Asymptotics for the Fredholm Determinant of the Sine Kernel on a Union of Intervals

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Abstract: In the bulk scaling limit of the Gaussian Unitary Ensemble of hermitian matrices the probability that an interval of length s contains no eigenvalues is the Fredholm determinant of the sine kernel $\frac{\sin(x-y)}{\pi(x-y)}$ over this interval. A formal asymptotic expansion for the determinant as s tends to infinity was obtained by Dyson. In this paper we replace a single interval of length s by sJ, where J is a union of m intervals and present a proof of the asymptotics up to second order. The logarithmic derivative with respect to s of the determinant equals a constant (expressible in terms of hyperelliptic integrals) times s, plus a bounded oscillatory function of s (zero if $m = 1$, periodic if $m = 2$, and in general expressible in terms of the solution of a Jacobi inversion problem), plus $o(1)$. Also determined are the asymptotics of the trace of the resolvent operator, which is the ratio in the same model of the probability that the set contains exactly one eigenvalue to the probability that it contains none. The proofs use ideas from orthogonal polynomial theory.

I. Introduction

The sine kernel

$$K(x, y) := \frac{\sin(x - y)}{\pi(x - y)}$$

arises in many areas of mathematics and mathematical physics. There is an extensive literature on the asymptotics of the eigenvalues of K_s, the operator with this kernel on an interval of length s, as $s \to \infty$, for example [4, 6, 12, 15], and asymptotic formulas of various kinds have been obtained. Some of these derivations were rigorous, others were more heuristic. The Fredholm determinant of this kernel, i.e., the operator determinant $\det(I - K_s)$, is of particular interest. In the bulk scaling limit of the Gaussian Unitary Ensemble of hermitian matrices it equals the probability that an interval of length s contains no eigenvalues.

The asymptotics of Fredholm determinants of convolution kernels $k(x - y)$ have a long history. (The history of their discrete analogue, Toeplitz determinants, is even longer, beginning with the 1915 paper [16] of Szegö.) If the Fourier transform $\hat{k}$ of k is smooth and less than 1, and if k satisfies some other conditions, then for the corresponding operator K_s one has as $s \to \infty$,

$$\log \det (I - K_s) = c_1 s + c_2 + o(1),$$
(1)

where c_1 and c_2 are explicitly determined constants [10]. One can generally sharpen this, and even find an asymptotic expansion if k is sufficiently small at infinity. (For the analogous refinements for Toeplitz determinants see [7, 20].) An example is the Fredholm determinant which gives the probability of the absence of particles in an interval for the impenetrable Bose gas. Conjugating the operator which arises there with the Fourier transform converts it to a convolution operator of the type described. The asymptotics of this Fredholm determinant were obtained in [8]. (See also [11]. It should be mentioned that these authors considered as well the asymptotics of determinants where $\hat{k} > 1$ on a full interval. The general theory alluded to above gives no information at all in such cases.)

If $\hat{k} < 1$ but is not smooth the asymptotics are different. If $\hat{k}$ has only jump discontinuities, then (1) becomes [3]

$$\log \det (I - K_s) = c_1 s + c_2 \log s + c_3 + o(1),$$
(2)

where c_1, c_2 and c_3 are again explicitly determined constants. Even for some cases where $\hat{k} < 1$ is violated at finitely many points a similar relation holds [1]. But for the sine kernel $\hat{k}$ is the characteristic function of the interval $[-1, 1]$ and so the condition is violated on a set of positive measure and the situation is more complicated.

To see the significance of the condition $\hat{k} < 1$ in the context of the sine kernel we introduce a real parameter λ and consider

$$\det (I - \lambda K_s) = \prod_{k=1}^{\infty} (1 - \lambda \lambda_k(s)),$$
(3)

where $\lambda_1(s) > \lambda_2(s) > \cdots$ are the eigenvalues of the sine kernel on an interval of length s. These eigenvalues are all positive and less than 1 and as $s \to \infty$ they fill up the interval $[0, 1]$, i.e., all points in the interval are limits of eigenvalues and no other points are. (In general the eigenvalues fill up the interval between inf $\hat{k}$ and sup $\hat{k}$ when $\hat{k}$ is real-valued.) When $\lambda < 1$ all factors in (3) are bounded away from 0 and the determinant simply approaches 0 exponentially. (The asymptotics are given by (2).) When $\lambda = 1$ the kth factor will tend to 0 for each k, even exponentially, with the result that the determinant tends to 0 much more rapidly than when $\lambda < 1$. And when $\lambda > 1$ the determinant will actually equal 0 for a sequence of s's, the solutions of $\lambda_k(s) = \lambda^{-1}$. (One can see this change of behavior in the results of [13], where a special case of the expansion obtained is that of the logarithmic derivative of this determinant; for $\lambda > 1$ a sequence of poles appears.) The present paper deals exclusively with the transitional value, $\lambda = 1$. (In the literature one often finds $1/\pi$ as the transitional value. We have incorporated the π into the kernel.)

The asymptotics of the determinant for the sine kernel were probably first investigated by Dyson [5]. He derived the asymptotic formula

$$\log \det (I - K_s) = -\frac{s^2}{8} - \frac{1}{4} \log s + 3\zeta'(-1) + \frac{1}{3} \log 2 + o(1) \tag{4}$$

by applying a scaling argument to a known asymptotic formula for Toeplitz determinants [19], and then using inverse scattering techniques to complete this to an asymptotic expansion. In 1980 Jimbo et al. [9] showed that the logarithmic derivative of this determinant satisfies a second-order differential equation which is reducible to a Painlevé V equation. The asymptotic expansion can also be obtained, starting from (4), by substituting a formal expansion into the equation and successively solving for the coefficients.

Another quantity of interest from the random matrix point of view is

$$\text{tr } K_s (I - K_s)^{-1} \, ,$$

which in the same model equals the probability that an interval of length s contains precisely one eigenvalue, divided by the probability that it contains none. (There are similar but more complicated formulas for any finite number of eigenvalues [14].) The asymptotics of this were derived in [2], where

$$\text{tr } K_s (I - K_s)^{-1} \sim \frac{e^s}{2\sqrt{2\pi s}} \tag{5}$$

was obtained by scaling an analogous result for Toeplitz matrices, which in turn was proved by exploiting its connection with orthogonal polynomials, whose asymptotics were known. This formula was then extended to a complete asymptotic expansion by using the differential equation.

These asymptotic results were not rigorously proved by the methods described. In [21], by studying the asymptotics of a continuous analogue of orthogonal polynomials, we were able to give a proof of (5) and of the first-order asymptotics in (4),

$$\log \det (I - K_s) \sim -\frac{s^2}{8} \, ,$$

actually the slightly stronger

$$\frac{d}{ds} \log \det (I - K_s) = -\frac{s}{4} + O(1) \, . \tag{6}$$

Refinements of these results are not the subject of this paper, but rather the extension from the one interval case to several intervals. Thus a single interval of length s is replaced by sJ, where J is a fixed union of intervals. Even these asymptotics are quite elaborate and involve hyperelliptic integrals and the Jacobi inversion problem for Abelian integrals.

