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Leaf-Preserving Quantizations of Poisson $SU(2)$ are not Coalgebra Homomorphisms

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Abstract: Although it has been found that some deformation quantizations of the Poisson $SU(2)$ preserve symplectic leaves and some preserve the group (i.e. coalgebra) operation, this paper shows that a quantization of $SU(2)$ cannot be both leaf-preserving and group-preserving.

1. Introduction

Among various examples in the theory of quantum groups, one of the most well known and well understood in both algebraic and analytic contexts is quantum $SU(2)$, which has been studied in many different aspects [5, 15, 17].

We recall that on $SU(2)$ there is a multiplicative Poisson structure and Drinfeld's work shows that quantum $SU(2)$ gives a consistent algebraic deformation quantization of both the Poisson structure and the group structure. On the other hand, there have been found two types of C*-algebraic deformation quantization [13, 14, 1], a concept introduced by Rieffel [10, 11, 12], of the multiplicative Poisson structure on $SU(2)$ which are compatible with Woronowicz's C*-algebraic deformation quantization of the group structure of $SU(2)$ [17], in the sense that the C*-algebras obtained in these two processes are isomorphic. These two types of C*-algebraic deformations of the Poisson structure have very different features. One is constructed in a geometrically natural way (inspired by the concept of foliation C*-algebras [4]) and is a leaf-preserving deformation (to be defined later), while the other based on the work [6] is constructed in a more algebraic way and is not a leaf-preserving deformation, but it deforms the generators in Woronowicz's way and is actually a coalgebra isomorphism.

Since the coalgebra structure gives the group (action) structure, a deformation like the latter one which respects the coalgebra structure is probably of more interest from the algebraic viewpoint. It is a very interesting question to see whether the former quantization is actually a coalgebra isomorphism, and if not, whether there exists a leaf-preserving quantization that also preserves the coalgebra structure [12].

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A direct study of the first problem seems to be very difficult because it involves the sophisticated pseudodifferential calculus. However in this paper, by working directly on the second more abstract problem, we are able to show that leaf-preserving quantizations cannot preserve the coalgebra structure, which gives a negative answer to both questions.

2. Quantum $SU(2)$

In this section we briefly recall some facts about the C*-algebraic function algebra $C(S_\mu U(2))$ of quantum $SU(2)$ introduced by Woronowicz [17].

Recall that [17] the universal C*-algebra $C(S_\mu U(2)), -1 < \mu \leqq 1$, generated by α and γ, subject to the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + \mu^2\gamma^*\gamma = 1, \quad \gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = \mu\gamma\alpha, \quad \alpha\gamma^* = \mu\gamma^*\alpha$$

is endowed with a coalgebra structure given by the comultiplication

$$\Phi(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma \quad \text{and} \quad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma .$$

For $n \geqq 0$ and $0 \leqq k \leqq n$, we have

$$\Phi(\alpha^{n-k}\gamma^{*k}) = \sum_{0 \leqq l \leqq n} \alpha^{n-l}\gamma^{*l} \otimes \omega_{\mu lk}^{(n+1)}$$

for some simplified finite $\mathbb{Z}$-linear sum $\omega_{\mu lk}^{(n+1)}$ of monomials in the (noncommutative) elements $\mu, \mu^{-1}, \alpha, \alpha^*, \gamma$, and γ^* in $C(S_\mu U(2))$. It is known [17] that the matrix $\omega_\mu^{(n)} = (\omega_{\mu lk}^{(n)})_{lk} \in M_n(\mathbb{C}) \otimes C(S_\mu U(2))$ represents an n-dimensional corepresentation of $C(S_\mu U(2))$ and all smooth irreducible corepresentations are equivalent to one of these $\omega_\mu^{(n)}$'s. (Here we mention that Nagy observed that Woronowicz's results indeed imply that all nondegenerate and hence all irreducible corepresentations are smooth [9].) The first three are given explicitly by $\omega_\mu^{(1)} = (1)$,

$$\omega_\mu^{(2)} = \begin{pmatrix} \alpha & \gamma^* \\ -\mu\gamma & \alpha^* \end{pmatrix},$$

and

$$\omega_\mu^{(3)} = \begin{pmatrix} \alpha^2 & \alpha\gamma^* & \gamma^{*2} \\ -(\mu + \mu^{-1})\alpha\gamma & \alpha\alpha^* - \gamma\gamma^* & (1 + \mu^{-2})\gamma^*\alpha^* \\ \mu^2\gamma^2 & -\mu\gamma\alpha^* & \alpha^{*2} \end{pmatrix}.$$

