Commun. Math. Phys. 170, 101–116 (1995)

Communications in
Mathematical
Physics

Dispersionless Toda Hierarchy and Two-Dimensional String Theory

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Received: 4 April 1994

Abstract: The dispersionless Toda hierarchy turns out to lie in the heart of a recently proposed Landau–Ginzburg formulation of two-dimensional string theory at self-dual compactification radius. The dynamics of massless tachyons with discrete momenta is shown to be encoded into the structure of a special solution of this integrable hierarchy. This solution is obtained by solving a Riemann–Hilbert problem. Equivalence to the tachyon dynamics is proven by deriving recursion relations of tachyon correlation functions in the machinery of the dispersionless Toda hierarchy. Fundamental ingredients of the Landau–Ginzburg formulation, such as Landau–Ginzburg potentials and tachyon Landau–Ginzburg fields, are translated into the language of the Lax formalism. Furthermore, a wedge algebra is pointed out to exist behind the Riemann–Hilbert problem, and speculations on its possible role as generators of "extra" states and fields are presented.

1. Introduction

Recently a Landau–Ginzburg model of two-dimensional strings at self-dual radius (i.e., $c = 1$ topological matter coupled to two-dimensional gravity) has been proposed and studied by several groups [1, 2, 3]. This model is in a sense a natural extrapolation of the topological A_{k+1} model to $k = -3$, and seems to inherit the remarkable properties of the A_{k+1} models such as: (i) an underlying structure of Lax equation [4], (ii) a period integral representation of correlation functions [5], (iii) an algebraic structure of gravitational primaries and descendents [6], etc. Although the status of the so-called special (discrete) states [7] still remains obscure, the dynamics of massless tachyons with discrete momenta is shown to be correctly described in this new framework.

The $c = 1$ model, however, differs from the A_{k+1} (and some other $c < 1$) models in several essential aspects. This seems to be eventually due to the difference of underlying integrable hierarchies. The A_{k+1} models are special solutions of the dispersionless KP (or generalized KdV) hierarchy [8, 9, 10]. Hanany et al. [2] suggested a similar link between the $c = 1$ model and the dispersionless Toda hierarchy [11].

In this paper, we demonstrate the suggestion of Hanany et al. in the machinery of dispersionless Toda hierarchy, and search for implications therefrom. Our basic observation is that the tachyon dynamics at self-dual radius is perfectly encoded into the structure of a special solution of this integrable hierarchy. In Sect. 2, we recall fundamental notions concerning the dispersionless Toda hierarchy, and in Sect. 3–4, reformulate several results of our previous work [11] in a more convenient form. The aforementioned special solution is constructed in Sect. 5 by solving a Riemann–Hilbert problem. A set of $w_{1+\infty}$-constraints (recursion relations) characterizing tachyon correlation functions are derived from the Riemann–Hilbert problem in Sect. 6. Since, as remarked by Hanany et al., those $w_{1+\infty}$-constraints determine tachyon correlation functions uniquely, we can conclude that our solution indeed describes the tachyon dynamics. In Sect. 7 we show the existence of a wedge algebra being the Riemann–Hilbert problem, and propose a speculative interpretation of this algebra as generators of "extra" states and fields in the $c = 1$ model. Section 8 is devoted to concluding remarks.

2. Fundamental Notions in Dispersionless Toda Hierarchy

The Lax formalism of the dispersionless Toda hierarchy is based on the two-dimensional Poisson bracket

$$\{A(p,s),B(p,s)\} = p\frac{\partial A}{\partial p}\frac{\partial B}{\partial s} - \frac{\partial A}{\partial s}p\frac{\partial B}{\partial p} \tag{1}$$

rather than the usual commutators. Fundamental quantities (counterparts of Lax operators) are two Laurent series $\mathscr{L}$ and $\bar{\mathscr{L}}$ of the form

$$\mathscr{L} = p + \sum_{n=0}^{\infty} u_{n+1}(t,\bar{t},s)p^{-n} , \tag{2}$$

$$\bar{\mathscr{L}}^{-1} = \bar{u}_0 p^{-1} + \sum_{n=0}^{\infty} \bar{u}_{n+1}(t,\bar{t},s)p^{n} , \tag{3}$$

where the coefficients depend on time variables of flows $t = (t_1, t_2, \dots)$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, \dots)$ as well as the spatial coordinate s. (We have slightly changed notations in the previous work [11].) Lax equations of these "Lax functions" are written

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial t_n} &= \{\mathscr{B}_n, \mathscr{L}\}, \quad \frac{\partial \mathscr{L}}{\partial \bar{t}_n} = \{\bar{\mathscr{B}}_n, \mathscr{L}\} , \\ \frac{\partial \bar{\mathscr{L}}}{\partial t_n} &= \{\mathscr{B}_n, \bar{\mathscr{L}}\}, \quad \frac{\partial \bar{\mathscr{L}}}{\partial \bar{t}_n} = \{\bar{\mathscr{B}}_n, \bar{\mathscr{L}}\} , \end{aligned} \tag{4}$$

where $\mathscr{B}_n$ and $\bar{\mathscr{B}}_n$ are given by

$$\begin{aligned} &\mathscr{B}_n = (\mathscr{L}^n)_{\geqq 0}, \quad \bar{\mathscr{B}}_n = (\bar{\mathscr{L}}^{-n})_{\leqq -1} , \\ &(\ )_{\geqq 0} : \quad \text{projection onto } p^0, p^1, \dots , \\ &(\ )_{\leqq -1} : \quad \text{projection onto } p^{-1}, p^{-2}, \dots . \end{aligned} \tag{5}$$

Furthermore, given such a pair $\mathcal{L}$ and $\bar{\mathcal{L}}$, one can find another pair of Laurent series

$$\mathcal{M} = \sum_{n=1}^{\infty} nt_n \mathcal{L}^n + s + \sum_{n=1}^{\infty} v_n(t,\bar{t},s)\mathcal{L}^{-n} ,$$

$$\bar{\mathcal{M}} = -\sum_{n=1}^{\infty} n\bar{t}_n \bar{\mathcal{L}}^{-n} + s - \sum_{n=1}^{\infty} \bar{v}_n(t,\bar{t},s)\bar{\mathcal{L}}^{n} \tag{6}$$

that satisfy the Lax equations

$$\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}, \quad \frac{\partial \mathcal{M}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \mathcal{M}\} ,$$

$$\frac{\partial \bar{\mathcal{M}}}{\partial t_n} = \{\mathcal{B}_n, \bar{M}\}, \quad \frac{\partial \bar{\mathcal{M}}}{\partial \bar{t}_n} = \{\bar{\mathcal{B}}_n, \bar{\mathcal{M}}\} \tag{7}$$

and the canonical Poisson relations

$$\{\mathcal{L}, \mathcal{M}\} = \mathcal{L}, \quad \{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = \bar{\mathcal{L}} . \tag{8}$$

It is rather these "extra" Lax functions that play a central role in our approach to two-dimensional strings.

