Commun. Math. Phys. 172, 1–11 (1995)

Communications in
Mathematical
Physics

BRS Cohomology of the Supertranslations in $D = 4$

J.A. Dixon[1], R. Minasian[2]

[1] Department of Physics and Astronomy, University of Calgary, 2500 University Drive NW, Calgary Alberta, Canada T2N 1N4
e-mail: JDIXON@acs.ucalgary.ca
[2] Center for Theoretical Physics, Physics Department, Texas A & M University, College Station, Texas 77843, USA
e-mail: RUBEN@PHYS.TAMU.EDU

Received: 10 August 1993/in revised form: 30 April 1994

Abstract: Supersymmetry transformations are a kind of square root of spacetime translations. The corresponding Lie superalgebra always contains the supertranslation operator $\delta = c^\alpha \sigma^\mu_{\alpha\dot\beta} \bar{c}^{\dot\beta} (\varepsilon^\mu)^\dagger$. We find that the cohomology of this operator depends on a spin-orbit coupling in an $SU(2)$ group and has a quite complicated structure. This spin-orbit type coupling will turn out to be basic in the cohomology of supersymmetric field theories in general.

1. Introduction

It is well known that anomalies in gauge theories can be classified as elements of the cohomology space of the BRS operator acting on the space of local polynomials [1,2]. Due to locality, all cohomology calculations are brought to an algebraic form, and so all details of the space-time topology may be neglected (see, e.g. [6]). Here we are interested in cohomology classes built by space-time independent "ghosts," so the requirement of locality is trivial, but we shall see many common features between our construction and cohomology classes of local polynomials. In particular, the present computation is similar to the analysis of the cohomology of supersymmetric theories which was commenced in [4].

We think the method describing the BRS cohomology of the supertranslation operator in four dimensions outlined in this paper is directly applicable to problems which lie at the heart of all four dimensional supersymmetric quantum field theories and superstring theory. There also may be a simple connection between the present results and the results in quantum field theory, the most recent discussion of which one may find in [7].

Since the supertranslation operator analysed here has a direct analogue for gauge theories, let us recall an essential point in the problem of the computation of the BRS cohomology of Yang–Mills theory before presenting our calculations. The golden nugget in that problem [1,3] is the cohomology of the following nilpotent operator

$$\delta = f^{abc} \omega^a \omega^b (\omega^c)^\dagger , \tag{1.1}$$

where

$$\{(\omega^a)^\dagger, \omega^b\} = \delta_a^b\,; \tag{1.2}$$

$(\omega^a)^\dagger$ is an annihilation operator and f^{abc} are the structure constants of a compact semisimple Lie algebra. The cohomology space of this operator is well known and is given by the kernel of the "Laplacian."

$$\Delta = (\delta + \delta^\dagger) = Y^a Y^a\,, \tag{1.3}$$

where

$$Y^a = (Y^a)^\dagger = i f^{abc} \omega^b (\omega^c)^\dagger \tag{1.4}$$

is the generator of rotations. The kernel consists of all solutions of

$$Y^a H = 0\,, \tag{1.5}$$

which are just the invariant antisymmetric tensors of the adjoint representation for the group.

By a straightforward mapping, these antisymmetric tensors in turn generate the cohomology space of Yang–Mills theory, which determines all possible anomalies of such theories. Gravity works in an analogous way – the relevant group there is SO(1, D-1) for D dimensional gravity.

The supertranslation operator which will be analysed in the present paper is:

$$\delta = c^\alpha \sigma^\mu_{\alpha\dot\beta} \bar c^{\dot\beta} (\varepsilon^\mu)^\dagger\,. \tag{1.6}$$

As mentioned, (1.6) is the analogue of (1.1) for supersymmetry. Similar operators occur in all supersymmetry theories in all dimensions, however, their cohomology is certainly dimension dependent. Here we concentrate on the $D = 4$ case, and it turns out that performing calculations in terms of two component Weyl spinors rather than in terms of Majorana spinors simplifies the problem significantly. A set of conventions which are particularly well suited to our problem is introduced in Appendix A. They preserve the symmetry of the theory under complex conjugation. This symmetry is obscure in most sets of conventions. We also give a number of identities in this notation for convenience and ease of the present and future work.

