0,1]}(x)| + \int_0^1 |x|^2 < \zeta^2$.

We choose $T(\zeta) = \mathcal{T}(\rho(\zeta)) + 1$, and the second part of Lemma 3.1 is easily checked. □

The following lemma concerns the action $\mathcal{S}_{\mathcal{J}}$ defined by formula (3.3):

Lemma 3.2. *Suppose ($\mathcal{H}1$) to ($\mathcal{H}4$) are true. Then, for any interval $\mathcal{J}$ of size greater than* 1, *the functional $\mathcal{S}_{\mathcal{J}}$ is well-defined and continuous on E, and for any $x \in E$,*

$$\int_{\mathbb{R}} \Omega_x(\dot{x}) = \frac{1}{2}\int_{\mathbb{R}} \langle Jx, \dot{x}\rangle, \quad \text{hence } \mathcal{S}_{\mathbb{R}}(x) = I(x)\,. \tag{3.5}$$

Moreover, for some $\kappa > 0$ independent of $x, \mathcal{J}$, we have

$$|\mathcal{S}_{\mathcal{J}}(x)| + \int_{\mathcal{J}} |x|^2 \leqq \kappa\|x\|_E^2\,. \tag{3.6}$$

Proof. From ($\mathscr{H}4$), $\frac{\Omega_x}{|x|}$ and the differential Ω_x' are uniformly bounded on $\mathbb{R}^{2N}$.

Now, given $x \in H^{1/2}(\mathbb{R}, \mathbb{R}^{2N}) = E$, there is $\bar{x} \in H^1(\mathbb{R}^2, \mathbb{R}^{2N})$ such that the trace of $\bar{x}$ on $\mathbb{R} \times \{0\}$ is x, and that $\|\bar{x}\|_{H^1}^2 \leqq K\|x\|_E^2$ for $K > 0$ independent of x. The following estimates prove that $x \to \Omega_x$ is well-defined as a function from E to E:

$$\|\Omega_x\|_E = O(\|\Omega_{\bar{x}}\|_{H^1}) = O(\|\Omega_{\bar{x}}\|_{L^2} + \|\Omega_{\bar{x}}' \cdot \nabla\bar{x}\|_{L^2})$$
$$= O(\|\bar{x}\|_{L^2} + \|\nabla\bar{x}\|_{L^2}) = O(\|x\|_E) .$$

In a similar way, and with the help of Lebesgue's convergence theorem, we find that $x \to \Omega_x$ is continuous from E to E. □

We now consider a smooth function $b : \mathbb{R} \to [0,1]$ whose support is a subset of $[-1,1]$, and such that $\int_{\mathbb{R}} b = 1$. We denote

$$\begin{cases} b_\varepsilon(t) = \dfrac{1}{\varepsilon} b\left(\dfrac{t}{\varepsilon}\right) , \\ B_\varepsilon(t) = \displaystyle\int_t^{+\infty} b_\varepsilon(\tau)\, d\tau . \end{cases} \tag{3.7}$$

We have the following result:

Lemma 3.3. *Assume ($\mathscr{H}1$ to 4) are true, and suppose there is a sequence $(x_n)_{n\geqq 0}$ in E satisfying*

$$(1 + \|x_n\|_E)\|I'(x_n)\|_{E'} \to 0 \quad \text{as } n \to +\infty .$$

Then, for any $\varepsilon > 0$ fixed, we have

$$\lim_{n\to+\infty} \|H(x_n) * b_\varepsilon\|_{L^\infty(\mathbb{R})} = 0 . \tag{3.8}$$

Proof. For a fixed ε, and a given $x \in E$, smooth and compactly supported, we have

$$\left|\int_{\mathbb{R}} H(x(\tau))b_\varepsilon(t-\tau)d\tau\right|$$

(integrating by parts)

$$= \left|\int_{\mathbb{R}} (H'(x(\tau)) \cdot \dot{x}(\tau))B_\varepsilon(t-\tau)\, d\tau\right|$$

(remarking that $\dot{x} = JH'(x) + JI'(x)$)

$$= \left|\int_{\mathbb{R}} (H'(x(\tau)) \cdot JI'(x)(\tau))B_\varepsilon(t-\tau)d\tau\right|$$
$$= \{H'(x(\tau))B_\varepsilon(t-\tau); JI'(x)(\tau)\}_{E\times E'}$$
$$\leqq C_\varepsilon^1 \|H'(x)\|_E \|I'(x)\|_{E'}$$
$$\leqq C_\varepsilon \|x\|_E \|I'(x)\|_{E'} , \quad \text{from (2.1), (2.2)} .$$

Here, $C_\varepsilon > 0$ is a constant independent of t,x. By density, the final inequality is true for any $x \in E$. Taking $x = x_n$, we obtain (3.8). □

As a consequence of Lemma 3.3, we now prove

Lemma 3.4. *Assume ($\mathcal{H}$1 to 4) are true, and that $(x_n)_{n \geqq 0}$ satisfies*

$$(1 + \|x_n\|_E)\|I'(x_n)\|_{E'} \underset{n \to +\infty}{\longrightarrow} 0 .$$

Then there is a constant $C > 0$ such that, for any $n \geqq 0$,

$$\sup_{t_0 \in \mathbb{R}} \|x_n \chi_{[t_0, t_0+1]}\|_{L^2} \leqq C . \tag{3.9}$$

Proof. Otherwise, by extraction and translation, we get a sequence $y_n = x_{\varphi(n)}(t - t_n)$ such that $\|y_n \chi_{[0,1]}\|_{L^2} \to +\infty$.

Writing $z_n = \frac{y_n}{\|y_n \chi_{[0,1]}\|_{L^2}}$, we have $\int_0^1 z_n^2 \geqq 1$, hence

$$\int_0^1 z_n^2 \chi_{\left\{|z_n| \geqq \frac{1}{\sqrt{2}}\right\}} \geqq \frac{1}{2} .$$

Now, recall that $R(x) = a|x|^2$ with $a > \frac{\|A\|}{2}$, for $|x|$ large enough. So, for $\frac{\|y_n \chi_{[0,1]}\|_{L^2}}{\sqrt{2}}$ large enough, we get

$$\int_0^1 R(y_n) \chi_{\left\{|z_n| \geqq \frac{1}{\sqrt{2}}\right\}} \geqq \frac{1}{2} \|y_n \, \chi_{[0,1]}\|_{L^2}^2 \left(a - \frac{\|A\|}{2} \right) .$$

Consider the finite bound $m = -\min_{x \in \mathbb{R}^{2N}} H(x)$, and fix $0 < \varepsilon < \frac{1}{2}$.
There is $n_0 > 0$ such that, if $n \geqq n_0$, then for any $\tau \in [-\varepsilon, \varepsilon]$,

$$\begin{aligned}\int_{\tau-\varepsilon}^{1+\tau+\varepsilon} R(y_n(t))\, dt &\geqq \frac{1}{2} \|y_n \, \chi_{[0,1]}\|_{L^2}^2 \left(a - \frac{\|A\|}{2} \right) - 2m \\ &\geqq 10, \quad \text{for instance.}\end{aligned}$$

So we will have

$$\int_{-\varepsilon}^{1+\varepsilon} (H(y_n) * b_\varepsilon)(t)\, dt = \int_{-\varepsilon}^{\varepsilon} d\tau \, b_\varepsilon(-\tau) \int_{\tau-\varepsilon}^{1+\tau+\varepsilon} H(y_n)(u)\, du \geqq 10 .$$

But remembering that (y_n) comes from (x_n), we see that Lemma 3.3 is contradicted.
Lemma 3.4 is thus proved. □

The following result deals with the "vanishing" situations:

