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Differential Calculus on $ISO_q(N)$, Quantum Poincaré Algebra and q-Gravity

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Abstract: We present a general method to deform the inhomogeneous algebras of the B_n, C_n, D_n type, and find the corresponding bicovariant differential calculus. The method is based on a projection from $B_{n+1}, C_{n+1}, D_{n+1}$. For example we obtain the (bicovariant) inhomogeneous q-algebra $ISO_q(N)$ as a consistent projection of the (bicovariant) q-algebra $SO_q(N+2)$. This projection works for particular multiparametric deformations of $SO(N+2)$, the so-called "minimal" deformations. The case of $ISO_q(4)$ is studied in detail: a real form corresponding to a Lorentz signature exists only for one of the minimal deformations, depending on one parameter q. The quantum Poincaré Lie algebra is given explicitly: it has 10 generators (no dilatations) and contains the *classical* Lorentz algebra. Only the commutation relations involving the momenta depend on q. Finally, we discuss a q-deformation of gravity based on the "gauging" of this q-Poincaré algebra: the lagrangian generalizes the usual Einstein–Cartan lagrangian.

1. Introduction

Perturbative quantum Einstein gravity is known to be mathematically inconsistent, since it is plagued by ultraviolet divergences appearing at two-loop order (the absence of one-loop divergencies was found in [1], whereas two-loop divergencies were explicitly computed in [2]). In supergravity the situation is only slightly better, the divergences starting presumably at three loops[1]. In the last fifteen years or so there have been various proposals to overcome this difficulty, and consistently quantize gravity either alone or as part of a unified theory of the fundamental interactions. Such a unified picture is provided by superstrings (see for a review [3]), where Einstein gravity arises as a low-energy effective theory, coupled more or less realistically to gauge fields and leptons, and regulated at the Planck scale by an

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[1] No explicit calculation like the one of ref. [2] exists, but there is no symmetry principle that excludes them.

infinite number of heavy particles (the superstring massive spectrum). How to make phenomenological predictions from superstrings is still the object of current research.

Another, more speculative, line of thought deals with the quantization of spacetime itself, whose smoothness under distances of the order of the Planck length $L_P \sim 10^{-33}$cm is really a mathematical assumption. Indeed if we probe spacetime geometry with a test particle, the accuracy of the measure depends on the Compton wavelength of the particle. For higher accuracy we need a higher mass m of the particle, and for $m \sim 1/L_P$ the mass significantly modifies the curvature it is supposed to measure (i.e. the curvature radius becomes of the order of the particle wavelength: the particle is no more a *test* particle).

Thus it is not inconceivable that spacetime has an intrinsic cell-like structure: lattice gravity, or Regge calculus may turn out to be something more fundamental than a regularization procedure. Another way to discretization is provided by non-commutative geometry: when spacetime coordinates do not commute the position of a particle cannot be measured exactly. The notion of spacetime *point* loses its physical meaning, and is to be replaced by the notion of spacetime cell; the question is whether this sort of lattice structure does indeed regularize gravity at short distances. References on non-commutative geometry and its uses for regularization can be found in [4].

Fundamental interactions are described by field theories with an underlying algebraic structure given by particular Lie groups, as for ex. unitary Lie groups for the strong and electroweak interactions and the Poincaré group for gravity. It is natural to consider the so-called quantum groups [5, 6, 7] (continuous deformations of Lie groups whose geometry is non-commutative) as the algebraic basis for generalized gauge and gravity theories. The bonus is that we maintain a rich algebraic structure, more general than Lie groups, in a theory living in a discretized space. This does not happen usually with lattice approaches, where one loses the symmetries of the continuum. For a review of non-commutative differential geometry on quantum groups see for ex. [8]. This subject, initiated in [9], has been actively developed in recent years: a very short list of references can be found in [10–15].

In this paper we address the problem of constructing a non-commutative deformation of Einstein gravity. For this we need a q-deformation of the Poincaré Lie algebra. We obtain it in Sect. 4 as a special case of the quantum inhomogeneous $ISO_q(N)$ algebras, whose differential calculus is presented in Sect. 3. These algebras are obtained as projections from particular multiparametric deformations of $SO(N+2)$, called "minimal" deformations. Their R matrix is *diagonal*, and the braiding matrix $\hat{R} = PR$ has unit square. On q-groups with diagonal R-matrices see for ex. [16] and references therein. Deformations of Lie algebras whose braiding matrix has unit square were considered some time ago by Gurevich [17].

The projective method to obtain the bicovariant differential calculus on inhomogeneous quantum groups was introduced in [18] for $IGL_q(N)$, and extended to the multiparametric q-groups $IGL_{q,r}(N)$ in [19]. References on inhomogeneous q-groups can also be found in [20].

A general discussion on the differential calculus on multiparametric q-groups is given in Sect. 2. In Sect. 5 we discuss the q-deformation of Cartan–Maurer equations, Bianchi identities, diffeomorphisms and propose a lagrangian for q-gravity, based on $ISO_q(3,1)$. Other deformations of the Poincaré algebra have been considered in recent literature [21]. Although interesting in their own right, none of these deformations corresponds to a bicovariant differential calculus on a quantum Poincaré group.

2. Bicovariant Calculus on Multiparametric Quantum Groups

We recall that (multiparametric) quantum groups are characterized by their R-matrix, which controls the noncommutativity of the quantum group basic elements T^a_b (fundamental representation):

$$R^{ab}{}_{ef}T^e{}_cT^f{}_d = T^b{}_fT^a{}_eR^{ef}{}_{cd} \tag{2.1}$$

and satisfies the quantum Yang–Baxter equation

$$R^{a_1b_1}{}_{a_2b_2}R^{a_2c_1}{}_{a_3c_2}R^{b_2c_2}{}_{b_3c_3} = R^{b_1c_1}{}_{b_2c_2}R^{a_1c_2}{}_{a_2c_3}R^{a_2b_2}{}_{a_3b_3}\,, \tag{2.2}$$

a sufficient condition for the consistency of the "RTT" relations (2.1). The R-matrix components $R^{ab}{}_{cd}$ depend continuously on a (in general complex) set of parameters q_{ab}, r. For $q_{ab} = q$, $r = q$ we recover the uniparametric q-groups of ref. [6]. Then $q_{ab} \to 1$, $r \to 1$ is the classical limit for which $R^{ab}{}_{cd} \to \delta^a_c\delta^b_d$: the matrix entries $T^a{}_b$ commute and become the usual entries of the fundamental representation. The multiparametric R matrices for the A, B, C, D series can be found in [22] (other refs. on multiparametric q-groups are given in [23]). For the B, C, D case they read:

$$\begin{aligned} R^{ab}{}_{cd} &= \delta^a_c\delta^b_d\left[\frac{r}{q_{ab}} + (r-1)\delta^{ab} + (r^{-1}-1)\delta^{ab'}\right](1-\delta^{an_2}) + \delta^a_{n_2}\delta^b_{n_2}\delta^{n_2}_c\delta^{n_2}_d \\ &\quad + (r-r^{-1})[\theta^{ab}\delta^b_c\delta^a_d - \varepsilon_a\varepsilon_c\theta^{ac}r^{\rho_a-\rho_c}\delta^{a'b}\delta_{c'd}]\,, \end{aligned} \tag{2.3}$$

where $\theta^{ab} = 1$ for $a > b$ and $\theta^{ab} = 0$ for $a \leqq b$; we define $n_2 \equiv \frac{N+1}{2}$ and primed indices as $a' \equiv N + 1 - a$. The indices run on N values (N = dimension of the fundamental representation $T^a{}_b$), with $N = 2n + 1$ for $B_n[SO(2n + 1)]$, $N = 2n$ for $C_n[Sp(2n)]$, $D_n[SO(2n)]$. The terms with the index n_2 are present only for the B_n series. The ε_a and ρ_a vectors are given by:

$$\varepsilon_a = \begin{cases} +1 & \text{for } B_n, D_n\,, \\ +1 & \text{for } C_n \text{ and } a \leqq n\,, \\ -1 & \text{for } C_n \text{ and } a > n\,. \end{cases} \tag{2.4}$$

$$(\rho_1, \ldots \rho_N) = \begin{cases} (\frac{N}{2}-1, \frac{N}{2}-2, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -\frac{N}{2}+1) & \text{for } B_n \\ (\frac{N}{2}, \frac{N}{2}-1, \ldots 1, -1, \ldots, -\frac{N}{2}) & \text{for } C_n \\ (\frac{N}{2}-1, \frac{N}{2}-2, \ldots, 1, 0, 0, -1, \ldots, -\frac{N}{2}+1) & \text{for } D_n \end{cases}\,. \tag{2.5}$$

Moreover the following relations reduce the number of independent q_{ab} parameters [22]:

$$q_{aa} = r, \quad q_{ba} = \frac{r^2}{q_{ab}}\,; \tag{2.6}$$

$$q_{ab} = \frac{r^2}{q_{ab'}} = \frac{r^2}{q_{a'b}} = q_{a'b'}\,, \tag{2.7}$$

where (2.7) also implies $q_{aa'} = r$. Therefore the q_{ab} with $a < b \leqq \frac{N}{2}$ give all the q's.

Remark 1. If we denote by q, r the set of parameters q_{ab}, r, we have

$$R^{-1}_{q,r} = R_{q^{-1},r^{-1}}\,. \tag{2.8}$$

The inverse R^{-1} is defined by $(R^{-1})^{ab}{}_{cd}R^{cd}{}_{ef} = \delta^a_e\delta^b_f = R^{ab}{}_{cd}(R^{-1})^{cd}{}_{ef}$. Equation (2.8) implies that for $|q| = |r| = 1$, $\bar{R} = R^{-1}$.

