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On Diagonalization in Map$(M, \mathbf{G})$

Matthias Blau[1], **George Thompson**[2]
ICTP, P.O. Box 586, I-34014 Trieste, Italy

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Abstract: Motivated by some questions in the path integral approach to (topological) gauge theories, we are led to address the following question: given a smooth map from a manifold M to a compact group $\mathbf{G}$, is it possible to smoothly "diagonalize" it, i.e. conjugate it into a map to a maximal torus $\mathbf{T}$ of $\mathbf{G}$?

We analyze the local and global obstructions and give a complete solution to the problem for regular maps. We establish that these can always be smoothly diagonalized locally and that the obstructions to doing this globally are non-trivial Weyl group and torus bundles on M. We explain the relation of the obstructions to winding numbers of maps into $\mathbf{G}/\mathbf{T}$ and restrictions of the structure group of a principal $\mathbf{G}$ bundle to $\mathbf{T}$ and examine the behaviour of gauge fields under this diagonalization. We also discuss the complications that arise in the presence of non-trivial $\mathbf{G}$-bundles and for non-regular maps.

We use these results to justify a Weyl integral formula for functional integrals which, as a novel feature not seen in the finite-dimensional case, contains a summation over all those topological $\mathbf{T}$-sectors which arise as restrictions of a trivial principal $\mathbf{G}$ bundle and which was used previously to solve completely Yang–Mills theory and the G/G model in two dimensions.

1. Introduction

One of the most useful properties of a compact Lie group $\mathbf{G}$ is that its elements can be "diagonalized" or, more formally, conjugated into a fixed maximal torus $\mathbf{T} \subset \mathbf{G}$. In this paper we investigate to which extent this property continues to hold for spaces of (smooth) maps from a manifold M to a compact Lie group $\mathbf{G}$. Thus, given a smooth map $g : M \to \mathbf{G}$, the first thing one would like to know is if it can be written as

$$g(x) = h(x)\, t(x)\, h^{-1}(x)\,, \tag{1.1}$$

[1] e-mail: blau@ictp.trieste.it
[2] e-mail: thompson@ictp.trieste.it

where $t : M \to \mathbf{T}$ and $h : M \to \mathbf{G}$ are smooth globally defined maps. It is easy to see (by examples) that this cannot be true in general, not even for loop groups ($M = S^1$), and we are thus led to ask instead the following questions:

1. Under which conditions can (1.1) be achieved locally on M?
2. Under which conditions will $t(x)$ be smooth (while possibly relaxing the conditions on h)?
3. What are the obstructions to representing g as in (1.1) globally?

We will not be able to answer these questions in full generality. For those maps, however, which take values in the dense set $\mathbf{G}_r$ of regular elements of $\mathbf{G}$ we provide complete answers to 1–3. We establish that conjugation into $\mathbf{T}$ can always be achieved locally and that non-trivial $\mathbf{T}$-bundles on M are the obstructions to finding smooth functions h which accomplish (1.1) globally. Furthermore we prove that if either $\mathbf{G}$ or M is simply connected the diagonalized map t will be smooth globally. These results confirm the intuition that (in $SU(n)$ language) obstructions to diagonalization can arise from the ambiguities in either the phase of h or in the ordering of the eigenvalues of t.

While these equations seem to be interesting in their own right, they also arise naturally within the context of gauge fixing in non-Abelian gauge theories. In [7], 't Hooft has argued that a "diagonalizing gauge" may not only be technically useful but also essential for unravelling the physical content of these theories. For us the motivation for looking at this issue arose originally in the context of low-dimensional gauge theories. In particular, in [1,2] we used a path integral version of the Weyl integral formula, which relates the integral of a conjugation invariant function over $\mathbf{G}$ to an integral over $\mathbf{T}$, to effectively abelianize non-Abelian gauge theories like 2d Yang–Mills theory and the G/G gauged Wess–Zumino–Witten model. The path integrals for the partition function and correlation functions on arbitrary two-dimensional closed surfaces Σ could then be calculated explicitly and straightforwardly. Formally this Abelianization was achieved by using the local conjugation (gauge) invariance of the action to impose the "gauge condition" $g(x) \in \mathbf{T}$ (or its Lie algebra counterpart in the case of Yang–Mills theory). The correct results emerged when the resulting Abelian theory was summed over all topological sectors of $\mathbf{T}$-bundles on Σ, even though the original $\mathbf{G}$-bundle was trivial. This method has been reviewed and applied to some other models recently in [12].

In light of the above, the occurrence of the sum over isomorphism classes of $\mathbf{T}$-bundles can now be understood as a consequence of the fact that the chosen gauge condition cannot necessarily be achieved globally on $M = \Sigma$ by smooth gauge transformations. But while it is certainly legitimate to use a change of variables in the path integral which is not a gauge transformation, one needs to exercise more care when keeping track of the consequences of such a change of variables. Thus to the above list of questions we add (with hindsight)

4. What happens to $\mathbf{G}$ gauge fields A under the possibly non-smooth gauge transformation $A \to A^h = h^{-1}Ah + h^{-1}dh$? In particular, does this give rise to $\mathbf{T}$ gauge fields on non-trivial $\mathbf{T}$ bundles on M?
5. What is the correct version of the path integral analogue of the Weyl integral formula taking into account the global obstructions to achieving (1.1) globally? In particular, does this explain the appearance of the sum over all isomorphism classes of $\mathbf{T}$ bundles?

It turns out that indeed connections on $\mathbf{T}$-bundles appear in that way and that the Weyl integral formula should include a sum over those topological sectors which appear as obstructions to diagonalization. When M and $\mathbf{G}$ are such that there are no non-trivial $\mathbf{G}$ bundles on M, all isomorphism classes of torus bundles appear as obstructions (because then all torus bundles are restrictions of the trivial $\mathbf{G}$ bundle). In particular, this takes care of the two- and three-dimensional models considered in [1, 2] (as the contributions from the non-regular maps are suppressed by the zeros of the Faddeev–Popov determinant).

The situation concerning non-regular maps is quite different and much murkier. For example, there are maps taking on non-regular values just at isolated points but which nevertheless cannot be smoothly diagonalized in *any* open neighbourhood of one of these points. Consequently, the (differential-topology) methods we use in this paper to investigate regular maps are inappropriate in the more general situation. We present some examples illustrating the difficulties and discuss why our present treatment fails in these cases.

This paper is organized as follows: In Sect. 2 we briefly recall the basic facts we need from the theory of Lie groups. In Sect. 3 we discuss three prototypical examples which illustrate the possible ways in which (1.1) can fail either locally or globally. The first of these, a smooth map from S^1 to $SU(2)$, shows that not even $t(x)$ is necessarily smooth in general. The second, a regular map from S^2 to $SU(2)$, can be smoothly diagonalized locally but not globally. It provides a preliminary identification of certain obstructions in terms of winding numbers of maps from M to $\mathbf{G}/\mathbf{T}$ and also shows quite clearly how and why connections on non-trivial $\mathbf{T}$-bundles emerge. Finally, the third example (a map into $SO(3)$) illustrates how global smoothness of t can fail even for regular g when both M and $\mathbf{G}$ are not simply connected.

Section 4 contains the main mathematical results of this paper. We prove that regular maps can be smoothly conjugated into the torus over any contractible open set in M and we identify the obstructions to doing this globally. These results are summarized in Propositions 1 and 2. Proposition 3 contains the corresponding statements for Lie algebra valued maps. We also explain how to extend the results to sections of a non-trivial adjoint bundle $\mathrm{Ad}P_G$ of a principal bundle P_G and how finding a solution to (1.1) is related to restricting the structure group of P_G to $\mathbf{T}$. In particular, we establish a relation between restrictions of P_G and regular sections of $\mathrm{Ad}P_G$.

Section 5 contains some additional results which are useful for the application of the previous considerations to gauge theories. We first look at what happens to gauge fields on P_G under restrictions of the structure group. For two-dimensional theories (and simply-connected $\mathbf{G}$) we explain the appearance of the obstructions in the form of non-trivial torus bundles by relating their Chern classes to winding numbers associated with regular maps (the space of which is, in contrast to the space of all maps, not connected). We also consider $SU(n)$-bundles on four-manifolds to illustrate the obstruction to restrictions of the structure group. Finally, we address the issue of genericity of regular maps and make some comments on the problem of conjugating non-regular maps into the torus.

In Sect. 6, we turn to applications of the above results. We use them to justify a version of the Weyl integral formula for functional integrals over spaces of maps into a simply connected group. As a novel feature not present in the finite dimensional (or quantum mechanical path integral) version this formula includes a sum over all those topological sectors of $\mathbf{T}$ bundles which arise as restrictions of a trivial principal **G** bundle, justifying the method used in [1, 2] to solve exactly some low-dimensional (topological) gauge theories.

