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Quantum Principal Commutative Subalgebra in the Nilpotent Part of $U_q\widehat{sl}_2$ and Lattice KdV Variables

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Abstract: We propose a quantum lattice version of B. Feigin and E. Frenkel's constructions, identifying the KdV differential polynomials with functions on a homogeneous space under the nilpotent part of $\widehat{sl}_2$. We construct an action of the nilpotent part $U_q\widehat{n}_+$ of $U_q\widehat{sl}_2$ on their lattice counterparts, and embed the lattice variables in a $U_q\widehat{n}_+$-module, coinduced from a quantum version of the principal commutative subalgebra, which is defined using the identification of $U_q\widehat{n}_+$ with its dual algebra.

Introduction

In [FF1, FF2], B. Feigin and E. Frenkel propose a new approach to the generalized KdV hierarchies. They construct an action of the nilpotent part $\widehat{n}_+$ of the affine algebra $\widehat{g}$ on differential polynomials in the Miura fields, connected to the action of screening operators. This enables them to consider these differential polynomials as functions on a homogeneous space of $\widehat{n}_+$, and to interpret in this way the KdV flows. They also suggest that analogous constructions should hold for the quantum KdV equations.

In this work we propose a quantum lattice version of part of these constructions. Following ideas of lattice W-algebras, we replace the differential polynomials by an algebra of q-commuting variables, set on a half-infinite line. The analogue of the action of [FF1] is then an action of the nilpotent part $U_q\widehat{n}_+$ of the quantum affine algebra $U_q\widehat{sl}_2$. Recall that the homogeneous space occurring in [FF1] is $\widehat{N}_+/A$, where $\widehat{N}_+$ and A are the groups corresponding to $\widehat{n}_+$ and its principal commutative subalgebra a. A natural question is then what the analogue of a is in the quantum situation.

We construct a quantum analogue of a in the following way: we use an isomorphism of $U_q\widehat{b}_+$ with the coordinate ring $\mathbf{C}[\widehat{B}_+]_q$ ([Dr, LSS]) and transport in the first algebra a twisted version of the well-known commutative family res $d\lambda\lambda^k \operatorname{tr} T(\lambda)$. We prove that this subalgebra of $U_q\widehat{b}_+$ gives Ua for $q = 1$. This proof uses characterizations of these algebras as centralizers of one element.

Using a realization of the coordinate ring $\mathbf{C}[\widehat{B}_+]_q$ in q-commuting variables, due to Volkov, we find explicit expressions for the representation of $U_q a$ in operators on the half line. A symmetry argument then shows the analogue of the result of Feigin and Frenkel: injection of the lattice variables in a module coinduced from $U_q a$ to $U_q \widehat{b}_+$.

1. The Approach of B. Feigin and E. Frenkel

Let us recall briefly the part of [FF1] we will be concerned with (in the $\widehat{sl}_2$ case). Let ϕ be the free field on $S^1 \{\phi(x), \phi(y)\} = \delta'(x-y)$, and $\varphi' = \phi$. There is an action of the upper nilpotent part of $\widehat{sl}_2$ on the algebra $\mathbf{C}[\phi(x), \phi'(x), \ldots]$ of polynomials in $\phi(x), \phi'(x), \ldots$, given by $Q_+ P(\phi(x), \phi'(x), \ldots) = e^{-\varphi(x)} \{\int_{S^1} e^{\varphi}, P(\phi(x), \phi'(x), \ldots)\}$ and $Q_- P(\phi(x), \phi'(x), \ldots) = e^{\varphi(x)} \{\int_{S^1} e^{-\varphi}, P(\phi(x), \phi'(x), \ldots)\}$; Q_+ and Q_- are the usual generators of $\widehat{n}_+ \subset \widehat{sl}_2$, satisfying the analogues of Serre relations.

There is a duality between $U\widehat{n}_+$ and $\mathbf{C}[\phi(x), \phi'(x), \ldots]$, given by

$$U\widehat{n}_+ \times \mathbf{C}[\phi(x), \phi'(x), \ldots] \to \mathbf{C}$$

$$T \times P \longmapsto \varepsilon(TP) = (TP)(\phi(x) = 0, \phi'(x) = 0, \ldots).$$

Here ε is the operation of suppression of all non-constant terms in a given differential polynomial.

Let $a \subset \widehat{n}_+$ be the principal commutative subalgebra, spanned by $Q_+ + Q_-$, $[Q_+ - Q_-, h(i)], i \geqq 1$, where $h(i)$ are inductively defined by $h(1) = [Q_+, Q_-]$, $h(i+1) = [Q_+, [Q_-, h(i)]]$. Then $\varepsilon(xP) = 0$, if $x \in a$. The pairing thus factors through a pairing $(\mathbf{C} \otimes_{Ua} U\widehat{n}_+) \times \mathbf{C}[\phi(x), \phi'(x), \ldots] \to \mathbf{C}$; it enables to identify $\mathbf{C}[\phi(x), \phi'(x), \ldots]$ with $\mathbf{C}[\widehat{N}_+/A]$ as $\widehat{n}_+$-module ($\widehat{N}_+$ and A being the groups corresponding to $\widehat{n}_+$ and a).

2. The Lattice Setting

Let us consider variables x_i, $i \leqq 0$, satisfying the relations $x_i x_j = q x_j x_i$ if $i < j$; they are thought of as analogues of variables $e^{\varphi(-i)}$ and polynomials $\prod_{i \leqq 0} x_i^{\alpha_i}$ with $\sum_{i \leqq 0} \alpha_i = 0$ as analogues of the differential polynomials in $\phi(x), \phi'(x), \ldots$, on the half infinite lattice $i \leqq 0$, i integer (the point 0 of this lattice corresponds to x in the continuous approach.)