The easiest way to see (1.6) arising is to consider the algebra of the well known supersymmetry translation operators Q:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu\,, \tag{1.7}$$

$$[Q_\alpha]^* = \bar Q_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \frac{1}{2}\bar\sigma^\mu_{\dot\alpha\beta}\theta^\beta\partial_\mu\,. \tag{1.8}$$

To close the algebra of these operators we must add an anticommuting translation "ghost" ε^μ and a commuting Weyl spinor ghost c_α and its complex conjugate $\bar c_{\dot\alpha}$. The resulting nilpotent "BRS" type operator is:

$$\delta = c^\alpha Q_\alpha + \bar c^{\dot\alpha}\bar Q_{\dot\alpha} - c^\alpha\sigma^\mu_{\alpha\dot\beta}\bar c^{\dot\beta}\varepsilon^{\mu\dagger} + \varepsilon^\mu\partial_\mu\,, \tag{1.9}$$

which summarizes the algebra of the Q's. Note than $\delta = \bar\delta$ is a "real" anticommuting operator. The following identities hold:

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = \sigma^\mu_{\alpha\dot\beta}\partial_\mu\,. \tag{1.10}$$

Now in a field theory, the operators Q_α and ∂_μ become functional derivative operators whose action is really quite complex and the cohomology of the operator δ becomes also quite complicated. However, the supertranslation part of this operator remains even in field theory, and using the present results and the method of spectral sequences the full BRS cohomology can now be found for chiral superfields in $D = 4$, $N = 1$ supersymmetry [5], although there are still unsolved difficulties when vector superfields (i.e. super Yang–Mills fields) are present.

2. Laplacian

The calculation of the cohomology space is based on the simple result:

$$H(\delta) = \ker\delta/\mathrm{im}\,\delta \approx [\ker\delta]^\perp \cap [\mathrm{im}\,\delta]^\perp = \ker\varDelta\,, \tag{2.1}$$

where $H(\delta)$ is the cohomology space of δ and $\varDelta$ is the Laplacian formed from δ:

$$\varDelta = [\delta + \delta^\dagger]^2 = \delta\delta^\dagger + \delta^\dagger\delta\,. \tag{2.2}$$

Though in this paper we identify H and $\ker\varDelta$, in general, it is important to keep in mind that the former is actually a factor space which is isomorphic with the latter.

Using various identities contained in Appendix A, we show that δ is an operator in a Fock space with positive definite metric. An essential point is that this can be done even though our spacetime in intrinsically Minkowskian. The detailed discussion of this point is presented in Appendix B. After some algebra, we find that the Hodge "Laplacian" operator for the operator (1.6) can be written in the form:

$$\varDelta = [n + \bar{n} + 2]N + 2n\bar{n} + 4[J_iL_i + \bar{J}_i\bar{L}_i]\,, \tag{2.3}$$

where we use the abbreviations:

$$n = c^\alpha(c^\alpha)^\dagger, \qquad \bar{n} = \bar{c}^{\dot\alpha}(\bar{c}^{\dot\alpha})^\dagger\,, \tag{2.4}$$

$$N = \varepsilon^\mu(\varepsilon^\mu)^\dagger\,, \tag{2.5}$$

$$J_i = \frac{1}{2}c^\alpha(\sigma_i)_\alpha^\beta(c^\beta)^\dagger, \qquad \bar{J}_i = \frac{1}{2}\bar{c}^{\dot\alpha}(\bar{\sigma}_i)_{\dot\alpha}^{\dot\beta}(\bar{c}^{\dot\beta})^\dagger\,, \tag{2.6}$$

$$L_i(\sigma_i)_\alpha^\beta = \frac{1}{2}(\sigma^{\mu\nu})_\alpha^\beta\varepsilon_\mu(\varepsilon^\nu)^\dagger\,. \tag{2.7}$$

The operator L_i can also be written more explicitly in the form:

$$L_i = -\frac{1}{2}(\varepsilon_0(\varepsilon_i)^\dagger + \varepsilon_i(\varepsilon_0)^\dagger + i\varepsilon_{ijk}\varepsilon_j(\varepsilon_k)^\dagger)\,. \tag{2.8}$$

It is easy to verify that L_i and J_i obey the commutation rules of the $SU(2)$ Lie algebra:

$$[J_i, J_j] = i\varepsilon^{ijk}J_k\,, \tag{2.9}$$

$$[L_i, L_j] = i\varepsilon^{ijk}L_k\,, \tag{2.10}$$

and we note

$$[J_i, L_j] = 0\,. \tag{2.11}$$

Initially, it was a surprise for us to find that this operator involves only angular momentum in the compact $SU(2)$ algebra even though the supertranslation operator is intrinsically defined in Minkowski space. But the result is reasonable. This happens because the Fock space is positive definite, so that any relevant group theory for our Laplacian must necessarily also be defined in a compact positive definite context. Anyway, regardless of the philosophy, one finds the above result. The same kind of thing happens for the BRS operators of string theory when the problem is formulated in this way.

