Asymptotic Solutions to the Knizhnik–Zamolodchikov Equation and Crystal Base

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Dedicated to the memory of Ansgar Schnizer

Abstract: The Knizhnik–Zamolodchikov equation associated with sl_2 is considered. The transition functions between asymptotic solutions to the Knizhnik–Zamolodchikov equation are described. A connection between asymptotic solutions and the crystal base in the tensor product of modules over the quantum group $U_q sl_2$ is established, in particular, a correspondence between the Bethe vectors of the Gaudin model of an inhomogeneous magnetic chain and the $\mathbb{Q}$-basis of the crystal base.

Introduction

In this work we describe transition functions between asymptotic solutions to the Knizhnik–Zamolodchikov (KZ) equation and establish a connection between asymptotic solutions and the crystal base in the tensor product of modules over a quantum group.

We consider the KZ equation associated with sl_2 and the quantum group $U_q sl_2$, general case can be considered similarly.

For a positive integer m, denote by $L(m)$ the sl_2 irreducible module with highest weight m. For positive integers $m_1, \ldots, m_n$, set $L = L(m_1) \otimes \cdots \otimes L(m_n)$.

Let $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e \in sl_2^{\otimes 2}$ be the Casimir operator. For $i \neq j$ denote by Ω_{ij} the linear operator on L which acts as Ω on the ith and j^{th} factors and as the identity on the other factors. The KZ equation on an L-valued function $\psi(z_1, \ldots, z_n)$ is the system of equations

$$\frac{\partial \psi}{\partial z_j} = \frac{1}{\kappa} \sum_{l \neq j} \frac{\Omega_{jl}}{z_j - z_l} \psi, \quad j = 1, \ldots, n,$$

where κ is a complex parameter. In this paper we assume that κ is not a rational number. The KZ equation is defined over $\mathcal{U}_n = \{z \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$.

For any permutation $\sigma \in S_n$, define an open convex unbounded polytope

$$D_\sigma = \{z \in \mathbb{R}^n | z_{\sigma(1)} < \cdots < z_{\sigma(n)}\}\,.$$

One can easily resolve singularities of D_σ and cover D_σ by local charts with local coordinates $u_1, \ldots, u_n$ such that:

$$u_n = z_1 + \cdots + z_n\,, \tag{0.1}$$

D_σ is defined in this chart by inequalities $u_1 > 0, \ldots, u_{n-1} > 0$, $\qquad$ (0.2)

The KZ equation in this chart has the form

$$\frac{\partial \psi}{\partial u_n} = 0, \quad \frac{\partial \psi}{\partial u_j} = \frac{1}{\kappa} H_j, \quad j = 1, \ldots, 1 - n\,, \tag{0.3}$$

where $H_j = \Omega_j / u_j + \mathrm{Reg}_j$, Ω_j is a constant operator on L, and Reg_j is an operator regular at $u = 0$.

We call such charts *asymptotic zones*; they are numerated by suitable trees T.

The operators $\Omega_1, \ldots, \Omega_{n-1}$ commute and have a common eigenbasis. We distinguish an eigenbasis $\mathscr{B} = (v_1(\kappa), \ldots, v_N(\kappa))$ for nonrational κ. For every j the vector $v_j(\kappa)$ is proportional to a fixed common eigenvector of $\Omega_1, \ldots, \Omega_{n-1}$, and the coefficient of the proportionality appropriately depends on κ, see (2.3.1).

For each $j = 1, \ldots, N$ and a nonrational κ, there exists a unique solution ψ_j to the KZ equation with parameter κ defined in D_σ and such that

$$\psi_j = (v_j(\kappa) + \mathscr{O}(u, k)) \prod_{l=1}^{n-1} u_l^{\mu_l/\kappa}\,, \tag{0.4}$$

where μ_l is the eigenvalue of Ω_l at v_j, $\mathscr{O}(u, \kappa)$ is a regular function of u at $u = 0$, and $\mathscr{O}(0, \kappa) = 0$. (Notice that all numbers μ_l are real.)

Thus, for each asymptotic zone (T, σ) we construct a fundamental system of solutions $\psi_{T, \sigma} = \{\psi_j\}$.

Our first main result gives explicit transition functions between these fundamental solutions $\psi_{T, \sigma}$ and $\psi_{T', \sigma'}$ in terms of the q-$6j$-symbols, where $q = q(\kappa)$ and $q(\kappa) = \exp(2\pi i/\kappa)$, see (2.4).

In particular, if $\kappa = is$, $s \in \mathbb{R}$, $s \to +0$, then $q(\kappa) \to \infty$, and our formulae show that for every σ, T, T' the transition function between the fundamental systems of solutions $\psi_{T, \sigma}$ and $\psi_{T', \sigma}$ has the form $\mathbf{1} + \mathscr{O}(q(\kappa)^{-\frac{1}{4}})$, where $\mathbf{1}$ is the unit matrix and the solutions composing $\psi_{T, \sigma}$ and $\psi_{T', \sigma}$ must be suitably enumerated, see (2.4) and (1.3.12).

Assume that $\kappa = is$, $s \in \mathbb{R}$ and $s \to +0$. For any asymptotic zone consider the fundamental system of solutions defined by (0.4). Our second main result, see (2.5), states that for every j the solution ψ_j has an asymptotic expansion of the form

$$\psi_j = \prod_{l=1}^{n-1} u_l^{-i\mu_l/s} \exp\left(-iS/s\right) \sum_{l=0}^{\infty} f_l s^l\,, \tag{0.5}$$

where S, f_l are real-analytic functions of u in our chart regular at $u = 0$, the function S is real valued, the functions f_l are L-valued, and, in particular,

$$\left| \prod_{l=1}^{n-1} u_l^{-i\mu_l/s} \exp(-iS/s) \right| = 1\,.$$

The first term f_0 is a common eigenvector of the operators $H_1, \ldots, H_{n-1}$. The first terms of asymptotics of the solutions ψ_j form an eigenbasis of the operators $H_1, \ldots, H_{n-1}$. We describe its limit at $u \to 0$.

If the KZ equation has an asymptotic solution of the form

$$\psi(u) = \exp(P(u)/\kappa) \sum_{l=0}^{\infty} g_l(u) \cdot \kappa^l,$$

one can show that $g_0(u)$ must be a common eigenvector of $H_1, \ldots, H_{n-1}$ and the function $g_0(u)$ is determined up to multiplication by a constant [RV]. One can deduce from this remark that a solution having an asymptotic expansion of the form described in (0.5) is unique up to multiplication by a function of s which has the form $1 + \mathcal{O}(s)$, where $\mathcal{O}(0) = 0$.

Our two main results have the following quantum group interpretation.

Let U_q be the quantum group corresponding to the sl_2 and $U_{q=q(\kappa)}$ its specialization at $q = q(\kappa)$. For a positive integer m denote by $L(m, q)$ (resp. $L(m, q = q(\kappa))$) the irreducible U_q (resp. $U_{q=q(\kappa)}$) module with highest weight m. For positive integers $m_1, \ldots, m_n$ and a permutation σ denote by $L^{\sigma}(q)$ the product $L(m_{\sigma(1)}, q) \otimes \cdots \otimes L(m_{\sigma(n)}, q)$. Denote by $L^{\sigma}(q = q(\kappa))$ the corresponding product for $U_{q=q(\kappa)}$. Denote by $\mathrm{Sol}(\kappa)_{\sigma}$ the space of solutions over D_{σ} to the KZ equation with parameter κ.

Our formulae for transition functions between the constructed fundamental system of solutions allows us to construct an isomorphism

$$\pi_{\sigma}(\kappa) : L^{\sigma}(q = q(\kappa)) \cong \mathrm{Sol}(\kappa)_{\sigma}$$

and show, once again, that the monodromy representation of the KZ equation is isomorphic to the R-matrix representation of the braid group, where R is the universal R-matrix of $U_{q=q(\kappa)}$, see [K, Dr, V1, FW]. Under the isomorphism $\pi_{\sigma}(\kappa)$ each of the constructed fundamental systems of solutions $\psi_{T, \sigma}$ gives a basis $\mathscr{B}_{T, \sigma}(\kappa)$ in $L^{\sigma}(q = q(\kappa))$, see (2.4).

In [Ka] the notion of a crystal base is defined. The module $L(m, q)$ has a standard crystal base when $q \to \infty$. The tensor product of the standard crystal bases gives a *distinguished crystal base* in $L^{\sigma}(q)$. Our isomorphism $\pi_{\sigma}(\kappa)$ has the following property:

For every T, the fundamental system of solutions $\psi_{T, \sigma}$ lifts to the same distinguished crystal base of $L^{\sigma}(q)$, see (2.6).

Therefore, we may conclude that the distinguished crystal base is given by (0.4) and normalized by property (0.5). This gives an "asymptotic" definition of the crystal base.

The proof of transition formulae between the fundamental solutions and the proof of property (0.5) are based on integral representations for solutions to the KZ equation, see [SV, V1]. Any solution in $\mathrm{Sol}(\kappa)_{\sigma}$ can be represented as a linear combination of solutions of the form

$$\psi(z) = \int_{\gamma(z)} \Phi M(t, z), \tag{0.6}$$

where M is a rational L-valued differential form of z and some variable $t = (t_1, \ldots, t_k)$, $\gamma(z)$ is a family of cycles in t-space,

$$\Phi(t,z) = \prod_{1 \le l < j \le n} (z_j - z_l)^{m_l m_j / 2\kappa} \prod_{1 \le l < j \le k} (t_l - t_j)^{2/\kappa} \prod_{l=1}^{k} \prod_{j=1}^{n} (z_j - t_l)^{-m_j/\kappa}.$$

In this construction cycles $\gamma(z)$ numerate solutions to the KZ equation. In [V1] we identified the homology group of such cycles for a fixed k with the subspace of singular vectors in $L^\sigma(q = q(\kappa))$ of a suitable weight, see precise statements in [V1]. This construction after suitable renormalization gives us the isomorphism π_σ, see (4.4).

If $\kappa = is$, $s \to +0$, then integral (0.6) is localized at critical points of the function Φ with respect to the variable t, and one can compute asymptotics of the solution applying the method of steepest descent, see [RV, V2]. One can push the cycle $\gamma(z)$ onto t-critical points of Φ. If $\gamma(z)$ "sits" on some of the t-critical points then each of the points gives its input into the asymptotic expansion of the solution.

The symmetric group S_k of permutations of coordinates $t_1, \ldots, t_k$ acts on the set of critical points of Φ.

In each asymptotic zone (T, σ), one can describe asymptotics of t-critical points of Φ. It turns out that the S_k orbits of t-critical points of Φ are in correspondence with solutions ψ_j composing the fundamental system $\psi_{T,\sigma}$. For any j we distinguish a cycle $\gamma_j(z)$ such that

(1) $\psi_j = \int_{\gamma_j(z)} \Phi M$,
(2) $\gamma_j(z)$ sits exactly on one S_k orbit of t-critical points,
(3) the asymptotic expansion of the integral at this orbit has property (0.5).

This gives a "topological" definition of the crystal base as the base in the homology group of cycles corresponding to a fixed z. This base is formed by the classes numerated by the orbits of t-critical points and each such a class has a representing chain sitting on one orbit. In this construction z must be in an asymptotic zone.

The value of the function M at a t-critical point $(t(z), z)$ of the function Φ gives the first term of asymptotics as $k = is$ and $s \to +0$. This value $M(t(z), z)$ is an eigenvector of the commuting operators $H_1(z), \ldots, H_{n-1}(z)$.

The algebraic Bethe Ansatz is a construction of eigenvectors for a system of commuting operators. One considers a vector valued function of a special form and determines its arguments in such a way that the value of this function is an eigenvector. The equations which determine these special values of arguments are called the Bethe equations. The eigenvector is called a Bethe vector, for more details see [FT].

One of the systems of commuting operators which can be diagonalized by the ABA is the Gaudin model of an inhomogeneous magnetic chain [G].

It turns out that the function $M(t,z)$ is exactly the special function which appear in the ABA for the Gaudin model and the Bethe equations for the Gaudin model coincide with the equations on t-critical points of the function Φ, see [B, BF, RV].

This relation to the ABA and the above formulated results give us a correspondence between the $\mathbb{Q}$-basis of the distinguished crystal base and the Bethe vectors of the Gaudin model, see (2.6.5), or, more generally, between the crystallization phenomenon in the theory of quantum groups and the Bethe ansatz construction in statistical mechanics. On the Bethe ansatz and the KZ equation see also [FFR].

Section 1 contains preliminary information on bases in L and $L^\sigma(q)$. In Sect. 2 we state the formulae for transition functions and the theorem about asymptotic expansion (0.5). In Sect. 3 we review integral representations for solutions and describe asymptotics of critical points and the corresponding Bethe vectors. Section 4 contains the homological part of the work.

1. Bases in Tensor Products

(1.1). Quantum Group $U_q sl_2$. The $\mathbb{Q}(q^{1/4})$-algebra $U_q = U_q sl_2$ is the algebra generated by the symbols e, f, h with the relations

$$[h, e] = 2e \,,$$

$$[h, f] = -2f \,,$$

$$[e, f] = q^{h/2} - q^{-h/2} \,. \tag{1.1.1}$$

Here $q^{1/4}$ is an indeterminate.

Remark. Usually one considers U_q as an $\mathbb{Q}(q)$ algebra. Our choice of $q^{1/4}$ instead of q is motivated by topological considerations in Sect. 4.

U_q has a Hopf algebra structure with the comultiplication Δ defined by

$$\Delta(h) = h \otimes 1 + 1 \otimes h \,,$$

$$\Delta(e) = e \otimes q^{h/2} + 1 \otimes e \,,$$

$$\Delta(f) = f \otimes 1 + q^{-h/2} \otimes f \,. \tag{1.1.2}$$

By Δ, the tensor product of U_q-modules has a structure of U_q-module.

Let

$$[n]_q = (q^{n/2} - q^{-n/n})/(q^{1/2} - q^{-1/2}) \,,$$

$$(n)_q = q^{n/2} - q^{-n/2} \,,$$

$$[n]_q! = \prod_{k=1}^n [k]_q \,,$$

$$e^{(n)} = \frac{e^n}{[n]_q!}, \qquad e^{(0)} = 1 \,,$$

$$f^{(n)} = \frac{f^n}{[n]_q!}, \qquad f^{(0)} = 1 \,. \tag{1.1.3}$$

Let V_1 and V_2 be U_q-modules. The modules $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are isomorphic. The isomorphism is given by the formula

$$V_1 \otimes V_2 \xrightarrow{R} V_1 \otimes V_2 \xrightarrow{P} V_2 \otimes V_1 \,, \tag{1.1.4}$$

where P is the transposition of the factors. $R \in U_q \widehat{\otimes} U_q$ is the universal R-matrix, and $\widehat{\otimes}$ denotes a proper completion of the tensor product. Denote the isomorphism PR by $\hat{R}$.

Denote by S_n the group of permutations of the set $\{1,\ldots,n\}$, by B_n the braid group on n strings with the standard generators $\sigma_1,\ldots,\sigma_{n-1}$, by $\tau\colon B_n \to S_n$, the natural epimorphism.

Let $V_1,\ldots,V_n$ be U_q-modules. For $j = 1,\ldots,n-1$, define an U_q-isomorphism.

$$R_{j,j+1}\colon V_1 \otimes \cdots \otimes V_n \to V_1 \otimes \cdots \otimes V_{j+1} \otimes V_j \otimes \cdots \otimes V_n , \qquad (1.1.5)$$

as the map which acts as $\hat{R}$ on $V_j \otimes V_{j+1}$ and as the identity on the other factors.

For any braid $\sigma \in B_n$ these isomorphisms induce a well defined isomorphism

$$R_\sigma\colon V_1 \otimes \cdots \otimes V_n \to V_{\tau\sigma(1)} \otimes \cdots \otimes V_{\tau\sigma(n)} . \qquad (1.1.6)$$

(1.2). Bases in Tensor Products. For $m \in \mathbb{Z}$ let $V(m)$ be the Verma module with highest weight m. $V(m)$ is generated by its singular vector v_m such that $ev_m = 0$ and $q^h v_m = q^m v_m$. The elements $f^{(j)}v_m$, $j \geq 0$, form a basis of $V(m)$. For a nonnegative m the vector $f^{(m+1)}v_m$ generates a proper submodule of $V(m)$. The quotient, $L(m)$, is the $(m+1)$-dimensional irreducible U_q-module with a basis generated by $v_m, f^{(1)}v_m, \ldots, f^{(m)}v_m$.

The tensor product of irreducible modules is the direct sum of irreducible modules:

$$L(m_1) \otimes L(m_2) = L(|m_1 - m_2|) \oplus L(|m_1 - m_2| + 2) \oplus \cdots \oplus L(m_1 + m_2) ,$$

and a singular vector of $L(m_1 + m_2 - 2k)$ has the form

$$(v_{m_1},{}_k v_{m_2}) = \sum_{p=0}^{k} (-1)^p \frac{(m_2 - k + 1)_q \cdots (m_2 - k + p)_q}{(m_1)_q \cdots (m_1 - p + 1)_q}$$

$$\cdot q^{-p(m_2 - 2k + p + 1)/2} f^{(p)} v_{m_1} \otimes f^{(k-p)} v_{m_2} . \qquad (1.2.1)$$

This decomposition gives a basis $\{f^{(l)}(v_{m_1},{}_k v_{m_2})\}$ in the tensor product.