Lemma 3.5. *Suppose ($\mathcal{H}$1 to 4) are true. There are a constant $\zeta_0 > 0$ and a function $v : \mathbb{R}_+^* \to \mathbb{R}_+^*$, such that, given $M > 0$, $x \in E$ and $m_1 < m_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$,*

if $\sup\{\|x \, \chi_{[t_0, t_0+1]}\|_{L^2}, t_0 \in \mathbb{R}\}$ *is smaller than M, and*

$$\zeta = \left(\sup \left\{ |\mathcal{S}_{[k,k+1]}(x)| + \int_{[k,k+1]} |x|^2, m_1 \leqq k \leqq m_2 - 1, \ k \in \mathbb{Z} \right\} \right)^{\frac{1}{2}}$$

is smaller than ζ_0, *then*:

1) *In the case* $m_1 = -\infty$, $m_2 = +\infty$, *we have*

$$\|x\|_E^2 \leqq \nu(\zeta_0)\left(\zeta^2 + \|I'(x)\|_{E'}^2\right) . \tag{3.10}$$

2) *In any case, we have*

$$|\mathscr{S}_{[m_1,m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 \leqq \nu(M)\left(\zeta^2 Ln\left(\frac{1}{\zeta}\right) + \|I'(x)\|_{E'}^2\right) . \tag{3.11}$$

Proof. We only treat the case $m_1, m_2 \in \mathbb{Z}$, which contains the essential difficulties.

To simplify the notations, we write Θ_p instead of $\Theta_{[m_1+p,m_2-p]}$. We have

$$\begin{aligned} \Theta_p x &= \Theta_p LR'(x) + \Theta_p LI'(x) \\ &= \Theta_p L(R'(x)\chi_{[m_1+p,m_2-p]}) + \Theta_p L(R'(x)\chi_{[m_1,m_1+p]\cup[m_2-p,m_2]}) \\ &\quad + \Theta_p L(R'(x)\chi_{\mathbb{R}\setminus[m_1,m_2]}) + \Theta_p LI'(x) . \end{aligned} \tag{3.12}$$

We recall that $L = (-J\frac{d}{dt} - A)^{-1}$ is an isometry between E' and E. Moreover, $\Theta_p L$ is continuous from E' to E, and for some $K^{(0)}$ independent of p,

$$\|\Theta_p L\|_{\mathscr{L}(E',E)} \leqq K^{(0)} .$$

L may in fact be defined as a convolution product, $Lx = \mathscr{L} * x$, where $\mathscr{L}(t)$ is the matrix $e^{JAt}(\chi_{\mathbb{R}_+}(t)\prod_s - \chi_{\mathbb{R}_-}(t)\prod_u)$. Here, $\prod_s$ and $\prod_u$ are the matrices of projection on the stable and unstable spaces of the flow e^{tJA}. We see that for some $K^{(1)}, \mu > 0, |\mathscr{L}'(t)| + |\mathscr{L}(t)| \leqq K^{(1)}e^{-\mu|t|}$ for $t \neq 0$. As a consequence,

$$\begin{aligned} \|L(R'(x)\chi_{\mathbb{R}\setminus[m_1,m_2]})(t) \cdot \Theta_p(t)\|_E &\leqq K^{(2)}e^{-\mu p} \sup_{t_0\in\mathbb{R}} \|R'(x)\chi_{[t_0,t_0+1]}\|_{L^2} \\ &\leqq K^{(3)}M\, e^{-\mu p} . \end{aligned} \tag{3.13}$$

The same argument gives

$$\|\Theta_p L(R'(x)\chi_{[m_1,m_1+p]\cup[m_2-p,m_2]})\|_E \leqq K^{(3)}\zeta . \tag{3.14}$$

Now, since $|R'(x)| \leqq K^{(4)}\min(|x|,|x|^{3/2})$ we get

$$|R'(x)|^2 \leqq K^{(5)} \cdot (|\zeta x|^3 + |(1-\zeta)x|^2)$$

for any $0 \leqq \zeta \leqq 1$, $K^{(5)}$ being independent of ζ. So we have

$$\begin{aligned} \int_{[m_1+p,m_2-p]} |R'(x)|^2 &\leqq K^{(5)}\left(\int_{[m_1+p,m_2-p]} |\Theta_p x|^3 + \int_{\substack{[m_1+p,m_1+p+1] \\ \cup[m_2-p-1,m_2-p]}} |(1-\Theta_p)x^2\right) \\ &\leqq K^{(5)}\left(2\zeta^2 + \sum_{k=m_1+p}^{m_2-p-1} \int_k^{k+1} |\Theta_p x|^3\right) . \end{aligned}$$

But we may write

$$\int_k^{k+1} |\Theta_p x|^3 \leqq \left(\int_k^{k+1} |\Theta_p x|^2\right)^{\frac{1}{2}} \left(\int_k^{k+1} |\Theta_p x|^4\right)^{\frac{1}{2}},$$

by the Cauchy–Schwartz inequality. So

$$\int_{[m_1+p,\, m_2-p]} |R'(x)|^2 \leqq K^{(5)} \left(2\zeta^2 + \zeta \sum_{k=m_1+p}^{m_2-p-1} \left(\int_k^{k+1} |\Theta_p x|^4\right)^{\frac{1}{2}}\right)$$
$$\leqq K^{(6)}(\zeta^2 + \zeta \|\Theta_p x\|_E^2),$$

by a classical Sobolev inequality. Hence

$$\|\Theta_p L(R'(x)\chi_{[m_1+p,\, m_2-p]})\|_E \leqq K^{(7)}(\zeta + \zeta^{1/2}\|\Theta_p x\|_E). \tag{3.15}$$

Combining (3.12), (3.13), (3.14), (3.15), we get

$$\|x\Theta_p\|_E \leqq K^{(7)}(\zeta + \zeta^{1/2}\|x\Theta_p\|_E) + K^{(3)}(Me^{-\mu p} + \zeta) + K^{(0)}\|I'(x)\|_{E'}.$$

So, if $\zeta < \zeta_0 = \frac{1}{(2K^{(7)})^2}$, then

$$\|x\Theta_p\|_E^2 \leqq K^{(8)}(\zeta^2 + e^{-2\mu p} + \|I'(x)\|_{E'}^2)(1+M)^2. \tag{3.16}$$

Writing

$$|\mathscr{S}_{[m_1, m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 \leqq |\mathscr{S}_{[m_1+p+1,\, m_2-p-1]}(x)| + \int_{m_1+p+1}^{m_2-p-1} |x|^2$$
$$+ \sum_{\substack{k \in [m_1, m_1+p] \\ \cup [m_2-p-1, m_2-1]}} \left(|\mathscr{S}_{[k, k+1]}(x)| + \int_k^{k+1} |x|^2\right),$$

and using the inequality (3.6) of Lemma 3.2, we get

$$|\mathscr{S}_{[m_1, m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 \leqq \kappa\|x\Theta_p\|_E^2 + 2(p+1)\zeta^2. \tag{3.17}$$

We also have the rough estimate

$$|\mathscr{S}_{[m_1, m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 \leqq (m_2 - m_1)\zeta^2. \tag{3.18}$$

So there are two possibilities:

1st case: $\frac{m_2-m_1}{2} - 1 \leqq \frac{1}{\mu} Ln\left(\frac{1}{\zeta}\right)$.

Then, from (3.18), $|\mathscr{S}_{[m_1, m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 \leqq (\frac{2}{\mu} Ln(\frac{1}{\zeta}) + 1)\zeta^2$.

2nd case: $\frac{m_2-m_1}{2} - 1 \geqq \frac{1}{\mu} Ln\left(\frac{1}{\zeta}\right)$.