We continue to denote our operator by K_s, and it is understood that it now acts on sJ. We shall show that as $s \to \infty$,

$$-\frac{d}{ds} \log \det (I - K_s) = c_1 s + c_2(s) + o(1) \, , \tag{7}$$

where c_1 is a constant and $c_2(s)$ is a certain bounded oscillatory function of s, and that

$$\operatorname{tr} K_s(I - K_s)^{-1} = \frac{e^{c_3 s}}{\sqrt{s}}(c_4(s) + O(s^{-\frac{1}{2}})),\qquad(8)$$

where c_3 is another constant and $c_4(s)$ is another bounded oscillatory function. The constants c_1 and c_3 are explicitly computable but the determination of the functions $c_2(s)$ and $c_4(s)$ requires the solution of a Jacobi inversion problem. (At least this is the case with this method. It is possible that a different approach might lead to simpler representations.) If J consists of m intervals then there is a curve, parametrized by s, in the $m - 1$-torus $\mathbf{T}^{m-1}$, and two real-valued functions defined on $\mathbf{T}^{m-1}$. These functions, when restricted to the curve, are $c_2(s)$ and $c_4(s)$. When $m = 2$ they are periodic, and we can write down an integral representation for the period.

Here is how we obtain the asymptotics. For the sequence of monic orthogonal polynomials $P_n(z)$ associated with a weight function w on the unit circle, the square of the $L_2(w)$-norm of P_n is the ratio of two consecutive Toeplitz determinants associated with w. The asymptotics of P_n gives information on the asymptotics of this ratio of determinants. Analogously the asymptotics of the continuous analogue of these polynomials, which we shall denote by $R(x)$, determine the asymptotics of the logarithmic derivative of the Fredholm determinant. The underlying weight function for $R(x)$ will be the characteristic function of the complement of J in $\mathbf{R}$, so we have the continuous analogue of the polynomials orthogonal on a union of arcs. The asymptotics of orthogonal polynomials such as these were obtained in [18] and the ideas of this paper were used in [21] and are used here. We find an entire function which comes close to satisfying the characteristic property of R given in (9) below by first finding another function (which is not entire, or even single-valued) which has this property exactly (the function h_s in Lemma 4), and then we approximate this by an entire function. The entire function we find in this way will necessarily be a good approximation to R. This gives a weaker form of (7) at first. It is strengthened by exploiting its relation to a certain extremal problem. Some details will be omitted here since the complete details for the analogous orthogonal polynomial case can be found in [18].

We obtain (8) by using a representation, which is essentially contained in [9], for the resolvent kernel of K_s in terms of $R(x)$. There are analogous representations for a large class of kernels like the sine kernel [8, 17].

In [9] it was shown that for general J the Fredholm determinant is governed by a system of partial differential equations with the end-points of J as the independent variables. The dependent variables are the values of the function $R(x)$ at the end-points of J, and the logarithmic derivative of the Fredholm determinant has a simple representation in terms of these values. We do not use this representation but rather the one alluded to above (formula (10) below), which makes its asymptotics, at least by this method, more accessible. Nevertheless, our results may give a hint of what the asymptotics of the solutions of the system of equations might involve.

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II. The Function $R(x)$

We shall denote the Fourier transform by a circumflex, as usual,

$$\hat{f}(x) := \int_{-\infty}^{\infty} e^{ixt} f(t) \, dt \, ,$$

and write $\mathscr{E}_s$ for the space of Fourier transforms of functions in $L_2(-s, s)$. This consists of entire functions which are $O\left(e^{s|\text{Im}\, x|}\right)$ and whose restrictions to the real line $\mathbf{R}$ belong to L_2. We also write E for the complement of J in $\mathbf{R}$ and Ω for the complement of E in $\mathbf{C}$.

Lemma 1. *There exists a unique function $R(x) \in e^{-isx} + \mathscr{E}_s$ such that*

$$\int_E \overline{(R(x) - e^{-isx})} \, g(x) \, dx = \int_J g(x) \, e^{isx} \, dx \quad \text{for all } g \in \mathscr{E}_s \, . \tag{9}$$

In terms of this function we have the representations

$$-\frac{d}{ds} \log \det(I - K_s) = \frac{1}{\pi}|J| + \frac{1}{\pi} \int_E \left| R(x) - e^{-isx} \right|^2 dx \, , \tag{10}$$

where $|J|$ is the measure of J, and

$$\text{tr}\, K_s (I - K_s)^{-1} = -\frac{1}{\pi} \text{Im} \int_J R'(x) \overline{R(x)} \, dx \, . \tag{11}$$

Remark. The set $e^{-isx} + \mathscr{E}_s$ is analogous to the set of monic polynomials of a given degree, and (9) implies that $R(x)$ is orthogonal on E to a dense subset of $\mathscr{E}_s$, those functions which are Fourier transforms of smooth functions on $[-s, s]$ vanishing at $-s$. (The integral expressing the orthogonality is a principal value integral at infinity.) Thus $R(x)$ is analogous to a monic polynomial of a family orthogonal on a union of circular arcs. The identity (10) is the analogue of the fact that the square of the norm of the monic polynomial is the ratio of two consecutive determinants.

Proof. Write

$$K_s(x, y) := \frac{\sin s(x - y)}{\pi(x - y)} \, ,$$

and denote by K_s the operator on all of $\mathbf{R}$ with kernel $K_s(x, y) \chi_J(y)$. Our determinant and trace are the same for this kernel as for the other. We define

$$R(x) := (I - K_s)^{-1} e^{-isx} \, .$$

Since $K_s(x, y)$ is real-valued this definition may be written in the equivalent form

$$\overline{R(x)} - \int_J K_s(x, y) \overline{R(y)} \, dy = e^{isx} \, .$$

To obtain (9) we use the fact that $K_s(x, y)$ is the kernel of the projection operator from $L_2(\mathbf{R})$ to $\mathscr{E}_s$; this is so because the Fourier transform of $\sin sx / \pi x$

equals the characteristic function of $[-s, s]$. This implies that $R(x) \in e^{-isx} + \mathscr{E}_s$ and that

$$\int_{\mathbf{R}} g(x) K_s(x, y) \, dx = g(y)$$

for any $g \in \mathscr{E}_s$. If also $g \in L_1(\mathbf{R})$ then multiplying both sides of the previous identity by $g(x)$ and integrating over $\mathbf{R}$ give (9) in this case. The extension to general g is straightforward.

To obtain the uniqueness assertion of the lemma observe that the difference between any two functions R satisfying (9) belongs to $\mathscr{E}_s$ and is orthogonal on E to $\mathscr{E}_s$, and so it vanishes on E. Since $\mathscr{E}_s$ consists of entire functions, this difference must be 0.

Next, we use the facts that the left side of (10) is equal to the trace of $(I - K_s)^{-1} K_s'$ and that the kernel of K_s' is

$$\frac{1}{\pi} \cos s(x - y) = \frac{1}{\pi} \operatorname{Re} e^{-is(x-y)},$$

from which it follows that

$$\operatorname{tr}(I - K_s)^{-1} K_s' = \frac{1}{\pi} \operatorname{Re} \int_J R(x) e^{isx} \, dx.$$

If we apply the identity (9) to the function $g(x) = R(x) - e^{-isx}$ it gives

$$\int_E \left| R(x) - e^{-isx} \right|^2 dx = \int_J R(x) e^{isx} \, dx - |J|,$$

and this establishes (10).