The U_q-isomorphism

$$\hat{R} : L(m_1) \otimes L(m_2) \to L(m_2) \otimes L(m_1) \qquad (1.2.2)$$

is given by

$$f^{(l)}(v_{m_1},{}_k v_{m_2}) \mapsto R(m_1, m_2; k)_q f^l(v_{m_2},{}_k v_{m_1}) , \qquad (1.2.3)$$

$$R(m_1, m_2; k)_q = (-1)^k q^{(m_1 m_2 - 4km_2 + 2k(k+1))/4} \frac{(m_2)_q \cdots (m_2 - k + 1)_q}{(m_1)_q \cdots (m_1 - k + 1)_q} .$$
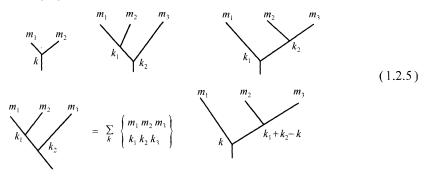
There are two ways to decompose $L(m_1) \otimes L(m_2) \otimes L(m_3)$ into irreducibles. These two ways give two bases in the triple tensor product: $\{f^{(l)}((v_{m_1},{}_{k_1} v_{m_2}),{}_{k_2} v_{m_3})\}$ and $\{f^{(l)}(v_{m_1},{}_{k_1}(v_{m_2},{}_{k_2} v_{m_3}))\}$.

The matrix elements of the matrix, connecting these bases, are called the *6j-symbols*:

$$f^{(l)}((v_{m_1},{}_{k_1} v_{m_2}),{}_{k_2} v_{m_3})$$

$$= \sum_k \begin{Bmatrix} m_1, & m_2, & m_3 \\ k_1, & k_2, & k \end{Bmatrix}_q f^{(l)}(v_{m_1},{}_k (v_{m_2},{}_{k_1 + k_2 - k} v_{m_3})) . \qquad (1.2.4)$$

There are numerous formulae for $6j$-symbols, see, for example, [KR].

The singular vectors $(v_{m_1},\,_k v_{m_2}), ((v_{m_1},\,_{k_1} v_{m_2}),\,_{k_2} v_{m_3}), (v_{m_1},\,_{k_1} (v_{m_2},\,_{k_2} v_{m_3}))$, and the matrix of $6j$-symbols can be visualized as follows.

$$(1.2.5)$$

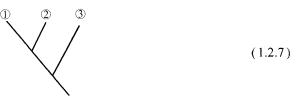$$\sum_k \left\{ \begin{matrix} m_1 & m_2 & m_3 \\ k_1 & k_2 & k_3 \end{matrix} \right\}$$

The braid group B_3 acts on the triple tensor products. The basis $((v_{m_1},\,_{k_1} v_{m_2}),\,_{k_2} v_{m_3})$ is an eigenbasis for $R_{1,2}$, and the basis $(v_{m_1},\,_{k_1}(v_{m_2},\,_{k_2} v_{m_3}))$ is an eigenbasis for $R_{2,3}$.

$$R_{12}: \qquad \mapsto R(m_1, m_2; k_1)_9$$

$$(1.2.6)$$

$$R_{23}. \qquad \mapsto R(m_2, m_3; k_2)_9$$

Consider the tensor product of n irreducible modules. We describe its bases, the connecting matrices, and the R-matrix action.

An *n-tree* is a planar tree with n tops, one root, and $(n-1)$ internal triple vertices. We numerate the tops by indices $1, \ldots, n$ from left to right.

See an example in (1.2.7):

$$(1.2.7)$$

For an n-tree we denote the set of its internal vertices by In_T. We say that an *n-tree has marked tops* if a nonnegative integer is assigned to every top. Denote by m_j the number assigned to the jth top.

For an n-tree T with marked tops, we construct a basis in $L(m_1) \otimes \cdots \otimes L(m_n)$, denoted by $\mathscr{B}_T$.

A *coloring* of T is a map $c\colon \mathrm{In}_T \to \mathbb{Z}_{\geq 0}$. Define the *weight of a coloring* as the number

$$m(c) = m_1 + \cdots + m_n - 2 \sum_{w \in \mathrm{In}_T} c(w). \tag{1.2.8}$$

Delete an internal vertex w, then the tree is decomposed into three trees: *the left branch*, *the right branch*, and the part containing the root.

Denote by $J^{le}(w)$ the set of indices of the tops lying in the left branch of w, denote by $I^{le}(w)$ the set of the internal vertices lying in the left branch of w. For a coloring c, define the weight of the left branch as the number

$$m^{le}(w) = \sum_{j \in J^{le}} m_j - 2 \sum_{u \in I^{le}} c(u).$$

Let $J^r(w)$, $I^r(w)$, and $m^r(w)$ be the sets and the weight defined for the right branch analogously.

Say that a coloring c is *admissible at a vertex w* if $c(w) \leqq \min(m^{le}(w), m^r(w))$. Say that a *coloring of an n-tree T with marked tops is admissible* it is admissible at every internal vertex.

The elements of the basis $\mathscr{B}_T$ are numerated by the pairs (c, l), where c is an admissible coloring and $l = 0, \ldots, m(c)$. The vector corresponding to (c, l) is defined inductively on n. Namely, let w be the internal vertex of T which is neighboring to the root of T. Let T^{le} and T^r be the left and right branches of w. The coloring c induces colorings c^{le} and c^r of T^{le} and T^r, respectively. Let $1, \ldots, i$ be the indices of the tops lying in the left branch of w. Let v^{le} be the vector in $L(m_1) \otimes \cdots \otimes L(m_i)$ corresponding to the pair $(c^{le}, 0)$ and v^r the vector in $L(m_{i+1}) \otimes \cdots \otimes L(m_n)$ corresponding to $(c^r, 0)$. If $i = 1$, then $v^{le} = v_{m_1}$, where v_{m_1} is the singular vector of $L(m_1)$. If $i = n - 1$, then $v^r = v_{m_n}$, where v_{m_n} is the singular vector of $L(m_n)$. Define the vector $v(c, l, T)$ by

$$v(c, l, T) = f^{(l)}(v^{le}, {}_{c(w)}v^r),\tag{1.2.9}$$

where $(\cdot, {}_{c(w)} \cdot)$ is defined by (1.2.1).

The U_q-action is given by
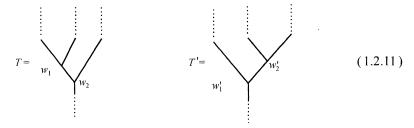
$$q^h v(c, l, T) = q^{m(c)-2l} v(c, l, T),$$

$$ev(c, l, T) = (m(c) - l + 1)_q v(c, l - 1, T) \quad \text{for } l > 0,$$

$$ev(c, 0, T) = 0,$$

$$fv(c, l, T) = [l + 1]_q v(c, l + 1, T) \quad \text{for } l < m(c),$$

$$fv(c, m(c), T) = 0.\tag{1.2.10}$$

Let T and T' be n-trees. Let w_1 and w_2 be internal vertices of T and w_1' and w_2' internal vertices of T'. Say that T and T' are *adjacent* at w_1, w_2, w_1', w_2', if in a neighborhood of the vertices the trees have the form shown in (1.2.11), and the trees are identical outside the neighborhood.

$$(1.2.11)$$

Let T and T' be adjacent. Let c and c' be admissible colorings of T and T' respectively. Say that the *colorings are similar*, $c \equiv c'$, if the colorings are equal at the corresponding internal vertices of T and T' different from w_1, w_2, w_1', w_2', and if $c(w_1) + c(w_2) = c'(w_1') + c'(w_2')$.

The matrix, connecting the bases $\mathscr{B}_T$ and $\mathscr{B}_{T'}$ of adjacent trees is given by

$$v(c, l, T) = \sum_{c' \equiv c} \left\{ \begin{matrix} m^{le}(w_1) & m^r(w_1) & m^r(w_2) \\ c(w_1) & c(w_2) & c'(w_1') \end{matrix} \right\}_q v(c', l, T'), \qquad (1.2.12)$$

where $\{ \ \}_q$ are $6j$-symbols.

The matrix connecting the bases $\mathscr{B}_T$ and $\mathscr{B}_{T'}$ for arbitrary T and T' can be written as a product of the matrices connecting adjacent bases.

For $j = 1, \ldots, n - 1$, we describe the action of the operator

$$R_{j,j+1} \colon L(m_1) \otimes \cdots \otimes L(m_n) \to L(m_1) \otimes \cdots L(m_{j+1}) \otimes L(m_j) \cdots \otimes L(m_n).$$

Denote the first tensor product by L and the second by L'.

We say that an n-tree T is *adjacent to the transposition* of j and $j + 1$ if there exists an internal vertex w such that the jth top of T forms the left branch of w, and the $(j + 1)^{\text{th}}$ top of T forms the right branch of w. Let T be adjacent to the transposition of j and $j + 1$.

Let $\{v(c, l, T; L)\}$ be the T-basis of L and $\{v(c, l, T; L')\}$ the T-basis of L'. Then for every (c, l) we have

$$R_{j,j+1} v(c, l, T; L) = R(m_j, m_{j+1}; c(w)) \cdot v(c, l, T; L'), \qquad (1.2.13)$$

where $R(m_j, m_{j+1}; c(w))$ is defined by (1.2.3). As an example see (1.2.3) and (1.2.6).

Formulae (1.2.12) and (1.2.13) describe the action of the braid group B_n on the tensor product of n irreducible U_q-modules.

(1.3). Crystal Base. The tensor product $L(m_1) \otimes \cdots \otimes L(m_n)$ has the trivial basis $f^{(l_1)} v_{m_1} \otimes \cdots \otimes f^{(l_n)} v_{m_n}$, $l_j = 0, \ldots, m_j$, $j = 1, \ldots, n$. It turns out that all the tree bases, constructed in (1.2), tend (in an appropriate sense) to the trivial basis as $q \to \infty$. This phenomenon is described in [K] and is called crystallization.

For a finite dimensional U_q-module M and $\lambda \in \mathbb{Z}$, let $M_\lambda = \{u \in M \,|\, q^h u = q^\lambda u\}$ be the weight space of weight λ. We have $M = \bigoplus_{\lambda \in \mathbb{Z}} M_\lambda$.

Any element u of M_λ is uniquely written in the form $u = \sum f^{(l)} u_l$, where $u_l \in \ker e \cap M_{\lambda + 2l}$. Following [K] define the endomorphisms $\tilde{e}$ and $\tilde{f}$ on M by

$$\tilde{e} u = \sum f^{(n-1)} u_n, \qquad \tilde{f} u = \sum f^{(n+1)} u_n.$$

Let A be the subring of $\mathbb{Q}(q^{1/4})$ consisting of the rational functions of $q^{1/4}$ regular at $q^{1/4} = \infty$.

A crystal base of M is a pair (L, B) satisfying the following conditions.

$$L \text{ is a free sub-}A\text{-module of } M \text{ such that } M = \mathbb{Q}(q^{1/4}) \underset{A}{\bigotimes} L, \qquad (1.3.1)$$

$$B \text{ is a base of the } \mathbb{Q}\text{-vector space } L/q^{-1/4} L, \qquad (1.3.2)$$

$$L = \bigoplus_{\lambda \in \mathbb{Z}} L_\lambda \text{ and } B = \coprod_{\lambda \in \mathbb{Z}} B_\lambda, \text{ where}$$

$$L_\lambda = L \cap M_\lambda, \qquad B_\lambda = B \cap (L_\lambda / q^{-1/4} L_\lambda), \qquad (1.3.3)$$

$$\tilde{e}L \subset L \text{ and } \tilde{f}L \subset L. \text{ Hence } \tilde{e} \text{ and } \tilde{f} \text{ operate on } L/q^{-1/4}L, \qquad (1.3.4)$$

$$\tilde{e}B \subset B \cup \{0\} \text{ and } \tilde{f}B \subset B \cup \{0\}, \qquad (1.3.5)$$

For $b, b' \in B$, we have $b' = \tilde{f}b$ if and only if $b = \tilde{e}b'$. $\qquad (1.3.6)$

For $b \in B$, set $\varepsilon(b) = \max\{k \geqq 0 | \tilde{e}^k b \in B\}$ and $\varphi(b) = \max\{k \geqq 0 | \tilde{f}^k b \in B\}$.

(1.3.7) Example. For $m \in \mathbb{Z}_{\geqq 0}$, consider the irreducible module $L(m)$ with the basis $f^{(l)}v_m$, $l = 0,\ldots,m$. We have $\tilde{e}f^{(l)}v_m = f^{(l-1)}v_m$ and $\tilde{f}f^{(l)}v_m = f^{(l+1)}v_m$. Let $\mathscr{L}(m) = \bigoplus_l A f^{(l)}v_m$ and $\mathscr{B}(m) = \{f^{(l)}v_m | l = 0,\ldots,m\} \subset L/q^{-1/4}L$. Then $(\mathscr{L}(m), \mathscr{B}(m))$ is a crystal base of $L(m)$.

(1.3.8) Theorem, [K]. *Let M_1 and M_2 be two finite dimensional U_q-modules and let (L_j, B_j) be a crystal base of M_j, $j = 1, 2$. Set $L = L_1 \otimes_A L_2$ and $B = \{b_1 \otimes b_2 \in L/q^{-1/4}L | b_j \in B_j\}$.*

(i) *Then (L, B) is a crystal base of $M_1 \otimes M_2$.*
(ii) *For $b_j \in B_j$, $j = 1, 2$, we have*

$$\tilde{f}(b_1 \otimes b_2) = \begin{cases} \tilde{f}b_1 \otimes b_2 & \text{if } \varphi(b_1) > \varepsilon(b_2) \\ b_1 \otimes \tilde{f}b_2 & \text{if } \varphi(b_1) \leqq \varepsilon(b_2), \end{cases}$$

$$\tilde{e}(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}b_2 & \text{if } \varphi(b_1) < \varepsilon(b_2) \\ \tilde{e}b_1 \otimes b_2 & \text{if } \varphi(b_1) \geqq \varepsilon(b_2). \end{cases}$$

(1.3.9) Theorem, [K]. *Let (L, B) be a crystal base of a finite dimensional U_q-module M. Then there exists an isomorphism $M \cong \bigoplus_j L(m_j)$ by which (L, B) is isomorphic to $\bigoplus_j (\mathscr{L}(m_j), \mathscr{B}(m_j))$.*

Consider the tensor product of n irreducible U_q-modules, $L = L(m_1) \otimes \cdots \otimes L(m_n)$. Let $\mathscr{L} = \mathscr{L}(m_1) \bigotimes_A \cdots \bigotimes_A \mathscr{L}(m_n)$, $\mathscr{B} = \mathscr{B}(m_1) \otimes \cdots \otimes \mathscr{B}(m_n)$. By (1.3.8), $(\mathscr{L}, \mathscr{B})$ is a crystal base of L. We call $(\mathscr{L}, \mathscr{B})$ the *distinguished crystal base* of the tensor product L.

For any n-tree introduce

$$\mathscr{L}_T = \bigoplus_{(c,l)} Av(c, l, T),$$

$$\mathscr{B}_T = \{v(c, l, T)\} \subset \mathscr{L}_T / q^{-1/4} \mathscr{L}_T. \qquad (1.3.10)$$

It follows from (1.2.10) that $(\mathscr{L}, \mathscr{B})$ is a crystal base of L.

(1.3.11) Proposition. *For any n-tree T, we have $(\mathscr{L}, \mathscr{B}) = (\mathscr{L}_T, \mathscr{B}_T)$.*

Proof. It suffices to prove the proposition for $n = 2$. Let $(v_{m_1}, {}_k v_{m_2})$ be the vector defined by (1.2.1). Then for any $k = 0,\ldots,\min(m_1, m_2)$ we have

$$(v_{m_1}, {}_k v_{m_2}) = v_{m_1} \otimes f^{(k)} v_{m_2} \mod q^{-1/4} \mathscr{L}. \qquad (1.3.12)$$

This equality easily implies that $\mathscr{L} = \mathscr{L}_T$ and then $\mathscr{B} = \mathscr{B}_T$.

(1.3.13) Corollary. *For $k_1 + k_2 \leqq m_2$,*

$$\begin{Bmatrix} m_1 & m_2 & m_3 \\ k_1 & k_2 & k \end{Bmatrix}_q = \begin{cases} 1 \mod q^{-1/4}A & \text{if } k = k_1 \\ 0 \mod q^{-1/4}A & \text{otherwise} . \end{cases}$$

For $k_1 + k_2 > m_2$,

$$\begin{Bmatrix} m_1 & m_2 & m_3 \\ k_1 & k_2 & k \end{Bmatrix}_q = \begin{cases} 1 \mod q^{-1/4}A & \text{if } k = m_2 - k_1 , \\ 0 \mod q^{-1/4}A & \text{otherwise} . \end{cases}$$

(1.3.14) Remark. For a finite dimensional U_q-module M, we have $\ker e = \bigoplus_\lambda \ker e \cap M_\lambda$.

A *crystal base of* $\ker e$ of M is a pair (L_λ, B_λ) for each λ such that $\ker e \cap M_\lambda \neq \{0\}$. A pair (L_λ, B_λ) has to satisfy the following conditions.

L_λ is a free sub-A-module of $\ker e \cap M_\lambda$ such that

$$\ker e \cap M_\lambda = \mathbb{Q}(q^{1/4}) \bigotimes_A L_\lambda . \tag{1.3.15}$$

B is a base of the $\mathbb{Q}$-vector space $L_\lambda / q^{1/4} L_\lambda$. $\tag{1.3.16}$

Obviously, a crystal base of $\ker e$ is uniquely extended to a crystal base of M. By (1.3.9), any crystal base of M is the extension of a crystal base on $\ker e$.

(1.4). Bases in Tensor Products of sl_2-Modules. Consider the Lie algebra $\mathfrak{g} = sl_2$ with the generators e, f, h such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f .$$

Denote by Ω the Casimir element

$$\frac{1}{2} h \otimes h + e \otimes f + f \otimes e \in \mathfrak{g} \otimes \mathfrak{g} .$$

Let $M_1, \ldots, M_n$ be $\mathfrak{g}$ modules. For $i < j$ let Ω_{ij} be the linear operator on $M_1 \otimes \cdots \otimes M_n$ acting as Ω on $M_i \otimes M_j$ and as the identity operator on the other factors.