3. Discussion of the Laplacian and the Cohomology Space

Since the Laplacian Δ in (2.3) consists solely of counting operators and coupled angular momentum operators, finding its kernel is an exercise in the theory of angular momentum.

It is well known that the eigenvalues of angular momentum operators J^2 are of the form $j(j+1)$, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots$. We rewrite the above Laplacian in the form:

$$\Delta = [n + \bar{n} + 2]N + 2n\bar{n} + 2[K^2 - J^2 - L^2] + 2[\bar{K}^2 - \bar{J}^2 - \bar{L}^2], \tag{3.1}$$

where we define the composite angular momentum operators

$$K_i = J_i + L_i, \qquad \bar{K}_i = \bar{J}_i + \bar{L}_i, \tag{3.2}$$

$$J^2 = J_i J_i, \qquad L^2 = L_i L_i. \tag{3.3}$$

What do the eigenvectors of these various angular momentum operators look like?

The eigenvectors of J^2 are:

$$J^2 c^{\alpha_1} c^{\alpha_2} \cdots c^{\alpha_m} = \frac{m}{2}\left(\frac{m}{2} + 1\right) c^{\alpha_1} c^{\alpha_2} \cdots c^{\alpha_m} \quad m \geqq 1. \tag{3.4}$$

These expressions are automatically symmetric under interchange of any pair of indices.

The eigenvectors of L^2 are best described using the variables

$$\varepsilon_{\alpha\dot{\beta}} = \sigma^{\mu}_{\alpha\dot{\beta}} \varepsilon_{\mu}. \tag{3.5}$$

The L^2 eigenvalue of a product of these variables is equal to the number of symmetrized undotted indices in the product. Since ε anticommutes, there are a very limited number of possibilities that are nonzero:

$$L^2 \varepsilon_{\mu} = \frac{3}{4}\varepsilon_{\mu}; \quad \left(l = \frac{1}{2}\right), \tag{3.6}$$

$$L^2 \varepsilon_{\mu}\varepsilon_{\nu}\sigma^{\mu\nu}_{\alpha\beta} = 2\varepsilon_{\mu}\varepsilon_{\nu}\sigma^{\mu\nu}_{\alpha\beta}; \quad (l = 1), \tag{3.7}$$

$$L^2 \varepsilon_{\mu}\varepsilon_{\nu}\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}} = 0; \quad (l = 0), \tag{3.8}$$

$$L^2 \varepsilon_{\mu}\varepsilon_{\nu}\varepsilon_{\lambda} = \varepsilon_{\mu\nu\lambda\rho} L^2\, W^{\rho} = \frac{3}{4}\varepsilon_{\mu\nu\lambda\rho} W_{\rho}; \quad \left(l = \frac{1}{2}\right), \tag{3.9}$$

with

$$W_\rho = -\frac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\varepsilon^\mu\varepsilon^\nu\varepsilon^\lambda\,, \tag{3.10}$$

where we note that a vector $W^{\alpha\dot{\beta}} = (\sigma^\rho)^{\alpha\dot{\beta}}W_\rho$ can only have one undotted spinor index. Finally,

$$L^2\varepsilon_\mu\varepsilon_\nu\varepsilon_\lambda\varepsilon_\rho = 0; \quad (l = 0)\,. \tag{3.11}$$

For the complex conjugate operators the discussion is identical except that only dotted indices are relevant.

For the combined operator K^2 the eigenvectors have spin k, where $2k$ is the number of symmetric (and consequently uncontracted) free undotted indices in the eigenvector. These indices may come from either c_α or $\varepsilon_{\alpha\dot{\beta}}$. So the eigenvectors of K^2 are of the form (for example):

$$K^2\varepsilon_\mu\sigma^\mu_{\alpha\dot{\beta}}c^\alpha = 0; \quad (k = 0)\,, \tag{3.12}$$

$$K^2(\varepsilon_{\alpha\dot{\beta}}c_\beta + \varepsilon_{\beta\dot{\beta}}c_\alpha) = 2(\varepsilon_{\alpha\dot{\beta}}c_\beta + \varepsilon_{\beta\dot{\beta}}c_\alpha); \quad (k = 1)\,, \tag{3.13}$$

etc.

As is known from the theory of angular momenta, the eigenvalues of K^2 run from $|j - l|$ to $(j + l)$. Now, finding all the possible combinations of eigenvalues that can give zero for the value of the Laplacian above allows the identification of the cohomology space. We do this simply by examining the various cases as a function of $N = N(\varepsilon)$ for $N = 0, 1, 2, 3, 4$. The analysis yields solutions

1. $N = 0,\ n\bar{n} = 0$.
2. $N = 1,\ n = \bar{n} = 1$.
3. $N = 1,\ k = \frac{n}{2} - \frac{1}{2},\ \bar{n} = 0$ (and the complex conjugate of this).
4. $N = 2,\ k = \frac{n}{2} - 1,\ \bar{n} = 0$ (and the complex conjugate of this).

