Commun. Math. Phys. 172, 223 – 262 (1995)

Communications in Mathematical Physics

Algebraic Index Theorem

Ryszard Nest[1], Boris Tsygan[2,★]

[1] University of Copenhagen

[2] Department of Mathematics, Eberly College of Science, Pennsylvania State University, 218 McAllister Building, University Park, PA 16802, USA

Received: 1 April 1993

Abstract: We prove the Atiyah–Singer index theorem where the algebra of pseudo-differential operators is replaced by an arbitrary deformation quantization of the algebra of functions on a symplectic manifold.

Introduction

This paper is the first in a series aimed at providing an algebraic insight to various Atiyah–Singer type index theorems.

In recent years much research was devoted to finding explicit local proofs of Atiyah–Singer and Riemann–Roch theorems. In the works of Bismut, Gillet, Soulé, Getzler, Witten and Fedosov, the index of an operator (usually Dirac operator) was expressed as the trace of a certain infinite dimensional operator; the asymptotic expansion of this trace provided a local formula. O'Brian, Toledo and Tong [TT, BTT] developed an explicit method of computing a certain cohomology class in the Čech complex. The integral of that class over a complex manifold is known to be the Euler characteristic of the analytic sheaf; thus the computation yields the explicit proof of the Riemann–Roch theorem. Arbarello, de Concini, Kac and Procesi [ACKP] suggested a method of proving the Riemann–Roch–Grothendieck theorem (inspired by string theory). One of the crucial steps of that method was the passage to a certain infinite dimensional manifold. The De Rham complex of that manifold is closely related to Gelfand–Fuks cohomology of the Lie algebra of vector fields; the computation was finally reduced to an explicit computation in Lie algebra cohomology. This method was further developed by Feigin and Tsygan [FT].

One of the useful tools in proving the index theorems is cyclic cohomology. This cohomology was independently introduced by Connes and Tsygan. In Connes' approach, it was intimately related to the index theory. The idea, going back partly to Helton and Howe [HH], is, roughly speaking, that the cyclic cocyles, being higher analogues of the traces, provide a natural algebraic framework for trace

★ Partially supported by NSF Grant DMS-9101817.

computations. This facilitates the explicit computation of the index as the trace of a certain operator.

Such a computation of the index of the Dirac operator is due ot Connes. Another proof of the Atiyah–Singer theorem, for the case when the manifold is a linear space, was given by Elliot, Natsume and Nest. This proof was more directly in line with the work of Helton and Howe. We will outline this proof later.

In this paper, we will state and prove a purely algebraic theorem which is parallel to the index theorem. It works in a much more general framework. Instead of the ring of pseudodifferential operators on a manifold we will consider any deformation of the algebra of functions on a sympletic manifold. (cf. [BFFLS]; it is well known that the ring of pseudodifferential operators can be viewed as a deformed algebra of functions on the cotangent bundle).

A similar theorem was proved by B. Fedosov (by completely different methods); cf. [F].

Note that the approach of this paper, though quite different, is closely related to the recent paper by Connes, Flato and Sternheimer [CFS].

The Atiyah–Singer theorem does not follow directly from the algebraic statement. But it can be proved along the same algebraic lines. We will do that in a separate paper. Here we will concentrate on the algebraic case.

First, to give an example (and also to explain why our algebraic theorem is indeed the algebraic form of the index theorem), let us recall the proof of the index theorem for $\mathbb{R}^n$ due to Elliot, Natsume and Nest. Let D be an elliptic differential operator on $C^\infty(\mathbb{R}^n)$; assume that $D = 1$ outside some compact. Let e_D be the orthogonal projection onto the graph of D in $L_2(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n)$. This is a pseudodifferential operator in the trivial bundle $\mathbb{C}^2$ over $\mathbb{R}^n$. One can construct an explicit homotopy $e(t)$, $0 \leqq t \leqq 1$, such that $e(1) = e_D$ and $e(0) = \begin{pmatrix} P & 0 \\ 0 & 1-Q \end{pmatrix}$, where P (resp. Q) is the orthogonal projection onto the kernel (resp. cokernel) of D. One has, obviously,

$$\text{index } D = \operatorname{Tr}\left(e(0 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) . \tag{0.1}$$

The operator $e_D - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not trace class. This suggests that, in order to compute the trace in (0.1) in terms of e_D, one should replace the trace with a suitable higher trace, or a cyclic cocycle. Let $a_0, a_1, a_2, \dots$ be integral operators with compactly supported kernels. Put

$$\Theta(a_0, \dots, a_{2n}) = \frac{1}{n!} \sum_{\sigma \in \Sigma_{2n}} \operatorname{sgn}(\sigma) \cdot \operatorname{Tr}(a_0[y_{\sigma 1}, a_1] \cdots [y_{\sigma(2n)}, a_{2n}]) , \tag{0.2}$$

where $(y_1, \dots, y_{2n}) = (\partial_{x_1}, x_1, \dots, \partial_{x_n}, x_n)$.

One checks that

(1) Θ is a cyclic cocyle cohomologous to the cocyle $\operatorname{Tr}(a_0 \cdots a_{2n})$. From this one deduces that

$$\operatorname{Tr}\left(e(0) - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \Theta(e(0), \dots, e(0)) ,$$

(2) $\Theta(e(0), \dots, e(0)) = \Theta(e(1), \dots, e(1))$. (0.3)

Note that the above formula of Connes is true for any cyclic cocyle Θ on an algebra A and for any two homotopic idempotents $e(0)$, $e(1)$ in A. The problem is that Θ is not defined on an algebra containing $e(1)$. So one has to check that:

i) the right-hand side of (0.3) makes sense;

ii) Connes' explicit calculation proving (0.3) makes sense for the explicit homotopy $e(t)$.

Now let us compute $\Theta(e_D, \ldots, e_D)$. Let h be a non-zero parameter. Let $D = D(x, \partial_x)$; $D(h) = D(x, h\partial_x)$. Then index $D =$ index $D(h)$ for any h; therefore, because of the well known formula for the trace,

$$\text{index } D = \frac{1}{n!} \lim_{h \to 0} \sum_\sigma \text{sgn}(\sigma) \text{Tr}\, e_{D(h)}[y_{\sigma 1}, e_{D(h)}] \cdots y_{\sigma(2n)}, e_{D(h)}]$$

$$= \frac{1}{n!} \lim_{h \to 0} \sum_\sigma \text{Tr} \int_{\mathbb{R}^{2n}} \sigma(e_D)\{y_{\sigma 1}, \sigma(e_D)\} \cdots \{y_{\sigma(2n)}, \sigma(e_D)\} d\xi_1 \cdots dx_n \, .$$

But $\sigma(e_D)(x, \xi) = e_{\sigma(D)(x,\xi)}$ – the orthogonal projection onto the graph of the 2×2 matrix $\sigma(D)(x, \xi)$. Clearly the last limit is equal to $\frac{1}{n!}\int_{\mathbb{R}^{2n}} e_{\sigma(D)} de_{\sigma(D)} \cdots de_{\sigma(D)}$.

Now, let X be a compact C^∞ manifold. Let D be an elliptic differential operator on X. Let e_D be the projection onto the graph of D in $L_2(X) \oplus L_2(X)$. There exists a homotopy between e_D and $e(0)$ as above. To compute the index of D one has to construct the analogue of the fundamental cocycle Θ. Then, rescaling the cotangent bundle by the transformation $\xi \mapsto \frac{1}{h}\xi$, one passes to a one-parameter family of projectors $e_D(h)$. Using the explicit formula for Θ, one has to show that

$$\text{index } D = \lim_{h \to 0} \text{index } D(h) = \int_{T^*X} ch\, e_{\sigma(D)} \cdot td(T_X \otimes \mathbb{C}) \, ,$$

where $\sigma(D)$ is the symbol of D, $\sigma(D)$ is the projection onto its graph and td is the Todd class.

In this paper we will prove the parallel algebraic theorem. Let (M, ω) be any symplectic manifold (instead fo T^*X). Let us define, following [BFFLS],

$$f * g = fg + \hbar\varphi_1(f, g) + \hbar^2\varphi_2(f, g) + \cdots \, ,$$

a formal deformation of the ring $C^\infty(M)$. We assume that $\lim_{h \to 0}(f * g - g * f) = \{f, g\}$, that $\varphi_i(f, g)$ are differential expressions in f, g and that $1 * f = f * 1 = f$. Consider the algebras $\mathbb{A}^\hbar(M) = (C^\infty(M)[[\hbar]], *)$ and $\mathbb{A}_0^\hbar(M) = (C_c^\infty(M)[[\hbar]], *)$. If B is an algebra over a ring k then $B^\sim = B + k$ is the algebra B with adjoined unit.

One can construct the canonical trace $\text{Tr} : \mathbb{A}_0^\hbar(M) \to \mathbb{C}[\hbar^{-1}, \hbar]]$ such that for $f \in C_c^\infty(M)$,

$$\text{Tr}(f) = \frac{1}{\hbar^n}\left(\frac{1}{n!}\int_M f \cdot \omega^n + \hbar\tau_1(f) + \hbar^2\tau_2(f) + \cdots\right)$$

and $\text{Tr}(f * g - g * f) = 0$, where $\tau_i(f)$ are local expressions in f. Also, consider an idempotent e in the matrix algebra $M_N(\mathbb{A}_0^\hbar(M)^\sim)$.

Let $e(\infty)$ be the constant value of e at infinity of M; $e_0 = e \mid \hbar = 0$; $e_0(\infty) = e(\infty) \mid \hbar = 0$.

In our idealized approach, $\mathbb{A}^\hbar(M)$ stands for all pseudodifferential operators; $\mathbb{A}_0^\hbar(M)$ stands for pseudodifferential operators of negative order; Tr stands for the operator trace; e stands for the projection e_D onto the graph of an elliptic differential operator D; e_0 stands for $e_{\sigma(D)}$; e_0 and $e_0(\infty)$ stand for $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $ch(e_0) = \sum_n \frac{1}{n!} \mathrm{tr}\, e_0(de_0)^n \in \Omega^{ev}(M)$. Aslo, let $td(\omega)$ be the Todd class of the reduction of the bundle of symplectic frames on M from $Sp(2n)$ to $U(n)$; let $c_i(\omega)$ be the Chern classes of that reduction.

For any deformation one can define a characteristic class $\theta \in H^2(M, \mathbb{C}[[\hbar]])$. (We do this in Sect. 5). One has $\theta = \omega + \hbar\theta_1 + \cdots + \hbar^i\theta_i + \cdots$, $\theta_i \in H^2(M, \mathbb{C})$.

Theorem 1.1.1.

$$\mathrm{Tr}(e - e(\infty)) = \int_M (ch(e_0) - \mathrm{tr}\, e_0(\infty)) \cdot \mathrm{td}(\omega) \cdot e^{-c_1(\omega)/2} e^{\theta/\hbar} .$$

For the special deformations this was proved by B. Fedosov [F].

The purpose of this paper is to prove this theorem using the method outlined above. In fact the paper is mainly devoted to constructing the analogue of the fundamental cyclic cocycle Θ.

First, in Sect. 2, we construct a Poincaré duality map in cyclic homology. Let A_0 be an algebra over commutative ring k, $\mathfrak{g}$ a Lie subalgebra of $\mathrm{Der}(A_0)$. Let $\tau: A_0/\mathfrak{g}A_0 \to k$ be a trace. We construct a map

$$\chi_\tau : C_*(\mathfrak{g}) \to CC_{\mathrm{per}}^{-*}(A_0) , \tag{0.4}$$

where C_* is the standard Lie algebra complex and CC_{per}^* is the periodic cyclic cochain complex.

Now, let A be an algebra over a ring k and A_0 an ideal in A; let $\tau: A_0/[A, A_0] \to k$ be a trace. Passing to the matrix algebras and using the well known connection between Lie algebra and cyclic homologies, we construct the map

$$\chi_\tau : \overline{CC}_{*-1}(A) \to CC_{\mathrm{per}}^{-*}(A_0) , \tag{0.5}$$

where $\overline{CC}_*$ is the reduced cyclic complex. Note that, to some extent, all known characteristic classes with values in periodic cyclic cohomology account for particular cases of the maps (0.4) and (0.5) (Sect. 2.4). We prove the crucial property of the map (0.5): there exists a map $CC_{*-2}(k) \to CC_{\mathrm{per}}^{-*}(A_0)$, making the diagram

$$\begin{array}{ccc} \overline{CC}_{*-1}(A) & \longrightarrow & CC_{\mathrm{per}}^{-*}(A_0) \\ \partial \big\downarrow & \nearrow & \\ CC_{*-2}(k) & & \end{array} \tag{0.6}$$

commute up to homotopy. Here ∂ is the boundary operator.

Apply the construction above to $A = \mathbb{A}^\hbar(M)$, $A_0 = \mathbb{A}_0^\hbar(M)$, $\tau = \mathrm{Tr}$. The diagram (0.6) is in fact a diagram of complexes of sheaves. We show in Sect. 3 that this diagram can be extended to the diagram

$$\begin{array}{ccc} \check{C}^*(M, \overline{CC}_{*-1}(\mathbb{A}^\hbar)) & \xrightarrow{\chi_{\mathrm{Tr}}} & CC_{\mathrm{per}}^{-*}(\mathbb{A}_0^\hbar) \\ \partial \big\downarrow & \nearrow & \\ \check{C}^*(M, CC_{*-2}(k)) & & \end{array} \tag{0.7}$$

where $\check{C}^*$ is the Čech complex.

Let $\mathscr{H}_*(M, \mathbb{A}^\hbar[\hbar^{-1}])$ be the hyperhomology of the double complex $\check{C}^*(M, \overline{CC}_{*-1}(\mathbb{A}^\hbar))[\hbar^{-1}]$. We show that $\mathscr{H}_{2n-1}$ is canonically isomorphic to $k = \mathbb{C}[\hbar^{-1}, \hbar]]$. Let Ω be the generator of $\mathscr{H}_{2n-1}$.

In the case when $M = \mathbb{R}^{2n}$, Ω is represented by the cochain $\frac{h^{-n}}{2n}\xi_1 \wedge x_1 \wedge \cdots \wedge \xi_n \wedge x_n$ (of course there is no need to involve Čech complexes). The fundamental cocycle Ω is equal to $\chi_{\mathrm{T}_1}(\Omega)$. The fact that Ω and $\mathrm{Tr}(a_0 \ldots a_{2n})$ are cohomologous follows from the commutativity of (0.6).

Now, note that the $(2n-2)^{\text{th}}$ hyperhomology group of $\check{C}^*(M, CC_*(k))$ is isomorphic to $H^{ev}(M, k)$. In Sect. 4 we show that Theorem 1.1.1 follows from the equality

$$\partial\Omega = \varepsilon(td(\omega) \cdot e^{-c_1(\omega)/2 + \theta/\hbar})^{-1}, \tag{0.8}$$

where $\varepsilon \mid H^{2k}(M) = (-1)^k$. Put $\Theta = \chi_{\mathrm{T}_1}(td(\omega) \cup \Omega)$.

Then

$$\mathrm{Tr}(e - e(\infty)) = \langle \Theta, ch(e) \rangle,$$

where $ch(e)$ is the Chern character in periodic cyclic homology. An explicit calculation shows that for any periodic cyclic α of $\mathbb{A}_0^\hbar(M)$,

$$\lim_{\hbar \to 0} \langle \Theta, \alpha \cdot e^{-\theta/\hbar} \rangle = \int_M \alpha \cdot td(\omega) \cdot e^{-c_1(\omega)/2}.$$

On the other hand, there is a rigidity property: if α is a periodic cyclic cycle of $\mathbb{A}_0^\hbar(M)$ *over* $\mathbb{C}$ (that is, if one considers $\mathbb{C}$, not $\mathbb{C}[[\hbar]]$, as the ring of scalars), then $\mathrm{Tr}(\alpha \cdot e^{-\theta/\hbar})$ does not depend on $\hbar$. The theorem follows.

It remains to prove (0.8). The proof occupies Sects. 5, 6. We reduce the problem to the local computation. Let $\widehat{\mathbb{A}}^\hbar$ be the standard deformation of the ring of power series $\mathbb{C}[[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]]$. Let $\mathfrak{g}$ be the Lie algebra $\hat{\mathbb{A}}^\hbar/\mathbb{C}[[\hbar]]$ with the bracket $[f, g] = \frac{1}{\hbar}(f * g - g * f)$. Let $\mathfrak{h}$ be the subalgebra $\{\sum a_{ij}x_i\xi_j \mid a_{ij} \in \mathbb{C}\}$. Let $\mathscr{H}_*(\mathfrak{g}, \mathfrak{h}; \hat{\mathbb{A}}^\hbar[\hbar^{-1}])$ be the hyperhomology of the double complex $C^*(\mathfrak{g}, \mathfrak{h}; \overline{CC}_*(\hat{\mathbb{A}}^\hbar))$. Then $\mathscr{H}_{2n-1}$ is one-dimensional; let Ω be the generator. There is the boundary map $\partial : C^*(\mathfrak{g}, \mathfrak{h}; \overline{CC}_*(\hat{\mathbb{A}}^\hbar)) \to C^*(\mathfrak{g}, \mathfrak{h}; CC_{*-1}(k))$. This provides the canonical element $\partial\Omega$ of $H^{ev}(\mathfrak{g}, \mathfrak{h}; k)$ and

$$\partial\Omega = \varepsilon(td(c_1, \ldots, c_n) \cdot e^{-c_1/2 + \theta/\hbar})^{-1}. \tag{0.9}$$

Here $\varepsilon \mid H^{2k} = (-1)^k$; c_i are the algebraic Chern classes in $H^{2i}(\mathfrak{g}, \mathfrak{h}; \mathbb{C})$; θ is the 2-cocycle coming from the central extension $0 \to \mathbb{C}[[\hbar]] \to \widehat{\mathbb{A}}^\hbar \to \mathfrak{g} \to 0$.

To reduce (0.8) to (0.9) (and to define the element θ of $H^2(M, \mathbb{C}[[\hbar]])$), we replace M by a certain infinite dimensional manifold of "non-linear frames" (compare with [ACKP, FT].

Finally, the local formula (0.9) is proved by a computation. This completes the proof of Theorem 1.1.1. In Sect. 7 we prove a conjecture of B. Feigin (from which the Riemann–Roch theorem can be easily deduced). Let M be an n-dimensional complex manifold; let $\mathscr{D}_M$ be the sheaf of holomorphic differential operators on M. Let $\check{C}^*(M, CC_*(\mathscr{D}_M))$ be the Čech hypercomplex; let $\mathbb{H}C_*(\mathscr{D}_M)$ be its hyperhomology. It was shown by Brylinski [B] that $\mathbb{H}C_0(\mathscr{D}_M) \xrightarrow{\sim} H^{2n}(M, \mathbb{C})$.

On the other hand, the trivial cocycle $1 \in \check{C}^0(M, CC_0)$ provides an element of $\mathbb{H}C_0(\mathscr{D}_M)$ and therefore an element $[1]$ of $H^{2n}(M, \mathbb{C})$. We prove that $[1] = td(M)_{2n}$. This is an easy consequence of (0.8) (with $\mathbb{A}^\hbar(T^*M)$ replaced by $\mathscr{D}_M$).

There is also a holomorphic version of Theorem 1.1.1. We will discuss it in another paper.

To construct the maps (0.4)–(0.7) and to prove (0.9), we use the operations on the cyclic complex. Let us discuss them in some more detail.

First, let X be a C^∞ manifold. Let $\mathfrak{g}$ be the algebra of vector fields on X. For $D \in \mathfrak{g}$, the operations L_D: $\Omega_X^* \to \Omega_X^*$ and i_D: $\Omega_X^* \to \Omega_X^{*-1}$ are defined, subject to standard relations (like $[i_D, d] = L_D$). Let ε be an odd formal parameter, $\varepsilon^2 = 0$; $\mathfrak{g}[\varepsilon] = \mathfrak{g} \otimes \mathbb{C}[\varepsilon]$. Let $\delta = \partial/\partial\varepsilon$: $\mathfrak{g}[\varepsilon] \to \mathfrak{g}[\varepsilon]$, $\delta(D + E\varepsilon) = E$. Then the set of standard relations for L_D, i_D is equivalent to the following:

the differential graded Lie algebra $(\mathfrak{g}[\varepsilon], \delta)$ acts on the complex Ω_X^* via $D + E\varepsilon \mapsto L_D + i_E$.