While we have used a global coordinate-independent approach to establishing the above results, in particular those of Sect. 4, they can of course also be obtained in a more pedestrian manner by working with coordinate patches, local trivializations and transition functions. At the referee's suggestion we primarily focus on the global approach in this paper and we refer the reader who likes to see things in local coordinates to the version of the paper available from the bulletin board (hep-th/9402097).

After having completed our investigations we came across a 1984 paper by Grove and Pedersen [5] in which the local obstructions we find in Sect. 4 are also identified, albeit using quite different techniques, see [5, Theorem 1.4]. The global issues which are our main concern in the present paper, in particular the relation between conjugation into the torus and restrictions of the structure group and the behaviour of gauge fields, are not addressed in [5], the emphasis there being on characterizing those spaces on which every continuous function taking values in normal matrices can be continuously diagonalized. These turn out to be so-called sub-Stonean spaces of dimension $\leqq 2$ satisfying certain additional criteria, [5, Theorem 5.6].

A final remark on terminology: we will (as above) occasionally find it convenient to use $SU(n)$ terminology even when dealing with a general compact Lie group **G**. In particular, we might say "diagonalize" when we should properly be saying "conjugate into the maximal torus" and we may loosely refer to the action of the Weyl group as "a permutation of the eigenvalues". We denote the space of maps from a manifold M into a group **G** by $\mathrm{Map}(M, \mathbf{G})$. Unless specified otherwise, these maps are taken to be smooth, although the topological results of this paper will of course continue to hold under less stringent requirements.

2. Background from the Theory of Lie Groups

We recall some basic facts from group theory we will need later on (see e.g. [3, 6]). Let **G** be a compact connected Lie group of rank r and **T** a maximal torus of **G**. We denote by $N(\mathbf{T})$ the normalizer of **T** in **G**, by W the Weyl group $W = N(\mathbf{T})/\mathbf{T}$, and by $\mathbf{G}_r$ and $\mathbf{T}_r = \mathbf{T} \cap \mathbf{G}_r$ the set of regular elements of **G** and **T** respectively, i.e. those lying in one and only one maximal torus of **G**. The non-regular elements of **G** form a set of codimension three in **G** and, although this set may not be a manifold, $\mathbf{G}_r$ and **G** have the same fundamental group, $\pi_1(\mathbf{G}_r) = \pi_1(\mathbf{G})$. Any element of **G** can be conjugated into **T**,

$$\forall g \in \mathbf{G}\, \exists h \in \mathbf{G} : h^{-1}gh \in \mathbf{T}\,. \tag{2.1}$$

For $g \in \mathbf{G}_r$, such an h is unique up to $h \to hn$, $n \in N(\mathbf{T})$, and if $h^{-1}gh = t \in \mathbf{T}$ then $(hn)^{-1}g(hn) = n^{-1}tn \in \mathbf{T}$ is one of the finite number of images $w(t)$ of t under the action of the Weyl group W. The conjugate map

$$\begin{aligned} q : \mathbf{G}/\mathbf{T} \times \mathbf{T}_r &\to \mathbf{G}_r \\ ([h], t) &\mapsto hth^{-1} \end{aligned} \tag{2.2}$$

is a $|W|$-fold covering onto $\mathbf{G}_r$. If **G** is simply connected, this W-bundle is trivial, and hence the Weyl group acts freely on each connected component $\mathbf{P}_r$ of $\mathbf{T}_r$

and simply transitively on the set of components. Thus we can identify $\mathbf{P}_r$, the image of a Weyl alcove under the exponential map, with a fundamental domain for the action of W on $\mathbf{T}_r$ and the restriction of q to $\mathbf{P}_r$ provides an isomorphism between $\mathbf{G}/\mathbf{T} \times \mathbf{P}_r$ and $\mathbf{G}_r$. In particular, one has $\pi_2(\mathbf{G}_r) = \mathbb{Z}^r$, to be contrasted with $\pi_2(\mathbf{G}) = 0$. In general, if one restricts q to $\mathbf{G}/\mathbf{T} \times \mathbf{P}_r$, it becomes a universal covering of $\mathbf{G}_r$ and the covering (2.2) is neither trivial nor connected. Nevertheless, the fact that, away from the non-regular points, the above map q is a smooth fibration (with discrete fibers) will be of utmost importance in our discussion in Sect. 4.

3. Examples: Obstructions to Globally Conjugating to the Torus

We will now take a look at three examples of maps which illustrate the obstructions to achieving (1.1) globally or smoothly. The first one, which we will only deal with briefly, illustrates what can go wrong with maps which pass through non-regular points of $\mathbf{G}$. We shall from then on (and until the end of Sect. 5) focus exclusively on regular maps and try to come to terms with them. The second example, a simple map from S^2 to $SU(2)$, allows us to detect an obstruction to globally and smoothly diagonalizing it more or less by inspection. This obstruction turns out to be a winding number associated with that map. Refining that winding number to include a gauge field contribution one can moreover read off directly that any attempt to force the map into the torus by a possibly non-smooth (discontinuous) h will give rise to non-trivial torus gauge fields. The third example, a map from the circle to $SO(3)$, highlights another obstruction which can only arise when neither $\mathbf{G}$ nor M is simply connected.

Example 1: A Map from S^1 to $SU(2)$. Let f be any smooth $\mathbb{R}$-valued function on the real line such that $f(x+2\pi) = -f(x)$. Then the map $g \in \text{Map}(S^1, SU(2))$ (the loop group of $SU(2)$) defined by

$$g(x) = \begin{pmatrix} \cos f(x) & -ie^{-ix/2} \sin f(x) \\ -ie^{ix/2} \sin f(x) & \cos f(x) \end{pmatrix} \tag{3.1}$$

is single-valued, $g(x+2\pi) = g(x)$, and smooth. As f is necessarily zero somewhere, g passes through the (non-regular) identity element. g can be diagonalized by a map h, $h^{-1}gh = t$, but for generic f neither h nor t are smooth. For instance, h can be chosen to be

$$h(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-ix/2} & 1 \\ -1 & e^{ix/2} \end{pmatrix}, \tag{3.2}$$

and t turns out to be

$$t(x) = \begin{pmatrix} e^{if(x)} & 0 \\ 0 & e^{-if(x)} \end{pmatrix}, \tag{3.3}$$

$$t(x+2\pi) = t^{-1}(x) \neq t(x). \tag{3.4}$$

What happens here is that, upon going around the circle, $t(x)$ comes back to itself only up to the action of the Weyl group, reflecting the ambiguity $h \to hn$ at the regular points of g mentioned in Sect. 2. Had g been regular everywhere to start off with, this ambiguity could have been consistently eliminated by giving a particular ordering prescription for the diagonal elements. Such a prescription, however, becomes ambiguous when two of the diagonal elements coincide (as at the identity element of the group). The fact that, when dealing with non-regular maps, one is leaving the realm of smooth or topological fiber bundles like (2.2) is illustrated by the observation that it is possible to conjugate g to a *continuous* and periodic map t', e.g.

$$t'(x) = \begin{pmatrix} e^{i|f(x)|} & 0 \\ 0 & e^{-i|f(x)|} \end{pmatrix} , \tag{3.5}$$

but that there is no differentiable choice of t', while any map h giving rise to a continuous t is necessarily discontinuous.

This illustrates clearly one of the difficulties one encounters when trying to diagonalize non-regular maps. Nevertheless, this difficulty disappears when one regards g as a smooth map from the real line to $SU(2)$, both h and t being smooth in that case. However, as we will see in Sect. 5, the procedure of diagonalization of non-regular maps is beset with rather more serious difficulties as well, with obstructions to smooth diagonalization appearing even on open and contractible sets.

Example 2: A Map from S^2 to $SU(2)$. A nice example (suggested to us by E. Witten) giving us a first idea of the possible obstructions in the case of regular maps and the role of non-trivial torus bundles is afforded by the following map from the two-sphere into $SU(2)$,

$$g(x) = \begin{pmatrix} ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & -ix_3 \end{pmatrix} , \tag{3.6}$$

where $x_1^2 + x_2^2 + x_3^2 = 1$. This map can also be written as $g(x) = \sum_k x_k\sigma_k$ which defines our conventions for the Pauli matrices σ_k. This map is clearly regular (the only non-regular elements of $SU(2)$ being plus or minus the identity element). It is a smooth map from the two-sphere to a two-sphere in $SU(2)$ and is, in fact, the identity map when one considers $SU(2) \sim S^3$ living inside $\mathbb{R}^4$ with cartesian co-ordinates (x_1,x_2,x_3,x_4) subject to $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. We represent elements of $SU(2)$ as $x_4\mathbf{1} + \sum_k x_k\sigma_k$ so that g maps the sphere to itself thought of as the equator of $S^3(x_4 = 0)$.