To summarize, we have shown that the following set of polynomials constitute all the vectors in the cohomology space of the operator (1.6).

$$c^{\alpha_1}c^{\alpha_2}\cdots c^{\alpha_m} \quad m \geqq 1\,, \tag{3.14}$$

$$\varepsilon_\mu\sigma^\mu_{\alpha\dot{\beta}}c^\alpha c^{\alpha_1}c^{\alpha_2}\cdots c^{\alpha_m} \quad m \geqq 0\,, \tag{3.15}$$

$$\varepsilon_\mu\varepsilon_\nu\sigma^{\mu\nu}_{\alpha\beta}c^\alpha c^\beta c^{\alpha_1}c^{\alpha_2}\cdots c^{\alpha_m} \quad m \geqq 0\,, \tag{3.16}$$

and their complex conjugates:

$$\bar{c}^{\dot{\alpha}_1}\bar{c}^{\dot{\alpha}_2}\cdots \bar{c}^{\dot{\alpha}_m} \quad m \geqq 1\,, \tag{3.17}$$

$$\varepsilon_\mu\bar{\sigma}^\mu_{\dot{\alpha}\beta}\bar{c}^{\dot{\alpha}}\bar{c}^{\dot{\alpha}_1}\bar{c}^{\dot{\alpha}_2}\cdots \bar{c}^{\dot{\alpha}_m} \quad m \geqq 0\,, \tag{3.18}$$

$$\varepsilon_\mu\varepsilon_\nu\bar{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}}\bar{c}^{\dot{\alpha}}\bar{c}^{\dot{\beta}}\bar{c}^{\dot{\alpha}_1}\bar{c}^{\dot{\alpha}_2}\cdots \bar{c}^{\dot{\alpha}_m} \quad m \geqq 0\,. \tag{3.19}$$

Finally, there is one more polynomial in the cohomology space which is equal to its complex conjugate:

$$\varepsilon_\mu c^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{c}^{\dot{\beta}}\,. \tag{3.20}$$

4. Projection Operators for Supertranslations

Since our actual interest is in the use of these results in finding the cohomology space for supersymmetric field theory where the use of spectral sequences will be necessary, we have found the orthogonal projection operators which project onto the space H. These operators all satisfy the relations

$$\Pi = \Pi^\dagger = \Pi^2 . \tag{4.1}$$

The projection operator onto (3.14) has the form $\Pi_{N=0}\Pi_{\bar{n}=0}\Pi_{n=r}(r \geqq 1)$, where

$$\Pi_{n=r} = \sum_{l=r}^{\infty} \frac{(-1)^{(l-r)}}{(l-r)!} n_l , \tag{4.2}$$

and we remind the reader that n is an operator defined in (2.4). The operators n_l are defined recursively by

$$n_{l+1} = (n-l)n_l \tag{4.3}$$

and are distinguished by the fact that they are normal ordered operators. For example,

$$n_2 = c^\alpha c^\beta [c^\alpha c^\beta]^\dagger . \tag{4.4}$$

This operator satisfies the relation

$$n\Pi_{n=r} \equiv n_1 \Pi_{n=r} = r\Pi_{n=r} \quad r = 0, 1, 2, \dots . \tag{4.5}$$

Next, we find the projection operator onto (3.20) $\varepsilon_\mu c^\alpha \sigma^\mu_{\alpha\dot{\beta}} \bar{c}^{\dot{\beta}}$. Here we must first take the combination of three operators of type (4.2) $\Pi_{N=1}\Pi_{\bar{n}=1}\Pi_{n=1}$ which selects the subspace $N = n = \bar{n} = 1$. Equation (3.20) is the totally contracted vector from this subspace. We find that the following operator performs the task of picking out the contracted part:

$$\Pi = \left(\frac{1}{4} - J \cdot L\right)\left(\frac{1}{4} - \bar{J} \cdot \bar{L}\right) \Pi_{N=1}\Pi_{\bar{n}=1}\Pi_{n=1} , \tag{4.6}$$

where

$$J \cdot L = J_i L_i . \tag{4.7}$$

The third projection operator is the one onto the vector $\varepsilon_\mu \sigma^\mu_{\alpha_1\dot{\beta}} c^{\alpha_1} c^{\alpha_2} c^{\alpha_3} \cdots c^{\alpha_r}$, for $r \geqq 1$. As above, first we project onto the space where $N = 1$, $n = r$, $\bar{n} = 0$. Then in this subspace we need

$$J \cdot L\Pi = -\left[\frac{r}{4} + \frac{1}{2}\right] \Pi . \tag{4.8}$$

The operator which accomplishes this is:

$$\Pi = \left[\frac{r}{2(r+1)} - \frac{2}{r+1} J \cdot L\right] \Pi_{N=1}\Pi_{\bar{n}=0}\Pi_{n=r} . \tag{4.9}$$