This algebra can be extended. Let $A = C^\infty(X)$; put $i_f(\omega) = f \cdot \omega$, $L_f(\omega) = df \wedge \omega$ for $f \in A$. Let $\underline{A}$ be the $\mathfrak{g}$-module A concentrated in the odd degree; let $\mathfrak{g} \ltimes \underline{A}$ be the semi-direct product. Then

the differential $\mathbb{Z}_2$-graded algebra $(\mathfrak{g} \ltimes \underline{A})[\varepsilon]$ acts on the $\mathbb{Z}_2$-graded complex Ω_X^* via $D + E\varepsilon \mapsto L_D + i_E$; $D, E \in \mathfrak{g} \ltimes \underline{A}$.

Now, let A be an arbitrary algebra; $\mathfrak{g} =$ Der(A). For $D \in \mathfrak{g}$, $a \in A$ one can define the operations L_D, i_a: $CC_*^{\mathrm{per}}(A) \to CC_*^{\mathrm{per}}(A)$ and i_D, L_a: $CC_*^{\mathrm{per}}(A) \to CC_{*-1}^{\mathrm{per}}(A)$. Also define the $\mathbb{Z}_2$-graded differential Lie algebra $((\mathfrak{g} \ltimes \underline{A})[\varepsilon], \delta + \delta_1)$, where $\delta_1(D + a) = ad(a)$ and δ_1 is $\mathbb{C}[\varepsilon]$-linear.

It is not true that this algebra acts on $CC_*^{\mathrm{per}}(A)$; for example, $[i_D, i_E] \neq 0$. However, it is true that:

the differential $\mathbb{Z}_2$*-graded algebra* $((\mathfrak{g} \ltimes \underline{A})[\varepsilon]; \delta + \delta_1)$ *acts on the* $\mathbb{Z}_2$*-graded complex* $CC_*^{\mathrm{per}}(A)$ *up to homotopy.*

We explain what this means and provide all the details in Appendix 1. The fact above is crucial for our approach. We do not need it in full strength for the proof of Theorem 1.1.1; it is needed, however, for the future applications of our method.

Let us outline some of these applications. First, there should be generalizations of Theorem 1.1.1 for families and for manifolds with boundary. Second, there probably is an analogue of 1.1.1 for any deformation of $C^\infty(M)$ for which the symplectic leaves form a foliation. This analogue should generalize the recent "higher index theorem" for the longitudinal Dirac operator on a certain class of foliations of codimension 1 (Moryoshi and Natsume).

There is also a possible application to algebraic geometry. Let M be a compact Kähler manifold. Assume that $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. Let ω be the symplectic form associated to the Kähler structure. Let L be an invertible sheaf such that $c_1(L) = \omega$. Let $-c_1(T_M)/2 = m \cdot \omega$.

Let $\mathbb{A}^\hbar(M)$ be the deformation with $\theta = \omega$ (such a deformation always exists; cf. [F]). Then, applying Theorem 1.1.1 and the Riemann–Roch theorem, one gets

$$\mathrm{Tr}(1) = P_M\left(L, \frac{1}{\hbar} + m\right),$$

where $P_M(L, k)$ is the Hilbert polynomial of L. This seems to identify $P_M(L, k)$ with the integral of some measure of a nature completely different from the usual form given by the Riemann–Roch theorem. It would be very interesting, for example, to find the canonical deformation when M is the moduli variety of rank n bundles on a curve (in connection to Verlinde formulas, [V]).

Let us mention another application. Let K be a compact Lie group. The quantized ring $\mathbb{C}[K]_q$ of algebraic functions on K ([S, FRT] defines the well known

Poisson structure on K [STS]. According to Soibelman, for $q < 1$ there is the natural way to assign a representation ρ_Ω of $\mathbb{C}[K]_q$ to any sympletic leaf $\Omega \subset K$. One can identify $\mathbb{C}[K]$ with $\mathbb{C}[K]_q$ as linear spaces; assume that $f \in M_N(\mathbb{C}[K]_q)$ and that $f|\partial\Omega \in GL_N$. Then

$$\operatorname{index}\rho_\Omega(f) = \frac{(-1)^{l-1}}{(2l-1)!!}\int_{\partial\Omega} \operatorname{tr}(df \cdot f^{-1})^{2l-1},$$

where $l = \frac{1}{2}\dim \Omega$. These matters will be discussed in a joint work of Soibelman and the second author.

Section 1. The Algebraic Index Theorem

Let M be a C^∞ manifold with a symplectic structure ω. Let $\{,\}$ be the Poisson bracket on $C^\infty(M)$ associated to ω. Let $\hbar$ be a formal parameter, and let $C^\infty(M)[[\hbar]]$ be the vector space of formal power series in $\hbar$ with coefficients in $C^\infty(M)$.

A deformation of $C^\infty(M)$ compatible to ω is by definition ([Ger, BFFLS]) a $\mathbb{C}[[\hbar]]$-bilinear, $(\hbar)$-adically continuous associative multiplication $*$ on $C^\infty(M)[[\hbar]]$ such that for $f, g \in C^\infty(M)$,

$$f * g = fg + \hbar\varphi_1(f,g) + \hbar^2\varphi_2(f,g) + \cdots,$$

where $\varphi_i(f,g) \in C^\infty(M)$ and $\varphi_1(f,g) - \varphi_1(g,f) = \{f,g\}$. We assume that

i) $\varphi_i(f,g)$ are bidifferential operators on $C^\infty(M)$;
ii) $1 * f = f * 1 = f$ for any $f \in C^\infty(M)$.

Given a deformation, we denote by $\mathbb{A}^\hbar(M)$ the algebra $C^\infty(M)$ with the product $*$. Also, by $\mathbb{A}_0^\hbar(M)$ we will denote the algebra $(C_c^\infty(M)[[\hbar]], *)$, where $C_c^\infty(M)$ is the space of compactly supported functions. Also, let $\widetilde{\mathbb{A}}_0^\hbar(M)$ be the algebra $\mathbb{A}_0^\hbar(M)$ with the unit adjoined.

By a trace on $\mathbb{A}_0^\hbar(M)$ we mean a $\mathbb{C}[[\hbar]]$-linear, $(\hbar)$-adically continuous functional $\operatorname{Tr} : \mathbb{A}_0^\hbar(M) \to \mathbb{C}[\hbar^{-1}, \hbar]]$ (the right-hand side stands for Laurent series) such that for f, g in $C_c^\infty(M)$

$$\operatorname{Tr}(f * g) = \operatorname{Tr}(g * f).$$

One can easily show that the traces form a one-dimensional vector space over $\mathbb{C}[\hbar^{-1}, \hbar]]$ (we prove a stronger statement in Appendix 2). One can obtain the canonical generator of this space as follows. First (cf. [BFFLS] and [G]) M admits a cover by open sets U such that $\mathbb{A}^\hbar(U)$ is isomorphic to the standard Weyl deformation (we recall the definition in (5.1.1)) $\mathbb{A}^\hbar(B_0)$, where $B_0 \subset \mathbb{R}^n$ is the open unit ball with the standard symplectic form ω_0. Let $g_U : \mathbb{A}^\hbar(U) \xrightarrow{\sim} \mathbb{A}^\hbar(B_0)$ be this isomorphism. For a function f with support inside U put

$$\operatorname{Tr} f = \frac{1}{n!}\hbar^{-n}\int_{B_0} g_U(f)\omega_0^n.$$

This does not depend on an isomorphism g_U because Tr is invariant under automorphisms of $\mathbb{A}^\hbar(B_0)$. For any element f of $\mathbb{A}_0^\hbar(M)$ put $\operatorname{Tr}(f) = \sum_U \operatorname{Tr}\rho_U * f$, where $\{\rho_U\}$ is the partition of unity subordinate to the cover. It is easy to see that for $f \in C_c^\infty(M)$,

$$\mathrm{Tr}(f) = \hbar^{-n}\left(\frac{1}{n!}\int_M f \cdot \omega^n + \hbar\tau_1(f) + \hbar^2\tau_2(f) + \cdots\right),$$

where $\tau_i(f)$ are linear local expressions in f. Clearly, Tr extends to the matrix algebra $M_N(\mathbb{A}_0^\hbar(M))$: $\mathrm{Tr}(f) = \sum_i \mathrm{Tr}(f_{ii})$.

Let e be an idempotent in $M_N(\widetilde{\mathbb{A}}_0^\hbar(M))$. Then $e = e_0 + \hbar e_1 + \cdots$, where $e_i \in M_N(\widetilde{C}_c^\infty(M))$. Obviously, e_0 is an idempotent in $M_N(\widetilde{C}_c^\infty(M))$. Here $\widetilde{C}_c^\infty(M)$ is the algebra $C_c^\infty(M)$ with adjoined unit. Let $e(\infty) = \sum_{i \geq 0} \hbar^i e_i(\infty)$, where $e_i(\infty)$ is the constant value of e_i outside some big enough compact in M. Then $e(\infty)$ is an idempotent in $M_N(\mathbb{C}[[\hbar]])$ and $e_0(\infty)$ is an idempotent in $M_N(\mathbb{C})$.

Let

$$ch(e_0) = \sum_{n \geq 0} \frac{1}{n!} \mathrm{tr}(e_0 \cdot (de_0)^{2n})$$

be the Chern character form of the idempotent e_0. This is an even-dimensional form on M. In fact $ch(e_0)$ is the Chern character form of the connection $e_0 \cdot d \cdot e_0$ in the vector bundle $e_0 \cdot \mathbb{C}^N$. The form $ch(e_0) - \mathrm{tr}(e_0(\infty))$ is compactly supported.

The symplectic form ω enables one to define the bundle $\mathscr{F}(M) \overset{Sp(2n)}{\to} M$ of symplectic frames. Since $Sp(2n)$ has the unitary group $U(n)$ as its maximal compact subgroup, one can reduce $\mathscr{F}(M)$ to the $U(n)$-principal bundle. Let $c_i(\omega)$ be the Chern classes of this bundle. Let $td(\omega)$ be the corresponding Todd class. This is an even-dimensional cohomology class of M with coefficients in $\mathbb{C}$.

In Subsect. 5.2, we define a characteristic class $\theta \in H^2(M, \mathbb{C}[[\hbar]])$ of the deformation $*$.

Theorem 1.1.1. *Under the notation and assumptions above, let e be an idempotent in* $M_N(\widetilde{\mathbb{A}_0^\hbar}(M))$. *Then*

$$\mathrm{Tr}(e - e(\infty)) = \int_M (ch(e_0) - ch(e_0(\infty))) \cdot \mathrm{td}(\omega) \cdot e^{-c_1(\omega)/2} \cdot e^{\theta/\hbar}.$$

Section 2. Characteristic Map Associated to a Trace

2.1. Cyclic (Co)homology and Lie Algebra (Co)homology in the Differential Graded Case. Suppose $A = \bigoplus_{i \geq 0} A_i$ is a differential graded algebra over a commutative ring $k \supset \mathbb{Q}$ with the differential $\delta : A_i \to A_{i-1}$; $\delta^2 = 0$; $\delta(ab) = \delta(a) \cdot (b) + (-1)^{|a|} a \cdot \delta(b)$. Here and below, given a homogeneous element a of A, we denote by $|a|$ the degree of a.

Let $C_{ml}(A) = (A \otimes A^{\otimes m})_l$ for $m, l \geq 0$. Define the cyclic permutation λ by

$$\lambda(a_0 \otimes \cdots \otimes a_m) = (-1)^{(|a_m|+1)\sum_{i=0}^{m-1}(|a_i|+1)} a_m \otimes a_0 \otimes \cdots \otimes a_{m-1}.$$

The Hochschild boundary operator b is given by

$$b(a_0 \otimes \cdots \otimes a_m) = \sum_{k=0}^{m-1} (-1)^{\varepsilon_k} a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_m + (-1)^{\varepsilon_m} a_m a_0 \otimes a_1 \otimes \cdots \otimes a_{m-1}, \qquad (2.1.2)$$

where

$$\varepsilon_k = \left(\sum_{i=0}^{k} |a_i|\right) + k, \quad k < n;$$

$$\varepsilon_m = (|a_m| + 1) \sum_{i<m} (|a_i| + 1) + |a_m| .$$

Set, on $A \otimes A^{\otimes m}$, $N = 1 + \lambda + \lambda^2 + \cdots + \lambda^m$;

$$B\alpha = (1 - \lambda)(1 \otimes N\omega) .$$

Lemma 2.1.1. $B^2 = (B + b)^2 = b^2 = 0$.

Proof. Direct computation. Cf. [LQ, Bu]. □

Put

$$C_m(A) = \coprod_{p+q=m} C_{pq}(A) .$$

We let the differential δ act on $C_*(A)$ by

$$\delta(a_0 \otimes \cdots \otimes a_m) = \sum_{k=0}^{m} (-1)^{\mu_k} a_0 \otimes \cdots \otimes \delta(a_k) \otimes \cdots \otimes a_m , \tag{2.1.3}$$

where

$$\mu_k = \sum_{i=1}^{k-1} (|a_i| + 1) .$$

Lemma 2.2.2. $(B + b + \delta)^2 = (b + \delta)^2 = 0$.

Proof. Direct computation. □

It is well known that the differential b preserves the image of $1 - \lambda$.

Definition 2.1.3. a) $CC_m(A) = C_m(A)/\mathrm{im}(1 - \lambda)$.

b) *The cyclic homology of A is the homology of the complex* $(CC_*(A), b + \delta)$. *It is denoted by* $HC_m(A)$, $m \geqq 0$.

c) *Put* $\bar{A} = A/k$; *for* $p \in \mathbb{Z}$, *put*

$$CC_{pq}^{\mathrm{per}}(A) = \prod_{j \equiv p (\mathrm{mod}\, 2)} \left(A \otimes \bar{A}^{\otimes j}\right)_q ,$$

$$CC_m^{\mathrm{per}}(A) = \prod_{p+q=m} CC_{pq}^{\mathrm{per}}(A) .$$

The homology of the complex $(CC_*^{\mathrm{per}}(A); b + B + \delta)$ *is called the periodic cyclic homology of A.*

(*It is easy to see that the differentials* b, B *and* δ *still make sense if one replaces* $A \otimes A^{\otimes j}$ *by* $A \otimes \bar{A}^{\otimes j}$).

d) *Put*

$$\overline{CC}_m(A) = \bigoplus_{p+q=m} \left(\bar{A}^{\otimes (p+1)}\right)_q \Big/ \mathrm{im}(1 - \lambda),$$

$$\overline{HC}_*(A) = H_*(\overline{CC}_*(A); b + \delta) .$$

$\overline{HC}_*(A)$ *is called reduced cyclic homology.*

Definition 2.1.4. *Put*

$$CC^m(A) = CC_m(A)' = \mathrm{Hom}_k(CC_m(A), k);$$

$$CC^m_{\mathrm{per}}(A) = \bigoplus_{p+q=m} \bigoplus_{j \equiv m (\mathrm{mod}\ 2)} (A \otimes \bar{A}^{\otimes j})' ;$$

the cyclic cohomology, resp. periodic cyclic cohomology, of A is the cohomology of the complex $CC^*(A)$, resp. $CC^*_{\mathrm{per}}(A)$, with the differential dual to $b+\delta$, resp. $b+B+\delta$.

Remark 2.1.5. In the two definitions above, we concentrated upon the versions of the cyclic theory which would be of main use in the sequel. The cyclic homology admits another definition, in the spirit of 2.1.3 c).

Put

$$C^{cy}_{pq}(A) = \bigoplus_{j \equiv p (\mathrm{mod}\ 2);\ j \leqq p} A \otimes \bar{A}^{\otimes j} ;$$

$$C^{cy}_m(A) = \bigoplus_{p+q=m} C^{cy}_{pq}(A) .$$

Then homology of the complex $(C^{cy}_*(A); b+B+\delta)$ is isomorphic to $HC_*(A)$ (recall that we assume $\mathbb{Q} \subset k$).

Also, we can replace $A \otimes \bar{A}^{\otimes j}$ by $A^{\otimes (j+1)}$ in all the definitions above. The (co)homology stays unchanged. We will denote corresponding complexes by $\widetilde{CC}^{\mathrm{per}}_*$, $\widetilde{CC}^*_{\mathrm{per}}$ or $\widetilde{C}^{cy}_*$.

There are obvious morphisms of complexes (shift to the left in case of cohomology, to the right in case of homology)

$$S : CC^{\mathrm{per}}_*(A) \to CC^{\mathrm{per}}_{*-2}(A); \qquad CC^*_{\mathrm{per}} \to CC^{*+2}_{\mathrm{per}} ;$$

$C^{cy}_* \to C^{cy}_{*-2}$. We shall denote the corresponding operators on (co)homology also by S. Clearly in case of periodic cyclic (co)homology S is an isomorphism.

Remark 2.1.5.1. At one point we will use yet another equivalent definition of cyclic homology. Let $b' : A^{\otimes m+1} \to A^{\otimes m}$ be same as in formula (2.1.2) but without the last summand. Then $b(1-\lambda) = (1-\lambda)b'$ and $b'N = Nb$ (this is why the image of $1-\lambda$ is invariant under b). Therefore the complex $NC_*(A); b')$ is isomorphic to $CC_*(A)$.

Definition 2.1.6. *Let A be an algebra over k; let $\widetilde{A}$ be the algebra A with adjoined unit. Then* $CC_*(A) = \ker(CC_*(\widetilde{A}) \to CC_*(k)), CC^*_{\mathrm{per}}(A) = \ker(CC^{\mathrm{per}}_*(\widetilde{A}) \to CC^{\mathrm{per}}_*(k))$ *etc.*

Remark 2.1.7. In this paper, k will be either $\mathbb{C}$ or $\mathbb{C}[[\hbar]]$ or $\mathbb{C}[\hbar^{-1}, \hbar]]$. The algebra A will be mainly $\mathbb{A}^\hbar(M)$ or $\mathbb{A}^\hbar_0(M)$. Then in definitions above $CC_*(A), C_*(A)$ will involve completed tensor products (both in C^∞ and $(\hbar)$-adic topology); also, $CC_*(A)$, $C_*(A)'$, etc. will mean topological duals. Sometimes we will write $CC^{(k)}_*(A)$, etc. to emphasize that the tensor products are taken over k.

Now, let $\mathfrak{g}$ be a differential graded Lie algebra with the differential δ of degree -1.

Define

$$\wedge(\mathfrak{g}) = T(\mathfrak{g})/\langle D_1 \otimes D_2 - (-1)^{(|D_1|+1)(|D_2|+1)} D_2 \otimes D_1 \rangle .$$

We define the grading on $\wedge(\mathfrak{g})$:

$$|D_1 \wedge \cdots \wedge D_m| = \sum_{i=1}^{m} (|D_i| + 1) .$$

Define a $U(\mathfrak{g})$-module map δ^{Lie} on $U(\mathfrak{g} \oplus \wedge(\mathfrak{g}))$,

$$\begin{aligned} &\delta^{\mathrm{Lie}}(X \otimes D_1 \wedge \cdots \wedge D_m) \\ &\quad = \sum_{i<j} (-1)^{\sigma_{ij}} X \otimes D_1 \wedge \cdots \wedge D_{i-1} \wedge [D_i, D_j] \wedge \cdots \wedge \widehat{D}_j \wedge \cdots \wedge D_n \\ &\qquad + \sum_{i=1}^{n} (-1)^{\sigma_i} X D_i \wedge D_1 \wedge \cdots \wedge \widehat{D}_i \wedge \cdots \wedge D_m , \end{aligned}$$

where

$$\sigma_{ij} = 1 + |X| + \sum_{i<k<j} (|D_k| + 1)(|D_j| + 1) + \sum_{k \leqq i} (|D_k| + 1) ,$$

$$\sigma_i = 1 + |X| + \sum_{k<i} (|D_k| + 1)(|D_i| + 1) .$$

Extend δ to the endomorphism of $U(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$ of degree -1:

$$\begin{aligned} \delta(X \otimes D_1 \wedge \cdots \wedge D_m) = &\delta(X) \otimes D_1 \wedge \cdots \wedge D_m + \sum_{j=1}^{m} (-1)^{|X| + \sum_{k<j}(|D_k|+1)} \\ &\times X \otimes D_1 \wedge \cdots \wedge \delta(D_j) \wedge \cdots \wedge D_m . \end{aligned}$$

One has $\delta^{\mathrm{Lie}} \cdot \delta + \delta \cdot \delta^{\mathrm{Lie}} = (\delta^{\mathrm{Lie}})^2 = \delta^2 = 0$. We denote the complex $(U(\mathfrak{g}) \otimes \wedge(\mathfrak{g}), \delta^{\mathrm{Lie}} + \delta)$ by $R_*(\mathfrak{g})$.