To detect a possible obstruction to diagonalizing g we proceed as follows. To any map f from the two sphere to the two sphere we may assign an integer, the winding number $n(f)$ of that map. This winding number is invariant under homotopies of f. Writing (as above) $f = \sum_k f_k\sigma_k$ with $\sum_k(f_k)^2 = 1$, an integral representation of its winding number is

$$n(f) = \frac{1}{32\pi} \int_{S^2} \mathrm{Tr} f[df, df] . \tag{3.7}$$

Clearly for (3.6) we have $n(g) = 1$, as it should be

Now suppose that one can smoothly conjugate the map g into a map $t : S^2 \to U(1)$ via some map h. As the space of maps from S^2 to $SU(2)$ is connected, g is homotopic to t and one has $n(g) = n(t)$. But, since $g^2 = -\mathbf{1}$, t is a constant map so that $n(t) = 0$, a contradiction.[1] More generally, if one has an $f : S^2 \to S^2 \subset S^3$

[1] t can be chosen to be either $t = \sigma_3$ or $t = (-\sigma_3)$. We fix on one of these throughout S^2 so that t is smooth. This is justified in the next section.

of the above form and one is able to smoothly conjugate this map to a map into $U(1)$, then one necessarily has $n(f) = 0$. So what we have learnt is that one may not, in general, smoothly conjugate into the maximal torus globally. We will see in the next section that this can be done locally in open neighbourhoods.

There is a disadvantage in simply considering the number (3.7) for it does not tell us how non-trivial $U(1)$ bundles will arise if we insist, in any case, on conjugating into $U(1)$, regardless of whether we can do so smoothly or not. There is a slight generalisation of the formula (3.7) which is not only a homotopy invariant, but for which conjugation (gauge) invariance can be established directly without any integration by parts. The advantage of such a formula is that it allows one to conjugate with arbitrary maps, not just smooth ones, and so to relate maps which are not homotopic.

Let A be a connection on the $SU(2)$ product bundle over the sphere. As the bundle is trivial such an A can be thought of as a Lie algebra valued one-form on S^2, $A \in \Omega^1(S^2, su(2))$. The number we want is

$$n(f,A) = -\frac{1}{32\pi}\int_{S^2} \mathrm{Tr}\, f[df, df] - \frac{1}{2\pi}\int_{S^2} \mathrm{Tr}[d(fA)], \tag{3.8}$$

and obviously coincides with (3.7) when both f and A are smooth. Furthermore $n(f,A)$ is gauge invariant, i.e. invariant under simultaneous transformation of f and A,

$$n(h^{-1}fh, A^h) = n(f,A), \tag{3.9}$$

where $A^h = h^{-1}Ah + h^{-1}dh$, even for discontinuous h. This is seen most readily by rewriting (3.8) in manifestly gauge invariant form,

$$n(f,A) = -\frac{1}{32\pi}\int_{S^2} \mathrm{Tr}\, f[d_A f, d_A f] - \frac{1}{2\pi}\int_{S^2} \mathrm{Tr}[fF_A], \tag{3.10}$$

with $d_A f = df + [A, f]$ and $F_A = dA + \frac{1}{2}[A, A]$.

Let us now choose h so that it conjugates our favourite map g into $U(1)$, say $g = h\sigma_3 h^{-1}$. Using (3.9) we find

$$n(g,A) = 1 = -\frac{1}{2\pi}\int_{S^2} \mathrm{Tr}\sigma_3 d(A^h). \tag{3.11}$$

In particular, if we introduce the Abelian gauge field $a = -\mathrm{Tr}\sigma_3 A^h$, we obtain

$$n(g,A) = 1 = \frac{1}{2\pi}\int_{S^2} da. \tag{3.12}$$

We now see the price of conjugating into the torus. The first Chern class of the $U(1)$ component of the gauge field A^h is equal to the winding number of the original map! We have picked up the sought for non-trivial torus bundles. In this case it is just the pull-back of the $U(1)$-bundle $SU(2) \to SU(2)/U(1) \sim S^2$ via g and this turns out to be more or less what happens in general.

As both g and its diagonalization $\pm\sigma_3$ may just as well be regarded as Lie algebra valued maps, this example establishes that obstructions to diagonalization will also arise in the (seemingly topologically trivial) case of Lie algebra valued maps.

As we will see in Sect. 5, a certain non-regular extension of this map provides us with an example of a map which cannot be smoothly diagonalized in any open neighbourhood of a non-regular point.

Example 3: *A Map from S^1 to $SO(3)$*. While we have seen in Example 1 that non-regularity is one obstruction to finding a globally well-defined smooth diagonalization t, even for regular g an obstruction to finding such a t may arise. We will establish in Sect. 4 that this can only happen when neither $\mathbf{G}$ nor M is simply connected. The raison d'être of this obstruction is the fact that diagonalization involves lifting a map into $\mathbf{G}_r$ to a map into $\mathbf{G}/\mathbf{T}\times\mathbf{T}_r$ which may not be possible if the fibration (2.2) is non-trivial. Here we illustrate this obstruction by a map from S^1 into $SO(3)_r$.

Consider first of all the following path in $SU(2)_r$,

$$\tilde{g}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{ix/2} & ie^{-ix/2} \\ ie^{ix/2} & e^{-ix/2} \end{pmatrix} . \tag{3.13}$$

As $\tilde{g}(2\pi) = -\tilde{g}(0)$, $\tilde{g}$ will project to a non-contractible loop $g \equiv \mathrm{Ad}(\tilde{g}) \in \mathrm{Map}(S^1, SO(3)_r)$. Explicitly, this g, satisfying $\tilde{g}^{-1}\sigma_k\tilde{g} = g_{kl}\sigma_l$ and $g(2\pi) = g(0)$, is given by

$$g(x) = \begin{pmatrix} 0 & 0 & 1 \\ \sin x & \cos x & 0 \\ -\cos x & \sin x & 0 \end{pmatrix} . \tag{3.14}$$

There is no obstruction to diagonalizing $\tilde{g}$, $\tilde{g} = \tilde{h}\tilde{t}\tilde{h}^{-1}$ and there are two solutions $\tilde{t}_\pm$ differing by a Weyl transformation (exchange of the diagonal entries). It can be checked that $\tilde{t}_\pm(2\pi)$ differs from $\tilde{t}_\pm(0)$ not only by a sign but also by a Weyl transformation,

$$\tilde{t}_\pm(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \pm i & 0 \\ 0 & 1 \mp i \end{pmatrix} = -\tilde{t}_\mp(2\pi) . \tag{3.15}$$

Hence $\tilde{t}$ will not project to a closed loop in $SO(3)$ and the diagonalization t of g will necessarily be discontinuous (non-periodic), as can also be checked directly. Choosing the torus $SO(2) \subset SO(3)$ to consist of elements of the form

$$\begin{pmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.16}$$

with the Weyl group acting as $y \to -y$, one finds that

$$t(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.17}$$

while

$$t(2\pi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.18}$$

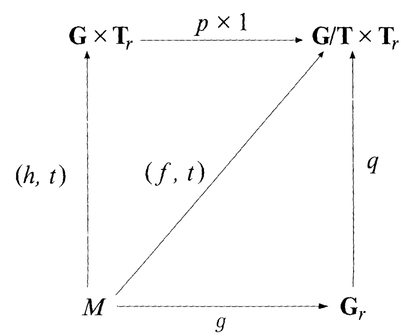
Hence the periodic regular map g cannot be diagonalized to a periodic map t and, regarded as map from S^1 into $SO(2)_r$, t will only be smooth locally.

This concludes our visit to the zoo of obstructions, and we will now establish some general local and global results concerning the diagonalizability of maps.

4. Local Conjugation to the Maximal Torus and Global Obstructions

In this section we will explore the diagonalizability of regular maps $g \in \mathrm{Map}(M, \mathbf{G}_r)$ and $g \in \mathrm{Map}(U, \mathbf{G}_r)$, where M is a smooth connected manifold and $U \subset M$ a contractible open set. The local considerations will of course apply equally well to local regular sections of the adjoint bundle $\mathrm{Ad}P_G$ of a non-trivial principal $\mathbf{G}$ bundle P_G over M.

Being able to locally conjugate smoothly into the maximal torus is the statement that we can find smooth maps $h_U \in \mathrm{Map}(U, \mathbf{G})$ and $t_U \in \mathrm{Map}(U, \mathbf{T})$ such that the restriction g_U of g to U can be written as $g_U = h_U t_U h_U^{-1}$. In other words, we are looking for a (local) lift of the map $g \in \mathrm{Map}(M, \mathbf{G}_r)$ to a map $(h, t) \in \mathrm{Map}(M, \mathbf{G}) \times \mathrm{Map}(M, \mathbf{T}_r)$. We will establish the existence of this lift in a two-step procedure indicated in diagram (4.1).