Remark. 2. For $r = 1$, $\hat{R}^2 = 1$, where $\hat{R} \equiv PR\,(\hat{R}^{ab}{}_{cd} \equiv R^{ba}{}_{cd})$.

Orthogonality (and symplecticity) conditions can be imposed on the elements $T^a{}_b$, consistently with the RTT relations (2.1):

$$
\begin{aligned}
&T^a{}_bC^{bc}T^d{}_c = C^{ad}\,,\\
&T^a{}_bC_{ac}T^c{}_d = C_{bd}\,,
\end{aligned}
\tag{2.9}
$$

where the (antidiagonal) metric is:

$$
C_{ab} = \varepsilon_a r^{-\rho_a}\delta_{ab'}\,, \tag{2.10}
$$

and its inverse C^{ab} satisfies $C^{ab}C_{bc} = \delta^a_c = C_{cb}C^{ba}$. We see that for the orthogonal series, the matrix elements of the metric and the inverse metric coincide, while for the symplectic series there is a change of sign.

The consistency of (2.9) with the RTT relations is due to the identities:

$$
C_{ab}\hat{R}^{bc}{}_{de} = (\hat{R}^{-1})^{cf}{}_{ad}C_{fe}\,, \tag{2.11}
$$

$$
\hat{R}^{bc}{}_{de}C^{ea} = C^{bf}(\hat{R}^{-1})^{ca}{}_{fd}\,. \tag{2.12}
$$

These identities hold also for $\hat{R} \to \hat{R}^{-1}$. The co-structures of the B, C, D multiparametric quantum groups have the same form as in the uniparametric case: the coproduct Δ, the counit ε and the coinverse κ are given by

$$
\Delta(T^a{}_b) = T^a{}_b \otimes T^b{}_c\,, \tag{2.13}
$$

$$
\varepsilon(T^a{}_b) = \delta^a_b\,, \tag{2.14}
$$

$$
\kappa(T^a{}_b) = C^{ac}T^d{}_cC_{db}\,. \tag{2.15}
$$

A conjugation (i.e. algebra antihomomorphism, coalgebra homomorphism and involution, satisfying $\kappa(\kappa(T^*)^*) = T$) can be defined trivially as $T^* = T$ or via the metric as $T^* = (\kappa(T))^t$. In the first case, compatibility with the RTT relations (2.1) requires $\bar{R}_{q,r} = R^{-1}_{q,r} = R_{q^{-1},r^{-1}}$, i.e. $|q| = |r| = 1$, and the corresponding real forms are $SO_{q,r}(n,n;\mathbf{R})$, $SO_{q,r}(n,n+1;\mathbf{R})$ (for N even and odd respectively) and $Sp_{q,r}(n;\mathbf{R})$. In the second case the condition on R is $\bar{R}^{ab}{}_{cd} = R^{dc}{}_{ba}$, which happens for $q_{ab}\bar{q}_{ab} = r^2$, $r \in \mathbf{R}$. The metric on a "real" basis has compact signature $(+,+,\ldots+)$ so that the real form is $SO_{q,r}(N;\mathbf{R})$.

There is also a third way to define a conjugation on the orthogonal quantum groups $SO_{q,r}(2n,\mathbf{C})$, which extends to the multiparametric case the one proposed by the authors of ref. [24] for $SO_q(2n,\mathbf{C})$. The conjugation is defined by:

$$
(T^a{}_b)^* = \mathscr{D}^a{}_cT^c{}_d\mathscr{D}^d{}_b\,, \tag{2.16}
$$

$\mathscr{D}$ being the matrix that exchanges the index n with the index $n+1$. This conjugation is compatible with the coproduct: $\Delta(T^*) = (\Delta T)^*$; for $|r| = 1$ it is also compatible with the orthogonality relations (2.9) (due to $\bar{C} = C^T$ and also $\mathscr{D}C\mathscr{D} = C$) and with the antipode: $\kappa(\kappa(T^*)^*) = T$. Compatibility with the RTT relations is easily seen to require

$$(\bar{R})_{n\leftrightarrow n+1} = R^{-1}\,, \tag{2.17}$$

where $()_{n\leftrightarrow n+1}$ means interchanging the indices n and $n+1$ in the expression in parentheses. Eq. (2.17) implies

i) $|q_{ab}| = |r| = 1$ for a and b both different from n or $n+1$;
ii) $q_{ab}/r \in \mathbf{R}$ when at least one of the indices a, b is equal to n or $n+1$.

Since later we consider the case $r = 1$ and $(R)_{n\leftrightarrow n+1} = R$ (and therefore $\bar{R} = R^{-1}$ because of (2.17)), the conditions on the parameters implied by (2.17) will be:

$$|q_{ab}| = 1 \text{ for } a \text{ and } b \text{ both different from } n \text{ or } n+1\,, \tag{2.18}$$

$$q_{ab} = 1 \text{ for } a \text{ or } b \text{ equal to } n \text{ or } n+1\,. \tag{2.19}$$

This last conjugation leads to the real form $SO_{q,r}(n+1, n-1; \mathbf{R})$, and will in fact be the one we need in order to obtain $ISO_q(3,1;\mathbf{R})$, as we discuss in Sect. 4.

A bicovariant differential calculus [9] on the multiparametric q-groups can be constructed in terms of the corresponding R matrix, in much the same way as for uniparametric q-groups (for which we refer to [11, 13, 8]). Here we concentrate on $SO_{q,r}(N+2)$, but everything holds also for $Sp_{q,r}(N+2)$. For later convenience we adopt upper case indices for the fundamental representation of $SO_{q,r}(N+2)$ and lower case indices for the fundamental representation of $SO_{q,r}(N)$.

The basic object is the braiding matrix

$$\Lambda_{A_1}{}^{A_2}{}_{D_1}{}^{D_2}|^{C_1}{}_{C_2}{}^{B_1}{}_{B_2} \equiv d^{F_2} d^{-1}_{C_2} R^{F_2 B_1}{}_{C_2 G_1} (R^{-1})^{C_1 G_1}{}_{E_1 A_1} (R^{-1})^{A_2 E_1}{}_{G_2 D_1} R^{G_2 D_2}{}_{B_2 F_2}\,, \tag{2.20}$$

which is used in the definition of the exterior product of quantum left-invariant one-forms $\omega_A{}^B$:

$$\omega_{A_1}{}^{A_2} \wedge \omega_{D_1}{}^{D_2} \equiv \omega_{A_1}{}^{A_2} \otimes \omega_{D_1}{}^{D_2} - \Lambda_{A_1}{}^{A_2}{}_{D_1}{}^{D_2}|^{C_1}{}_{C_2}{}^{B_1}{}_{B_2}\, \omega_{C_1}{}^{C_2} \otimes \omega_{B_1}{}^{B_2} \tag{2.21}$$

and in the q-commutations of the quantum Lie algebra generators $\chi^A{}_B$:

$$\chi^{D_1}{}_{D_2} \chi^{C_1}{}_{C_2} - \Lambda_{E_1}{}^{E_2}{}_{F_1}{}^{F_2}|^{D_1}{}_{D_2}{}^{C_1}{}_{C_2}\, \chi^{E_1}{}_{E_2} \chi^{F_1}{}_{F_2} = \mathbf{C}^{D_1}{}_{D_2}{}^{C_1}{}_{C_2}|_{A_1}{}^{A_2} \chi^{A_1}{}_{A_2}\,, \tag{2.22}$$

where the structure constants are explicitly given by:

$$\mathbf{C}^{A_1}{}_{A_2}{}^{B_1}{}_{B_2}|_{C_1}{}^{C_2} = \frac{1}{r - r^{-1}} \left[-\delta^{B_1}_{B_2} \delta^{A_1}_{C_1} \delta^{C_2}_{A_2} + \Lambda_B{}^B{}_{C_1}{}^{C_2}|^{A_1}{}_{A_2}{}^{B_1}{}_{B_2} \right]. \tag{2.23}$$

The d^A vector in (2.20) is defined via the diagonal matrix $D^A{}_B$ as $d^A = D^A{}_A$ (no sum on A), with $D = CC^t$, or

$$D^A{}_B = C^{AC} C_{BC}\,. \tag{2.24}$$

A graphical representation of the braiding matrix (2.20) is given in Appendix A.

Remark. 3. For $r = 1$ we have $\Lambda^2 = 1$. This is due to $\hat{R}^2 = 1$ and $D^A{}_B = \delta^A_B$.

The braiding matrix Λ and the structure constants $\mathbf{C}$ defined in (2.23) satisfy the conditions

$$\mathbf{C}_{ri}{}^n \mathbf{C}_{nj}{}^s - \Lambda^{kl}{}_{ij} \mathbf{C}_{rk}{}^n \mathbf{C}_{nl}{}^s = \mathbf{C}_{ij}{}^k \mathbf{C}_{rk}{}^s \quad (q\text{-Jacobi identities})\,, \tag{2.25}$$

$$\Lambda^{nm}{}_{ij} \Lambda^{ik}{}_{rp} \Lambda^{js}{}_{kq} = \Lambda^{nk}{}_{ri} \Lambda^{ms}{}_{kj} \Lambda^{ij}{}_{pq} \quad (\text{Yang–Baxter})\,, \tag{2.26}$$

$$\mathbf{C}_{mn}{}^{i}\Lambda^{ml}{}_{rj}\Lambda^{ns}{}_{lk} + \Lambda^{il}{}_{rj}\mathbf{C}_{lk}{}^{s} = \Lambda^{pq}{}_{jk}\Lambda^{is}{}_{lq}\mathbf{C}_{rp}{}^{l} + \mathbf{C}_{jk}{}^{m}\Lambda^{is}{}_{rm}\,, \tag{2.27}$$

$$\mathbf{C}_{rk}{}^{m}\Lambda^{ns}{}_{ml} = \Lambda^{ij}{}_{kl}\Lambda^{nm}{}_{ri}\mathbf{C}_{mj}{}^{s}\,, \tag{2.28}$$

where the index pairs ${}_{A}{}^{B}$ and ${}^{A}{}_{B}$ have been replaced by the indices ^{i} and ${}_{i}$ respectively. These are the so-called "bicovariance conditions," see refs. [9, 10, 8], necessary for the existence of a consistent bicovariant differential calculus, as we discuss further in Appendix B.