Let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}_0 \subset \mathfrak{g}$ which is annihilated by δ. We assume that $\mathfrak{h}$ acts semisimply on $\mathfrak{g}$. Define the right action of $\mathfrak{h}$ on $R_*(\mathfrak{g})$:

$$(X \otimes D_1 \wedge \cdots \wedge D_m)h = Xh \otimes D_1 \wedge \cdots \wedge D_m + \sum_{i=1}^{m} X \otimes D_1 \wedge \cdots \wedge [D_i, h] \wedge \cdots \wedge D .$$

It is easy to see that the differentials δ^{Lie} and δ together with the right action of $\mathfrak{h}$ are well defined on $(U(\mathfrak{g}) \otimes \wedge(\mathfrak{g})/\mathfrak{h}))_{\mathfrak{h}}$. (If $\mathfrak{a}$ is any Lie algebra and V is any $\mathfrak{a}$-module, we write $V_{\mathfrak{a}} = V/V\mathfrak{a}$ (or $V/\mathfrak{a}V$)). Put

$$R_*(\mathfrak{g}, \mathfrak{h}) = ((U(\mathfrak{g}) \otimes \wedge(\mathfrak{g}/\mathfrak{h}))_{\mathfrak{h}}; \delta^{\mathrm{Lie}} + \delta) .$$

The algebra $U(\mathfrak{g})$ is naturally graded; we write $|X \otimes \alpha| = |X| + |\alpha|$ for $x \in U(\mathfrak{g})$, $\alpha \in \wedge(\mathfrak{g}/\mathfrak{h})$.

Let M_* be a differential graded right $\mathfrak{g}$-module with differential δ_M of degree -1; $\delta_M(m \cdot D) = \delta_M(m) \cdot D + (-1)^{|m|} m \cdot \delta(D)$. Put

$$C_*(\mathfrak{g}, \mathfrak{h}; M_*) = M_* \otimes_{U(\mathfrak{g})} R_*(\mathfrak{g}, \mathfrak{h}) .$$

For $m \in M,\ \alpha \in R_*(\mathfrak{g},\mathfrak{h})$ put

$$d(m \otimes \alpha) = \delta_M m \otimes \alpha + (-1)^{|m|} m \otimes (\partial^{\mathrm{Lie}} + \delta)\alpha\,.$$

We obtain a complex $C_*(\mathfrak{g},\mathfrak{h};M)$ with differential of degree -1. The homology of this complex is relative Lie algebra homology. It is denoted by $H_*(\mathfrak{g},\mathfrak{h};M_*)$.

Let N^* be a differential graded left $\mathfrak{g}$-module with differential δ_N of degree $+1$. Put

$$\mathrm{Hom}_{\mathfrak{g}}(R_*(\mathfrak{g},\mathfrak{h}),N^*) = \{\varphi : R_*(\mathfrak{g},\mathfrak{h}) \to N^* | \varphi(D\alpha) = (-1)^{|\varphi| \cdot |D|} D\varphi(\alpha)\}\,.$$

For α from $R_*(\mathfrak{g},\mathfrak{h})$ put

$$(d\varphi)(\alpha) = \delta_N \varphi(\alpha) - (-1)^{|\varphi|}\varphi((\delta^{\mathrm{Lie}} + \delta)\alpha)\,.$$

Put

$$C^*(\mathfrak{g},\mathfrak{h};N^*) = \mathrm{Hom}_{\mathfrak{g}}(R_*(\mathfrak{g},\mathfrak{h}),N^*);d)\,.$$

This is a complex with differential of degree $+1$. Its cohomology is denoted by $H^*(\mathfrak{g},\mathfrak{h};N^*)$.

If $\mathfrak{h} = 0$, we will write $C_*(\mathfrak{g},M_*), H_*(\mathfrak{g},M_*), C^*(\mathfrak{g},N^*)$, etc. for $C_*(\mathfrak{g},\mathfrak{h};M_*)$, etc. If $M_* = k$, the ground ring concentrated in dimension zero with trivial action, we will write $C_*(\mathfrak{g},\mathfrak{h})$ for $C_*(\mathfrak{g},\mathfrak{h};M_*)$, etc.

Proposition 2.1.8. *The following maps are well defined morphisms of complexes*:

$$CC_{*-1}(A) \to C_*(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}\,;$$

$$\overline{CC}_{*-1}(A) \to C_*(\mathfrak{gl}(A),\mathfrak{gl}(k))\,;$$

$$a_0 \otimes \cdots \otimes a_m \mapsto E_{01}(a_0) \wedge E_{12}(a_1) \wedge \cdots \wedge E_{m0}(a_m)\,.$$

The proof follows from direct computation; cf. [Bu, LQ, T].

Remark 2.1.9. Let $k[\varepsilon]$ be a two-dimensional Grassmann algebra: $|\varepsilon| = 1,\ \varepsilon^2 = 0$. Put $\mathfrak{g}[\varepsilon] = \mathfrak{g} \otimes_k k[\varepsilon]$. By the Poincaré–Birkhoff–Witt theorem $U(\mathfrak{g}[\varepsilon]) \widetilde{\to} U(\mathfrak{g}) \otimes \wedge(\mathfrak{g})$ as a k-module; let $\frac{\partial}{\partial \varepsilon}$ be the derivation of degree -1 sending ε to 1 and D to 0, $D \in \mathfrak{g}$. Also, the differential δ extends to $\mathfrak{g}[\varepsilon]$ (put $\delta(\varepsilon) = 0$). Under the isomorphism above, these two differentials become $-\partial^{\mathrm{Lie}}$ and δ. Also, $R_*(\mathfrak{g},\mathfrak{h}) \widetilde{\to} U(\mathfrak{g}[\varepsilon]) \otimes_{U(\mathfrak{h}[\varepsilon])} k$.

For an associative algebra A, put $A\{\varepsilon\} = A *_k k[\varepsilon]$. Then $\overline{CC}_{*-1}(A) \widetilde{\to} A\{\varepsilon\}/ (A + [A\{\varepsilon\},A\{\varepsilon\}])$. Under this isomorphism, $\frac{\partial}{\partial \varepsilon}$ and δ become b and δ from formulas (2.1.2, 2.1.3).

2.2. The Characteristic Map. Suppose that A is a unital associative algebra ($\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded) over a commutative unital ring k of characteristic zero. Let A_0 be a homogeneous ideal of A.

Remark 2.2.1. Our main examples will be $A = \mathbb{A}^\hbar(M)$, $A_0 = \mathbb{A}_0^\hbar(M)$ (or may be matrices over these rings). The ideal A_0 is NOT the zero degree part of A.

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Der}(A)$ which includes $ad(A)$ and preserves A_0. Suppose, moreover, that τ is a $\mathfrak{g}$-invariant trace on A_0.

Let $\underline{A}$ be another copy of A but with shifted grading: $\underline{A}_i = A_{i-1}$. Consider the differential graded Lie algebra $\mathfrak{g} \ltimes \underline{A}$,

$$\mathfrak{g} \ltimes \underline{A} = \mathfrak{g} \dot{+} \underline{A}; \quad [\underline{A}, \underline{A}] = 0\,;$$

$$[D, \underline{a}] = \underline{[D, a]} \quad \text{for } D \in \mathfrak{g},\ \underline{a} \in \underline{A}\,;$$

$$\delta(D) = 0,\ D \in \mathfrak{g}; \quad \delta(\underline{a}) = (-1)^{|a|} ad(a)\,.$$

For an element X of $\mathfrak{g} \ltimes A$ define the operator

$$i_X : CC_*^{\mathrm{per}}(A_0) \to CC_{*+|X|-1}^{\mathrm{per}}(A_0)$$

by

$$i_D(a_0 \otimes \cdots \otimes a_m) = (-1)^{(|a_0|+1)|D|+a_0|} a_0 \cdot D(a_1) \otimes a_2 \otimes \cdots \otimes a_m\,,$$

$$i_{\underline{a}}(a_0 \otimes \cdots \otimes a_m) = (-1)^{(|a_0|+1)(|a|+1)+|a_0|} a_0 \cdot a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m\,.$$

Lemma 2.2.1. *For any X in $\mathfrak{g} \ltimes \underline{A}$,*

$$[b, i_X] + i_{\delta X} = 0\,.$$

Proof. Direct computation.

Definition 2.2.2. *Let χ_τ be the map*

$$\chi_\tau : C_*(\mathfrak{g} \ltimes \underline{A}) \to C_{\mathrm{per}}^{-*}(A_0)$$

given by

$$\chi_\tau(X_1 \wedge \cdots \wedge X_m)(\alpha) = \frac{1}{m!} \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \tau(i_{X_{\sigma(1)}} \ldots i_{X_{\sigma(m)}} \alpha)\,,$$

where $\alpha \in CC_*^{\mathrm{per}}(A_0)$ and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ of graded elements $X_1, \ldots, X_m$ determined by

$$\mathrm{sgn}\begin{pmatrix} X_{i_1} & X_{i_2} \\ X_{i_2} & X_{i_1} \end{pmatrix} = (-1)^{(|X_{i_1}|+1)(|X_{i_2}|+1)}\,.$$

Proposition 2.2.3.

$$\chi_\tau \cdot \partial^{\mathrm{Lie}} = B\chi_\tau\,;$$

$$\chi_\tau \cdot \delta = b\chi_\tau\,.$$

Proof. Direct computation.

Corollary 2.2.4. *The following is a commutative diagram of morphisms of complexes:*

$$\begin{array}{ccc} C_*(\mathfrak{g}) & \xrightarrow{\chi_\tau} & C_{\mathrm{per}}^{-*}(A_0) \\ \downarrow & \nearrow_{\chi_\tau} & \\ C_*(\mathfrak{g} \ltimes \underline{A}) & & \end{array}$$

2.3. Poincaré Duality. Suppose now that A is a unital algebra over k, A_0 is an ideal in A and τ is an ad(A)-invariant trace on A_0. Replacing A, A_0 by $M_N(A)$, $M_N(A_0)$ and τ by $\sum \tau(a_{ii})$, we can apply the construction of χ_τ to $\mathfrak{g} = \mathfrak{gl}_N(A)$ and get the commutative triangle of morphisms of complexes

$$\begin{array}{ccc} C^*(\mathfrak{gl}_N(A)) & \xrightarrow{\chi_\tau} & CC^{-*}_{\mathrm{per}}(M_N(A_0)) \\ \downarrow & \nearrow_{\chi_\tau} & \\ C^{\mathrm{Lie}}_*(\mathfrak{gl}_N(A) \ltimes \underline{M_N(A)}) & & \end{array}$$

Clearly, $\mathfrak{gl}_N(k)$ acts on all three complexes, and all the maps are $\mathfrak{gl}_N(k)$-equivariant. Therefore we get a similar diagram of complexes of coinvariants. Note that $\mathfrak{gl}_N(A) \ltimes \underline{M_N(A)} = \mathfrak{gl}_N(A[\eta])$ with the differential $\partial/\partial\eta$, where

$$k[\eta] = k\{\eta\}/(\eta^2); \quad \deg \eta = 1; \quad A[\eta] = A \otimes k[\eta]\,.$$

Making N big enough and applying Proposition 2.1.8 we obtain a map

$$CC_{*-1}(A[\eta]) \longrightarrow CC^{-*}_{\mathrm{per}}(M_N(A_0))_{\mathfrak{gl}_N(k)}$$

(defined when $* < N$); now notice two simple facts following immediately from definitions.

i) by restriction from $M_N(A_0)$ to subalgebra $\{a \cdot \delta_{ij}\} \xrightarrow{\sim} A_0$ one gets a map $CC^{-*}_{\mathrm{per}}(M_N(A_0))_{\mathfrak{gl}(N)} \longrightarrow CC^{-*}_{\mathrm{per}}(A_0)$;

ii) under this restriction, the elements of A appear only through the corresponding inner derivations $ad(a)$. Hence the composition $CC_{*-1}(A[\eta]) \longrightarrow CC^{-*}_{\mathrm{per}}(A_0)$, which we still denote by χ_τ, descends to the reduced cyclic complexes $\overline{CC}_{*-1}(A[\eta])$.

Therefore we end up with the following commutative diagram of morphisms of complexes:

$$\begin{array}{ccc} \overline{CC}_{*-1}(A) & \xrightarrow{\chi_\tau} & CC^{-*}_{\mathrm{per}}(A_0) \\ \downarrow & \nearrow_{\chi_\tau} & \\ \overline{CC}_{*-1}(A[\eta]) & & \end{array}$$

Definition 2.3.1. *The induced map* $\chi_\tau : \overline{HC}_{*-1}(A) \to HC^*_{\mathrm{per}}(A_0)$ *is called the Poincaré duality map associated to a trace* τ.

Recall that there is the long exact sequence

$$\cdots \to HC_n(k) \to HC_n(A) \to \overline{HC}_n(A) \xrightarrow{\partial} HC_{n-1}(k) \to \cdots\,.$$

Theorem 2.3.2. *The Poincaré duality map* χ_τ *factorizes through the boundary map* ∂:

$$\begin{array}{ccc} \overline{HC}_{*-1}(A) & \xrightarrow{\chi_\tau} & HC^{-*}_{\mathrm{per}}(A_0) \\ \partial\downarrow & \nearrow_{\widehat{\chi}_\tau} & \\ HC_{*-2}(k) & & \end{array}$$

where $\widehat{\chi}_\tau\left(\frac{(2n-2)!}{(n-1)!}1^{\otimes(2n-1)}\right) = S^{-n}\tau.$

Proof. Note that χ_τ factorizes through the characteristic map for the differential graded algebra $A[\eta]$. But $(A[\eta], \delta)$ is a contractible comples:

$$0 \longrightarrow A\eta \xrightarrow{\delta} A \longrightarrow 0\,,$$

and therefore the cyclic homology of $A[\eta]$ is trivial, as can be seen from the spectral sequence associated to the double complex $(CC_*(A[\eta]; b, \delta)$. In particular, the boundary map

$$\partial : \overline{HC}_{*-1}(A[\eta]) \longrightarrow HC_{*-2}(k)$$

is an isomorphism. This proves that there exists χ_τ making the above diagram commute. It remains to calculate χ_τ.

Lemma 2.3.3. i) *the chain* $\eta^{\otimes n}$ *defines a reduced cyclic homology class in* $\overline{HC}_{2n-1}(A[\eta])$,

ii) $\chi_\tau(\eta^{\otimes n}) = \frac{1}{(n-1)!} S^{-n}\tau$ *in* $\overline{HC}_{2n-1}(A[\eta])$,

iii) $\partial(\eta^{\otimes n}) = \frac{1}{(n-1)!} \cdot \frac{(2n-2)!}{(n-1)!} 1^{\otimes 2n-2}$ *in* $HC_{2n-2}(k)$.

Note that the theorem follows from the above lemma immediately.

Proof of 2.3.3. i) is obvious (both b and ∂ vanish at $\eta^{\otimes n}$);

ii) follows from Definition 2.2.2 of χ_τ and Proposition 2.1.8; note that for α in $C^*_{\mathrm{per}}(A_0)$ $\chi_\tau(E_{\sigma(0),\overline{\sigma(0)+1}} \cdot \cdots \cdot E_{\sigma(n-1),\overline{\sigma(n-1)+1}} \cdot \alpha)$ equal $\tau(\alpha)$ if σ is a power of the cycle $(0\ 1 \ldots n-1)$ and equals zero otherwise. (Here $E_{i,\overline{i+1}} = E_{i,i+1}$ for $i < n-1$ and $E_{n-1,0}$ for $i = n-1$.)

Let us prove (iii). Recall the definition of the cyclic complex C^{cy}_* (Remark 2.1.5). Note that in $C^{\mathrm{cy}}_*(A[\eta])$,

$$b(\eta^{\otimes m}) = 0; \qquad \delta(\eta^{\otimes m}) = 1 \otimes \eta^{\otimes m-1}; \qquad B(\eta^{\otimes m}) = m \cdot (1 \otimes \eta^{\otimes m})$$

for any m. Therefore the homology class of $\eta^{\otimes n} \in CC_{2n-1}(A[\eta])$ is represented in C^{cy}_* by a cycle whose component in $A[\eta] \otimes \overline{A[\eta]}^{\otimes 0}$ is equal to $(-1)^{n-1}\frac{1}{(n-1)!} \cdot 1$. Note that $C^{\mathrm{cy}}_{2m}(k) = k$; one checks easily that the homology class of $1 \in C^{\mathrm{cy}}_{2m}(k)$ is represented by $(-1)^m \cdot \frac{(2m)!}{m!} \cdot 1^{\otimes(2m+1)} \in CC_{2m}(k)$. From this we conclude that

$$\partial(\eta^{\otimes n}) = \frac{1}{(n-1)!} \cdot \frac{(2n-1)!}{(n-1)!} 1^{\otimes 2n-1};$$

$$\chi_\tau(\eta^{\otimes n}) = \frac{1}{(n-1)!} S^{-n}\tau,$$

whence the statement of the lemma. □

Let us recall that (reduced) cyclic homology carries the cup product

$$\cup : \overline{HC}_a(A) \otimes \overline{HC}_b(B) \to \overline{HC}_{a+b+1}(A \otimes B).$$

At the level of the complex $(NC_*(A), b' + \delta)$ (Remark 2.1.5.1) it is given by the standard shuffle of tensors.

Proposition 2.3.4. *There exists the product*

$$\bullet : HC_a(k) \otimes HC_b(k) \to HC_{a+b+2}(k)$$

such that:

i) $1^{\bullet(n+1)} = \frac{(2n)!}{n!} 1^{\otimes(2n+1)}$ *for any* $n \geqq 0$;

ii) *for* $\alpha \in \overline{HC}_a(A)$, $\beta \in \overline{HC}_b(B)$

$$\partial(\alpha \cup \beta) = \partial\alpha \bullet \partial\beta \,.$$

Remark 2.3.5. Formula (2.3.4) i) is equivalent to $S(1^{\bullet n}) = -1^{\bullet(n-1)}$ for any $n > 0$.

Proof of 2.3.4. The composition

$$\overline{CC}_a(A[\eta]) \otimes \overline{CC}_b(B[\eta]) \to \overline{CC}_{a+b+1}(A \otimes B[\eta_1, \eta_2]) \overset{\eta_1 = \eta_1}{\longrightarrow} \overline{CC}_{a+b+1}((A \otimes B)[\eta])$$

defines a product for which $\eta^{\cup n} = (n-1)!\eta^{\otimes n}$. Now, as we have seen above, the boundary map ∂ induces the isomorphisms $\overline{HC}_a(A[\eta]) \to HC_{a-1}(k)$, etc., and $\partial\eta^{\cup(n+1)} = \frac{(2n)!}{n!} 1^{\otimes(2n+1)}$. □

Remark 2.3.6. It is easy to write the explicit formula for the Poincaré duality map χ_τ. Let $x_1 \otimes \cdots \otimes x_p \in \overline{CC}_{p-1}(A)$, $a_0 \otimes \cdots \otimes a_p \in CC_p(A_0)$. Let $\lambda = (1\,2 \ldots p)$ be the cyclic permutation. Then

$$\langle \chi_\tau(x_1 \otimes \cdots \otimes x_p), a_0 \otimes \cdots \otimes a_p \rangle = (-1)^{p(p-1)/2} \sum_{i=1}^{p-1} \operatorname{sgn}(\lambda^i) \cdot \tau(a_0[x_{\lambda^i 1}, a_1] \ldots [x_{\lambda^i p}, a_p]) \,.$$

2.4. Examples. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded algebra. Let $F \in A_1$ such that $F^2 = 1$. Let $\tau_s : A \to k$ be an even supertrace on A; i.e., $\tau_s(ab) = (-1)^{|a||b|}\tau_s(ba)$. Note that $F^{\otimes 2n}$ is a cycle of $\overline{CC}_{2n-1}(A)$. It is easy to see that $\partial F^{\otimes 2n} = 2n \cdot 1^{\bullet n}$. One has

$$\frac{1}{2n}\chi_{\tau_s}(F^{\otimes 2n})(a_0, \ldots, a_{2n}) = \frac{1}{(2n)!}\tau_s(a_0[F, a_1] \cdots [F, a_{2n}])$$

for $a_0, \ldots, a_{2n} \in A_0$. Thus we obtain Connes' formula for the Chern character of a Fredholm module. Theorem 2.3.2 implies that the cocycle in the formula above is cohomologous to $S^{-n}\tau_s$.

Now, let us keep the same assumptions as above except $F^2 = 1$. There are two odd elements in the algbera $\mathfrak{g} \ltimes \underline{A}$, where $\mathfrak{g} = \operatorname{Der}(A)$; one is $\operatorname{ad}(F)$, the other is $\underline{F^2}$.

Therefore there is the subspace $\wedge^*(k \cdot \underline{F^2} + k \cdot \operatorname{ad}(F))$ in $C_*(\mathfrak{g} \ltimes \underline{A})$; this subspace consists of all *symmetric* polynomial functions of $\operatorname{ad}(F)$ and $\underline{F^2}$.