Before going forward, a few comments on formal residue calculus are in order. We consider residues as being defined for 1-forms as:

$$\operatorname{res} \sum a_n z^n dz = a_{-1} . \tag{9}$$

Residues of more general 1-form are to be evaluated by the standard rule of exterior differential calculus:

$$\operatorname{res} f(z)dg(z) = \operatorname{res} f(z)\frac{dg(z)}{dz}dz . \tag{10}$$

Residues thus defined are invariant under coordinate transformations $z \to w = h(z)$ sending $\infty \to \infty$ or $0 \to 0$.

We can now define four fundamental potentials $\phi, F, \mathcal{S}$ and $\bar{\mathcal{S}}$ as follows.

The first potential $\phi = \phi(t,\bar{t},s)$ is defined by the equation

$$d\phi = \sum_{n=1}^{\infty} \operatorname{res}(\mathcal{L}^n d\log p)dt_n - \sum_{n=1}^{\infty} \operatorname{res}(\bar{\mathcal{L}}^{-n} d\log p)d\bar{t}_n + \log \bar{u}_0 ds , \tag{11}$$

where "d" means total differentiation in $(t,\bar{t},s)$, and of course $d\log p = dp/p$. The right-hand side is a closed form as far as $\mathcal{L}$ and $\bar{\mathcal{L}}$ are subject to Lax equations (4). This potential ϕ satisfies the second-order equation

$$\frac{\partial^2 \phi}{\partial t_1 \partial \bar{t}_1} + \frac{\partial}{\partial s} \exp\left(\frac{\partial \phi}{\partial s}\right) = 0 . \tag{12}$$

This is the well known dispersionless (or long-wave) limit of the two-dimensional Toda field equation.

The second potential $F = F(t,\bar{t},s)$ is defined by the equation

$$dF = \sum_{n=1}^{\infty} v_n dt_n - \sum_{n=1}^{\infty} \bar{v}_n d\bar{t}_n + \phi ds . \tag{13}$$

Again, the right-hand side is a closed form as far as $\mathscr{L}, \mathscr{M}, \bar{\mathscr{L}}$ and $\bar{\mathscr{M}}$ are subject to Lax equations (4, 7, 8). This potential F plays the role of a "generating function" – all other quantities $u_n, \bar{u}_n, v_n, \bar{v}_n$ and ϕ can be reproduced from F by differentiation with respect to $t, \bar{t}$ and s. This is obviously reminiscent of the role of partition functions with external sources in usual field theories. In our earlier work [11, 1991], F was defined as the logarithm of the "tau function" of the dispersionless Toda hierarchy, but it was later recognized that F is also connected with the tau function $\tau(\hbar, t, \bar{t}, s)$ of the full Toda hierarchy by $\hbar$-expansion [11, 1993]:

$$\log \tau(\hbar, t, \bar{t}, s) = \hbar^{-2} F(t, \bar{t}, s) + O(\hbar^{-1}) . \tag{14}$$

The last two potentials $\mathscr{S} = \mathscr{S}(t, \bar{t}, s, p)$ and $\bar{\mathscr{S}} = \bar{\mathscr{S}}(t, \bar{t}, s, p)$ can be defined rather directly as:

$$\begin{aligned} \mathscr{S} &= \sum_{n=1}^{\infty} t_n \mathscr{L}^n + s \log \mathscr{L} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial F}{\partial t_n} \mathscr{L}^{-n} , \\ \bar{\mathscr{S}} &= \sum_{n=1}^{\infty} \bar{t}_n \bar{\mathscr{L}}^{-n} + s \log \bar{\mathscr{L}} + \phi - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial F}{\partial \bar{t}_n} \bar{\mathscr{L}}^{n} . \end{aligned} \tag{15}$$

We call $\mathscr{S}$ and $\bar{\mathscr{S}}$ "potentials" because they can also be characterized as:

$$\begin{aligned} d\mathscr{S} &= \mathscr{M} d \log \mathscr{L} + \log p ds + \sum_{n=1}^{\infty} \mathscr{B}_n dt_n + \sum_{n=1}^{\infty} \bar{\mathscr{B}}_n d\bar{t}_n , \\ d\bar{\mathscr{S}} &= \bar{\mathscr{M}} d \log \bar{\mathscr{L}} + \log p ds + \sum_{n=1}^{\infty} \mathscr{B}_n dt_n + \sum_{n=1}^{\infty} \bar{\mathscr{B}}_n d\bar{t}_n , \end{aligned} \tag{16}$$

where "d" now means total differentiation in $(t, \bar{t}, s)$ and p. An immediate consequence of (16) is the following expressions of $\mathscr{B}_n$ and $\bar{\mathscr{B}}_n$:

$$\begin{aligned} \mathscr{B}_m &= \mathscr{L}^m - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 F}{\partial t_m \partial t_n} \mathscr{L}^{-n} \\ &= \frac{\partial^2 F}{\partial t_m \partial s} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 F}{\partial t_m \partial \bar{t}_n} \bar{\mathscr{L}}^{n} , \\ \bar{\mathscr{B}}_m &= - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 F}{\partial \bar{t}_m \partial t_n} \mathscr{L}^{-n} \\ &= \bar{\mathscr{L}}^{-m} + \frac{\partial^2 F}{\partial \bar{t}_m \partial s} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial^2 F}{\partial \bar{t}_m \partial \bar{t}_n} \bar{\mathscr{L}}^{n} . \end{aligned} \tag{17}$$

3. Spectral Parameters λ and $\bar{\lambda}$

We now introduce two new variables λ and $\bar{\lambda}$, and reformulate the setting of the previous section by replacing

$$\mathscr{L} \to \lambda, \quad \bar{\mathscr{L}} \to \bar{\lambda} . \tag{18}$$

By this substitution, $\mathcal{S}$ and $\bar{\mathcal{S}}$ are replaced by

$$\mathcal{S}(\mathcal{L} \to \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n + s \log \lambda - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial F}{\partial t_n} \lambda^{-n} ,$$

$$\bar{\mathcal{S}}(\bar{\mathcal{L}} \to \bar{\lambda}) = \sum_{n=1}^{\infty} \bar{t}_n \bar{\lambda}^{-n} + s \log \bar{\lambda} + \phi - \sum_{n=1}^{\infty} \frac{1}{n} \frac{\partial F}{\partial \bar{t}_n} \bar{\lambda}^{n} . \tag{19}$$

In the language of the full Toda hierarchy, these quantities are just the leading terms in $\hbar$-expansion of logarithm of two Baker–Akhiezer functions $\Psi(\hbar, t, \bar{t}, s, \lambda)$ and $\bar{\Psi}(\hbar, t, \bar{t}, s, \bar{\lambda})$ [11]:

$$\Psi(\hbar, t, \bar{t}, s, \lambda) = \exp[\hbar^{-1} \mathcal{S}(\mathcal{L} \to \lambda) + O(\hbar^0)] ,$$

$$\bar{\Psi}(\hbar, t, \bar{t}, s, \bar{\lambda}) = \exp[\hbar^{-1} \bar{\mathcal{S}}(\bar{\mathcal{L}} \to \bar{\lambda}) + O(\hbar^0)] . \tag{20}$$

The new variables λ and $\bar{\lambda}$ are thus nothing but the spectral parameters of the full Toda hierarchy. In the usual setting, actually, one does not have to distinguish between λ and $\bar{\lambda}$; in the present setting, they correspond to the two Lax functions $\mathcal{L}$ and $\bar{\mathcal{L}}$. Furthermore, in our interpretation of the Landau–Ginzburg formulation, they do arise in a different form as we shall see later. These are the main reasons that we use the two different spectral parameters.