Then we choose $p \in [\frac{1}{\mu} Ln(\frac{1}{\zeta}) - 1, \frac{1}{\mu} Ln(\frac{1}{\zeta})]$, and from (3.16), (3.17), we get

$$\begin{aligned}|\mathscr{S}_{[m_1,m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 &\leqq \kappa K^{(8)}(1+M)^2(\zeta^2 + e^{-2\mu p} + \|I'(x)\|_{E'}^2) + 2(p+1)\zeta^2 \\ &\leqq \kappa K^{(8)}(1+M)^2(\zeta^2 + e^{2\mu}\zeta^2 + \|I'(x)\|_{E'}^2) \\ &\quad + \left(\frac{2}{\mu} Ln\left(\frac{1}{\zeta}\right) + 2\right)\zeta^2 .\end{aligned}$$

So, in both cases,

$$|\mathscr{S}_{[m_1,m_2]}(x)| + \int_{m_1}^{m_2} |x|^2 \leqq v(M)\left(\zeta^2 Ln\left(\frac{1}{\zeta}\right) + \|I'(x)\|_E^2\right), \tag{3.19}$$

and Lemma 3.5 is proved. □

The following result deals with "non-vanishing" situations:

Lemma 3.6. *Assume that ($\mathscr{H}1$ to 4) hold, and $(1+\|x_n\|_E)I'(x_n) \to 0$ as $n \to \infty$. Take $\zeta > 0$. There is $N(\zeta) > 0$ such that: given $t_0 \in \mathbb{R}$ and $n \geqq N(\zeta)$, if $|\mathscr{S}_{[t_0,t_0+1]}(x)| + \int_{t_0}^{t_0+1} |x|^2 \geqq \zeta^2$, then $\mathscr{S}_{[t_0-T(\zeta),t_0+T(\zeta)]}(x_n) > \gamma$ and $\mathscr{S}_{\mathscr{I}}(x_n) > 0$ for any interval $\mathscr{I}$ such that $[t_0, t_0+1] \subset \mathscr{I} \subset [t_0 - T(\zeta), t_0 + T(\zeta)]$.*

Here, the constant γ and the integer $T(\zeta)$ are the same as in Lemma 3.1.

Proof. Otherwise, there would be $\zeta > 0$, a subsequence $x_{\varphi(n)}$ and a sequence of translations $(t_n)_{n\geqq 0}$ such that, denoting $y_n = x_{\varphi(n)}(t+t_n)$, $|\mathscr{S}_{[0,1]}(y_n)| + \int_0^1 |y_n|^2 \geqq \zeta^2$, and either $\mathscr{S}_{[-T(\zeta),T(\zeta)]}(y_n) \leqq \gamma$, or $\mathscr{S}_{\mathscr{I}_n}(y_n) \leqq 0$ for some interval $\mathscr{I}_n$, with $[0,1] \subset \mathscr{I}_n \subset [-T(\zeta), T(\zeta)]$.

By Lemma 3.4, we have $\sup_{t_0\in\mathbb{R}} \|y_n \chi_{[t_0,t_0+1]}\|_{L^2} \leqq C$.

So $y_n = LR'(y_n) + LI'(y_n)$ is precompact in L^2_{lcc}.

After extraction, we get $y_n \underset{n\to\infty}{\to} y_\infty$ in L^2_{loc}, and, by a standard bootstrap argument, $\beta y_n \underset{n\to\infty}{\to} \beta y_\infty$ in E, for any fixed smooth function β, compactly supported.

As a consequence, $y_\infty = LR'(y_\infty)$. So y_∞ is a solution of (1.1). Moreover, $|\mathscr{S}_{[0,1]}(y_\infty)| + \int_0^1 |y_\infty|^2 \geqq \zeta^2$, and either $\mathscr{S}_{[-T(\zeta),T(\zeta)]}(y_\infty) \leqq \gamma$, or $\mathscr{S}_{\mathscr{I}_\infty}(y_\infty) \leqq 0$, by continuity in E. Here, $\mathscr{I}_\infty$ is an interval whose end points are the limits of the end points of $(\mathscr{I}_n)$, after extraction. Now, from Lemma 3.3, for any $\varepsilon > 0$ and $t \in \mathbb{R}$, we have

$$[H(y_\infty) * b_\varepsilon](t) = \lim_{n\to+\infty} [H(y_n) * b_\varepsilon](t) = 0 .$$

So $H(y_\infty) = 0$, and y_∞ lies on Σ. So we get a contradiction with Lemma 3.1, and Lemma 3.6 is proved. □

Thanks to the two proceding lemmas, we are now ready to prove the key proposition of this section:

Proposition 3.7. *Suppose that ($\mathscr{H}1$ to 4) are true. Let $(x_n)_{n\geqq 0}$ be such that $(1+\|x_n\|_E)\|I'(x_n)\|_{E'} \to 0$ and $I(x_n) \to c \in \mathbb{R}$ as $n \to +\infty$.*

Then the sequence $(\|x_n\|_{L^2})_{n\geqq 0}$ is bounded.

Proof. ν, ζ_0 and γ being those of Lemmas 3.1 and 3.5, we take $\zeta_1 < \zeta_0$ such that:

$$\nu(C)\,\zeta_1^2\, Ln\left(\frac{1}{\zeta_1}\right) < \frac{\gamma}{4}, \tag{3.20}$$

where C is the constant of Lemma 3.4.

From (3.20) and Lemma 3.5, there is n_0 such that, for any $n \geqq n_0$ and any $m_1 < m_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$:

$$\begin{cases} \text{If} \quad |\mathscr{S}_{[k,k+1]}(x_n)| + \int_k^{k+1} |x_n|^2 < \zeta_1^2, \ \forall k \in \mathbb{Z}, \\ \text{then} \quad \|x_n\|_E \leqq \nu(\zeta_0)\zeta_0^2 . \end{cases} \tag{3.21}$$

$$\begin{cases} \text{if} \quad |\mathscr{S}_{[k,k+1]}(x_n)| + \int_k^{k+1} |x_n|^2 < \zeta_1^2, \ \forall k \in \mathbb{Z} \cap [m_1, m_2 - 1], \\ \text{then} \quad |\mathscr{S}_{[m_1,m_2]}(x_n)| + \int_{m_1}^{m_2} |x_n|^2 < \frac{\gamma}{4} . \end{cases} \tag{3.22}$$

We take $n \geqq \max(n_0, N(\zeta_1))$, where the function $N(\zeta)$ is the same as in Lemma 3.6.

Proposition 3.7 is an immediate consequence of (3.21) in the case

$$|\mathscr{S}_{[k,k+1]}(x_n)| + \int_k^{k+1} |x_n|^2 < \zeta_1^2, \quad \forall k \in \mathbb{Z} .$$

So we may assume that $|\mathscr{S}_{[k,k+1]}(x_n)| + \int_k^{k+1} |x_n|^2 \geqq \zeta_1^2$ for some $k \in \mathbb{Z}$. Then there is a partition $\mathscr{P}^n = \{\mathscr{I}_l^n, l \in A^n\}$ of $\mathbb{R}$, where A^n is a subset of $\mathbb{N}$, and each set $\mathscr{I}_l^n$ is an interval of one of the two following types:

Type 1. $\mathscr{S}_{\mathscr{I}_l^n}(x_n) > \gamma$, and $2T(\zeta_1) \leqq |\mathscr{I}_l^n| \leqq 3T(\zeta_1)$, where the function $T(\zeta)$ is the same as in Lemmas 3.1 and 3.6.