Let ∇ be an odd element of a different graded Lie algebra $\mathfrak{a}$. Formally, we can consider *any* symmetric function in ∇ as an element in $C_*(\mathfrak{a}\ )$.

Lemma 2.4.1. *If*

$$\delta\nabla + \frac{1}{2}[\nabla, \nabla] = 0$$

in $\mathfrak{a}$, *then* e^∇ *is a cycle in* $C_*(\mathfrak{a}\ )$.

Lemma 2.4.2. *Put* $\nabla = \operatorname{ad}(F) - \underline{F^2}$; *then*

$$\delta\nabla + \frac{1}{2}[\nabla, \nabla] = 0$$

in $\mathfrak{g} \ltimes \underline{A}$.

The proof follows from direct computations. Expand e^{∇} as a power series in $\mathrm{ad}(F)$ and $\underline{F^2}$, and apply the definition of χ_{τ_s} to this infinite sum of chains of $C_*(\mathfrak{g} \ltimes \underline{A})$. It is easy to see that for $a_0, \ldots, a_n \in A$,

$$\chi_{\tau_s}(\exp(\mathrm{ad}(F) - \underline{F^2}))(a_0, \ldots, a_n) = \int_{\Delta^n} \tau_s(a_0 e^{-t_0 F^2}[F, a_1]e^{-t_1 F^2} \times [F, a_2] \ldots [F, a_n]e^{-t_n F^2}) \times dt_1 \cdots dt_n \, ;$$

here $\Delta^n = \{(t_0, \ldots, t_n) | \sum t_i = 1; t_i \geqq 0\}$.

We thus recover the formula of Jaffe, Lesniewski and Osterwalder [JLO]. Of course, if the argument $a_0, \ldots, a_n$ are even, then the components of the periodic cyclic cocycle above are zero whenever n is odd.

Section 3. Globalization and the Fundamental Class

3.1. The Čech-cyclic Complexes and the Characteristic Maps. Let, as in Sect. 1, (M, ω) be a symplectic C^∞ manifold and let $\mathbb{A}^\hbar(M)$ be a formal deformation of $C^\infty(M)$ such that $\varphi_1(f,g) - \varphi_1(g,f) = \{f,g\}$ is the Poisson bracket associated to the symplectic structure ω. Since, by hypothesis, $\varphi_i(f,g)$ are bidifferential, one has in fact a well-defined sheaf of algebras $\mathbb{A}^\hbar(U)$ over $\mathbb{C}[[\hbar]]$, where U is any open subset of M. Choose an open cover $\{U_i\}$ of M such that $U_{i_0} \cap \cdots \cap U_{i_p}$ is either empty or symplectomorphic to a bounded open contractible subset of $\mathbb{R}^{2n}$ for any $i_0, \ldots, i_p$. From now on, $(\check{C}^*(M, \cdot\,), \check{\partial})$ will always mean the Čech complex associated to the cover $\{U_i\}$. Put

$$\mathscr{C}_{-p,m} = \check{C}^p(M, \overline{CC}_m(\mathbb{A}^\hbar))$$

$$C_n = \prod_{m-p=n} C_{-p,m} \, ;$$

the differential in this complex is $b + (-1)^m \check{\partial}$

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow \check{\partial} & & \uparrow \check{\partial} & & \\ \check{C}^1(M, \overline{CC}_0(\mathbb{A}^\hbar)) & \xleftarrow{b} & \check{C}^1(M, \overline{CC}_1(\mathbb{A}^\hbar)) & \longleftarrow & \cdots \\ \uparrow \check{\partial} & & \uparrow & & \\ \check{C}^0(M, \overline{CC}_0(\mathbb{A}^\hbar)) & \xleftarrow{b} & \check{C}^0(M, \overline{CC}_1(\mathbb{A}^\hbar)) & \longleftarrow & \cdots \end{array}$$

We will denote the homology of the complex $\mathscr{C}_*$ by $\mathscr{H}_n$ or $\mathscr{H}_n(M, \mathbb{A}^\hbar)$. Let $k = \mathbb{C}[[\hbar]]$.

Clearly, the boundary map $\partial : \overline{CC}_*(\mathbb{A}^\hbar) \to CC_{*-1}(k)$ extends to a well-defined morphism in the derived category of sheaves, therefore $\check{C}^*(M, \overline{CC}_*(\mathbb{A}^\hbar)) \xrightarrow{\partial} \check{C}^*(M, CC_{*-1}(k))$ is well defined in the derived category of complexes.

Proposition 3.1.1. *There exist the morphisms of complexes $\chi_\tau, \widehat{\chi}_\tau$ such that:*

i) *the diagram below commutes in the derived category:*

$$\begin{array}{ccc} \check{C}(M, CC_{*-1}(\mathbb{A}^\hbar)) & \xrightarrow{\chi_\tau} & CC^{-*}_{\rm per}(\mathbb{A}^\hbar_0(M)) \\ \partial \downarrow & \nearrow \widehat{\chi}_\tau & \\ \check{C}^*(M, CC_{*-2}(k)) & & \end{array}$$

ii) *the composition of χ_τ, resp. $\widehat{\chi}_\tau$ with the morphism $\overline{CC}_*(\mathbb{A}^\hbar(M)) \to \check{C}^*(M, \overline{CC}_*(\mathbb{A}^\hbar))$ resp. $CC_*(k) \to \widehat{C}^*(M, CC_*(k))$, is equal to χ_τ, resp. $\widehat{\chi}_\tau$ which were defined in Theorem 2.3.2.*

Proof. First of all, it is easy to construct the commutative diagram

$$\begin{array}{ccc} \check{C}^*(M, \overline{CC}_{*-1}(\mathbb{A}^\hbar)) & \xrightarrow{\chi_\tau} & \check{C}^*(M, CC^{-*}_{\rm per}(\mathbb{A}^\hbar_0(M)) \\ \partial \downarrow & \nearrow \widehat{\chi}_\tau & \\ \check{C}^*(M, CC_{*-2}(k)) & & \end{array}$$

(since χ_τ, $\widehat{\chi}_\tau$ are given by explicit local formulas). Now, note that the sheaf $U \mapsto CC^{\rm per}_*(\mathbb{A}^\hbar_0)$ admits partition of unity (unlike $U \mapsto \overline{CC}_*(\mathbb{A}^\hbar(U))$. For example, take $\{\rho_{U_i}\}$ to be the partition of unity subordinated to $\{U_i\}$ and let $I_{\rho_{U_i}}$ be the operators defined in Appendix 1. Thus, there is the quasi-isomorphism

$$\check{C}^*(M, CC^*_{\rm per}(\mathbb{A}^\hbar_0)) \to CC^*_{\rm per}(\mathbb{A}^\hbar_0(M))\,;$$

composing it with χ_τ, we get the morphism $\chi_\tau : \check{C}^*(M, \overline{CC}_{*-1}(\mathbb{A}^\hbar)) \to CC^{-*}_{\rm per}(\mathbb{A}^\hbar_0(M))$ given by the explicit formula

$$\{s_{U_0 \dots U_p}\} \mapsto \sum_{U_0, \dots, U_p} \chi_\tau(s)(I_{\rho_{U_0}}[B+b, I_{\rho_{U_1}}] \dots [B+b, I_{\rho_{U_p}}]\bullet)\,.$$

In the same fashion we get the morphism $\widehat{\chi}_\tau : \check{C}^*(M, CC_{*-2}(k)) \to CC^{-*}_{\rm per}(\mathbb{A}^\hbar_0(M))$ given by

$$\{c_{U_0 \dots U_p}\} \cdot 1^{\bullet n} \mapsto \sum_{u_0, \dots, u_p} c_{U_0 \dots U_p}(S^{n-1}\tau) I_{\rho_{U_0}}[b+B, I_{\rho_{U_1}}] \times \dots \times [B+b, I_{\rho_{U_p}}])\,. \quad \square$$

Corollary 3.1.2. *There exists a commutative triangle*

$$\begin{array}{ccc} \mathscr{H}_{m-1}(M, \mathbb{A}^\hbar) & \xrightarrow{\chi_\tau} & HC^{-m}_{\rm per}(\mathbb{A}^\hbar_0(M))) \\ \partial \downarrow & \nearrow \widehat{\chi}_\tau & \\ \bigoplus_{p \geqq \frac{m}{2}} H^{2p-m}(M, k) \cdot 1^{\bullet p} & & \end{array}$$

for any m.

Remark 3.1.3. Clearly, the cup product defines the $H^*(M,k)$-module structures on all three k-modules above (since $CC^*_{\rm per}(\mathbb{A}^\hbar)$ and $\check{C}^*(M, CC^*_{\rm per})$ are quasi-isomorphic), and all the morphisms above are in fact morphisms of $H^*(M,k)$ modules.

The same construction may be carried out if one replaces $\mathbb{A}^\hbar(U)$ by $\mathbb{A}^\hbar(U)[\hbar^{-1}] = \mathbb{A}^\hbar(U)\otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}[\hbar^{-1},\hbar]]$ and puts $k=\mathbb{C}[\hbar^{-1},\hbar]]$. We denote the bicomplex by $\check{C}^*(M,\overline{CC}_*(\mathbb{A}^\hbar[\hbar^{-1}]))$ or $\mathscr{C}^*(M,\mathbb{A}^\hbar[\hbar^{-1}])$ and its homology by $\mathscr{H}_*(M,\mathbb{A}^\hbar[\hbar^{-1}])$.

The next two subsections are devoted to the proof that $\mathscr{H}_{2n-1}(M,\mathbb{A}^\hbar(U)[\hbar^{-1}])$ is canonically one-dimensional. In 3.2 we show this when $M=\mathbb{R}^{2n}$; in 3.3 we prove this for arbitrary M.

3.2. Local Computation of $\mathscr{H}_{2n-1}$.

Theorem 3.2.1. $\overline{HC}_i(\mathbb{A}^\hbar(\mathbb{R}^{2n})[\hbar^{-1}])\simeq k$ *if* $i=1,3,\dots,2n-1$; $\overline{HC}_i(\mathbb{A}^\hbar(\mathbb{R}^{2n})[\hbar^{-1}]) = 0$ *otherwise.*

Proof. It is well known [BFFLS] that any formal deformation of $\mathbb{R}^{2n}$ corresponding to the standard symplectic structure is isomorphic to the standard deformation

$$f*g=\sum_\alpha \frac{\hbar^{|\alpha|}}{|\alpha|!}\partial_\xi^\alpha f\cdot\partial_x^\alpha g\,,$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\geqq 0$. Consider the $(\hbar)$-adic filtration of the Hochschild complex $C_*(\mathbb{A}^\hbar)$. The corresponding spectral sequence converges because the filtration is complete. One has $E^1=HH_*(C^\infty(\mathbb{R}^{2n}))=\Omega^*(\mathbb{R}^{2n})$; the differential in E^1 acts from Ω^* to Ω^{*-1}; if one identifies Ω^i with Ω^{2n-i} via the form ω, then the differential becomes the de Rham differential d (cf. [B1]). Therefore, the spectral sequence collapses at E^2 and $HH_{2n}(\mathbb{A}^\hbar)\cong k$; $HH_i(\mathbb{A}^\hbar)=0$, $i\neq 2n$. Now, from the Hochschild to cyclic spectral sequence [LQ] we get that $HC_i(\mathbb{A}^\hbar)\cong k$ if $i=2n+2p$, $p\geqq 0$; $HC_i(\mathbb{A}^\hbar)=0$ otherwise. It is clear that the chain $1\otimes(\xi_1\wedge\cdots\wedge\xi_n\wedge x_1\wedge\cdots\wedge x_n)$ in $\mathbb{A}^\hbar\otimes(\overline{\mathbb{A}^\hbar})^{\otimes 2n}$ is the generator of HH_{2n}. An easy computation in $(B+b)$-complex C_*^{cy} shows that this chain is homologous to a non-zero multiple of $1\in\mathbb{A}^\hbar\otimes(\overline{\mathbb{A}}^\hbar)^{\otimes 0}$. Thus, the morphisms $HC_{2n+2i}(k)\to HC_{2n+2i}(\mathbb{A}^\hbar)$ are isomorphisms. Now, the theorem follows from the exact sequence

$$\cdots\to HC_m(k)\to HC_m(\mathbb{A}^\hbar)\to\overline{HC}_m(\mathbb{A}^\hbar)\to HC_{m-1}(k)\to\cdots$$

(cf. [LQ]). □

Remark 3.2.2. Clearly, the chain $\frac{1}{\hbar}(\xi_1\otimes x_1)$ or the homologous chain $\frac{1}{2\hbar}(\xi_1\otimes x_1 - x_1\otimes\xi_1))$ represents the generator Ω_1 of $\overline{HC}_1(\mathbb{A}^\hbar(\mathbb{R}^1))$, and $\partial\Omega_1=1$. Now, put $\Omega_n=\Omega_1^{\cup n}$; then Ω_n is represented by the chain $\frac{1}{\hbar^n}\cdot\frac{1}{2n}(\xi_1\wedge\xi_n\wedge\cdots\wedge\xi\wedge x_n)$ and $\partial\Omega_n=1^{\bullet n}$ because of Proposition 2.3.4.

3.3. Computation of $\mathscr{H}_{2n-1}$ *and the Fundamental Class.* Consider the spectral sequence associated to the double complex $\mathscr{C}_{-p,m}$ from 3.1. The first differential is b; thanks to Theorem 3.2.1, one has

$$E^2_{-p,m}=\check{H}^p(M,k),\quad m=1,3,\dots,2n-1\,;$$

$$E^2_{-p,m}=0\qquad\text{otherwise}\,.$$

In particular, $E^2_{-p,2n-1}=k$ and $E^2_{-p,m}=0$, $m>2n-1$. Therefore there is the edge homomorphism

$$\check{H}^p(M,k)\to\mathscr{H}_{2n-1-p}(M,\mathbb{A}^\hbar[\hbar^{-1}])\,;\tag{3.3}$$

its restriction to $\check{H}^0$ is an isomorphism. Let Ω be the image of 1. This is the generator of $\mathscr{H}_{2n-1}(M, \mathbb{A}^\hbar[\hbar^{-1}])$. We normalized it requiring that its restriction to any coordinate open subset be represented by $\frac{1}{2n\hbar^n}(\xi_1 \wedge x_1 \wedge \cdots \wedge \xi_n \wedge x_n)$.

Note that the edge homomorphism (3.3.1) is given by

$$\check{H}^p \ni c \mapsto c \cdot \Omega \in \mathscr{H}_{2n-1-p} ;$$

$c \cdot \Omega$ means the cup product; we use the symbol $\cdot$ to distinguish from the cup product in cyclic homology.

Section 4. Algebraic Index Theorem for Formal Deformations

4.1. Boundary Operator at the Fundamental Class. Recalls that we had defined the generator Ω of $\mathscr{H}_{2n-1}(M, \mathbb{A}^\hbar[\hbar^{-1}])$ (cf. 3.3) and the boundary operator

$$\partial : \mathscr{H}_{m-1}(M, \mathbb{A}^\hbar) \to \bigoplus_{p \geqq \frac{m}{2}} H^{2p-m}(M, K) \cdot 1^{\bullet p}$$

(cf. 3.1). The following theorem will be proved in Sects. 5, 6.

Theorem 4.1.1. i) *the fundamental class Ω may be represented by a cochain in $\sum_{i \geqq 0} \hbar^{-(n+i)} \check{C}^i(M, \overline{CC}_{2n-1+i})$; modulo $\hbar \cdot \sum h^{-(n+i)} \check{C}^i(M, \overline{CC}_{2n+i-1})$, this cochain may be chosen in*

$$\sum \hbar^{-(i+n)} \check{C}^i(M, \wedge^{2n+i} \overline{\mathbb{A}^\hbar})) ;$$

ii) *One has*

$$\partial \Omega = \sum_k T_k \cdot 1^{\bullet n+k}$$

where, by definition,

$$(e^{\theta/\hbar} \cdot e^{-c_1(\omega)/2} \cdot td(\omega))^{-1} = \sum (-1)^k T_k, \quad T_k \in H^{2k}(M) .$$

4.2. Proof of the Main Theorem. First of all note the following fact which follows from Theorem 4.1.1.

Corollary 4.2.1. $\sum_{k \geqq 0} S^{-k} \chi_{\mathrm{Tr}}(\Omega \cdot T_k') = S^{-n+1} Tr$, *where* $e^{\theta/\hbar} e^{-c_1(\omega)/2} td(\omega) = \sum (-1)^k T_k'$.

Proof. One has

$$\begin{aligned}
\sum_{k \geqq 0} S^{-k} \chi_{\mathrm{Tr}}(T_k' \cdot \Omega) &= \sum_{k \geqq 0} S^{-k} \cdot T_k \cdot \widehat{\chi}_{\mathrm{Tr}}(\partial \Omega) \\
&= \sum S^{-k} \cdot T_k' \cdot \check{\chi}_{\mathrm{Tr}} \left(\sum_l T_k \cdot 1^{\bullet l+n} \right) \\
&= \sum_{k,l} S^{-k} \cdot T_k' \cdot T_k \check{\chi}_{\mathrm{Tr}}(1^{\bullet l+n}) \\
&= \sum_{k,l} S^{-k} \cdot T_k' \cdot T_l S^{1-n-l} \mathrm{Tr} = S^{1-n} \mathrm{Tr} . \qquad \square
\end{aligned}$$

Note that we have used Theorem 2.3.2 at the end.

Let $\mu : CC_*^{\mathrm{per}}(\mathbb{A}_0^\hbar(M)) \to \Omega_c^*(M)$ be the linear map such that

$$\mu(a_0 \otimes \cdots \otimes a_m) = \frac{1}{m!} a_0^{(0)} da_1^{(0)} \ldots da_m^{(0)} ,$$

where $a_i^{(0)} = \lim_{\hbar \to 0} a_i$. This is a morphism of complexes.

Theorem 4.2.2. *Let a be a periodic cyclic cycle of $\mathbb{A}_0^\hbar(M)$. Then*

i) $\langle \chi_{\mathrm{Tr}}(\Omega), a\rangle$ *has no singularity at* $\hbar = 0$, *and*

$$\lim_{\hbar \to 0} \langle \chi_{\mathrm{Tr}}(\Omega), a\rangle = (-1)^n \int_M \mu(a) ;$$

ii) $\mathrm{Tr}(a \cdot e^{-\theta/\hbar})$ *has no singularity at* $\hbar = 0$, *and*

$$\lim_{\hbar \to 0} \mathrm{Tr}(a \cdot e^{-\theta/\hbar}) = \int_M \varepsilon(\mu(a)) \cdot td(\omega) \cdot e^{-c_1(\omega)/2} ,$$

where $\varepsilon / H^k(M) = (-1)^k$.

iii) *Let a be a cycle of $CC_*^{\mathrm{per},\mathbb{C}}(\mathbb{A}^\hbar(M))$ (cf. Remark 3.1.3). Then*

$$\frac{\partial}{\partial \hbar} \mathrm{Tr}(a) = \mathrm{Tr}\left(a \cdot \frac{\partial}{\partial \hbar}(\theta/\hbar) \right) ,$$

and thus

$$\frac{\partial}{\partial \hbar} \mathrm{Tr}(a \cdot e^{-\theta/\hbar}) = 0 .$$

Proof. One has

$$\hbar^n \cdot \langle \chi_{\mathrm{Tr}}(\Omega), a\rangle = \sum_U \left\langle \chi_{\mathrm{Tr}} \left(\frac{1}{2n}(\xi_1 \wedge x_1 \wedge \cdots \wedge \xi_n \wedge x_n), I_{\rho_U} \right) a \right\rangle + * ,$$

where $*$ means a collection of similar terms which involve instead of $(\xi_1 \wedge \cdots \wedge x_n)$ some expressions of tensor degree higher than $2n$. But, because of the explicit formula for χ_{Tr} (Remark 2.3.6), any tensor factor will contribute a commutator which is divisible by $\hbar$. In particular we see, using Theorem 4.1.1. e), that $\langle \chi_{\mathrm{Tr}}(\Omega), a\rangle$ is regular at $\hbar = 0$ because the expression under χ_{Tr} contributes at least a $\hbar^{2n}$ factor. We also see that $\lim_{\hbar\to 0}\langle \chi_{\mathrm{Tr}}(\Omega), a\rangle$ depends only on a (mod $\hbar$). Let $(y_1, \ldots, y_{2n}) = (\xi_1, x_1, \ldots, \xi_n, x_n)$. The explicit formula for χ_{Tr} yields that

$$\begin{aligned}
&\lim_{\hbar \to 0} \left\langle \chi_{\mathrm{Tr}} \left(\frac{\hbar^{-n}}{2n} \xi_1 \wedge x_1 \wedge \cdots \wedge \xi_n \wedge x_n \right), a \right\rangle \\
&= \lim_{\hbar \to 0} \sum_\sigma \frac{(-1)^n}{\hbar^{2n}} \int \frac{1}{(2n)!} a_0 [y_{\sigma_1}, a_1] \ldots [y_{\sigma(2n)}, a_{2n}] dy_1 \ldots dy_{2n} \\
&= \sum_\sigma (-1)^n \int \frac{1}{(2n)!} a_0^{(0)} \{y_{\sigma_1}, a_1^{(0)}\} \ldots \{y_{\sigma(2n)}, a_{2n}^{(0)}\} dy_1 \ldots dy_{2n} \\
&= (-1)^n \int a_0^{(0)} da_1^{(0)} \ldots d\, a_{2n}^{(0)} ,
\end{aligned}$$

since $\{\xi_i, \cdot\} = \partial/\partial x_i$ and $\{x_i, \cdot\} = -\partial/\partial \xi_i$.