For the vectors $\varepsilon_\mu \varepsilon_\nu c^\alpha \sigma^{\mu\nu}_{\alpha\beta} c^\beta c^{\alpha_1} \cdots c^{\alpha_{r-2}}$ for $r \geqq 2$, we must first project onto the relevant subspace with $\Pi_{N=2,\, n=r,\, \tilde{n}=0}$, then onto the subspace in which $L^2 = 2$. This can be accomplished with the operator:

$$\Pi_{L^2=2} = \left[\frac{1}{2} + \frac{1}{8}(R + R^\dagger)\right] , \tag{4.10}$$

where

$$R = 2i\varepsilon_{ijk}\varepsilon_0 \varepsilon_i \varepsilon_j^\dagger \varepsilon_k^\dagger \, . \tag{4.11}$$

To see that this has the desired properties, we note that

$$L^2 \Pi_{L^2=2} = \left[1 + \frac{1}{4}(R + R^\dagger)\right] \Pi_{L^2=2} = 2\Pi_{L^2=2} \, . \tag{4.12}$$

This can be shown using the results that

$$RR^\dagger \Pi_{N=2} = 16\varepsilon_0 \varepsilon_0^\dagger \Pi_{N=2} \, , \tag{4.13}$$

$$R^\dagger R \Pi_{N=2} = 16(1 - \varepsilon_0 \varepsilon_0^\dagger)\Pi_{N=2} \, . \tag{4.14}$$

The next step is the projection onto the subspace where the eigenvalue $k(k+1)$ of K^2 is $k = \frac{r}{2} - 1$, or equivalently, we need:

$$J \cdot L\Pi = -\left[\frac{r}{2} + 1\right]\Pi \, . \tag{4.15}$$

The necessary operator is:

$$\Pi = \left[\frac{r-1}{4(r+1)} - \frac{r-1}{r(r+1)} J \cdot L + \frac{2}{r(r+1)} L_{ij} J_{ij}\right] \Pi_{N=2}\Pi_{\tilde{n}=0}\Pi_{n=r}\Pi_{L^2=2} \, , \tag{4.16}$$

where we define the normal ordered expressions J_{ij} and L_{ij} by:

$$J_i J_j = \frac{1}{4} n\delta_{ij} + \frac{1}{2} i\varepsilon_{ijk} J_k + J_{ij} \, , \tag{4.17}$$

$$L_i L_j = \frac{1}{4} N\delta_{ij} + \frac{1}{2} i\varepsilon_{ijk} L_k + L_{ij} \, , \tag{4.18}$$

and explicitly

$$J_{ij} = c^\alpha c^\beta (\sigma_i)^\gamma_\alpha (\sigma_j)^\delta_\beta (c^\gamma)^\dagger (c^\delta)^\dagger \, . \tag{4.19}$$

The total projection operator for the cohomology space of the supertranslation operator in $D=4$ is the sum of (4.2), (4.6), (4.9), (4.16) and the complex conjugates (where necessary).

5. Conclusion

The Laplacian operator for the supertranslation operator can be written in a very simple and transparent form (2.3) which contains only counting and angular momentum operators. Then it is straightforward, though somewhat involved, to deduce the form of the kernel of this Laplacian and so the cohomology space. The cohomology space is described in Eqs. (3.14–3.20). Its complexity is a reflection of the complexity of the cohomology of supersymmetric theories in general.

Surprisingly, the cohomology calculation involves only the compact Lie algebra $SU(2)$ even though supersymmetry is a Minkowski space algebra. This is a consequence of the positive definite metric on Fock space. Very similar things happen for the general BRS cohomology of the chiral superfield in supersymmetric quantum field theory, and we intend to treat various such theories in forthcoming papers.

The cohomology space of super Yang–Mills involves vector superfields as well as chiral superfields, and we have not yet been able to find the cohomology of that theory. The difficulty is that in the cohomology problem for super Yang–Mills theory, the gauge transformations and the supersymmetry transformations seem to become entangled in a complicated way, even though they are rather well separated in the relevant δ operator.

The BRS cohomology of supergravity is, of course, a separate problem and a very complicated one. But the supertranslation operator treated here naturally appears in supergravity and we hope that the present results will be helpful there.