Type 2. $|\mathscr{I}_l^n| \in \mathbb{N}^* \cup \{+\infty\}$, and, if we subdivide $\mathscr{I}_l^n$ into $|\mathscr{I}_l^n|$ disjoint intervals $\mathscr{I}'(q)$ of length 1, then for any $q, (1 \leqq q \leqq |\mathscr{I}_l^n|)$, we have

$$|\mathscr{S}_{\mathscr{I}'(q)}(x_n)| + \int_{\mathscr{I}'(q)} |x_n|^2 < \zeta_1^2 .$$

Moreover, any interval $\mathscr{I}$ of type 2 is adjacent to two intervals of type 1 if $\mathscr{I}$ is bounded, and one otherwise.

Proposition 3.7 follows from the existence of $\mathscr{P}^n$.

Indeed, if we call A_1^n, A_2^n the subsets of A^n associated to the intervals of type 1, 2 respectively, we have Card $A_2^n \leqq 1 +$ Card A_1^n.

So, from (3.22),

$$\mathscr{S}_{\mathscr{I}_l^n}(x_n) + \|x_n \chi_{\mathscr{I}_l^n}\|^2 \leqq \frac{\gamma}{4} \ \forall l \in A_2^n, \tag{3.23}$$

hence

$$I(x_n) \geqq \sum_{l\in A_1^n} \mathscr{S}_{\mathscr{I}_l^n}(x_n) - \sum_{l'\in A_2^n} |\mathscr{S}_{\mathscr{I}_{l'}^n}(x_n)|$$

$$\geqq \frac{\gamma}{2}\,\mathrm{Card}\,(A_1^n) \geqq \frac{\gamma}{4}(\mathrm{Card}\,(A^n) - 1)\,. \tag{3.24}$$

But $I(x_n) \to c$. So (Card A^n) is a bounded sequence. Now, if $l \in A_1^n$, then $\|x_n \chi_{\mathscr{I}_l^n}\|_{L^2}^2 \leqq 3T(\zeta_1)C^2$, where C is the constant of Lemma 3.4. Combining this with (3.23), we find that the norm

$$\|x_n\|_{L^2}^2 = \sum_{l\in A_1^n} \|x_n \chi_{\mathscr{I}_l^n}\|^2 + \sum_{l\in A_2^n} \|x_n \chi_{\mathscr{I}_l^n}\|^2$$

is bounded independently of n.

So, to end the proof of Proposition 3.7, we just have to give the construction of $\mathscr{P}^n$. We write T instead of $T(\zeta_1)$ to simplify the notations, and we proceed as follows:

We choose $k_0 \in \mathbb{Z}$ minimal such that $|\mathscr{S}_{[k_0,k_0+1]}(x_n)| + \int_{k_0}^{k_0+1} |x_n|^2 \geqq \zeta_1^2$. From Lemma 3.6, we have $\mathscr{S}_{[k_0-T,k_0+T]}(x_n) > \gamma$.

Moreover, the interval $\mathscr{I}_0^n =]-\infty, k_0 - T]$ is of type 2, by minimality of k_0.

Now, two possibilities may occur:

1st case. $[k_0 + 2T, +\infty[$ *is of type* 2. *We take* $\mathscr{I}_2^n = [k_1, +\infty]$, *with* $k_1 \in \mathbb{Z} \cap [k_0 + T, k_0 + 2T]$ *minimal such that* $\mathscr{I}_2^n$ *is of type* 2, *and we take* $\mathscr{I}_1^n = [k_0 - T, k_1]$.

Since $|\mathscr{S}_{[k,k+1]}(x_n)| + \int_k^{k+1} |x_n|^2 \geqq \zeta_1^2$ for $k \in \{k_0, k_1 - 1\}$, Lemma 3.6 implies that $\mathscr{I}_1^n$ is of type 1.

So, in the first case, $(\mathscr{I}_0^n, \mathscr{I}_1^n, \mathscr{I}_2^n)$ is the desired partition.

2nd case. There is minimal integer $k_1 \geqq k_0 + 2T$ *such that* $|\mathscr{S}_{[k_1,k_1+1]}(x_n)| + \int_{k_1}^{k_1+1} |x_n|^2 \geqq \zeta_1^2$.

If for some integer $k_0' \in [k_0 + T, \min(k_0 + 2T, k_1 - T) - 1], |\mathscr{S}_{[k_0',k_0'+1]}(x_n)| + \int_{k_0'}^{k_0'+1} |x_n|^2 \geqq \zeta_1^2$, then we choose $\mathscr{I}_1^n = [k_0 - T, \min(k_0 + 2T, k_1 - T)]$. From Lemma 3.6, $\mathscr{I}_1^n$ is of type 1. When $k_0 + 2T < k_1 - T$, we also take $\mathscr{I}_2^n = [k_0 + 2T, k_1 - T]$. This interval is of type 2, by minimality of k_1.

If no such k_0' exists, we take $\mathscr{I}_1^n = [k_0 - T, k_0 + T], \mathscr{I}_2^n = [k_0 + T, k_1 - T]$. $\mathscr{I}_1^n$ is again of type 1, $\mathscr{I}_2^n$ is of type 2.

To continue the construction, we apply the preceding procedure to k_1 instead of k_0. Iteratively, we find the desired partition, and Proposition 3.7 is proved. □

So we know that any Cerami sequence $(x_n)_{n\geqq 0}$ is bounded in global L^2-norm. Moreover, $x_n = LR'(x_n) + LI'(x_n)$, with $LI'(x_n) \to 0$ in E, and $|R'(x)| = 0(|x|)$. So, by classical arguments of the Concentration-Compactness theory, (see [5, 17]), Theorem 2 is true ("vanishing" is forbidden by Lemma 3.5).

A more direct way to end the proof of Theorem 2 is to make use of the construction given in the proof of Proposition 3.7.

This construction implies that, after extraction, there are an integer p $(0 \leqq p \leqq \frac{2c}{\gamma} + 1)$, p sequences $t_n^1, \ldots, t_n^p$ of real numbers, with $(i \neq j \Rightarrow |t_n^i - t_n^j| \to +\infty)$, and, for each n, a partition of $\mathbb{R}$ by intervals,

$$Q^n = Q_1^n \cup Q_2^n\,, \quad \text{with } Q_1^n = \{[t_n^i - d_n^i, t_n^i + d_n^i], 1 \leqq i \leqq p\}\,,$$

$$T(\zeta_1) \leqq d_n^i \leqq \left(\frac{3c}{\gamma} + 1\right) T(\zeta_1)\,, \quad \text{Card}(Q_2^n) = p + 1\,,$$

and with:

$$\text{for any } \mathscr{I} \in Q_1^n\,, \quad \mathscr{S}_{\mathscr{I}}(x_n) > \gamma\,, \tag{3.25}$$

$$\text{for any } \mathscr{I} \in Q_2^n\,, \quad |\mathscr{S}_{\mathscr{I}}(x_n)| + \int_{\mathscr{I}} |x_n|^2 < \frac{\gamma}{4}\,. \tag{3.26}$$