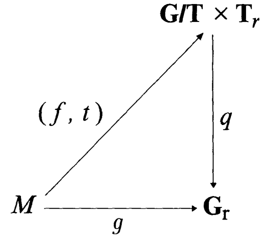


(4.1)

In the first step we lift g along the diagonal, i.e. we construct a pair (f, t), where $f \in \mathrm{Map}(M, \mathbf{G}/\mathbf{T})$, which projects down to g via the projection q introduced in (2.2). The obstruction to doing this globally is related to the possibility of having non-trivial W bundles on M (as in Examples 1 and 3 of the previous section) but only arises if neither $\mathbf{G}$ nor M is simply connected.

In the second step, dealing with the upper triangle, we will lift f locally to $\mathrm{Map}(M, \mathbf{G})$, and the obstruction to doing this globally is given by non-trivial $\mathbf{T}$ bundles on M (as in Example 2).

The First Lifting-Problem: *W-Bundles.* We begin by recalling that the conjugation map $q : \mathbf{G}/\mathbf{T} \times \mathbf{T}_r \to \mathbf{G}_r$, given by $([h], t) \mapsto hth^{-1}$, is a smooth $|W|$-fold covering of $\mathbf{G}_{/}$ so that $\mathbf{G}/\mathbf{T} \times \mathbf{T}_r$ is the total space of a principal fibre bundle over $\mathbf{G}_{/}$ with fibre and structure group W and projection q. Given the map g into $\mathbf{G}_r$, the base space of this bundle, we would like to lift this to a map into the total space, i.e. we want to find a pair $(f, t) \in \mathrm{Map}(M, \mathbf{G}/\mathbf{T} \times \mathrm{Map}(M, \mathbf{T}_r)$ such that diagram (4.2) commutes.



(4.2)

That such a map indeed exists locally is a consequence of the following fundamental result on the lifting of maps (see e.g. [8] for this and most of the other topological results used in this paper): If P is a (smooth) principal fiber bundle with base space B and f is a (smooth) map from a manifold X to B then f can be lifted to a (smooth) map into P if and only if the pull-back bundle f^*P over X is trivial. It is indeed easy to see that there is a direct correspondence between lifts of f and trivializing sections of f^*P.

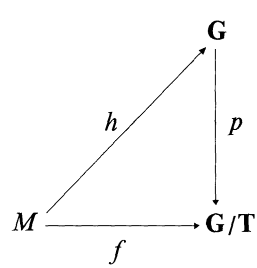
The first implication of this result is that locally, i.e. over some contractible open set $U \subset M$, the desired lift can always be found as the pull-back bundle will certainly be trivializable over U.

However, in certain cases we can sharpen this statement to establish the existence of a global lift. Consider e.g. the case when $\mathbf{G}$ is simply connected. As the principal W-bundle $\mathbf{G}/\mathbf{T} \times \mathbf{T}_r \to \mathbf{G}_r$ is then trivial, so is its pull-back to M via any map $g \in \text{Map}(M\, \mathbf{G}_r)$. Hence a lift (f, t) making the above diagram commute exists globally on M. There is an obvious $|W|$-fold ambiguity in the choice of such a lift.

Even if $\mathbf{G}$ is not simply connected but M is, the pull-back bundle is necessarily trivial over M (otherwise it would be a non-trivial covering of M) and again a lift (f, t) will exist globally.

Finally, there is a class of maps for which the W-obstruction does not arise regardless of what M and $\mathbf{G}$ are. This class consists of those maps g which are conjugate to a constant map t into $\mathbf{T}$. We will have more to say about these maps and why they are interesting in Sect. 6.

The Second Lifting Problem: $\mathbf{T}$-*Bundles.* It remains to lift the $\mathbf{G}/\mathbf{T}$ valued map f to $\mathbf{G}$. Thus we are looking for a $h \in \text{Map}(M, \mathbf{G})$ making the following diagram commute (with the replacement of M by U if only the local existence of (f, t) could be established):



(4.3)

Here p is the projection of the principal fibration $p : \mathbf{G} \to \mathbf{G}/\mathbf{T}$. By construction this map will then satisfy $g = hth^{-1}$. However, by the same result on the lifting of maps quoted above there will be an obstruction to finding such an h globally. As $\mathbf{G}$ can be regarded as the total space of a principal $\mathbf{T}$-bundle over $\mathbf{G}/\mathbf{T}$, the same reasoning as above leads us to conclude that such a lift exists iff $f^*\mathbf{G}$ is a trivial(izable) $\mathbf{T}$ bundle over M. Whether or not this is the case will depend on the interplay between the homotopy class of f and the classification of torus bundles on M. We will discuss both this torus bundle and the issue of its triviality in more detail below (see the discussion following (4.4) and Sect. 5). However, if we restrict f to $U \subset M$, then a lift h_U of f over U will always exist as the pull-back bundle is certainly trivializable over the contractible set U. The upshot of this is that, for a regular map g we can always locally find smooth $\mathbf{G}$-valued functions h_U such that $h_U^{-1} g_U h_U$ takes values in $\mathbf{T}_r$.

We summarize the results about the possibility to conjugate a map (or section) locally into a maximal torus in

Proposition 1. *Let* $\mathbf{G}$ *be a compact Lie group,* $\mathbf{T}$ *a maximal torus,* M *a smooth manifold,* $U \subset M$ *a contractible open set in* M, P_G *a principal* $\mathbf{G}$ *bundle over* M *and* g *a section of* $\mathrm{Ad}P_G$. *If* $g|_U \equiv g_U$ *is regular, then it can be smoothly conjugated into* $\mathbf{T}$. *In other words, under these circumstances there exist smooth functions* $t_U \in \mathrm{Map}(U, \mathbf{T}_r)$ *and* $h_U \in \mathrm{Map}(U, \mathbf{G})$ *such that* $g_U = h_U t_U h_U^{-1}$.

Of course, we already know a little bit more than that, for instance that under certain conditions the diagonalized map t will exist globally. We can also be more precise about the obstruction occurring in the second lifting problem, as torus bundles are classified by $H^2(M, \mathbb{Z}^r)$, where $r = \dim \mathbf{T}$ is the rank of $\mathbf{G}$. We have therefore established the following results concerning global obstructions to conjugating a map $g : M \to \mathbf{G}_r$ into the torus:

Proposition 2. *Let* $g : M \to \mathbf{G}_r$ *be a smooth regular map. Then a smooth map* $t : M \to \mathbf{T}_r$ *satisfying* $g = hth^{-1}$ *for some (not necessarily smooth) map* $h : M \to \mathbf{G}$ *exists globally if* $g^*(\mathbf{G}/\mathbf{T} \times \mathbf{T}_r)$ *is the total space of a trivial* W*-bundle over* M. *If, furthermore,* $f^*(\mathbf{G})$ *(where* f *is the* $\mathbf{G}/\mathbf{T}$*-part of the lift of* g*) is a trivial* $\mathbf{T}$ *bundle over* M, *then* h *can be chosen to be smooth globally.*

Corollary 1. *If either* M *or* $\mathbf{G}$ *is simply connected, a smooth diagonalization* $t \in \mathrm{Map}(M, \mathbf{T}_r)$ *of a regular* g *will exist globally. If, moreover* $H^2(M, \mathbb{Z}) = 0$, *then a smooth regular map* g *can be smoothly conjugated into a maximal torus, i.e. there exists a smooth function* $h \in \mathrm{Map}(M, \mathbf{G})$ *such that* $g = hth^{-1}$.

As loop groups are a particularly interesting and well studied class of spaces of group valued maps [9], we also mention separately the following immediate consequence of the above considerations:

Corollary 2. *If* $\mathbf{G}$ *is simply connected, every regular element of the group* $\mathbf{LG}$ *of smooth loops in* $\mathbf{G}$ *can be smoothly diagonalized.*

Examples 1 and 3 of Sect. 3 show that both regularity and simple connectivity are necessary conditions. What we have shown is that they are also sufficient.

Restriction of the Structure Group and Non-Trivial $\mathbf{T}$ *Bundles.* In order to deal with the question of diagonalizability of sections of non-trivial bundles as well as with the question of what happens to gauge fields under diagonalization, it will turn out to be convenient to look at the above constructions from a slightly different point of view, namely in terms of restrictions of the structure group of a principal $\mathbf{G}$ bundle P_G ($P_G \sim M \times \mathbf{G}$ in the above) to $\mathbf{T}$. In the following we will assume for simplicity that $\mathbf{G}$ is simply-connected, so that there are no obstructions to the first lifting problem.[2] Let $E_{G/T}$ be the homogeneous bundle associated to P_G ($E_{G/T} \sim M \times \mathbf{G}/\mathbf{T}$ if P_G is trivial). There is a bijective correspondence between sections of $E_{G/T}$ and restrictions of the structure group $\mathbf{G}$ of P_G to that of a principal $\mathbf{T}$ bundle $P_T \subset P_G$, this correspondence being given by pulling back the $\mathbf{T}$ bundle $P_G \to P_G/\mathbf{T} \sim E_{G/T}$ to M via a section $s : M \to E_{G/T}$ of $E_{G/T}$.