For a $\mathfrak{g}$-module M and $\lambda \in \mathbb{C}$, let $M_\lambda = \{u \in M | hu = \lambda u\}$ be the space of weight λ and $\mathrm{Sing} M_\lambda = \ker e \cap M_\lambda$ the space of singular elements of weight λ.

For a nonnegative integer m, let $L(m)$ be the $(m+1)$ dimensional irreducible $\mathfrak{g}$-module. The module is generated by its singular element v_m such that $ev_m = 0$ and $hv_m = mv_m$. The elements $v_m, fv_m, \ldots, f^m v_m$ form a basis of $L(m)$.

The *Shapovalov form* on $L(m)$ is the bilinear form B^m such that

$$B^m(f^k v_m, f^k v_m) = k! m! / (m-k)!, \quad B^m(f^k v_m, f^l v_m) = 0 \quad \text{for } k \neq l .$$

The tensor product of irreducible $\mathfrak{g}$ modules is the direct sum of irreducible modules: $L(m_1) \otimes L(m_n) = L(|m_1 - m_2|) \otimes L(|m_1 - m_2| + 2) \otimes \cdots \otimes L(m_1 + m_2)$, and a singular vector of $L(m_1 + m_2 - 2k)$ can be chosen of the form

$$\{v_{m_1}, {}_k v_{m_2}\} = \sum_{p=0}^{k} (-1)^p \frac{\prod_{j=0}^{k-1}(m_1 + m_2 - 2k + j + 2)}{\prod_{j=0}^{p-1}(m_1 - j) \prod_{j=0}^{k-p-1}(m_2 - j)} f^p v_{m_1} \otimes f^{k-p} v_{m_2} . \tag{1.4.1}$$

This decomposition gives a basis $\{f^l\{v_{m_1},\,_k\,v_{m_2}\}\}$ in the tensor product. We have

$$\Omega f^l\{w_{m_1},\,_k\,v_{m_2}\} = \mu(m_1, m_2; k) \cdot f^l\{w_{m_1},\,_k\,v_{m_2}\},$$

$$\mu(m_1, m_2; k) = \frac{1}{2}m_1 m_2 - k(m_1 + m_2) + k(k-1) . \qquad (1.4.2)$$

For nonnegative integers $m_1,\ldots,m_n$, consider the tensor product $L = L(m_1) \otimes \cdots \otimes L(m_n)$. For any n-tree T the tensor product has a distinguished basis $\mathscr{B}_T = \{v(c, l, T)\}$. The elements of $\mathscr{B}_T$ are numerated by the pairs (c, l), where c is an admissible coloring of T and $l = 0,\ldots,m(c)$, see (1.3). The vector $v(c, l, T)$ is defined inductively on n as in (1.3) but we use the formula

$$v(c, l, T) = f^l\{v^{le},\,_{c(w)}\,v^r\} \qquad (1.4.3)$$

instead of formula (1.2.9).

For any n-tree T and a permutation $\sigma \in S_n$ define a basis $\mathscr{B}_{T,\sigma} = \{v(c, l, T, \sigma)\}$ in $L(m_1) \otimes \cdots \otimes L(m_n)$. Namely, let $P_\sigma : L(m_1) \otimes \cdots \otimes L(m_n) \to L(m_{\sigma(1)}) \otimes \cdots \otimes L(m_{\sigma(n)})$ be the permutation of factors. Let $\mathscr{B}_T = \{v(c, l, T)\}$ be the basis in $L(m_{\sigma(1)}) \otimes \cdots \otimes L(m_{\sigma(n)})$ corresponding to the tree T with tops marked by $m_{\sigma(1)},\ldots,m_{\sigma(n)}$. Set $v(c, l, T, \sigma) = P_\sigma^{-1} v(c, l, T)$ for all c, l.

For any T, σ and an internal vertex w of T, define an operator $\Omega_{w, T, \sigma}$ acting on $L(m_1) \otimes \cdots \otimes L(m_n)$. Namely, for an internal vertex w, let J be the set of indices of the tops of T which belong to the right branch or the left branch of w. Set

$$\Omega_{w, T, \sigma} = \sum_{\substack{i<J \\ i, j \in \sigma^{-1}(J)}} \Omega_{ij} . \qquad (1.4.4)$$

For every T, σ, w the basis $\mathscr{B}_{T,\sigma}$ is an eigenbasis for the operator $\Omega_{w, T, \sigma}$:

$$\Omega_{w, T, \sigma} v(c, l, T, \sigma) = \mu(c, T, \sigma, w) \cdot v(c, l, T, \sigma) \qquad (1.4.5)$$

for all c, l and suitable numbers $\mu(c, T, \sigma, w)$, cf. (1.4.2).

(1.5). The Selberg Integral and Normalizing Constants. For any colored tree we define a function called the *normalizing constant*.

The Selberg integral is the integral

$$I_k(a, b; \kappa) = k! \int_\Delta \prod_{J=1}^k t_j^{-a/\kappa}(1 - t_j)^{-b/\kappa} \prod_{1 \leqq \iota < j \leqq k} (t_j - t_i)^{2/\kappa} dt_1 \wedge \cdots \wedge dt_k ,$$

where $\Delta = \{t \in \mathbb{R}^k | 0 < t_1 < \cdots < t_k < 1\}$. The integral can be computed explicitly [M]:

$$I_k(a, b; \kappa) = \prod_{j=0}^{k-1} \frac{\Gamma\left(\frac{-a+j}{\kappa} + 1\right) \Gamma\left(\frac{-b+j}{\kappa} + 1\right) \Gamma\left(\frac{j+1}{\kappa} + 1\right)}{\Gamma\left(\frac{-a-b+2k-j-2}{\kappa} + 2\right) \Gamma\left(\frac{1}{\kappa} + 1\right)} . \qquad (1.5.1)$$

Set

$$q(\kappa) = \exp(2\pi i/\kappa) .$$

Define $J_k(a, b; \kappa)$ by

$$J_k(a, b; \kappa) = (2\pi\kappa)^{-k/2} q(\kappa)^{\frac{-kb}{2} + \frac{3}{4}k(k-1)} I_k(a, b, \kappa)$$

$$\cdot \prod_{j=0}^{k-1} (1 - \kappa/(a + b - 2k + j + 2)) \cdot (b - j)_{q(\kappa)} , \qquad (1.5.2)$$

where q-numbers $(\ )_q$ are defined in (1.1.3).

Let T be an n-tree with tops marked by some nonnegative integers $m_1, \ldots, m_n$. Let c be an admissible coloring of c.

For an internal vertex $w \in \text{In}_T$ define its *normalizing constant* $C(w, c, T, \kappa)$ by

$$C(w, c, T, \kappa) = J_{c(w)}(m^{le}(w), m^r(w); \kappa) ,$$

where $c(w)$ is the color of w with respect to c, and $m^{le}(w)$ (resp. $m^r(w)$) is the weight of the left (resp. right) branch of w, see (1.2).

Define the *normalizing constant*, $C(c, T, \kappa)$, of a coloring c by the rule

$$C(c, T, \kappa) = \left(\sum_{w \in \text{In}_T} c(w) \right)! \prod_{w \in \text{In}_T} C(w, c, T, \kappa)/c(w)! . \qquad (1.5.3)$$

2. Asymptotic Solutions to the Knizhnik–Zamolodchikov Equation

(2.1). Knizhnik–Zamolodchikov Equation. Let $M_1, \ldots, M_n$ be $\mathfrak{g} = sl_2$ modules, $M = M_1 \otimes \cdots \otimes M_n$. The Knizhnik–Zamolodchikov equation (KZ) on an M-valued function $\psi(z_1, \ldots, z_n)$ is the system of equations

$$\kappa \frac{\partial \psi}{\partial z_i} = H_i \psi, \quad i = 1, \ldots, n ,$$

where κ is a parameter of the equation and

$$H_i = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} .$$

The KZ equation defines a connection on the trivial bundle $M \times \mathbb{C}^n \to \mathbb{C}^n$ with singularities at diagonal hyperplanes. This connection is flat. Parallel translation with respect to this connection commutes with the $\mathfrak{g}$ action on M.

For a solution ψ to the KZ equation we have

$$\left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n} \right) \psi = 0 ,$$

hence ψ depends only on differences $z_i - z_j$ of variables.

For nonnegative integers $m_1, \ldots, m_n$, let $L = L(m_1) \otimes \cdots \otimes L(m_n)$. For any λ the KZ equation with values in L preserves $\text{Sing} L_\lambda$. The KZ equation with values in $\bigoplus_\lambda \text{Sing} L_\lambda$ determines the KZ equation on L.

Denote by B the bilinear form $B^{m_1} \otimes \cdots \otimes B^{m_n}$ on L, where B^m is the Shapovalov form on $L(m)$, see (1.4). The operators $H_1, \ldots, H_n$ are B-symmetric:

$$B(H_i x, y) = B(x, H_i y)$$

for all i and all $x, y \in L$ [RV].

(2.2). Asymptotic Zones. For $\sigma \in S_n$ let

$$D_\sigma = \{z \in \mathbb{R}^n | z_{\sigma(1)} < \cdots < z_{\sigma(n)}\} \ .$$

For $\sigma \in S_n$ and an n-tree T, we define a diffeomorphism

$$u_{T,\sigma} : D_\sigma \to \mathbb{R} \times (\mathbb{R}_{>0})^{n-1}$$

called an *asymptotic zone.*

The first coordinate of $u_{T,\sigma}$ is $z_1 + \cdots + z_n$, the other coordinates are numerated by internal vertices of T. Denote by $u_{\omega, T, \sigma}$ the coordinate corresponding to a vertex ω.

Let $\sigma = id$. The coordinate $u_{\omega, T, \mathrm{id}}$ is defined inductively. For an internal vertex ω, let W be the set of internal vertices of T belonging to the shortest path in T from ω to the root. W can be empty. Assume that for any $y \in W$ the coordinate $u_{\omega, T, \mathrm{id}}$ is already defined. Let J^{le} (resp. J^r) be the set of the tops of T belonging to the left (resp. right) branch of w. Set $j^{le} = \max\{j \in J^{le}\}$, $j^r = \min(j \in J^r)$,

$$u_{w, T, \mathrm{id}} = (z_{j^r} - z_{j^{le}}) \Big/ \prod_{y \in W} u_{y, T, \mathrm{id}} \ . \tag{2.2.1}$$

For $\sigma \in S_n$, let

$$\tau_\sigma : D_\sigma \to D_{\mathrm{id}}, \quad (z_1, \ldots, z_n) \mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(n)}) \ .$$

Set

$$u_{w, T, \sigma} = u_{w, T, \mathrm{id}} \circ \tau_\sigma \ .$$

Let

$$\mathscr{U}_n = \{z_1, \ldots, z_n \in \mathbb{C}^n | z_i \neq z_j \text{ for all } i, j\} \ .$$

(2.2.2). Proposition. *For every σ, T the functions $z_1 + \cdots + z_n, \{u_{w, T, \sigma}\}_{w \in \mathrm{In}_T}$ define a diffeomorphism $u_{T,\sigma}$ of D_σ onto $\mathbb{R} \times (\mathbb{R}_{>0})^{n-1}$. The map $u_{T,\sigma}$ is extended to a biholomorphic map of $\mathscr{U}_n$ to $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$. Furthermore, $u_{T,\sigma}^{-1}$ is extended to a polynominal map of $\mathbb{C}^n$ to $\mathbb{C}^n$.*

Now we will construct curves connecting the domains D_σ and lying in $\mathscr{U}_n$.

For $j \in \{1, \ldots, n-1\}$ let τ be the transposition of j and $j+1$. Let T be a tree adjacent to τ, see (1.2), and w the internal vertex of T such that the jth top of T (resp. the $(j+1)^{\mathrm{th}}$) forms the left (resp. right) branch of w.

For any permutation σ, the biholomorphisms $u_{T,\sigma}$ and $u_{T,\sigma\tau}$ have the property:

$$u_{w, T, \sigma} = -u_{w, T, \sigma\tau} \ ,$$

$$u_{y, T, \sigma} = u_{y, T, \sigma\tau} \text{ for every internal vertex } y \text{ different from } w \ .$$

Define a curve $\gamma_{T,\sigma,\tau} : [0,1] \to \mathscr{U}_n$ by the rule:

$$u_{w,T,\sigma} = \exp(it) \text{ for } t \in [0,1] \, ,$$
$$u_{y,T,\sigma} = 1 \text{ for all } t \in [0,1] \text{ and all } y \in \mathrm{In}_T \text{ such that } y \neq w \, .$$

We have $\gamma(0) \in D_\sigma$ and $\gamma(1) \in D_{\tau\sigma}$. The curve $\gamma_{T,\sigma,\tau}$ forms a half circle connecting D_σ and $D_{\sigma\tau}$. The complementary half circle is formed by the curve $\gamma_{T,\sigma\tau,\tau}$.

For every T,σ the biholomorphism $u_{T,\sigma}$ resolves singularities of the union of hyperplanes in the following sense. Let

$$f : \mathbb{C}^n \to \mathbb{C}, \quad (z_1,\dots,z_n) \mapsto \prod_{i<j}(z_i - z_j) \, .$$

Then for every σ, T we have

$$f \circ u_{T,\sigma}^{-1} = \prod_{w \in \mathrm{In}_T} (u_{w,T,\sigma})^{a_w} \cdot (1 + \mathcal{O}(u))$$

for suitable positive integers $\{a_w\}$.

Consider the KZ equation with values in $V = V_1 \otimes \cdots \otimes V_n$.

(2.2.3). Proposition. *For any σ, T, the KZ equation with respect to variables $z_1 + \cdots + z_n$ and $\{u_{w,T,\sigma}\}$ has the form*

$$\left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_1} \right) \psi = 0 \, ,$$

$$\frac{\partial \psi}{\partial u_{w,T,\sigma}} = \frac{1}{\kappa} \left(\frac{\Omega_{w,T,\sigma}}{u_{w,T,\sigma}} + \mathrm{Reg} \right) \psi \, ,$$

where Reg *is an* $\mathrm{End}(V)$*-valued function holomorphic at* $u_{T,\sigma} = 0$, *and* $\Omega_{w,T,\sigma}$ *is the constant operator defined in* (1.4.4).

Example. For the tree in (1.2.7) and the identity permutation, the coordinates are $z_1 + z_2 + z_3$, $u_1 = z_3 - z_2$, $u_2 = (z_2 - z_1)/(z_3 - z_2)$, and the KZ equation has the form

$$\frac{\partial \psi}{\partial u_1} = \frac{1}{\kappa} \left(\frac{\Omega_{12} + \Omega_{13} + \Omega_{23}}{u_1} + \mathrm{Reg} \right), \qquad \frac{\partial \psi}{\partial u_2} = \frac{1}{\kappa} \left(\frac{\Omega_{12}}{u_2} + \mathrm{Reg} \right) \, .$$

(2.3). Asymptotic Solutions. For $m_1,\dots,m_n \in \mathbb{N}$ and a nonnegative integer λ, consider the tensor product of irreducible sl_2 modules $L = L(m_1) \otimes \cdots \otimes L(m_n)$ and the KZ equation with values in $\mathrm{Sing}\,L_\lambda$. Throughout the remainder of the paper we assume that the parameter κ of the KZ equation is not a rational number.

For any n-tree T and a permutation $w \in S_n$ we will construct a fundamental system of solutions, $\psi_{T,\sigma}$, to the KZ equation.

Namely, consider the basis $\mathscr{B}_{T,\sigma}$ of L. Then.

$$\mathscr{B}_{T,\sigma,\lambda} := \mathscr{B}_{T,\sigma} \cap \mathrm{Sing}\,L_\lambda$$

is a basis of $\mathrm{Sing}\,L_\lambda$. Vectors of $B_{T,\sigma,\lambda}$ have the form $v(c,0,T,\sigma)$, see (2.1), where c is an admissible coloring of the tree T with tops marked by $m_{\sigma(1)},\dots,m_{\sigma(n)}$ and the coloring c has weight λ, see (1.2.8). Denote the set of such colorings by

$\mathrm{Adm}(T, \sigma, \lambda)$. A vector $v(c, 0, T, \sigma)$ is an eigenvector of the operators $\{\Omega_{w, T, \sigma}\}_{w \in \mathrm{In}_T}$ with eigenvalues denoted by $\{\mu(c, T, \sigma, w)\}$, resp., see (1.4.5).

For every such a vector there exists a unique solution, $\psi_{c, T, \sigma}$, to the KZ equation restricted to D_σ such that

$$\psi_{c, T, \sigma} = C(c, T, \kappa) \left(\prod_{w \in \mathrm{In} T} (u_{w, T, \sigma})^{\mu(c, T, \sigma, w)/\kappa} \right) \cdot \left(v(c, 0, T, \sigma) + \mathcal{O}(u_{T, \sigma}, \kappa) \right) ,$$

$$(2.3.1)$$

where $C(c, T, \kappa)$ is the normalizing constant of the colored tree T with tops marked by $m_{\sigma(1)}, \ldots, m_{\sigma(n)}$, $\mathcal{O}(u_{T, \sigma}, \kappa)$ is a $\mathrm{Sing} L_\lambda$-valued function which is regular function of $u_{T, \sigma}$ at $u_{T, \sigma} = 0$ and which tends to zero as all coordinates $u_{T, \sigma}$ tend to zero, univalued branches of the functions $\{u^\mu\}$ are chosen by the rule: $\arg(u_{w, T, \sigma}) = 0$ for all $w \in \mathrm{In}_T$.

The collection of these solutions form a fundamental system of solutions. It will be called the *asymptotic solution corresponding to the asymptotic zone* $u_{T, \sigma}$, and will be denoted by $\psi_{T, \sigma}$.

The asymptotic solution can be analytically continued to a system of multivalued solutions over $\mathcal{U}_n$.

(2.4). Transition Functions Between Asymptotic Solutions. The first main result of this paper describes transition functions between asymptotic solutions.