Similarly, $\mathcal{M}$ and $\bar{\mathcal{M}}$ are replaced by

$$\mathcal{M}(\mathcal{L} \to \lambda) = \sum_{n=1}^{\infty} n t_n \lambda^n + s + \sum_{n=1}^{\infty} \frac{\partial F}{\partial t_n} \lambda^{-n} ,$$

$$\bar{\mathcal{M}}(\bar{\mathcal{L}} \to \bar{\lambda}) = - \sum_{n=1}^{\infty} n \bar{t}_n \bar{\lambda}^{-n} + s - \sum_{n=1}^{\infty} \frac{\partial F}{\partial \bar{t}_n} \bar{\lambda}^{n} , \tag{21}$$

where we have rewritten v_n and $\bar{v}_n$ into derivatives of F. By comparing (21) with (19), one can readily find that

$$\mathcal{M}(\mathcal{L} \to \lambda) = \lambda \frac{\partial}{\partial \lambda} \mathcal{S}(\mathcal{L} \to \lambda) ,$$

$$\bar{\mathcal{M}}(\bar{\mathcal{L}} \to \bar{\lambda}) = \bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} \bar{\mathcal{S}}(\bar{\mathcal{L}} \to \bar{\lambda}) . \tag{22}$$

These equations can be derived from (16), too.

Lastly, applying the same substitution rule to (17), we can define four quantities $\mathcal{B}_n(\mathcal{L} \to \lambda), \mathcal{B}_n(\bar{\mathcal{L}} \to \bar{\lambda}), \bar{\mathcal{B}}_n(\mathcal{L} \to \lambda), \bar{\mathcal{B}}_n(\bar{\mathcal{L}} \to \bar{\lambda})$. Equation (16) imply that these quantities, too, can be written as derivatives of $\mathcal{S}(\mathcal{L} \to \lambda)$ and $\bar{\mathcal{S}}(\bar{\mathcal{L}} \to \bar{\lambda})$:

$$\mathcal{B}_n(\mathcal{L} \to \lambda) = \frac{\partial}{\partial t_n} \mathcal{S}(\lambda), \qquad \bar{\mathcal{B}}_n(\mathcal{L} \to \lambda) = \frac{\partial}{\partial \bar{t}_n} \mathcal{S}(\lambda) ,$$

$$\mathcal{B}_n(\bar{\mathcal{L}} \to \bar{\lambda}) = \frac{\partial}{\partial t_n} \bar{\mathcal{S}}(\bar{\lambda}), \qquad \bar{\mathcal{B}}_n(\bar{\mathcal{L}} \to \bar{\lambda}) = \frac{\partial}{\partial \bar{t}_n} \bar{\mathcal{S}}(\bar{\lambda}) . \tag{23}$$

An immediate consequence of (22) and (23) is the following identities:

$$\frac{\partial}{\partial\lambda}\mathscr{B}_n(\mathscr{L}\to\lambda) = \frac{\partial}{\partial t_n}\mathscr{M}(\mathscr{L}\to\lambda)\lambda^{-1},$$

$$\frac{\partial}{\partial\lambda}\bar{\mathscr{B}}_n(\mathscr{L}\to\lambda) = \frac{\partial}{\partial \bar{t}_n}\mathscr{M}(\mathscr{L}\to\lambda)\lambda^{-1},$$

$$\frac{\partial}{\partial(\bar{\lambda}^{-1})}\mathscr{B}_n(\bar{\mathscr{L}}\to\bar{\lambda}) = -\frac{\partial}{\partial t_n}\bar{\mathscr{M}}(\bar{\mathscr{L}}\to\bar{\lambda})\bar{\lambda},$$

$$\frac{\partial}{\partial(\bar{\lambda}^{-1})}\bar{\mathscr{B}}_n(\bar{\mathscr{L}}\to\bar{\lambda}) = -\frac{\partial}{\partial \bar{t}_n}\bar{\mathscr{M}}(\bar{\mathscr{L}}\to\bar{\lambda})\bar{\lambda}. \qquad (24)$$

We shall show later that these quantities are fundamental ingredients of the Landau–Ginzburg formulation of two-dimensional strings.

4. Symmetries of Dispersionless Toda Hierarchy

Given two functions $\mathscr{A} = \mathscr{A}(\mathscr{L}, \mathscr{M})$ and $\bar{\mathscr{A}} = \bar{\mathscr{A}}(\bar{\mathscr{L}}, \bar{\mathscr{M}})$, one can construct an infinitesimal symmetry $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ of the dispersionless Toda hierarchy [11]. More precisely, $\mathscr{A}$ and $\bar{\mathscr{A}}$ are assumed to be a "good" function, such as a polynomial of $(\mathscr{L}, \mathscr{L}^{-1}, \mathscr{M})$ and $(\bar{\mathscr{L}}, \bar{\mathscr{L}}^{-1}, \bar{\mathscr{M}})$, respectively, with constant coefficients. We here explain how these symmetries are actually defined, and present several formulas that we shall use crucially in the subsequent sections.

Let us consider the ring $\mathscr{R}$ generated by $t, \bar{t}, s, F$ and all its derivatives. In this setting, F and its derivatives have to be considered abstract "symbols" rather than actual functions of $(t, \bar{t}, s)$. By "derivation" we mean a linear map $\delta : \mathscr{R} \to \mathscr{R}$ satisfying the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$. One can define the derivations $\partial/\partial t_n, \partial/\partial \bar{t}_n$, and $\partial/\partial s$ as derivations on $\mathscr{R}$ in an obvious manner:

$$\frac{\partial}{\partial t_n}F = v_n, \quad \frac{\partial}{\partial \bar{t}_n}F = -\bar{v}_n, \quad \frac{\partial}{\partial s}F = \phi,$$
$$\frac{\partial}{\partial t_n}t_m = \delta_{nm}, \quad \ldots\text{etc}\ldots. \qquad (25)$$

Differential equations satisfied by F and its derivatives (which include differential equations of $v_n, \bar{v}_n$ and ϕ, too) are thus encoded into these differential-algebraic structures of $\mathscr{R}$.