We have seen in the proof of Lemma 3.6 that, given a sequence $\sigma_n \in \mathbb{R}$, $x_n(t + \sigma_n)$ has at least one limit point x_∞ in the L^2_{loc}-topology, and that x_∞ is an orbit of (1.1) on Σ. We can say more: if $\delta_n = \text{dist}(\sigma_n, \{t_n^i, 1 \leqq i \leqq p\})$ is a bounded sequence, and if x_∞ is a limit point of $x_n(t + \sigma_n)$, then (3.25) implies that $x_\infty \neq 0$. Moreover,

$$\int_{-a}^{a} \Omega_{x_\infty}(\dot{x}_\infty) \leqq \overline{\lim_{n \to \infty}} \int_{-a+\sigma_n}^{+a+\sigma_n} [\Omega_{x_n}(\dot{x}_n) - H(x_n)] \leqq c + \frac{\gamma}{2}\,,$$

for any $a \geqq \sup(\delta_n)_{n \geqq 0} + (\frac{3c}{\gamma} + 1)T(\zeta_1)$. So x_∞ is a homoclinic orbit. If $\delta_n \to +\infty$, then, from (3.26) and Lemma 3.1, $x_\infty \equiv 0$. So, using Lemma 3.5, we find that, for any $\varepsilon > 0$, there are $\Delta(\varepsilon) > 0$ and $n_\varepsilon > 0$ such that

$$\|x_n \chi_{\{\text{dist}(t, \{t_n^i, 1 \leqq i \leqq p\}) > \Delta(\varepsilon)\}}\|_{L^2} \leqq \varepsilon \quad \text{for } n \geqq n_\varepsilon\,.$$

Finally, after extraction, $x_n(t + t_n^i) \to Z^i$ in L^2_{loc}, with Z^i homoclinic, and $\|x_n - \sum_{i=1}^p Z^i(t - t_n^i)\|_{L^2} \to 0$. By a bootstrap argument, $\|x_n - \sum_{i=1}^p Z^i(t - t_n^i)\|_E \to 0$, and Theorem 2.1 is proved. □

IV. Proof of Theorem 2.3

In this part, we shall use the notations given in Sect. II by formulas (2.3) to (2.22).

We define a function $e_1 \in \mathscr{S}(\mathbb{R}, \mathbb{R}^{2N})$, ($\mathscr{S}$ being the Schwartz class of C^∞ functions f such that $(\partial^\alpha f)P$ is bounded on $\mathbb{R}$ for any $\alpha \in \mathbb{N}$, and P polynomial), by

$$\hat{e}_1(\xi) = \frac{1}{\sqrt{2}} \hat{\psi}(\xi) X(\xi)\,, \tag{4.1}$$

where $X(\xi) = (1, 0, \ldots, 0, i\,\text{sign}(\xi), 0, \ldots, 0)$ is an eigenvector of $-iJ$ with eigenvalue sign (ξ), and ψ is some function whose Fourier transform $\hat{\psi} \in \mathscr{S}(\mathbb{R}, \mathbb{R})$ is non-zero and even, with $\hat{\psi} \equiv 0$ on a neighborhood of 0.

The essential properties of $e_1(\lambda t)$ are given in the following lemma:

Lemma 4.1.

$$\lambda^\alpha \|\, |L|^{\frac{\alpha}{2}} P_+(e_1(\lambda t))\|_E^2 \to \int |\xi|^{1-\alpha} |\hat{\psi}|^2 \quad \text{as } \lambda \to +\infty\,, \tag{4.2}$$

$$\|\, |L|^{\frac{\alpha}{2}} P_-(e_1(\lambda t))\|_E^2 = O\left(\frac{1}{\lambda^{\alpha+1}}\right) \quad \text{as } \lambda \to +\infty\,. \tag{4.3}$$

Proof. As $|\xi| \to \infty$, we have

$$\hat{D}(\xi) = i\xi J + O(1), \qquad |\hat{D}(\xi)| = |\xi| 1 + O(1),$$

hence

$$\pi_+(\xi) = \frac{1}{2}(1 - i\,\mathrm{sign}(\xi)J) + O\left(\frac{1}{|\xi|}\right),$$

$$\pi_-(\xi) = \frac{1}{2}(1 + i\,\mathrm{sign}(\xi)J) + O\left(\frac{1}{|\xi|}\right).$$

So

$$\pi_+(\xi)X(\xi) = X(\xi) + O\left(\frac{1}{|\xi|}\right),$$

$$\pi_-(\xi)X(\xi) = O\left(\frac{1}{|\xi|}\right).$$

From there, by straightforward calculations, we get (4.2) and (4.3). □

From (4.2), for λ greater than some constant λ_0, we can define

$$e_\lambda(t) = \frac{P_+(e_1(\lambda t))}{\|P_+(e_1(\lambda t))\|_E}. \tag{4.4}$$

We are going to choose $\overline{e} = e_\lambda$ in Theorem 2.3, for some λ large enough.[2] This strategy is going to work because the support of $\hat{e}_\lambda$ is included in $(-\lambda)\mathscr{I} \cup \lambda\mathscr{I}$, where $\mathscr{I}$ in a fixed closed interval of $\mathbb{R}$ with $0 \notin \mathscr{I}$, so that, for λ large, $\widehat{|L|k(x)}$ is "small" on support $(\hat{e}_\lambda)$ for $x \in B_-(\overline{\sigma})$. As a consequence, e_λ is "almost" orthogonal to $\Phi(B_-(\overline{\sigma}))$. We will also use the fact that for λ large, $e_\lambda(t)$ "looks like" $e_1(\lambda t)$.

The following result corresponds to (i) in Theorem 2.3.

Lemma 4.2. *Assume that ($\mathscr{H}$1–4) are true. Then there are $\overline{\sigma}, \lambda_1 > 0$ such that, for any $\lambda \geqq \lambda_1$, exists an I-admissible functional $\tilde{I}_\lambda$, with*

$$\tilde{I}_\lambda(\partial\mathscr{M}_-(\overline{\sigma}, e_\lambda)) < 0. \tag{4.5}$$

Proof. From formula (2.17), it is easy to check that $k: E \to E$ is smooth and has a differential $k'(x)$ uniformly bounded on E. We also have $k(0) = 0, k'(0) = 0$, hence

$$\|k(x)\|_E \leqq C\|x\|_E, \ \forall x \in E \quad \text{and} \quad \lim_{\|x\|_E \to 0} \frac{\|k(x)\|_E}{\|x\|_E} = 0. \tag{4.6}$$

This estimate will be used repeatedly.

The proof of Lemma 4.2 is divided in several steps.

First step. There is $C > 0$ such that, for any λ large enough, and all $\sigma > 0$, $(x, s) \in B_-(\sigma) \times [0, \sigma]$,

$$I(\Phi(x) + se_\lambda) \leqq I \circ \Phi(x) + \frac{s^2}{2} + C\frac{s\sigma}{\lambda}. \tag{4.7}$$

[2] The introduction of e_λ has been inspired by some familiarity of the author with the wavelets theory. The interested reader is referred to the book of Yves Meyer, [M].

Proof of the first step. For $x \in B_-(\sigma), s \in [0, \sigma]$, we have

$$I(\Phi(x) + se_\lambda) = \frac{1}{2}\|P_+|L|k(x) + se_\lambda\|_E^2 - \frac{1}{2}\|x + P_-|L|k(x)\|_E^2 - \int R(\Phi(x) + se_\lambda)\,. \tag{4.8}$$

We write

$$\begin{aligned}\frac{1}{2}\|P_+|L|k(x) + se_\lambda\|_E^2 &- \frac{1}{2}(\|P_+|L|k(x)\|_E^2 + s^2) \\ &= s(k(x), |L|e_\lambda)_E \leqq s\|k(x)\|_E\|Le_\lambda\|_E \\ &\leqq C s\sigma\frac{1}{\lambda}, \quad \text{from (4.3) and (4.6)}\,.\end{aligned} \tag{4.9}$$

We also have

$$\begin{aligned}\int R(\Phi(x) + se_\lambda) - \int R(\Phi(x)) &= s\int_{\mathbb{R}} dt \int_0^1 dh\; R'(\Phi(x) + hs\,e_\lambda)\cdot e_\lambda(t) \\ &= s\int_0^1 dh(R'(\Phi(x) + hs\,e_\lambda), e_\lambda)_{L^2} \\ &\leqq s\int_0^1 dh\|R'(\Phi(x) + hs\,e_\lambda)\|_E\|L\,e_\lambda\|_E \\ &\leqq C\,s(\sigma+s)\frac{1}{\lambda}, \quad \text{from } (\mathscr{H}4), \text{(4.3) and (4.6)}.\end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), one easily proves (4.7).