A metric can be defined in the adjoint representation of the B_n, C_n, D_n q-groups as follows:

$$C_{ij} \equiv C^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} = C^{c_1 f}(R^{-1})^{b_1 e}{}_{f c_2} C_{b_2 e}\,, \tag{2.29}$$

$$C^{ij} \equiv C_{a_1}{}^{a_2}{}_{c_1}{}^{c_2} = C_{a_1 e}R^{e a_2}{}_{c_1 f}C^{c_2 f}\,, \tag{2.30}$$

and satisfies the relations:

$$C_{ij}C^{jk} = \delta_i^k = C^{kj}C_{ji}\,, \tag{2.31}$$

and

$$C^{ik}\Lambda^{sl}{}_{kr} = (\Lambda^{-1})^{is}{}_{rj}C^{jl}\,, \tag{2.32}$$

$$\Lambda^{rj}{}_{is}C_{jl} = C_{ik}(\Lambda^{-1})^{kr}{}_{sl}\,, \tag{2.33}$$

i.e. the analogue of Eqs. (2.11)–(2.12). These relations allow to define consistent orthogonality relations for the q-group matrix elements in the adjoint representation (see Appendix B).

Remark. 4. When $r = 1$ ($\Rightarrow \hat{R}^2 = 1$, $\Lambda^2 = 1$), the following useful identities hold:

$$D^a{}_b \equiv C^{ac}C_{bc} = \delta^a_b, \quad (D^{-1})^a_b \equiv C^{ca}C_{cb} = \delta^a_b\,, \tag{2.34}$$

$$D^i_j \equiv C^{ik}C_{jk} = \delta^i_j, \quad (D^{-1})^i_j \equiv C^{ki}C_{kj} = \delta^i_j\,, \tag{2.35}$$

$$\hat{R}^{ab}{}_{cd}\hat{R}^{ce}{}_{af} = \delta^b_f\delta^e_d = \hat{R}^{ba}{}_{dc}\hat{R}^{ec}{}_{fa}\,, \tag{2.36}$$

$$\Lambda^{ri}{}_{sl}\Lambda^{sk}{}_{rj} = \delta^i_j\delta^k_l = \Lambda^{ir}{}_{ls}\Lambda^{ks}{}_{jr}\,. \tag{2.37}$$

The first two $*$-conjugations (the "usual ones") on the T's we have discussed earlier in this section can be extended to the dual space spanned by the q-Lie algebra generators χ as in the uniparametric case. The consistent extension of the third conjugation to the χ space is treated in Appendix C, for the case of minimal deformations ($r = 1$) of $SO(2n)$. We find that

$$(\chi^a{}_b)^* = -\mathscr{D}^a{}_c\chi^c{}_d\mathscr{D}^d{}_b \tag{2.38}$$

is compatible with the bicovariant differential calculus if the Λ and $\mathbf{C}$ tensors are invariant under the exchange of the indices n and n+1, and if the following relation holds:

$$\bar{\mathbf{C}}_{ij}{}^{k} = -\mathbf{C}_{ji}{}^{k}\,. \tag{2.39}$$

3. Inhomogeneous Quantum Groups and Their Differential Calculus

In this section we present a general method of quantizing inhomogeneous groups whose homogeneous subgroup belongs to the BCD series. In particular we concentrate on the q-deformations of the $ISO(N)$ groups, as these are the groups relevant for the construction of q-gravity theories.

The idea is to project $SO_q(N+2)$ and its differential calculus on $ISO_q(N)$, much as we did for $IGL_q(N)$ in ref. [18] (see also [19]), where we projected from $GL_q(N+1)$.

For this we have to consider the multiparametric deformations of the orthogonal groups $SO_{q,r}(N+2)$ with $r=1$ (minimal deformations). Only for $r=1$ we can obtain a consistent projection on $ISO_q(N)$.

We know that the $R^{ab}{}_{cd}$ matrix of $SO_{q,r}(N)$ is contained in the $R^{AB}{}_{CD}$ matrix of $SO_{q,r}(N+2)$: more precisely it is obtained from the "mother" R matrix by restricting its indices to the values A, B,.. $= 2, 3, \dots \mathrm{N}-1$. We therefore split the capital indices as $\mathrm{A} = (\circ, a = 1, \dots N, \bullet)$. Then the R matrix of $SO_{q,r}(N+2)$ can be rewritten in terms of $SO_{q,r}(N)$ quantities:

$$
R^{AB}_{CD} = \begin{array}{c} \\ \circ\circ \\ \circ\bullet \\ \bullet\circ \\ \bullet\bullet \\ \circ b \\ \bullet b \\ a\circ \\ a\bullet \\ ab \end{array}
\begin{array}{c} \begin{array}{ccccccccc} \circ\circ & \circ\bullet & \bullet\circ & \bullet\bullet & \circ d & \bullet d & c\circ & c\bullet & cd \end{array} \\
\begin{pmatrix}
r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & f(r) & r^{-1} & 0 & 0 & 0 & 0 & 0 & -C_{cd}\lambda r^{-\frac{N}{2}} \\
0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{r}{q_{\circ b}}\delta^b_d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{q_{\circ b}}{r}\delta^b_d & 0 & \lambda\delta^b_c & 0 \\
0 & 0 & 0 & 0 & \lambda\delta^a_d & 0 & \frac{q_{\circ a}}{r}\delta^a_c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{r}{q_{\circ a}}\delta^a_c & 0 \\
0 & -C^{ba}\lambda r^{-\frac{N}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & R^{ab}{}_{cd}
\end{pmatrix} \end{array}, \tag{3.1}
$$

where C_{ab} is the $SO_{q,r}(N)$ metric, $\lambda \equiv r - r^{-1}$ and $f(r) \equiv \lambda(1 - r^{-N})$.

It is not difficult to reexpress the Λ and $\mathbf{C}$ tensors in our index convention. Less trivial is to find a subset of these components, containing the Λ and $\mathbf{C}$ tensors of $SO_{q,r}(N)$, that satisfies the bicovariance conditions (2.25)–(2.28). This subset in fact exists for $r=1$ and is given by:

$$\Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2} = R^{f_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a_1} (R^{-1})^{a_2 e_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 f_2}\,, \tag{3.2}$$

$$\Lambda_{a_1}{}^{\circ}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{\circ} = \frac{q_{\circ d_1}}{q_{\circ d_2}} R^{d_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{d_1 a_1}\,, \tag{3.3}$$

$$\Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{\circ}|^{c_1}{}_{\circ}{}^{b_1}{}_{b_2} = \frac{q_{\circ b_2}}{q_{\circ b_1}} (R^{-1})^{c_1 b_1}{}_{e_1 a_1} (R^{-1})^{a_2 e_1}{}_{b_2 d_1}\,, \tag{3.4}$$

$$\Lambda_{a_1}{}^{\circ}{}_{d_1}{}^{\circ}|^{c_1}{}_{\circ}{}^{b_1}{}_{\circ} = \frac{q_{\circ d_1}}{q_{\circ b_1}} (R^{-1})^{c_1 b_1}{}_{d_1 a_1}\,, \tag{3.5}$$

$$\Lambda_{\bullet}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{\bullet}{}_{b_2} = \frac{q_{\circ c_1}}{q_{\circ c_2}} (R^{-1})^{a_2 c_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 c_2}\,, \tag{3.6}$$

$$\Lambda_{a_1}{}^{a_2}{}_{\bullet}{}^{d_2}|^{\bullet}{}_{c_2}{}^{b_1}{}_{b_2} = \frac{q_{\circ a_2}}{q_{\circ a_1}} R^{f_2 b_1}{}_{c_2 a_1} R^{a_2 d_2}{}_{b_2 f_2}\,, \tag{3.7}$$

$$\Lambda_{\bullet}{}^{a_2}{}_{\bullet}{}^{d_2}|^{\bullet}{}_{c_2}{}^{\bullet}{}_{b_2} = \frac{q_{\circ a_2}}{q_{\circ c_2}} R^{a_2 d_2}{}_{b_2 c_2}\,, \tag{3.8}$$

$$\Lambda_{\bullet}{}^{a_2}{}_{d_1}{}^{\circ}|^{c_1}{}_{\circ}{}^{\bullet}{}_{b_2} = q_{\circ c_1} q_{\circ b_2} (R^{-1})^{a_2 c_1}{}_{b_2 d_1}\,, \tag{3.9}$$

$$\Lambda_{a_1}{}^{\circ}{}_{\bullet}{}^{d_2}|^{\bullet}{}_{c_2}{}^{b_1}{}_{\circ} = \left(q_{\circ a_1} q_{\circ d_2}\right)^{-1} (R^{-1})^{d_2 b_1}{}_{c_2 a_1}\,, \tag{3.10}$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}|_{d_1}{}^{d_2} = \text{structure constants of } SO_{q,r=1}(N)\,, \tag{3.11}$$