3. Poisson $SU(2)$ and Quantum Leaves

Recall that $SU(2)$ has a canonical Poisson Lie group structure, i.e. a Poisson structure which is multiplicative [8]. From a geometric point of view, the most important object associated with a Poisson manifold is its singular foliation by maximal symplectic submanifolds, called its symplectic leaves [16]. The symplectic leaves of the Poisson $SU(2)$ are known [15] to be either singletons in $\mathbb{T} = U(1)$ or the 2-dimensional leaves

$$L_0 = \left\{ \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} : \alpha \in \mathbb{D},\ \gamma = \sqrt{1 - |\alpha|^2 e^{i\theta}} \right\},$$

where $\mathbb{D}$ is the unit disc and $\theta \in [0, 2\pi)$.

In [15], it is shown that symplectic leaves L of the Poisson $SU(2)$ are in a canonical one-to-one correspondence with irreducible representations π_L of the C*-algebra $C(S_\mu U(2))$ of quantum $SU(2)$. In [7], the quotient algebra $C(L_\mu) = \pi_L(C(S_\mu U(2)))$ (with $\dim(L) = 2$) of $C(S_\mu U(2))$, realized as the universal unital C*-algebra generated by $(1 + zz^*)^{-1/2} z$ subject to the relation $1 + zz^* = \mu^2(1 + z^* z)$, is viewed as a deformation of the commutative function algebra of the leaf L which is parametrized by complex numbers z as a symplectic manifold, and π_L is viewed as the "restriction" map from the quantum group $S_\mu U(2)$ to a "quantum leaf" L_μ. (For $\dim(L) = 0$, we take $C(L_\mu) = \mathbb{C}$.)

Originated from quantum mechanics, the theory of deformation quantization of symplectic or more generally Poisson structures has become a mathematical subject studied in algebraic, analytic, or geometric context [5, 10, 2]. Motivated by the study of C*-algebraic deformation quantization of Poisson structure introduced by Rieffel [10, 12], our result applies to a more general setting. In fact, we shall loosely refer the term "quantization" of $SU(2)$ (or of L) to any linear map $w : C(SU(2))^\infty \to C(S_\mu U(2))$ (or $\rho : C(L)^\infty \to C(L_\mu)$, where $C(SU(2))^\infty$ is the algebra of regular functions on $SU(2)$ [15], and $C(L)^\infty$ is the algebra of restrictions $f|_L$ of $f \in C(SU(2))^\infty$) with $w(\mathbb{C}) = \mathbb{C}$, where $|\mu| < 1$. We say that w is leaf-preserving if $f|_L = 0$ implies $\pi_L(w(f)) = 0$ for any $f \in C(SU(2))^\infty$ and any symplectic leaf L of $SU(2)$. Clearly if w is leaf-preserving then $f = g$ on L implies $\pi_L(w(f)) = \pi_L(w(g))$, so $\pi_L(w(f))$ is determined by $f|_L$ and w induces a "quantization" ρ of L defined by $\rho(f|_L) = \pi_L(w(f))$. Thus w and ρ are related by the commuting diagram

$$\begin{array}{ccccc} C(SU(2))^\infty = & C(S_1 U(2))^\infty & \xrightarrow{|_L} & C(L_1)^\infty & = C(L)^\infty \\ & \downarrow w & & \downarrow \rho & \\ & C(S_\mu U(2)) & \xrightarrow{\pi_L} & C(L_\mu) & \end{array}$$

When this happens, we say that w induces a compatible deformation on all symplectic leaves. Conversely, if a quantization w of $SU(2)$ satisfies the above commuting diagram for some quantization ρ of L for each leaf L, then w is leaf-preserving.

We mention the following fact about C*-algebraic deformation quantization (which is a special case of our general quantization). It is known [13, 14] (see [1, 9] for some generalizations) that by applying Weyl calculus to the restriction $f|_L$ of an element f of $C^\infty(SU(2))$ to the 2-dimensional leaves L one can get a leaf-preserving C*-algebraic deformation quantization W_h, while on the other hand the Weyl transformation $\mathscr{W}_h$ [6] constructed from the Fourier expansion on $SU(2)$ gives us a non-leaf-preserving one which is a coalgebra homomorphism. Both deformations give rise to the algebra $C(S_\mu U(2))$ (with μ and h suitably related). We remark that strictly speaking, the quantization W_h only preserves the 2-dimensional leaves, but the unitarily equivalent one $W'_h (= W_1(\cdot\ _h)[13])$ studied in [1] preserves all leaves.