On the space $\mathbf{C}[x_i x_0^{-1}]_{i<0}$ of degree zero polynomials, we define the operators Q_+, Q_- and K by

$$Q_+ P = \left[\sum_{i<0} x_i, P\right] x_0^{-1}, Q_- P = \left[\sum_{i<0} x_i, P\right] x_0, KP = x_0 P x_0^{-1}.$$

Lemma 1. *The operators Q_+, Q_- satisfy the q-Serre–Chevalley relations*

$$Q_\pm^3 Q_\mp - (q^2 + 1 + q^{-2}) Q_\pm^2 Q_\mp Q_\pm + (q^2 + 1 + q^{-2}) Q_\pm Q_\mp Q_\pm^2 - Q_\mp Q_\pm^3 = 0,$$

and the relations $KQ_\pm = q^{\mp 1}Q_\pm K$. *So they define an action of* $U_q\widehat{b}_+ \subset U_q\widehat{sl}_2$ *on* $\mathbf{C}[x_i x_0^{-1}]_{i<0}$ (*the level of* $U_q\widehat{sl}_2$ *is taken to be zero*).

Proof. We have

$$Q_+\left(\prod_{i\leqq 0} x_i^{\alpha_i}\right) = \sum_{j<0}\left(q^{-\sum_{s<j}\alpha_s} - q^{\sum_{s\leqq j}\alpha_s}\right)\prod_{i\leqq 0} x_i^{\alpha_i+\delta_{ij}} x_0^{-1}$$

and

$$Q_-\left(\prod_{i\leqq 0} x_i^{\alpha_i}\right) = \sum_{j<0}\left(q^{\sum_{s<j}\alpha_s} - q^{-\sum_{s\leqq j}\alpha_s}\right)\prod_{i\leqq 0} x_i^{\alpha_i-\delta_{ij}} x_0\,,$$

if $\sum_{i\leqq 0}\alpha_i = 0$ (the products are written with lower indices at the left, e.g. $\prod_{i\leqq 0} x_i^{\alpha_i} = \dots x_n^{\alpha_n}\dots x_0^{\alpha_0}$).

Let us associate to $\prod_{i\leqq 0} x_i^{\alpha_i}$ the element $e^{\sum_{i<0}\alpha_i\xi_i}$ in the (commutative) algebra $\mathbf{C}[e^{\pm\xi_i}, i<0]$. In this representation, $Q_\pm$ can be written

$$Q_\pm = \sum_{j<0} e^{\pm\xi_j}\left(q^{\mp\sum_{s<j}\frac{\partial}{\partial\xi_s}} - q^{\pm\sum_{s\leqq j}\frac{\partial}{\partial\xi_s}}\right).$$

Pose

$$\theta_j = e^{\xi_j} q^{\sum_{s\leqq j}\frac{\partial}{\partial\xi_s}},\ \theta_j^- = e^{-\xi_j} q^{-\sum_{s\leqq j}\frac{\partial}{\partial\xi_s}},\ \theta'_j = e^{\xi_j} q^{-\sum_{s<j}\frac{\partial}{\partial\xi_s}},\ \theta_j'^- = \theta_j'^{-1}\,.$$

Then

$$Q_+ = -\sum_{j<0}\theta_j + \sum_{j<0}\theta'_j,\quad Q_- = -\sum_{j<0}\theta_j^- + \sum_{j<0}\theta_j'^{-1}\,.$$

Remark that if $j>k$, $\theta_j\theta_k = q\theta_k\theta_j$, $\theta_k\theta'_{k'} = q\theta'_{k'}\theta_k$, for all k and k', and $\theta'_{k'}\theta'_{j'} = q\theta'_{j'}\theta'_{k'}$ if $k'<j'$. The two first relations can then be deduced from the following result ([F, KP]):

Lemma 2 (B. Feigin). *If* $s_i^\pm$, $i\in\mathbf{Z}$ *are variables such that for* $i<j$, $s_i^\varepsilon s_j^{\varepsilon'} = q^{\varepsilon\varepsilon'} s_j^{\varepsilon'} s_i^\varepsilon, \varepsilon,\varepsilon' = \pm 1$, *then* $s^\pm = \sum_{i\in\mathbf{Z}} s_i^\pm$ *satisfy the q-Serre relations of* $U_q\widehat{sl}_2$.

Proof. (Note that we may have only a finite number of non-vanishing $s_i^\pm$.) Iterated application of the coproduct of $U_q\widehat{n}_+$ gives an algebra morphism $U_q\widehat{n}_+ \to (U_q\widehat{n}_+)^{\underline{\otimes}\mathbf{Z}}$, where $\underline{\otimes}$ denotes the twisted (w.r.t. root graduation) tensor product: $(a\underline{\otimes}b)(c\underline{\otimes}d) = q^{|b||c|}ac\underline{\otimes}bd$; in $U_q\widehat{n}_+$ the degrees are defined by $|Q_+| = -|Q_-| = 1$. We then have algebra morphisms $U_q\widehat{n}_+ \to \mathbf{C}[s_i^\pm]$, defined by $Q_\pm \mapsto s_i^\pm$, and $(U_q\widehat{n}_+)^{\underline{\otimes}\mathbf{Z}} \to \mathbf{C}[s_i^\pm, i\in\mathbf{Z}]$ (because $\mathbf{C}[s_i^\pm]^{\underline{\otimes}\mathbf{Z}} = \mathbf{C}\langle s_i^\pm, i\in\mathbf{Z}\rangle/(s_i^\varepsilon s_j^{\varepsilon'} - q^{\varepsilon\varepsilon'} s_j^{\varepsilon'} s_i^\varepsilon$ if $i<j$)).