$$\mathbf{C}^{c_1}{}_{\circ}{}^{b_1}{}_{b_2}|_{d_1}{}^{\circ} = \lim_{r\to 1} \frac{1}{r-r^{-1}} \left[-\delta^{b_1}_{b_2} \delta^{c_1}_{d_1} + \frac{q_{\circ b_2}}{q_{\circ b_1}} (R^{-1})^{c_1 b_1}{}_{e_1 a} (R^{-1})^{a e_1}{}_{b_2 d_1} \right], \tag{3.12}$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{\circ}|_{d_1}{}^{\circ} = R^{g_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a} (R^{-1})^{a e_1}{}_{g_2 d_1}\,, \tag{3.13}$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{\circ}|_{\bullet}{}^{d_2} = q^{-1}_{\circ d_2} C^{a e_1} R^{d_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a}\,, \tag{3.14}$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{\bullet}{}_{b_2}|_{\bullet}{}^{d_2} = \frac{q_{\circ c_1}}{q_{\circ c_2}} R^{c_1 d_2}{}_{b_2 c_2}\,, \tag{3.15}$$

$$\mathbf{C}^{c_1}{}_{c_2}{}^{\bullet}{}_{b_2}|_{d_1}{}^{\circ} = -\frac{q_{\circ c_1}}{q_{\circ c_2} q_{\circ d_2}} C_{b_2 c_2} \delta^{c_1}_{d_1}\,, \tag{3.16}$$

$$\mathbf{C}^{\bullet}{}_{c_2}{}^{b_1}{}_{b_2}|_{\bullet}{}^{d_2} = \lim_{r\to 1} \frac{1}{r-r^{-1}} \left[-\delta^{b_1}_{b_2} \delta^{d_2}_{c_2} + R^{f_2 b_1}{}_{c_2 a} R^{a d_2}{}_{b_2 f_2} \right]. \tag{3.17}$$

This is the key result of this section, and enables the consistent projection on the $ISO_q(N)$ algebra by setting:

$$\chi^{\circ}{}_{b} = \chi^{a}{}_{\bullet} = \chi^{\circ}{}_{\circ} = \chi^{\bullet}{}_{\bullet} = \chi^{\circ}{}_{\bullet} = \chi^{\bullet}{}_{\circ} = 0\,. \tag{3.18}$$

The $ISO_q(N)$ Lie algebra is given explicitly in Table 1. The reason we call this the $ISO_q(N)$ Lie algebra will be explained below. The limits in eqs. (3.12) and (3.17) are finite, since the numerators behave as $0(r-r^{-1})$.

We prove now that the components (3.2)–(3.17) indeed satisfy the bicovariance conditions (2.25)–(2.28). We label by the letter H the subset of indices present in Eqs. (3.2)–(3.17), i.e. $_H = {}^{a}{}_{b}, {}^{a}{}_{\circ}, {}^{\bullet}{}_{b}$ $\left({}^H = {}_{a}{}^{b}, {}_{a}{}^{\circ}, {}_{\bullet}{}^{b}\right)$, and by the letter K all the other composite indices. We have to prove that, setting equal to H all free indices in (2.25)–(2.28), only H indices enter in the index sums (and therefore the H-tensors of (3.2)–(3.17) satisfy *by themselves* the bicovariance conditions). This is true i) for the quantum Yang–Baxter Eqs. (2.26) since the tensor $P\Lambda$ is diagonal for $r=1$, so that

$$\Lambda^{HH}{}_{HK} = \Lambda^{HH}{}_{KH} = \Lambda^{HH}{}_{KK} = \Lambda^{HK}{}_{HH} = \Lambda^{KH}{}_{HH} = \Lambda^{KK}{}_{HH} = 0\,; \tag{3.19}$$

Table 1. $ISO_q(N)$ Lie algebra

$$\chi^{c_1}{}_{c_2}\chi^{b_1}{}_{b_2} - \Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}\chi^{a_1}{}_{a_2}\chi^{d_1}{}_{d_2} = \mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}|_{d_1}{}^{d_2}\chi^{d_1}{}_{d_2} \quad [SO_{q,r=1}(N) \text{ Lie algebra}]$$

$$\chi^{c_1}{}_{c_2}\chi^{b_1}{}_{\circ} - \frac{q_{\circ}d_1}{q_{\circ}d_2} R^{d_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{d_1 a_1} \chi^{a_1}{}_{\circ}\chi^{d_1}{}_{d_2} = R^{g_2 b_1}{}_{c_2 g_1} (R^{-1})^{c_1 g_1}{}_{e_1 a} (R^{-1})^{a e_1}{}_{g_2 d_1} \chi^{d_1}{}_{\circ}$$

$$\chi^{c_1}{}_{\circ}\chi^{b_1}{}_{\circ} - \frac{q_{\circ}d_1}{q_{\circ}b_1} (R^{-1})^{c_1 b_1}{}_{d_1 a_1} \chi^{a_1}{}_{\circ}\chi^{d_1}{}_{\circ} = 0$$

$$\chi^{c_1}{}_{c_2}\chi^{\bullet}{}_{b_2} - \frac{q_{\circ}c_1}{q_{\circ}c_2} (R^{-1})^{a_2 c_1}{}_{g_2 d_1} R^{g_2 d_2}{}_{b_2 c_2} \chi^{\bullet}{}_{a_2}\chi^{d_1}{}_{d_2} = \frac{q_{\circ c_1}}{q_{\circ c_2}} R^{c_1 d_2}{}_{b_2 c_2} \chi^{\bullet}{}_{d_2}$$

$$\chi^{\bullet}{}_{c_2}\chi^{\bullet}{}_{b_2} - \frac{q_{\circ a_2}}{q_{\circ c_2}} R^{a_2 d_2}{}_{b_2 c_2} \chi^{\bullet}{}_{a_2}\chi^{\bullet}{}_{d_2} = 0$$

$$\chi^{c_1}{}_{\circ}\chi^{\bullet}{}_{b_2} - q_{\circ c_1} q_{\circ b_2} (R^{-1})^{a_2 c_1}{}_{b_2 d_1} \chi^{\bullet}{}_{a_2}\chi^{d_1}{}_{\circ} = 0$$

ii) for the q-Jacobi Eqs. (2.25) because $\mathbf{C}_{HH}{}^{K}$ can be different from zero only when ${}^{K} = {}_{\bullet}{}^{\circ}$, and $\mathbf{C}_{HK}{}^{H} = \mathbf{C}_{KH}{}^{H} = 0$ when ${}_{K} = {}^{\bullet}{}_{\circ}$; iii) for the last two bicovariant conditions (2.27)–(2.28) again because of (3.19). □

Thus far we have shown that there is a subset of $\chi^{A}{}_{B}$ (the generators of the q-Lie algebra of $SO_{q,r=1}(N+2)$) closing on the q-algebra of Table 1, namely $\chi^{a}{}_{b}, \chi^{a}{}_{\circ}$ and $\chi^{\bullet}{}_{b}$. This algebra is bicovariant, in the sense that the corresponding Λ and $\mathbf{C}$ tensors satisfy (2.25)–(2.28). It wold seem that the number of momenta is twice what we need, since there are two kinds of "momentum" generators, $\chi^{a}{}_{\circ}$ and $\chi^{\bullet}{}_{b}$. However by examining in some detail the q-algebra we can conclude that only N combinations of these momenta do survive, and if we rewrite the algebra of Table 1 in terms of these combinations we precisely obtain a deformation of $ISO(N)$. Let us prove this.

Consider the structure constants $\mathbf{C}^{c_1}{}_{\circ}{}^{b_1}{}_{b_2}|_{d_1}{}^{\circ}$. It is not difficult to see from (3.12) that for $c_1 = b_2$, $b_1 = b_2'$ these constants are vanishing for any value of r (use the explicit expression (2.3)), and thus in particular for $r = 1$. On the other hand the structure constants $\mathbf{C}^{b_1}{}_{b_2}{}^{c_1}{}_{\circ}|_{d_1}{}^{\circ}$ and $\mathbf{C}^{b_1}{}_{b_2}{}^{c_1}{}_{\circ}|_{\bullet}{}^{d_2}$ are *not* vanishing for the same values of c_1, b_1, b_2, but:

$$\mathbf{C}^{b_1}{}_{b_2}{}^{c_1}{}_{\circ}|_{d_1}{}^{\circ} = \delta^{c_1}_{d_1}\,, \tag{3.20}$$

$$\mathbf{C}^{b_1}{}_{b_2}{}^{c_1}{}_{\circ}|_{\bullet}{}^{d_2} = C^{c_1 b_1} q^{-1}_{\circ d_2} \delta^{d_2}_{b_2}\,. \tag{3.21}$$

Thus we have the two commutations:

$$\chi^{c_1}{}_{\circ}\chi^{b_1}{}_{b_2} - \Lambda_{e_1}{}^{e_2}{}_{f_1}{}^{\circ}|^{c_1}{}_{\circ}{}^{b_1}{}_{b_2}\chi^{e_1}{}_{e_2}\chi^{f_1}{}_{\circ} = 0\,, \tag{3.22}$$

$$\chi^{b_1}{}_{b_2}\chi^{c_1}{}_{\circ} - \Lambda_{e_1}{}^{\circ}{}_{f_1}{}^{f_2}|^{b_1}{}_{b_2}{}^{c_1}{}_{\circ}\chi^{e_1}{}_{\circ}\chi^{f_1}{}_{f_2} = \chi^{b_1}{}_{\circ} + q^{-1}_{\circ b_1'}\chi^{\bullet}{}_{b_1'} \tag{3.23}$$

with $b_1' \equiv N + 1 - b_1$. Next we remark that for $\Lambda^2 = I$ (as is the case for $r = 1$) the two left-hand sides of the above equations are equal up to a minus sign, so that finally we have:

$$\chi^{b_1}{}_{\circ} + q^{-1}_{\circ b_1'}\chi^{\bullet}{}_{b_1'} = 0\,. \tag{3.24}$$

These N equations reduce the number of independent momenta to N. We can easily rewrite the algebra in Table 1 in terms of the redefined momenta:

$$\chi^{a} \equiv q^{\frac{1}{2}}_{\circ a'}\chi^{a}{}_{\circ} - q^{-\frac{1}{2}}_{\circ a'}\chi^{\bullet}{}_{a'}\,, \tag{3.25}$$

and we have done so in Table 2.