To obtain (11) we express the resolvent kernel $R_s(x, y)$, the kernel of the operator $K_s(I - K_s)^{-1}$, in terms of $R(x)$. If M denotes multiplication by x then the commutator $[M, K_s]$ has kernel

$$\frac{1}{2\pi i} \left(e^{isx} e^{-isy} - e^{-isx} e^{isy} \right).$$

It follows from this upon left- and right-multiplying by $(I - K_s)^{-1}$ that the commutator $[M, (I - K_s)^{-1}]$ has kernel

$$\frac{\overline{R(x)} R(y) - R(x) \overline{R(y)}}{2\pi i}.$$

From this we see that for $x, y \in J$

$$R_s(x, y) = \frac{\overline{R(x)} R(y) - R(x) \overline{R(y)}}{2\pi i (x - y)},$$

and so

$$R_s(x, x) = -\frac{1}{\pi} \operatorname{Im} R'(x) \overline{R(x)},$$

which gives (11). $\quad\square$

III. Green Functions, Neumann Functions and a Reproducing Function for E

In [21], where $J = [-1, 1]$, $E = (\infty, -1] \cup [1, \infty)$ and $\Omega = \mathbf{C} \backslash E$, the procedure was to show first that (9) holds if $R(x)$ is replaced by the sum of the two limiting values on E of the function

$$h(x) := e^{-is\sqrt{x^2 - 1}} \sqrt{\frac{x + \sqrt{x^2 - 1}}{2\sqrt{x^2 - 1}}} \tag{12}$$

defined in Ω with appropriate branches of the square roots. Notice that $R(x)$ and $h(x)$ have the same asymptotic behavior as $x \to \infty$ in Ω. One approximates $h(x)$ by an entire function, for which (9) holds approximately, and then deduces that this function must be a good approximation to $R(x)$.

The function $\sqrt{x^2 - 1}$ in (12) was chosen to be that branch of the square root which is analytic in Ω and asymptotically equal to x as $x \to \infty$ in the upper half-plane. Its imaginary part vanishes on E and equals $|\mathrm{Im}\, x| + O(1)$ in Ω, and so is a kind of Green function for Ω. We begin by constructing its analogue for general

$$J = \bigcup_{k=1}^{m} [\alpha_k, \beta_k].$$

Let

$$q(x) = \prod_{k=1}^{m} (x - \alpha_k)(x - \beta_k)$$

and let $\sqrt{q(x)}$ denote that branch which is analytic in Ω and asymptotic to x^m as $x \to \infty$ in the upper half of Ω. This has purely real limiting values on E. Let $p(x)$ be the monic m^{th} degree polynomial determined by the m equations

$$\int_{\alpha_k}^{\beta_k} \frac{p(y)}{\sqrt{q(y)}}\, dy = 0, \quad (k = 1, \ldots, m). \tag{13}$$

Then define

$$G_0(x) := \int_{\alpha_1}^{x} \frac{p(y)}{\sqrt{q(y)}}\, dy, \quad (x \in \Omega). \tag{14}$$

This is multiple-valued when $m > 1$ because of the presence of the intervals $[\beta_k, \alpha_{k+1}]$ $(k \leqq m - 1)$ one might integrate around and get a nonzero result. If C_k is a curve going around this interval and none of the others then the periods

$$\Delta_{C_k} G_0 = \oint_{[\beta_k, \alpha_{k+1}]} \frac{p(y)}{\sqrt{q(y)}}\, dy \tag{15}$$

are purely real, so that $\mathrm{Im}\, G_0(x)$ is a single-valued harmonic function in Ω. The conditions (13) guarantee that $\mathrm{Im}\, G_0(x) \to 0$ as $x \to E$.

For some real constants a_1' and a_2 we have

$$\frac{p(x)}{\sqrt{q(x)}} = 1 + \frac{a_1'}{x} + \frac{a_2}{x^2} + O\left(\frac{1}{x^3}\right) \quad \text{as } x \uparrow \infty.$$

Here the notation $x \uparrow \infty$ means that $x \to \infty$ in the upper half-plane of Ω. (Similarly $x \downarrow \infty$ will mean that $x \to \infty$ in the lower half-plane of Ω.) The integral

$$\int_{-\infty+0i}^{\infty+0i} \left(\frac{p(x)}{\sqrt{q(x)}} - 1 \right) dx$$

equals $-\pi i a'_1$. But it follows from (13) that it is also real. Hence $a'_1 = 0$ and so

$$\frac{p(x)}{\sqrt{q(x)}} = 1 + \frac{a_2}{x^2} + O\left(\frac{1}{x^3}\right) \quad \text{as } x \uparrow \infty. \tag{16}$$

It follows readily that

$$G_0(x) = x + \left[\int_{\alpha_1}^{\infty+0i} \left(\frac{p(y)}{\sqrt{q(y)}} - 1 \right) dy - \alpha_1 \right] - \frac{a_2}{x} + O\left(\frac{1}{x^2}\right),$$

where a_1, the expression in brackets, is purely real. We now define

$$G(x) := G_0(x) - a_1.$$

$G(x)$ has the same Green function characteristics as $G_0(x)$ and has the behavior

$$G(x) = x - \frac{a_2}{x} + O\left(\frac{1}{x^2}\right) \quad \text{as } x \uparrow \infty. \tag{17}$$

Since G_0 is purely imaginary on J, we have $G_0(\bar{x}) = -\overline{G_0(x)}$, and so from the above we deduce

$$G(x) = -x - 2a_1 + \frac{a_2}{x} + O\left(\frac{1}{x^2}\right) \quad \text{as } x \downarrow \infty. \tag{18}$$

The function $e^{isG(x)}$ is multiple-valued and analytic in Ω and has single-valued absolute value. Such a function can be thought of as a section of a holomorphic line bundle over Ω, but instead we proceed as follows. Denote by $\mathscr{H}$ the set of all (single-valued) analytic functions f in the complement of $(-\infty, \alpha_1] \cup [\beta_1, \infty)$ in $\mathbf{C}$ for which the limits

$$\lim_{\varepsilon \to 0+} \frac{f(x+i\varepsilon)}{f(x-i\varepsilon)}, \quad x \in [\alpha_k, \beta_k], \quad (k = 2, \ldots, m) \tag{19}$$

are constants of absolute value 1. Each such function continues to a multiple-valued function in Ω with single-valued absolute value. Two functions in $\mathscr{H}$ are said to belong to the same class if the corresponding limits (19) for them are the same. A class is denoted by Γ and the set of classes can be identified with the $m-1$-torus. We denote by $\mathscr{H}(\Gamma)$ the functions of class Γ. Occasionally we shall allow our functions to have poles. An example is the exponential Green function $\Phi(x, x_0)$ described below. We shall write our classes additively, and so each will be an element of $\mathbf{T}^{m-1} := (\mathbf{R}/2\pi\mathbf{Z})^{m-1}$. If Γ_0 is the class of $e^{iG(x)}$, then $s\Gamma_0$ is the class of $e^{isG(x)}$. This is our curve in $\mathbf{T}^{m-1}$.

Next we have to find an analogue of the second factor in (12). There will be many, one for each class $s\Gamma_0$, and so we will find one for a general class Γ. We begin by recalling the characteristic properties of the Green and Neumann functions

for Ω. Given x_0, Green's function with pole at x_0, denoted by $g(x, x_0)$, is harmonic for $x \in \Omega$, except at x_0 where it has the singularity $\log |x - x_0|^{-1}$, and it has limit 0 as $x \to E$. Denoting by $\tilde{g}(x, x_0)$ its (multiple-valued) harmonic conjugate, we define the exponential Green function by

$$\Phi(x, x_0) := e^{g(x, x_0) + i\tilde{g}(x, x_0)} .$$

Neumann's function $N(x, x_0, x_1)$ is harmonic except at $x = x_0$ and $x = x_1$, where it has the singularities $\log |(x - x_1)/(x - x_0)|$, and on E

$$\frac{\partial N(x, x_0, x_1)}{\partial n_x} = 0 .$$

Here $\partial/\partial n_x$ denotes the two derivatives normal to E. We set

$$\Psi(x, x_0, x_1) := e^{N(x, x_0, x_1) + i\tilde{N}(x, x_0, x_1)} .$$

This has a simple pole at $x = x_0$ and a simple zero at $x = x_1$, unless the two are equal, in which case the function equals 1. The function $\Phi(x, x_0)$ is multiple-valued with single-valued absolute value, and is determined only up to an arbitrary constant factor of absolute value 1. The function $\Psi(x, x_0, x_1)$ is single-valued but determined only up to an arbitrary nonzero constant factor. The analogue of the second factor in (12) will be built out of these functions.