The image of $Q_\pm$ by this last morphism is the image of $\sum\cdots\underline{\otimes}Q_\pm\underline{\otimes}1\cdots$, i.e. $\sum s_i^\pm$. ■

The two last relations are obvious. ■

Remark. The operators $Q_\pm$, K, defined on the space $\mathbf{C}[x_i^{\pm 1}]_{i\leqq 0}$ of arbitrary polynomials by $Q_\pm P = [\sum_{i<0} x_i^{\pm 1}, P]_q x_0^{\mp 1}$, $K = Ad\,x_0$ (where $[a,b]_q = ab - q^{|a||b|}ba$, and

$|\prod_{i\leqq 0} x_i^{\alpha_i}| = \sum_{i\leqq 0}\alpha_i$), satisfy also the relations of Lemma 1. Note also that the formulas for the non-modified screening action on the half line $[\sum x_{i\pm 1}, \cdot\,]_q$, are also expressed by the formulas giving $Q_\pm$ in variables ξ_i, so that the following results are also valid for the non-modified screening action.

3. Classical Results on the Lattice

From Lemma 1 follows that the vector fields $Q_\pm^{cl} = \mp\sum_{j<0} e^{\pm\xi_j}(\frac{\partial}{\partial\xi_j} + 2\sum_{s<j}\frac{\partial}{\partial\xi_s})$, acting on $\mathbf{C}[e^{\pm\xi_i}, i<0]$, satisfy the usual affine sl_2 Serre relations. Let σ be the automorphism of $\mathbf{C}[e^{\pm\xi_i}, i<0]$ defined by $\sigma(e^{\pm\xi_i}) = e^{\mp\xi_i}$. Then $\sigma_* Q_\pm^{cl} = Q_\mp^{cl}$ (σ_* of a vector field denotes its conjugation by σ.) So, $\sigma_*(Q_+^{cl} + Q_-^{cl}) = Q_+^{cl} + Q_-^{cl}$. Similarly, $\sigma_*([Q_+^{cl}, Q_-^{cl}]) = -[Q_+^{cl}, Q_-^{cl}]$; posing as in 1, $h(1) = [Q_+^{cl}, Q_-^{cl}]$, $h(i+1) = [Q_+^{cl}, [Q_-^{cl}, h(i)]]$, we show by induction that $\sigma_* h(i) = -h(i)$; if it is true for $h(i)$ then $\sigma_* h(i+1) = [Q_-^{cl}, [Q_+^{cl}, -h(i)]] = -h(i+1)$ (by Jacobi identity and $[h(i), h(1)] = 0$). Then $\sigma_*[Q_+^{cl} - Q_-^{cl}, h(i)] = [Q_-^{cl} - Q_+^{cl}, -h(i)]$ and so $[Q_+^{cl} - Q_-^{cl}, h(i)]$ is σ-invariant. In conclusion, all vectors fields of the subalgebra $a \subset \hat{n}_+$, spanned by $Q_+^{cl} + Q_-^{cl}$, and the $[Q_+^{cl} - Q_-^{cl}, h(i)]$, $i \geqq 1$, are σ-invariant.

Note that if the vector field $X = \sum_{i<0} X(\xi_j)\frac{\partial}{\partial\xi_i}$ is σ-invariant, we have $X_i(-\xi_j) = -X_i(\xi_j)$, so $X_i(0) = 0$. Let then $\varepsilon : \mathbf{C}[e^{\pm\xi_i}, i<0] \to \mathbf{C}$ be the map of evaluation at $\xi_i = 0$. We have shown that $\varepsilon(xP) = 0$, if $x \in a$, $P \in \mathbf{C}[e^{\pm\xi_i}, i<0]$, and so the pairing

$$U\hat{n}_+ \times \mathbf{C}[e^{\pm\xi_i}, i<0] \to \mathbf{C},$$

$$(T, P) \mapsto \varepsilon(TP)$$

factors through $(\mathbf{C} \otimes_{Ua} U\hat{n}_+) \times \mathbf{C}[e^{\pm\xi_i}, i<0]$.

Let us now show that the resulting morphism of $\hat{n}_+$-modules $\mathbf{C}[e^{\pm\xi_i}, i<0] \to (\mathbf{C} \otimes_{Ua} U\hat{n}_+)^*$ is an injection. For this, it is enough to show that the Lie algebra generated by Q_+^{cl} and Q_-^{cl} contains vector fields $X^{(n)} = \sum_{k\geqq 1} X_k^{(n)}(\xi_{-1}, \cdots, \xi_{-k})\frac{\partial}{\partial\xi_{-k}}$ with $X_k^{(n)}(0) = 0$ for $k<n$, $X_n^{(n)}(0) \neq 0$ for any $n \geqq 1$.

We can take $X^{(1)} = Q_+^{cl}$, and $X^{(n+1)} = [Q_+^{cl} + Q_-^{cl}, X^{(n)}] - 2X^{(n)}$. By combinations of products of the $X^{(n)}$, it is then possible to construct in the algebra generated by Q_+^{cl} and Q_-^{cl}, differential operators of the form $\sum f_{\alpha_1,\cdots\alpha_N}(\xi)(\frac{\partial}{\partial\xi_{-1}})^{\alpha_1}\cdots(\frac{\partial}{\partial\xi_{-N}})^{\alpha_N}$ + left ideal generated by $\frac{\partial}{\partial\xi_{-N-k}}$, $k \geqq 1$; with $f_{\alpha_1,\cdots,\alpha_N}(0) = \delta_{\alpha_1,\cdots,\alpha_N;\beta_1,\cdots,\beta_N}$, for any fixed $N \geqq 1$ and $\beta_1, \cdots, \beta_N \geqq 0$. Then, any non-zero combination $\sum_\gamma \lambda_\gamma e^{\sum_{i=1}^N \gamma_i\xi_{-i}}$ will have non-zero pairing with a combination of the operators constructed above. We have thus shown:

Proposition 1. *The pairing defined above between $U\hat{n}_+$ and $\mathbf{C}[e^{\pm\xi_i}, i<0]$ defines an injection of the latter space in the space of formal series at the origin of $\widehat{N}_+/A$, which is an algebra and $\hat{n}_+$-module morphism.*

Remark that the image of this injection does not contain $\mathbf{C}[\widehat{N}_+/A]$, because the latter space contains an element (x_1, or ϕ in the formalism of [FF1]) such that $Q_+^{cl}x_1 = Q_-^{cl}x_1 = 1$, and such an element does not exist in $\mathbf{C}[e^{\pm\xi_i}, i<0]$.