Table 2. $ISO_q(N)$ Lie algebra in the $\chi^{a} \equiv q^{\frac{1}{2}}_{\circ a'}\chi^{a}{}_{\circ} - q^{-\frac{1}{2}}_{\circ a'}\chi^{\bullet}{}_{a'}$ basis ($a' \equiv N + 1 - a$)

$$\chi^{c_1}{}_{c_2}\chi^{b_1}{}_{b_2} - \Lambda_{a_1}{}^{a_2}{}_{d_1}{}^{d_2}|^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}\chi^{a_1}{}_{a_2}\chi^{d_1}{}_{d_2} = \mathbf{C}^{c_1}{}_{c_2}{}^{b_1}{}_{b_2}|_{d_1}{}^{d_2}\chi^{d_1}{}_{d_2} \quad [SO_{q,r=1}(N) \text{ Lie algebra}]$$

$$\chi^{c_1}{}_{c_2}\chi^{b_1} - \frac{q_{\circ d_1}}{q_{\circ d_2}} R^{d_2 b_1}{}_{c_2 g_1}(R^{-1})^{c_1 g_1}{}_{d_1 a_1}\chi^{a_1}\chi^{d_1}{}_{d_2} = \frac{q^{\frac{1}{2}}_{\circ b_1'}}{q^{\frac{1}{2}}_{\circ d_1'}} R^{g_2 b_1}{}_{c_2 g_1}(R^{-1})^{c_1 g_1}{}_{e_1 a}(R^{-1})^{a e_1}{}_{g_2 d_1}\chi^{d_1}$$

$$\chi^{c_1}\chi^{b_1} - \frac{q_{\circ d_1}}{q_{\circ b_1}}(R^{-1})^{c_1 b_1}{}_{d_1 a_1}\chi^{a_1}\chi^{d_1} = 0$$

This was possible because the q-commutator of $\chi^a{}_\circ$ with a given generator χ is the same as the q-commutator of $-q_{\circ a'}^{-1}\chi^\bullet{}_{a'}$ with χ, because the constraint (3.24) is consistent with the q-Lie algebra of Table 1. Another way to see it is to remark that the algebra of Table 1 satisfies the q-Jacobi identities (2.25). Then we have an explicit matrix representation of the q-generators χ: the adjoint representation $(\chi_i)^j{}_k \equiv \mathbf{C}_{ki}{}^j$. Equation (3.24) means that the generators $q_{\circ a'}^{\frac{1}{2}}\chi^a{}_\circ$ and $-q_{\circ a'}^{-\frac{1}{2}}\chi^\bullet{}_{a'}$ have the *same* matrix representative, and hence the same commutations with the other generators.

The q-Lie algebra of Table 2 satisfies the bicovariant conditions (2.25)–(2.28). As discussed in Appendix B, these define a (bicovariant) differential calculus on the quantum group generated by the elements $M_i{}^j$ (adjoint representation).

Note however that we do not have an invertible adjoint metric any more (but only a submetric C_{ij} with i,j along the $SO_q(N)$ directions). Then the existence of the antipode of $M_i{}^j$ is not ensured via Eq. (B.4): the quantum group generated by $M_i{}^j$ is a Hopf algebra only if we manage to find an inverse $(M^{-1})_i{}^j$. Otherwise we have a bialgebra.

Let us examine now the dual algebra generated by χ_i, $f^i{}_j$, $f^{-i}{}_j$. In this case we can find an antipode, without reference to an adjoint metric (see the discussion at the end of Appendix B). For the argument, Eqs. (2.37) were crucial: do they hold also in the "projected" case? The answer is yes, due to the matrix $P\Lambda$ being diagonal for $r = 1$. Then the algebra generated by the χ_i, $f^i{}_j$, $f^{-i}{}_j$ is a *bonafide* Hopf algebra, which we call the quantum $ISO_q(N)$ bicovariant algebra (we reserve the name of q-Lie algebra to the one generated only by the χ_i).

Finally, we come to the $*$-conjugation on the generator space induced by the rule (2.38), for $ISO_q(2n)$. Recall that this rule is consistent when the Λ and $\mathbf{C}$ tensors are $n \leftrightarrow n+1$ invariant and condition (2.39) is satisfied. The question is whether the projected $*$-conjugation is still compatible with the projected differential calculus. This indeed happens: the Λ and $\mathbf{C}$ tensors corresponding to the algebra in Table 2, satisfying the bicovariance conditions (2.25)–(2.28), are still invariant under the exchange of the (fundamental) indices n and n+1. If the structure constants $\mathbf{C}$ satisfy (2.39), the result of Appendix C holds also for $ISO_q(2n)$.

Then we have the (Hopf algebra) conjugation:

$$(\chi^a{}_b)^* = -\mathscr{D}^a{}_c\chi^c{}_d\mathscr{D}^d{}_b\,, \tag{3.26}$$

$$(\chi^a)^* = -\mathscr{D}^a{}_b\chi^b\,, \tag{3.27}$$

whose consistency can be checked explicitly in the example of the next Section.

4. $ISO_q(3,1)$ and the Quantum Poincaré Lie Algebra

We come now to applying the preceding formalism to the case of $ISO(4)$. We know from the discussion of the previous section that a real form exists corresponding to a $(3,1)$ signature. Let us consider the "mother" R-matrix of $SO_{q,r=1}(6)$. According to (2.6) and (2.7) there are three independent deformation parameters, i.e. $q_{\circ 1}$, $q_{\circ 2}$ and q_{12} (in the index convention $a = \circ, 1, 2, 3, 4, \bullet$). It is not difficult to see that this R matrix has the $2 \leftrightarrow 3$ symmetry only if $q_{\circ 2} = 1$ and $q_{12} = 1$. Therefore we are left with the only parameter $q_{\circ 1} \equiv q$. Note that q_{12} is the deformation parameter of the

Lorentz subalgebra, and $q_{12} = 1$, means that this subalgebra is *classical.* Moreover the condition (2.18) becomes $|q| = 1$.

Consider now the $ISO_q(4)$ algebra one deduces by specializing $N = 4$, $q_{\circ 2} = 1$, $q_{\circ 1} \equiv q$ in Table 2.

Besides the q-commutations between the generators, one finds a set of relations similar to (3.24):

$$q_{12}\chi^1{}_2 + \chi^3{}_4 = 0, \quad \chi^1{}_3 + q_{12}\chi^2{}_4 = 0, \quad \chi^2{}_1 + q_{12}\chi^4{}_3 = 0\,, \tag{4.1}$$

$$q_{12}\chi^3{}_1 + \chi^4{}_2 = 0, \quad \chi^1{}_1 - \chi^2{}_2 - \chi^3{}_3 + \chi^4{}_4 = 0\,, \tag{4.2}$$

$$[\chi^1{}_1 + \chi^2{}_2 + \chi^3{}_3 + \chi^4{}_4, \text{any } \chi] = 0\,. \tag{4.3}$$

We see from (4.3) that $\sum_a \chi^a{}_a$ decouples: we can consistently set it equal to zero and obtain a reduced bicovariant q-Lie algebra (in fact this can be explicitly verified on Table 3, see later). We can therefore introduce the basis $\chi_{ab} \equiv \frac{1}{2}[C_{ac}\chi^c{}_b - C_{bc}\chi^c{}_a]$: for $q_{12} = 1$ the metric C_{ab} is the classical antidiagonal metric $C_{14} = C_{23} = C_{32} = C_{41} = 1$ (otherwise 0), and χ_{ab} is antisymmetric in a and b.

Table 3 gives the commutations of the $ISO_q(4)$ generators in the new basis χ_{ab}, $\chi_a \equiv C_{ab}\chi^b$. The invariance under the index exchange $2 \leftrightarrow 3$ is explicit, and the condition (2.39) is easily seen to hold.

Then we can define a consistent $*$-conjugation on the χ, according to (3.26), (3.27):

$$\begin{aligned} \chi^*_{\alpha\beta} &= -\chi_{\alpha\beta} \quad (\alpha, \beta \neq 2, 3)\,, \\ \chi^*_{2\beta} &= -\chi_{3\beta}\,, \\ \chi^*_{3\beta} &= -\chi_{2\beta}\,, \\ \chi^*_{23} &= \chi_{23}\,, \end{aligned} \tag{4.4}$$

$$(\chi_1)^* = -\chi_1, \quad (\chi_2)^* = -\chi_3, \quad (\chi_3)^* = -\chi_2, \quad (\chi_4)^* = -\chi_4\,, \tag{4.5}$$

whose compatibility with the commutations of Table 3 can be directly verified.