If P_G is trivial, there are no *a priori* obstructions to restrictions of the structure group and such sections correspond to maps from M to $\mathbf{G}/\mathbf{T}$. In particular, the solution of the first lifting problem provides one with such a map, namely f, and hence with the (possibly non-trivial) torus bundle

$$P_T \sim (\mathrm{Id} \times f)^*(M \times \mathbf{G} \to M \times \mathbf{G}/\mathbf{T})\,. \tag{4.4}$$

The relevance of this bundle lies in the fact (already mentioned above) that its non-triviality is the obstruction to finding a global diagonalizing map h lifting f. Moreover, connections on P_G will give rise to connections on P_T after diagonalization. It is therefore important to determine, which isomorphism classes of $\mathbf{T}$ bundles can arise in this way. This can readily be done when M is two-dimensional, since $\mathbf{G}/\mathbf{T}$ is then a classifying space for $\mathbf{T}$ bundles and the isomorphism class of P_T can be identified with the homotopy class of f. We will come back to this below.

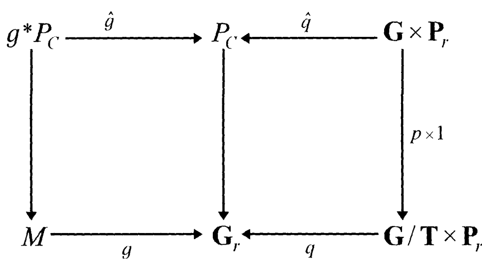
When P_G is non-trivial, there may be obstructions to restricting its structure group to $\mathbf{T}$ and one may wonder how much of the above then carries over to that case. It turns out that this obstruction is also the obstruction to finding regular sections of the adjoint bundle $\mathrm{Ad}P_G$ so that, as long as we restrict our attention to regular maps and sections (as we have been doing), the situation concerning non-trivial bundles is indeed exactly analogous to that for trivial bundles. We will also have a little bit more to say about this below.

There is a slightly more canonical way of describing the torus bundle P_T and the results obtained in Proposition 2, one which does not depend on the (arbitrary) choice of a maximal torus $\mathbf{T}$ of $\mathbf{G}$. We first observe that over $\mathbf{G}_r$ there is a natural torus bundle P_C (the centralizer bundle) with total space

$$P_C = \{(g_r, \hat{g}) \in \mathbf{G}_r \times \mathbf{G} : \hat{g} \in C(g_r)\}\,, \tag{4.5}$$

($C(g_r) \sim \mathbf{T}$ denoting the centralizer of g_r in $\mathbf{G}$) and projection $(g_r, \hat{g}) \mapsto g_r$. For any map $g \in \mathrm{Map}(M, \mathbf{G}_r)$ this bundle can be pulled back to a torus bundle g^*P_C over M and it is the possible non-triviality of this bundle which is the obstruction to finding a globally smooth h accomplishing the diagonalization. To make contact with the previous construction, we note that under the isomorphism $q : \mathbf{G}/\mathbf{T} \times \mathbf{P}_r \to \mathbf{G}_r$ the bundle P_C pulls back to the $\mathbf{T}$-bundle $\mathbf{G} \times \mathbf{P}_r \to \mathbf{G}/\mathbf{T} \times \mathbf{P}_r$, while the lift (f, t) in diagram (4.2) can be written as $(f, t) = q^{-1} \circ g$. This is illustrated in the diagram below.

[2] Similar considerations, however, apply in general, the non-triviality of (the pull-back of) the W-bundle being the obstruction to reducing the structure group from $N(\mathbf{T})$ to $\mathbf{T}$.



(4.6)

While not canonically a principal $\mathbf{T}$ bundle, under the non-canonical identification $C \sim \mathbf{T}$ the bundle P_C can be identified with the torus bundle P_T of (4.4).

Diagonalization of Sections of Non-Trivial Ad-Bundles. We consider now the situation where the bundle P_G is non-trivial. We furthermore assume the existence of a regular section g of $\mathrm{Ad}P_G$.[3] As, for simply-connected $\mathbf{G}$, the conjugation map q of (2.2) provides an isomorphism between $\mathbf{G}_r$ and $\mathbf{G}/\mathbf{T} \times \mathbf{P}_r$, a regular section of $\mathrm{Ad}P_G$ is the same thing as a section of $E_{G/T} \times \mathbf{P}_r$. Hence a regular section of $\mathrm{Ad}P_G$ will exist if and only if P_G can be restricted to a principal $\mathbf{T}$ bundle P_T. If this is the case, a smooth diagonalization t of g, a section of the trivial adjoint bundle $\mathrm{Ad}P_T$, will exist globally.

It may be instructive to see how these conclusions can be reached from a patching argument in terms of local data. Thus we assume that P_G is characterized by a set of transition functions $\{g_{\alpha\beta}\}$ with respect to a contractible open covering $\{U_\alpha\}$ of the base space M. Since g is a section of the adjoint bundle, its local representatives g_α are related on overlaps $U_\alpha \cap U_\beta$ by $g_\alpha = g_{\alpha\beta} g_\beta g_{\alpha\beta}^{-1}$. Locally, i.e. over each contractible open set U_α, the situation is exactly as in the case of $\mathbf{G}_r$-valued maps, and hence we can use the results of Proposition 1 to deduce the existence of smooth local diagonalizing functions $h_\alpha \in \mathrm{Map}(U_\alpha, \mathbf{G})$ such that $h_\alpha^{-1} g_\alpha h_\alpha = t_\alpha$ takes values in $\mathbf{T}_r$. It then follows that on overlaps the t_α are related by

$$t_\alpha = (h_\alpha^{-1} g_{\alpha\beta} h_\beta) t_\beta (h_\alpha^{-1} g_{\alpha\beta} h_\beta)^{-1} \ . \tag{4.7}$$

As the t_α are regular, (4.7) implies that the (transition) functions $h_\alpha^{-1} g_{\alpha\beta} h_\beta$ take values in $N(\mathbf{T})$ (otherwise t_α would be contained in two distinct maximal tori $\mathbf{T}$ and $(h_\alpha^{-1} g_{\alpha\beta} h_\beta)\mathbf{T}(h_\alpha^{-1} g_{\alpha\beta} h_\beta)^{-1}$ – a contradiction). Moreover, if $\mathbf{G}$ is simply-connected one can use the ambiguity $h_\alpha \to h_\alpha n_\alpha$ with $n_\alpha : U_\alpha \to N(\mathbf{T})$ to conjugate all the t_α into the same fundamental domain $\mathbf{P}_r \sim \mathbf{T}_r/W$. Thus the $h_\alpha^{-1} g_{\alpha\beta} h_\beta$ can actually be chosen to take values in $\mathbf{T}$,

$$h_\alpha^{-1} g_{\alpha\beta} h_\beta : U_\alpha \cap U_\beta \to \mathbf{T} \tag{4.8}$$

(hence reducing the structure group to $\mathbf{T}$). Then the locally defined diagonalized maps t_α piece together to a globally well defined $\mathbf{T}_r$-valued function $t = \{t_\alpha\}$,

$$t_\alpha = t_\beta \quad \text{on} \quad U_\alpha \cap U_\beta \ . \tag{4.9}$$

[3] As $\mathbf{G}_r$ is invariant under conjugation, the notion of a regular section is independent of the choice of local trivialization and hence well defined.

The h_α's, on the other hand, also define the corresponding section of $E_{G/T}$ and local sections of P_G in the trivialization determined by (4.8).

Conjugation of **g**-*valued Maps into the Cartan Subalgebra.* The question of diagonalizability of Lie algebra valued maps (the case of interest in e.g. Yang–Mills or Chern–Simons theory) can be addressed in complete analogy with the analysis for group valued maps performed above. It will turn out that the only substantial difference between the two is that the first obstruction (non-trivial W-bundles) does not arise. That the second obstruction, related to non-trivial torus bundles, persists can already be read off from Example 2 of Sect. 3 as the map $g = \sum_k x_k \sigma_k$ and its diagonalization $t = \pm\sigma_3$ considered there can equally well be regarded as Lie algebra valued maps.

Let us denote by $\mathbf{g}$ and $\mathbf{t}$ (a Cartan subalgebra of $\mathbf{g}$) the Lie algebras of $\mathbf{G}$ and $\mathbf{T}$ respectively and by $\mathbf{g}_r$ and $\mathbf{t}_r$ their regular elements. As in (2.2) there is a smooth $|W|$-fold covering

$$q' : \mathbf{G}/\mathbf{T} \times \mathbf{t}_r \to \mathbf{g}_r ,$$

$$q'([h], \tau) = h\tau h^{-1} . \tag{4.10}$$

However, $\mathbf{g}$ is a vector space and hence simply connected. As a consequence $\mathbf{g}_r$ is simply connected as well. Therefore this W-bundle is necessarily trivial and the first lifting problem can always be solved globally on M. This establishes the global existence of a lift (f, τ) of a smooth map $\phi \in \mathrm{Map}(M, \mathbf{g}_r)$ to $\mathbf{G}/\mathbf{T} \times \mathbf{t}_r$. In particular, a smooth global diagonalization $\tau \in \mathrm{Map}(M, \mathbf{t}_r)$ of ϕ always exists.