Lemma 2. *The function* $1 + p(x)/q(x)^{\frac{1}{2}}$ *is nonzero (i.e., has nonzero limits) on* E *and has* $m - 1$ *zeros in* Ω.

Proof. We know that

$$\mathrm{Im}(x + G(x)) = 0 \text{ on } E . \tag{20}$$

From (17) and (18) we deduce that

$$\lim_{x \to \infty} \mathrm{Im}(x + G(x)) \geqq 0 .$$

Therefore

$$\mathrm{Im}(x + G(x)) > 0 \text{ in } \Omega . \tag{21}$$

If

$$1 + \frac{p(x)}{\sqrt{q(x)}} = (x + G(x))'$$

had limit zero at some point of E this would contradict (20) and (21) since at a critical point an analytic function maps a local half-disc to a full disc.

For the second statement let C_k be the contours in (15) and $C = C_- \cup C_+$, where C_- starts at $-\infty - 0i$, goes around α_1, and ends at $-\infty + 0i$, while C_+ starts at $\infty + 0i$, goes around β_m, and ends at $\infty - 0i$. All the contours are described so that the parts below E are traversed to the right, the parts above E to the left. We find that

$$\Delta_C \arg\left(1 + \frac{p(x)}{\sqrt{q(x)}}\right) = 0 , \quad \Delta_{C_k} \arg\left(1 + \frac{p(x)}{\sqrt{q(x)}}\right) = -2\pi, \ (k = 1, \ldots, m - 1) .$$

This and the argument principle establish the lemma. $\quad\square$

Denote the zeros of $1 + p(x)/q(x)^{\frac{1}{2}}$ by $x_1{}^*, \ldots, x_{m-1}{}^*$. It follows from a discussion in [18] that for each class Γ there are unique points $x_1, \ldots, x_{m-1}$, and unique numbers $\varepsilon_1, \ldots, \varepsilon_{m-1}$ each equal to ± 1, such that

$$\prod_{k=0}^{m-1} \Psi(x, x_k{}^*, x_k) \geqq 0 \quad \text{on } E \tag{22}$$

(with an appropriate normalization of the Ψ's) and such that

$$k_\Gamma(x) := \left[\frac{1}{2} \left(1 + \frac{p(x)}{\sqrt{q(x)}} \right) \prod_{k=0}^{m-1} \Psi(x, x_k{}^*, x_k) \, \Phi(x, x_k)^{\varepsilon_k} \right]^{\frac{1}{2}} \in \Gamma. \tag{23}$$

The x_k are given by the solution of a certain Jacobi inversion problem. The details, which we shall not present, can be found in Sect. 6 of [18].

One possibility, which occurs for certain Γ, is that all $x_k = x_k{}^*$ so the Ψ's don't appear at all. In general, though, they do appear and the zeros of $1 + p(x)/q(x)^{\frac{1}{2}}$ are cancelled by the poles of the product of the Ψ's. The zero x_k of $\Psi(x, x_k{}^*, x_k)$ is cancelled by the pole of $\Phi(x, x_k)^{\varepsilon_k}$ if $\varepsilon_k = 1$ and reinforced to a double zero if $\varepsilon_k = -1$. So there is no extra multiple-valuedness introduced by taking the square root. Observe, though, that we have not quite defined k_Γ because of the nonuniqueness of the exponential Green and Neumann functions. We make them unambiguous by the requirements

$$N(x, x_k{}^*, x_k), \tilde{N}(x, x_k{}^*, x_k), \tilde{g}(x, x_k{}^*, x_k) \to 0 \quad \text{as } x \uparrow \infty, \tag{24}$$

and then choose the square root so that $k_\Gamma(x) \to 1$ as $x \uparrow \infty$.

Our eventual replacement for $h(x)$ will be

$$h_s(x) := e^{-isG(x)} k_{s\Gamma_0}(x). \tag{25}$$

Lemma 3. *Suppose $\psi \in \mathcal{H}(\Gamma)$ has the behavior*

$$\psi(x) = \frac{c}{x} + O\left(\frac{1}{x^2}\right) \text{ as } x \uparrow \infty, \quad \psi(x) = O\left(\frac{1}{x}\right) \quad \text{as } x \downarrow \infty.$$

Then

$$\oint_E \overline{k_\Gamma(x)} \psi(x) |dx| = -\pi i c.$$

Remark 1. The conjugate-analytic function $\overline{k}_\Gamma$ belongs to the class $-\Gamma$ so that $\overline{k_\Gamma(x)} \psi(x)$ is single-valued in Ω. The notation $\oint_E \cdots |dx|$ indicates that E is not oriented but is taken twice, using the two limiting values of the function.

Remark 2. It is because of this characteristic property of k_Γ, reminiscent of that of reproducing kernels, that we call it the "reproducing function" associated with Γ and Ω: integrating ψ against $\overline{k}_\Gamma$ yields the constant determining the behaviour of ψ as $x \uparrow \infty$.

Proof. Let $j_\Gamma(x)$ be the function obtained by replacing each ε_k by $-\varepsilon_k$ in (23). Then

$$|j_\Gamma(x)| = |k_\Gamma(x)| \quad \text{on } E \tag{26}$$

and, by (22),

$$j_\Gamma(x) k_\Gamma(x) \text{ is real-valued on } E \text{ with the same sign as } 1 + \frac{p(x)}{\sqrt{q(x)}}. \qquad (27)$$

From the fact that $\text{Im}(x + G(x))$ is zero on E and positive in Ω it follows that

$$1 + \frac{p(x)}{\sqrt{q(x)}} = (x + G(x))'$$

has positive limit on E from above and negative limit from below. Alternatively, if we denote the "upper" (resp. "lower") part of E by E^+ (resp. E^-), then

$$\text{sgn}\left(1 + \frac{p(x)}{\sqrt{q(x)}}\right) = \begin{cases} 1 & \text{on } E^+ \\ -1 & \text{on } E^-. \end{cases}$$

Hence, from (26) and (27),

$$k_\Gamma \, \overline{k_\Gamma} = k_\Gamma \, j_\Gamma \times \begin{cases} 1 & \text{on } E^+ \\ -1 & \text{on } E^-, \end{cases}$$

since both sides have the same absolute value and are positive on E. Dividing by k_Γ we see that the integral in the statement of the lemma equals

$$\oint_E j_\Gamma(x) \psi(x) dx,$$

where now E is oriented so that E^+ is traversed to the right, E^- to the left. Observe that $j_\Gamma \in \mathcal{H}(-\Gamma)$ and so $j_\Gamma(x)\psi(x)$ extends to a single-valued analytic function in Ω. Moreover

$$j_\Gamma(x) \, \psi(x) = \begin{cases} cx^{-1} + O(x^{-2}) & \text{as } x \uparrow \infty, \\ O(x^{-2}) & \text{as } x \downarrow \infty \end{cases} \qquad (28)$$

(the second because $1 + p(x)/\sqrt{q(x)} = O(1/x)$ as $x \downarrow \infty$). The integral is equal to the limit of the sum of the integrals over two semicircles, one in the upper half-plaine and one in the lower, as their radii tend to infinity. By (28) the integral over the lower semi-circle tends to 0 while the integral over the upper tends to $-\pi i c$. □

Now recall that we define $h_s(x)$ by (25).