Table 3. $ISO_q(3{,}1)$ Lie algebra in the $\chi_a \equiv C_{ab}\chi^b, \chi_{ab} \equiv \frac{1}{2}[C_{ac}\chi^c{}_b - C_{bc}\chi^c{}_a]$ basis

$[\chi_{ab}, \chi_{cd}] = C_{bc}\chi_{ad} + C_{ad}\chi_{bc} - C_{bd}\chi_{ac} - C_{ac}\chi_{bd}$

$[\chi_{12}, \chi_a]_{q^{-1}} = q^{-\frac{1}{2}}C_{2a}\chi_1 - q^{-\frac{1}{2}}C_{1a}\chi_2$

$[\chi_{13}, \chi_a]_{q^{-1}} = q^{-\frac{1}{2}}C_{3a}\chi_1 - q^{-\frac{1}{2}}C_{1a}\chi_3$

$[\chi_{14}, \chi_a] = C_{4a}\chi_1 - C_{1a}\chi_4$

$[\chi_{23}, \chi_a] = C_{3a}\chi_2 - C_{2a}\chi_3$

$[\chi_{24}, \chi_a]_q = q^{\frac{1}{2}}C_{4a}\chi_2 - q^{\frac{1}{2}}C_{2a}\chi_4$

$[\chi_{34}, \chi_a]_q = q^{\frac{1}{2}}C_{4a}\chi_3 - q^{\frac{1}{2}}C_{3a}\chi_4$

$[\chi_1, \chi_2]_{q^{-1}} = 0, \quad [\chi_1, \chi_3]_{q^{-1}} = 0$

$[\chi_1, \chi_4]_{q^{-2}} = 0, \quad [\chi_2, \chi_3] = 0$

$[\chi_2, \chi_4]_{q^{-1}} = 0, \quad [\chi_3, \chi_4]_{q^{-1}} = 0$

with $[A, B]_s \equiv AB - sBA$, $C_{14} = C_{22} = C_{33} = C_{41} = 1$ (otherwise 0).

This conjugation allows the definition of "antihermitian" quantum generators ξ:

$$\xi_{\alpha\beta} = \chi_{\alpha\beta}\,,$$
$$\xi_{2\beta} = \frac{1}{\sqrt{2}}(\chi_{2\beta} + \chi_{3\beta})\,,$$
$$\xi_{3\beta} = \frac{1}{\sqrt{2}}i(\chi_{2\beta} - \chi_{3\beta})\,,$$
$$\xi_{23} = -i\chi_{23}\,, \tag{4.6}$$
$$\xi_{\alpha} = \chi_{\alpha}\,,$$
$$\xi_{2} = \frac{1}{\sqrt{2}}(\chi_{2} + \chi_{3})\,,$$
$$\xi_{3} = \frac{1}{\sqrt{2}}i(\chi_{2} - \chi_{3})\,. \tag{4.7}$$

On this basis the metric in Table 3 becomes

$$C_{ab} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.8}$$

with the desired signature $(+, +, +, -)$.

5. Cartan–Maurer Equations, q-Diffeomorphisms and q-Gravity

In this section we discuss the q-generalization of Poincaré gravity based on the deformed Poincaré algebra of Table 3. As in the classical case we start by defining the curvatures. To do so, we first need the deformed Cartan–Maurer equations [9, 8]

$$d\omega^i + C_{jk}{}^i \omega^j \wedge \omega^k = 0\,, \tag{5.1}$$

where the ω are the left-invariant one-forms discussed in Appendix B. The C structure constants appearing in the Cartan–Maurer equations are in general related to the $\mathbf{C}$ constants of the q-Lie algebra [8]:

$$\mathbf{C}_{jk}{}^i = C_{jk}{}^i - \Lambda^{rs}{}_{jk} C_{rs}{}^i\,. \tag{5.2}$$

In the particular case $\Lambda^2 = I$ it is not difficult to see that in fact $C = \frac{1}{2}\mathbf{C}$, which is a worthwhile simplification.

The procedure we have advocated in refs. [25] for the "gauging" of quantum groups essentially retraces the steps of the group-geometric method for the gauging of usual Lie groups, described for instance in refs. [26].

We consider one-forms ω^i which are not left-invariant any more, so that the Cartan–Maurer equations are replaced by:

$$R^i = d\omega^i + C_{jk}{}^i \omega^j \wedge \omega^k\,, \tag{5.3}$$

where the curvatures R^i are now non-vanishing, and satisfy the q-Bianchi identities:

$$dR^i - C_{jk}{}^i R^j \wedge \omega^k + C_{jk}{}^i \omega^j \wedge R^k = 0 \tag{5.4}$$

due to the Jacobi identities on the structure constants C [8]. As in the classical case we can write the q-Bianchi identities as $\nabla R^i = 0$, which define the covariant derivative ∇.

Equation (5.3) can be taken as the definition of the curvature R^i. We apply it to the q-Poincaré algebra of Table 3: the one-forms are $\omega^i \equiv V^a$, ω^{ab} and the corresponding curvatures read (we omit wedge symbols):

$$\begin{aligned}
R^1 &= dV^1 + q^{-\frac{1}{2}}\omega^{12}V_2 + q^{-\frac{1}{2}}\omega^{13}V_3 + \omega^{14}V_4\,,\\
R^2 &= dV^2 - q^{-\frac{1}{2}}\omega^{12}V_1 + \omega^{23}V_3 + q^{\frac{1}{2}}\omega^{24}V_4\,,\\
R^3 &= dV^3 - q^{-\frac{1}{2}}\omega^{13}V_1 - \omega^{23}V_2 + q^{\frac{1}{2}}\omega^{34}V_4\,,\\
R^4 &= dV^4 - \omega^{14}V_1 - q^{\frac{1}{2}}\omega^{24}V_2 - q^{\frac{1}{2}}\omega^{34}V_3\,,
\end{aligned} \tag{5.5}$$

$$R^{ab} = d\omega^{ab} + C_{cd}\omega^{ac}\omega^{db}\,, \tag{5.6}$$

where $V_a \equiv C_{ab}V^b$, C_{ab} being given in (4.8). We have rescaled ω^{ab} by a factor $\frac{1}{2}$ to obtain standard normalizations. R^{ab} is the q-Lorentz curvature, coinciding with the classical one (as a function of ω^{ab}), and R^a is the q-deformed torsion.

From the definition (B.20) of the exterior product we see that for $\Lambda^2 = I$ the one-forms ω^i q-commute as:

$$\omega^i\omega^j = -\Lambda^{ij}{}_{kl}\omega^k\omega^l\,. \tag{5.7}$$

Inserting the Λ tensor corresponding to Table 3 we find:

$$\begin{aligned}
V^a\omega^{12} &= -q^{-1}\omega^{12}V^a\,,\\
V^a\omega^{13} &= -q^{-1}\omega^{13}V^a\,,\\
V^a\omega^{14} &= -\omega^{14}V^a\,,\\
V^a\omega^{23} &= -\omega^{23}V^a\,,\\
V^a\omega^{24} &= -q\omega^{24}V^a\,,\\
V^a\omega^{34} &= -q\omega^{34}V^a\,,
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
V^2V^1 &= -q^{-1}V^1V^2\,,\\
V^3V^1 &= -q^{-1}V^1V^3\,,\\
V^4V^1 &= -q^{-2}V^1V^4\,,\\
V^3V^2 &= -V^2V^3\,,\\
V^4V^2 &= -q^{-1}V^2V^4\,,\\
V^4V^3 &= -q^{-1}V^3V^4\,.
\end{aligned} \tag{5.9}$$

and usual anticommutations between the ω^{ab} (components of the Lorentz spin connection). The exterior product of two identical one-forms vanishes (this is not true in general when $\Lambda^2 \neq I$).

We are now ready to write the lagrangian for the q-gravity theory based on $ISO_q(3,1)$. The lagrangian looks identical to the classical one, i.e.:

$$\mathscr{L} = R^{ab} V^c V^d \varepsilon_{abcd} \,. \tag{5.10}$$

The Lorentz curvature R^{ab}, although defined as in the classical case, has non-trivial commutations with the q-vielbein:

$$\begin{aligned} V^a R^{12} &= q^{-1} R^{12} V^a \,, \\ V^a R^{13} &= q^{-1} R^{13} V^a \,, \\ V^a R^{14} &= R^{14} V^a \,, \\ V^a R^{23} &= R^{23} V^a \,, \\ V^a R^{24} &= q R^{24} V^a \,, \\ V^a R^{34} &= q R^{34} V^a \,, \end{aligned} \tag{5.11}$$

deducible from the definition (5.6). As in ref. [25, 8], we make the assumption that the commutations of $d\omega^i$ with the one-forms ω^l are the same as those of $C_{jk}{}^i \omega^j \omega^k$ with ω^l, i.e. the same as those valid for $R^i = 0$. For the definition of ε_{abcd} in (5.10) see below.

We discuss now the notion of q-diffeomorphisms. It is known that there is a consistent q-generalization of the Lie derivative (see refs. [8, 27, 15]) which can be expressed as in the classical case as:

$$\ell_{t_i} = i_{t_i} d + d i_{t_i} \,, \tag{5.12}$$

where i_{t_i} is the q-contraction operator defined in refs. [8, 27], with the following properties:

$$\begin{aligned} &\text{i)}\ i_V(a) = 0, \quad a \in A,\ V \text{ generic tangent vector}\,, \\ &\text{ii)}\ i_{t_i} \omega^j = \delta_i^j I\,, \\ &\text{iii)}\ i_{t_i}(\theta \wedge \omega^k) = i_{t_r}(\theta) \omega^l \Lambda^{rk}{}_{li} + (-1)^p \theta\ \delta_i^k \quad \theta \text{ generic } p\text{-form}\,, \\ &\text{iv)}\ i_V(a\theta + \theta') = a i_V(\theta) + i_V \theta', \quad \theta, \theta' \text{ generic forms}\,, \\ &\text{v)}\ i_{\lambda V} = \lambda i_V, \quad \lambda \in \mathbf{C}\,, \\ &\text{vi)}\ i_{\varepsilon V}(\theta) = i_V(\theta)\varepsilon, \quad \varepsilon \in A\,. \end{aligned} \tag{5.13}$$