A. Conventions and Useful Formulae

Though many different sets of conventions exist in literature (see the useful discussion in [8]), we do not quite follow them because of our desire to have symmetry under complex conjugation. Our Lorentz metric is defined by the relation:

$$x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \,. \tag{A.1}$$

Lorentz transformations Λ^μ_ν are real matrices which preserve this quadratic form when they are used to transform the vectors:

$$x'^\mu = \Lambda^\mu_\nu x^\nu \,, \tag{A.2}$$

so they satisfy:

$$\Lambda^\mu_\nu \Lambda^\sigma_\tau \eta_{\mu\sigma} = \eta_{\nu\tau} \,. \tag{A.3}$$

Another way to preserve this quadratic form is to consider the following action of the group SL$(2, C)$ of all of 2×2 complex matrices M of determinant 1 on a 2×2 complex hermitian matrix:

$$\sigma^\mu x'_\mu = M \sigma^\mu x_\mu M^\dagger \,, \tag{A.4}$$

where σ^μ are a basis of the set of 2×2 complex hermitian matrices, which are chosen to be Pauli matrices. The transformation (A.4) defines a Lorentz transformation on the four vector

$$x'^\mu = \Lambda^\mu_\nu x^\nu \,, \tag{A.5}$$

since the determinant is preserved by this transformation and it is equal to (minus) the quadratic form (A.1):

$$\det[M\sigma^\mu x_\mu M^\dagger] = \det[\sigma^\mu x'_\mu] = \det[\sigma^\mu x_\mu] = x'_0 x'_0 - x'_i x'_i = x_0 x_0 - x_i x_{i,} . \tag{A.6}$$

We will write the indices of the matrix M as M_α^β and its complex conjugate as $(M_\alpha^\beta)^* = \bar{M}_{\dot\alpha}^{\dot\beta}$. The hermitian conjugate has the form $(M_\alpha^\beta)^{*T} = (M_\alpha^\beta)^\dagger = (\bar{M}^T)_{\dot\alpha}^{\dot\beta}$.

The notation A^* is interchangeable with the notation $\bar{A}$. Both mean simply complex conjugation. It follows that the position and kind of indices of the matrices σ^μ are now determined $\sigma^\mu_{\alpha\dot{\beta}} = (1, \sigma^i)_{\alpha\dot{\beta}}$, and since the σ matrices are hermitian, the complex conjugate matrices are $(\sigma^\mu_{\alpha\dot{\beta}})^* = \bar{\sigma}^\mu_{\dot{\alpha}\beta} = \sigma^\mu_{\beta\dot{\alpha}}$.

Contrary to the usual convention, we do not reverse the order of (anticommuting) spinors when taking the complex conjugate. Such a change of order spoils the natural symmetry of supersymmetry under complex conjugation and makes all computations more tricky. (Reversing the order of commuting spinors makes no difference of course.) Indices are raised and lowered as follows:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = -\varepsilon_{\alpha\beta}\psi^\beta\,, \tag{A.7}$$

$$\bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}, \qquad \bar{\psi}_{\dot{\alpha}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\,, \tag{A.8}$$

where the ε tensors are real antisymmetric matrices with:

$$\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \varepsilon^\alpha_\beta = -\varepsilon^\alpha_\beta = \delta^\alpha_\beta\,, \tag{A.9}$$

where $\delta^\alpha_\beta = 1$ if $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ if $\alpha \neq \beta$ (Same for $\delta^{\dot{\alpha}}_{\dot{\beta}}$). We take:

$$\varepsilon^{\alpha\beta} = i(\sigma_2)^{\alpha\beta}\,. \tag{A.10}$$

We also have:

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\theta^2\varepsilon^{\alpha\beta}, \qquad \theta_\alpha\theta^\beta = -\frac{1}{2}\theta^2\delta^\beta_\alpha\,, \tag{A.11}$$

where $\theta^2 = \theta^\alpha\theta_\alpha$. Using this rule for raising and lowering indices, we get:

$$\bar{\sigma}^{\mu\dot{\alpha}\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\beta\gamma}\bar{\sigma}^\mu_{\dot{\delta}\gamma} = (1, -\sigma^i)^{\dot{\alpha}\beta} \tag{A.12}$$

so that:

$$\bar{\sigma}^{\dot{\alpha}\beta}_\mu = -(1, \sigma^i)^{\dot{\alpha}\beta}\,, \tag{A.13}$$

where we use $\sigma_2(\sigma_i)^*\sigma_2 = -\sigma_i$. The well known relation for Pauli matrices