Second step. We consider a non-zero function θ in $\mathscr{S}(\mathbb{R},\mathbb{R})$ such that $\hat{\theta}$ is even, and such that $|\eta| \leqq \frac{|\xi|}{2}$ for any $\eta \in$ support$(\hat{\theta})$ and $\xi \in$ support$(\hat{e}_1)$.

There are $b, B > 0$ such that, for λ large enough, and all $\sigma > 0$, $(x,s) \in B_-(\sigma) \times [0,\sigma]$,

$$\int_{\mathbb{R}} |\Phi(x) + se_\lambda|^2(t)\,\theta(\lambda t) \geqq \frac{s^2}{\lambda}b - B\frac{s\sigma}{\lambda^{3/2}}\,. \tag{4.11}$$

Proof of the second step. Roughly speaking, this proof is based on the idea that $\Phi(x)$ and e_λ are "almost orthogonal" in L^2.

Denoting $f(t) = \theta(t)\,e_1(t)$, we see that $\hat{f}$ is of the form $\frac{1}{\sqrt{2}}\hat{g}(\xi)\,X(\xi)$, where $g \in \mathscr{S}(\mathbb{R},\mathbb{R})$ is non-zero and even, and satisfies $\hat{g} \equiv 0$ in a neighborhood of 0. So Lemma 4.1 can be applied to f.

We also denote $n_\lambda = \|P_+(e_1(\lambda t))\|_E$. From Lemma 4.1, n_λ tends to a non-zero limit as λ tends to infinity.

We write

$$\Phi(x) + se_\lambda = y + \frac{s}{n_\lambda}e_1(\lambda t)\,, \tag{4.12}$$

with the notation $y = \Phi(x) - \frac{s}{n_\lambda}P_-(e_1(\lambda t))$.

We have

$$\int_{\mathbb{R}} |\Phi(x) + se_\lambda|^2(t)\,\theta(\lambda t) \geqq \frac{s^2}{n_\lambda^2}\frac{\|f\|_{L^2}^2}{\lambda} - \frac{2s}{n_\lambda}(y(t), f(\lambda t))_{L^2}\,. \tag{4.13}$$

Now,

$$
\begin{aligned}
(y(t), f(\lambda t))_{L^2} &= (|L|k(x), P_+(f(\lambda t)))_{L^2} + (y(t), P_-(f(\lambda t)))_{L^2} \\
&\leqq \|k(x)\|_E \| \, |L|^2 P_+(f(\lambda t))\|_E + \|y\|_E \| |L| P_-(f(\lambda t))\|_E \\
&= O\left(\frac{\sigma}{\lambda^2}\right) + O\left(\frac{\sigma}{\lambda^{\frac{3}{2}}}\right) = O\left(\frac{\sigma}{\lambda^{\frac{3}{2}}}\right) , \qquad (4.14)
\end{aligned}
$$

and (4.11) is proved by combining (4.13) and (4.14).

Third step. We consider a non-zero function $\theta \in C^0 \cap L^1(\mathbb{R}, \mathbb{R})$ *such that* $|\theta(t)| \leqq 1$ *for all t, and* $\|\theta\|_{L^1} \leqq 1$.

Given $r_0 > 0$, *if a function* $X \in E$ *is such that*

$$\int_{\mathbb{R}} |X(t)|^2 \, \theta(\lambda t) dt \geqq \frac{2r_0^2}{\lambda} , \qquad (4.15)$$

then

$$\int |X(t)|^2 \chi_{\{|X(t)| \geqq r_0\}} dt \geqq \frac{1}{2} \int_{\mathbb{R}} |X(t)|^2 \, \theta(\lambda t) dt . \qquad (4.16)$$

Proof of the third step. Given two non-negative functions f, g with f in $L^1(\mathbb{R})$ and g in $C^0 \cap L^1(\mathbb{R})$, it is well-known that

$$\int f g \chi_{\left\{ f \geqq \frac{\int fg}{2\int g} \right\}} \geqq \frac{1}{2} \int fg . \qquad (4.17)$$

We choose $f(t) = |X(t)|^2$ and $g(t) = |\theta(\lambda t)|$. The assumptions on θ and the condition (4.15) imply

$$g \chi_{\left\{ f \geqq \frac{\int fg}{2\int g} \right\}} \leqq \chi_{\{ f \geqq r_0^2 \}} . \qquad (4.18)$$

So $\int f \chi_{\{ f \geqq r_0^2 \}} \geqq \frac{1}{2} \int fg$, and (4.16) is proved.

Conclusion. We recall the notation $l(\sigma) = \sup I \circ \Phi(S_-(\sigma))$. From Corollary 2.2, we have $\lim_{\sigma \to +\infty} l(\sigma) = -\infty$.

We fix a function θ satisfying the assumptions of the second and third steps. We fix r_0 such that $|x| \leqq \frac{r_0}{2}$ for any $x \in \Sigma$.

We choose s_0 such that $\frac{s_0^2}{4} b \geqq r_0^2$, and $\overline{\sigma} \geqq s_0$ such that $l(\overline{\sigma}) \leqq -s_0^2$. We then take $\lambda_1 \geqq \lambda_0$ such that $C \frac{s_0 \overline{\sigma}}{\lambda_1} \leqq \frac{s_0^2}{4}$, and $B \frac{\overline{\sigma}^2}{\lambda_1^{1/2}} \leqq \frac{s_0^2}{2} b$. Here, C, B, b are the constants of the first and second steps.

If $\lambda \geqq \lambda_1$ and $(\Phi(x) + s e_\lambda) \in \partial \mathcal{M}_-(\overline{\sigma}, e_\lambda)$, there are two possibilities:

1) $s \in [0, s_0]$, and $x \in S_-(\overline{\sigma})$. Then, from the first step,

$$
\begin{aligned}
I(\Phi(x) + s e_\lambda) &\leqq l(\overline{\sigma}) + \frac{s_0^2}{2} + C \frac{s_0 \overline{\sigma}}{\lambda_0} \\
&\leqq -\frac{s_0^2}{4} < 0 . \qquad (4.19)
\end{aligned}
$$

2) $s \in [s_0, \overline{\sigma}]$, and $x \in B_-(\overline{\sigma})$. The first step gives

$$I(\Phi(x) + se_\lambda) \leqq \frac{\overline{\sigma}^2}{2} + C\frac{\overline{\sigma}^2}{\lambda_0} = M\ . \tag{4.20}$$

But from the second step,

$$\int |\Phi(x) + se_\lambda|^2(t)\,\theta(\lambda t) \geqq \frac{1}{\lambda}\left(s^2 b - B\frac{s\overline{\sigma}}{\lambda^{1/2}}\right)$$
$$\geqq \frac{1}{\lambda}\left(s_0^2 b - B\frac{\overline{\sigma}^2}{\lambda^{1/2}}\right) \geqq \frac{s_0^2}{2\lambda} b \geqq \frac{2r_0^2}{\lambda}\ . \tag{4.21}$$

So, using the third step,

$$\int |\Phi(x) + se_\lambda|^2 \chi_{\{|\Phi(x)+se_\lambda| \geqq r_0\}} dt \geqq \frac{1}{2}\int |\Phi(x) + se_\lambda|^2(t)\,\theta(\lambda t) \geqq \frac{r_0^2}{\lambda}\ . \tag{4.22}$$