The second lifting problem depends only on the $\mathbf{G}/\mathbf{T}$-part f of the lift and is identical with that for group valued maps. Therefore the situation concerning Lie algebra valued maps is the following:

Proposition 3. *Let $\phi \in \mathrm{Map}(M, \mathbf{g}_r)$ be a smooth regular map into the Lie algebra $\mathbf{g}$ of a compact Lie group. Then a smooth diagonalization $\tau \in \mathrm{Map}(M, \mathbf{t}_r)$ exists globally. If $f^*\mathbf{G}$ is the total space of a trivial principal $\mathbf{T}$-bundle over M, then there exists a smooth functions $h \in \mathrm{Map}(M, \mathbf{G})$ such that $\phi = h\tau h^{-1}$ globally.*

Corollary 3. *If $H^2(M, \mathbb{Z}) = 0$, any $\phi \in \mathrm{Map}(M, \mathbf{g}_r)$ can be smoothly diagonalized.*

5. Connections, Winding Numbers and Non-Regular Maps

In this section we will briefly discuss a variety of topics related to the issue of diagonalization of maps and relevant to the application of the above results to the gauge theories which provided the original motivation for this investigation. In particular, we will look at what happens to gauge fields under diagonalization and the accompanying restriction of the structure group. We illustrate these considerations in the case of two-dimensional manifolds (relating the Chern classes of P_T to winding numbers of $g \in \mathrm{Map}(M, G_r)$) and $SU(n)$-bundles on four-manifolds. We end with some non-conclusive comments on non-regular maps.

Relation between Connections on **G** *and* **T** *Bundles.* Let $\mathbb{A}$ be a connection one-form on P_G (we use $\mathbb{A}$ to distinguish it from the one-form A on the base manifold M we will use to represent a connection on a trivial **G** bundle). Then the torus part

$\mathbb{A}^{\mathbf{t}}$ of $\mathbb{A}$ is a connection on the $\mathbf{T}$ bundle $P_G \to E_{G/T}$. Let s be a section of $E_{G/T}$, P_T the corresponding restricted bundle and $s_* : P_T \hookrightarrow P_G$ the corresponding bundle morphism. Then $(s_*)^* \mathbb{A}^{\mathbf{t}}$ is a connection on P_T.

If P_G is trivial, we represent $\mathbb{A}$ as a one-form on $M \times \mathbf{G}$ as $\mathbb{A} = g^{-1}Ag + g^{-1}dg \, (g \in \mathbf{G})$ and

$$((\mathrm{Id} \times f)_*)^*(g^{-1}Ag + g^{-1}dg)^{\mathbf{t}}$$

is a connection on P_T. By choosing local lifts $h_\alpha \in \mathrm{Map}(U_\alpha, \mathbf{G})$ of $f \in \mathrm{Map}(M, \mathbf{G/T})$, one obtains the local representatives

$$a_\alpha = (h_\alpha^{-1}Ah_\alpha + h_\alpha^{-1}dh_\alpha)^{\mathbf{t}} \tag{5.1}$$

on M of this connection on the possibly non-trivial bundle P_T. In a slightly cavalier fashion we will also denote by $a = (A^h)^{\mathbf{t}}$ the possibly singular representative of this connection on M obtained by choosing a possibly discontinuous lift of f to $\mathbf{G}$. The $\mathbf{k}$-part $B = (A^h)^{\mathbf{k}}$ of A^h (where we orthogonally decompose the Lie algebra $\mathbf{g}$ as $\mathbf{g} = \mathbf{t} \oplus \mathbf{k}$), on the other hand, transforms as a section of the associated bundle $P_T \times_{\mathbf{T}} \mathbf{k}$. *Mutatis mutandis* the same conclusions can be reached in the case of non-trivial P_G.

Torus Bundles on Two-Manifolds and Winding Numbers. In this subsection we consider the case when $M = \Sigma$ is a two-manifold. The obstruction to finding a globally defined h accomplishing the diagonalization of some $g \in \mathrm{Map}(\Sigma, \mathbf{G}_r)$ is encoded in the Chern class

$$c_1(P_T) \in H^2(\Sigma, \mathbb{Z}^r) \sim \mathbb{Z}^r \tag{5.2}$$

of the corresponding torus bundle P_T. One may wonder, how this topological information is encoded in the original map g as, after all, the space of maps $\mathrm{Map}(\Sigma, \mathbf{G})$ to a simply-connected group $\mathbf{G}$ is connected. The point is that, while this is true, the space of regular maps is not connected. Recalling the isomorphism $\mathbf{G}_r \sim \mathbf{G/T} \times \mathbf{P}_r$ one finds that

$$\pi_0(\mathrm{Map}(\Sigma, \mathbf{G}_r)) \sim \pi_0(\mathrm{Map}(\Sigma, \mathbf{G/T})) \sim \pi_2(\mathbf{G/T}) \sim \mathbb{Z}^r \,. \tag{5.3}$$

One thus expects the Chern classes of P_T to represent the winding numbers of the map $g \in \mathrm{Map}(\Sigma, \mathbf{G}_r)$.

Abstractly this can be seen by noting that, for simply connected $\mathbf{G}$, $\mathbf{G/T}$ is a classifying space for $\mathbf{T}$ bundles on Σ, so that $\mathbf{T}$ bundles are classified by homotopy classes of maps from Σ to $\mathbf{G/T}$. Furthermore, regular maps in $\mathrm{Map}(\Sigma, \mathbf{G}_r)$ are regularly homotopic (i.e. homotopic in $\mathrm{Map}(\Sigma, \mathbf{G}_r)$) iff their lifts to $\mathrm{Map}(\Sigma, \mathbf{G/T})$ are homotopic so that $\mathbf{T}$ bundles can alternatively be classified by homotopy classes of maps into $\mathbf{G}_r$. In particular, this establishes that all isomorphism classes of $\mathbf{T}$ bundles on Σ will arise upon diagonalization of elements of $\mathrm{Map}(\Sigma, \mathbf{G}_r)$. This holds more generally if there are no non-trivial $\mathbf{G}$ bundles on M.

Concretely, one can establish a correspondence between the Chern–Weil representatives of $c_1(P_T)$ and integral representations of winding numbers of $f \in \mathrm{Map}(\Sigma, \mathbf{G/T})$. Denoting by $a = -a^l \lambda_l$, the $\{\lambda^l\}$ a set of fundamental weights of $\mathbf{G}$, the l^{th} component of the torus connection a, the Chern classes of P_T can be represented by

$$c_1^{(l)}(a) = \frac{1}{2\pi} \int_\Sigma da^l \,. \tag{5.4}$$

Winding numbers of maps f from Σ to **G/T**, on the other hand, are also characterized by an r-tuple of integers, associated with the pull-back of the r fundamental Kirillov–Kostant symplectic forms on the regular coadjoint orbit **G/T**. An integral representation for these winding numbers is

$$n^{(l)}(f) = \frac{1}{4\pi} \int_{\Sigma} \mathrm{Tr}\lambda^l [h^{-1}dh, h^{-1}dh], \tag{5.5}$$

where we have identified f with the map $h\lambda^l h^{-1}$. Notice that the integrand is exact (and hence the winding number is zero) if h is globally defined. The relevant formula relating these expressions is

$$c_1^{(l)}(a) = n^{(l)}(f = h\lambda^l h^{-1}) - \frac{1}{2\pi} \int_{\Sigma} d(\mathrm{Tr} f A), \tag{5.6}$$

where $(A^h)^{\mathbf{t}} = a^l \alpha_l$ and $\{\alpha_l\}$ are the set of simple roots dual to $\{\lambda_l\}$. The boundary term is automatically zero in the case of simply-connected groups we have been considering, as then both A and f are globally defined. The advantage of adding this boundary term to the winding number is that the resulting expression is invariant even under discontinuous gauge transformations which would change the ordinary winding number of f. The reason for this is that, in terms of the fields A and h, the original gauge symmetry $A \to A^{\tilde{g}}$, $g \to \tilde{g}^{-1} g \tilde{g}$ reads $A \to A^{\tilde{g}}$, $h \to \tilde{g}^{-1} h$. Hence A^h and the right-hand side of (5.6), which can be thought of as a generalized winding number $n^{(l)}(f, A)$ of f, are manifestly gauge invariant. As such they should provide integral representations for the magnetic numbers introduced in [4] in a related context. Using some trace identities it can be checked that in the case $\mathbf{G} = SU(2)$ the various expressions given above for the winding numbers reduce to those given in Example 2 of Sect. 3.