4. Quantum Principal Commutative Subalgebra

Let us assume q to be generic and denote by $U_q\widehat{b}_+$ the algebra generated by K, $Q_\pm$, subject to the relations of Lemma 1; $U_q\widehat{b}_+$ is a Borel subalgebra of the full quantum algebra $U_q\widehat{sl}_2$ (at level zero). Denoting by $U_q\widehat{b}_-$ the opposite Borel subalgebra, we then have an algebra injection $U_q\widehat{b}_+ \hookrightarrow (U_q\widehat{b}_-)^*$ ([D]). The coordinate ring corresponding to $U_q\widehat{b}_-$, denoted $\mathbf{C}[\widehat{B}_-]_q$, is the algebra generated by $t_{ij;n}$, $i, j = 1, 2, n \geqq 0$, with $t_{12;0} = 0$ and relations

$$R(\lambda,\mu)T^{(1)}(\lambda)T^{(2)}(\mu) = T^{(2)}(\mu)T^{(1)}(\lambda)R(\lambda,\mu), \text{ and } \det_q T(\lambda) = 1$$

(see [T]), where $T(\lambda) = (t_{ij}(\lambda))_{1\leqq i,j\leqq 2} = (\sum_{n\geqq 0} t_{ij;n}\lambda^n)_{1\leqq i,j\leqq 2}$, and $R(\lambda,\mu)$ is proportional to the R-matrix of [J]:

$$R(\lambda,\mu) = \frac{1+q^{1/2}}{2}(\lambda-\mu q^{1/2}) + \frac{1-q^{1/2}}{2}(\lambda+\mu q^{1/2})h\otimes h - (q-1)(\lambda f\otimes e + \mu e\otimes f),$$

with $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We will show:

Lemma 3. *The injection $U_q\widehat{b}_+ \hookrightarrow (U_q\widehat{b}_-)^*$ induces an algebra isomorphism between $U_q\widehat{b}_+$ and $\mathbf{C}[\widehat{B}_-]_q$.*

Proof. The pairing between $\mathbf{C}[\widehat{B}_-]_q$ and $U_q\widehat{b}_-$ is given by $\langle t_{ij;n}, x\rangle = \operatorname{res}_{\lambda=\infty}\lambda^{n-1}$ $\langle i|\pi\circ T_\lambda(x)|j\rangle d\lambda$, $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is the notations of [LSS], app. This enables to identify η_1, η_2, $e^{\hbar\xi_1}$ of *loc. cit.*, 7 with $t_{12;1}$, $t_{21;0}$, $t_{11;0} = t_{22;0}^{-1}$, respectively. The statement can be seen inductively from the relations defining $\mathbf{C}[\widehat{B}_-]_q$ (for example, the relation $(1-q)\lambda(t_{22}(\lambda)t_{11}(\mu) - t_{22}(\mu)t_{11}(\lambda)) = q^{\frac{1}{2}}(\lambda-\mu)[t_{21}(\mu), t_{12}(\lambda)]$ gives $(1-q)(t_{22;1}t_{11;0} - t_{22;0}t_{11;1}) = q^{\frac{1}{2}}[t_{21;0}, t_{12;1}]$, and the determinant relation gives $\alpha t_{11;0}t_{22;1} + \beta t_{11;1}t_{22;0} = t_{21;0}t_{12;1}$, with $\alpha, \beta \to 1$ when $q \to 1$, so combinations of these relations give $t_{11;1}$ and $t_{22;1}$ in terms of the generators). ■

Remark the difference with the classical situation, where $\mathbf{C}[\widehat{B}_-]_q$ is not finitely generated; though as Poisson algebra it is generated by $t_{11;0}^{\pm 1}$, $t_{12;1}$ and $t_{21;0}$. Note also that $U_q\widehat{n}_+$ can be considered as possessing two classical limits, one being the non-commutative algebra $U\widehat{n}_+$ and the other being the Poisson algebra generated by Q_+, Q_- and relations $\{Q_\pm, \{Q_\pm, \{Q_\pm, Q_\mp\}\}\} = Q_\pm^2\{Q_\pm, Q_\mp\}$ (it is the limit for $\hbar \to 0$ of the q-Serre relations, with $\{a,b\} = \lim_{\hbar\to 0}\frac{1}{\hbar}[a,b]$) and $q = e^\hbar$); these relations are satisfied in particular for $Q_\pm = \int_{S^1} e^{\pm\varphi}, \varphi$ classical free field.

We will now construct a quantum analogue of the principal commutative subalgebra of $\widehat{sl}_2$.

Proposition 2. *For $u(\lambda) = d, Ad(\Lambda = e + \lambda f$, d any diagonal matrix, independent of λ), the set of coefficients of $\lambda^k (k \geqq 0)$ in $\operatorname{tr} u(\lambda)T(\lambda)$ forms a commutative family in $\mathbf{C}[\widehat{B}_+]_q$. For $u = \Lambda\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, with $d_1 d_2 \neq 0$, the classical limit of the corresponding family in $U_q\widehat{b}_+$ is the subalgebra of $U\widehat{b}_+$ generated by the principal commutative subalgebra spanned by $d_2 e(i) + d_1 f(i+1)$, $i \geqq 0(e(0)$*

and $e(1)$ *denote the elements of* $\widehat{Ub_+}$ *corresponding to* η_1, η_2 *of* [*LSS*], *and* $e(i+1) = [e(0),[f(1),e(i)]]$, $f(i+1) = [f(1),[e(0),f(i)]]$ *for* $i \geqq 1$).