The symmetry $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ is defined to be an additional derivation of $\mathscr{R}$ with the following properties [11]:

- The action of $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ on F is given by

$$\delta_{\mathscr{A},\bar{\mathscr{A}}}F = -\operatorname{res}\left(\int_0^{\mathscr{M}(\mathscr{L}\to\lambda)}\mathscr{A}(\lambda,\mu)d\mu\right)d\log\lambda + \operatorname{res}\left(\int_0^{\bar{\mathscr{M}}(\bar{\mathscr{L}}\to\bar{\lambda})}\bar{\mathscr{A}}(\bar{\lambda},\bar{\mu})d\bar{\mu}\right)d\log\lambda. \qquad (26)$$

- $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ acts trivially on $t, \bar{t}$ and s as:

$$\delta_{\mathscr{A},\bar{\mathscr{A}}}t_n = 0, \quad \delta_{\mathscr{A},\bar{\mathscr{A}}}\bar{t}_n = 0, \quad \delta_{\mathscr{A},\bar{\mathscr{A}}}s = 0. \qquad (27)$$

- $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ commutes with $\partial/\partial t_n$, $\partial/\partial \bar{t}_n$ and $\partial/\partial s$:

$$\left[\delta_{\mathscr{A},\bar{\mathscr{A}}}, \frac{\partial}{\partial t_n}\right] = \left[\delta_{\mathscr{A},\bar{\mathscr{A}}}, \frac{\partial}{\partial \bar{t}_n}\right] = \left[\delta_{\mathscr{A},\bar{\mathscr{A}}}, \frac{\partial}{\partial s}\right] = 0 . \tag{28}$$

The last property implies, in particular, that $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ commutes with all flows of the dispersionless Toda hierarchy, a condition characterizing a symmetry!

Furthermore, these symmetries satisfy the following commutation relations [11]:

$$\begin{aligned}\left[\delta_{\mathscr{A},\bar{\mathscr{A}}}, \delta_{\mathscr{B},\bar{\mathscr{B}}}\right] = \delta_{\{\mathscr{A},\mathscr{B}\},\{\bar{\mathscr{A}},\bar{\mathscr{B}}\}} + \operatorname{res}\big(&\mathscr{A}(\lambda,0)d\mathscr{B}(\lambda,0) \\ &- \bar{\mathscr{A}}(\bar{\lambda},0)d\bar{\mathscr{B}}(\bar{\lambda},0)\big)\partial_F ,\end{aligned} \tag{29}$$

where ∂_F is yet another derivation on $\mathscr{R}$ defined by

$$\partial_F F = 1, \quad \partial_F\,(\text{any other generator of } \mathscr{R}) = 0 , \tag{30}$$

which accordingly commute with all other derivations $\partial/\partial t_n$, $\partial/\partial \bar{t}_n, \partial/\partial s$ and $\delta_{\mathscr{A},\bar{\mathscr{A}}}$. Thus an underlying Lie algebra is a central extension of $w_{1+\infty} \oplus w_{1+\infty}$; note that $w_{1+\infty}$ is now realized as the Lie algebra of Poisson brackets.

The action of $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ on other fundamental quantities such as $v_n, \bar{v}_n$ and ϕ, etc. can be read off from the above construction, because they all are derivatives of F. For $v_n, \bar{v}_n$ and ϕ, we have the following formulas (and, actually, the above formula for F was first discovered by "integrating" these formulas [11]):

$$\begin{aligned}\delta_{\mathscr{A},\bar{\mathscr{A}}} v_n &= \operatorname{res}\left(-\mathscr{A}(\mathscr{L},\mathscr{M}) + \bar{\mathscr{A}}(\bar{\mathscr{L}},\bar{\mathscr{M}})\right)\, d\mathscr{B}_n , \\ \delta_{\mathscr{A},\bar{\mathscr{A}}} \bar{v}_n &= \operatorname{res}\left(+\mathscr{A}(\mathscr{L},\mathscr{M}) - \bar{\mathscr{A}}(\bar{\mathscr{L}},\bar{\mathscr{M}})\right)\, d\bar{\mathscr{B}}_n , \\ \delta_{\mathscr{A},\bar{\mathscr{A}}} \phi &= \operatorname{res}\left(-\mathscr{A}(\mathscr{L},\mathscr{M}) + \bar{\mathscr{A}}(\bar{\mathscr{L}},\bar{\mathscr{M}})\right)\, d\log p .\end{aligned} \tag{31}$$

Furthermore, since $\mathscr{M}(\mathscr{L} \to \lambda)$ and $\bar{\mathscr{M}}(\bar{\mathscr{L}} \to \bar{\lambda})$ are generating functions of v_n and $\bar{v}_n$, one should be able to rewrite the first two of (31) in terms of these generating functions. This indeed results in the following formulas:

$$\begin{aligned}\delta_{\mathscr{A},\bar{\mathscr{A}}}\mathscr{M}(\mathscr{L} \to \lambda) &= \lambda\frac{\partial}{\partial\lambda}\left[\left(\mathscr{A}(\mathscr{L},\mathscr{M}) - \bar{\mathscr{A}}(\bar{\mathscr{L}},\bar{\mathscr{M}})\right)_{\leqq -1}(\mathscr{L} \to \lambda)\right] , \\ \delta_{\mathscr{A},\bar{\mathscr{A}}}\bar{\mathscr{M}}(\bar{\mathscr{L}} \to \bar{\lambda}) &= \bar{\lambda}\frac{\partial}{\partial\bar{\lambda}}\left[\left(-\mathscr{A}(\mathscr{L},\mathscr{M}) + \bar{\mathscr{A}}(\bar{\mathscr{L}},\bar{\mathscr{M}})\right)_{\geqq 0}(\bar{\mathscr{L}} \to \bar{\lambda})\right] ,\end{aligned} \tag{32}$$

where $\delta_{\mathscr{A},\bar{\mathscr{A}}}$ is understood to act trivially on λ and $\bar{\lambda}$ (i.e., $\delta_{\mathscr{A},\bar{\mathscr{A}}}\lambda = 0$ and $\delta_{\mathscr{A},\bar{\mathscr{A}}}\bar{\lambda} = 0$); inside "[ ]" on the right-hand side, we first take the projection with respect to powers of p, then reexpand the results into powers of $\mathscr{L}$ and $\bar{\mathscr{L}}$ instead of p, and finally replace them by λ and $\bar{\lambda}$.

5. Riemann–Hilbert Problem

We are now in a position to apply the general machinery of the preceding sections to two-dimensional string theory. In this section, we solve a Riemann–Hilbert problem to construct a special solution of the dispersionless Toda hierarchy. In the next section, we prove that it indeed describes the tachyon dynamics at self-dual radius by showing that its F potential satisfies $w_{1+\infty}$-constraints of tachyon correlation functions.