Now, we claim that the contribution of the summand $*$ (formula (4.2.1)) to $\lim_{\hbar\to 0}\langle\chi_{\mathrm{Tr}}(\Omega), a\rangle$ is zero. Indeed, because of Theorem 4.1.1 i), if supp $a \subset U_0 \cap \cdots \cap U_p$, this contribution is equal to

$$\pm\frac{1}{(2n+m)!}\sum_{\sigma\in S_{2n+m}}\int a_0\{X_{\sigma_1}, a_1\}\cdots\{X_{\sigma(2n+m)}, a_{2n+m}\}d\xi dx = 0$$

(where X_i are some functions on $U_0\cap\cdots\cap U_p$).

This proves part i) of Theorem 4.2.2.

Lemma 4.2.3. *The morphism* $\mu : HC_*^{\mathrm{per}}(\mathbb{A}_0^{\hbar}(M)) \to H_c^*(M)$ *commutes with the action of* $H^*(M,\mathbb{C})$.

Proof. Let $c = (c_{U_0\dots U_p}) \in \check{C}^p(M,\mathbb{C})$. Let $a = a_0\otimes\cdots\otimes a_m$. Then

$$\mu(c\cdot a) = \mu\left(\sum c_{U_0\dots U_p} I_{\underline{\rho}_{U_0}}[B+b, I_{\underline{\rho}_{U_1}}]\cdots[B+b, I_{\underline{\rho}_{U_p}}]a\right).$$

We have

$$I_{\underline{\rho}}(a_0\otimes a_1\otimes\cdots) = a_0\underline{\rho}\otimes a_1\otimes\cdots + 1\otimes\cdots;$$

the second summand is irrelevant because the value of μ at it is an exact form. Also,

$$[B+b, I_{\underline{\rho}}] = L_{\underline{\rho}} + I_{ad(\rho)} \equiv L_{\underline{\rho}}(\mathrm{mod}\,\hbar).$$

Thus

$$\mu(c\cdot a) = \mu\left(\sum c_{U_0\cdots U_p}\sum(\pm a_0\rho_{U_0}\otimes\cdots\otimes\rho_{U_{\sigma_1}}\otimes\cdots\rho_{U_{\sigma_p}}\otimes\cdots)\right)$$
$$= \frac{1}{(m+p)!}\frac{(m+p)!}{m}\sum c_{U_0\cdots U_p}\rho_{U_0}d\rho_{U_1}\cdots d\rho_{U_p}$$
$$\cdot\, a_0^{(0)}da_1^{(0)}\cdots da_m^{(0)} = c\cdot\mu(a). \qquad\square$$

We obtain from 4.2.1 and 4.2.3 that

$$\mathrm{Tr}(ae^{-\theta/\hbar}) = \langle\chi_{\mathrm{Tr}}(\Omega), a\cdot\varepsilon(td(\omega)\cdot e^{-c_1(\omega)/2})\rangle$$

and

$$\lim_{\hbar\to 0}\mathrm{Tr}(ae^{-\theta/\hbar}) = \int\varepsilon\mu(a)\cdot td(\omega)e^{-c_1(\omega)/2}.$$

This proves (4.2.2 ii). Part iii) follows from Corollary 6.5.2. $\square$

To prove Theorem 1.1.1, consider the Chern character in cyclic homology [C, K]). For an idempotent e in $M_N(\tilde{\mathbb{A}}_0^{\hbar}(M))$, put

$$ch(e) = \sum_{k\geqq 0}(-1)^k\cdot\frac{(2k)!}{k!}\mathrm{Tr}\left(\left(e-\tfrac{1}{2}\right)\otimes e^{\otimes 2k}\right).$$

Clearly, $\mu(ch(e)) = \varepsilon(ch(e_0))$ up to an exact form.

Theorem 1.1.1 follows from Theorem 4.2.2 if we apply it to $a = (ch(e) - ch(e(\infty)))\cdot\theta^m$, $m\geqq 0$.

Section 5. Gelfand–Fuks Cohomology

5.1. The Space of "Nonlinear Frames." In this subsection, we will replace the space $\mathscr{F}(M)$ of symplectic frames by the infinite dimensional manifold of "nonlinear frames." For this, we need some information about local structure of deformations.

Recall that, if U is a contractible subset of $\mathbb{R}^{2n}$ and $\omega = \sum d\xi_i \wedge dx_i$, then any deformation of $C^\infty(U)$ corresponding to ω is isomorphic to the standard Weyl deformation:

$$(f * g)(\xi, x) = \exp\left(\frac{\hbar}{2} \cdot \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial y_i} - \frac{\partial}{\partial \eta_i}\frac{\partial}{\partial x_i}\right) f(\xi, x) g(\eta, y)\Big|_{\substack{\xi_i=\eta_i \\ x_i=y_i}} . \quad (5.1.1)$$

From now on we will denote this standard deformation by $\mathbb{A}^\hbar(U)$. Let x be a point of U. By $\widehat{\mathbb{A}}^\hbar(U)_x$ we will denote the algebra of ∞-jets of elements of $\mathbb{A}^\hbar(U)$ at x. We view this as a topological algebra; the topology is given by powers of the ideal I generated by $\hbar, x_1, \dots, x_n, \ \xi_1, \dots, \xi_n$.

Proposition 5.1.1. *Let g be a continuous $\mathbb{C}[[\hbar]]$-linear automorphism of the algebra $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$. Then there exists an element $\phi = \phi_0 + \hbar\phi_1 + \cdots$ in $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$ such that*:

i) $\phi_0 \in \langle x_1, \dots, x_n, \ \xi_1, \dots, \xi_n\rangle^2$;
ii) $g(f) = e^{\frac{1}{\hbar}\phi} \cdot f \cdot e^{-\frac{1}{\hbar}\phi}$

for any $f \in \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$.

Proof. The element ϕ can be easily constructed step by step; (cf. [BFFLS and G]). Note that the condition i) means that the Hamiltonian vector field corresponding to ϕ_0 preserves the point zero.

Now, consider a symplectic manifold (M, ω) together with a deformation $\mathbb{A}^\hbar(M)$. Denote by $\mathbb{A}^\hbar(M)_m$ the topological algebra of ∞-jets of elements of $\mathbb{A}^\hbar(M)$ at the point $m \in M$. Clearly, any such algebra is isomorphic to $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$.

Put

$$\widetilde{M} = \{(m, \varphi) | m \in M; \ \varphi : \widehat{\mathbb{A}}^\hbar(M)_m \xrightarrow{\sim} \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0\},$$

where φ is a continuous isomorphism of $\mathbb{C}[[\hbar]]$-algebras and φ mod $\hbar$ is real (i.e., commutes with complex conjugation).

Consider the group $G_0 = \operatorname{Aut} \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$ of continuous $\mathbb{C}[[\hbar]]$-linear automorphisms, real mod $\hbar$. Let K be the subgroup consisting of the transformations $Ad(e^\Phi)$, $\Phi \in \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$. The correspondence $Ad(e^\Phi) \leftrightarrow \Phi$ mod $\mathbb{C}[[\hbar]] \cdot 1$ is clearly bijective. We endow K with the corresponding topology. It is easy to see that K is normal in G_0.

Now, the quotient group G_0/K is isomorphic to the group H_0 of formal symplectomorphisms of $\mathbb{R}^{2n}$. Note that $Sp(2n) \subset H_0$ and $H_0/Sp(2n)$, as a set, is in one-to-one correspondence with $\langle x_1, \dots, x_n, \ \xi_1, \dots, \xi_n\rangle^3$. Therefore there is the obvious topology on H_0 and whence on G_0. It is easy to see that G_0 becomes a Lie group with model space $\mathbb{R}^\infty$. As a manifold, G_0 is a projective limit of finite dimensional manifolds.

The symplecting group $Sp(2n)$ is a subgroup of G_0. It acts by linear symplectic coordinate changes.

Let $\mathfrak{g}_0$ be the Lie algebra of G_0;

$$\mathfrak{g}_0 = \{\phi = \phi_0 + \hbar\phi_1 + \cdots \in \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0 | \phi_0 \in \langle x_i, \xi_i\rangle^2\}/\mathbb{C}[[\hbar]] \cdot 1, \phi_0 \text{ real}\}$$

with the bracket $[\phi, \psi] = \frac{1}{\hbar}(\phi * \psi - \psi * \phi)$. This Lie algebra contains a subalgebra linearly generated by $x_i x_j$, $x_i \xi_j$, $\xi_i \xi_j$ which is the Lie algebra $\mathfrak{sp}(2n)$ of $Sp(2n)$.

It is clear that the embedding $Sp(2n) \subset G_0$ is a homotopy equivalence. Thus, one has homotopy equivalences

$$\widetilde{M}/Sp(2n) \overset{\sim}{\to} M;$$
$$\widetilde{M}/U(n) \overset{\sim}{\to} M.$$

Let $(m, \varphi) \in \widetilde{M}$. Let $\varphi = \varphi_0 + \hbar\varphi_1 + \cdots$; the lineralization of φ_0 is a symplectic isomorphism $\mathbb{R}^{2n} \overset{\sim}{\to} T_m M$. This provides us with the map $\widetilde{M} \to \mathscr{F}(M)$, where $\mathscr{F}(M)$ is the bundle of symplectic frames. This map is clearly a homotopy equivalence. Thus, we can conclude with the following.

Lemma 5.1.2. *The principal bundle $\widetilde{M} \to \widetilde{M}/Sp(2n)$ is homotopically equivalent to the principal bundle $\mathscr{F}(M) \overset{Sp(2n)}{\to} M$, where $\mathscr{F}(M)$ is the space of symplectic frames on M.*

5.2. Lie Algebra Cohomology, Cohomology of $\widetilde{M}/Sp(2n)$ and Characteristic Classes. Let $\mathfrak{g} = \{\phi \in \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0/\mathbb{C}[[\hbar]] \cdot 1 | \phi_0 \text{ real}\}$ with the bracket $[\phi, \psi] = \frac{1}{\hbar}(\phi * \psi - \psi * \phi)$. Clearly, $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{g}$.

Lemma 5.2.1. *There exists the natural action of $\mathfrak{g}$ on $\widetilde{M}$ which extends the action of $\mathfrak{g}_0$. It induces the isomorphism $\mathfrak{g} \overset{\sim}{\to} T_{(m,\varphi)}\widetilde{M}$ for any point (m, φ) of $\widetilde{M}$.*

Proof. Consider a point $m \in M$ and an open neighborhood U_m of m. Consider an ∞-jet of a symplectomorphism $g_0 : V \overset{\sim}{\to} U_m$, where V is an open neighborhood of 0 in $\mathbb{R}^{2n}$ and $g_0(0) = m$. Consider a one parameter family of ∞-jets of symplectomorphisms $h_t : W_0 \overset{\sim}{\to} V_0$, where W_0 is an open neighborhood of 0 in $\mathbb{R}^{2n}$, $h_0 = id$ locally and $h_t(z_t) = 0$ for some $z_t \in W_0$. Assume that we have continuous isomorphisms

$$\widehat{\mathbb{A}}^\hbar(U_m)_m \overset{\varphi}{\to} \widehat{\mathbb{A}}^\hbar(V_0)_0 \overset{\varphi_t}{\to} \widehat{\mathbb{A}}^\hbar(W_0)_{z_t}$$

such that $\varphi(f) = f \cdot g_0 + \hbar \ldots$, $\psi_t(f) = f \cdot h_t + \hbar \ldots$, $\psi_0 = id$.

The set G of continuous isomorphisms $\widetilde{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0 \overset{\sim}{\to} \widetilde{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_z$, where z varies, has obvious smooth structure (in fact $G = \widetilde{R^{2n}}$). Assume that $\{\psi_t\}$ is a smooth family. Then $\frac{\partial}{\partial t}|_{t=0}(\psi_t \cdot \varphi)$ defines a tangent vector to M at the point (m, φ). Clearly, this vector depends only on $\frac{\partial}{\partial t}\Big|_{t=0} \varphi_t$. It is also clear that the tangent space to G is equal to $\mathfrak{g}$. Thus, we have a homomorphism $\alpha : \mathfrak{g} \to \operatorname{Vect} \widetilde{M}$. It is easy to see that this homomorphism is in fact a Lie algebra homomorphism and that $\mathfrak{g} \overset{\alpha}{\underset{\sim}{\to}} T_{(m,\varphi)}\widetilde{M}$; this is because the map $\psi \mapsto (m, \psi \circ \varphi)$ is in fact a local diffeomorphism between G and $\widetilde{M}$. $\square$

Let $C^*_{\rm cont}(\mathfrak{g})$ be the complex of continuous cochains of Lie algebra $\mathfrak{g}$ with coefficients in the trivial module $\mathbb{C}$. We define a homomorphism

$$\mu : C^*_{\rm cont}(\mathfrak{g}) \to \Omega^*(\widetilde{M})\,; \tag{5.2.1}$$

if $\eta \in C^n_{\rm cont}(\mathfrak{g})$ and $X_1, \ldots, X_n \in T_{(m,\varphi)}\widetilde{M}$, then

$$(\mu(\eta))(X_1, \ldots, X_n) = \eta(\alpha^{-1}X_1, \ldots, \alpha^{-1}X_n)\,, \tag{5.2.2}$$

where $\alpha : \mathfrak{g} \to \mathrm{Vect}\,\widetilde{M}$ was defined above. Clearly, μ is a map of complexes. Let $U(n)$ be the maximal compact subgroup of $Sp(2n) \subset G_0$, then μ descends to a map

$$\mu : C^*_{\rm cont}(\mathfrak{g}, \mathfrak{u}(n); \mathbb{C}) \to \Omega^*(\widetilde{M}/U(n)) \tag{5.2.3}$$

which induces a homomorphism

$$\mu : H^*(\mathfrak{g}, \mathfrak{u}(n); \mathbb{C}) \to H^*(M, \mathbb{C}) \tag{5.2.4}$$

(because $\widetilde{M}/U(n)$ is homotopically equivalent to M).

Now recall the definition of the Chern classes in relative Lie algebra cohomology ([Fu]). Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a reductive subalgebra such that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is reductive. Then there exists an $\mathfrak{h}$-equivariant splitting $\nabla : \mathfrak{g} \to \mathfrak{h}$. Put

$$R(X, Y) = \nabla([X, Y]) - [\nabla(X), \nabla(Y)];$$

this is an $\mathfrak{h}$-valued two-cochain of $\mathfrak{g}$. For any invariant polynomial $P \in S^k[\mathfrak{h}']^{\mathfrak{h}}$ put

$$c_P = P(R) \in C^{2k}(\mathfrak{g}, \mathfrak{h}; \mathbb{C})\,.$$

It is not hard to show that c_P is a cocycle and its relative cohomology class does not depend on ∇. If $\mathfrak{h} = \mathfrak{u}(n)$ then we get, using the polynomials $P_k = \mathrm{Tr} \wedge^k X$, the Chern classes $c_k \in H^{2k}(\mathfrak{g}, \mathfrak{u}(n))$.

Lemma 5.2.2. $\mu(c_k) = c_k(\omega) \in H^{2k}(M, \mathbb{C})$.

Proof. Indeed, a splitting $\nabla : \mathfrak{g} \to \mathfrak{u}(n)$ provides a connection in the principal bundle $\widetilde{M} \to \widetilde{M}/U(n)$; the forms $\mu(c_k)$ are the Chern–Weyl forms of this connection. But this principal bundle is homotopically equivalent to the bundle of symplectic frames. □

Now, let θ_0 be the 2-cocycle corresponding to the central extension

$$0 \to \mathbb{R} + \hbar\mathbb{C}[[\hbar]] \to \widehat{\mathbb{A}}^\hbar \to \mathfrak{g} \to 0\,.$$

Put $\theta = \mu(\theta_0)$; $\theta \in H^2(M, \mathbb{C}[[\hbar]])$.

5.3. Fundamental Class in Lie Algebra Cohomology. Consider the double complexes

$$C_{-p,q}(\mathfrak{g}, \mathfrak{u}(n); \widehat{\mathbb{A}}^\hbar) = C^p(\mathfrak{g}, \mathfrak{u}(n); \overline{CC}_q(\widehat{\mathbb{A}}^\hbar)(\mathbb{R}^{2n}))$$

and

$$C_{-p,q}(\mathfrak{g}, \mathfrak{u}(n); (\widehat{\mathbb{A}}^\hbar[\hbar^{-1}]) = C_{-p,q} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}[\hbar^{-1}, \hbar]]$$

with the differentials $\partial^{\mathrm{Lie}} : C_{-p,q} \to C_{-p-1,q}$, and $b : C_{-p,q} \to C_{-p,q-1}$. Let $C_m = \prod_{q-p=m} C_{-p,q}$ be the complex with differential $b + (-1)^q \partial^{\mathrm{Lie}}$. We denote its homology by $\mathscr{H}_*(\mathfrak{g}, u(n); \widehat{\mathbb{A}}^\hbar)$.

As in 3.1, one defines the boundary map

$$\mathscr{H}_{m-1}(\mathfrak{g}, \mathfrak{u}(n); \widehat{\mathbb{A}}^\hbar[\hbar^{-1}]) \xrightarrow{\partial} \bigoplus_{p \geqq \frac{m}{2}} H^{2p-m}(\mathfrak{g}, \mathfrak{u}(n); k) \cdot 1^{\bullet p},$$

$k = \mathbb{C}[\hbar^{-1}, \hbar]])$. By the same spectral sequence argument as in 3.2, $\mathscr{H}_i(\mathfrak{g}, \mathfrak{u}(n); \widehat{\mathbb{A}}^\hbar [\hbar^{-1}]) = 0$ for $i \geqq 2n$ and is one-dimensional for $i = 2n-1$. Denote by Ω_L the generator of $\mathscr{H}_{2n-1}$ for which $\partial \Omega_L = 1^{\bullet n} + \cdots$.

Section 6. Proof of Theorem 4.1.1

6.1. Reduction to Lie Algebra Cohomology. Our aim is to reduce the computation of $\partial\Omega$ in Čech cohomology to the computation in Lie algebra cohomology. To do this, we, as usual, introduce the third complex which involves both Lie and Čech structures and serves as a link. Put

$$\mathscr{D}(M, \overline{CC}_*(\mathbb{A}^\hbar)) = \check{C}^*(M, C^*(\mathfrak{g}, \mathfrak{u}(n);\ C^\infty(\widetilde{M}, \overline{CC}_*(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})[\hbar^{-1}])))\,.$$

More precisely, $\mathscr{D}$ is the Čech complex of M with coefficients in the sheaf whose space of sections over U is the Lie algebra cochain complex of $\mathfrak{g}$ relatively to $\mathfrak{u}(n)$ with coefficients in $\mathfrak{g}$-module of C^∞ maps from $\widetilde{U}$ to $\overline{CC_*}(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0)$. Note that $\mathfrak{g}$ acts on $\widetilde{U}$ (this action was defined in Sect. 5), and $\mathfrak{g}$ acts on $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0$ by derivations. Therefore $\mathfrak{g}$ acts on $C^\infty(\widetilde{U}, \overline{CC_*}(\widehat{\mathbb{A}}(\mathbb{R}^{2n})_0))$; the space of $\mathfrak{g}$-invariant sections coincides with $\overline{CC_*}(\mathbb{A}^\hbar(M))$.