To compare two asymptotic solutions $\psi_{T, \sigma}$ and $\psi_{T', \sigma'}$ we have to distinguish a curve from D_σ to D'_σ, lying in $\mathcal{U}_n$, then analytically continue $\psi_{T, \sigma}$ along the curve and express the analytic continuation in terms of $\psi_{T', \sigma'}$.

Let T and T' be n-trees. Let w_1 and w_2 be internal vertices of T and w'_1 and w'_2 internal vertices of T'. Assume that T and T' are adjacent at w_1, w_2, w'_1, w'_2, see (1.2).

For any $\sigma \in S_n$, the asymptotic solutions $\psi_{T, \sigma}$ and $\psi_{T', \sigma}$ are defined over the same D_σ.

(2.4.1). Theorem. *For any $c \in \mathrm{Adm}(T, \sigma, \lambda)$ we have*

$$\psi_{c, T, \sigma} = \sum_{\substack{c' \equiv c \\ c' \in \mathrm{Adm}(T', \sigma, \lambda)}} \left\{ \begin{matrix} m^{le}(w_1) & m^r(w_1) & m^r(w_2) \\ c(w_1) & c(w_2) & c'(w'_1) \end{matrix} \right\}_{q(\kappa)} \psi_{c', T', \sigma} .$$

Here $\{ \ \}_{q(\kappa)}$ are the $6j$-symbols for $q(\kappa) = \exp(2\pi i/\kappa)$. For the definition of the arguments of the $6j$-symbols, see (1.2).

The theorem is proved in Sect. 4.

For $j \in \{1, \ldots, n-1\}$, let τ be the transposition of j and $j + 1$. Let T be an n-tree adjacent to τ, see (1.2), and w the internal vertex of T such that the jth tops of T (resp. the $(j+1)$) forms the left (resp. right) branch of w.

For any $\sigma \in S_n$, the curve $\gamma_{T, \sigma, \tau}$, defined in (2.2), connects D_σ and $D_{\sigma\tau}$. Continue $\psi_{T, \sigma}$ along the curve and express the continuation in terms of $\psi_{T, \sigma\tau}$.

(2.4.2). Proposition. *For any $c \in \mathrm{Adm}(T, \sigma, \lambda)$ we have*

$$\psi_{c, T, \sigma} = R(m_{\sigma(j)}, m_{\sigma(j+1)}; c(w))_{q(\kappa)} \psi_{c, T, \sigma\tau} ,$$

where R is defined in (1.2.3).

The proposition easily follows from the definition of the normalizing constant.

Statements (2.4.1) and (2.4.2) allow us to compare two arbitrary asymptotic solutions.

By (2.4.1) and (2.4.2), the transition functions between asymptotic solutions to the KZ equation are exactly the same as the transition functions between the bases in tensor products of irreducible modules over the quantum group $U_{q=q(\kappa)}$, where $U_{q=q(\kappa)}$ is the $\mathbb{C}$-algebra obtained from U_q by specializing the indeterminate $q^{1/4}$ to $q(\kappa/4)$. Namely, for natural $m_1, \ldots, m_n$, let $L(m_1), \ldots, L(m_n)$ be the irreducible sl_2-modules with highest weights $m_1, \ldots, m_n$, resp. For a nonrational $\kappa \in \mathbb{C}$ and an integer λ, consider the KZ equation with the parameter κ and the values in $\operatorname{Sing} L_\lambda$. For an n-tree T and a permutation $\sigma \in S_n$, consider the asymptotic solution $\psi_{T,\sigma}(\kappa) = \{\psi_{c,T,\sigma}\}$.

Let $L(m_1, q = q(\kappa)), \ldots, L(m_n, q = q(\kappa))$ be irreducible $U_{q=q(\kappa)}$ modules with highest weight $m_1, \ldots, m_n$, resp. For a permutation $\sigma \in S_n$, consider the tensor product

$$L^\sigma = L(m_{\sigma(1)}, q = q(\kappa)) \otimes \cdots \otimes L(m_{\sigma(n)}, q = q(\kappa)) \ , \qquad (2.4.3)$$

and the subspace

$$\operatorname{Sing} L^\sigma(q = q(\kappa)) = \{v \in L^\sigma | q^h v = q(\kappa)^\lambda v, \, ev = 0\} \ . \qquad (2.4.4)$$

Let $\mathscr{B}_{T,\sigma}(q = q(\kappa)) = \{v(c, 0, T)\}$ be the basis in $\operatorname{Sing} L^\sigma(q = q(\kappa))_\lambda$ corresponding to an n-tree T, see (1.2).

Define a map

$$\pi_{T,\sigma}(\kappa) : \mathscr{B}_{T,\sigma}(q = q(\kappa)) \to \psi_{T,\sigma}(\kappa), \quad v(c, 0, T) \mapsto \psi_{c,T,\sigma} \ . \qquad (2.4.5)$$

(2.4.6). Corollary of (1.2.12) and (2.4.1). *For every $\sigma \in S_n$ the maps $\{\pi_{T,\sigma}(\kappa)\}$ induce a well defined isomorphism $\pi_\sigma(\kappa)$ of $\operatorname{Sing} L^\sigma(q = q(\kappa))_\lambda$ and the space of solutions to the KZ equation over D_σ with parameters κ and values in $\operatorname{Sing} L_\lambda$.*

(2.4.7). Corollary of (1.2.13) and (2.4.2). *Under the isomorphisms described in (2.4.6), the R-matrix action on $\{\operatorname{Sing} L^\sigma(q = q(\kappa))_\lambda\}_{\sigma \in S_n}$ given by (1.2.13) is isomorphic to the monodromy of the KZ equation with the parameter κ and with values in $\operatorname{Sing} L_\lambda$.*

Kohno [Ko] and Drinfeld [D] proved existence of an isomorphism between the R-matrix action and the monodromy representation of the KZ equation for generic κ. The case $\kappa = 1/l$ for a natural l is described in [V, Sects. 13–14]. In Sect. 3 we will construct isomorphisms (2.4.6) and (2.4.7) geometrically in terms of integral representations for solutions to the KZ equation, see [SV, V]

(2.5). Quasiclassical Asymptotics. Let D be a ball, $\pi \colon \mathbb{C}^N \otimes D \to D$ projection. Let

$$\nabla_\kappa = \kappa d - \omega$$

be a holomorphic connection in π depending on the parameter κ. Here

$$\omega = H_1 dz_1 + \cdots + H_n dz_n \ ,$$

where $\{H_i\}$ are matrix valued functions. Assume that for every κ the connection is integrable, $\kappa d\omega + \omega \wedge \omega = 0$, or

$$\frac{\partial H_i}{\partial z_j} = \frac{\partial H_j}{\partial z_i}, \quad [H_i, H_j] = 0 \qquad (2.5.1)$$

for all i and j.

An *asymptotically flat section* is a section of the form

$$F = \exp(S/\kappa)(f_0 + \kappa f_1 + \cdots) \quad \text{for } \kappa \to 0 \ . \tag{2.5.2}$$

Here $S(z_1, \ldots, z_n)$ is a function, $\{f_j(z_1, \ldots, z_n)\}$ are sections of π, and F must be a formal solution of the equation $\nabla_\kappa F = 0$.

We call $\exp(S/\kappa)f_0$ an *asymptotically flat section of the first order* if there exists a power series (2.5.2) which provides an asymptotic solution to the equation $\nabla_\kappa F = 0$ modulo terms of order κ^2.

Assume that the linear operators $\{H_j\}$ are simultaneously diagonalizable for each $z \in D$: there exists a basis $\{v_l(z)\}$ in $\mathbb{C}^N$ such that

$$H_j(z)v_l(z) = \lambda_{jl}(z) \cdot v_l(z) \tag{2.5.3}$$

for all j, l. Assume that the spectrum of $\{H_j\}$ separates elements of the basis: for every z and every l, m, there exists j such that $\lambda_{jl}(z) \neq \lambda_{jm}(z)$.

Let $F = \exp(S/\kappa)f_0$ be an asymptotically flat section of the first order, then f_0 is an eigenvector of the operators $\{H_i\}$,

$$H_j(z)f_0(z) = \lambda_j(z)f_0(z), \quad j = 1, \ldots, n \ , \tag{2.5.4}$$

moreover, $\lambda_j = \frac{\partial S}{\partial z_j}$, see [RV].

Let $\exp(S/\kappa)f_0$ and $\exp(T/\kappa)g_0$ be asymptotically flat sections of the first order corresponding to the same eigenvector of $\{H_i\}$, then

$$\exp(S/\kappa)f_0 = \text{const} \cdot \exp(T/\kappa)g_0 \ , \tag{2.5.5}$$

see [RV].

Assume that there exists a symmetric bilinear form $B : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$ and the operators $H_1, \ldots, H_n$ are symmetric with respect to $B : B(H_i x, y) = B(x, H_i y)$ for all x, y, i. Assume that $\exp(S/\kappa)(f_0 + \cdots)$ and $\exp(T/\kappa)(g_0 + \cdots)$ are two asymptotically flat sections.

(2.5.6). Lemma [RV]. *If* $S - T \neq \text{const}$, *then* $B(f_0, g_0) \equiv 0$, *if* $S - T = \text{const}$ *then* $B(f_0, g_0) \equiv \text{const}$.

The KZ equation gives an example of a family of flat connections. The KZ operators $\{H_i\}$ are symmetric with respect to the Shapovalov form.

For natural $m_1, \ldots, m_n$, let $L(m_1), \ldots, L(m_n)$ be the irreducible sl_2 modules with highest weights $m_1, \ldots, m_n$. For a nonrational number κ and an integer λ, consider the KZ equation with parameter κ and with values in $\text{Sing} L_\lambda$. For an n-tree T and a permutation $\sigma \in S_n$ consider the asymptotic solution $\psi_{T,\sigma} = \{\psi_{c,T,\sigma}\}$. Consider the coordinates $\{u_{w,T,\sigma}\}_{w \in \text{In}_T}$. These coordinates take positive values on D_σ.

For $\varepsilon > 0$, let

$$D_{\sigma,T,\varepsilon} = \{p \in D_\sigma | u_{w,T,\sigma}(p) < \varepsilon \quad \text{for all } w \in \text{In}_T\} \ .$$

Our second main result is the following theorem.

(2.5.7). Theorem. *Assume that* $\kappa = is$ *and* $s \to +0$. *Then there exists* $\varepsilon > 0$ *such that any solution* $\psi_{c,T,\sigma} \in \psi_{T,\sigma}$, *restricted to* $D_{\sigma,T,\varepsilon}$, *has an asymptotic expansion*

$$\psi_{c,T,\sigma} \sim \prod_{w \in \text{In}_T} (u_{w,T,\sigma})^{-i\mu(c,T,\sigma,w)/s} \exp(-iS/s) \sum_{j=0}^{\infty} f_j s^j \ , \tag{2.5.8}$$

where $\{\mu\}$ *are the numbers described in* (2.3.1) *and* (1.4.5);

$$S = S_{c,T,\sigma} : D_{\sigma,T,\varepsilon} \to \mathbb{R}, \quad f_j = f_{j,c,T,\sigma} : D_{\sigma,T,\varepsilon} \to \operatorname{Sing} L_\lambda$$

are suitable real analytic functions, all the functions $\{S, f_j\}$ *depend only on* $\{u_{w,T,\sigma}\}_{w\in\ln_T}$ *and can be analytically continued to real analytic functions in a neighborhood of the set* $\{u_{w,T,\sigma} = 0 | w \in \ln_T\}$.

The asymptotic expansion can be differentiated an arbitrary number of times.

The asymptotic expansion means that for any N,

$$\left| \psi_{c,T,\sigma} \prod_{w\in\ln_T} (u_w)^{i\mu(w)/s} \exp(iS/s) - \sum_{j=0}^{\infty} f_j s^j \right| = \mathcal{O}(s^{N+1})$$

uniformly in $D_{\sigma,T,\varepsilon}$.

The theorem has the following appendix.

(2.5.9). Appendix.

(1) *For any* $p \in D_{\sigma,T,\varepsilon}$ *the first terms* $\{f_{0,c,T,\sigma}(p)\}$ *of the above asymptotic expansions form a basis in* $\operatorname{Sing} L_\lambda$ *orthogonal with respect to the Shapovalov form.*

(2) *For any* c, $f_{0,c,T,\sigma}(p)$ *tends to*

$$(-1)^k k! \cdot \left(\prod_{w\in\ln_T} (c(w)!)^3 \prod_{j=0}^{c(w)-1} \frac{(m^{le}(w) + m^r(w) - 2c(w) + j + 2)^3}{(m^{le}(w) - j)(m^r(w) - j)} \right)^{-1/2}$$

$$\cdot v(c, 0, T, \sigma)$$

as $u_{w,T,\sigma}(p) \to 0$ *for all* $w \in \ln_T$. *Here* T *is an* n*-tree with vertices marked by* $m_{\sigma(1)}, \ldots, m_{\sigma(n)}$. *The numbers* $c(w), m^{le}(w)$, *and* $m^r(w)$ *for an* n*-tree with marked tops are defined in* (1.2). *The vector* $v(c, 0, T, \sigma)$ *is defined by* (2.3.1), $k = (m_1 + \cdots + m_n - \lambda)/2$.

The theorem and the appendix are proved in Sect. 4.

(2.6). *Quasiclassical Asymptotics and Crystal Base.* For natural $m_1, \ldots, m_n$, let $L(m_1, q), \ldots, L(m_n, q)$ be irreducible U_q modules with highest weights $m_1, \ldots, m_n$, resp. For any permutation $\sigma \in S_n$, let

$$L^\sigma(q) = L(m_{\sigma(1)}, q) \otimes \cdots \otimes L(m_{\sigma(n)}, q).$$

Statements (2.4.6) and (2.5.7) allow us to give a construction of a crystal base in $L^\sigma(q)$ purely in terms of quasiclassical asymptotics of solutions to the KZ equation.

This "quasiclassical" crystal base coincides with the distinguished crystal base of the tensor product defined in (1.3). This statement can be considered as a quasiclassical characterization of the distinguished crystal base.

We will give a construction for $\sigma = id$, since for an arbitrary permutation the construction is the same.

To construct a crystal base in $L(q) = L(m_1, q) \otimes \cdots \otimes L(m_n, q)$ it suffices to construct a crystal base in

$$\operatorname{Sing} L(q)_\lambda = \{v \in L(q) | q^h v = q^\lambda v, ev = 0\}$$

for every λ.

Let $\mu: \mathbb{Q}(q^{\frac{1}{4}}) \to \frac{1}{4}\mathbb{Z}$ be the valuation map which assigns to a function $g(q^{\frac{1}{4}})$ the order of its pole at $q = \infty$.

Using the KZ equation, we will construct a map

$$\mu: \operatorname{Sing} L(q)_\lambda \to \frac{1}{4}\mathbb{Z}$$

such that

$$\mu(v) = 0 \quad \text{iff } v = 0\,,$$

$$\mu(v + w) \leqq \max(\mu(v), \mu(w))\,,$$

$$\mu(gv) = \mu(g) + \mu(v) \qquad\qquad (2.6.1)$$

for all $v, w \in \operatorname{Sing} L(q)_\lambda$, $g \in \mathbb{Q}(q^{\frac{1}{4}})$.

Let $v \in \operatorname{Sing} L(q)_\lambda$,

$$v = \sum_{k_1, \ldots, k_n} a_K f^{k_1} v_1 \otimes \cdots \otimes f^{k_n} v_n\,,$$

where v_j is the generating vector of $L(m_j, q)$, $j = 1, \ldots, n$, and the coefficients $\{a_K\}$ are rational functions in $q^{\frac{1}{4}}$.

For $\kappa \in \mathbb{C}$, let $\operatorname{Sing} L(q = q(\kappa))_\lambda$ be the space defined by (2.4.4). For any κ such that $q = q(\kappa)$ is not a pole of the functions $\{a_K\}$, let $v(q = q(\kappa)) \in \operatorname{Sing} L(q = q(\kappa))$ be the specialization of v at $q = q(\kappa)$.

Consider the KZ equation with parameter κ and values in $\operatorname{Sing} L_\lambda$, see (2.3). Let $\operatorname{Sol}(\kappa)_\lambda$ be the space of its solutions over D_{id}, and let $\pi(\kappa): \operatorname{Sing} L(q = q(\kappa)) \to \operatorname{Sol}(\kappa)_\lambda$ be the isomorphism defined in (2.4.6). Then $y(\kappa) = \pi(\kappa)(v(q = q(\kappa))) \in \operatorname{Sol}(\kappa)_\lambda$ is a solution over D_{id} to the KZ equation with parameter κ.

For any n-tree T, let $\psi_{T,\mathrm{id}}(\kappa) = \{\psi_{c,T,\mathrm{id}}\}$ be the asymptotic solution, then

$$y(\kappa) = \sum_c b_c(q(\kappa)^{\frac{1}{4}}) \cdot \psi_{c,T,\sigma}\,,$$

where $\{b_c(q^{\frac{1}{4}})\}$ are some rational functions.

(2.6.2). Proposition.

1. *Let* $\|\ \|: \operatorname{Sing} L_\lambda \to \mathbb{R}_{\geqq 0}$ *be any norm on* $\operatorname{Sing} L_\lambda$. *Let* $\kappa = is$, $s \in \mathbb{R}$, *and* $s \to +0$. *For any n-tree T, let* $D_{\mathrm{id},T,\varepsilon} \subset D_{\mathrm{id}}$ *be the domain described in (2.5.7). Then for every* $p \in D_{\mathrm{id},T,\varepsilon}$ *there exists a limit of* $s \cdot \ln \|y(is)(p)\|$. *This limit is an element of* $\frac{1}{4}\mathbb{Z}$, *and does not depend on p and T. Denote this limit by* $\mu(v)$.

2. *The map* μ, *defined by this rule, has properties (2.8.1).*

Proof. The proposition easily follows from (2.5.7), (2.5.9), (2.4.1), and (1.3.12).