In general, Riemann–Hilbert problems for solving the dispersionless Toda hierarchy can be written

$$\bar{\mathscr{L}} = f(\mathscr{L}, \mathscr{M}), \quad \bar{\mathscr{M}} = g(\mathscr{L}, \mathscr{M}), \tag{33}$$

where $\mathscr{L}, \mathscr{M}$, $\bar{\mathscr{L}}$ and $\bar{\mathscr{M}}$ are required to be Laurent series of p of the form assumed in (3,6); $f = f(\lambda, \mu)$ and $g = g(\lambda, \mu)$ ("Riemann–Hilbert data") are functions satisfying the area-preserving condition

$$\lambda \frac{\partial f}{\partial \lambda} \frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \mu} \lambda \frac{\partial g}{\partial \lambda} = f \tag{34}$$

(which means that the map $(\log \lambda, \mu) \to (\log f, g)$ is area-preserving in the ordinary sense) and some additional condition on its analyticity. A general theorem [11] ensures that if (33) has a unique solution, then $\mathscr{L}, \mathscr{M}$, $\bar{\mathscr{L}}$ and $\bar{\mathscr{M}}$ satisfy all relevant equations (4, 7, 8) of the Lax formalism. Theoretically, one can thus obtain all solutions of the dispersionless Toda hierarchy. Practically, explicit solutions of such a Riemann–Hilbert problem is rarely available. Note that (33) is just a compact expression of an infinite number of highly nonlinear relations between the two sets of variables (u_n, v_n) and $(\bar{u}_n, \bar{v}_n)$ (in which $t, \bar{t}$ and s enter as parameters); solving these equations looks as difficult as solving the hierarchy directly! Fortunately, the Riemann–Hilbert problem we consider below, is relatively easy to handle.

The Riemann–Hilbert problem to be considered is the following:

$$\mathscr{L} = \bar{\mathscr{M}} \bar{\mathscr{L}}, \quad \bar{\mathscr{L}}^{-1} = \mathscr{M} \mathscr{L}^{-1}. \tag{35}$$

Apparently this does not take the form of (33), but can be readily rewritten in that form. This non-standard (but more symmetric) expression is rather suited for recognizing a wedge algebra structure. The area-preserving condition, too, can be easily checked. This Riemann–Hilbert problem can be solved by almost the same method as used for the A_{k+1} models [10]. Actually, details of calculations are rather similar to the case of the D_ℓ models [12]; the integrable hierarchy underlying these models, too, has four Lax functions, and Riemann–Hilbert problems take the same form as (33).

Solving (35) consists of several steps. The first step is to split each equation of (35) into two pieces by applying $(\ )_{\geqq 0}$ and $(\ )_{\leqq -1}$. This gives the following four equations:

$$(\mathscr{L})_{\geqq 0} = -\sum_{k=2}^{\infty} k\bar{t}_k (\bar{\mathscr{L}}^{-k+1})_{\geqq 0} - \bar{t}_1 + s\bar{\mathscr{L}} + \sum_{n=1}^{\infty} \bar{v}_n \bar{\mathscr{L}}^{n+1}, \tag{36}$$

$$(\mathscr{L})_{\leqq -1} = -\sum_{k=2}^{\infty} k\bar{t}_k (\bar{\mathscr{L}}^{-k+1})_{\leqq -1}, \tag{37}$$

$$(\bar{\mathscr{L}}^{-1})_{\geqq 0} = \sum_{k=2}^{\infty} k t_k (\mathscr{L}^{k-1})_{\geqq 0} + t_1, \tag{38}$$

$$(\bar{\mathscr{L}}^{-1})_{\leqq -1} = \sum_{k=2}^{\infty} k t_k (\mathscr{L}^{k-1})_{\leqq -1} + s\mathscr{L}^{-1} + \sum_{n=1}^{\infty} v_n \mathscr{L}^{-n-1}. \tag{39}$$

The second step is to decompose each equation into an infinite number of equations not including p, by taking residue pairing of both hand sides with suitable 1-forms. For instance, by taking the residue pairing of both hand sides of (36, 37) with (i) $\bar{u}_0 p^{-1} d\log p$, (ii) $p^{n-1} d\log p$, (iii) $\bar{\mathscr{L}}^{-n-1} d\log\bar{\mathscr{L}}$, respectively, we can obtain the equations

$$\bar{u}_0 = -\sum_{k=2}^{\infty} k\bar{t}_k\bar{u}_0 \operatorname{res}[\bar{\mathscr{L}}^{-k+1} p^{-1} d\log p] + s\,, \tag{40}$$

$$u_n = -n\bar{t}_n\bar{u}_0^{n-1} - \sum_{k=n+1}^{\infty} k\bar{t}_k \operatorname{res}[\bar{\mathscr{L}}^{-k+1} p^{n-1} d\log p]\,, \tag{41}$$

$$\begin{aligned}\bar{v}_n = {} & \operatorname{res}[(\mathscr{L})_{\geqq 0}\bar{\mathscr{L}}^{-n-1} d\log\bar{\mathscr{L}}] \\ & + \sum_{k=2}^{\infty} k\bar{t}_k \operatorname{res}[(\bar{\mathscr{L}}^{-k+1})_{\geqq 0}\bar{\mathscr{L}}^{-n-1} d\log\bar{\mathscr{L}}]\end{aligned} \tag{42}$$

for $n = 1, 2, \ldots$. Here trivial equations of the form $0 = 0$ have been omitted. It should be noted that this process is reversible, because the 1-forms (i)–(iii) used in the residue pairing form a complete set. Similarly, from (38, 39), we obtain another infinite set of equations

$$\bar{u}_0 = \sum_{k=2}^{\infty} kt_k \operatorname{res}[\mathscr{L}^{k-1} d\log p] + s\,, \tag{43}$$

$$\bar{u}_n = nt_n + \sum_{k=n+1}^{\infty} kt_k \operatorname{res}[\mathscr{L}^{k-1} p^{-n+1} d\log p]\,, \tag{44}$$

$$\begin{aligned}v_n = {} & \operatorname{res}[(\bar{\mathscr{L}}^{-1})_{\leqq -1}\mathscr{L}^{n+1} d\log\mathscr{L}] \\ & - \sum_{k=2}^{\infty} kt_k \operatorname{res}[(\mathscr{L}^{k-1})_{\leqq -1}\mathscr{L}^{n+1} d\log\mathscr{L}]\end{aligned} \tag{45}$$

for $n = 1, 2, \ldots$. This process, too, is reversible. Therefore we now have only to solve these equations for $u_n, v_n, \bar{u}_n$ and $\bar{v}_n$.

The third and final step is to solve these equations by Taylor expansion. Equations (40, 41, 43, 44) include only u's and $\bar{u}$'s. By expanding these unknown functions into Taylor series of $(t, \bar{t})$ at $(t, \bar{t}) = (0, 0)$, one can convert these equations into (very complicated) recursion relations of Taylor coefficients. By the standard power counting method, one can show that these recursion relations uniquely determine u's and $\bar{u}$'s as:

$$\begin{aligned}\bar{u}_0 &= s + \text{ higher order terms,} \\ u_n &= -n\bar{t}_n s^{n-1} + \text{ higher order terms,} \\ \bar{u}_n &= nt_n + \text{ higher order terms } (n \geqq 1)\,.\end{aligned} \tag{46}$$

Once u's and $\bar{u}$'s are thus determined, the remaining two equations (42, 45) give v_n and $\bar{v}_n$ explicitly. Thus our Riemann–Hilbert problem turns out to have a unique solution.