As a consequence, the q-Lie derivative satisfies:

$$\begin{aligned} &\text{i)}\ \ell_V a = i_V(da) \equiv V(a)\,, \\ &\text{ii)}\ \ell_V d\theta = d\ell_V \theta\,, \\ &\text{iii)}\ \ell_V(\lambda\theta + \theta') = \lambda \ell_V(\theta) + \ell_V(\theta')\,, \\ &\text{iv)}\ \ell_{\varepsilon V}(\theta) = (\ell_V \theta)\varepsilon - (-1)^p i_V(\theta) d\varepsilon, \quad \theta \text{ generic } p\text{-form}\,, \\ &\text{v)}\ \ell_{t_i}(\theta \wedge \omega^k) = (\ell_{t_r}\theta) \wedge \omega^l \Lambda^{rk}_{li} + \theta \wedge \ell_{t_i} \omega^k\,. \end{aligned} \tag{5.14}$$

In analogy with the classical case, we define the q-diffeomorphism variation of the fundamental field ω^i as

$$\delta\omega^k \equiv \ell_{\varepsilon^i t_i} \omega^k \,, \tag{5.15}$$

where according to iv) in (5.14):

$$\ell_{\varepsilon^i t_i}\omega^k = (i_{t_i}d\omega^k + di_{t_i}\omega^k)\varepsilon^i + d\varepsilon^k = (i_{t_i}d\omega^k)\varepsilon^i + d\varepsilon^k . \tag{5.16}$$

Notice that if we postulate:

$$\begin{aligned} \Lambda^{rk}{}_{li}\omega^l\varepsilon^i &= \varepsilon^r\omega^k , \\ \Lambda^{rk}{}_{li}\omega^l \wedge d\varepsilon^i &= -d\varepsilon^r \wedge \omega^k , \end{aligned} \tag{5.17}$$

we find

$$\delta(\omega^j \wedge \omega^k) = \delta\omega^j \wedge \omega^k + \omega^j \wedge \delta\omega^k , \tag{5.18}$$

i.e. a rule that any "sensible" variation law should satisfy. To prove (5.18) use iv) and v) of (5.14). The q-commutations (5.17) were already proposed in [25] in the context of q-gauge theories.

As in the classical case, there is a suggestive way to write this variation:

$$\ell_{\varepsilon^i t_i}\omega^k = i_{\varepsilon^i t_i}R^k + \nabla\varepsilon^k , \tag{5.19}$$

where

$$\begin{aligned} \nabla\varepsilon^k &\equiv d\varepsilon^k - C_{rs}{}^k i_{t_i}(\omega^r \wedge \omega^s)\varepsilon^i \\ &= d\varepsilon^k - C_{rs}{}^k\varepsilon^r\omega^s + C_{rs}{}^k\omega^r\varepsilon^s . \end{aligned} \tag{5.20}$$

Proof. Use (5.3), (5.17) and iii) in (5.13).

We have now all the tools we need to investigate the invariances of the q-gravity lagrangian (5.10). These are discussed in ref. [28]. The result is analogous to the classical one: after imposing the horizontality conditions $i_{t_{ab}}R^{cd} = i_{t_{ab}}R^c = 0$ along the Lorentz directions one finds that, *provided* the ε tensor in (5.10) is appropriately defined, the lagrangian is invariant under q-diffeomorphisms and local Lorentz rotations. The correct definition of the q-alternating tensor is:

$$\begin{array}{llll} \varepsilon_{1234} = 1, & \varepsilon_{1243} = -q, & \varepsilon_{1324} = -1, & \varepsilon_{1342} = q, \\ \varepsilon_{1423} = q^{\frac{3}{2}}, & \varepsilon_{1432} = -q^{\frac{3}{2}}, & \varepsilon_{2134} = -1, & \varepsilon_{2143} = q, \\ \varepsilon_{3124} = 1, & \varepsilon_{3142} = -q, & \varepsilon_{4123} = -q^{\frac{3}{2}}, & \varepsilon_{4132} = q^{\frac{3}{2}}, \\ \varepsilon_{2314} = q^{\frac{1}{2}}, & \varepsilon_{2341} = -q^{\frac{5}{2}}, & \varepsilon_{2413} = -q^2, & \varepsilon_{2431} = q^3, \\ \varepsilon_{3214} = -q^{\frac{1}{2}}, & \varepsilon_{3241} = q^{\frac{5}{2}}, & \varepsilon_{4213} = q^2, & \varepsilon_{4231} = -q^3, \\ \varepsilon_{3412} = q^2, & \varepsilon_{3421} = -q^3, & \varepsilon_{4312} = -q^2, & \varepsilon_{4321} = q^3. \end{array} \tag{5.21}$$

Note 1. Using the general formula (B.22) one sees that the rule (5.17) follows from postulating the following coproduct on ε^i:

$$\Delta(\varepsilon^i) = \varepsilon^j \otimes M_j{}^i . \tag{5.22}$$

Note 2. The q-Lie derivative was defined along left-invariant vectors in ref. [8], and extended to a Lie derivative along any tangent vector in ref. [27]. In both these references, formulas are given where t_i and ω^j are left-invariant. In (5.13) and (5.14) we have generalized these formulas to non-left invariant t_i and ω^j, with the t_i still dual to the ω^j.

Note 3. The $D = 2$ bicovariant q-Poincaré algebra proposed in the first of refs. [14] coincides with the one obtained from $SO_{q,r=1}(4)$ via the procedure of Sect. 3.

Note added in proof: We have treated in this paper the $ISO_q(N)$ *algebras.* We refer to [30] for the R-matrix formulation of the quantum inhomogenous *groups* $ISO_{q,n}(N)$ and $ISp_{q,n}(N)$.

A. Change of Basis and Graphical Representation of Λ

Consider (2.22) with adjoint indices:

$$\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l = \mathbf{C}_{ij}{}^k\chi_k \,. \tag{A.1}$$

Under a (nonsingular) change of basis

$$\chi_i = S_i^j\xi_j \,, \tag{A.2}$$

the q-Lie algebra transforms into:

$$\xi_i\xi_j - \tilde\Lambda^{kl}{}_{ij}\xi_k\xi_l = \tilde{\mathbf{C}}_{ij}{}^k\xi_k \tag{A.3}$$

with

$$\tilde\Lambda^{kl}{}_{ij} = (S^{-1})_i^{i'}(S^{-1})_j^{j'}\Lambda^{k'l'}{}_{i'j'}S_{k'}{}^kS_{l'}{}^l \,, \tag{A.4}$$

$$\tilde{\mathbf{C}}_{ij}{}^k = (S^{-1})_i^{i'}(S^{-1})_j^{j'}\mathbf{C}_{i'j'}{}^{k'}S_{k'}{}^k \,, \tag{A.5}$$

which amounts to say that Λ and $\mathbf{C}$ transform as tensors under (A.2). Therefore we have the

Theorem. The transformed $\tilde\Lambda$ and $\tilde{\mathbf{C}}$ tensors satisfy the bicovariance relations (2.25)–(2.28).

Proof. Obvious since (2.25)–(2.28) are tensor relations.

There is a particular change of basis for the χ that allows a graphical representation for the braiding matrix Λ. In the case of the B, C, D series the new ξ are defined via the metric C:

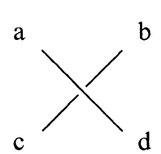
$$\xi_{ab} = C_{ac}\chi^c{}_b \,, \tag{A.6}$$

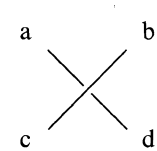
and the Λ tensor takes the form:

$$\Lambda^{a_1a_2d_1d_2}{}_{c_1c_2b_1b_2} = (\hat R^{-1})^{hg_1}{}_{b_1c_2}(\hat R^{-1})^{ea_1}{}_{g_1c_1}\hat R^{d_1a_2}{}_{fe}\hat R^{d_2f}{}_{b_2h} \,. \tag{A.7}$$

To prove this, one uses the definition (2.24) inside (2.20), and the identities (2.11)–(2.12).

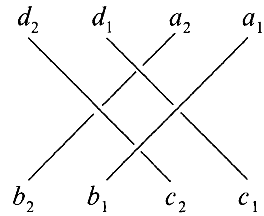
If we represent $\hat R$ and $\hat R^{-1}$ as

$\hat R^{ab}{}_{cd} =$ 

$(\hat R^{-1})^{ab}{}_{cd} =$ 

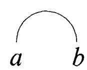
then the braiding matrix Λ is represented by (cf. also [12]):

$$\Lambda^{a_1 a_2 d_1 d_2}{}_{c_1 c_2 b_1 b_2} =$$

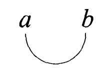


The metric C_{ab} is represented as

$C_{ab} =$



$C^{ab} =$



and, as an amusing exercise, the reader can draw the graphical representation of the adjoint metric C_{ij} given in (2.29) and its inverse, and of the relations (2.11)–(2.12) and (2.32)–(2.33). The metric C_{ab} allows to close the braids into knots, and the graphical representation of this section yields a knot invariant for any B_n, C_n, D_n q-group. The three Reidemeister moves hold because of

i) $C_{ab}\hat{R}^{ab}{}_{cd} \propto C_{cd}$,
ii) the definition of the crossings corresponding to $\hat{R}$ and $\hat{R}^{-1}$,
iii) the quantum Yang–Baxter equations for $\hat{R}$.

On the connection between knot theory and quantum groups see refs. [29, 12].