Proof. For the first part, we first check that $u(\lambda) \otimes u(\mu)$ commutes with $R(\lambda,\mu)$ in the two cases. Then

$$\begin{aligned}\operatorname{tr} u(\lambda)T(\lambda)\operatorname{tr} u(\mu)T(\mu) &= \operatorname{tr} u(\lambda) \otimes u(\mu)T^{(1)}(\lambda)T^{(2)}(\mu)\\ &= \operatorname{tr} u(\lambda) \otimes u(\mu)R(\lambda,\mu)^{-1}T^{(2)}(\mu)T^{(1)}(\lambda)R(\lambda,\mu)\\ &= \operatorname{tr} u(\lambda) \otimes u(\mu)T^{(2)}(\mu)T^{(1)}(\lambda) = \operatorname{tr} u(\mu)T(\mu)\operatorname{tr} u(\lambda)T(\lambda).\end{aligned}$$

To prove the second part, we first observe that the enveloping algebra of the principal commutative subalgebra is exactly the centralizer in $\widehat{Ub_+}$ of $d_2e(0) + d_1f(1)$. This can be seen in the associated graded algebra $\mathbf{C}[\widehat{b}_+^*]$; in the basis $z_i =$ image of $h(i)$, $i \geqq 0$ [$h(0)$ is the element of $\widehat{Ub_+}$ corresponding to ξ_1 of [LSS], and $h(i+1) = [e(0),[f(1),h(i)]]$ for $i \geqq 0$], $x_i =$ image of $d_2e(i) + d_1f(i+1)$, $i \geqq 0$, and $y_i =$ image of $d_2e(j) - d_1f(i+1)$, $i \geqq 0$, the Poisson bracket with x_0 is the vector field $\sum_{i\geqq 0} 2(-1)^{i+1}y_i\frac{\partial}{\partial h_i} + (-1)^{i+1}d_1d_2h_{i+1}\frac{\partial}{\partial y_i}$; ordering the basis as $(z_i, h_0, y_0, h_1, y_1, \cdots)$, we see that the only polynomials in z_i, h_i, y_i in the kernel of the vector field are those depending on z_i only.

The image in $\widehat{Ub_+}$ (by the specialisation $q = 1$) of the commutative subalgebra of $U_q\widehat{b}_+$ corresponding to $\operatorname{tr} u(\lambda)T(\lambda)$ is commutative, and it contains $d_2e(0) + d_1f(1)$. It remains to see that the subalgebra generated by $\operatorname{tr} u(\lambda)T(\lambda)$ is maximal as a commutative subalgebra of $\mathbf{C}[\widehat{B}_+]_q$. We will show it for the corresponding Poisson subalgebra of $\mathbf{C}[\widehat{B}_+]$. Denote $T(\lambda) = (t_{ij}(\lambda)) = \begin{pmatrix} a(\lambda) & c(\lambda) \\ b(\lambda) & d(\lambda)\end{pmatrix}$, with $a(\lambda) = \sum_{n\geqq 0} a_n\lambda^n$, etc. ($b_0 = 0$). The Poisson brackets between the variables $a_n, b_n, \cdots$, are given by $\{T(\lambda), \otimes T(\mu)\} = [r(\lambda,\mu), T(\lambda) \otimes T(\mu)]$, with $r(\lambda,\mu) = \frac{1}{2}\frac{\lambda+\mu}{\lambda-\mu}h \otimes h + \frac{2\lambda}{\lambda-\mu}f \otimes e + \frac{2\mu}{\lambda-\mu}e \otimes f$ (trigonometric r-matrix). Let us prove that the polynomials in $a_n, b_n, \cdots$, commuting with $b_1 - c_0$ (to simplify; the proof with $d_2b_1 + d_1c_0$ instead is similar[1], are exactly the polynomials in $b_{n+1} - c_n (n \geqq 0)$. By specializing for $\mu = 0$ the formulas for $\{a(\lambda), b(\mu)\}, \cdots$, we get $\{b_1 - c_0, a(\lambda)\} = (b_1 + c_0)a(\lambda) - 2a_0(\frac{b(\lambda)}{\lambda} + c(\lambda))$, $\{b_1 - c_0, \frac{b(\lambda)}{\lambda} + c(\lambda)\} = \frac{4}{\lambda}(d_0a(\lambda) - a_0d(\lambda))$, $\{b_1 - c_0, \frac{b(\lambda)}{\lambda} - c(\lambda)\} = 0$.

So, $\{b_1 - c_0, a_n\} = (b_1 + c_0)a_n - 2a_0(b_{n+1} + c_n)$, $\{b_1 - c_0, b_{n+1} + c_n\} = 4(d_0a_{n+1} - a_0d_{n+1})$, $\{b_1 - c_0, b_{n+1} - c_n\} = 0$, for $n \geqq 0$.