Lemma 4. *Suppose $g(x)$ is single-valued and analytic in Ω and satisfies*

$$e^{isx} g(x) = \frac{c}{x} + O\left(\frac{1}{x^2}\right) \text{ as } x \uparrow \infty, \quad e^{-isx} g(x) = O\left(\frac{1}{x}\right) \quad \text{as } x \downarrow \infty.$$

Then

$$\oint_E \overline{h_s(x)} \, g(x) \, |dx| = -\pi i c.$$

Proof. Write the integral as

$$\oint_E \overline{k_{s\Gamma_0}(x)} \, e^{isG(x)} g(x) \, |dx|.$$

The second factor belongs to class $s\Gamma_0$ and by (17) and (18) has the properties of $\psi(x)$ in the statement of Lemma 3. The assertion follows. $\square$

IV. Asymptotics of $R(x)$

Define, for $x \in E$,
$$h_\pm(x) := \lim_{\varepsilon \to 0\pm} h_s(x + i\varepsilon).$$

We shall deduce from Lemma 4 and the first statement of Lemma 1 that R is well approximated on E by $h_+ + h_-$. To do this we first replace $h_+(x) + h_-(x)$ by an entire function with the same general behavior as R and which is close to $h_+ + h_-$ on E. Such a function will be $e^{-isx} + q(x)$, where

$$q(x) := \frac{e^{-isx}}{2\pi i} \int_{\iota a - \infty}^{\iota a + \infty} \frac{e^{isy} h(y)}{y - x} \, dy \tag{29}$$

with $a < \min(0, \operatorname{Im} x)$. (We have dropped the subscript s in the notation for the function h_s.) To prove that we do get a good approximation we derive an integral representation for $q(x)$ which requires deforming the path of integration in a somewhat involved way, as well as the analytic continuation of $h(y)$.

Recall that the exponential Green functions in (23) have absolute value 1 on E and by (22) the product of exponential Neumann functions is positive there. Thus both of these continue to the Riemann surface obtained by joining Ω to a copy $\overline{\Omega}$ in the usual way, by indentifying $E^\pm$ in Ω with $E^\mp$ in $\overline{\Omega}$. Similarly $G(y)$ continues into $\overline{\Omega}$ and its continuation has *negative* imaginary part there. The resulting continuation of $h(y)$ is not single-valued because of the presence of branch points at the α_k and β_k. We shall deform our path of integration to a system of contours, all lying in $\overline{\Omega}$ where $\operatorname{Im} G(y) < 0$, and all lying in the upper half-plane where the factor e^{isy} is small.

Before stating the main lemma we describe the contours A_i, $(i = 1, \ldots, 2m)$. If $i = 2k - 1$, then A_i begins by running vertically from $\alpha_k + i\infty$ to $\alpha_k + i0$; it makes two infinitesimal counterclockwise circuits of α_k and then runs vertically from $\alpha_k + i0$ back to $\alpha_k + i\infty$. If $i = 2k$ then A_i begins and ends at $\beta_k + i\infty$ but goes around β_k, when it gets to it, clockwise instead of counterclockwise. Except for one infinitesimal loop around each point of ∂J, these contours lie entirely in the upper half-plane of $\overline{\Omega}$, and so have the sought properties.

The function $h(y)$ in the integrand below will be a continuation of the original $h(y)$. (This will be clear during the course of the derivation.) It can be described as that continuation of h which, at the initial points of the contours A_i, is the function obtained by continuing our original h from below E across it into the upper half-plane; on the rest of the A_i we take the continuation along A_i of this one.

Lemma 5. *For $x \in E$ we have*

$$e^{-isx} + q(x) = h_+(x) + h_-(x) + e^{-isx} \sum_{i=1}^{2m} \frac{(-1)^{i+1}}{2\pi i} \int_{A_i} \frac{e^{isy} h(y)}{y - x} \, dy. \tag{30}$$

Proof. The path of integration in (29) may be deformed to the union of the contours C and C_k which occur in the proof of Lemma 2 (all described so that E^- is traversed to the right and E^+ to the left) plus an infinitely large semi-circle in the

upper halfplane traversed clockwise. Since $e^{isy}h(y) = 1 + O(|y|^{-1})$ as $y \uparrow \infty$ the integral over the semi-circle equals $-\pi i$, and so

$$e^{isx}q(x) = -\frac{1}{2} + \frac{1}{2\pi i} \int_{C \cup C_1 \cup \cdots \cup C_{m-1}} \frac{e^{isy}h(y)}{y - x} \, dy. \tag{31}$$

Suppose now, for definiteness, that $x \in (-\infty, \alpha_1)$, and consider C_-, the left part of C, which goes from $-\infty - 0i$, around α_1 counterclockwise, and then to $-\infty + 0i$. We move the upper part of C_- through $(-\infty, \alpha_1)$ into the lower half-plane and the lower part of C_- through $(-\infty, \alpha_1)$ into the upper half-plane. The result is a contour which runs from $\alpha_1 + i\infty$ to α_1, around α_1 counterclockwise, and then down to $\alpha_1 - i\infty$, with $h(y)$ in the integrand replaced by its continuation. If we keep in mind how $G(y)$ and k_Γ continue we see that the continuation of $h(y)$ from the lower half-plane to the upper satisfies

$$e^{isy}h(y) = O\left(\frac{1}{|y|}\right),$$

while its continuation from the upper to the lower satisfies

$$e^{isy}h(y) = 1 + O\left(\frac{1}{|y|}\right).$$

This shows that the deformation is valid if we add to the integral the integral over an infinitely large quarter-circle in the lower half-plane, which is $-\pi i/2$, as well as the contributions from the pole at $y = x$, which is passed twice during the deformation. It follows from this discussion that if we replace C_- by this new contour, which we call B_1, then we must add

$$-\frac{1}{4} + e^{isx}h_+(x) + e^{isx}h_-(x)$$

to the right side of (31). Next, the contour C_1, which runs counterclockwise around $[\beta_1, \alpha_2]$, we deform by pushing its upper part down through (β_1, α_2) into the lower half-plane and its lower part up through (β_1, α_2) into the upper half-plane. Both continuations of $e^{isy}h(y)$ are bounded at ∞ and the result is that we can replace C_1 by $-B_2 + B_3$, where B_2 (resp. B_3) starts at $\beta_1 + i\infty$ (resp. $\alpha_2 + i\infty$), loops around β_1 clockwise (resp. α_2 counterclockwise), and ends at $\beta_1 - i\infty$ (resp. $\alpha_2 - i\infty$). During this deformation no extra terms are picked up. We continue analogously with the remaining contours C_k and then end with C_+, the right part of C, deforming it as we did C_-. [The reader is advised to draw a picture.] Another integral over an infinitely large quarter-circle is picked up during this last deformation. The result at this stage is

$$e^{isx}q(x) = -1 + e^{isx}h_+(x) + e^{isx}h_-(x) + \sum_{i=1}^{2m} \frac{(-1)^{i+1}}{2\pi i} \int_{B_i} \frac{e^{isy}h(y)}{y - x} \, dy. \tag{32}$$

Finally the lower parts of B_1 and $-B_2$ can be joined into a single contour, and the part of this new contour which runs from α_1 to β_1 in the lower half-plane can be pushed up through (α_1, β_1) into the upper half-plane, resulting in the contour $A_1 - A_2$ as in the statement of the lemma. This deformation is valid since $G(x)$

continues to have negative imaginary part during it. The remaining contours are deformed similarly, and this completes the demonstration of the lemma. $\square$