This last remark shows that there is a well defined morphism of complexes

$$\rho : \check{C}^*(M, \overline{CC_*}(\mathbb{A}^\hbar)[\hbar^{-1}]) \to \mathscr{D}(M, \overline{CC}_*(\mathbb{A}^\hbar))\,.$$

This morphism is induced by the following morphism of sheaves: a section of s_U of $\overline{CC_*}(\mathbb{A}^\hbar(U))$ maps to a section $((m, \varphi) \mapsto$ (image of $\mathrm{jet}_\infty(s_U)_m$ under φ)) which is a closed cochain in $C^0(\mathfrak{g}, \mathfrak{u}(n); C^\infty(\widetilde{U}, \overline{CC_*}(\mathbb{A}^\hbar(\mathbb{R}^{2n})_0)[\hbar^{-1}]))$. On the other hand, one has the morphism of complexes $\mu' : C^*(\mathfrak{g}, \mathfrak{u}(n); \overline{CC_*}(\mathbb{A}^\hbar(\mathbb{R}^{2n})_0[\hbar^{-1}])) \to \mathscr{D}(M, \overline{CC_*}(\mathbb{A}^\hbar))$.

The morphism μ' is induced by the map which sends an element $\eta \in \overline{CC_*}(\widehat{\mathbb{A}}^\hbar, (\mathbb{R}^{2n})_0)$ to the constant function on $\widetilde{M}$ with value η.

Similarly, we can construct the complex $\mathscr{D}(M, CC_*(k))$, $k = \mathbb{C}[\hbar^{-1}, \hbar]]$, and the morphisms μ' and ρ. We get a diagram of complexes

$$\begin{array}{ccccc}
C^*(\mathfrak{g}, \mathfrak{u}(n); \overline{CC}_{*-1}(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0[\hbar^{-1}])) & \xrightarrow{\mu'} & \mathscr{D}(M, \overline{CC}_{*-1}(\mathbb{A}^\hbar)) & \xrightarrow{\rho} & \check{C}^*(M, \overline{CC}_{*-1}[\hbar^{-1}]) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
C^*(\mathfrak{g}, \mathfrak{u}(n); CC_{*-2}(k)) & \xrightarrow{\mu'} & \mathscr{D}(M, CC_{*-2}(k)) & \xrightarrow{\rho} & \check{C}^*(M, CC_{*-2}(k))
\end{array}$$

Note that ρ on the bottom of the diagram is a quasi-isomorphism. Indeed, $C^*(\mathfrak{g}, \mathfrak{u}(n); C^\infty(\widetilde{U}, \mathbb{C}))$ is isomorphic to de Rham complex $\Omega^*(\widetilde{U}/U(n))$. But $\widetilde{U}/U(n)$ is contractible; therefore the sheaf of complexes $U \mapsto \Omega^*(\widetilde{U}/U(n))$ is quasi-isomorphic to the constant sheaf $U \mapsto \mathbb{C}$.

The same spectral sequence argument as in 3.2 shows that there exists a unique fundamental class $\Omega_{\mathscr{G}}$ in $H_{2n-1}(\mathscr{D})$ such that $\partial\Omega = 1^{\bullet n} + \cdots$ and a unique fundamental class Ω_L in $\mathbb{H}_{2n-1}(C^*(\mathfrak{g}, \mathfrak{u}(n); \overline{CC}_{*-1}))$ with the same property. Clearly, $\mu(\Omega_L) = \rho(\Omega) = \Omega_{\mathscr{G}}$.

Therefore $\partial\Omega = \rho^{-1}\mu'(\partial\Omega_L)$; it is easy to see from definitions that $\rho^{-1}\mu'$ induces the same map on cohomology as $\mu : H^*(\mathfrak{g}, \mathfrak{u}(n)) \to H^*(M, \mathbb{C})$ from Sect. 5. Therefore we have reduced the proof of Theorem 4.1.1 to the following.

Theorem 6.1.1. i) *the fundamental class Ω_L may be represented by a cochain in $\sum_{i \geqq 0} \hbar^{-(i+n)} C^i(\mathfrak{g}, u(n); \overline{CC}_{2n+i-1}(\widehat{\mathbb{A}}^\hbar))$; modulo $\hbar \sum_{i \geqq 0} \hbar^{-(i+n)} C^i(\mathfrak{g}, u(n); \overline{CC}_{2n+i-1})$, this chain may be chosen in $\sum_{i \geqq 0} \hbar^{-(i+n)} C^i(\mathfrak{g}, u(n); \wedge^{2n+i} \widehat{\mathbb{A}}^\hbar)$;*

ii) $\partial\Omega_L = \sum_{k=0}^\infty T_k \cdot 1^{\bullet(n+k)}$,

where $T_k \in H^{2k}(\mathfrak{g}, \mathfrak{u}(n); \mathbb{C}[[\hbar]])$ and

$$e^{-\frac{1}{\hbar}\theta_0 + c_1/2} \cdot td^{-1}(c_1, \ldots, c_n) = \sum_k (-1)^k T_k \,.$$

The proof will occupy the next three subsections. Part ii) will be proved in 6.2–6.3; part i) will be proved in 6.4.

6.2. Computation of $\partial\Omega_L$; Reduction to One-Dimensional Case. First of all, consider the Lie subalgebra $\mathfrak{g}l(n)$ in $\mathfrak{g}$: the matrix (a_{ij}) is represented by the element $\sum a_{ij} x_i \xi_j$. It is clear that $\mathfrak{g}l(n)_{\mathbb{C}}$ is conjugate to $\mathfrak{u}(n)_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ by a matrix from $GL(2n, \mathbb{C})$. Thus it is enough to prove that, for the fundamental class $\Omega_L \in \mathbb{H}^{2n-1}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}l(n)_{\mathbb{C}}; \overline{CC}_*(\widehat{\mathbb{A}}^\hbar))$,

$$\partial\Omega_L = \sum_k T_k \cdot 1^{\bullet n+k} \,,$$

where $T_k \cdot (-1)^k$ are H^{2k} components of the class $td^{-1} \cdot e^{-\frac{1}{\hbar}\theta_0} \cdot e^{c_1/2}$ constructed from the Chern classes $c_1, \ldots, c_n$ corresponding to $\mathfrak{g}l(n)_{\mathbb{C}}$.

We shall denote the subalgebra $\mathfrak{g}l(n)_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ by $\mathfrak{h}_n$, or simply by $\mathfrak{h}$.

Our first step is to enlarge the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ acting on $\overline{CC}_*(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})_0)$. Let $\widehat{\mathfrak{g}}_{\mathbb{C}}$ be the Lie algebra $\{\phi \in \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n}) | \phi \bmod \hbar \text{ is real }\}_{\mathbb{C}}$. One has the central extension $0 \to (\mathbb{R} + \hbar\mathbb{C}[[\hbar]])_{\mathbb{C}} \to \widehat{\mathfrak{g}}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} \to 0$. Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C} \cdot 1$. Clearly, $C^*(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}; M) = C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}, \widehat{\mathfrak{h}}; M)$ for any $\mathfrak{g}_{\mathbb{C}}$-module M. Let $\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon]$ be as in Remark 2.1.9. Then $\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon]$ acts on $\overline{CC}_*$ as follows:

$$L_D(a_0 \otimes \cdots \otimes a_n) = \frac{1}{\hbar} \sum_{i=0}^n (a_0 \otimes \cdots \otimes [D, a_i] \otimes \cdots \otimes a_n);$$

$$L_{\varepsilon D}(a_0 \otimes \cdots \otimes a_n) = \frac{1}{\hbar} \sum_{i=0}^n (-1)^i (a_0 \otimes \cdots \otimes a_i \otimes D \otimes a_{i+1} \otimes \cdots \otimes a_n) \,.$$

One checks (cf. Appendix 1) that these operators indeed define an action of the differential graded algebra $\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon]$.

We will need some information about relative cohomology of $\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon]$. Let $\mathfrak{g}$ be any Lie algebra, $\mathfrak{h}$ – a reductive subalgebra of $\mathfrak{g}$ such that $ad \mid \mathfrak{h}$ is semisimple on $\mathfrak{g}$. Consider the differential graded Lie algebra $\mathfrak{g}[\varepsilon]$, as above. For any invariant polynomial $P \in S^k[\mathfrak{h}']^{\mathfrak{h}}$ define the characteristic cocyle $c_P = P((\partial^{\mathrm{Lie}} + \delta)\nabla + \nabla^2)$ in $C^{2k}(\mathfrak{g}[\varepsilon], \hbar; \mathbb{C})$, where $\nabla : \mathfrak{g}[\varepsilon] \to \mathfrak{h}$ is an $\mathfrak{h}$-equivariant splitting (such that $\nabla(\mathfrak{g}\varepsilon) = 0$). Recall the following proposition (cf., for example, [Na]).

Lemma 6.2.1. *The map* $S^*[\mathfrak{h}']^{\mathfrak{h}} \to H^{2*}(\mathfrak{g}[\varepsilon], \mathfrak{h})$, $P \mapsto c_P$, *is an isomorphism*; *also*, $H^{2*+1}(\mathfrak{g}[\varepsilon], \mathfrak{h}) = 0$.

For the readers' convenience let us sketch the proof. Since $(\mathfrak{g}[\varepsilon], \delta)$ is contractible as a complex, the cohomology of the differential induced by δ on $C^*(\mathfrak{g}[\varepsilon], \mathfrak{h})$ is isomorphic to $S^*[\mathfrak{h}']^{\mathfrak{h}}$. The isomorphism sends $\wedge^P \mathfrak{g} \otimes \wedge^q \mathfrak{g}\varepsilon$ to zero for $p > 0$; for $p = 0$ it sends an element of $\wedge^q(\varepsilon\mathfrak{g}')^{\mathfrak{h}} = S^q(\mathfrak{g}')^{\mathfrak{h}}$ to its restriction to $\mathfrak{h}$. Thus, the spectral sequence of the double complex $(C^*, \partial^{\mathrm{Lie}}, \delta)$ collapses. □

We now have a slight technical difficulty because the action of $\varepsilon\widehat{\mathfrak{g}}_{\mathbb{C}}$ does not preserve the subcomplex $CC_*(k)$. It preserves, however, the subcomplex $CC_*^{(1)}(\widehat{\mathbb{A}}^{\hbar})$ which is the linear span of tensors $a_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_n$. Note that $CC_*/CC_*^{(1)} = \overline{CC_*}$. Thus one can define a boundary map

$$\partial : C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon], \widehat{\mathfrak{h}}; \overline{CC_*}(\widehat{\mathbb{A}}^{\hbar})[\hbar^{-1}]) \to C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon], \widehat{\mathfrak{h}}; CC_{*-1}^{(1)}(\widehat{\mathbb{A}}^{\hbar})[\hbar^{-1}]) .$$

Lemma 6.2.2. *There exists a quasi-isomorphism*

$$C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon], \widehat{\mathfrak{h}}; CC_*(k)) \xrightarrow{\alpha} C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon], \widehat{\mathfrak{h}}; CC^{(1)}(\widehat{\mathbb{A}}^{\hbar})) \tag{6.2.1}$$

which extends the obvious embedding

$$C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}, \widehat{\mathfrak{h}}; CC_*(k)) \hookrightarrow C^*(\widehat{\mathfrak{g}}_{\mathbb{C}}, \widehat{\mathfrak{h}}; CC^{(1)}(\widehat{\mathbb{A}}^{\hbar})) . \tag{6.2.2}$$

Proof. Consider the spectral sequence of the double complex $(\partial^{\mathrm{Lie}} + \delta; b)$. One has $E^2_{-p,q} = H^p(\widehat{\mathfrak{g}}_{\mathbb{C}}, \widehat{\mathfrak{h}}; \mathbb{C}) \otimes_{\mathbb{C}} CC_q(k)$ for both complexes in (6.2.1), resp. $H^p(\widehat{\mathfrak{g}}_{\mathbb{C}}, \widehat{\mathfrak{h}}; \mathbb{C}) \otimes_{\mathbb{C}} CC_q(k)$ for both complexes in (6.2.2). Moreover, since $H^{\mathrm{odd}}(\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon], \widehat{\mathfrak{h}}) = 0$, the spectral sequences for (6.2.1) degenerate at E^2. This implies the statement. □

By the same argument as above, one can extend the fundamental class Ω_L to the class in $\mathbb{H}_{2n-1}(\widehat{\mathfrak{g}}_{\mathbb{C}}, \widehat{\mathfrak{h}}; \overline{CC_*}(\widehat{\mathbb{A}}^{\hbar})[\hbar^{-1}])$. To prove 6.1.1. ii), it suffices to show that

$$\alpha^{-1}(\partial\Omega_L) = \sum T_k \cdot 1^{\bullet n+k} , \tag{6.2.3}$$

where α was constructed in Lemma 6.1.3.

Now one can see the advantage of passing to $\widehat{\mathfrak{g}}_{\mathbb{C}}[\varepsilon]$ from $\mathfrak{g}_{\mathbb{C}}$. Indeed, by virtue of Lemma 6.1.2 and by uniqueness of the fundamental class, it suffices to prove (6.2.3) if Ω_L is the fundamental class in $\mathbb{H}_{2n-1}(\widehat{\mathfrak{h}}[\varepsilon], \widehat{\mathfrak{h}}; CC_*(\widehat{\mathbb{A}}^{\hbar})[\hbar^{-1}])$ and the right-hand side is in $\mathbb{H}_*(\widehat{\mathfrak{h}}[\varepsilon], \widehat{\mathfrak{h}}; CC_*(k))$.

Now we can make further reduction. Let $\mathfrak{d}_n \subset \mathfrak{h}_n$ be the subalgebra of diagonal matrices. Let $\widehat{\mathfrak{d}}_n = \mathfrak{d}_n \oplus \mathbb{C} \cdot 1$. By Lemma 6.2.2, the restriction homomorphism $C^*(\widehat{\mathfrak{h}}[\varepsilon], \widehat{\mathfrak{h}}; CC_{*-1}(k)) \to C^*(\widehat{\mathfrak{d}}[\varepsilon], \widehat{\mathfrak{d}}; CC_{*-1}(k))$ is a monomorphism (any invariant polynomial on $\mathfrak{h}$ is determined by its restriction to $\mathfrak{d}$).

Our next step is to reduce the problem to the case $n = 1$. Note that the products $\smile$ and $\bullet$ on cyclic complexes (cf. Subsect. 2.3) extend to hypercohomology (one can combine them with the standard multiplication of the Lie algebra cochains); $\partial(a \smile b) = \partial a \bullet \partial b$. Denote $\mathfrak{gl}(1)_{\mathbb{C}}$ by $\mathfrak{d}_1$; then $\mathfrak{d} = \mathfrak{d}_n = \mathfrak{d}_1^{\oplus n}$. In fact the algebra $\widehat{\mathfrak{d}}_1^{\oplus n}[\varepsilon]$ acts on $\overline{CC_*}(\widehat{\mathbb{A}}^\hbar)$ (via the homomorphism $\widehat{\mathfrak{d}}_1^{\oplus n}[\varepsilon] \to \widehat{\mathfrak{d}}_n[\varepsilon]$; $(X_i + \lambda_i 1) \mapsto (x_1, \ldots, x_n, \sum \lambda_i \cdot 1))$. Denote by $\Omega_L(1)$ the fundamental class in $\mathbb{H}_{2n-1}(\widehat{\mathfrak{d}}_1[\varepsilon], \widehat{\mathfrak{d}}_1; (\widehat{\mathbb{A}}^\hbar)(\mathbb{R}^1)_0[\hbar^{-1}])$. Then, by the multiplicativity formula above and by uniqueness of the fundamental class, one has $\Omega_L = \Omega_L(1)^{\smile n}$; if we know that

$$\partial\Omega_L(1) = 1^{\bullet 1} \bullet \sum T_k \cdot 1^{\bullet k} , \tag{6.2.4}$$

then

$$\partial\Omega_L = (\partial\Omega_L(1))^{\bullet n} = 1^{\bullet n} \bullet \sum (T_k \cdot 1^{\bullet k})^{\bullet n} ,$$

and Lemma 6.2.3. ii) follows from (6.1.5) because of multiplicativity of our cohomology class.

6.3. Proof for $n = 1$. First, compute $\partial\Omega_L$ restricted to the subalgebra $\mathbb{C} \cdot (x\xi - \frac{1}{2}) = \tilde{\mathfrak{d}}_1$. In fact our computation will be carried out over the following subring D_1 of $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^1)_0$. Let $\partial_x = \frac{1}{\hbar}\xi$; then put $D_1 = \mathbb{C}[x, \partial_x]$. Let us examine the complex $\check{C}^*(\tilde{\mathfrak{d}}_1[\varepsilon], \tilde{\mathfrak{d}}_1; \overline{CC_*}(D_1))$ in more detail.

First of all, by definition, the first Chern class c_1 is represented by the following cochain: $c_1(\varepsilon x \partial_x) = 1$. Thus $c_1^n(\varepsilon x\partial_x, \ldots, \varepsilon x \partial_x) = n!$. Therefore, for a chain η of $\overline{CC_*}(D_1)$,

$$\partial^{\mathrm{Lie}}(\eta) = c_1 \cdot L_{\varepsilon x \partial_x} \eta ;$$

$$\partial^{\mathrm{Lie}}(c_1^n \cdot \eta) = c_1^{n+1} L_{\varepsilon x \partial_x} \eta .$$

Thus, the complex $C^*(\tilde{\mathfrak{d}}_1[\varepsilon], \tilde{\mathfrak{d}}_1; \overline{CC_*}(D_1))$ is of the following form:

$$\begin{array}{ccccccccc}
c_1^2\overline{CC}_0 & \overset{b}{\leftarrow} & c_1^2\overline{CC}_1 & \overset{b}{\leftarrow} & c_1^2\overline{CC}_2 & \overset{b}{\leftarrow} & c_1^2\overline{CC}_3 & \overset{b}{\leftarrow} & \cdots\cdots \\
 & & \uparrow L & & \uparrow L & & \uparrow L & & \\
 & & c_1\overline{CC}_0 & \overset{b}{\leftarrow} & c_1\overline{CC}_1 & \overset{b}{\leftarrow} & c_1\overline{CC}_2 & \overset{b}{\leftarrow} & c_1\overline{CC}_3 \quad \cdots \\
 & & & & \uparrow L & & \uparrow L & & \uparrow L \\
 & & & & \overline{CC}_0 & \overset{b}{\leftarrow} & \overline{CC}_1 & \overset{b}{\leftarrow} & \overline{CC}_2 \quad \cdots
\end{array}$$

where $\overline{CC}_i = \overline{CC}_i(D_1)$ and $L = L_{\varepsilon x \partial_x}$.

Lemma 6.3.1. *The fundamental class $\Omega(1)$ is represented by the following chain*:

$$\Omega(1) = -\sum_n \frac{1}{n} c_1^{n-1}(x \otimes \partial_x)^{\otimes n} . \tag{6.3.1}$$

Proof. One checks that

$$(b + c_1 L)\Omega(1) = \sum_{n \geqq 0} c_1^n \cdot 1 \otimes (\partial_x \otimes x)^{\otimes n} , \tag{6.3.2}$$

where $\Omega(1)$ is the right-hand side of (6.3.1). But the right-hand side of (6.3.2) is in the subcomplex $\overline{CC}^{(1)}(D_1)$.