The solutions $u_n, \bar{u}_n, v_n$ and $\bar{v}_n$ of the above equations turn out to have good scaling properties. Note that each equation of (35) is invariant under the following formal rescaling of variables included therein:

$$\begin{aligned} t_n \to c^{-n} t_n, &\quad \bar{t}_n \to c^n \bar{t}_n , \\ s \to s, &\quad p \to c^{-1} p , \\ u_n \to c^n u_n, &\quad \bar{u}_n \to c^{-n} \bar{u}_n , \\ v_n \to c^n v_n, &\quad \bar{v}_n \to c^{-n} \bar{v}_n . \end{aligned} \tag{47}$$

Since the Riemann–Hilbert problem has a unique solution, this means that $u_n, \bar{u}_n, v_n$ and $\bar{v}_n$ indeed have the above scaling property as functions of $(t, \bar{t}, s)$. In other words, if we define a weight (U(1)-charge) of $t, \bar{t}, s$ as

$$\mathrm{wt}(t_n) = -n, \quad \mathrm{wt}(\bar{t}_n) = n, \quad \mathrm{wt}(s) = 0 , \tag{48}$$

then $u_n, \bar{u}_n, v_n$ and $\bar{v}_n$ become quasi-homogeneous functions of degree $n, -n, n$ and $-n$, respectively. Accordingly, the functions ϕ and F, which are defined by (11, 13), become quasi-homogeneous function of degree 0.

Three remarks are now in order:

First, we have in fact two equations (40) and (43) that include $\bar{u}_0$ as a main term; apparently this is redundant. Actually, one may select one of them arbitrarily, and solve them along with (41, 44). This eventually leads to the same result, as one can verify by returning to (36–39) and reexamining the derivation of the above equations therefrom.

Second, in the final step of the above consideration, we have Taylor-expanded all unknown functions at $(t, \bar{t}) = (0, 0)$, but s is left free. Namely, we do not need Taylor expansion in s, and can set it to any constant value. This is also reflected in the fact that the weight (U(1)-charge) of s is zero. This is a desirable property, because s is interpreted to be the cosmological constant of two-dimensional strings, and an advantage of the Landau–Ginzburg formulation lies in the fact that it describes the theory with non-zero cosmological constant.

Third, we have not specified any explicit expression of $u_n, v_n, \bar{u}_n$ and $\bar{v}_n$; they should be very complicated, and we actually do not need such explicit formulas. We just have to prove that the Riemann–Hilbert problem has a unique solution. The general machinery of the dispersionless Toda hierarchy can work only after this fact is confirmed. Once the existence of such a solution is proven, one can derive $w_{1+\infty}$-constraints to the F potential therefrom, and identify it with the generating function of tachyon correlation functions, as we shall show in the next section. All relevant information on the tachyon dynamics is now encoded into the F potential.

6. Constraints to F Potential

Let us now derive $w_{1+\infty}$-constraints to F. To this end, we start from the relations

$$\mathscr{L}^n = \bar{\mathscr{M}}^n \bar{\mathscr{L}}^n, \quad \bar{\mathscr{L}}^{-n} = \mathscr{M}^n \mathscr{L}^{-n}, \quad n = 1, 2, \dots , \tag{49}$$

which are an obvious consequence of (35). Just as we derived (40) etc. in the previous section, we now take residue pairing of both hand sides of (49) with $d\mathcal{B}_m, d\bar{\mathcal{B}}_m$ and $d\log p\,(m = 1,2,\ldots)$. This results in the following relations:

$$
\begin{aligned}
\mathrm{res}[\mathcal{L}^n d\mathcal{B}_m] &= \mathrm{res}[\bar{\mathcal{M}}^n \bar{\mathcal{L}}^n d\mathcal{B}_m]\,, \\
\mathrm{res}[\mathcal{L}^n d\bar{\mathcal{B}}_m] &= \mathrm{res}[\bar{\mathcal{M}}^n \bar{\mathcal{L}}^n d\bar{\mathcal{B}}_m]\,, \\
\mathrm{res}[\mathcal{L}^n d\log p] &= \mathrm{res}[\bar{\mathcal{M}}^n \bar{\mathcal{L}}^n d\log p]\,, \\
\mathrm{res}[\bar{\mathcal{L}}^{-n} d\mathcal{B}_m] &= \mathrm{res}[\mathcal{M}^n \mathcal{L}^{-n} d\mathcal{B}_m]\,, \\
\mathrm{res}[\bar{\mathcal{L}}^{-n} d\bar{\mathcal{B}}_m] &= \mathrm{res}[\mathcal{M}^n \mathcal{L}^{-n} d\bar{\mathcal{B}}_m]\,, \\
\mathrm{res}[\bar{\mathcal{L}}^{-n} d\log p] &= \mathrm{res}[\mathcal{M}^n \mathcal{L}^{-n} d\log p]\,.
\end{aligned}
\tag{50}
$$

Note that these relations conversely imply (49), because this residue pairing is complete (i.e., $\mathrm{res}[f d\mathcal{B}_m] = \mathrm{res}[f d\bar{\mathcal{B}}_m] = \mathrm{res}[f d\log p] = 0$ for all $m = 1,2,\ldots$ if and only if $f = 0$). We can now apply (31) to each equation of (50) to rewrite them as:

$$
\begin{aligned}
\frac{\partial}{\partial t_m}\delta_{\mathcal{L}^n,\bar{\mathcal{M}}^n\bar{\mathcal{L}}^n}F = \frac{\partial}{\partial \bar{t}_m}\delta_{\mathcal{L}^n,\bar{\mathcal{M}}^n\bar{\mathcal{L}}^n}F = \frac{\partial}{\partial s}\delta_{\mathcal{L}^n,\bar{\mathcal{M}}^n\bar{\mathcal{L}}^n}F = 0\,, \\
\frac{\partial}{\partial t_m}\delta_{\mathcal{M}^n\mathcal{L}^{-n},\bar{\mathcal{L}}^{-n}}F = \frac{\partial}{\partial \bar{t}_m}\delta_{\mathcal{M}^n\mathcal{L}^{-n},\bar{\mathcal{L}}^{-n}}F = \frac{\partial}{\partial s}\delta_{\mathcal{M}^n\mathcal{L}^{-n},\bar{\mathcal{L}}^{-n}}F = 0\,.
\end{aligned}
\tag{51}
$$

These equations show that $\delta_{\mathcal{L}^n,\bar{\mathcal{M}}^n\bar{\mathcal{L}}^n}F$ and $\delta_{\mathcal{M}^n\mathcal{L}^{-n},\bar{\mathcal{L}}^{-n}}F$ are constant. Actually, this constant should vanish: If one recalls the aforementioned scaling properties of $v_n, \bar{v}_n$ and ϕ, and apply them to general formula (26), one will be able to see that $\delta_{\mathcal{L}^n,\bar{\mathcal{M}}^n\bar{\mathcal{L}}^n}F$ and $\delta_{\mathcal{M}^n\mathcal{L}^{-n},\bar{\mathcal{L}}^{-n}}F$ are quasi-homogeneous of degree -1. This means that the constant values should be zero. Thus we can conclude that F satisfies the equations

$$
\delta_{\mathcal{L}^n,\bar{\mathcal{M}}^n\bar{\mathcal{L}}^n}F = 0, \quad \delta_{\mathcal{M}^n\mathcal{L}^{-n},\bar{\mathcal{L}}^{-n}}F = 0, \quad n = 1,2,\ldots\,. \tag{52}
$$

Furthermore, by carefully examining the above derivation, one can see that this derivation is reversible; Eqs. (49) (therefore the original Riemann–Hilbert problem) can be derived conversely from (52).