Lemma 6. *We have, as $s \to \infty$,*

$$\int_E |e^{-isx} + q(x) - h_+(x) - h_-(x)|^2 dx = O(s^{-1}).$$

Proof. We must show that the norm in $L_2(E)$ of the sum in (30) is $O(s^{-\frac{1}{2}})$. Consider the integral over A_1, which can be written as a single integral parameterized as $y = \alpha_1 + it$, $t \geqq 0$. (The integral over the infinitesimal loops around α_1 vanish.) For each $\delta > 0$ the integral over $t \geqq \delta$ is bounded by a quantity which is exponentially small in s, uniformly in x, times $O((1 + |x|)^{-1})$, so the $L_2(E)$ norm of this part of the integral is exponentially small. In the neighborhood of $t = 0$, $\text{Im}\, G(\alpha_1 + it)$ is asymptotically a *negative* constant times $t^{\frac{1}{2}}$ (see the remark below), while

$$|k_{s\Gamma_0}(\alpha_1 + it)| = O(t^{-1/4}). \tag{33}$$

(This last comes from the behavior of $1 + p/\sqrt{q}$.) Since for $y \in A_1$,

$$\frac{1}{|x - y|} = O\left(\frac{1}{|x - \alpha_1| + t}\right),$$

we see that this part of the integral is bounded by a constant times

$$\int_0^\delta \frac{e^{-\eta s \sqrt{t}}}{t^{1/4}(|x - \alpha_1| + t)}\, dt \tag{34}$$

for some $\eta > 0$ and it is a simple exercise to show that the $L_2(\mathbf{R})$ norm of this is $O(s^{-\frac{1}{2}})$. The other A_i are treated in a similar manner. $\square$

Remark. Since the function $k_{s\Gamma_0}$ in (33) depends on the parameter s we have to know that the estimate holds uniformly in s. The reason it does is that the Green and Neumann functions appearing in (23) are smooth, uniformly in their parameters, away from their poles in the Riemann surface $\Omega \cup \overline{\Omega}$. (See, for example, Lemma 4.1 of [18].) The local coordinate for the surface near $x = \alpha_1$ is $(x - \alpha_1)^{\frac{1}{2}}$ and it follows that $k_{s\Gamma_0}$ and $G(x)$ are smooth functions of $(x - \alpha_1)^{\frac{1}{2}}$. This shows that (33) holds uniformly in s and also accounts for the $\sqrt{t}$ in the exponent of the integrand in (34).

Lemma 7. *We have, as $s \to \infty$,*

$$\int_E |R(x) - h_+(x) - h_-(x)|^2 dx = O(s^{-1}).$$

Proof. The integrand equals

$$(R(x) - h_+(x) - h_-(x))\overline{(e^{-isx} + q(x) - h_+(x) - h_-(x))}$$

$$+(R(x) - h_+(x) - h_-(x))\overline{(R(x) - e^{-isx} - q(x))}.$$

Let us look at the last term. It follows from Lemmas 1 and 4 that the integral over E of the first factor times the complex conjugate of the Fourier transform of any smooth function in $L_2(E)$ vanishes, and by continuity that Fourier transform does not have to be of a smooth function. Now $R(x) - e^{-isx}$ is the Fourier transform of a function in $L_2(E)$, and so is $q(x)$: it is easily seen from its definition (29) that $q(x)$ is an entire function of exponential type s and it follows from Lemma 6 that it belongs to $L_2(\mathbf{R})$. Thus the integral over E of the second term above vanishes. By Lemma 6 and Schwarz's inequality the integral of the first term is $O(s^{-\frac{1}{2}})$ times the square root of the integral we started with. This establishes the lemma. □

This is all we shall need for (7) and the reader interested only in this can go on to the next section. A few more estimates will be needed for the proof of (8).

Lemma 8. (a) $e^{-isx} + q(x) = h(x) + O(s^{-3/2})$ *uniformly on compact subsets of Ω.*
(b) *For every $\varepsilon > 0$ there exists a neighborhood of ∂J in which $q(x) = O(e^{\varepsilon s})$.*

Proof. There is a modification of (30), proved in the same way, which holds for $x \in \Omega$: the sum $h_+(x) + h_-(x)$ is replaced by $h(x)$ and the contours A_i are adjusted so as to avoid x (or the compact subset of Ω in which they may lie), but they still lie in the upper half-plane of $\overline{\Omega}$. This is easily seen to give assertion (a). It follows as a special case of this that if δ is sufficiently small then $q(x) = O(e^{\varepsilon s})$ on the circular arc

$$x = \alpha_k + \delta\, e^{i\theta}, \qquad -\frac{3\pi}{4} < \theta < \frac{3\pi}{4}.$$

For the rest of the circle we use the continuation of (30), as originally stated, from $x \in E$ onto that arc and obtain a similar estimate. Since q is analytic inside the circle it is $O(e^{\varepsilon s})$ in the full disc. A similar argument applies, of course, to neighborhoods of the β_k. □

Lemma 9. (a) $R(x) = h(x) + O(s^{-\frac{1}{2}} e^{s\,\mathrm{Im}\,G(x)})$ *uniformly on compact subsets of the interior of J.*
  (b) *For every $\varepsilon > 0$ there exists a neighborhood of ∂J in $\mathbf{C}$ in which $R(x) = O(e^{\varepsilon s})$.*

Proof. The function

$$\eta(x) := |(R(x) - e^{-isx} - q(x))\, e^{isG(x)}|^2 \tag{35}$$

is bounded and subharmonic in Ω and it follows from Lemmas 6 and 7 that its boundary function satisfies

$$\oint \eta(x)\, |dx| = O(s^{-1}).$$

Hence, from general considerations, $\eta(x) = O(s^{-1})$ uniformly on compact subsets of Ω. Combining this with the estimate of Lemma 8(a) proves our first assertion. (Observe that the error term in the statement is exponentially large and so dominates the term e^{-isx}.) To prove the second we replace $G(x)$ in (35) by the analogous function where J is replaced by J minus small neighborhoods of its boundary points. We then get an estimate for $R(x) - e^{-isx} - q(x)$ also in a neighborhood of ∂J, apply Lemma 8(b), and deduce the second assertion. □

V. Demonstration of (7)

Let us first evaluate asymptotically the integral

$$\mathscr{I} := \int_E |h_+(x) + h_-(x) - e^{-isx}|^2 dx \ . \tag{36}$$

It can be written as

$$\int_E \overline{h_+(x)}(h_+(x) - e^{-isx})dx + \int_E \overline{h_-(x)}(h_-(x) - e^{-isx})dx$$

$$- \int_E (h_+(x)e^{isx} + h_-(x)e^{isx} - 1)dx + \int_E (\overline{h_+(x)}h_-(x) + \overline{h_-(x)}h_+(x))dx \ .$$

Since this is real it equals the real part of what is obtained by replacing the third integral by its complex conjugate, so the above equals

$$\mathrm{Re}\left\{ \oint_E \overline{h(x)}(h(x) - e^{-isx})dx - \int_E (\overline{h_+(x)}e^{-isx} + \overline{h_-(x)}e^{-isx} - 1)dx \right\}$$

$$+ 2\mathrm{Re}\int_E \overline{h_+(x)}h_-(x)dx = \mathrm{Re}(\mathscr{I}_1 + \mathscr{I}_2) \ ,$$

say. Thus $\mathscr{I} = \mathrm{Re}\,(\mathscr{I}_1 + \mathscr{I}_2)$. We shall evaluate $\mathscr{I}_1$ exactly and show that $\mathscr{I}_2$ is $O(s^{-1})$.