B. A Note on Bicovariance Conditions and q-Groups Defined in the Adjoint Representation

Whenever we have a set of Λ and $\mathbf{C}$ components satisfying the bicovariance conditions (2.25)–(2.28), and an invertible metric C_{ij} satisfying (2.32), (2.33), we can define

i) A quantum group generated by the matrix elements $M_i{}^j$. The "RTT" relations become now "ΛMM" relations:

$$M_i{}^j M_r{}^q \Lambda^{ir}{}_{pk} = \Lambda^{jq}{}_{ri} M_p{}^r M_k{}^i \,. \tag{B.1}$$

The co-structures on M are similar to the ones defined on the T (Eqs. (2.13)–(2.15)):

$$\Delta(M_i{}^j) = M_i{}^k \otimes M_k{}^j \,, \tag{B.2}$$

$$\varepsilon(M_i{}^j) = \delta_i^j \,, \tag{B.3}$$

$$\kappa(M_i{}^j) = C_{ik} M_l{}^k C_{lj} \,. \tag{B.4}$$

Moreover, we can impose the orthogonality relations:

$$M_i{}^j M_k{}^l C^{ik} = C^{jl} \,, \tag{B.5}$$

$$M_i{}^j M_k{}^l C_{jl} = C_{ik} \,, \tag{B.6}$$

These are compatible with the AMM relations (B.1) because of Eqs. (2.32) and (2.33).

ii) Functionals χ_i and $f^i{}_j$ via their action on M:

$$\chi_j(M_i{}^k) = \mathbf{C}_{ij}{}^k\,, \tag{B.7}$$

$$f^i{}_l(M_k{}^j) = \Lambda^{ij}{}_{kl}\,, \tag{B.8}$$

whose co-structures are given by:

$$\Delta'(\chi_i) = \chi_j \otimes f^j{}_i + I' \otimes \chi_i\,, \tag{B.9}$$

$$\varepsilon'(\chi_i) = 0\,, \tag{B.10}$$

$$\kappa'(\chi_i) = -\chi_j \kappa'(f^j{}_i)\,, \tag{B.11}$$

$$\Delta'(f^i{}_j) = f^i{}_k \otimes f^k{}_j\,, \tag{B.12}$$

$$\varepsilon'(f^i{}_j) = \delta^i_j\,, \tag{B.13}$$

$$\kappa'(f^i{}_j) = f^i_j \circ \kappa\,. \tag{B.14}$$

The action of χ_i and f^i_j on products of M elements is defined in the usual way via the coproduct Δ', i.e. $\chi_i(ab) = \Delta'(\chi_i)(a \otimes b)$, etc.

These functionals satisfy the relations:

$$\chi_i\chi_j - \Lambda^{kl}{}_{ij}\chi_k\chi_l = \mathbf{C}_{ij}{}^k\,, \tag{B.15}$$

$$\Lambda^{nm}{}_{ij} f^i{}_p f^j{}_q = f^n{}_i f^m{}_j \Lambda^{ij}{}_{pq}\,, \tag{B.16}$$

$$\mathbf{C}_{nm}{}^i f^m{}_j f^n{}_k + f^i{}_j \chi_k = \Lambda^{pq}{}_{jk}\chi_p f^i{}_q + \mathbf{C}_{jk}{}^l f^i{}_l\,, \tag{B.17}$$

$$\chi_k f^n{}_l = \Lambda^{ij}{}_{kl} f^n{}_i \chi_j\,, \tag{B.18}$$

which are the operatorial equivalents of the bicovariance relations (2.25)–(2.28), cf. [10]. Indeed the latter can be obtained by applying the former to $M_r{}^s$. We recall that products of functionals are convolution products, for example

$$\chi_i\chi_j \equiv (\chi_i \otimes \chi_j)\Delta\,. \tag{B.19}$$

The algebra generated by the χ and f modulo the relations (B.15)–(B.18) is a Hopf algebra, and defines a bicovariant differential calculus on the q-group generated by the M elements. For example, one can introduce left-invariant one-forms ω^i as duals to the "tangent vectors" χ_i, an exterior product

$$\omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}{}_{kl}\omega^k \otimes \omega^l\,, \tag{B.20}$$

an exterior derivative on $Fun_q(M_i{}^j)$ (the quantum group generated by the $M_i{}^j$) as

$$da = (id \otimes \chi_i)\Delta(a)\omega^i, \quad a \in Fun_q(M_i{}^j) \tag{B.21}$$

and so on. The commutations between one-forms and elements $a \in Fun_q(M_i{}^j)$ are given by:

$$\omega^i a = (id \otimes f^i{}_j)\Delta(a)\omega^j \, . \tag{B.22}$$

Note. When $\Lambda^2 = 1$ there is a way to define the antipode of $f^i{}_j$ without reference to an adjoint metric. This we manage by enlarging the algebra, adding to χ_i and $f^i{}_j$ also the functionals $f^{-i}{}_j$ defined by:

$$f^{-i}{}_l(M_k{}^j) = \Lambda^{ji}{}_{lk} \, . \tag{B.23}$$

An antipode for $f^i{}_j$ can be defined as

$$\kappa'(f^i{}_j) = f^{-i}{}_j \, , \tag{B.24}$$

since

$$f^{-i}{}_s f^s{}_j(M_l{}^k) = (f^{-i}{}_s \otimes f^s{}_j)\Delta(M_l{}^k) = f^{-i}{}_s(M^r{}_l) f^s{}_j(M_r{}^k) = \Lambda^{ri}{}_{sl}\Lambda^{sk}{}_{rj} = \delta^i_j \delta^k_l \, , \tag{B.25}$$

the last equality being due to Eq. (2.37). Then we see that

$$f^{-i}{}_s f^s{}_j = \delta^i_j \varepsilon \, , \tag{B.26}$$

so that $f^{-i}{}_j$ is a good inverse for $f^i{}_j$. Similarly we can prove that

$$f^i{}_s f^{-s}{}_j(M_l{}^k) = \delta^i_j \delta^k_l \, , \tag{B.27}$$

and therefore

$$\kappa'(f^{-i}{}_j) = f^i_j \, , \tag{B.28}$$

which means that i) $\kappa'^2 = 1$, ii) the algebra generated by $\chi_i, f^i{}_j, f^{-i}{}_j$ is closed under all the co-structures (actually the set of the generators itself is closed).

Using Eq. (B.24), the relations (B.15)–(B.18) can be easily extended to include the $f^{-i}{}_j$.

C. Extension of the $*$-Conjugation to the Dual Algebra Generated by the χ and f functionals

Here we want to show that the conjugation:

$$(\chi_i)^* = \mathscr{D}_i{}^{i'} \chi_{i'} \, , \tag{C.1}$$

$$(f^i{}_j)^* = \mathscr{D}_j{}^{j'} f^{i'}{}_{j'} \mathscr{D}_{i'}{}^i \, , \tag{C.2}$$

where $\mathscr{D}$ is the matrix that permutes the fundamental indices $n \leftrightarrow n+1$, is compatible with the operatorial bicovariance conditions (B.15)–(B.18). We begin by taking the $*$-conjugate of (B.15), which yields

$$\chi_j^* \chi_i^* - \bar{\Lambda}^{kl}{}_{ij} \chi_l^* \chi_k^* = \bar{\mathbf{C}}_{ij}{}^k \chi_k^* . \tag{C.3}$$

Using now (C.1) and the fact that $\bar{\Lambda}$ is invariant under the index permutation $n \leftrightarrow n+1$, we can rewrite the above equation as:

$$\chi_j \chi_i - \bar{\Lambda}^{kl}{}_{ij} \chi_l \chi_k = -\bar{\mathbf{C}}_{ij}{}^m \chi_m . \tag{C.4}$$

Remark. For $|q| = 1$ we have

$$\bar{\Lambda}^{kl}{}_{ij} = \Lambda^{lk}{}_{ji} . \tag{C.5}$$

Indeed rewrite (B.15) as

$$\chi_j \chi_i - \Lambda^{lk}{}_{ji} \chi_l \chi_k = \mathbf{C}_{ji}{}^m \chi_m . \tag{C.6}$$

For $r = 1$, $\Lambda^{lk}{}_{ji}$ is proportional to $\delta_j^k \delta_i^l$ (and contains only one term, as we can see from the defining formula (2.20) and the fact that R is diagonal with only one term in the diagonal entries) and it is easy to deduce that (C.6) is compatible with (B.15) only if $\Lambda^{lk}{}_{ji} = (\Lambda^{kl}{}_{ij})^{-1}$ (the inverse of the matrix *element*). For $|q| = 1$ we have $\bar{\Lambda}^{kl}{}_{ij} = (\Lambda^{kl}{}_{ij})^{-1}$ so that (C.5) is proved.

Notice now that (C.4) would reproduce exactly (C.6), proving the compatibility of the conjugation rule (C.1) with the q-Lie algebra, if we had also:

$$\bar{\mathbf{C}}_{ij}{}^m = -\mathbf{C}_{ji}{}^m . \tag{C.7}$$

We could not prove (C.7) on general grounds: presumably one can always find a basis for the χ_i generators such that it holds. This is indeed the case for $ISO_q(3,1)$.

In a similar way one shows the compatibility of the conjugation (C.1), (C.2) with the remaining bicovariance operator conditions (B.16)–(B.18). One just needs to use property (C.7) and a useful identity valid for $\Lambda^2 = I$:

$$\mathbf{C}_{ij}{}^m = -\Lambda^{kl}{}_{ij} \mathbf{C}_{kl}{}^m , \tag{C.8}$$

i.e. the structure constants are Λ-antisymmetric. This is easily proved by remarking that the left-hand side of (B.15) does not change under multiplication by the projector $P_A = (I - \Lambda)/2$, since $P_A(I - \Lambda) = I - \Lambda$ (for $\Lambda^2 = I$), and thus

$$\frac{1}{2}(I - \Lambda)^{kl}{}_{ij} \mathbf{C}_{kl}{}^m = \mathbf{C}_{ij}{}^m , \tag{C.9}$$

which is just Eq. (C.8). Again this can be immediately checked to hold for $ISO_q(3,1)$.

Finally, it is a simple matter to check that

$$\kappa'(\kappa'(\chi_i^*)^*) = \chi_i, \quad \Delta'(\chi_i^*) = [\Delta'(\chi_i)]^* \tag{C.10}$$

(and similar for $f^i{}_j, f^{-i}{}_j$) showing that (C.1)–(C.2) is a Hopf algebra conjugation.

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