Eqs. (52) are, actually, just a disguise of the $w_{1+\infty}$-constraints of Hanany et al. By general formula (26), one can rewrite (52) into a more explicit form:

$$
\begin{aligned}
v_n - \frac{1}{n+1}\,\mathrm{res}[\bar{\mathcal{M}}^{n+1}\bar{\mathcal{L}}^n d\log\bar{\mathcal{L}}] = 0\,, \\
\bar{v}_n + \frac{1}{n+1}\,\mathrm{res}[\mathcal{M}^{n+1}\mathcal{L}^{-n} d\log\mathcal{L}] = 0\,.
\end{aligned}
\tag{53}
$$

One can then substitute $v_n = \partial F/\partial t_n$ and $\bar{v}_n = -\partial F/\partial \bar{t}_n$ to write the left hand side in terms of derivatives of F. Furthermore, one can introduce a new variable X and, as in (18), rewrite the residues in terms of X by replacing $\mathscr{L} \to X, \bar{\mathscr{L}} \to X^{-1}$. Thus, eventually, (52) turn into the following form:

$$\frac{\partial F}{\partial t_n} - \frac{1}{n+1} \operatorname{res} \left[\left(\frac{\bar{\mathscr{M}}(\bar{\mathscr{L}} \to X^{-1})}{X} \right)^{n+1} dX \right] = 0 ,$$

$$\frac{\partial F}{\partial \bar{t}_n} + \frac{1}{n+1} \operatorname{res} \left[\left(\frac{\mathscr{M}(\mathscr{L} \to X)}{X} \right)^{n+1} dX \right] = 0 , \tag{54}$$

which become exactly the $w_{1+\infty}$-constraints of Hanany et al. if we interpret their two Landau–Ginzburg potentials $W(X), \bar{W}(X)$ and tachyon correlation functions $\langle\langle T_n \rangle\rangle$ as:

$$W(X) = -\frac{\mathscr{M}(\mathscr{L} \to X)}{X}, \quad \bar{W}(X) = -\frac{\bar{\mathscr{M}}(\bar{\mathscr{L}} \to X^{-1})}{X} , \tag{55}$$

$$\langle\langle T_n \rangle\rangle = \frac{1}{n}\frac{\partial F}{\partial t_n}, \quad \langle\langle T_0 \rangle\rangle = \frac{\partial F}{\partial s} ,$$

$$\langle\langle T_{-n} \rangle\rangle = -\frac{1}{n}\frac{\partial F}{\partial \bar{t}_n}, \quad (n = 1,2,\ldots) . \tag{56}$$

The extra numerical factors on the right-hand side emerge because our $(t, \bar{t}, s)$ are slightly different from the background sources of Hanany et al. Our results agree with theirs if we interpret the correlator $\langle\langle \mathscr{O} \rangle\rangle$ as:

$$\langle\langle \mathscr{O} \rangle\rangle = \left\langle \mathscr{O} \exp\left(\sum_{n=1}^{\infty} n t_n T_n + s T_0 - \sum_{n=1}^{\infty} n \bar{t}_n T_{-n} \right) \right\rangle . \tag{57}$$

Actually, in place of (49), one can consider even more general combinations of the fundamental Riemann–Hilbert relation as:

$$\mathscr{M}^k \mathscr{L}^{n-k} = \bar{\mathscr{M}}^n \bar{\mathscr{L}}^{n-k}, \quad k,n = 0,1,\ldots . \tag{58}$$

Then, by the same reasoning as above, the following constraints can be obtained:

$$\delta_{\mathscr{M}^k \mathscr{L}^{n-k}, \bar{\mathscr{M}}^n \bar{\mathscr{L}}^{n-k}} F = 0 . \tag{59}$$

In terms of the Landau–Ginzburg potential, more explicitly, these constraints can be written

$$\frac{1}{k+1} \operatorname{res} \left[(-W(X))^{k+1} X^n dX \right] = \frac{1}{n+1} \operatorname{res} \left[(-\bar{W}(X))^{n+1} X^k dX \right] . \tag{60}$$

Of course, as also noted by Hanany et al., their $w_{1+\infty}$-constraints are in themselves powerful enough to determine the tachyon correlation functions completely. In this respect, the above constraints are redundant. These extra constraints, however, turn out to stem from underlying higher symmetries, as we shall discuss in the next section.

7. States and Fields Generated by Wedge Algebra

We first note that both hand sides of (58) are generators of a wedge algebra. To clarify this fact, we introduce nonnegative half-integer indices (j, m) in the "wedge" $|m| \leqq j$ by the usual convention

$$k = j - m, \quad n = j + m , \tag{61}$$

and write both hand sides of (58) as w_{jm}:

$$w_{jm} = \mathscr{L}^n(\mathscr{M}\mathscr{L}^{-1})^k = (\bar{\mathscr{M}}\bar{\mathscr{L}})^n\bar{\mathscr{L}}^{-k} . \tag{62}$$

Since

$$\{\mathscr{L}, \mathscr{M}\mathscr{L}^{-1}\} = \{\bar{\mathscr{M}}\bar{\mathscr{L}}, \bar{\mathscr{L}}^{-1}\} = 1 , \tag{63}$$

w_{jm} indeed form a wedge algebra with respect to the Poisson bracket. In the following, we propose a speculative interpretation of this wedge algebra as generators of "extra" states and fields of two-dimensional strings.