From $\det T(\lambda) = 1$, we obtain $a_0d_{n+1} - d_0a_{n+1} = -2d_0a_{n+1} + c_0b_{n+1} + c_nb_1 +$ terms in $b_i, i \leqq n, c_i, i \leqq n-1, a_i, i \leqq n$. Note that $c_0b_{n+1} + c_nb_1 = \frac{1}{2}[(c_0 + b_1)(c_n + b_{n+1}) - (b_1 - c_0)(b_{n+1} - c_n)]$. Pose for $i \geqq 0$, $z_i = b_{i+1} - c_i$ and $x_i = b_{i+1} + c_i$. The polynomials in a_n, b_n, c_n, d_n are then the polynomials in a_0^{-1}, $a_i, x_i, z_i (i \geqq 0)$. In this basis the vector field $\partial = \{b_1 - c_0, \ \}$ is expressed by $\partial(a_n) = x_0a_n - 2a_0x_n$, $\partial(x_n) = 2d_0a_{n+1} + \frac{1}{2}x_0x_n +$ terms in $a_i, i \leqq n, x_i, i \leqq n-1, z_i$, and $\partial(z_n) = 0$. The same argument as above can then be applied, with ordering $(z_i, a_0^{\pm 1}, x_1, a_1, x_2, \cdots)$. Explicitly, let $P(z_i, a_i, x_i)$ be a polynomial, and x_i (or a_i) be the greatest terms on which P depends non-trivially ; then the terms in d_0a_{i+1} (resp. a_0x_i) of ∂P will

[1] (here d_i denote the coefficients of the diagonal matrix)

be $2\frac{\partial P}{\partial x_i} d_0 a_{i+1}$ (resp. $-2\frac{\partial P}{\partial a_i} a_0 x_i$ if $i \neq 0$, and $-\frac{\partial P}{\partial a_0} a_0 x_0$ else); $\partial P = 0$ implies then $\frac{\partial P}{\partial x_i} = 0$, (resp. $\frac{\partial P}{\partial a_0} = 0$), contradiction.

As a by-product of this proof, we obtain:

Corollary. *For q generic or $q = 1$, the centralizer of $Q_+ - Q_-$ forms a maximal commutative subalgebra of $U_q\widehat{b}_+$.*

Proof. For $q = 1$, it is the first part of the proof above. For q generic, we translate the statement for $\mathbf{C}[\widehat{B}_+]_q$, and use the limit $q \to 1$ and the second part of the proof above. ■

We will call this subalgebra of $U_q\widehat{b}_+$ its *quantum principal commutative subalgebra* and denote it $U_q a$; note that $U_q a$ is not a Hopf subalgebra of $U_q\widehat{b}_+$ (a is already not a subbialgebra of $\widehat{b}_+$).

5. Realisation of $U_q a$ in q-Commuting Variables

Let us go back to the setting of Lemma 2. It gives an algebra morphism $U_q\widehat{n}_+ \to \mathbf{C}[s_i^{\pm}]$, and also by composition $U_q\widehat{n}_+ \to \mathbf{C}[s_i^{\pm}]/(s_i^{\pm}s_i^{\mp} = q_i)$, q_i being invertible scalars. Let us describe the image of $U_q a$ by this morphism. For this we need to construct the morphism $\mathbf{C}[\widehat{B}_+]_q \to \mathbf{C}[k, s_i^{\pm}]$ deduced from $U_q\widehat{b}_+ \to \mathbf{C}[k, s_i^{\pm}]$ by the isomorphism $U_q\widehat{b}_+ \simeq \mathbf{C}[\widehat{B}_+]$ (k is an additional variable, with $ks_i^{\pm} = q^{\mp\frac{1}{2}} s_i^{\pm} k$, and we prolongate $U_q\widehat{n}_+ \to \mathbf{C}[s_i^{\pm}]$ by $K \mapsto k$). From Lemma 3, we see that it is defined by $t_{11;0} \mapsto k$, $t_{22;0} \mapsto k^{-1}$, $t_{12;1} \mapsto \sum s_i^+$, $t_{21;0} \mapsto \sum s_i^-$.

Let k_i, $u_i^{\pm}$ be auxiliary variables, with $k_i u_i^{\pm} = q^{\mp\frac{1}{2}} u_i^{\pm} k_i$, other relations being commutation relations, and $\prod k_i = k$, $\prod_{j<i} k_j^{\pm 1} u_i^{\mp} \prod_{j>i} k_j^{\mp 1} = s_i^{\mp}$. Note that we may impose that $u_i^{\pm} u_i^{\mp} = q_i$. Following Volkov ([V0]), we remark that the matrices $\frac{1}{(1-\lambda q q_i)^{1/2}} \begin{pmatrix} k_i & u_i^- \\ \lambda u_i^+ & k_i^{-1} \end{pmatrix}$, and hence also the matrix $T'(\lambda) = \prod_{i=-\infty}^{+\infty} \frac{1}{(1-\lambda q q_i)^{1/2}} \begin{pmatrix} k_i & u_i^- \\ \lambda u_i^+ & k_i^{-1} \end{pmatrix}$, satisfy the relations $R(\lambda,\mu) T'(\lambda)^{(1)} T'(\mu)^{(2)} = T'(\mu)^{(2)} T'(\lambda)^{(1)} R(\lambda,\mu)$, $\det_q T'(\lambda) = 1$. Denote $T'(\lambda) = (t'_{ij}(\lambda))$, $t'_{ij}(\lambda) = \sum_{n \geqq 0} t'_{ij;n} \lambda^n$. The mapping from $\mathbf{C}[\widehat{B}_+]$ to $\mathbf{C}[k, s_i^{\pm}]$, sending $t_{ij;n}$ to $t'_{ij;n}$ thus extends to an algebra morphism; since $t'_{11;0}$, $t'_{22;0}$, $t'_{12;1}$ and $t'_{21;0}$ are respectively k, k^{-1}, $\sum s_i^+$, $\sum s_i^-$, this morphism is the desired composition $\mathbf{C}[\widehat{B}_+]_q \simeq U_q\widehat{b}_+ \to \mathbf{C}[k, s_i^{\pm}]$.