4. Leaf Preserving Quantization

In this section, we show a coalgebra homomorphism cannot be leaf-preserving.

Let $w : C(SU(2))^\infty \to C(S_\mu U(2))$, with $|\mu| < 1$, be a coalgebra homomorphism (preserving $\mathbb{C}$). Then clearly $(w \otimes I_n)(\omega_1^{(n)})$ gives rise to a corepresentation of $C(S_\mu U(2))$ [18]. It is known that all nondegenerate smooth corepresentations of a compact matrix pseudogroup are completely reducible [18]. We observe that in fact Proposition 4.6 and the proof of Theorem 5.8 in [18] can be used to show that when the Haar measure is faithful, all corepresentations (nondegenerate or not) are completely reducible to a (finite) direct sum of (smooth) irreducible corepresentations and a zero corepresentation, and hence are all smooth. (Nagy kindly pointed out to the author that this observation had already been made in [3] independently.) From this and the fact that the Haar measure of $C(S_\mu U(n))$ is faithful [9], we have $(w \otimes I_3)(\omega_1^{(3)})$, if nondegenerate, equivalent to $\omega_\mu^{(1)} \oplus \omega_\mu^{(1)} \oplus \omega_\mu^{(1)}, \omega_\mu^{(1)} \oplus \omega_\mu^{(2)}$, or $\omega_\mu^{(3)}$, i.e. for some invertible $T \in M_3(\mathbb{C})$,

$$(w \otimes I_3)\begin{pmatrix} \alpha^2 & \alpha\bar{\gamma} & \bar{\gamma}^2 \\ -2\alpha\gamma & \alpha\bar{\alpha} - \gamma\bar{\gamma} & 2\overline{\gamma\alpha} \\ \gamma^2 & -\gamma\bar{\alpha} & \bar{\alpha}^2 \end{pmatrix}$$

is equal to $T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \gamma^* \\ 0 & -\mu\gamma & \alpha^* \end{pmatrix}T^{-1}$, or

$$T\begin{pmatrix} \alpha^2 & \alpha\gamma^* & \gamma^{*2} \\ -(\mu+\mu^{-1})\alpha\gamma & \alpha\alpha^* - \gamma\gamma^* & (1+\mu^{-2})\gamma^*\alpha^* \\ \mu^2\gamma^2 & -\mu\gamma\alpha^* & \alpha^{*2} \end{pmatrix}T^{-1}.$$

When $(w \otimes I_3)(\omega_1^{(3)})$ is degenerate, we get similar equalities with some zero diagonal block in the matrix being conjugated by T.

If w is also leaf-preserving, then since the irreducible representations of $C(S_\mu U(2))$ associated with the 0-dimensional leaves $z \in U(1) = \mathbb{T} \subset SU(2)$, when applied to a noncommutative polynomial of α, γ, α^*, and γ^*, are exactly the evaluation of the corresponding commutative polynomial of $\alpha, \gamma, \bar{\alpha}$, and $\bar{\gamma}$ at z (cf. Theorem 3.2, Proposition 3.3, and the discussion following it in [15]), we get, when restricting everything to $U(1)$, the possibilities:

$$\begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} = T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} = T\begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{z} \end{pmatrix}T^{-1},$$

$$\begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} = T\begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix}T^{-1},$$

or similar equations with a degenerate left-hand side, for all $z \in \mathbb{T}$. Clearly the first two cases as well as the degenerate cases are not possible, while the third one, equivalent to

$$T\begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-2} \end{pmatrix} T$$

for all $z \in \mathbb{T}$, implies that T is diagonal, say,

$$T = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} .$$

Let $b_j = a_j^{-1}$. Then from

$$(w \otimes I_3)\begin{pmatrix} \alpha^2 & \alpha\bar{\gamma} & \bar{\gamma}^2 \\ -2\alpha\gamma & \alpha\bar{\alpha} - \gamma\bar{\gamma} & 2\overline{\gamma\alpha} \\ \gamma^2 & -\gamma\bar{\alpha} & \bar{\alpha}^2 \end{pmatrix}$$

$$= T\begin{pmatrix} \alpha^2 & \alpha\gamma^* & \gamma^{*2} \\ -(\mu + \mu^{-1})\alpha\gamma & \alpha\alpha^* - \gamma\gamma^* & (1 + \mu^{-2})\gamma^*\alpha^* \\ \mu^2\gamma^2 & -\mu\gamma\alpha^* & \alpha^{*2} \end{pmatrix} T^{-1}$$