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k \tag{A.14}$$

results in a number of other relations such as:

$$\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta} = -2\eta^{\mu\nu}\delta^\beta_\alpha\,, \tag{A.15}$$

which yields

$$\varepsilon^{\dot{\gamma}\dot{\delta}}\sigma^\mu_{\alpha\dot{\gamma}}\sigma^\nu_{\beta\dot{\delta}}\partial_\mu\partial_\nu = -\varepsilon_{\alpha\beta}\Box\,, \tag{A.16}$$

$$\sigma^\mu_{\alpha\dot{\beta}}\bar{\sigma}^{\dot{\gamma}\delta}_\mu = -2\delta^\delta_\alpha\delta^{\dot{\gamma}}_{\dot{\beta}}\,. \tag{A.17}$$

We define:

$$\sigma^{\mu\nu}_{\alpha\beta} = \sigma^{\mu\nu}_{\beta\alpha} = -\sigma^{\nu\mu}_{\alpha\beta} = \frac{1}{2}[\sigma^\mu_{\alpha\dot{\gamma}}\bar{\sigma}^{\nu\dot{\gamma}}_\beta - \sigma^\nu_{\alpha\dot{\gamma}}\bar{\sigma}^{\mu\dot{\gamma}}_\beta]\,. \tag{A.18}$$

Then,

$$(\sigma^{\mu\nu})_{\alpha\beta} = \sigma^{\mu}_{\alpha\dot\beta}\bar\sigma^{\nu\dot\beta}_{\gamma} + \eta^{\mu\nu}\varepsilon_{\alpha\beta}\,, \tag{A.19}$$

$$\varepsilon^{\kappa\lambda}_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} = 2i(\sigma^{\kappa\lambda})_{\alpha\beta}\,, \tag{A.20}$$

$$\sigma^{\mu\nu}_{\alpha\beta}\sigma^{\lambda\beta}_{\dot\gamma} = \sigma^{\mu}_{\alpha\dot\gamma}\eta^{\nu\lambda} - \sigma^{\nu}_{\alpha\dot\gamma}\eta^{\mu\lambda} + i\varepsilon^{\mu\nu\lambda\rho}\sigma_{\rho\alpha\dot\gamma}\,, \tag{A.21}$$

$$\begin{aligned}(\sigma^{\mu\nu})_{\alpha\beta}(\sigma^{\lambda\tau})^{\beta}_{\gamma} &= 2\eta^{\mu\lambda}(\sigma^{\nu\tau})_{\alpha\gamma} + 2\eta^{\nu\tau}(\sigma^{\mu\lambda})_{\alpha\gamma} - 2\eta^{\mu\tau}(\sigma^{\nu\lambda})_{\alpha\gamma}\\ &\quad - 2\eta^{\nu\lambda}(\sigma^{\mu\tau})_{\alpha\gamma} + [\eta^{\mu\lambda}\eta^{\nu\tau} - \eta^{\mu\tau}\eta^{\nu\lambda}) + i\varepsilon^{\mu\nu\lambda\tau}]\varepsilon_{\alpha\gamma}\,.\end{aligned} \tag{A.22}$$

Note that σ^{0i} is not independent of σ^{ij}:

$$(\sigma^{0i})^{\beta}_{\alpha} = -(\sigma^{i})^{\beta}_{\alpha} = -\frac{1}{2}i\varepsilon^{ijk}(\sigma^{ij})^{\beta}_{\alpha}\,, \tag{A.23}$$

$$(\sigma^{i})^{\beta}_{\alpha}(\sigma^{i})^{\delta}_{\gamma} = 2\delta^{\delta}_{\alpha}\delta^{\beta}_{\gamma} - \delta^{\beta}_{\alpha}\delta^{\delta}_{\gamma}\,. \tag{A.24}$$

Formulae involving products of invariant tensors can be reduced using the basic relations:

$$\sigma^{\mu}\cdot\bar\sigma^{\nu}\cdot\sigma^{\lambda} + \sigma^{\lambda}\cdot\bar\sigma^{\nu}\cdot\sigma^{\mu} = 2\eta^{\mu\lambda}\sigma^{\nu} - 2\eta^{\mu\nu}\sigma^{\lambda} - 2\eta^{\lambda\nu}\sigma^{\mu}\,, \tag{A.25}$$

$$\sigma^{\mu}\cdot\bar\sigma^{\nu}\cdot\sigma^{\lambda} - \sigma^{\lambda}\cdot\bar\sigma^{\nu}\cdot\sigma^{\mu} = -2i\varepsilon^{\mu\nu\lambda\rho}\sigma_{\rho}\,, \tag{A.26}$$

where we define $\varepsilon^{0ijk} = \varepsilon^{ijk}$ and, for example, $(\sigma^{\mu}\cdot\bar\sigma^{\nu}\cdot\sigma^{\lambda})_{\alpha\dot\delta} = (\sigma^{\mu})_{\alpha\dot\beta}(\bar\sigma^{\nu})^{\dot\beta\gamma}(\sigma^{\lambda})_{\gamma\dot\delta}$,