The image of $U_q a$ is then generated by

$$
\begin{aligned}
t'_{21;n} - t'_{12;n+1} = \sum_{i_1 < \cdots < i_{2n+1}} & \left(\prod_{i<i_1} k_i\right) u_{i_1}^+ \left(\prod_{i_1<i<i_2} k_i^{-1}\right) u_{i_2}^- \cdots u_{i_{2n+1}}^+ \left(\prod_{i>i_{2n+1}} k_i\right) \\
& - \left(\prod_{i<i_1} k_i^{-1}\right) u_{i_1}^- \left(\prod_{i_1<i<i_2} k_i\right) u_{i_2}^+ \cdots u_{i_{2n+1}}^- \left(\prod_{i>i_{2n+1}} k_i^{-1}\right) \\
& + \sum_{p<n} \text{scalars (analogous expression with } n \text{ replaced by } p)
\end{aligned}
$$

for $n \geqq 0$, which can be written

$$t'_{21;n} - t'_{12;n+1} = \sum_{i_1 < \cdots < i_{2n+1}} s^+_{i_1} s^-_{i_2} \cdots s^+_{i_{2n+1}} - s^-_{i_1} s^+_{i_2} \cdots s^-_{i_{2n+1}}$$

$$+ \sum_{p<n} \text{scalars (analogous expression with } n \text{ replaced by } p).$$

We have proved:

Lemma 4. *The image of the principal commutative subalgebra of* $U_q\widehat{n}_+$, *by the mapping defined in Lemma 2, is the subalgebra of* $\mathbf{C}[s_i^\pm]$ *generated by*

$$\sum_{i_1 < \cdots < i_{2n+1}} s^+_{i_1} s^-_{i_2} \cdots s^+_{i_{2n+1}} - s^-_{i_1} s^+_{i_2} \cdots s^-_{i_{2n+1}}, \quad \textit{for } n \geqq 0 .$$

Note that in the case where there is only a finite number N of $s_i^\pm$ the image of $U_q a$ is finitely generated (the sums vanish for $n \geqq [\frac{N+1}{2}]$). One may think that the elements $t'_{21;n} - t'_{12;n+1}$, for $n \geqq [\frac{N+1}{2}]$, generate the kernel of the morphism $U_q\widehat{n}_+ \to \mathbf{C}[s_i^\pm]$, and that this morphism is injective if there is an infinite number of $s_i^\pm$.

6. The Pairing Between $U_q\widehat{n}_+$ and the Lattice KdV Variables

Recall that in Sect. 2, $K = Adx_0 = q^{-\sum_{s<0} \frac{\partial}{\partial \xi_s}} = \theta'_0$ (posing $\xi_0 = 0$). The arguments of Sect. 2 show that the operators $\overline{Q}_+ = -\sum_{j<0} \theta_j + \sum_{j \leqq 0} \theta'_j = Q_+ + \theta'_0$, and $\overline{Q}_- = -\sum_{j<0} \theta'_j + \sum_{j \leqq 0} \theta'^-_{j'}$ (where $\theta'^-_0 = \theta'^{-1}_0$) satisfy the q-Serre relations.

Let us consider the algebra mapping $\varepsilon : \mathbf{C}[e^{\pm\xi_i}] \to \mathbf{C}$, defined by $e^{\pm\xi_i} \mapsto 1$. We can compose it with the action of $U_q\widehat{n}_+$ (by $\overline{Q}_+$ and $\overline{Q}_-$) on $\mathbf{C}[e^{\pm\xi_i}]$, and obtain a pairing between $U_q\widehat{n}_+$ and $\mathbf{C}[e^{\pm\xi_i}]$.

Let us show that for any polynomial $P \in \mathbf{C}[e^{\pm\xi_i}]$, and $n \geqq 0$, $\varepsilon((t_{21;n} - t_{12;n+1})P) = 0$. Ordering the θ_i, θ'_j by $(\theta_{-1}, \theta_{-2}, \cdots, \cdots, \theta'_{-1}, \theta'_0)$, Lemma 4 shows that

$$(t_{21;n} - t_{12;n+1})P = \left(\sum_{i_1 < \cdots < i_{2n+1}} \varphi^+_{i_1} \varphi^-_{i_2} \cdots \varphi^+_{i_{2n+1}} - \sum_{i_1 < \cdots < i_{2n+1}} \varphi^-_{i_1} \varphi^+_{i_2} \cdots \varphi^-_{i_{2n+1}} \right) P,$$

$\varphi_i^\pm$ is the list $(\theta^\pm_{-1}, \cdots, \theta'^\pm_0)$. We split each of these sums in two parts: the terms such that for some α, $\varphi_{i_1} = \theta_\alpha$,[2] and $\varphi_{i_{2n+1}} = \theta'_{\alpha+1}$ and the other terms for the first sum, and the terms such that $\varphi^-_{i_1} = \theta^-_\alpha$ and $\varphi^-_{i_{2n+1}} = \theta'_{\alpha+1}$ and the other terms for the second. We can define a bijection between the sets of remaining terms in the following way: to $\varphi_{i_1} \varphi'^-_{i_2} \cdots \varphi_{i_{2n+1}}$, with $\varphi_{i_1} = \theta_\alpha$ and $\varphi_{i_{2n+1}} = \theta'_{\beta+1}$ we associate $\varphi^-_{i_2} \varphi_{i_3} \cdots \varphi_{i_{2n+1}} \theta'^-_{\alpha+1}$ if $\alpha > \beta$, and $\theta^-_\beta \varphi_{i_1} \varphi^-_{i_2} \cdots \varphi^-_{i_{2n}}$ if $\alpha < \beta$. In both cases, $\varepsilon((\varphi_{i_1} \varphi'^-_{i_2} \cdots \varphi_{i_{2n+1}} -$ its associated term$) P) = 0$. Indeed, in the first case $\varphi_{i_1} = e^{\xi_\alpha} q^{\sum_{s \leqq \alpha} \frac{\partial}{\partial \xi_s}}$, and $\theta'^-_{\alpha+1} = e^{-\xi_{\alpha+1}} q^{\sum_{s \leqq \alpha} \frac{\partial}{\partial \xi_s}}$. Since $\alpha + 1$ is larger than all indices occurring in $\varphi^-_{i_2} \cdots \varphi_{i_{2n+1}}$, $e^{-\xi_{\alpha+1}}$ can be translated to the left (in the