Let $A \subset \mathbb{Q}(q^{\frac{1}{4}})$ be the subalgebra of functions regular at $q = \infty$. Let

$$\mathscr{L}_\lambda^0 = \{v \in \operatorname{Sing} L(q)_\lambda | \mu(v) \leqq 0\}\,.$$

(2.6.3). Corollary. $\mathscr{L}_\lambda^0$ *is an A-submodule of* $\operatorname{Sing} L(q)_\lambda$.

Let $(\mathscr{L}, \mathscr{B})$ be the distinguished crystal base in $L(q)$ defined in (1.3). Let $(\mathscr{L}_\lambda, \mathscr{B}_\lambda)$ be the crystal base in $\operatorname{Sing} L(q)_\lambda$ induced by $(\mathscr{L}, \mathscr{B})$.

(2.6.4). Proposition. $\mathscr{L}_\lambda^0 = \mathscr{L}_\lambda$.

The proposition easily follows from (2.5.7) and (2.5.9).

For any n-tree T, let $\psi_{T,\,\text{id}} = \{\psi_{c,\,T,\,\text{id}}\}$ be the asymptotic solution considered in (2.5.7) and (2.5.9), let

$$\left\{ \prod_w (u_{w,\,T,\,\text{id}})^{-\mu(c,\,T,\,\text{id},\,w)/s} \exp(-iS_{c,\,T,\,\text{id}}/s) \cdot f_{0,\,c,\,T,\,\text{id}} \right\}$$

be the collection of its first terms of asymptotics.

Denote by Asym_T the $\mathbb{Q}$-module of $\mathbb{Q}$-linear combinations of the first terms of asymptotics.

(2.6.5). Proposition. *The above construction induces the canonical isomorphism of the $\mathbb{Q}$ modules $\mathscr{L}_\lambda^0/q^{-\frac{1}{4}}\mathscr{L}_\lambda^0$ and Asym_T.*

The module Asym_T has a canonical basis generated by the first terms of asymptotics. This basis is uniquely determined by the fact that each element of the basis is the first term of an asymptotic expansion of the form (2.5.8) of a solution to the KZ equation and by the normalizing condition (2.5.9.2). The canonical basis in Asym_T induces a basis in $\mathscr{L}_\lambda^0/q^{-\frac{1}{4}}\mathscr{L}_\lambda^0$. This induced basis coincides with the basis $\mathscr{B}_\lambda$.

3. Integral Representations for Solutions to the KZ Equation and the Bethe Vectors

(3.1). Local System. Let $m_1, \ldots, m_n \in \mathbb{C}$, $\kappa \in \mathbb{C}^*$, let k be a nonnegative integer. Set

$$\Phi(t,z) = \prod_{1 \leq l < j \leq n} (z_j - z_l)^{m_l m_j/2\kappa} \prod_{1 \leq l < j \leq k} (t_l - t_j)^{2/\kappa} \cdot \prod_{l=1}^{k}\prod_{j=1}^{n} (z_j - t_l)^{-m_j/\kappa}. \quad (3.1.1)$$

Φ is a multivalued holomorphic function on

$$\mathscr{U}_{k+n} = \{(t,z) \in \mathbb{C}^{k+n} | t_l \neq t_m, \; t_l \neq z_m, \; z_l \neq z_m, \; \text{for all } l, m\}.$$

Univalued branches of Φ over open subsets of $\mathscr{U}_{k+n}$ generate a complex one-dimensional local system over $\mathscr{U}_{k+n}$ denoted by $\mathscr{S}(\kappa)$.

Let $\mathscr{U}_n = \{z \in \mathbb{C}^n | z_l \neq z_j \text{ for all } l, j\}$, and let

$$pr_{k,\,n} \colon \mathscr{U}_{k+n} \to \mathscr{U}_n, \quad (t,z) \mapsto z,$$

be a projection. Denote its fiber $pr^{-1}(z)$ by $\mathscr{U}_{k,\,n}(z)$. Denote by $\mathscr{H}_{k,\,n}(\kappa)$ the complex vector bundle over $\mathscr{U}_n$ with fiber $H_k(\mathscr{U}_{k,\,n}(z), \mathscr{S}(\kappa))$ over $z \in \mathscr{U}_n$. The bundle has a canonical flat connection called the *Gauss–Manin connection*.

The symmetric group S_k acts on $\mathscr{U}_{k+n}$ by permutations of coordinates $t_1, \ldots, t_k$ preserving fibers of the projection. The function Φ is symmetric with respect to this action. Therefore, S_k naturally acts on the singular chains in $\mathscr{U}_{k,\,n}(z)$ with coefficients in $\mathscr{S}(\kappa)$. This action induces an action of S_k on $H_k(\mathscr{U}_{k,\,n}(z), \mathscr{S}(\kappa))$. This action on fibers of $\mathscr{H}_{k,\,n}(\kappa)$ commutes with the Gauss–Manin connection.

Denote by $H_k(\mathscr{U}_{k,\,n}(z), \mathscr{S}(\kappa))_-$ the skew symmetric part of this action:

$$H_k(\mathscr{U}_{k,\,n}(z), \mathscr{S}(\kappa))_- = \{v \in H_k | \sigma v = (-1)^{|\sigma|} v \text{ for all } \sigma \in S_k\}.$$

(3.2). Integral Representations. For $m_1, \ldots, m_n \in \mathbb{N}$ and a nonnegative integer λ, consider the tensor product of irreducible sl_2 modules, $L = L(m_1) \otimes \cdots \otimes L(m_n)$, and the KZ equation with parameter κ and values in $\mathrm{Sing}\, L_\lambda$. We will describe integral representations for solutions to the KZ equation.

Set $k = (m_1 + \cdots + m_n - \lambda)/2$. The number k is a nonnegative integer. A *monomial of weight* λ is an element of L_λ of the form

$$f_K = f^{k_1} v_1 \otimes \cdots \otimes f^{k_n} v_n, \tag{3.2.1}$$

where $K = (k_1, \ldots, k_n)$, $k_1 + \cdots + k_n = k$. For a monomial f_K define a differential k-form in t and z:

$$\eta(f_K) = A_K(t,z) dt_1 \wedge \cdots \wedge dt_k,$$

$$A_k = \sum_{\sigma \in S(k;\, k_1, \ldots, k_n)} \prod_{i=1}^{k} \frac{1}{(t_1 - z_{\sigma(i)})}. \tag{3.2.2}$$

The sum is over the set $S(k; k_1, \ldots, k_n)$ of maps σ from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$ such that for all m the cardinality of $\sigma^{-1}(m)$ is k_m.

Consider the L_λ-valued form

$$N = \sum_{f_K \in L_\lambda} \Phi(t,z) \cdot \eta(f_K) \otimes f_K, \tag{3.2.3}$$

where Φ is the function defined by (3.1.1). N is a multi-valued holomorphic k-form on $\mathcal{U}_{k+n}$.

In [SV] it is proved that:

For every j, the form

$$\left(\kappa \frac{\partial}{\partial z_j} - \sum_{l \neq j} \frac{\Omega_{j,l}}{z_j - z_l} \right) N \tag{3.2.4}$$

is a sum of the differential of a suitable $(k-1)$-form and a form which has zero restriction to fibers of the projection $pr_{k,n}$.

The form $eN = \sum \Phi \cdot \eta(f_K) \otimes e f_K$ is a sum of the differential of a suitable $(k-1)$-form and a form which has zero restriction to fibers of $pr_{k,n}$. (3.2.5)

The forms mentioned in (3.2.4) and (3.2.5) have the shape $\sum \Phi w(M) \otimes M$, where the sum is over monomials in L, and $\{w(M)\}$ are suitable rational forms, regular on $\mathcal{U}_{k+n}$. (3.2.6)

Assume that $\gamma(z) \in H_k(\mathcal{U}_{k,n}(z), \mathcal{S}(\kappa))$. Assume that the map $z \mapsto \gamma(z)$ forms a flat section of the bundle $\mathcal{H}_{k,n}(\kappa)$ when z runs through an open subset of $\mathcal{U}_n$. Then the function

$$\Psi(z) = \int_{\gamma(z)} N \tag{3.2.7}$$

takes values in $\mathrm{Sing}\, L_\lambda$ and satisfies the KZ equation with parameter κ [SV, V].

(3.2.8). Remark. The group S_k naturally acts on the space of differential forms on $\mathcal{U}_{k+n}$ by permutations of $t_1, \ldots, t_k$. The differential form N is skew symmetric

with respect to this action. Let $v : H_k(\mathscr{U}_{k,n}(z), \mathscr{S}(\kappa)) \to H_k(\mathscr{U}_{k,n}(z), \mathscr{S}(\kappa))_-$ be the canonical projection, then for any flat section $z \mapsto \gamma(z)$ of $\mathscr{H}_{k,n}(\kappa)$ we have

$$\int\limits_{\gamma(z)} N = \int\limits_{v(\gamma(z))} N \,.$$

(3.3). *Quasiclassical Aysmptotic Solutions to the KZ Equation.* Let $\kappa \to 0$. We use the form N defined in (3.2.3) to construct quasiclassical asymptotic solutions to the KZ equation.

The function Φ can be written in the form

$$\Phi(t,z) = \exp(S(t,z)/\kappa) \,,$$

where

$$S(t,z) = \sum_{1 \le l < j \le n} \frac{m_l m_j}{2} \ln(z_j - z_l) + \sum_{1 \le l < j \le n} 2\ln(t_l - t_j)$$

$$- \sum_{l=1}^{k} \sum_{j=1}^{n} m_j \ln(z_j - t_l). \tag{3.3.1}$$

Set

$$\operatorname{Hess}_t(-S) = \det\left(-\frac{\partial^2 S}{\partial t_l \partial t_j}\right). \tag{3.3.2}$$

For a fixed $z \in \mathscr{U}_n$, consider the equation of critical points of S in $\mathscr{U}_{k,n}(z)$:

$$\frac{\partial S}{\partial t_j} = 0, \quad j = 1,\dots,k \,. \tag{3.3.3}$$

Let $t = t(z)$ be a nondegenerate solution of (3.3.3) holomorphically depending on z in a neighborhood of a point $z^0 \in \mathscr{U}_n$.

Let $B(z) \subset \mathscr{U}_{k,n}(z)$ be a small ball with center at $(t(z),z)$. Set

$$B_-(z) = \{(t,z) \in \mathscr{U}_{k,n}(z) \,|\, \operatorname{Im}(S(t,z)) < \operatorname{Im}(S(t(z),z))\} \,. \tag{3.3.4}$$

It is known that $H_k(B(z), B_-(z), \mathscr{S}(\kappa))$ is one-dimensional, see Sect. 11 in [AGV]. The Gauss–Manin connection identifies these groups for neighboring z's.

A generator of the homology group (for all κ simultaneously) can be chosen as follows. Fix a branch of arguments of all functions $t_j - t_l, t_j - z_l, z_l - z_m$ for all l, m in a neighborhood of the point $(t(z^0), z)$. This choice determines a branch, $\beta(\kappa)$, of Φ and a branch of S in a neighborhood of $(t(z^0), z^0)$. The branch $\beta(\kappa)$ gives a section of $\mathscr{S}(\kappa)$ for all κ.

There exist local coordinates $u_1,\dots,u_k$ in $\mathscr{U}_{k,n}(z)$ centered at $t(z)$ such that the coordinates holomorphically depend on z in a neighborhood of z^0 and

$$S(t(u),z) = -i(u_1^2 + \cdots + u_n^2) + g(z)$$

for some function $g(z)$. For these coordinates and a small $\varepsilon > 0$, denote by $d(z)$ the disc $\{(u,z) \in \mathscr{U}_{k,n}(z) \,|\, u_1,\dots,u_k \in \mathbb{R}, \, u_1^2 + \cdots + u_k^2 \le \varepsilon\}$.

The homology class of the cycle

$$\delta(z,\kappa) = (d(z), \beta(\kappa)|_{d(z)}) \tag{3.3.5}$$

gives a generator of $H_k(B(z), B_-(z), \mathscr{S}(\kappa))$. The classes $[\delta(z, \kappa)]$ form a flat section of the Gauss–Manin connection of the vector bundle over a neighborhood of z^0 with fiber $H_k(B(z), B_-(z), \mathscr{S}(\kappa))$.

Set

$$\Psi(z, \kappa) = (2\pi\kappa)^{-\frac{k}{2}} \int\limits_{\delta(z,\kappa)} N \ . \tag{3.3.6}$$

(3.3.7). Theorem [RV]. *Let* $\kappa = is$, $s \in \mathbb{R}$, *and* $s \to +0$.

1. *Then the function* $\Psi(z)$ *has an asymptotic expansion*

$$\Psi(z, is) \sim \exp(-iS(t(z), z)/s) \sum_{j=0}^{\infty} g_j(z) s^j \ ,$$

where $\{g_j\}$ *are* $\mathrm{Sing} L_\lambda$-*valued holomorphic functions defined in a neighborhood of* z^0.

2. *The function* $\Psi(z, \kappa)$ *gives a quasiclassical asymptotic solution to the KZ equation in the sense of* (2.5).

3.
$$g_o(z) = \mathrm{Hess}_t(-S(t(z), z))^{-\frac{1}{2}} \sum_{f_K \in L_j} A_K(t(z), z) f_K \ .$$

(3.3.8). Remark. Let $\delta^0(z, \kappa)$ be a singular chain in $B(z)$ with coefficients in $\mathscr{S}(\kappa)$ such that the boundary of $\delta'(z, \kappa)$ lies in $B_-(z)$ and the class of $\delta'(z, \kappa)$ in $H_k(B(z), B_-(z), \mathscr{S}(\kappa))$ coincides with $[\delta(z, \kappa)]$. Then the function

$$\Psi^0(z, \kappa) = (2\pi\kappa)^{-\frac{k}{2}} \int\limits_{\delta^0(z,\kappa)} N$$

has the same asymptotic expansion as the function $\Psi(z, \kappa)$.

The vector

$$g(t(z), z) = \sum_{f_K \in L_\lambda} A_K(t(z), z) f_k \tag{3.3.9}$$

is called the *Bethe vector*, see [B, Ba, BF, FFR, G, R, TV].

It is shown in [V2], that

$$B(g(t(z), z),\ g(t(z), z)) = \mathrm{Hess}_t(S(t(z), z)) \ , \tag{3.3.10}$$

where B is the Shapovalov form on $\mathrm{Sing} L_\lambda$.

The group of permutations of coordinates $t_1, \ldots, t_k$ acts on the set of t-critical points of S. The Bethe vectors corresponding to t-critical points of the same orbit are identical. If two critical points $t = t^1(z)$ and $t = t^2(z)$ lie in different orbits, then the corresponding Bethe vectors are orthogonal with respect to the Shapovalov form,

$$B(g(t^1(z), z), g(t^2(z), z)) = 0 \ . \tag{3.3.11}$$

Moreover, for generic $z \in \mathscr{U}_n$, there are exactly $\dim \mathrm{Sing} L_\lambda$ different orbits of non-degenerate t-critical points, and, consequently, the corresponding Bethe vectors form a basis in $\mathrm{Sing} L_\lambda$, see [RV].

(3.4). *Homology Class Sitting on a Critical Point.* Under the assumptions of Sect. (3.3), assume that for almost all κ a homology class $[\gamma(\kappa)] \in H_k(\mathcal{U}_{k,n}(z^0), \mathcal{S}(\kappa))$ is given. Assume that a chain

$$\gamma(\kappa) = \sum_{j=1}^{M} (c_j, \alpha_j(\kappa)) \qquad (3.4.1)$$

is given, such that:

For every j, $c_j \subset \mathcal{U}_{k,n}(z^0)$ is a singular cell.   (3.4.2)

For every j, $\alpha_j(\kappa)$ is a section of $\mathcal{S}(\kappa)$ over c_j which has the following form:

Fix a branch of arguments of all functions $t_j - t_l$, $t_j - z_l$, $z_l - z_m$ for all l, m in a neighborhood of c_j. This choice determines a branch $\beta_j(\kappa)$ of Φ over c_j. Then $\alpha_j(\kappa) = \varepsilon_j(\kappa)\beta_j(\kappa)$, where $\varepsilon_j(\kappa)$ is a rational function of $q(\kappa)^{\frac{1}{4}}$.   (3.4.3)

If $\kappa \in \mathbb{C}$ is such that $\varepsilon_1(\kappa), \ldots, \varepsilon_m(\kappa)$ are defined, then the chain $\gamma(\kappa)$ is a cycle representing $[\gamma(\kappa)]$.   (3.4.4)

In this case we will say that the *class $[\gamma(\kappa)]$ is flat with respect to κ, and $\gamma(\kappa)$ is a flat chain representative.*

Assume that $[\gamma(\kappa)]$ is flat. Assume that $[\gamma(\kappa)]$ has a representing chain $\gamma(\kappa)$ of the form (3.4.1)–(3.4.4) and such that

$(c_1, \alpha_1(\kappa))$ has the form $(d(z^0), \varepsilon_1(\kappa)\beta(\kappa)|_{d(z^0)})$, where $(d(z^0), \beta(\kappa)|_{d(z^0)})$ is described in (3.3.5) and $\varepsilon_1(\kappa)$ is a rational function of $q(\kappa)^{\frac{1}{4}}$.   (3.4.5)

For $\kappa = is$, $s \in \mathbb{R}_{>0}$, and for every $j = 2, \ldots, M$, we have

$$\lim_{s \to +0} \sup_{c_j} |\alpha_j(\kappa)| / |\varepsilon_1(\kappa)\beta(\kappa)(t(z^0), z^0)| = 0 . \qquad (3.4.6)$$

In this case we will say that *the class $[\gamma(\kappa)]$ sits on the t-critical point* $(t(z^0), z^0)$.

Assume that $[\gamma(\kappa)]$ sits on $(t(z^0), z^0)$. Let $[\gamma(\kappa)] \in H_k(\mathcal{U}_{k,n}(z), \mathcal{S}(\kappa))$ be the class such that for a fixed κ the map $z \mapsto [\gamma(\kappa)]$ forms a flat section of the Gauss–Manin connection over a neighborhood of z^0 and $[\gamma(z^0, \kappa)] = [\gamma(\kappa)]$.