Evaluation of $\mathscr{I}_1$. We write our integrals over E as limits as $r \to \infty$ of the corresponding integrals over $E_r := E \cap [-r,r]$. These integrals can be combined, and we obtain

$$\mathscr{I}_1 = \lim_{r \to \infty} \left\{ \oint_{E_r} \overline{h(x)}(h(x) - 2e^{-isx})dx + 2r \right\} - |J| \ . \tag{37}$$

Now we can almost, but not quite, apply Lemma 4 here with

$$g(x) = h(x) - 2e^{-isx} \ .$$

The problem is that this function does not satisfy the first hypothesis of the lemma, but we shall just modify its proof. Recall the notation there; for convenience we drop all subscripts Γ. We have

$$\oint_{E_r} \overline{h(x)}(h(x) - 2e^{-isx})dx = \oint_{E_r} j(x)e^{isG(x)}(e^{isG(x)}k(x) - 2e^{-isx})dx$$

$$= \int_{C_r^+ \cup C_r^-} j(x)(k(x) - 2e^{is(G(x)-x)})dx \ ,$$

where C_r^+ and C_r^- are semi-circles described clockwise in the upper and lower half-planes, respectively, joining $-r$ and r. Since $j(x)$ and $k(x)$ are $O(x^{-1})$ as $x \downarrow \infty$ and $\mathrm{Im}\,(G(x) - x) > 0$ there, the integral over C_r^- tends to zero as $r \to \infty$. As for C_r^+, both $j(x)$ and $k(x)$ tend to 1 and $x \uparrow \infty$, and are even analytic there. Suppose

$$j(x) = 1 + \frac{j_0}{x} + O\left(\frac{1}{x^2}\right), \quad k(x) = 1 + \frac{k_0}{x} + O\left(\frac{1}{x^2}\right) \quad \text{as } x \uparrow \infty \ .$$

If we use (17) then we find that the integrand equals

$$-1 + \frac{2ia_2s}{x} + \frac{k_0 - j_0}{x} + O\left(\frac{1}{x^2}\right)$$

as $x \uparrow \infty$. Hence the integral over C_r^+ equals $-2r + \pi(2a_2s + i(j_0 - k_0)) + o(1)$ as $r \to \infty$ and (37) gives

$$\mathscr{I}_1 = 2\pi a_2 s + \pi i(j_0 - k_0) - |J|,$$

whence

$$\operatorname{Re} \mathscr{I}_1 = 2\pi a_2 s + \pi \operatorname{Im}(k_0 - j_0) - |J|.$$

Observe that

$$k_0 = \lim_{x \uparrow \infty} x(k(x) - 1) = \lim_{x \uparrow \infty} x \log k(x),$$

and similarly for $j(x)$. Thus

$$\operatorname{Im}(k_0 - j_0) = \lim_{x \to \infty + 0i} x \arg \frac{k(x)}{j(x)}.$$

Now $j(x)$ is obtained from $k(x)$ by changing the signs of the ε_k in (23), so that

$$\frac{k(x)}{j(x)} = \prod_{k=0}^{m-1} \Phi(x, x_k)^{\varepsilon_k}.$$

Recalling the normalizations (24) we deduce that

$$\operatorname{Im}(k_0 - j_0) = \sum_{k=0}^{m-1} \varepsilon_k \lim_{x \to \infty + 0i} x \tilde{g}(x, x_k),$$

and so

$$\operatorname{Re} \mathscr{I}_1 + |J| = 2\pi a_2 s + \pi \sum_{k=0}^{m-1} \varepsilon_k \lim_{x \to \infty + 0i} x \tilde{g}(x, x_k). \tag{38}$$

Proof that $\mathscr{I}_2 = O(s^{-1})$: Since $G_0(\bar{x}) = -\overline{G_0(x)}$ and $G(x) = G_0(x) - a_1$ we have, with an obvious notation,

$$\overline{h_+(x)}h_-(x) = \overline{k_+(x)}k_-(x)e^{2is(G_+(x) + a_1)}.$$

Again we use the facts that the product of the exponential Green and Neumann functions in (23) is smooth, uniformly in s, and that the local behavior of the other functions arising here is determined locally. In particular $\overline{k_+(x)}k_-(x)$ is $O(|x|^{-1})$ at infinity, in fact for each n its nth derivative is $O(|x|^{-n-1})$, and it has the behavior of $(x - \alpha_k)^{-\frac{1}{2}}$ neat α_k and analogous behavior near β_k. The function $G_+(x)$ has the behavior of x near infinity by (17), and

$$G'(x) = \frac{p(x)}{\sqrt{q(x)}},$$

which is positive on E^+. It follows easily from these facts that if $\varphi(x)$ is any C^∞ function with compact support which is identically 1 on a neighborhood of ∂J then

$$\int_E (1 - \varphi(x))\overline{k_+(x)}k_-(x)e^{2isG_+(x)}dx = O(s^{-n})$$

for any n, while

$$\int_E \varphi(x)\overline{k_+(x)}k_-(x)e^{2isG_+(x)}dx$$

has the behavior of

$$\int_0^\infty \psi(t)\frac{e^{is\sqrt{t}}}{\sqrt{t}}dt \, ,$$

where ψ is a C^∞ function with compact support. Such an integral is $O(s^{-1})$.

Recapitulation: We have shown that $\mathscr{I}$, the integral in (36), equals

$$2\pi a_2 s + \pi \sum_{k=0}^{m-1} \varepsilon_k \lim_{x\to\infty+0i} x\, \tilde{g}(x, x_k) - |J| + O(s^{-1}) \, .$$

In particular $\mathscr{I} = O(s)$ and so from Lemma 7 we get

$$\int_E |R(x) - e^{isx}|^2 dx = \mathscr{I} + O(1) \, ,$$

and this and (10) give

$$-\pi \frac{d}{ds}\log\det(I - K_s) = \mathscr{I} + |J| + O(1) \, .$$

If the error team $O(1)$ here were $o(1)$ we would have established (7) with

$$c_1 = 2a_2, \quad c_2(s) = \sum_{k=0}^{m-1} \varepsilon_k \lim_{x\to\infty+0i} x\, \tilde{g}(x, x_k) \, . \tag{39}$$

As it stands, of course, we have only proved the weaker statement

$$-\frac{d}{ds}\log\det(I - K_s) = c_1 s + O(1) \, .$$

Conclusion: Here is how to get the stronger statement. The details of the analogous argument for orthogonal polynomials can be found in [18].

It follows from (9) that

$$\left| \int_J g(x)e^{isx}dx \right| \leqq \| g \| \, \| R(x) - e^{-isx} \|$$

for all $g \in \mathscr{E}_s$, the norms being that of the space $L_2(E)$, and that this becomes an equality when $g(x) = R(x) - e^{-isx}$. From this and (10) we deduce

$$-\pi\frac{d}{ds}\log\det(I - K_s) = |J| + \max_{g\in\mathscr{E}_s} \frac{|\int_J g(x)e^{isx}dx|^2}{\| g \|^2} \, .$$

Denote the maximum on the right by M_0. It follows from Lemma 4, in the form

$$\int_E \overline{(h_+(x) + h_-(x) - e^{-isx})} g(x)dx = \int_J g(x)e^{isx}dx$$

(first for Fourier transforms of smooth functions in $L_2(-s,s)$ and then by continuity for all such functions), that

$$M_0 \leqq \int_E |h_+(x) + h_-(x) - e^{-isx}|^2 dx .$$

The right side is just what we have called $\mathscr{I}$ and so we know that

$$M_0 \leqq \pi c_1 s + \pi c_2(s) - |J| + O(s^{-1}) . \tag{40}$$

Now we modify the extremal problem by replacing $\| g \|$ by the norm of g in $L_2(E, w_\delta)$ where the weight function $w_\delta(x)$ is equal to 1 except on the 2δ-neighborhood of ∂E, equal to $q(x)^{-\frac{1}{2}}$ in the δ-neighborhood of ∂E and, say, linear in between. Denote by M_δ the corresponding extremum for the weight function w_δ. Clearly