[2] we note also $\varphi^+_i = \varphi_i, \theta^+_i = \theta_i$

expression $\varphi_{i_2}^- \cdots \varphi_{i_{2n+1}} \theta'^-_{\alpha+1}$) without changing the result, and there is also no correction due to the transport of $q^{\Sigma_{s \leqq \alpha} \frac{\partial}{\partial \xi_s}}$ to the left, because it has to cross the same number of e^{ξ_i} and $e^{-\xi_j}$, with all these i and j less than α. In conclusion, we can identify $\varphi_{i_1}\varphi'^-_{i_2} \cdots \varphi_{i_{2n+1}}$ with $e^{\xi_\alpha + \xi_{\alpha+1}}$.(its associated term). Similarly, in case $\alpha < \beta$, $\varphi_{i_1}\varphi'^-_{i_2} \cdots \varphi_{i_{2n+1}}$ is identified with $e^{-\xi_\beta - \xi_{\beta+1}}$.(its associated term) so if $\alpha \neq \beta$, $\varepsilon((\varphi_{i_1}\varphi'^-_{i_2} \cdots \varphi_{i_{2n+1}} -$ associated term$)P) = 0$.

For the first parts of the sums, we divide them in partial sums Σ_α, with $\varphi_{i_1} = \theta_\alpha$ and $\varphi_{i_{2n+1}} = \theta'_{\alpha+1}$ (resp. $\varphi'^-_i = \theta^-_\alpha$ and $\varphi^-_{i_{2n+1}} = \theta'^-_{\alpha+1}$). Then $\theta_\alpha \varphi^-_{i_2} \varphi_{i_3} \cdots \varphi^-_{i_{2n}} \theta'_{\alpha+1} = e^{\xi_{\alpha+1} + \xi_{\alpha+1}} \varphi^-_{i_2} \varphi_{i_3} \cdots \varphi^-_{i_{2n}}$, and $\theta^-_\alpha \varphi_{i_2} \varphi^-_{i_3} \cdots \varphi_{i_{2n}} \theta'^-_{\alpha+1} = e^{-\xi_\alpha - \xi_{\alpha+1}} \varphi_{i_2} \varphi^-_{i_3} \cdots \varphi^-_{i_{2n-1}} \varphi_{i_{2n}}$. So $\varepsilon(\Sigma_\alpha.P) =$

$$= \varepsilon \left[\left(\sum_{i_1(\alpha) < i_2 \cdots < i_{2n} < i_{2n+1}(\alpha)} \varphi^-_{i_2} \varphi_{i_3} \cdots \varphi^-_{i_{2n}} - \varphi_{i_2} \varphi^-_{i_3} \cdots \varphi_{i_{2n}} \right) P \right] ;$$

this is an expression of the same type that the expression we started with, with smaller degree. So we can use an induction argument to show that these expressions vanish.

So $\varepsilon((t_{21;n} - t_{12;n+1})P) = 0$ as claimed. And we can state the first part of:

Theorem. *The pairing between $U_q\widehat{n}_+$ and $\mathbf{C}[e^{\pm\xi_i}]$, given by*

$$U_q\widehat{n}_+ \times \mathbf{C}[e^{\pm\xi_i}] \to \mathbf{C}[e^{\pm\xi_i}] \xrightarrow{\varepsilon} \mathbf{C} ,$$

where the first map is the action of $U_q\widehat{n}_+$ on $\mathbf{C}[e^{\pm\xi_i}]$, factors through a pairing

$$(\mathbf{C} \otimes_{U_q a} U_q\widehat{n}_+) \times \mathbf{C}[e^{\pm\xi_i}] \to \mathbf{C} ,$$

which induces an injection of $U_q\widehat{n}_+$-modules $\mathbf{C}[e^{\pm\xi_i}] \hookrightarrow (\mathbf{C} \otimes_{U_q a} U_q\widehat{n}_+)^*$.

To prove the injection statement, we note that the classical limit of the operator $\overline{Q}_\pm$ is $\overline{Q}^{cl}_\pm = Q^{cl}_\pm + 1$. Let φ be a function on $\widehat{N}_+$ such that $Q_+\varphi = Q_-\varphi = 1$; φ is (up to an additive constant) the function assigning to $\exp(\alpha_0 e(0)) \exp(\beta_1 f(1)) \exp(\alpha_1 e(1)) \exp(\beta_2 f(2)) \cdots \in \widehat{N}_+, e^{\alpha_0 + \beta_1}$ (in the notations of Prop. 2). Denoting by ι the injection $\mathbf{C}[e^{\pm\xi_i}] \to (\mathbf{C}_{\otimes_{Ua}} U\widehat{n}_+)^*$ provided by the operators $Q^{cl}_\pm$, the analogous mapping $\bar{\iota}$, provided by $\overline{Q}^{cl}_\pm$, will be $\bar{\iota} = \varphi\iota$ (composition of ι with the multiplication by φ), and so will also be an injection. Since by [LSS], the family $U_q\widehat{n}_+$ is flat at $q = 1$ (PBW result), and by Prop. 2, the limit of $U_q a$ is Ua, the quantum mapping $\mathbf{C}[e^{\pm\xi_i}] \to (\mathbf{C} \otimes_{U_q a} U_q\widehat{n}_+)^*$ has for limit the classical mapping $\mathbf{C}[e^{\pm\xi_i}] \xrightarrow{\bar{\iota}} (\mathbf{C} \otimes_{U_q a} U\widehat{n}_+)^*$, which is injective, and so is injective.

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