Now, a simple calculation shows that, in the double complex above,

$$c_1^n \cdot 1 \otimes (\partial_x \otimes x)^{\otimes n} \sim c_1^n \frac{(2n)!}{n!(n+1)!} 1^{\otimes(2n+1)}$$

$$= \frac{1}{(n+1)!} \cdot c_1^n \cdot 1^{\bullet n+1};$$

thus,

$$\partial\Omega(1) = \sum \frac{c_1^n}{(n+1)!} \cdot 1^{\bullet n+1};$$

comparing this with

$$td^{-1} = \frac{1-e^{-c_1}}{c_1} = \sum \frac{(-1)^n c_1^n}{(n+1)!}. \qquad \square$$

To complete the computation of $\partial\Omega_L(1)$, consider the algebra $\widehat{\mathfrak{d}}_1$. We view the element $\alpha^{-1}(\partial\Omega_L(1))$ as a power series on $\widehat{\mathfrak{d}}_1$ with values in $CC_*(k)$. (Note that α is defined in Lemma 6.1.3.) Our statement will follow from

Lemma 6.3.2. *For any $\lambda \in \mathbb{C}$ and $X \in \mathfrak{d}_1$,*

$$\alpha^{-1}\partial\Omega_L(1)(X + Y \cdot 1) = \alpha^{-1}\partial\Omega_L(1)(X) \bullet \sum_{m \geqq 0} \frac{\lambda^m}{m!} 1^{\bullet m}.$$

Proof. Consider the following isomorphisms in derived category:

$$C^*(\widehat{\mathfrak{d}}_1[\varepsilon], \widehat{\mathfrak{d}}_1; CC_{*-1}(k)) \xrightarrow{\gamma} C^*(\widehat{\mathfrak{d}}_1[\varepsilon], \widehat{\mathfrak{d}}_1; CC_*^{(1)}(\widehat{\mathbb{A}}^\hbar))$$

$$\xrightarrow{i} C^*(\widehat{\mathfrak{d}}_1[\varepsilon], \widehat{\mathfrak{d}}_1; CC_*^{(1)}(\widehat{\mathbb{A}}^\hbar[\eta])) \xleftarrow{\mathfrak{d}} C^*(\widehat{\mathfrak{d}}_1[\varepsilon], \widehat{\mathfrak{d}}_1; \overline{CC_*}(\widehat{\mathbb{A}}[\eta])).$$

One can define the composition $\partial^{-1} \cdot i \cdot \alpha$ explicitly:

$$\frac{1}{k!}(\partial^{-1} \cdot i \cdot \alpha)(1^{\bullet m})(\underbrace{\varepsilon X, \ldots, \varepsilon X}_{k \text{ times}}) = \frac{(-1)^k}{k!} L^k_{\underline{\eta X}}((m-1)!\eta^{\otimes m})$$

$$= \frac{(-1)^k}{k!} L^k_{\underline{\eta X \hbar^{m-1}}} L^{m-1}_{\underline{\eta}}(\eta).$$

Now, to define $\partial^{-1} i(\partial\Omega_L(1))$, one has to define it on the algebra $\mathfrak{d}_1$ and extend it trivially to $\widehat{\mathfrak{d}}_1$. The lemma follows easily. $\square$

6.4. Proof of Theorem 6.1.1 i). Let us construct the fundamental class Ω_L explicitly. Let $C_*^{(1)} = C^*(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h};\ \mathrm{End}\,\overline{CC}_*(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})))$, where $\mathfrak{g}_{\mathbb{C}}$-module structure on $\mathrm{End}\,\overline{CC}_*$ is given by multiplication by L_D from the left. The differential in $C_*^{(1)}$ is given by $B + b + \partial^{\mathrm{Lie}}$. Let $C_*^{(2)} = C^*(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}; \overline{CC}_*(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})))$, where $\mathfrak{g}_{\mathbb{C}}$-module structure on $\overline{CC}_*$ is trivial. The differential in $C_*^{(2)}$ is denoted by $B + b + \partial^{\mathrm{Lie}}$. Multiplication of

Lie algebra cochains and action of End $\overline{CC}_*$ on $\overline{CC}_*$ induce the multiplication

$$C_*^{(1)} \otimes C_*^{(2)} \longrightarrow C_*^{(2)} .$$

Consider the cochain

$$\alpha = \alpha_0 + \alpha_1 + \cdots$$

of $C_*^{(1)}$; $\alpha_m(X_1, \ldots, X_m) = L_{\underline{X_1 - \nabla X_1}} \ldots L_{\underline{X_m - \nabla X_m}}$.

Let $\Omega_L^{(0)}$ be the fundamental class in $C^*(\mathfrak{h}[\varepsilon], \mathfrak{h}; \overline{CC}_*(\widehat{\mathbb{A}}^\hbar))$. Let $R(X, Y)$ be the curvature form. Let εR be the two-cochain $X \wedge Y \mapsto \varepsilon R(X, Y)$. Consider the $\wedge^*(\varepsilon\mathfrak{h})$-valued cochain $\sum_{m \geqq 0} \frac{(\varepsilon R)^m}{m!}$. Put

$$\beta = \Omega_L^{(0)} \left(\sum_{m \geqq 0} \frac{(\varepsilon R)^n}{m!} \right) .$$

This is a cochain of $C_*^{(2)}$.

Proposition 6.4.1. *The cochain $\alpha \cdot \beta$ represents the fundamental class Ω_L.*

Proof. Clearly, the lowest component of $\alpha \cdot \beta$ is equal to $\frac{1}{2n \cdot \hbar^n}(\xi_1 \wedge x_1 \wedge \cdots \wedge \xi_n \wedge x_n)$. It remains to show that $\alpha \cdot \beta$ is a cocycle. Let L_∇ and $L_{\underline{R}}$ be cochains of $C_*^{(1)}$: $L_\nabla(X) = L_{\nabla(X)}$ and $L_{\underline{R}(X,Y)} = L_{\underline{R}(X,Y)}$. It is easy to check that

$$(B + b + \partial^{\mathrm{Lie}})\alpha = \alpha \cdot (-L_\nabla + L_{\underline{R}});$$
$$(B + b + \partial_0^{\mathrm{Lie}})\beta = (L_\nabla - L_{\underline{R}})\beta$$

(we use the Bianchi identity $\partial_0^{\mathrm{Lie}} R = [\nabla, R]$).

Therefore $\alpha \cdot \beta$ is a cocycle and represents the class Ω_L. Finally, it is easy to check (compare to 6.3.1) that $\Omega_L^{(0)}$ may be represented by a cochain from $\sum_l C^{2l}(\mathfrak{h}[\varepsilon], \mathfrak{h}; \hbar^{-l-n}\overline{CC}_{2(l+n)})$. $\square$

Corollary 6.4.2. *The fundamental class Ω_L (resp. Ω) may be represented by a cochain in $C^*(\mathfrak{g}, \mathfrak{u}(n); \overline{CC}_*^{(\mathbb{C})}(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})))$, resp. $\check{C}^*(M, \overline{CC}_*^{(\mathbb{C})}(\widehat{\mathbb{A}}^\hbar))$.*

6.5. One-Dimensional Families. Let $\mathbb{R}_+ = \{\lambda \in \mathbb{R} | \lambda > 0\}$. Let $\mathbb{A}^{\hbar,\lambda}(M \times \mathbb{R}_+)$ be the deformation of $C^\infty(M \times \mathbb{R}_+)$ given by

$$f * g = \sum_n (\hbar\lambda)^n \varphi_n(f, g) .$$

If U is a coordinate neighborhood in M such that $\mathbb{A}^\hbar(U)$ is isomorphic to the Weyl deformation then $\mathbb{A}^{\hbar,\lambda}(U \times \mathbb{R}_+) \xrightarrow{\sim} \mathbb{A}^\hbar(U) \widehat{\otimes} C^\infty(\mathbb{R}_+)$. The isomorphism is given by $f(x, \xi) \mapsto f(x, \lambda\xi)$. Let $\widetilde{M \times \mathbb{R}_+}$ be the space of pairs (x, φ), where x is a point of $M \times \mathbb{R}_+$ and φ is an ∞-jet of an isomorphism $\mathbb{A}^{\hbar,\lambda}(U_x) \xrightarrow{\sim} \widehat{\mathbb{A}}^\hbar(\mathbb{R}^2)[[\lambda]]$. The group G_0^λ of automorphisms of $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^2)[[\lambda]]$ acts on $\widetilde{M \times \mathbb{R}_+}$; it contains a subgroup $O(1) \times U(n)$ (the generator of $O(1)$ sends λ to $-\lambda$), and $\widetilde{M \times \mathbb{R}_+}/O(1) \times U(n)$

is homotopically equivalent to $M \times \mathbb{R}_+$. Let $\mathfrak{g}_0^\lambda$ be the Lie algebra of G_0^λ. As in 5.2.1, one can extend the action of $\mathfrak{g}_0^\lambda$ to the Lie algebra $\mathfrak{g}^\lambda$:

$$\mathfrak{g}_{\mathbb{C}}^\lambda = W_1 \ltimes (\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})[[\lambda]]/\mathbb{C}[[\lambda]] \cdot 1),$$

where $W_1 = \{f(\lambda)\frac{\partial}{\partial\lambda} | f(\lambda) \in \mathbb{C}[[\lambda]]\}$ and the bracket on the factor on the right is equal to $\frac{1}{\hbar}(f * g - g * f)$. One can construct the morphisms

$$\mu : C^*(\mathfrak{g}_{\mathbb{C}}^\lambda,\ O(1) \times \mathfrak{u}(n)_{\mathbb{C}}; \mathbb{C}) \to C^*(M, \mathbb{C}); \tag{6.5.1}$$

$$\mu' : C^*(\mathfrak{g}_{\mathbb{C}}^\lambda,\ O(1) \times \mathfrak{u}(n)_{\mathbb{C}};\ \operatorname{End} CC_*^{\mathrm{per}}(\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n}))[[\lambda]][d\lambda])$$
$$\longrightarrow C^*(M, \underline{\operatorname{End}} CC_*^{\mathrm{per}}(\mathbb{A}^{\hbar,\lambda}(M) \otimes_{C^\infty(\mathbb{R}_+)} \Omega^*(\mathbb{R}_+))), \tag{6.5.2}$$

where *End* in the right-hand side of (6.5.2) stands for the sheaf of complexes with the differential $[B + b,\ \cdot\ ]$.

Let $\nabla_0 : \widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})[[\lambda]] \longrightarrow \mathbb{C}[[\lambda]]$ be evaluation at $x = \xi = 0$.

Lemma 6.5.1. *The following two cochains of the left-hand side of* (6.5.2) *are cohomologous*:

i) $\omega_0 \in C^0(\mathfrak{g}_{\mathbb{C}}, \ldots)$; $\omega_0 = L_{\frac{\partial}{\partial\lambda} d\lambda}$,

ii) $\omega_2 \in C^2(\mathfrak{g}_{\mathbb{C}}, \ldots)$; $\omega_2(X, Y) = 0$ *if* X *or* Y *is in* W^1; $\omega_2(X, Y) = -\frac{1}{\hbar}\nabla_0(X*Y - Y * X)$ *for* X, Y *in* $\widehat{\mathbb{A}}^\hbar(\mathbb{R}^{2n})[[\lambda]]/\mathbb{C}[[\lambda]] \cdot 1$.

Proof. Let $\eta_0 \in C^0(\mathfrak{g}_{\mathbb{C}}^\lambda, \ldots)$; $\eta_0 = I_{\frac{\partial}{\partial\lambda} d\lambda}$,

$$\eta_1 \in C^1(\mathfrak{g}_{\mathbb{C}}^\lambda, \ldots);\ \eta_1(X) = 0,\ X \in W_1; \eta_1(X) = -\frac{1}{\hbar} \underline{I_{\frac{\partial}{\partial\lambda}(X - \nabla_0(X))d\lambda}}\,.$$

One checks that

$$(\partial^{\mathrm{Lie}} B + b)(\eta_0 + \eta_1) = \omega_0 - \omega_2\,. \qquad \square$$

Consider the $C^\infty(\mathbb{R}_+)$-valued trace on $\mathbb{A}_C^{\hbar,\lambda}(M \times \mathbb{R}_+)$:

$$\operatorname{Tr}(f) = \frac{1}{(\lambda\hbar)^n n!} \cdot \left(\textstyle\int f \cdot \omega^n + (\lambda\hbar)\tau_1(f) + \cdots\right)\,.$$

If U is a coordinate neighborhood in M and we identify $\mathbb{A}^{\hbar,\lambda}(U \times \mathbb{R}_+)$ with $\mathbb{A}^\hbar(U)\widehat{\otimes} C^\infty(\mathbb{R}_+)$ then Tr becomes the old trace tensored by $id_{C^\infty(\mathbb{R}_+)}$. We deduce from Lemma 6.4.1 the following.

Corollary 6.5.2. *For* $a \in HC_*^{\mathrm{per}}(\mathbb{A}_c^{\hbar,\lambda}(M \times \mathbb{R}_+))$,

$$\frac{\partial}{\partial\lambda}\operatorname{Tr} a - \operatorname{Tr}\left(a \cdot \frac{\partial}{\partial\lambda}\left(\frac{\theta(\lambda\hbar)}{\lambda\hbar}\right)\right) = 0\,.$$

Therefore

$$\frac{\partial}{\partial\lambda}\operatorname{Tr}\left(a \cdot e^{-\frac{\theta(\lambda\hbar)}{\lambda\hbar}}\right) = 0\,.$$

Proof. Indeed, $\operatorname{Tr} \mu'(\omega_0 - \omega_2) = \frac{\partial}{\partial\lambda}\operatorname{Tr} a - \operatorname{Tr}\left(a \cdot \frac{\partial}{\partial\lambda}\left(\frac{\theta(\lambda\hbar)}{\lambda\hbar}\right)\right) = 0$. $\square$

Section 7. An Application

In this section we shall prove a conjecture of B. Feigin.

Let M be a complex analytic manifold, $\mathscr{D}_M$ – the sheaf of holomorphic differential operators on M. Consider the double complex $\check{C}^*(M, CC_*(\mathscr{D}_M))$. This is a complex of modules over $\check{C}^*(M, \mathbb{C})$. Brylinski proved that

$$\mathbb{H}_*(\check{C}^*(M, CC_*(\mathscr{D}_M)) \simeq \bigoplus_{k \geqq 0} H^*(M, \mathbb{C}) \cdot 1^{\bullet(n+k)}) .$$

In particular, $\mathbb{H}_0 \simeq H^{2n}(M, \mathbb{C})$, where $n = \dim_{\mathbb{C}} M$.

On the other hand, one has the obvious cycle $1 \in \check{C}^0(M, CC_0(\mathscr{D}_M))$. A question arises how to identify the image [1] of this cycle under the isomorphism above.

Theorem 7.1.1. $[1] = (td(T_M))_{2n}$.

Proof. We shall use an analogue of Theorem 4.1.1. Consider the double complex $\check{C}^*(M, \overline{CC}_*(\mathscr{D}_M))$. There is the canonical $(2n-1)$-dimensional fundamental class Ω in cohomology of this double complex. Consider the boundary map

$$\partial : \check{C}^*(M, \overline{CC}_*(\mathscr{D}_M)) \to \check{C}^*(M, CC_*(\mathbb{C})) .$$

Theorem 7.1.2. $\partial\Omega = \sum td_k^{-1} \cdot 1^{\bullet n+k}$, *where* $td_k^{-1} \in H^{2k}(M, \mathbb{C})$ *and* $td_k^{-1}(T_M) = \sum (-1)^k td_k^{-1}$.

Proof. The proof of Theorem 4.1.1. works word for word. One has to replace $\mathbb{A}^\hbar$ by $\mathscr{D}_M$ and the Lie algebra $\mathfrak{g}$ from Sects. 5, 6 by the Lie algebra W_n of formal vector fields. □

Note that the Bott periodicity homomorphism S acts on $\check{C}^*(M, \overline{CC}_*(\mathscr{D}_M))$ and commutes with the action of $\check{C}^*(M, \mathbb{C})$. It maps $1^{\bullet(k+1)}$ to $-1^{\bullet k}$. Put $x^{n-k} = (-1)^{n-k-1} 1^{\bullet(n+k)}$ for any k. From Theorem 7.1.2 we get that

$$x^n + \sum_{k \geqq 1} (-1)^k td_k^{-1} \cdot x^{n-k} = 0 .$$

Indeed, the cycle representing the left-hand side is equal to the differential of the fundamental class. The operator S acts as multiplication by x. Clearly, under the identification above, $[1] = x^{2n-1}$. Denote $a_k = (-1)^k td_k^{-1}$; recall that, by definition, $\sum a_k = 1/td(T_M)$. We conclude that $[1] = b \cdot x^n$, where

$$x^{2n-1} \equiv bx^{n-1} + \cdots (\bmod x^n + a_1 x^{n-1} + \cdots + a_n) .$$

Let $1 + b_1 + b_2 + \cdots$ be the element of $H^{ev}(M, \mathbb{C})$ inverse to $1 + a_1 + a_2 + \cdots$ (i.e., $1 + b_1 + \cdots = td(T_M)$). Note that

$$(x^n + a_1 x^{n-1} + \cdots + a_n)(x^{n-1} + b_1 x^{n-1} + \cdots + b_{n-1})$$

is a polynomial of the form $x^{2n-1} - b_n x^{n-1} + \cdots$, therefore $b = b_n$. □

Appendix 1. Operations on the Cyclic Complex

Let A be an associative unitary algebra over an associative commutative unitary ring k. Let $\bar{A} = A/k \cdot 1$; $C_m(A) = A \otimes \bar{A}^{\otimes m}$. For any k-linear map $D : A \to A$, $D(1) = 0$, define the operations

$$L_D : C_m(A) \to C_m(A); \quad i_D : C_m(A) \to C_{m-1}(A);$$

$$S_D : C_m(A) \to C_{m+1}(A) :$$

$$L_D(a_0 \otimes \cdots \otimes a_m) = \sum_{i=0}^{m} (-1)^{\eta_{i-1}|D|} a_0 \otimes \cdots \otimes D(a_i) \otimes \cdots \otimes a_m ,$$

where $\eta_i = \sum_{j=0}^{i-1}(|a_j| - 1)$;

$$i_D(a_0 \otimes \cdots \otimes a_m) = (-1)^{(|a_0|+1)|D|+|a_0|} a_0 \cdot D(a_1) \otimes a_2 \otimes \cdots \otimes a_m ,$$

$$S_D(a_0 \otimes \cdots \otimes a_m) = \sum_{i=1}^{m+1} \sum_{j=1}^{i-1} (-1)^{(\eta_{m+1}-1)\eta_i + |D|(\eta_{m+1}-\eta_i-\eta_j-1)}$$

$$\times (1 \otimes a_i \otimes \cdots \otimes a_m \otimes a_0 \otimes \cdots \otimes D(a_j) \otimes \cdots a_{i-1}) .$$

For any $a \in A$, define the operations

$$L_{\underline{a}} : C_m(A) \to C_{m+1}(A); \quad i_{\underline{a}} : C_m(A) \to C_m(A);$$

$$S_{\underline{a}} : C_m(A) \to C_{m+1}(A);$$

$$L_{\underline{a}}(a_0 \otimes \cdots \otimes a_m) = \sum_{i=0}^{m} (-1)^{\eta_i(|a|+1)} a_0 \otimes \cdots \otimes a_{i-1} \otimes a \otimes \cdots \otimes a_m;$$

$$i_{\underline{a}}(a_0 \otimes \cdots \otimes a_m) = (-1)^{(|a_0|+1)(|a|+1)+|a_0|} a_0 a \otimes a_1 \otimes \cdots \otimes a_m;$$

$$S_{\underline{a}}(a_0 \otimes \cdots \otimes a_m) = \sum_{i=1}^{m+1} \sum_{j=1}^{i} (-1)^{(\eta_{m+1}-1)\eta_i + (|a|+1)(\eta_{m+1}-\eta_i-\eta_j-1)}$$

$$\times (1 \otimes a_i \otimes \cdots \otimes a_m \otimes a_0 \otimes \cdots \otimes a_{(j-1)} \otimes a \otimes \cdots \otimes a_{i-1}) .$$

Let $\mathfrak{g} = \mathrm{Der}(A)$ be the graded Lie algebra of derivations of A. Let $\underline{A}$ be the usual $\mathfrak{g}$-module A with the grading shifted by 1 : for $a \in A$, we write $\underline{a} \in \underline{A}$ and $|\underline{a}| = |a| + 1$. Let $\mathfrak{g} \ltimes \underline{A}$ be the semi-direct product;

$$[D + \underline{a}, E + \underline{b}] = [D, E] + \underline{D(b)} + (-1)^{|a||E|+1} \underline{E(a)} .$$

Define the derivation $\delta : \mathfrak{g} \ltimes \underline{A} \to \mathfrak{g} \ltimes \underline{A}$;

$$\delta(D + \underline{a}) = (-1)^{|a|} ad(a) .$$

Let $U(\mathfrak{g} \ltimes \underline{A})$ be the universal enveloping algebra of $\mathfrak{g} \ltimes \underline{A}$. The grading of $\mathfrak{g} \ltimes \underline{A}$ and the derivation δ induce the structure of a differential graded algebra on

$U(\mathfrak{g} \ltimes \underline{A})$. We need the following map

$$U(\mathfrak{g} \ltimes \underline{A}) \to \mathrm{Hom}_k(\overline{A}, A) \dot{+} \underline{A} \, ; \tag{A1}$$

for $D_i \in \mathfrak{g}$ and $a \in A$, the image of $D_1 \dots D_p$ is $D_1 \circ \cdots \circ D_p : A \to A$ and the image $D_1 \dots D_p \underline{a}$ is $\underline{D_1 \circ \cdots \circ D_p(a)} \in \underline{A}$.

For $Y \in U(\mathfrak{g} \ltimes \underline{A})$, we shall denote by $\overline{Y}$ the image of Y under the map (A1). Put

$$S_Y = S_{\overline{Y}}; \qquad i_Y = i_{\overline{Y}}; \qquad L_Y = L_{\overline{Y}} \, .$$

Finally, put $I_Y = i_Y + (-1)^{|Y|} S_Y$.

Let $\Delta : U(\mathfrak{g} \ltimes \underline{A}) \to U(\mathfrak{g} \ltimes \underline{A})^{\otimes 2}$ be the standard comultiplication. We denote

$$\begin{aligned} \Delta Y &= \sum Y_1 \otimes Y_2 = Y \otimes 1 + \sum Y_1^+ \otimes Y_2 \\ &= 1 \otimes Y + \sum Y_1 \otimes Y_2 = Y \otimes 1 + 1 \otimes Y + \sum Y_1^+ \otimes Y_2^+ \, . \end{aligned}$$

Let b be the Hochschild differential and B be the cyclic differential from Sect. 2.