(3.4.7). Theorem. *Under the above conditions consider the* $\mathrm{Sing}L_\lambda$*-valued function*

$$\Psi(z, \kappa) = \int\limits_{[\gamma(z, \kappa)]} N ,$$

which for a fixed κ gives a solution to the KZ equation with parameter κ. Assume that $\kappa = is$, $s \in \mathbb{R}$, and $s \to +0$. Then

1. *The function Ψ has an asymptotic expansion*

$$\Psi(z, is) \sim F(\kappa)\exp(-iS(t(z), z)/s) \sum_{j=0}^{\infty} g_j(z)s^j ,$$

where $\{g_j\}$ are $\mathrm{Sing}L_\lambda$*-valued holomorphic functions defined in a neighborhood of z^0.*

2. *The function Ψ gives a quasiclassical aysmptotic solution to the KZ equation in the sense of* (2.5).

3. $g_0(z)$ *is given by* (3.3.7.3).

The theorem is a corollary of (3.3.7).

Consider the natural action of S_k on singular chains and homology classes. Let $[\gamma(z,\kappa)]$ be as above. Define a new skew symmetric class $[\gamma(z,\kappa)]_- \in H_k(\mathcal{U}_{k,n}(z), \mathcal{S}(\kappa))_-$ by

$$[\gamma(z,\kappa)]_- = \sum_{\sigma \in S_k} (-1)^{|\sigma|} \sigma[\gamma(z,\kappa)]. \qquad (3.4.8)$$

We have

$$\int_{[\gamma(z,\kappa)]_-} N = k! \int_{[\gamma(z,\kappa)]} N. \qquad (3.4.9)$$

(3.4.10). Corollary. *The intergral in* (3.4.9) *has an asymptotic expansion as $\kappa = is$, $s \in \mathbb{R}$, and $s \to +0$. The asymptotic expansion is equal to the asymptotic expansion in* (3.4.7.1) *multiplied by $k!$.*

We will say that $[\gamma(z,\kappa)]_-$ sits on the S_k orbit of the t-critical point $(t(z),z)$.

(3.5). Example of a Bethe Vector. Let

$$\Phi_{k,m_1,m_2,\kappa}(t) = \prod_{j=1}^{k} (-t_j)^{-\frac{m_1}{\kappa}}(1-t_j)^{-\frac{m_2}{\kappa}} \cdot \prod_{1 \leqq j < l \leqq k} (t_j - t_l)^{2/\kappa},$$

$$S(t) = S_{k,m_1,m_2}(t,\kappa) = \kappa \ln\Phi(t). \qquad (3.5.1)$$

(3.5.2). Theorem, [V2], cf. [Sz, Sect. 6.7].

1. *If $t^0 = (t_1,\ldots,t_k)$ is a critical point of Φ, then the symmetric functions*

$$\lambda_1 = t_1 + \cdots + t_k, \quad \lambda_2 = \sum t_l t_j, \quad \ldots, \quad \lambda_k = t_1 \cdot \cdots \cdot t_k$$

are given by

$$\lambda_l = \binom{k}{l} \prod_{j=1}^{l} \frac{(m_1 + j - k)}{(m_1 + m_2 + j + 1 - 2k)}$$

for all l.

2. *The Bethe vector is $\mathrm{Sing}(L(m_1) \otimes L(m_2))_{m_1+m_2-2k}$ corresponding to the critical point t^0 is given by*

$$\sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{\prod_{j=0}^{k-1}(m_1 + m_2 - 2k + j + 2)}{\prod_{j=0}^{p-1}(m_1 - j)\prod_{j=0}^{k-p-1}(m_2 - j)} f^p v_1 \otimes f^{k-p} v_2.$$

3. $\qquad \mathrm{Hess}(S(t^0)) = k! \prod_{j=0}^{k-1} \frac{(m_1 + m_2 - 2k + j + 2)^3}{(m_1 - j)(m_2 - j)}.$

(3.6). Asymptotics of Critical Points and Bethe Vectors. In (2.2) we have constructed asymptotic zones. In this section we will describe asymptotics of t-critical points and Bethe vectors in an asymptotic zone, cf. [RV].

Asymptotic zones are numerated by n-trees and elements of the symmetric group S_n. We will consider the case of the identity permutation. The case of an arbitrary permutation is treated similarly.

For an n-tree T consider the asymptotic zone

$$u_{T, \mathrm{id}} = \{z_1 + \cdots + z_n, \{u_{w, T, \mathrm{id}}\}_{w \in \mathrm{In}_T}\} : D_{\mathrm{id}} \to \mathbb{R} \times (\mathbb{R}_{>0})^{n-1} . \tag{3.6.1}$$

Consider the function $\Phi(t, z)$ defined by (3.1.1).

Mark the tops of T by $m_1, \ldots, m_n$. Let the set Adm be the set of all admissible colorings of T having weight $\lambda = m_1 + \cdots + m_n - 2k$, see (1.2).

(3.6.2). Theorem. *Assume that $z \in D_{\mathrm{id}}$ and $u_{w, T, \mathrm{id}}(z) \ll 1$ for all $w \in \mathrm{In}_T$. Then we have the following three statements.*

1. *The number of S_k orbits of nondegenerate t-critical points of $\Phi(t, z)$ is equal to $\dim \mathrm{Sing} L_\lambda$, see* [RV].

2. *The S_k orbits of nondegenerate t-critical points can be numerated by elements of Adm in such a way that the S_k-orbit corresponding to a coloring $c \in \mathrm{Adm}$ has a t-critical point $(t^c(z), z)$ of the following form:*

Let $\{M_w\}_{w \in \mathrm{In}_T}$ be an arbitrary partition of $\{1, \ldots, k\}$ into a union of disjoint subsets such that $\#M_w = c(w)$. For each M_w, fix its arbitrary ordering $M_w = \{j_1(w), \ldots, j_{c(w)}(w)\}$. For any $w \in \mathrm{In}_T$ define a number $l(w)$ as follows. The tops of the tree T are numerated by $1, \ldots, n$ from left to right. The number $l(w)$ is the maximum of the indices of the tops lying on the left branch of the tree T at w. Let $m^{le}(w)$ and $m^r(w)$ be the numbers defined in (1.2) for a vertex w of a colored tree T with marked tops. Consider the function $S_{c(w), m^{le}(w), m^r(w)}$ defined by (3.5.1). Let $(a_1(w), \ldots, a_{c(w)}(w))$ be coordinates of a critical point of the function $S_{c(w), m^{le}(w), m^r(w)}$. Then the critical point $(t^c(z), z)$ has the form:

$$t^c_{j_p(w)}(z) - z_{l(w)} = (a_p(w) + \mathcal{O}(u_{T, \mathrm{id}}(z))) \prod_y u_{y, T, \mathrm{id}} .$$

The product is over all internal vertices y lying on the shortest path in T connecting w and the root of T (the vertex w is included). The function $\mathcal{O}$ is a function of $u_{x, T, \mathrm{id}}$, $x \in \mathrm{In}_T$, holomorphic in a neighborhood of the set $\{u_{x, T, \mathrm{id}} = 0, x \in \mathrm{In}_T\}$, and such that $\mathcal{O}(u = 0) = 0$.

3. *The Bethe vector corresponding to $(t^c(z), z)$ has the following form:*

$$g(t^c(z), z) = (v(c, 0, T, \mathrm{id}) + \mathcal{O}(u(z))) \prod_{w \in \mathrm{In}_T} (u_{w, T, \mathrm{id}})^{-c(w) - b(w)} ,$$

where $b(w) = \sum_y c(y)$, the sum is over $y \in \mathrm{In}_T$ such that y lies on the right or on the left branch of T at w. The function $\mathcal{O}(u)$ has the same structure as in (3.6.2.2).

To prove the theorem we consider new variables $\{x_{j_p(w)}, u_{T, \mathrm{id}}\}$ defined by

$$t_{j_p(w)} - z_{l(w)} = x_{j_p(w)} \prod_y u_{y, T, \mathrm{id}} .$$

$$z_m = z_m(u_{T, \mathrm{id}}) , \tag{3.6.3}$$

where the product is the same as in (3.6.2.2), and the second formula is determined by (3.6.1). It is easy to see that

$$\Phi(t(x, u), z(u)) = (-1)^{A/\kappa} \prod_{w \in \mathrm{In}_T} (u_{w,\, T,\, \mathrm{id}})^{\mu(c,\, T,\, \mathrm{id},\, w)/\kappa}$$

$$\cdot \left(\prod_{w \in \mathrm{In}_T} \Phi_{c(w),\, m^l(w),\, m^r(w),\, \kappa}(\{x_{j_p(w)}\}) + \mathcal{O}(u) \right), \quad (3.6.4)$$

where A is some constant, $\{\mu(c, T, id, w)\}$ are the eigenvalues of the vector $v(c, 0, T, id)$ with respect to the operators $\{\Omega_{w,\, T,\, \mathrm{id}}\}$, see (2.3.1), $\mathcal{O}$ is a function of x, u holomorphic in a neighborhood of the set

$$X = \{x_{j_p(w)} = a_p(w), u_{w,\, T,\, \mathrm{id}} = 0 \quad \text{for all } w, p\}, \quad (3.6.5)$$

and which is zero on X. This statement implies (3.6.2.2) and (3.6.2.3). Statement (3.6.2.1) is proved in [RV].

(3.6.6) *Remark.* There are integral representations for solutions to the KZ equation with values in a tensor product of modules over an arbitrary Kac–Moody Lie algebra $\mathfrak{g}$, see [SV, V1]. There are obvious $\mathfrak{g}$ analogs of (3.3.7), (3.4.4) and (3.6.2), cf. [RV].

4. Integral Representations for Solutions to the KZ Equation and the Quantum Group U_q

(4.1). *Complex* [*V1, Sect. 5*]. For $\kappa \in \mathbb{C}$, consider the $\mathbb{C}$-algebra $U_{q=q(\kappa)}$. For $k > 0$, let

$$(U^-_{q=q(\kappa)})_k = \mathbb{C} f^k \subset U_{q=q(\kappa)} \quad \text{and} \quad U^-_{q=q(\kappa)} = \bigoplus_{k>0} \mathbb{C} f^k.$$

The map $\mu\colon U^- \to U^- \otimes U^-$,

$$\mu\colon f^k \mapsto \sum_{l=1}^{k-1} \binom{k}{l}_{q(\kappa)} f^l \otimes f^{k-l} \quad (4.1.1)$$

defines a coalgebra structure on U^-, here

$$\binom{k}{l}_{q(\kappa)} = \frac{(k)_{q(\kappa)}!}{(l)_{q(\kappa)}!(k-l)_{q(\kappa)}!}$$

is the q-binomial coefficient.

For a $U_{q=q(\kappa)}$-module M with highest weights, the map $v\colon M \to U^-_{q=q(\kappa)} \otimes M$,

$$v\colon u \mapsto \sum_{l>0} f^l \otimes \frac{(q^{-\frac{h}{4}} e)^l}{[l]_{q(\kappa)}!} u, \quad (4.1.2)$$

defines a $U^-_{q=q(\kappa)}$-comodule structure on M.

For $k \geqq 0$, set

$$C^k(M) = (U^-_{q=q(\kappa)})^{\otimes k} \otimes M.$$

Define $d: C^k \to C^{k+1}$ by

$$d: a_k \otimes \cdots \otimes a_1 \otimes u \mapsto - a_k \otimes \cdots \otimes a_1 \otimes v(u)$$

$$+ \sum_{l=1}^{k} (-1)^{l+1} a_k \otimes \cdots \otimes \mu(a_l) \otimes \cdots \otimes a_1 \otimes u \, .$$

$$(4.1.3)$$

For any λ and k, set

$$C_\lambda^k(M) = \bigoplus_{l_1, \ldots, l_k} (U^-)_{l_1} \otimes \cdots \otimes (U^-)_{l_k} \otimes M_{\lambda - l_1 - \cdots - l_k} \, . \qquad (4.1.4)$$

Then $C^k = \bigoplus_\lambda C_\lambda^k$. The differential d preserves the grading. For any λ we have a complex $(C_\lambda^\bullet(M), d)$ and

$$H^0(C_\lambda^\bullet, d) = \ker e \cap M_\lambda \qquad (4.1.5)$$

for a nonrational κ. Assume that κ is not a rational number.

For an integer m, let $V(m, q = q(\kappa))$ (resp. $L(m, q = q(\kappa))$) be the $U_{q=q(\kappa)}$ Verma module (resp. irreducible module) with highest weight m.

For any $\sigma \in S_n$, $\lambda \in \mathbb{Z}$, and natural $m_1, \ldots, m_n$, set

$$V^\sigma(q = q(\kappa)) = V(m_{\sigma(1)}, q = q(\kappa)) \otimes \cdots \otimes V(m_{\sigma(n)}, q = q(\kappa)) \, ,$$

$$\mathrm{Sing}\, V^\sigma(q = q(\kappa))_\lambda = (V^\sigma)_\lambda \cup \ker e \, . \qquad (4.1.6)$$

Let $L^\sigma(q = q(\kappa))$ and $\mathrm{Sing}\, L^\sigma(q = q(\kappa))_\lambda$ be the corresponding objects constructed from the irreducible modules, Denote by

$$pr: \mathrm{Sing}\, V^\sigma(q = q(\kappa))_\lambda \to \mathrm{Sing}\, L^\sigma(q = q(\kappa))_\lambda \qquad (4.1.7)$$

the natural epimorphism.

For any $j = 1, \ldots, n - 1$, we have an isomorphism

$$R_{j, j+1}: (U^-)^{\otimes k} \otimes V^\sigma \to (U^-)^{\otimes k} \otimes V^{\sigma \tau} \, , \qquad (4.1.8)$$

where $\tau \in S_n$ is the transposition of j and $(j + 1)$. $R_{j, j+1}$ commutes with the differential and induces an isomorphism of cohomology groups of the corresponding complexes.

(4.2). The Main Result. Assume that $\kappa \in \mathbb{C}$ is not a rational number. Consider the function $\Phi(t, z)$ defined by (3.1.1) and the objects associated with the function $\Phi(t, z)$ in (3.1) and (3.2).

For $z \in \mathcal{U}_n$, let $\mathscr{C}_\bullet(\mathcal{U}_{k,n}(z), \mathscr{S}(\kappa))$ be the complex of singular chains in $\mathcal{U}_{k,n}(z)$ with coefficients in $\mathscr{S}(\kappa)$. The group S_k of permutations of corrdinates $t_1, \ldots, t_k$ acts on $\mathscr{C}_\bullet$. Denote by $\mathscr{C}_\bullet(\mathcal{U}_{k,n}(z), \mathscr{S}(\kappa))_-$ the skew symmetric part of the action.

(4.2.1). Theorem. *For any $\sigma \in S_n$ and any $z \in D_\sigma$ there exists a monomorphism of complexes*

$$v(z, \kappa): C_\lambda^\bullet(V^\sigma(q = q(\kappa))) \to C_{k-\bullet}(\mathcal{U}_{k,n}(z), \mathscr{S}(\kappa))_- \, ,$$

where $\lambda = m_1 + \cdots + m_n - 2k$. *Denote the image of* v *by*

$$C_\bullet(\mathcal{U}_{k,n}(z), \mathcal{S}(\kappa))_- \subset \mathcal{C}_\bullet(\mathcal{U}_{k,n}(z), \mathcal{S}(\kappa))_- .$$

The monomorphism has the following six properties:

1. $v(z, \kappa)$ *is a quasiisomorphism. Denote by the same symbol* $v(z, \kappa)$ *the induced isomorphism of homology groups:* $H^\bullet(C_\lambda^\bullet(V^\sigma(q = q(\kappa))), d) \cong H_{k-\bullet}(\mathcal{U}_{k,n}(z), \mathcal{S}(\kappa))_-.$

2. $v(z, \kappa)$ *is flat with respect to the Gauss–Manin connection. Namely, if*

$$P(z^1, z^2): H_\bullet(\mathcal{U}_{k,n}(z^1), \mathcal{S}(\kappa))_- \to H_\bullet(\mathcal{U}_{k,n}(z^2), \mathcal{S}(x))_-$$

is the isomorphism of the Gauss–Manin connection along a curve in D_σ *from a point* $z^1 \in D_\sigma$ *to a point* $z^2 \in D_\sigma$, *then* $v(z^2, \kappa) = P(z^1, z^2) \cdot v(z^1, \kappa)$.

3. *Let* τ *be the transposition of* j *and* $j + 1$ *for* $j = 1, \ldots, n - 1$. *Connect the domains* D_σ *and* $D_{\sigma\tau}$ *by a curve* $\gamma: [0, 1] \to \mathcal{U}_n$ *of the following form. Let* $(z_1^0, \ldots, z_n^0) \in D_\sigma$, $z_{\sigma(1)}^0 < \cdots < z_{\sigma(n)}^0$. *Assume that* $z_{\sigma(j)}$ *and* $z_{\sigma(j+1)}$ *are very close. Set* $z_l(\gamma(s)) = z_l^0$ *for* $l \neq \sigma(j+1)$, *set* $z_{\sigma(j+1)}(\gamma(s)) = \exp(\pi i s)(z_{\sigma(j+1)}^0 - z_{\sigma(j)}^0) + z_{\sigma(j)}^0$. *Let*

$$P_\gamma: H_\bullet(U_{k,n}(\gamma(0)), \mathcal{S}(\kappa))_- \to H_\bullet(\mathcal{U}_{k,n}(\gamma(1)), \mathcal{S}(\kappa))_-$$

be the isomorphism of the Gauss–Manin connection along γ. *Let*

$$R_{j,j+1}: H^\bullet(C_\lambda^\bullet(V^\sigma(q = q(\kappa)), d)) \to H^\bullet(C_\lambda^\bullet(V^{\sigma\tau}(q = q(\kappa)), d))$$

be the R-matrix isomorphism. Then $P_\gamma \cdot v(\gamma(0), \kappa) = v(\gamma(1), \kappa) \cdot R_{j,j+1}$.