We choose $\tilde{a}(\lambda) = \lambda\frac{2M}{r_0^2}$ and we construct a smooth nonnegative function $\tilde{R}_\lambda$ such that $\tilde{R}_\lambda(x) = 0$ when $|x| \leqq \frac{r_0}{2}$ and $\tilde{R}_\lambda(x) = \tilde{a}(\lambda)|x|^2$ when $|x| \geqq r_0$. The functional $\tilde{I}_\lambda(x) = I(x) - \int \tilde{R}_\lambda(x)$ is I-admissible, and from (4.20), (4.22), we have

$$\tilde{I}_\lambda(\Phi(x) + se_\lambda) \leqq -M < 0\ . \tag{4.23}$$

Lemma 4.2 is thus proved. □

We define, for $\lambda \geqq \lambda_1$,

$$F_\lambda : B_-(\overline{\sigma}) \times [0, \overline{\sigma}] \to E_- \times \mathbb{R}_+$$
$$(x, s) \mapsto (P_-(\Phi(x) + se_\lambda)\,,\, \|\Phi(x) + se_\lambda\|_E^2)\ . \tag{4.24}$$

We have the following result:

Lemma 4.3. *Assume ($\mathscr{H}$1) to ($\mathscr{H}$4) are true. Then there exist $\overline{\lambda} \geqq \lambda_1$ and $\overline{\rho} > 0$ such that ($\lambda_1, \overline{\sigma}$ are those of Lemma 4.2):*

- *For any $x \in B_+(\overline{\rho})$, $\quad I(x) \geqq \frac{\|x\|_E^2}{4}$.*
- *The equation $F_{\overline{\lambda}}(x, s) = (0, \overline{\rho}^2)$ has a unique solution in $B_-(\overline{\sigma}) \times [0, \overline{\sigma}]$, $(x, s) = (0, \overline{\rho})$.*
- *The differential of $F_{\overline{\lambda}}$ at $(0, \overline{\rho})$ is an automorphism of $E_- \times \mathbb{R}$.*

The geometric meaning of the two last properties is that

$$S_+(\overline{\rho}) \cap \mathscr{M}_-(\overline{\sigma}, e_{\overline{\lambda}}) = \{\overline{\rho} e_{\overline{\lambda}}\}\ ,$$

and that this intersection is transversal.

Proof. We first take $\rho < \overline{\sigma}$ and $\lambda \geqq \lambda_1$ arbitrary.

The equation $F_\lambda(x, s) = (0, \rho^2)$ has at least one solution in the set $B_-(\overline{\sigma}) \times [0, \overline{\sigma}]$, $(x, s) = (0, \rho)$. At that point, the differential of F_λ is $dF_\lambda(0, \rho) = (dx, 2\rho\, ds)$. It is an automorphism of E_-.

The tangent space to $\mathscr{W}_-$ at 0 is E_-, and the differential of Φ at 0 is the identity of E_-. So there is $\varepsilon > 0$ such that the only solution of the equation $P_- \circ \Phi(x) = 0$ in $B_-(\varepsilon)$ is $x = 0$. As a consequence, the equation $F_\lambda(x,s) = (0,\rho^2)$ admits $(0,\rho)$ as unique solution in $B_-(\varepsilon) \times [0,\overline{\sigma}]$.

We now have to study the equation $F_\lambda(x,s) = (0,\rho^2)$ in the set $(B_-(\overline{\sigma})\backslash B_-(\varepsilon)) \times [0,\overline{\sigma}]$.

Let $\eta = \inf\{\|\Phi(x)\|_E,\ x \in E_-\backslash B_-(\varepsilon)\}$. η is strictly positive, because $l(\varepsilon) = \sup I \circ \Phi(S_-(\varepsilon))$ is strictly negative.

There is $K > 0$ such that, for any $x \in B_-(\overline{\sigma})\backslash B_-(\varepsilon)$ and $s \in [0,\overline{\sigma}]$,

$$\begin{aligned}
\|\Phi(x) + se_\lambda\|_E^2 &= \|\Phi(x)\|_E^2 + 2s(\Phi(x), e_\lambda)_E + s^2 \\
&\geqq \eta^2 + 2s(|L|k(x), e_\lambda)_E + s^2 \\
&\geqq \eta^2 - K\frac{s\overline{\sigma}}{\lambda} + s^2, \quad \text{from (4.2) and (4.6)} \\
&\geqq \eta^2 - K\frac{\overline{\sigma}^2}{\lambda}\,. \qquad (4.25)
\end{aligned}$$

Let us fix $\overline{\lambda} \geqq \lambda_1$ such that $K\frac{\overline{\sigma}^2}{\overline{\lambda}} \leqq \frac{\eta^2}{2}$. We get

$$\|\Phi(x) + se_{\overline{\lambda}}\|_E^2 \geqq \frac{\eta^2}{2}\,. \qquad (4.26)$$

So, if $\rho^2 < \frac{\eta^2}{2}$, the equation $F_{\overline{\lambda}}(x,s) = (0,\rho^2)$ has no solution in the set $(B_-(\overline{\sigma})\backslash B_-(\varepsilon)) \times [0,\overline{\sigma}]$.

To end the proof of Lemma 4.3, we choose $\overline{\rho} < \frac{\eta}{\sqrt{2}}$ such that for any $x \in B_+(\overline{\rho})$,

$$\int_{\mathbb{R}} R(x) + \tilde{R}_{\overline{\lambda}}(x) \leqq \frac{\|x\|_E^2}{4}\,. \qquad \square \qquad (4.27)$$

In Theorem 2.3, the constants $\overline{\sigma}$ and $\overline{\rho}$ will be the same as in Lemmas 4.2 and 4.3, and we shall take $(\bar{e},\tilde{I}) = (e_{\overline{\lambda}}, \tilde{I}_{\overline{\lambda}})$. We define

$$\begin{aligned}
&\mathscr{F} : B_-(\overline{\sigma}) \times [0,\overline{\sigma}] \times \mathbb{R}_+ \to E_- \times \mathbb{R}_+ \\
&(x,s,\tau) \mapsto (P_-\tilde{\phi}_\tau(\Phi(x) + se_\lambda)\,,\, \|\tilde{\phi}_\tau(\Phi(x) + se_\lambda)\|_E^2)\,. \qquad (4.28)
\end{aligned}$$

Here, $\tilde{\phi}_\tau$ is the flow of $-\nabla\tilde{I}$.

We have the following result:

Lemma 4.4. *Suppose* ($\mathscr{H}1$) *to* ($\mathscr{H}4$) *true. Then* $\mathscr{F}$ *is a smooth Fredholm operator of index* 1, *and there is* $\varepsilon > 0$ *such that for any* $T > 0$, *the restriction of* $\mathscr{F}$ *to* $U_{\varepsilon,T} = \mathscr{F}^{-1}(B_-(\varepsilon) \times [(\overline{\rho}-\varepsilon)^2, (\overline{\rho}+\varepsilon)^2]) \cap (B_-(\overline{\sigma}) \times [0,\overline{\sigma}] \times [0,T])$ *is proper.*

Proof. From the form of the differentials of ∇I, $\nabla\tilde{I}$,

$$\begin{aligned}
(\nabla I)' \cdot h &= P_+h - P_-h - |L|(R''(x) \cdot h)\,, \\
(\nabla\tilde{I})' \cdot h &= (\nabla I)' \cdot h - |L|(\tilde{R}''(x) \cdot h)\,, \qquad (4.29)
\end{aligned}$$

we find by the method of variation of the constant that $\mathscr{F}$ is a smooth Fredholm mapping. Since

$$(\mathscr{F})'(0) \cdot (h, ds, d\tau) = (h, 0), \tag{4.30}$$

$\mathscr{F}$ is of index 1.