Let us show how such "states" emerge in our framework. Let W_{jm} denote the following symmetries of the dispersionless Toda hierarchy:

$$W_{jm} = \delta_{\mathscr{L}^n(\mathscr{M}\mathscr{L}^{-1})^k, 0} = -\delta_{0, (\bar{\mathscr{M}}\bar{\mathscr{L}})^n\bar{\mathscr{L}}^{-k}} . \tag{64}$$

These symmetries are understood to be acting on the ring $\mathscr{R}$ of Sect. 4. The two expressions on the right-hand side give the same symmetry because of (59). Furthermore, by (29), W_{jm} obey the same commutation relations as the Poisson commutation relations of w_{jm}; the central terms disappear, as usual, on a wedge. The action of those sitting on the "edge" of the wedge, $(j, m) = (n/2, \pm n/2)$ generate the tachyon correlation functions:

$$W_{n/2, n/2}F = -\frac{\partial F}{\partial t_n} = -n\langle\langle T_n \rangle\rangle ,$$

$$W_{n/2, -n/2}F = \frac{\partial F}{\partial \bar{t}_n} = -n\langle\langle T_{-n} \rangle\rangle . \tag{65}$$

In view of this, we propose to consider the action of other W's, too, as insertion of a "state" W_{jm} into the correlator:

$$W_{j_1, m_1} \cdots W_{j_r, m_r}F = \langle\langle W_{j_1, m_1} \cdots W_{j_r, m_r} \rangle\rangle . \tag{66}$$

Commutation relations (29) of our symmetries will then reproduce the $w_{1+\infty}$ Ward identities in the matrix model approach [13] (now in the presence of tachyon backgrounds).

What about "fields"? A set of fields $\phi_n(X)$ and $\bar{\phi}_n(X)$ are introduced by Hanany et al. [2] as $c = 1$ analogues of $c < 1$ chiral ring generators and gravitational descendents. In our interpretation of $(t, \bar{t})$ as background sources, $\phi_n(X)$ are given by

$$\phi_n(X) = -\frac{1}{n}\frac{\partial W(X)}{\partial t_n}, \quad \phi_{-n}(X) = \frac{1}{n}\frac{\partial W(X)}{\partial \bar{t}_n} \quad (n = 1, 2, \ldots) , \tag{67}$$

and $\bar{\phi}_n(X)$ by similar derivatives of $\bar{W}(X)$. Since the Landau–Ginzburg potentials are written in terms of $\mathscr{M}(\mathscr{L} \to X)$ and $\bar{\mathscr{M}}(\bar{\mathscr{L}} \to X^{-1})$ as shown in (55), these "fields" are exactly the same quantities as emerging on the right-hand side of (24), i.e., derivatives of the flow generators $\mathscr{B}_n$ and $\bar{\mathscr{B}}_n$ with respect to the Landau–Ginzburg field variable X. Note that this is parallel to the construction of chiral ring generators in the A_{k+1} models [4, 8, 9]. These "fields" are Landau–Ginzburg counterparts of tachyon "states" $W_{n/2,\pm n/2}$. To find other "fields," let us note that $\phi_n(X)$ can also be written

$$\phi_n(X) = X^{n-1} + \frac{1}{n}\delta_{\mathscr{L}^n,0}W(X), \quad \phi_{-n}(X) = -\frac{1}{n}\delta_{0,\bar{\mathscr{L}}^{-n}}W(X). \tag{68}$$

Here we have used (32), recalling the correspondence (55) between the Landau–Ginzburg potential and the Lax functions. The somewhat strange extra term X^{n-1} is due to the presence of tachyon backgrounds. Since the symmetries on the right-hand side are just $W_{n/2,\pm n/2}$, we are naturally led to conjecture that "fields" $\Phi_{jm}(X)$ corresponding to the "states" W_{jm} are to be given by

$$\Phi_{jm}(X) = W_{jm}W(X). \tag{69}$$

Similarly the action of W_{jm} on $\bar{W}(X)$ will give another set of extra "fields" $\bar{\Phi}_{jm}(X)$. In principle, one can find an explicit form of these "extra fields" from (32), though it will become considerably complicated in general. To push forward this speculation further, we will have to examine if the period integral representation of tachyon correlation functions and the contact algebra of $\phi_n(X)$ and $\bar{\phi}_n(X)$ [1, 2, 3] can be extended to our $\Phi_{jm}(X)$ and $\bar{\Phi}_{jm}(X)$.

8. Conclusion

Inspired by the suggestion of Hanany et al., we have considered the integrable structure of two-dimensional string theory at self-dual compactification radius. Our main conclusion is that the dispersionless Toda hierarchy is a very convenient framework for studying the tachyon sector of this theory. We have been able to identify a special solution of this integrable hierarchy in which full data of the tachyon dynamics is encoded. The $w_{1+\infty}$-constraints of tachyon correlation functions can be indeed reproduced from the construction (Riemann–Hilbert problem) of this solution. The Landau–Ginzburg formulation, too, turns out to be closely related to the Lax formalism of the dispersionless Toda hierarchy. Furthermore, we have pointed out the existence of a wedge algebra structure behind the Riemann–Hilbert problem, and proposed a speculative interpretation of this algebra as generators of "extra" states and fields in this model of two-dimensional strings. The last issue deserves to be pursued in more detail.

We conclude this paper with several remarks.

1) In the context of two-dimensional gravity, the dispersionless Toda hierarchy is zero-genus limit of the full Toda hierarchy. A full-genus analysis in the language of the Toda hierarchy is done by Dijkgraaf et al. [14]. We will be able to extend the results of this paper to that case.

2) As already mentioned, the integrable hierarchy underlying the topological D_ℓ models [12] resembles the dispersionless Toda hierarchy. This hierarchy is related to the Drinfeld–Sokolov hierarchy of D-type. It is intriguing that Danielsson [3] pointed out a link between a deformed Landau–Ginzburg model and the Drinfeld–Sokolov hierarchy of D-type.

3) Our method for solving a Riemann–Hilbert problem can be extended to more general cases such as:

$$\mathscr{L}^N = \bar{\mathscr{M}}\bar{\mathscr{L}}^{\bar{N}}/\bar{N}, \quad \bar{\mathscr{L}}^{-\bar{N}} = \mathscr{M}\mathscr{L}^{-N}/N, \tag{70}$$

where N and $\bar{N}$ are nonzero integers. In this paper, we have considered the simplest case, $N = \bar{N} = 1$; other cases, too, may have interesting physical interpretations. For instance, the work of Dijkgraaf et al. [14] implicitly shows that if the compactification radius (β in their notation) is a positive integer, the dynamics of tachyons in zero-genus limit can be described by the solution of (70) with $N = \bar{N} = \beta$. Thus we can deal with a discrete series of theories at non-self-dual ($\beta > 1$) radii in much the same way; a full genus analysis will become possible in the full Toda hierarchy.

4) Discrete states and quadratic Ward identities in the free field approach [15] are still beyond our scope. Our approach by the dispersionless Toda hierarchy is at most an effective theory in the tachyon sector, though we can anyhow reproduce the wedge algebra symmetries acting on tachyon states. Presumably, a suitable integrable extension of the dispersionless (or full) Toda hierarchy will provide a framework for dealing with this issue.

Acknowledgement. The author is very grateful to Hiroaki Kanno and Takashi Takebe for many useful comments. This work is partially supported by the Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan.

Note added in proof. Hiroaki Kanno informed the author that Tohru Eguchi independently arrived at the same Riemann–Hilbert relation as ours (35)

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Communicated by M. Jimbo