$$(A\cdot\sigma^{\mu}\cdot\bar B)^{*} = B\cdot\sigma^{\mu}\cdot\bar A = \bar A\cdot\bar\sigma^{\mu}\cdot B\,, \tag{A.27}$$

and, in particular,

$$(A\cdot\sigma^{\mu}\cdot\bar A)^{*} = A\cdot\sigma^{\mu}\cdot\bar A = \bar A\cdot\bar\sigma^{\mu}\cdot A \tag{A.28}$$

is a real quantity. The Fierz identity takes the form:

$$A\cdot\sigma^{\mu}\cdot\bar B C\cdot\sigma_{\mu}\cdot\bar D = -2A\cdot C\bar B\cdot\bar D \tag{A.29}$$

for commuting spinors, with appropriate change of sign for the anticommuting case.

B. Inner Product

Here we discuss the inner product used in the text. It is at first surprising that a positive metric in the Fock space can be defined while preserving the non-compact Lorentz invariant metric. The reason this can be done is that the two metrics are not connected in any way. One is a Fock space metric defined for arbitrary polynomials, and the other is a restriction on the space of polynomials. Here are some examples.

We define the adjoint spinor $(c^{\alpha})^{\dagger}$ to satisfy:

$$[(c^{\alpha})^{\dagger}, c^{\beta}] = \delta^{\beta}_{\alpha}, \qquad [(\bar c^{\dot\alpha})^{\dagger}, \bar c^{\dot\beta}] = \delta^{\dot\beta}_{\dot\alpha}, \qquad [(\bar c^{\dot\alpha})^{\dagger}, c^{\beta}] = 0\,. \tag{B.1}$$

Now consider for example the following expression:

$$\langle 0|(c^\alpha \sigma^\mu_{\alpha\dot\beta}\bar\lambda^{\dot\beta})^\dagger (c^\gamma \sigma^\nu_{\gamma\dot\delta}\bar\lambda^{\dot\delta})|0\rangle = (\sigma^\mu_{\alpha\dot\beta})^\dagger \sigma^\nu_{\gamma\dot\delta}\delta^\gamma_\alpha \delta^{\dot\delta}_{\dot\beta} = (-\bar\sigma^{\dot\beta\alpha}_\mu)\sigma^\nu_{\alpha\dot\beta} = +2\delta^\nu_\mu \,. \tag{B.2}$$

Writing the indices in this way enables one to see that the positive definite inner product preserves the Lorentz invariance of Lorentz invariant expressions.

Another example is the following. The Minkowski metric controls the way that indices are contracted and this is preserved by the operators. The Fock space metric assigns a positive number to each vector. Take for example the state

$$\langle p_\mu p^\mu | p_\nu p^\nu \rangle = \langle 0|(p^\mu)^\dagger (p_\mu)^\dagger p_\nu p^\nu |0\rangle = 2\eta_{\mu\nu}\eta^{\mu\nu} = 8\,, \tag{B.3}$$

where we use

$$[p^\dagger_\mu, p_\nu] = \delta^\mu_\nu \,. \tag{B.4}$$

References

1. Becchi, C., Rouet, A., Stora, R.: Renormalization of gauge theories. Ann. Phys. **98**, 287–321 (1976)
2. Bonora, L., Cotta-Ramusino, P.: Some remarks on BRS transformations, anomalies and the cohomology of the Lie algebra of the group of gauge transformations. Commun. Math. Phys. **87**, 589–603 (1983)
3. Dixon, J.A.: Calculation of BRS cohomology with spectral sequences. Commun. Math. Phys. **139**, 495–526 (1991)
4. Dixon, J.A.: BRS cohomology of the chiral superfield. Commun. Math. Phys. **140**, 169–201 (1991)
5. Dixon, J.A., Minasian, R., Rahmfeld, J.: Higher Spin BRS Cohomology of Supersymmetric Chiral Matter in $D = 4$. Preprint CTP-TAMU-20/93 (hep-th@xxx9308013)
6. Dubois-Violette, M., Tallon, M., Viallet, C.M.: BRS algebras, analysis of the consistency equations of gauge theory. Commun. Math. Phys. **102**, 105–122 (1985)
7. Dubois-Violette, M., Henneaux, M., Tallon, M., Viallet, C.M.: Some results on local cohomologies in field theory. Phys. Lett. **B267**, 81–87 (1991)
8. Sohnius, M.: Introducing supersymmetry. Phys. Rep. **128**, 39–204 (1985)

Communicated by G. Felder