Returning to our problem of conjugating maps into the torus, we can now read off directly from the above that a smooth map $g \in \mathrm{Map}(\Sigma, \mathbf{G}_r)$ can be smoothly conjugated into the torus iff the (generalized) winding number of f is zero. Furthermore, if one insists on conjugating into the torus nevertheless, albeit by a non-continuous h, the resulting map f is a constant map (with winding number zero) but $n^l(f, A)$ will remain unchanged, measuring the obstruction to doing this smoothly.

$SU(n)$-Bundles on Four-Manifolds. We recall the observation made in Sect. 4 that a principal **G** bundle P_G admits a restriction to **T** if and only if its adjoint bundle $\mathrm{Ad}P_G$ has a regular section. It explains the intimate relationship we found between diagonalization and restriction of the structure group and highlights the crucial role played by the assumption of regularity.

In general, the question whether either of these two assertions has an affirmative answer (Is there a restriction? Is there a regular section?) has to be tackled by the methods of obstruction theory. In four dimensions, however, necessary and sufficient conditions for the existence of restrictions of $SU(n)$ bundles can be read off more or less by inspection and this gives some insight into the nature of this problem.

We recall first that $SU(n)$ bundles P on a compact oriented four-manifold are completely classified by the second Chern class $c_2(P) \in H^4(M, \mathbb{Z}) \sim \mathbb{Z}$. In terms of the curvature F_A of a connection A on P the Chern–Weil representative of $c_2(P)$ is

$$c_2(P) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\, F_A F_A \tag{5.7}$$

(with the trace normalized to $\mathrm{Tr}\,\tau^a\tau^b = 2\delta^{ab}$, the τ^a a basis of the Lie algebra of $SU(n)$). Torus bundles $P_T, \mathbf{T} \sim U(1)^{n-1}$, on the other hand are classified by $H^2(M, \mathbb{Z}^{n-1})$. As all $\mathbf{T}$ bundles can be regarded as $SU(n)$ bundles, they will all arise as the restriction of some $SU(n)$ bundle but not necessarily as restrictions of the trivial $SU(n)$ bundle. Moreover, some $SU(n)$ bundles may have no restrictions at all while others may admit several inequivalent restrictions. In this four-dimensional context it is straightforward to find obstructions to such an Abelianization. For a principal $SU(n)$ bundle which admits a restriction to a $\mathbf{T}$ bundle P_T, its second Chern class is related to the curvature of a connection a on P_T by

$$c_2(P) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\, da\, da\,. \tag{5.8}$$

By looking at some concrete examples of four-manifolds we will see that this relation can impose severe constraints on $c_2(P)$.

Let us, for instance, take M to be the four-sphere $M = S^4$. Then there are no non-trivial $\mathbf{T}$ bundles on M as $H^2(M, \mathbb{Z}) = 0$, and the right-hand side of (5.8) is zero as the integrand is then necessarily globally exact. Hence we reach the conclusion that only the trivial $SU(n)$ bundle on S^4 admits a restriction to a $\mathbf{T}$ bundle (the trivial $\mathbf{T}$ bundle in this case). This may also be seen in a different way by noting that, on any n-sphere, the bundle is characterized by the glueing (transition) function h from the equator $\sim S^{n-1}$ to the group $\mathbf{G}$. If h takes values in $\mathbf{T}$, then its winding number is zero ($\pi_{n-1}(\mathbf{T}) = 0$ for $n > 2$) and hence

$$8\pi^2 c_2(P) = \int_{S^3} \mathrm{Tr}(h^{-1}dh)^3 = 0\,. \tag{5.9}$$

Thus we conclude that the adjoint bundles of non-trivial $SU(n)$ bundles over S^4 have no regular sections whatsoever.

This is not to mean that only trivial $SU(n)$ bundles can be reduced to $\mathbf{T}$ bundles. As another example consider $M = \mathbb{CP}^2$ and $\mathbf{G} = SU(2)$. In this case, $H^2(M, \mathbb{Z}) \sim H^4(M, \mathbb{Z}) \sim \mathbb{Z}$, generated by the Kähler form ω. Thus there are non-trivial torus and $SU(2)$ bundles on $\mathbb{CP}^2$. The curvature of the connection on a $U(1)$ bundle is cohomologous to $k\omega$ for $k \in \mathbb{Z}$ and, as $\omega^2[\mathbb{CP}^2] = 1$, a necessary condition for an $SU(2)$ bundle P to be reducible to $U(1)$ is that $c_2(P) = k^2$ for some $k \in \mathbb{Z}$. As any $U(1)$ bundle with first Chern class k is the reduction of some $SU(2)$ bundle, this condition is also sufficient and for every non-trivial $SU(2)$ bundle on $\mathbb{CP}^2$ with $c_2(P) = k^2$ there are two inequivalent reductions to $U(1)$, characterized by the first Chern class $\pm k$.

This situation is more or less the same for all compact four-manifolds. If a torus bundle, thought of as an $SU(n)$ bundle, has second Chern class $c_2 = m$, then it can be obtained as the reduction of this $SU(n)$ bundle. Conversely, if an integer m does not arise as the second Chern class of some torus bundle, the corresponding $SU(n)$ bundle with $c_2(P) = m$ cannot be Abelianized. As a consequence of the above result such bundles have no regular sections whatsoever.

Non-Regular Maps. While we have seen above that non-trivial adjoint bundles may admit no regular sections at all, which forces us to face the task of diagonalizing non-regular sections, one may have hoped that at least for trivial bundles regular maps are, in some sense, generic. If this were so, then "most" maps could indeed be conjugated into smooth torus-valued functions by the results of Sect. 4, at least via locally defined or discontinuous diagonalizing functions h. But, as the set of non-regular points is of codimension three in $\mathbf{g}$ or $\mathbf{G}$, a dimension counting argument shows that this will not be the case if the dimension of M is larger than two. Worse than that, in the case of $\mathbf{G}$-valued maps there may be entire connected components of Map(M,$\mathbf{G}$) not containing a single regular map. To see that, let us consider a simple example the space of maps from $M = S^3$ to $\mathbf{G} = SU(2) \sim S^3$. This space consists of an infinite number of connected components labelled by the winding number of the map in $\pi_3(SU(2)) = \mathbb{Z}$. As the only non-regular elements of $SU(2)$ are plus or minus the identity, regular maps are those which avoid the north and south poles of the target S^3. But any map in one of the non-trivial winding number sectors has, in particular, the property that its image is the entire $SU(2)$, covered an appropriate number of times. Hence, no map with a non-trivial winding number can be regular.

The fact that even for trivial bundles there may be too many non-regular maps for comfort provides an additional impetus for coming to terms with the diagonalization of these maps.

As a simple example, consider the extension of the map of Example 2 of Sect. 3 to the identity map from the three-sphere to $SU(2)$, $g(x) = x_4\mathbf{1} + \sum_k x_k\sigma_k$. This map takes on non-regular values only at $x_4 = \pm 1$. There is clearly no smooth diagonalization of the restriction of this map to any open set containing the north-pole $\{x_4 = 1\}$. If there were, this would in particular imply the existence of a global smooth diagonalization of a map from S^2 to $SU(2)$ which is regularly homotopic to that of Example 2 – a contradiction.

There are two conclusions that can be drawn from this example and Example 1 of Sect. 3. The first is that, in general, a non-regular smooth (or continuous) map cannot be smoothly (or continuously) diagonalized even on open contractible sets. As a consequence, the second conclusion one can draw is that the framework of locally trivializable bundles is simply not suitable for addressing the question of diagonalization of non-regular maps.

The source of the problem is, of course, that the conjugation map from $\mathbf{G}/\mathbf{T}\times\mathbf{T}$ to $\mathbf{G}$ is not proper at non-regular points of $\mathbf{G}$. This is reflected in the fact that the quotient

$$\mathbf{G}/\mathrm{Ad}\mathbf{G} \sim \mathbf{T}/W\,, \tag{5.10}$$

unlike its regular counterpart $\mathbf{T}_r/W$, is not a smooth manifold (but the closure of a Weyl alcove or, rather, its image under the exponential map), and that the fiber of $\mathbf{G}\to\mathbf{G}/\mathrm{Ad}\mathbf{G}$ above a singular (non-regular) point is strictly smaller than that at a regular point. Clearly this is a rather singular situation to consider and different methods are needed to make some headway here.

To end this section on a positive note we mention that there is one rather special type of non-regular maps to which the considerations of this paper continue to apply. These are maps g whose degree of non-regularity is constant (meaning that the centralizers $C(g(x))$ are isomorphic to some fixed $C(g) \supset \mathbf{T}$ for all $x \in M$). All that one needs to do in that case is to replace $\mathbf{G}/\mathbf{T}$ in the fundamental fibration (2.2) by the appropriate smaller non-regular coadjoint orbit $\mathbf{G}/C(g)$. Typically, it is this type of non-regular maps that one encounters in topological field theory (see the remarks at the end of Sect. 6).