4. *Consider the KZ equation with parameter* κ *and with values in* $\mathrm{Sing}\, L_\lambda$, *see* (3.2). *Let* $\mathrm{Sol}(\kappa)_{\lambda,\sigma}$ *be the space of solutions to the KZ equation over* D_σ. *By* (4.2.1.1) *and* (4.2.1.2), *for any* $v \in \mathrm{Sing}\, V^\sigma(q = q(\kappa))_\lambda$, *the function*

$$\psi_v: D_\sigma \to \mathrm{Sing}\, L_\lambda, \qquad z \mapsto \int\limits_{v(z,\kappa)v} N \tag{4.2.2}$$

is a solution to the KZ equation. Here N *is defined by* (3.2.3). *Hence, by* (4.2.1.1) *and* (4.2.1.2), *we have a homomorphism*

$$\mathcal{V}_\sigma(\kappa): \mathrm{Sing}\, V^\sigma(q = q(\kappa))_\lambda \to \mathrm{Sol}(\kappa)_{\lambda,\sigma} . \tag{4.2.3}$$

Let $\mathrm{pr}: \mathrm{Sing}\, V^\sigma(q = q(\kappa))_\lambda \to \mathrm{Sing}\, L^\sigma(q = q(\kappa))_\lambda$ *be the canonical epimorphism. We claim that* $\ker \mathrm{pr} \subset \ker \mathcal{V}_\sigma(\kappa)$. *Hence we have a homomorphism*

$$\pi_\sigma(\kappa): \mathrm{Sing}\, L^\sigma(q = q(\kappa))_\lambda \to \mathrm{Sol}(\kappa)_{\lambda,\sigma} . \tag{4.2.4}$$

5. *For an n-tree* T, *let* $\mathcal{B}_{T,\sigma}(\kappa) = \{v(c, 0, T, \sigma, \kappa)\}$ *be the basis in* $\mathrm{Sing}\, L^\sigma(q = q(\kappa))_\lambda$ *corresponding to the tree* T *with tops marked by* $m_\sigma(1), \ldots, m_\sigma(n)$, *see* (1.2). *Then for any vector* $v \in \mathcal{B}_{T,\sigma}(\kappa)$ *we have*

$$\pi_\sigma(\kappa)(v(c, 0, T, \sigma, \kappa)) = \psi_{c,T,\sigma} .$$

Here $\psi_{c,T,\sigma}$ *is the asymptotic solution defined by* (2.3.1).

Corollary. $\pi_\sigma(\kappa)$ *is an isomorphism.*

6. *Let* $u_{T,\sigma}$ *be the asymptotic zone constructed in* (2.2). *If* $z \in D_\sigma$ *and* $\{u_{w,T,\sigma}(z)\}_{w\in \ln_T}$ *are small enough, then for any* $v(c, 0, T, \sigma, \kappa) \in \mathcal{B}_{T,\sigma}$ *the class*

$v(z,\kappa)v(c,0,T,\sigma,\kappa)$ sits on the S_k-orbit of the t-critical point $(t^c(z),z)$ corresponding to the coloring c, see (3.6.2). Moreover, the asymptotic expansion (3.4.10), (3.4.7.1), and (3.3.7.1) of $\pi_\sigma(\kappa)v(c,0,T,\sigma,\kappa)$ has the form (2.5.8) and properties described in (2.5.7) and (2.5.9).

Theorem (4.2.1) implies (2.4.1), (2.5.7), and (2.5.9). Theorem (4.2.1) is proved in (4.4)–(4.6).

(4.3). *Another Form of $U_q sl_2$.* Define the $\mathbb{Q}(q^{\frac{1}{4}})$-algebra $\widetilde{U}_q$ as the algebra generated by the symbols E, F, H with the relations

$$[H,E] = 2E ,$$
$$[H,F] = -2F ,$$
$$[E,F] = q^{H/2} - q^{-H/2} . \tag{4.3.1}$$

$\widetilde{U}_q$ has a Hopf algebra structure with the comultiplication Δ defined by

$$\Delta(H) = H \otimes 1 + 1 \otimes H ,$$
$$\Delta(E) = E \otimes q^{H/4} + q^{-H/4} \otimes E ,$$
$$\Delta(F) = F \otimes q^{H/4} + q^{-H/4} \otimes F . \tag{4.3.2}$$

Let $\widetilde{U}_{q=q(\kappa)}$ be the $\mathbb{C}$-algebra obtained from $\widetilde{U}_q$ by specializing the indeterminate $q^{1/4}$ to $q(\kappa/4)$.

For $k > 0$, set $(\widetilde{U}^-)_k = \mathbb{C}F^k$ and $\widetilde{U}^- = \bigoplus_{k>0} \mathbb{C}F^k$. The map $\tilde{\mu} \colon \widetilde{U}^- \to \widetilde{U}^- \otimes \widetilde{U}^-$ given by (4.1.1), in which f is replaced by F, defines a coalgebra structure. For a $\widetilde{U}_{q=q(\kappa)}$-module $\tilde{M}$ with highest weights, the map

$$\tilde{v} \colon u \mapsto \sum_{l>0} F^l \otimes E^{(l)}u \tag{4.3.3}$$

defines a $\widetilde{U}^-$-comodule structure on $\tilde{M}$. For any λ define a complex $(\widetilde{C}_\lambda^\bullet(\tilde{M}), \tilde{d})$ as in (4.1).

The Hopf algebras U_q and $\widetilde{U}_q$ are isomorphic, an isomorphism $\pi \colon U_q \to \widetilde{U}_q$ is given by $\pi(h) = H$, $\pi(e) = q^{H/4}E$, $\pi(f) = Fq^{-H/4}$. In particular, these formulae give an isomorphism of $U_{q=q(\kappa)}$ and $\widetilde{U}_{q=q(\kappa)}$.

Assume that κ is not a rational number.

For an integral m, let $V(m, q = q(\kappa))$ be the $U_{q=q(\kappa)}$ Verma module with the highest weight m, the generating vector v_m, and the basis $f^{(l)}v_m$, $l \geq 0$. Let $\widetilde{V}(m, q = q(\kappa))$, $\tilde{v}_m$, $F^{(l)}\tilde{v}_m$ be the corresponding objects over $\widetilde{U}_{q=q(\kappa)}$. Let

$$\pi_m \colon V(m, q = q(\kappa)) \to \widetilde{V}(m, q = q(\kappa)), \quad f^{(l)}v_m \mapsto \frac{(Fq^{-H/4})^l}{[l]_{q(\kappa)}!} \tilde{v}_m .$$

Then for any $a \in U_{q=q(\kappa)}$ and $u \in V(m, q = q(\kappa))$ we have $\pi_m(au) = \pi(a)\pi_m(u)$.

For natural $m_1, \ldots, m_n$ and $\sigma \in S_n$, let

$$V^\sigma(q = q(\kappa)) = V(m_{\sigma(1)}, q = q(\kappa)) \otimes \cdots \otimes V(m_{\sigma(n)}, q = q(\kappa))$$

and
$$\widetilde{V}^\sigma(q = q(\kappa)) = \widetilde{V}(m_{\sigma(1)}, q = q(\kappa)) \otimes \cdots \otimes \widetilde{V}(m_{\sigma(n)}, q = q(\kappa)).$$

For any k and λ, define a map

$$\Psi \colon C_\lambda^k(V^\sigma(q = q(\kappa))) \to \widetilde{C}_\lambda^k(\widetilde{V}^\sigma(q = q(\kappa))), \qquad (4.3.4)$$

$$f^{l_1} \otimes \cdots \otimes f^{l_k} \otimes u_1 \otimes \cdots \otimes u_n \mapsto F^{l_1} \otimes \cdots \otimes F^{l_k} \otimes \pi_{m_{\sigma(1)}}(u_1) \otimes \cdots \otimes \pi_{m_{\sigma(n)}}(u_n).$$

Then

$$\Psi \text{ defines an isomorphism of } C_\lambda^\bullet(V^\sigma(q = q(\kappa))) \text{ and } \widetilde{C}_\lambda^\bullet(\widetilde{V}^\sigma(q = q(\kappa))). \qquad (4.3.5)$$

For a nonnegative integer m denote by $\widetilde{L}(m, q = q(\kappa))$ the irreducible $\widetilde{U}_{q=q(\kappa)}$-module with highest weight m. Set

$$\text{Sing}\, \widetilde{V}(q = q(\kappa))_\lambda = \{v \in \widetilde{V}^\sigma(q = q(\kappa)) | q^H v = q^\lambda(\kappa)v, \ Ev = 0\},$$

$$\text{Sing}\, \widetilde{L}^\sigma(q = q(\kappa))_\lambda = \{v \in \widetilde{L}^\sigma(q = q(\kappa)) | q^H v = q^\lambda(\kappa)v, \ Ev = 0\}.$$

(4.4). Construction of the Monomorphism v. Assume that $\kappa, m_1, \ldots, m_n, \lambda, k$ are the same as in (4.2).

In [V1] a monomorphism

$$\tilde{v}(z, \kappa) \colon \widetilde{C}_\lambda^\bullet(\widetilde{V}^\sigma(q = q(\kappa))) \to \mathscr{C}_{k-\bullet}(\mathscr{U}_{k,n}(z), \mathscr{S}(\kappa))_- \qquad (4.4.1)$$

is constructed for any $\sigma \in S_n$ and any $z \in D_\sigma$. Denote the image of the monomorphism by $\widetilde{C}_\bullet(\mathscr{U}_{k,n}(z), \mathscr{S}(\kappa))_- \subset \mathscr{C}_\bullet(\mathscr{U}_{k,n}(z), \mathscr{S}(\kappa))_-$. It is proved in [V1] that the monomorphism has the following four properties:

$\tilde{v}(z, \kappa)$ is a quasiisomorphism, cf. (4.2.1.1) and [V1, Sect. 8.2]. Denote by the same symbol $\tilde{v}(z, \kappa)$ the induced isomorphism in homology groups.   (4.4.2)

$\tilde{v}(z, \kappa)$ is flat with respect to the Gauss–Manin connection, cf. (4.2.1.2) and [V1, Sect. 8.2].   (4.4.3)

$\tilde{v}(z, \kappa)$ transforms the R-matrix action on $\widetilde{C}_\lambda^\bullet(\widetilde{V}^\sigma(q = q(\kappa)))$ into the monodromy action on $\widetilde{C}_{k-\bullet}(\mathscr{U}_{k,n}(z), \mathscr{S}(\kappa))_-$, cf. the precise statement in (4.2.1.3) and [V1, Sect. 8.14].   (4.4.4)

For any $v \in \text{Sing}\, \widetilde{V}^\sigma(q = q(\kappa))_\lambda$, the function

$$\tilde{\psi} \colon D_\sigma \to \text{Sing}\, L_\lambda, \qquad z \mapsto \int_{\tilde{v}(z, \kappa)v} N, \qquad (4.4.5)$$

is a solution to the KZ equation. By (4.4.3) we have a homomorphism

$$\widetilde{\mathscr{V}}_\sigma(\kappa) \colon \text{Sing}\, \widetilde{V}(q = q(\kappa))_\lambda \to \text{Sol}(\kappa)_{\lambda, \sigma}.$$

Let $pr \colon \text{Sing}\, \widetilde{V}(q = q(\kappa))_\lambda \to \text{Sing}\, \widetilde{L}(q = q(\kappa))_\lambda$ be the canonical projection. It is proved in [V1, Sect. 12.2] that $\ker pr \subset \ker \widetilde{\mathscr{V}}$. Hence, we have a homomorphism

$$\tilde{\pi}_\sigma \colon \text{Sing}\, \widetilde{L}^\sigma(q = q(\kappa))_\lambda \to \text{Sol}(\kappa)_{\lambda, \sigma}.$$

Let $\Psi: C_{\lambda}^{k}(V^{\sigma}(q = q(\kappa))) \rightarrow \widetilde{C}_{\lambda}^{k}(\widetilde{V}^{\sigma}(q = q(\kappa)))$ be the isomorphism constructed in (4.3.4). We set

$$v(z, \kappa) = (2\pi\kappa)^{-\frac{k}{2}} \widetilde{v}(z, \kappa)\Psi . \qquad (4.4.6)$$

Obviously, this monomorphism has properties (4.2.1.1)–(4.2.1.4). We will prove that the monomorphism has properties (4.2.1.5) and (4.2.1.6).

(4.5). *Proof of Theorem* (4.2.1) *for* $n = 2$. For $n = 2$ we have

$$\Phi(t, z) = (z_2 - z_1)^{m_1 m_2/\kappa} \prod_{1 \leqq l < j \leqq \kappa} (t_l - t_j)^{2/\kappa} \prod_{l=1}^{k} \prod_{j=1}^{2} (z_j - t_l)^{-m_j/\kappa} . \qquad (4.5.1)$$

There are two permutations in S_2: the identity permutation *id* and permutation $(2,1)$. We will prove properties (4.2.1.5) and (4.2.1.6) for D_{id}. There is only one 2-tree T, and therefore, there is only one asymptotic zone in D_{id}:
$u_{T,\mathrm{id}} = \{z_1 + z_2, \ u = z_2 - z_1\}$.
Introduce the following new coordinates $x_1, \ldots, x_l$:

$$t_l - z_1 = u \cdot x_l, \quad l = 1, \ldots, k ,$$

cf. (3.6.3). Then

$$\Phi(t(x, u), z(u)) = (-1)^{m_1 m_2/2\kappa} u^{(\frac{m_1 m_2}{2} - k(m_1 + m_2) + k(k-1))/\kappa} \Phi_{k, m_1, m_2, \kappa}(x) , \qquad (4.5.2)$$

where the function $\Phi_{k, m_1, m_2, \kappa}$ is given by (3.5.1).

Let $\Delta = \{x \in \mathbb{R}^k | 0 < x_1 < \cdots < x_k < 1\}$.

Let $v_{m_j} \in L(m_j)$ be the generating vector of the sl_2 irreducible module. Let $\widetilde{v}_{m_j} \in \widetilde{L}(m_j, q = q(\kappa))$ be the generating vector of the $\widetilde{U}_{q=q(\kappa)}$ irreducible module. Let

$$v = \sum_{p=0}^{k} (-1)^p \frac{(m_2 - k + 1)_{q(\kappa)} \ldots (m_2 - k + p)_{q(\kappa)}}{(m_1)_{q(\kappa)} \ldots (m_1 - p + 1)_{q(\kappa)}}$$

$$\times q(\kappa)^{-p(m_2 - 2k + p + 1)/2} F^{(p)} \widetilde{v}_{m_1} \otimes F^{(k-p)} \widetilde{v}_{m_2} \qquad (4.5.3)$$

be the generating vector of $\mathrm{Sing}\widetilde{L}_{\lambda}^{\mathrm{id}}(q = q(\kappa))$. Set $\gamma(z) = \widetilde{v}(z, \kappa)v \in H_k(\mathcal{U}_{2,k}(z), \mathcal{S}(\kappa))_-$ for $z \in D_{\mathrm{id}}$.

According to the explicit construction of $v(z, \kappa)$ [V1], we have

$$\int_{\gamma(z)} N(t, z) = q(\kappa)^{-\frac{m_2}{4} + \frac{k(k-1)}{4}} (m_2)_{q(\kappa)} \ldots (m_2 - k + 1)_{q(\kappa)} \cdot u^{(\frac{m_1 m_2}{2} - k(m_1 + m_2) + k(k-1))/\kappa}$$

$$\times k! \int_{\Delta} \prod_{j=1}^{k} x_j^{-m_1/\kappa} (1 - x_j)^{-m_2/\kappa} \prod_{1 \leqq j < l \leqq k} (x_l - x_j)^{2/\kappa} A(x) dx_1 \wedge \cdots \wedge dx_k$$

$$\cdot f^l v_{m_1} \otimes f^{k-l} v_{m_2} ,$$

and

$$A_l(x) = \sum_{\sigma \in S(k; l, k-l)} \prod_{i=1}^{k} \frac{1}{x_l - y_{\sigma(i)}} , \qquad (4.5.3)$$

where $y_1 = 0$, $y_2 = 1$, and $S(k; l, k - l)$ is defined in (3.2.2).

The value of the integral over Δ is considered in the sense of analytic continuation from the domain of parameters where κ is positive and m_1, m_2 are negative.

This formula is the decisive property of $\tilde{v}(z, \kappa)$ which allows us to prove (4.2.1.5) and (4.2.1.6).

Consider the single basic vector $v(c, 0, T, id, \kappa) \in \mathscr{B}_{T, \mathrm{id}}(\kappa)$ of the one dimensional space $\mathrm{Sing} L^{\mathrm{id}}_\lambda(q = q(\kappa))$. (4.4.6) implies that

$$\int_{v(z, \kappa)v(c, 0, T, \mathrm{id}, \kappa)} N = (2\pi\kappa)^{-k/2} q(\kappa)^{-\frac{m_2}{2} + \frac{3}{4}k(k-1)}$$

$$\times u^{(\frac{m_1 m_2}{2} - k(m_1 + m_2) + k(k-1))/\kappa} k! I_k(m_1, m_2; \kappa)$$

$$\times \prod_{j=0}^{k-1} \left(\left(1 - \frac{\kappa}{m_1 + m_2 - 2k + j + 2} \right)(m_2 - j)_{q(\kappa)} \right) \cdot \{v_1, {}_k v_2\},$$

(4.5.4)

where the vector $\{v_1, {}_k v_2\} \in \mathrm{Sing}(L(m_1) \otimes L(m_2))_\lambda$ is given by (1.4.1) and $I_k(m_1, m_2; \kappa)$ is the Selberg integral given by (1.5.1). By (1.5.2) and (1.5.3), we have

$$\int_{v(z, \kappa)v(c, 0, T, \mathrm{id}, \kappa)} N = C(c, T, \kappa) \cdot u^{(\frac{m_1 m_2}{2} - k(m_1 + m_2) + k(k-1))/\kappa} \{v_1, {}_k v_2\}.$$

(4.5.5)

The right-hand side is exactly the asymptotic solution defined by (2.3.1). This proves (4.2.1.5).