Let us take $\varepsilon > 0$ and a sequence (x_n, s_n, τ_n) in $U_{\varepsilon,T}$ such that $\mathscr{F}(x_n, s_n, \tau_n) = (y_n, \rho_n^2)$ converges to a limit $(y_*, \rho_*^2) \in B_-(\varepsilon) \times [(\overline{\rho} - \varepsilon)^2, (\overline{\rho} + \varepsilon)^2]$. To end the proof of Lemma 4.4, we have to show that for ε small enough, (x_n) is precompact in E_-.

If we could write $\tilde{\phi}_\tau = \Lambda_\tau + K_\tau$ with Λ_τ a linear isomorphism, and K_τ a compact mapping, as is the case in periodic problems, then the compactness of (x_n) would be trivial. Unfortunately, this classical decomposition is impossible here, because we work with a space of function defined on the time domain $\mathbb{R}$, and $\tilde{\phi}_\tau$ is equivariant by translations in time.

This difficulty has been solved by Hofer and Wysocki in ([H-W], proof of Lemma 4.4). In our framework, their idea can be expressed as follows:

First of all, by the method of variation of the constant, we may write

$$\tilde{\phi}_\tau(\Phi(x) + s\bar{e}) = e^\tau(x + |L|K(x, s, \tau)), \tag{4.31}$$

where K is uniformly continuous from $B_-(\overline{\sigma}) \times [0, \overline{\sigma}] \times [0, T]$ to E.

We have $x_n = e^{-\tau_n} y_n - e^{-\tau_n} P_- |L| K(x_n, s_n, \tau_n) = e^{-\tau_n} y_n - z_n$, where (z_n) is bounded in $H^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}^{2N})$. After extraction, we impose that (s_n, τ_n) converges to a limit (s_*, τ_*). From the compact Sobolev embedding $H^{\frac{1}{2}}_{\mathrm{loc}} \subset H^{\frac{3}{2}}$, we see that after a new extraction, there is $z_* \in E$ such that $\beta z_n \to \beta z_*$ in E as $n \to \infty$, for any compactly supported function $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$.

Let us denote $x_n^1 = x_n - e^{-\tau_*} y_* - z_*$. Clearly, $x_n^1 \in E_-$, and it is not very difficult to prove that $P_- \circ \tilde{\phi}_{\tau_n} \circ \Phi(x_n^1) \to 0$ in E and that $\|\tilde{\phi}_{\tau_n} \circ \Phi(x_n^1)\|_E$ is smaller than $\overline{\rho} + 2\varepsilon$ for n large enough. So, choosing ε small enough, we find from Lemma 4.3 that

$$\liminf I \circ \Phi(x_n^1) \geqq \liminf \tilde{I} \circ \tilde{\phi}_{\tau_n} \circ \Phi(x_n^1) \geqq 0. \tag{4.32}$$

Since $l(r) = \sup I \circ \Phi(S_-(r))$ is a strictly decreasing function of r, this inequality implies that $x_n^1 \to 0$ in E. The sequence x_n is thus convergent in E, and Lemma 4.4 is proved. □

We now end the proof of the theorems.

Given $T \geqq 0$, we denote

$$V_{\varepsilon,T} = \mathscr{F}^{-1}(B_-(\varepsilon) \times [(\overline{\rho} - \varepsilon)^2, (\overline{\rho} + \varepsilon)^2]) \cap (B_-(\overline{\sigma}) \times [0, \overline{\sigma}] \times \{T\}). \tag{4.33}$$

The mapping $\mathscr{F}_T = \mathscr{F}(\cdot, \cdot, T)$ is Fredholm of index 0, and its restriction to $V_{\varepsilon,T}$ is proper, from Lemma 4.4. Moreover, $(0, \overline{\rho}) \notin \mathscr{F}_T(\partial V_{\varepsilon,T})$. So, following Smale [Sm], we can define a $\mathbb{Z}_2$ degree $d_T = \deg(\mathscr{F}_T, V_{\varepsilon,T}, (0, \overline{\rho}))$.

From Lemma 4.3, we have $d_0 = 1$, and from Lemma 4.4, $d_T = d_0$. As a consequence, the equation $\mathscr{F}_T(x, s) = (0, \overline{\rho}^2)$ has always a solution in $V_{\varepsilon,T}$. This proves (iii) of Theorem 2.3. By Lemmas 4.2 and 4.3, (i) and (ii) are also true, so Theorem 2.3 is proved. This ends the proof of Theorem 1.1. □

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References

[A-B] Ambrosetti, A., Bertotti, M.: Homoclinics for second order conservative systems. To appear in the Proceedings of a Conference in honour of L. Nirenberg

[B] Bolotin, S.V.: Libration motions of natural dynamical systems. Moscow University Bulletin 6 (1978)

[B-G] Benci, V., Giannoni, F.: Homoclinic orbits on compact manifolds. J. Math. Ann. Appl. **157**, 568–576 (1991)

[C] Cerami, G.: Un criterio di existenza per i punti critici su varietá illimitati. Rend. dell'Academia di Sci. Lett. dell'Inst. Lombardo **112**, 332–336 (1978)

[CZ-E-S] Coti Zelati, V., Ekeland, I., Séré, E.: A variational approach to homoclinic orbits in Hamiltonian systems. Math. Ann. **288**, 133–160 (1990)

[E] Ekeland, I.: Convexity Methods in Hamiltonian Mechanics. Berlin-Heidelberg-New York: Springer, 1989

[H-W] Hofer, H., Wysocki, K.: First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems. Math. Ann. **288**, 483–503 (1990)

[H-Z] Hofer, H., Zehnder, E.: Periodic solutions on hypersurfaces and a result by C. Viterbo. Invent. Math. **90**, 1–9 (1987)

[K] Kozlov, V.V.: Calculus of variations in the large and classical mechanics. Russ. Math. Surv. **40**, 35–71 (1985)

[L] Lions, P.L.: The concentration-compactness principle in the Calculus of Variations. Revista Iberoamericana **1**, 145–201 (1985)

[M] Meyer, Y.: Ondelettes et Opérateurs I. Paris: Hermann, 1990

[R] Rabinowitz, P.: Homoclinic orbits for a class of Hamiltonian systems. Proc. Roy. Soc. Edinburgh **114A**, 33–38 (1990)

[R-T] Rabinowitz, P., Tanaka, K.: Some results on connecting orbits for a class of Hamiltonian systems. Math. Zeitschrift **206**, 473–499 (1991)

[Sc] Scholze, R.: Homoclinic orbits on a certain class of energy surfaces. Preprint Rhür-Universität Bochum, (1992)

[Se1] Séré, E.: Existence of infinitely many homoclinic orbits in Hamiltonian systems. Math. Zeitschrift **209**, 27–42 (1992)

[Se2] Séré, E.: Looking for the Bernoulli Shift. Ann. Inst. Henri Poincaré, Analyse non linéaire, **10**, no 5, 561–590 (1993)

[Sm] Smale, S.: An infinite dimensional version of Sard's Theorem. Am. J. Math. **87**, 861–866 (1965)

[St] Struwe, M.: A global compactness result for elliptic boundary value problems involving limiting nonlinearities. Math. Zeitschrift **187**, 511–517 (1984)

[T] Tanaka, K.: Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonics. Journ. Diff. Eq. **94**, 315–339 (1991)

[V] Viterbo, C.: A proof of Weinstein's conjecture in $\mathbb{R}^{2N}$. Annales de l'Institut Henri Poincaré: Analyse non linéaire **4**, 337–356 (1987)

[W] Weinstein, A.: On the hypotheses of Rabinowitz's periodic orbit theorems. J. Differ. Eqs. **33**, 353–358 (1979)

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