$$= \begin{pmatrix} * & * & a_1 b_3 \gamma^{*2} \\ * & \alpha\alpha^* - \gamma\gamma^* & * \\ \mu^2 a_3 b_1 \gamma^2 & * & * \end{pmatrix}$$

we have

$$\begin{aligned} w((\alpha\bar{\alpha} - \gamma\bar{\gamma}) + 2\bar{\gamma}^2 - 1) &= (\alpha\alpha^* - \gamma\gamma^*) + 2a_1 b_3 \gamma^{*2} - 1 \\ &= -(1 + \mu^2)\gamma\gamma^* + 2a_1 b_3 \gamma^{*2} . \end{aligned}$$

Recall that the representation $\pi_0 := \pi_{L_0}$ corresponding to the leaf L_0 is the prominent representation on $\ell^2(\mathbb{N} \cup \{0\})$ that sends α to the weighted shift $\sum_{k \geqq 1} \sqrt{1 - \mu^{2k}}\, e_{k,k-1}$ and γ to the self-adjoint diagonal operator $\sum_{k \geqq 0} \mu^k e_{k,k}$ (cf. Theorem 3.2, Proposition 3.3, and the discussion following it in [15]). From $\pi_0(\gamma) = \pi_0(\gamma^*)$, we get

$$\pi_0[w((\alpha\bar{\alpha} - \gamma\bar{\gamma}) + 2\bar{\gamma}^2 - 1)] = [-(1 + \mu^2) + 2a_1 b_3]\pi_0(\gamma)^2 .$$

On the other hand, we also have

$$w(\bar{\gamma}^2 - \gamma^2) = a_1 b_3 \gamma^{*2} - \mu^2 a_3 b_1 \gamma^2 = a_1 b_3 \gamma^{*2} - \mu^2 (a_1 b_3)^{-1} \gamma^2 ,$$

and hence

$$\pi_0[w(\bar{\gamma}^2 - \gamma^2)] = [a_1 b_3 - \mu^2 (a_1 b_3)^{-1}]\pi_0(\gamma)^2 .$$

We claim that w cannot be a leaf-preserving map and hence get a contradiction. In fact, if it is, then since both $(\alpha\bar{\alpha} - \gamma\bar{\gamma}) + 2\bar{\gamma}^2 - 1$ and $\bar{\gamma}^2 - \gamma^2$ vanish on the leaf L_0, we have $\pi_0[w((\alpha\bar{\alpha} - \gamma\bar{\gamma}) + 2\bar{\gamma}^2 - 1)] = 0$ and $\pi_0[w(\bar{\gamma}^2 - \gamma^2)] = 0$, which imply that $-(1 + \mu^2) + 2a_1 b_3 = 0$ and $(a_1 b_3)^2 = \mu^2$. So $(1 + \mu^2)/2 = a_1 b_3 = \pm\mu$ and hence $\mu = \pm 1$. A contradiction, since $|\mu| < 1$.

Concluding the above discussion, we get

Theorem 1. *A quantization (i.e. a linear map preserving* $\mathbb{C}$*)* $w : C(SU(2))^{\infty} \to C(S_{\mu}U(2))$ *with* $|\mu| < 1$ *can not be both leaf-preserving and a coalgebra homomorphism.*

We remark that any coalgebra homomorphism w with nonzero $w(1)$ preserves $\mathbb{C}$.

Corollary 2. *If a quantization of* $SU(2)$ *is a coalgebra homomorphism, then it does not induce a compatible quantization on all symplectic leaves.*

Since the coalgebra structure reflects the group structure, we may call a quantization $w : C(SU(2))^{\infty} \to C(S_{\mu}U(2))$ group-preserving if it is a coalgebra homomorphism. The above result shows that although a deformation quantization of the Poisson structure on $SU(2)$ may either preserve the (geometric) symplectic leaf structure (like W_h') or preserve the (algebraic) group structure (like $\mathscr{W}_h$), it cannot be simultaneously leaf-preserving and group-preserving. The same conclusion holds for higher dimensional cases which will be dealt with in a separate paper.

Finally we remark that the above results hold if we only require a quantization to be inducing quantizations of the leaf L_0 and the 0-dimensional leaves (which form the boundary of L_0), since the proof only relies on this assumption.

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