$$M_\delta \leqq M_0 . \tag{41}$$

There exist reproducing functions analogous to k_Γ in (23) for any weight function satisfying quite general conditions, so that $|dx|$ in the statement of Lemma 3 can be replaced by $w_\delta(x)|dx|$. The extra ingredient is the introduction of a certain nonzero function in $\mathscr{H}$ whose absolute value on E equals w_δ. We then proceed exactly as before, defining the analogous h_s by (25) and q by (29). The main point is that because w_δ has the behavior of $q(x)^{-\frac{1}{2}}$ near ∂E the corresponding reproducing function k_Γ has the behavior of $q(x)^{\frac{1}{2}}$, with the result that the factor $t^{1/4}$ does not appear in the denominator in (34). This improves the error estimate in Lemma 6 to $O(s^{-2})$ in this case, and so in the end we obtain

$$M_\delta = \pi c_1 s + \pi c_{2,\delta}(s) - |J| + O(s^{-1}) ,$$

where $c_{2,\delta}(s)$ is the function in (39) associated with the weight function w_δ. Because of the continuity of this function in the class Γ as well as the weight function w, we deduce that for any $\varepsilon > 0$ there is a δ such that

$$M_\delta \geqq \pi c_1 s + \pi c_2(s) - |J| - \varepsilon + o(1) .$$

Putting this together with (40) and (41) gives the desired result. (The details of a similar argument for orthogonal polynomials can be found in Sect. 9 of [18].)

If J is a single interval of length 1, say $J = \left[-\frac{1}{2}, \frac{1}{2}\right]$, then $G(x) = \sqrt{x^2 - \frac{1}{4}}$, and we find that $c_1 = 1/4$, $c_2 = 0$. Thus (7) in this case is (6) with the improved error term $o(1)$. Notice that this is consistent with (4) since the first power of s does not appear there.

It is easy to see that $c_2(s)$ is periodic when $m = 2$ and to compute its period. For its value depends only on the class of $e^{isG(x)}$, which in turn depends only on the value modulo 2π of s times the quantity $\Delta_{C_1} G_0$ given by (15). Hence the period of $c_2(s)$ equals

$$\left\{ \frac{1}{\pi} \int_{\beta_1}^{\alpha_2} \frac{p(x)}{\sqrt{q(x)}} dx \right\}^{-1} . \tag{42}$$

VI. Demonstration of (8)

It follows from Lemma 9(b) that for each $\varepsilon > 0$ the contribution to the integral in (11) of a sufficiently small neighborhood of ∂J is $O(e^{\varepsilon s})$. The main contribution will be from certain interior points of J, as we shall see. Outside any neighborhood of ∂J the asymptotics of $R(x)$ are given by Lemma 9(a), which may be rewritten as

$$R(x) = e^{-isG(x)}(k(x) + O(s^{-\frac{1}{2}})) .$$

It follows from general considerations that the derivative of the O term above is also $O(s^{-\frac{1}{2}})$. We use the formula

$$(uv)'\overline{uv} = |u|^2 \left(\frac{u'}{u}|v|^2 + v'\bar{v} \right)$$

with u equal to the exponential factor above and v the other to write

$$R'(x)\overline{R(x)} = e^{2s \, \mathrm{Im} \, G(x)} \left\{ -isG'(x)|k(x) + O(s^{-\frac{1}{2}})|^2 + k'(x)\overline{k(x)} + O(s^{-\frac{1}{2}}) \right\} .$$

Since $G'(x)$ is purely imaginary in J the first term in the braces is purely real, and so does not contribute to the right side of (11), and we are left with the computation of

$$\int e^{2s \, \mathrm{Im} \, G(x)} (\mathrm{Im} \, k'(x)\overline{k(x)} + O(s^{-\frac{1}{2}}))dx . \tag{43}$$

The integration is taken over J with its little neighborhood of ∂J removed. The main contribution to the integral will come from the point or points where $\mathrm{Im} \, G(x)$ achieves its maximum, and these will be among the zeros of $p(x)$. Denote these zeros by $z_1, \ldots, z_m$, one lying in each interval of J. Since at each z_i

$$\frac{d^2}{dx^2} \mathrm{Im} \, G(x) = \mathrm{Im} \, \frac{d}{dx} \frac{p(x)}{\sqrt{q(x)}} = -\frac{|p'(x)|}{\sqrt{|q(x)|}} , \tag{44}$$

we find by standard asymptotics that (43) equals

$$\sqrt{\frac{\pi}{s}} \sum_{i=1}^{m} e^{2s \, \mathrm{Im} \, G(z_i)} \left\{ \frac{|q(z_i)|^{1/4}}{|p'(z_i)|^{\frac{1}{2}}} \mathrm{Im} \, k'(z_i)\overline{k(z_i)} + O(s^{-\frac{1}{2}}) \right\} .$$

We have from (23)

$$|k(z_i)|^2 = \frac{1}{2} \prod_{k=1}^{m-1} |\Psi(z_i, x_k^* x_k)\Phi(z_i, x_k)^{\varepsilon_k}| ,$$

and, using in addition to (23) the very last part of (44), we find that

$$\mathrm{Im} \, \frac{k'(z_i)}{k(z_i)} = \frac{1}{2} \left\{ -\frac{|p'(z_i)|}{\sqrt{|q(z_i)|}} + \sum_{k=1}^{m-1} (\tilde{N}'(z_i, x_k^* x_k) + \varepsilon_k \, \tilde{g}'(z_i, x_k^* x_k)) \right\} ,$$

where the primes on the right denote differentiation in the real direction with respect to the first variable. Combining these last two formulas gives $\mathrm{Im} \, k'(z_i)\overline{k(z_i)}$. Comparing with (11) shows that (8) is established, with

$$c_3 = 2 \max_k \mathrm{Im} \, G(z_k)$$

and

$$c_4(s) = \frac{1}{4\sqrt{\pi}} \sum_i \left\{ \frac{|q(z_i)|^{1/4}}{|p'(z_i)|^{\frac{1}{2}}} \left[\frac{|p'(z_i)|}{\sqrt{|q(z_i)|}} - \sum_{k=1}^{m-1} (\tilde{N}'(z_i, x_k{}^* x_k) + \varepsilon_k \, \tilde{g}'(z_i, x_i{}^* x_k)) \right] \right.$$

$$\left. \times \prod_{k=1}^{m-1} |\Psi(z_i, x_k{}^* x_k) \Phi(z_i, x_k)^{\varepsilon_k}| \right\} , \tag{45}$$

where the outer sum is taken over those i for which

$$\operatorname{Im} G(z_i) = \max_k \operatorname{Im} G(z_k) .$$

There is an explicit representation for the quantities $\operatorname{Im} G(z_k)$. It follows from (13) and (14) that for $z \in (\alpha_k, \beta_k)$,

$$\operatorname{Im} G(z) = \int_{\alpha_k}^{z} \frac{\pm p(x)}{\sqrt{|q(x)|}} dx ,$$

the sign being that of $p(\alpha_k)$. The maximum of this, its value at z_k, is the integral to z_k, while the integral from z_k to β_k is the negative of this since the sum of the two is zero. Hence

$$\operatorname{Im} G(z_k) = \frac{1}{2} \int_{\alpha_k}^{\beta_k} \frac{|p(x)|}{\sqrt{|q(x)|}} dx ,$$

and so

$$c_3 = \max_k \int_{\alpha_k}^{\beta_k} \frac{|p(x)|}{\sqrt{|q(x)|}} dx .$$

When J is a single interval this equals $|J|$.

In this same special case (45) becomes

$$\frac{1}{4\sqrt{\pi}} \frac{|p'(z)|^{\frac{1}{2}}}{|q(z)|^{1/4}} ,$$

simply, and when the length of J is 1 this is $1/2\sqrt{2\pi s}$ and we recover the formula (5). When $m = 2$ the function $c_4(s)$ is periodic with the same period (42) as $c_2(s)$.

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