Theorem A1. *Let $D, E \in \mathfrak{g} \ltimes \underline{A}$. Then*

$$[L_D, L_E] = L_{[DE]}; \;\; [I_Y, L_D] = I_{[Y, D]}; \tag{A2}$$

$$[B + b, L_D] = L_{\delta D}; \tag{A3}$$

$$[B + b, I_Y] + I_{\delta Y} = -\sum (-1)^{|Y_1|} I_{Y_1^+} \cdot I_{Y_2^+} + L_Y \, , \tag{A4}$$

where, for $D_i \in \mathfrak{g} \ltimes \underline{A}$, $L_{D_1 \cdots D_k} = L_{D_1} \cdots L_{D_k}$.

The proof follows from the direct computations.

Let us say a few words about the meaning of this theorem. If $[I_D, I_E]$ were equal to zero, then there would be no need in the operators I_Y, where Y is in $U(\mathfrak{g} \ltimes \underline{A})$ but not in $\mathfrak{g} \ltimes \underline{A}$. Let ε be an odd formal parameter, $\varepsilon^2 = 0$; let $(\mathfrak{g} \ltimes A)[\varepsilon] = (\mathfrak{g} \ltimes A) \otimes_k k[\varepsilon]$; put $(\partial/\partial\varepsilon)(D + \varepsilon E) = -E$; then $((\mathfrak{g} \ltimes \underline{A})[\varepsilon]; \; \delta + \partial/\partial\varepsilon)$ is a differential graded Lie algebra. If $[I_D, I_E]$ were equal to zero, this would mean together with (A2), (A3) that the differential $\mathbb{Z}_2$-graded algebra $((\mathfrak{g} \ltimes \underline{A})[\varepsilon]; \; \delta + \partial/\partial\varepsilon)$ acts on the $\mathbb{Z}_2$-graded complex $(CC_*^{\mathrm{per}}(A); b + B)$.

In reality $[I_D, I_E] \neq 0$. We claim that, because of (A4), $(\mathfrak{g} \ltimes \underline{A})[\varepsilon]$ acts on $CC_*^{\mathrm{per}}(A)$ up to homotopy. What we will in fact show is that another differential graded algebra, which is homotopicallyat equivalent to $U(\mathfrak{g} \ltimes \underline{A})$, acts on $CC_*^{\mathrm{per}}(A)$.

Let $(\mathfrak{a}, \delta)$ be a differential graded Lie algebra. Define $B_0(\mathfrak{a})$ to be the free algebra with the generators I_Y, $Y \in U(\mathfrak{a})^+$ (the augmentation ideal); we assume that I_Y is linear in Y and that $|I_Y| = |Y| + 1$. Let d be the derivation of $B_0(\mathfrak{a})$ such that

$$dI_Y = -I_{\delta Y} - \sum (-1)^{|Y_1|} I_{Y_1^+} \cdot I_{Y_2^+} \, . \tag{A5}$$

The algebra $U(\mathfrak{a})$ acts on $B_0(\mathfrak{a})$ in the obvious way. Let $U(\mathfrak{a}) \ltimes B_0(\mathfrak{a})$ be the semidirect product. The differentials d on B_0 and δ on $U(\mathfrak{a})$ combine to provide the obvious differential d on $U(\mathfrak{a}) \ltimes B_0(\mathfrak{a})$. We modify this differential a little bit; put

$$d'Y = \delta Y, \quad Y \in U(\mathfrak{a}); \quad d'I_Y = dI_Y + Y \, . \tag{A6}$$

We denote the differential graded algebra $(U(\mathfrak{a}) \ltimes B_0(\mathfrak{a}); d')$ by $B(\mathfrak{a})$.

Theorem A2. *The differential $\mathbb{Z}_2$-graded algebra $B(\mathfrak{g} \ltimes \underline{A})$ acts on the $\mathbb{Z}_2$-graded complex $CC_*^{\mathrm{per}}(A)$.*

Proof. This is obvious from (A2)–(A5).

Theorem A3. *The embedding $k \cdot 1 \hookrightarrow B(\mathfrak{a})$ is a quasi-isomorphism for any $\mathfrak{a}$.*

Proof. The algebra $(B_0(\mathfrak{a}), d)$ is a partial case of the well known construction; this is the cobar construction of the chain complex of the Abelian algebra $\mathfrak{a} \cdot \varepsilon \subset \mathfrak{a}[\varepsilon]$. It is well known that there is a quasi-isomorphism $B_0(\mathfrak{a}) \to U(\mathfrak{a} \cdot \varepsilon) = \wedge^*(\mathfrak{a})$. The theorem follows easily. □

We can conclude that the algebra $B_0(\mathfrak{a})$ is a free $\mathfrak{a}$-module resolution of the trivial $\mathfrak{a}$-module k. Let $R_*(\mathfrak{a})$ be the standard resolution. Then there is the chain map $R_*(\mathfrak{a}) \to B(\mathfrak{a})$.

Combining this with the action of $B(\mathfrak{g} \ltimes \underline{A})$ on $CC_*^{\mathrm{per}}(A)$ we obtain the crucial functorial chain map

$$R_*(\mathfrak{g} \ltimes \underline{A}) \otimes CC_*^{\mathrm{per}}(A) \xrightarrow{L} CC_*^{\mathrm{per}}(A) . \tag{A7}$$

This map is $(\mathfrak{g} \ltimes \underline{A})$-equivariant ($\mathfrak{g} \ltimes \underline{A}$ acts only on the tensor factor $R_*(\mathfrak{g} \ltimes \underline{A})$ of the left-hand side; it acts on the right-hand side by the operators L_D).

For example, let $\tau : A/\mathfrak{g}A \to k$ be a trace. Then τ extends to a $(\mathfrak{g} \ltimes \underline{A})$-invariant functional on $CC_*^{\mathrm{per}}(A)$. Consider the composition

$$C^*(\mathfrak{g} \ltimes \underline{A}) \otimes CC_*^{\mathrm{per}}(A) = k \otimes_{U(\mathfrak{g} \ltimes \underline{A})} R_* \otimes CC_*^{\mathrm{per}} \to k \otimes_{U(\mathfrak{g} \ltimes \underline{A})} CC_*^{\mathrm{per}} \xrightarrow{\tau} k ;$$

we obtain the map

$$\chi_\tau : C^*(\mathfrak{g} \ltimes \underline{A}) \to CC_{\mathrm{per}}^{-*}(A)$$

from Subsect. 2.3.

Let us make a few concluding remarks. Let (C_*, δ) be a differential graded coalgebra. Let Cobar (C_*) be the tensor algebra of C_* with shifted grading; we denote the free generators by J_c, $c \in C_*$, and $|J_c| = |c| + 1$. Let d be the derivation of Cobar (C_*) such that

$$dJ_c = -J_{\delta c} - \sum (-1)^{|c_1|} J_{c_1} \cdot J_{c_2} ,$$

where $\Delta c = \sum c_1 \otimes c_2$ is the comultiplication in C_*. Then $d^2 = 0$ and Cobar (C_*, d) is a differential graded algebra (cf. [Q]). Let $\mathfrak{a}$ be a differential graded Lie algebra, $C_*(\mathfrak{a})$ the standard chain complex of $\mathfrak{a}$ (Sect. 2); $C_*^+(\mathfrak{a}) = \bigoplus_{i>0} \wedge^i \mathfrak{a}$; then $C_*^+(\mathfrak{a})$ is a coalgebra. The homomorphism of differential graded algebras

$$\varphi : \text{ Cobar } C_*^+(\mathfrak{a}) \to U(\mathfrak{a}) ,$$

$$\varphi(J_D) = D, \quad D \in \mathfrak{a} ; \quad \varphi(J_c) = 0, \quad c \in \wedge^{>1}\mathfrak{a} ,$$

is a homotopy equivalence of complexes.

Theorem A4. *There exists an action of the differential graded algebra Cobar $C_*^+(\mathfrak{g} \ltimes \underline{A})$ on the $\mathbb{Z}_2$-graded complex $CC_*^{\mathrm{per}}(A)$ such that*:

i) *for $D \in \wedge^1(\mathfrak{g} \ltimes \underline{A})$, the generator J_D acts via the operator L_D;*
ii) *for $\varepsilon D \in \wedge^1(\varepsilon(\mathfrak{g} \ltimes \underline{A}))$, the generator $J_{\varepsilon D}$ acts via the operator I_D.*

Sketch of the proof. First, note that (for any $\mathfrak{a}$), as a coalgebra and as an $\mathfrak{a}$ -module, $U(\mathfrak{a}) \simeq S(\mathfrak{a})$. Let us modify the definition of the algebra $B(\mathfrak{a})$ above. The only difference is in the differential; if d is the differential in $U(\mathfrak{a}) \ltimes B_0(\mathfrak{a})$, then put

$$d''Y = dY, \quad Y \in U(\mathfrak{a}); \quad d''I_Y + dI_Y + L_Y, \quad Y \in S^1\mathfrak{a};$$

$$d''I_Y = L_Y, \; Y \in S^{>1}\mathfrak{a}.$$

The algebra $(U(\mathfrak{a}) \ltimes B_0(\mathfrak{a}), d'')$ is denoted by $B_1(\mathfrak{a})$. One checks easily that $B_0(\mathfrak{a})$ and $B_1(\mathfrak{a})$ are in fact isomorphic and that $B_1(\mathfrak{a})$ is a homomorphic image of Cobar $C_*^+(\mathfrak{a})$. □

Saying that $C_*^+(\mathfrak{a})$ acts on a complex C_* is the same as saying that there is a twisting cochain of $\mathfrak{a}$ with coefficients in End C_*. A twisting cochain is an odd cochain $\rho \in C^*(\mathfrak{a}; \text{ End } C_*)$ such that

$$\partial\rho + \frac{1}{2}[\rho, \rho] = 0.$$

The operators L_D, i_D, S_D, where $D \in \mathrm{Der}(\mathfrak{g})$ were introduced by Rinehart [R]; he proved that $[B + b, I_D] = L_D$.

From our point of view, the map given by (A7) is crucial for understanding differential geometric nature of periodic cyclic homology. If the periodic complex is the non-commutative analogue of the de Rham complex then (A7) is the analogue for the classical formula

$$(d\omega)(X_1, \dots, X_n) = \sum (-1)^{i-1} X_i\omega(X_1, \dots, \widehat{X}_i, \dots)$$
$$+ \sum (-1)^{i+j} \omega([X_i X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots).$$

Finally we would like to mention that our approach to operations is close to Getzler's, cf. [Ge]; the algebra $\mathfrak{g} \ltimes \underline{A}$ can be extended to the algebra of all Hochschild cochains ([Ge, DT]). This extension plays an important role in Getzler's construction of Gauss–Manin connection in periodic cyclic homology, in Nistor's study of the bivariant Chern character, [N], and in [CFS].

Appendix 2. Cyclic Homology of Deformations

Let $\mathbb{A}_0^\hbar(M)$ be as in Sect. 1.

Theorem A2.1.

$$HH_i(\mathbb{A}_0^\hbar(M)) \xrightarrow{\sim} H_c^{2n-i}(M) \otimes \mathbb{C}[\hbar^{-1}, \hbar]];$$

$$HC_i(\mathbb{A}_0^\hbar(M)[\hbar^{-1}]) \xrightarrow{\sim} \bigoplus_{p \geqq 0} H_c^{i+2p}(M) \otimes \mathbb{C}[\hbar^{-1}, \hbar]].$$

Proof. Essentially, this statement is contained in [B1]. Let $C^*(\mathbb{A}_0^\hbar(M)[\hbar^{-1}])$ be the standard Hochschild complex. Consider $(\hbar)$-adic filtration on this complex. Then $E^0_{ij} = \Omega^i_c(M)$; the differential $d_1 : E^0_{ij} \to E^0_{i+1,j-1}$ is described by Brylinski in [B1]. If we identify Ω^i_c with Ω^{2n-i}_c using ω^n, then d_1 becomes de Rham differential d. Thus $E^1_{ij} = H^{2n-i}_c(M)$. We need to show that the spectral sequence degenerates at E^1. If not, then for some i $\dim_{\mathbb{C}[\hbar^{-1},\hbar]]} HH_i(\mathbb{A}_0^\hbar[\hbar^{-1}]) < b^{2n-i}_c(M)$, where

$$b^{2n-i}_c(M) = \dim H^{2n-i}_c(M)\,.$$

From this, using Hochschild to cyclic spectral sequence [C], we will conclude that

$$\dim_{\mathbb{C}[\hbar^{-1},\hbar]]} HC^{\mathrm{per}}_*(\mathbb{A}_0^\hbar[\hbar^{-1}]) < \sum_i b^i_c(M)\,.$$

It remains to show that this is impossible.

Let A_0 be any algebra over $\mathbb{C}$. Let

$$a * b = \sum_i \hbar^i \varphi_i(a,b); \quad \varphi_0(a,b) = ab$$

be any formal deformation of A_0. Let $A_0^\hbar = A_0[[\hbar]]$ with multiplication $*$; also, let $A_0[[\hbar]]$ be the algebra of power series with standard multiplication.

Theorem A2.2. (*Goodwillie–Getzler*, [Ge]). *The complexes $CC^{\mathrm{per}}_*(A_0[[\hbar]])$ and $CC^{\mathrm{per}}_*(A_0^\hbar)$ are canonically isomorphic.*

Proof. The proof rests on the same algebraic idea that we were using throughout this paper. A Hochschild k-cochain of an algebra A is a linear map $A^{\otimes k} \to A$. The space of all such cochains is denoted by $C^k(A,A)$. Let $m \in C^2(A,A)$; $m(a,b) = ab$. One can construct (cf. [Ger]) a $\mathbb{Z}$-graded Lie algebra structure on $\oplus C^k(A,A)$, where, for $D \in C^k$, $|D| = k-1$; if $D, E \in C^1$, then $[D,E] = D \circ E - E \circ D$. One has $[m,m] = 0$; put $\delta D = [m,D]$. Then $\delta^2 = 0$; δ is the Hochschild cohomological differential, $\delta : C^k \to C^{k+1}$. If $\mathfrak{g} = \mathrm{Der}(A)$, then the algebra $\mathfrak{g} \ltimes \underline{A}$ (Appendix 1) is the subalgebra of $C^*(A,A)$; $\mathfrak{g} \ltimes \underline{A} = C^0 + \mathrm{Ker}(\delta : C^1 \to C^2)$.

Having $D \in C^k(A,A)$, one can construct the operations

$$L_D : CC^{\mathrm{per}}_*(A) \to CC^{\mathrm{per}}_{*-k+1}(A);$$

$$I_D : CC^{\mathrm{per}}_*(A) \to CC^{\mathrm{per}}_{*-k}(A)$$

such that

$$[L_D, L_E] = L_{[D,E]}; \qquad [I_D, L_E] = I_{[DE]}\,;$$

$$[B+b, I_D] = I_{[m,D]} + L_D\,;$$

$$L_m = b\,,$$

where b is the Hochschild homological differential. Then the formulas above show that

$$\frac{d}{d\hbar}(B+b) = [B+b, I_{dm/d\hbar}]\,.$$

Thus,

$$B + b = X(B + b)|_{\hbar=0} X^{-1} ,$$

where

$$\frac{dX}{d\hbar} \cdot X^{-1} = -I_{dm/dh} .$$

This shows that X is an isomorphism between $CC_*^{\mathrm{per}}(A_0[[\hbar]])$ and $CC_*^{\mathrm{per}}(A_0^\hbar)$.

Corollary A2.3. $\dim HC_*^{\mathrm{per}}(A^{\hbar_0}) = \sum_i b_c^i(M)$.

Corollary A2.4. *The space of* $\mathbb{C}[[\hbar]]$*-linear continuous functionals* $\mathrm{Tr} : (A_0^\hbar)(M) \to \mathbb{C}[\hbar^{-1}, \hbar]]$ *such that* $\mathrm{Tr}(f * g) = \mathrm{Tr}(g * f)$ *is one-dimensional.*

Acknowledgements. We would like to thank P. Baum, A. Connes, B. Fedosov, M. Flato, E. Getzler, A. Goncharov, T. Natsume, V. Nistor, D. Quillen, I. Singer, Y. Soibelman, D. Sternheimer, A. Szenes and N. Reshetikhin for very helpful discussions. We thank B. Fedosov for helping us find an error in the original version. The second author would especially like to thank Boris Feigin, in joint work with whom many of the methods of this paper take their origin.

References

[ACKP] Arbarello, E., de Concini, C., Kac, V., Procesi, C.: Moduli spaces of curves and representation theory. Commun. Math. Phys. **117**, 1–36 (1988)

[B] Brylinski, J.-L.: Some examples of cyclic and Hochschild homologies. Lect. Notes Math., **1271**, 33–72 (1987)

[B1] Brylinski, J.-L.: A differential complex for Poisson manifold. J. Diff. Geom. **28**, 93–114 (1988)

[BFFLS] Bayen, F., Flato, M., Fronsdal, C., Lichnerowiscz, A., Sternheimer, D.: Ann. Phys. **111**, 61–151 (1978)

[Bu] Burghelea, D.: Cyclic homology and algebraic K-theory of topological spaces. Contemp. Math. **55**, 89–115 (1983)

[BTT] O'Brien, N., Toledo, D., Tong, Y.: A Grothendieck-Riemann-Roch formula for maps of complex manifolds. Math. Ann. **271**, 493–526 (1985)

[C] Connes, A.: Noncommutative differential geometry. Publ. Math. IHES **62** (1986)

[CFS] Connes, A., Flato, M., Sternheimer, D.: Closed star products and cyclic homology. Preprint, Paris, 1991

[DT] Daletskı, Y., Tsygan, B.: Operations on Hochschild and cyclic complexes. Preprint, 1992

[F] Fedosov, B.: Index theorems. Itogi Nauki i tekhniki **65** (1991)

[FRT] Faddeev, L., Reshetikhin, N., Takhtajan, L.: Quantization of Lie groups and algebras. Algebra and Analysis **1**, 178–206 (1978)

[FT] Feigin, B., Tsygan, B.: Lie algebra cohomology and Riemann-Roch theorem. Rend. Sci. Palermo, **I** (1987)

[Ge] Getzler, E.: Cartan homotopy formulas and Gauss-Manin connection in cyclic homology. Preprint (1992)

[GeS] Getzler, E., Szenes, A.: On the Chern character of a theta-summable Fredholm module. J. Funct. Anal. **84**, 2, 343–357 (1989)

[Ger] Gerstenhaber, M.: The cohomology structure on an associative ring. Ann. Math. **78**, 59–103 (1963)

[G] Gut, S.: Thesis, Mets, 1980

[HH] Helton, J., Howe, R.: Integral operators, commutators, traces, index and homology. Proc. of Conf. on operator theory, Lect. Notes in Math. **345** (1973)

[JLO] Jaffe, A., Lesniewski, A., Osterwalder, K.: Quantum K-theory, I, K-theory **2**, 675–682 (1989)

[K] Karoubi, M.: Homologie cyclique et K-theorie. Asterisque **149** (1987)

[L] Losik, M.V.: Characteristic classes of structures of the manifolds. Funct. An. **21**, 3, 38–52 (1987)
[LQ] Loday, J.L., Quillen, D.: Cyclic homology and Lie algebra homology of matrices. Commun. Math. Helv. **59**, 565–591 (1984)
[MN] Moryoshi, H., Natsume, T.: To appear
[N] Nistor, V.: A bivariant Chern character for p-summable quasi-homomorphisms. K-theory **5**, 193–211 (1991)
[S] Soibelman, Y.: Irreducible representations of functıons algebra on the quantum group SU(n) and Schubert cells. Dokl. AN SSSR **40**, 34–38 (1990)
[STS] Semyonov-Tian-Shansky, M.: Dressing transformation and Poisson group actions. Publ. RIMS **21**, 1237–1260 (1985)
[T] Tsygan, B.: Homology of matrix Lie algebras over rings and Hochschild homology. Uspekhi Mat. Nauk **38**, 2, 217–218 (1983)
[TT] Toledo, D., Tong, Y.: Parametrix for $\bar{\partial}$ and Riemann-Roch in Čech theory. Topology **15**, 273–301 (1976)
[V] Verlinde, E.: Fusion rules and modular transformations in 2d conformal field theory. Nucl. Phys. B **300**, 360–376 (1988)
[Wo] Wodzicki, M.: Cyclic homology of pseudodifferential operators and noncommutative Euler class. C.R.A.S., Ser. 1, 321–325 (1988)

Communicated by N. Yu. Reshetikhın