6. Applications: A Weyl Integral Formula for Path Integrals

In the previous sections we have analyzed the problem of diagonalizing maps from a manifold M into a compact Lie group $\mathbf{G}$ or its Lie algebra $\mathbf{g}$. As mentioned in the Introduction, this problem arose in a field theoretic context when we attempted to exploit the rather large local gauge symmetry present in certain low-dimensional non-Abelian gauge theories to abelianize (and hence more or less trivialize) the theories via diagonalization [1, 2, 12]. Assuming that the contributions from non-regular maps can indeed be neglected in these examples (and we have nothing to add to the arguments put forward in [2] to that effect), the analysis of the present paper can be regarded as a topological justification for the formal path integral version of the Weyl integral formula we used to solve these theories.

The Weyl integral formula expresses the integral of a smooth (real or complex valued) function over $\mathbf{G}$ in terms of an integral over $\mathbf{T}$ and $\mathbf{G}/\mathbf{T}$, using the conjugation map q (2.2) to pull back the Haar measure on $\mathbf{G}$ to $\mathbf{G}/\mathbf{T}\times\mathbf{T}$ and reads

$$\int_{\mathbf{G}} dg\, f(g) = \int_{\mathbf{T}} dt \Delta(t) \int_{\mathbf{G}/\mathbf{T}} dg\, f(g^{-1}tg)\,. \tag{6.1}$$

Here $\Delta(t)$, the Weyl determinant, is the Jacobian of q. Its precise form will not interest us here and we just note that it vanishes precisely at the non-regular points of $\mathbf{T}$ (this being the mechanism by which contributions from non-regular points should be suppressed in the functional integral). For an explanation of the standard proof of (6.1) and for a derivation in the spirit of the Faddeev-Popov trick see [1,2]. The case of interest to us is when the function f is conjugation invariant (a class function), i.e. when f satisfies

$$f(h^{-1}gh) = f(g) \quad \forall g, h \in \mathbf{G}\,. \tag{6.2}$$

In that case, since any element of $\mathbf{G}$ is conjugate to some element of $\mathbf{T}$, both f and its integral over $\mathbf{G}$ are determined by their restriction to $\mathbf{T}$ and the Weyl integral formula reflects this fact,

$$\int_{\mathbf{G}} dg\, f(g) = \int_{\mathbf{T}} dt \Delta(t) f(t)\,. \tag{6.3}$$

It is this formula which we would like to generalize to functional integrals, i.e. to a formula which relates an integral over a space of maps into $\mathbf{G}$ to an integral over a space of maps into $\mathbf{T}$.

For concreteness, consider a local functional $S[g; A]$ (the "action") of maps $g \in$ Map$(M, \mathbf{G})$ and gauge fields $A \in \Omega^1(M, \mathbf{g})$, i.e. of sections of Ad$P_G$ and connections on a trivial principal $\mathbf{G}$ bundle $P_G \sim M \times \mathbf{G}$ (a dependence on other fields could be included as well). Assume that $\exp iS[g; A]$ is gauge invariant,

$$\exp iS[g; A] = \exp iS[h^{-1}gh; A^h] \quad \forall h \in \mathrm{Map}(M, \mathbf{G})\,, \tag{6.4}$$

at least for smooth h. If e.g. a partial integration is involved in establishing the gauge invariance (as in Chern–Simons theory), this may fail for non-smooth h's and more care has to be exercised when such a gauge transformation is performed. Then the functional $F[g]$ obtained by integrating $\exp iS[g; A]$ over A,

$$F[g] := \int D[A] \exp iS[g; A]\,, \tag{6.5}$$

is conjugation invariant,

$$F[h^{-1}gh] = F[g]\,. \tag{6.6}$$

It is then tempting to use a formal analogue of (6.3) to reduce the remaining integral over g to an integral over maps taking values in the Abelian group **T**. In field theory language this amounts to using the gauge invariance (6.4) to impose the "guage condition" $g(x) \in \mathbf{T}$. The first modification of (6.3) will then be the replacement of the Weyl determinant $\Delta(t)$ by a functional determinant $\Delta[t]$ of the same form which needs to be regularized appropriately (see the Appendix of [2]).

However, the main point of this paper is that this is of course not the whole story. We already know that this "gauge condition" cannot necessarily be achieved smoothly and globally. Insisting on achieving this "gauge" nevertheless, albeit via non-continuous field transformations, turns the **t**-component a of the transformed gauge field A^h into a gauge field on a possibly non-trivial **T** bundle P_T (while the **k**-component transforms as a section of an associated bundle). Moreover we know that all those **T** bundles will contribute which arise as restrictions of the (trivial) bundle P_G. Let us denote the set of isomorphism classes of these **T** bundles by $[P_T;\, P_G]$. Hence the "correct" (meaning correct modulo the analytical difficulties inherent in making any field theory functional integral rigorous) version of the Weyl integral formula, capturing the topological aspects of the situation, is one which includes a sum over the contributions from the connections on all the isomorphism classes of bundles in $[P_T;\, P_G]$.

Let us denote the space of connections on P_G and on a principal **T** bundle P_T^l representing an element $l \in |P_T;\, P_G|$ by $\mathscr{A}$ and $\mathscr{A}[l]$ respectively and the space of one-forms with values in the sections of $P_T^l \times_{\mathbf{T}} \mathbf{k}$ by $\mathscr{B}[l]$. Then, with

$$Z|P_G| = \int_{\mathscr{A}} D[A] \int D[g] \exp iS[g;\, A]\,, \tag{6.7}$$

the Weyl integral formula for functional integrals reads

$$Z|P_G| = \sum_{l \in [P_T;\, P_G]} \int_{\mathscr{A}[l]} D[a] \int_{\mathscr{B}[l]} D[B] \int D[t] \Delta[t] \exp iS[t;\, a, B] \tag{6.8}$$

(modulo a normalization constant on the right-hand side). The t-integrals carry no l-label as the spaces of sections of $\mathrm{Ad}P_T^l$ are all isomorphic to the space of maps into **T**. There is an exactly analogous formula generalizing the Lie algebra version of the Weyl integral formula. On the basis of the results established in this paper it is also possible to write down a functional integral version of (6.1), in which the summation over the topological sectors in (6.8) will have to be replaced by a sum over integrals of the connected components (winding number sectors) of $\mathrm{Map}(M, \mathbf{G}_r)$. This integral formula can then be applied to theories having less or no gauge symmetry (like the G/H gauged Wess–Zumino–Witten models for $H \subset G$).

In the examples considered in [1,2], Chern–Simons theory on three-manifolds of the form $\Sigma \times S^1$, 2d Yang–Mills theory and the **G/G** gauged Wess–Zumino–Witten model, the fields B entered purely quadratically in the reduced action $S[t; a, B]$ and could be integrated out directly, leaving behind an effective Abelian theory depending on the fields t and a with a measure determined by $\Delta[t]$ and the (inverse) functional determinant coming from the B-integration. The general structure of these terms and the "quantum corrections" coming from the regularization has been determined in [12].

A further property these models were found to have is that they localize onto reducible connections and their isotropy groups (in the case of the **G/G** model) respectively algebras (for Yang–Mills theory) so that, in practice, the necessity only ever arose to diagonalize these maps. This is possible globally even if the group is not simply connected (when, as we recall from Sect. 4, the existence of a globally smooth diagonalized map t or τ is not guaranteed *a priori*). The reason for this is the following (for group valued maps – the Lie algebra case is entirely analogous).

The reducibility condition $A^g = A$ implies that Tr g^n is constant for all n. This allows one to determine that g is conjugate to a t which is constant globally and (of course) unique up to an overall W-transformation. This provides the $\mathbf{T}_r$ part of the lift in diagram (4.2). Furthermore, the constancy of the traces implies that g can itself be regarded as a map into $\mathbf{G}/\mathbf{T}$ (or $\mathbf{G}/C(g)$) and hence furnishes the $\mathbf{G}/\mathbf{T}$-part f of the lift. At this point the argument can then proceed as in the simply-connected case. The fact that isotropy groups of connections are indeed conjugate to subgroups of $\mathbf{G}$ (thought of as spaces of constant maps) is well known. What seems to be less generally appreciated is the fact that the conjugation itself cannot necessarily be done globally.

We have also applied this formula to several other models like BF theories in three dimensions (related to $3d$ gravity) and the supersymmetric Chern–Simons models of Rozansky and Saleur [10]. The formula can also be used to go some way towards evaluating the generating functional for Donaldson theory on Kähler manifolds with the action as in [11]. These results will be presented elsewhere.

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