Now assume that $\kappa = is$, $s \in \mathbb{R}$, and $s \to +0$. We compute the asymptotic expansion of the right-hand side of (4.5.4) using (1.5.1) and the Stirling formula. This shows that the asymptotic expansion of $\pi_{\mathrm{id}}(\kappa)v(c, 0, T, id, \kappa)$ has the form (2.5.8) and the properties described in (2.5.7) and (2.5.9).

According to the construction in [V1], the class $v(z, \kappa)v(c, 0, T, id, \kappa)$ is flat with respect to κ, see (3.4). The function $\Phi(t, z)$ has exactly one S_k-orbit of t-critical points, see (3.5). To show that the class sits on this orbit we need the following lemma.

(4.5.6). Lemma. *Let $[\gamma(\kappa)] \in H_k(\mathscr{U}_{k, 2}(z), \mathscr{S}(\kappa))$ be a class flat with respect to κ. Then $[\gamma(\kappa)]$ sits on the orbit of t-critical points of $\Phi(t, z)$ or for any $P > 0$ there exist a chain*

$$\gamma(\kappa) = \sum_{j=1}^{M} (c_j, \alpha_j(\kappa))$$

representing $[\gamma(\kappa)]$, flat with respect to κ, and such that, for every $j = 1, \ldots, m$, we have

$$\lim_{\substack{\kappa = is \\ s \to +0}} q(\kappa)^P \sup_{c_j} |\alpha_j(\kappa)| = 0.$$

(4.5.7)

(4.5.8). Corollary. *If $[\gamma(\kappa)]$ does not sit on the $\mathscr{S}_k$ orbit then*

$$\lim_{\substack{\kappa = is \\ s \to +0}} q(\kappa)^P \int_{[\gamma(\kappa)]} N = 0$$

for all P.

Proof. First, remark that Φ^κ is a rational function on $\mathscr{U}_{k, 2}(z)$.

Now consider the compactification of $\mathcal{U}_{k,2}(z)$ in $(\mathbb{C}P^1)^k$. Let $X \subset (\mathbb{C}P^1)^k$ be the divisor of singularities of Φ. A resolution of singularities of $X \subset (\mathbb{C}P^1)^k$ is a proper analytic map $F:(A \subset B) \to (X \subset (\mathbb{C}P^1)^k)$ such that A is a divisor with normal crossings in a nonsingular B and $F|_{B-A}: B - A \to (\mathbb{C}P^1)^k - X$ is a biholomorphism. The condition $k \leqq \min(m_1, m_2)$ implies that there exists a resolution having the following property:

For any point $a \in A$, there exists local coordinates $u_1, \ldots, u_k$ on B centered at a and such that A is defined by equation

$$u_1 \cdot \cdots \cdot u_r = 0$$

for some $r \in \{1, \ldots, k\}$ and

$$\Phi^\kappa \circ F(u) = u_1^{l_1} \cdot \cdots \cdot u_r^{l_r}, \tag{4.5.9}$$

where $l_1 \cdot \cdots \cdot l_r \neq 0$.

Let $S = \kappa \ln \Phi$. Then $S = \sum_{j=1}^r l_j(\ln|u_j| + i \arg u_j)$. Introduce the new real coordinates $v_j = \ln|u_j|$ and $w_j = \arg u_j$. Let $H_1, \ldots, H_r$ be arbitrary real numbers such that $H_1 l_1 + \cdots + H_r l_r = -1$. Set $Y = \sum_j H_j \frac{\partial}{\partial w_j}$. The vector field Y decreases $\operatorname{Im} S$ in a neighborhood of the point a.

To prove the lemma we start with an arbitrary representing chain $\gamma(\kappa)$, flat with respect to κ. Then we deform $\gamma(\kappa)$ into the direction of decrease of $\operatorname{Im} S$. To deform the chain we use local fields $\{Y\}$ constructed above in a neighborhood of X and we use the field $\operatorname{grad}(\operatorname{Im} S)$ in the "finite" part of the $(\mathbb{C}P^1)^k \backslash X$. This procedure will push the chain onto the orbit of t-critical points or will create a new representing chain with property (4.5.7).

The lemma and Theorem (4.2.1) for $n = 2$ are proved.

(4.6). Proof of Theorem (4.2.1). To simplify notation we will prove the theorem for the case $n = 3$, $\sigma = id \in S_3$, and for the 3-tree shown in (4.6.1). The general case is completely similar.

$$\tag{4.6.1}$$

In this case the asymptotic zone has the form $u = u_{T,\text{id}} = \{z_1 + z_2 + z_3, u_1, u_2\}$, where $z_3 - z_2 = u_2$, $z_2 - z_1 = u_1 u_2$. If u_1 and u_2 are small, then $z_2 - z_1 \ll z_3 - z_2$.

Let the tops of T be marked by nonnegative integers m_1, m_2, m_3 and let c be an admissible coloring of T having level $\lambda = m_1 + m_2 + m_3 - 2k$. Set $c_1 = c(w_1)$, $c_2 = c(w_2)$, then $c_1 + c_2 = k$.

Let v_m be the generating vector of the sl_2 module $L(m)$. Let $\bar{v}_m$ be the generating vector of the $U_{q=q(\kappa)}$ module $L(m, q = q(\kappa))$.

Set $v = \{\{v_{m_1}, {}_{c_1} v_{m_2}\}, {}_{c_2} v_{m_3}\}$, $\bar{v} = ((\bar{v}_{m_1}, {}_{c_1} \bar{v}_{m_2}), {}_{c_2} \bar{v}_{m_3})$, see notations in (1.2.1) and (1.4.1)

Consider the solution to the KZ equation given by

$$\psi: z \mapsto \int\limits_{v(z, \kappa)\bar{v}} N, \tag{4.6.2}$$

where $z = (z_1, z_2, z_3)$, $z_1 < z_2 < z_3$. Our problem is to show the following statements (4.6.3) and (4.6.4):

$$\psi(z(u)) = C(c, T, \kappa) \cdot u_1^{\mu_1/\kappa} u_1^{\mu_2/\kappa}(v + \mathcal{O}(u)),$$

$$C(c, T, \kappa) = \frac{k!}{c_1! c_2!} \cdot J_{c_1}(m_1, m_2; \kappa) \cdot J_{c_2}(m^1, m_3; \kappa), \qquad (4.6.3)$$

where $m^1 = m_1 + m_2 - 2c_1$, see (1.5.3), and the real numbers μ_1, μ_2 are defined in (2.3.1).

(4.6.4). Theorem. *Assume that $\kappa = is$, $s \to +0$. Then for small u_1, u_2 the class $v(z, \kappa)\overline{v}$ sits on the orbit of the t-critical point $(t^c(z), z)$ such that*

$$t_p^c(z) - z_1 = (a_p + \mathcal{O}(u))u_1 u_2, \quad p = 1, \ldots, c_1,$$

$$t_{p+c_1}^c(z) - z_2 = (b_1 + \mathcal{O}(u))u_2, \quad p = 1, \ldots, c_2,$$

where $(a_1, \ldots, a_{c_1})$ (resp. $(b_1, \ldots, b_{c_2})$) is a critical point of $\Phi_{c_1, m_1, m_2, \kappa}$ (resp. $\Phi_{c_2, m^1, m_3, \kappa}$), see (3.6.2). Moreover, the asymptotic expansion of $\psi(z(u))$ as $s \to +0$ has the form described in (2.5.6) and the properties described in (2.5.7) and (2.5.9).

Proof. To prove these statements we will choose a special chain representation for the class $v(z, \kappa)\overline{v}$. We will use the construction of iterated cycles described in [V1, Sect. 14].

Consider the space $\mathbb{C}^{2+c_1}$ with coordinates $z_1, z_2, t_1, \ldots, t_{c_1}$. Let

$$pr_I: \mathbb{C}^{2+c_1} \to \mathbb{C}^2, \ (z_1, z_2, t) \mapsto (z_1, z_2),$$

$$\Phi_I = (z_2 - z_1)^{\frac{m_1 m_2}{2\kappa}} \prod_{1 \le l < j \le c_1} (t_l - t_j)^{2/\kappa} \cdot \prod_{l=1}^{c_1} \prod_{j=1}^{2} (z_j - t_l)^{-m_j/\kappa}. \qquad (4.6.5)$$

By (4.4.6) we have a map

$$v_I(z_1, z_2, \kappa) : \mathrm{Sing}(L(m_1, q = q(\kappa)) \otimes L(m_2, q = q(\kappa)))_{m^1 = m_1 + m_2 - 2c_1}$$

$$\to H_{c_1}(pr_I(z_1 z_2)^{-1}, \mathscr{S}_I(\kappa))_-,$$

where $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$, and $\mathscr{S}_I(\kappa)$ is the local system defined by Φ_I.

For $z_1 = 0$, $z_2 = 1$, fix a chain representative γ_I for the class $[\gamma_I(0, 1)] = v_I(0, 1, \kappa)(\overline{v}_{m_1}, c_1 \overline{v}_{m_2})$, flat with respect to κ, cf. (3.4.1).

For fixed numbers z_1^0, z_2^0 the map

$$T: (z_1, z_2, t_1, \ldots, t_{c_1}) \mapsto ((z_2^0 - z_1^0)z_1 + z_1^0, (z_2^0 - z_1^0)z_2 + z_1^0, (z_2^0 - z_1^0)t_1 + z_1^0, \ldots)$$

sends $pr_I^{-1}(0, 1)$ to $p_I^{-1}(z_1^0, z_2^0)$. Hence, the image of γ_I under T gives a representative

$$\gamma_I(z_1^0, z_2^0) = \sum_{j=1}^{M_I} (C_j^I(z_1^0, z_2^0), \alpha_j^I(\kappa)) \qquad (4.6.6)$$

of the class $v_I(z_1^0, z_2^0, \kappa)(\overline{v}_{m_1}, c_1 \overline{v}_{m_2})$ for arbitrary $z_1^0 < z_2^0$. This representation is flat with respect to κ.

Similarly, consider the space $\mathbb{C}^{2+c_2}$ with coordinates $z_2, z_3, t_{c_1+1}, \ldots, t_k$. Let

$$pr_{II} : \mathbb{C}^{2+c_2} \to \mathbb{C}^2, \quad (z_2, z_3, t) \mapsto (z_2, z_3),$$

$$\Phi_{II} = (z_3 - z_2)^{m^1 m_3 / 2\kappa} \prod_{c_1 < l < j \leq k} (t_l - t_j)^{2/\kappa} \prod_{l=c_1+1}^{k} (z_2 - t_l)^{-m^1/\kappa} (z_3 - t_l)^{-m_3/\kappa} .$$

By (4.4.6) we have a map

$$v_{II}(z_2, z_3, \kappa) : \operatorname{Sing}(L(m^1, q = q(\kappa)) \otimes L(m_3, q = q(\kappa))_{m^1 + m_3 - 2c_2}$$

$$\to H_{c_2}(pr_{II}^{-1}(z_2, z_3), \mathscr{S}_{II}(\kappa))_- ,$$

where $z_2, z_3 \in \mathbb{R}$, $z_2 < z_3$, and $\mathscr{S}_{II}(\kappa)$ is defined by Φ_{II}. As above, construct a chain representation

$$\gamma_{II}(z_2, z_3) = \sum_{l=1}^{M_{II}} (C_l^{II}(z_2, z_3), \alpha_l^{II}(\kappa)) \tag{4.6.7}$$

of the class $v_{II}(z_2, z_3)(\bar{v}_{m^1}, {}_{c_2}\bar{v}_{m_3})$ which is flat with respect to κ.

Using the chains γ_I, γ_{II}, we will construct a chain representing $v(z, \kappa)$ $((\bar{v}_{m_1}, {}_{c_1}\bar{v}_{m_2}), {}_{c_2}\bar{v}_{m_3})$.

Let u_1, u_2 be small, then $z_2 - z_1 \ll z_3 - z_2$.

Identify $pr_I^{-1}(z_1, z_2)$ with $\mathbb{C}^{c_1}$ using the coordinates $t_1, \ldots, t_{c_1}$, identify $pr_{II}^{-1}(z_2, z_3)$ with $\mathbb{C}^{c_2}$ using the coordinates $t_{c_1+1}, \ldots, t_k$, and identify $pr_{3,k}^{-1}(z_1, z_2, z_3)$ with $\mathbb{C}^k = \mathbb{C}^{c_1} \times \mathbb{C}^{c_2}$.

Using these identifications, for any $j = 1, \ldots, M_I$, $l = 1, \ldots, M_{II}$, define a k-cell $C_{jl}(z_1, z_2, z_3)$ in $pr_{3,k}^{-1}(z_1, z_2, z_3)$:

$$C_{jl}(z_1, z_2, z_3) = C_j(z_1, z_2) \times C_l(z_2, z_3) .$$

Using α_j^{II} and α_l^{II} we will define a coefficient of C_{jl} in the local system $\mathscr{S}(\kappa)$ defined by the function Φ described in (3.1.1) for $n = 3$.

α_j^I has the form $\varepsilon_j^I \cdot \beta_j^I$, where ε_j^I is a rational function of $q(\kappa)^{\frac{1}{4}}$ and β_j^I is a branch of Φ_I over C_j^I. The coefficient α_l^{II} has the form $\varepsilon_l^{II} \cdot \beta_l^{II}$, where ε_l^{II} is a rational function of $q(\kappa)^{\frac{1}{4}}$ and β_l^{II} is a branch of Φ_{II} over C_l^{II}. We define the coefficient of C_{jl} in $\mathscr{S}(\kappa)$ by

$$\alpha_{jl} = \varepsilon_j^I \cdot \varepsilon_l^{II} \cdot \beta_{ij} ,$$

where β_{ij} is the branch of Φ over C_{jl} defined below.

Any branch of Φ_I, Φ_{II}, or Φ is defined by determining arguments of all differences $t_i - t_j$, $z_j - t_l$, $z_l - z_j$ in the formulae for Φ_I, Φ_{II}, and Φ. Therefore, we assume that over C_j^I (resp. C_l^{II}) the argument of every difference in Φ_I (resp. Φ_{II}) is fixed. Let us determine the argument of every difference in Φ by the following rule:

Choose an argument of $z_3 - z_2$, $z_2 - z_1$, $z_b - t_a$ (for $a > c_1$, $b = 2, 3$), $z_b - t_a$ (for $a < c_1$ and $b = 1, 2$), $t_a - t_b$ (for $1 \leq a < b \leq c_1$ and for $c_1 < a < b \leq k$) the same as those in β_j^I and β_l^{II}. The function $z_1 - t_a$ for $a > c_1$ is approximately equal to $z_2 - t_a$ on C_{jl}. Choose the argument of $z_1 - t_a$ which is close to the argument of $z_2 - t_a$ in β_l^{II}.

The function $z_3 - t_a$ for $a \leq c_1$ is approximately equal to $z_3 - z_2$ on C_{jl}. Choose the argument of $z_3 - t_a$ which is close to the argument of $z_3 - z_2$ chosen in β_l^{II}.

The function $t_a - t_b$ for $a \leqq c_1$ and $b > c_1$ is approximately equal to $z_2 - t_b$ on C_{jl}. Choose the argument of $t_a - t_b$ which is close to the argument of $z_2 - t_b$ on C_{jl}.

It is easy to see that

$$\tau(z_1 z_2 z_3) = \sum_j^{M_I} \sum_l^{M_{II}} (C_{jl}, \alpha_{jl})$$

is a cycle. The group S_k acts on the space of chains with coefficients in $\mathscr{S}(\kappa)$. Set

$$\gamma(z_1, z_2, z_3) = \frac{1}{c_1! c_2!} \sum_{\chi \in S_k} (-1)^{|\chi|} \chi \cdot \tau(z_1, z_2, z_3).$$

According to [V1, Sect. 14] we have

The chain $\gamma(z_1, z_2, z_3)$ represents the class $v(z_1, z_2, z_3)\bar{v}$.   (4.6.8)

Now introduce the new coordinates $x_1, \ldots, x_k$ in $pr_{3,k}^{-1}(z_1, z_2, z_3)$:

$$t_p - z_1 = x_p u_1 u_2, \quad p = 1, \ldots, c_1,$$

$$t_p - z_2 = x_p u_2, \quad p = c_1 + 1, \ldots, k.$$

The cells C_{jl} of the chain τ written in coordinates $x_1, \ldots, x_p$ do not depend on u_1, u_2. The dependence of Φ in those coordinates has the form

$$\Phi = u_1^{\mu_1/\kappa} u_2^{\mu_2/\kappa} (\Phi_{c_1, m_1, m_2, \kappa}(x_1, \ldots, x_{c_1}) \cdot \Phi_{c_2, m^1, m_3, \kappa}(x_{c_1+1}, \ldots, x_k) + \mathcal{O}(u)). \quad (4.6.9)$$

Knowing that $\gamma_I(0, 1)$ sits on $(a_1, \ldots, a_{c_1})$ and $\gamma_{II}(0, 1)$ sits on $(b_1, \ldots, b_{c_2})$, we may conclude that γ sits on the t-critical point described in (4.6.4). Knowing that integrals over γ_I and γ_{II} have the desired asymptotic expansion, see (4.5), we may conclude that the asymptotic expansion of the function ψ has the form described in (2.5.6) and the properties described in (2.5.7). Formula (4.6.9) also shows that the integral

$$\int_{\gamma(z_1, z_2, z_3)} N$$

is the asymptotic solution $\psi_{c, T, \mathrm{id}}$.

Theorem (4.2.1) is proved.

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